

An Observability Criterion for Dynamical Systems Governed by Riccati Differential Equations

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Abstract—In this paper we present an observability criterion for systems whose state is governed by a matrix Riccati differential equation and whose output is given by an affine transformation.

I. INTRODUCTION

Count J. Riccati, born in Venice in 1676, is known to be the first researcher who studied differential equations which included quadratic nonlinearities.

By the matrix Riccati differential equation (RDE), we understand the quadratic differential equation

$$\dot{X} = A_{2,1} + A_{2,2}X - XA_{1,1} - XA_{1,2}X$$

where X is an $m \times n$ matrix and where $A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}$ are matrices of appropriate size. There exists an impressive set of applications ranging from optimal control theory to H_∞ optimization and stochastic realization whose solution is governed by the solution of an RDE. We refer readers interested in this material to the recent special volume [1] on Riccati equations, where many different research directions are surveyed by experts in the field.

In this paper we will investigate the question of observability of the state $X(t)$ if the output $Y(t)$ is related to $X(t)$ through an affine transformation and more generally through a linear fractional transformation. In Theorem 2.1 we will provide a new sufficiency criterion, and in Theorem 2.2 we will provide a new necessary criterion. The proof of those results and several illustrative examples are given in the last section. In the last section we also establish the connection to the recent paper by Ghosh and the author [2] where the main technical result needed in this paper has been derived.

II. MAIN RESULTS

We will treat, in our presentation, the real and the complex situation simultaneously. For this, let \mathbb{K} be either the field of real ($\mathbb{K} = \mathbb{R}$) or the field of complex ($\mathbb{K} = \mathbb{C}$) numbers.

Let $A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}$ be matrices of size $n \times n, n \times m, m \times n$ and $m \times m$, respectively, and assume that the state variables $X(t)$ are given through a matrix RDE

$$\dot{X} = A_{2,1} + A_{2,2}X - XA_{1,1} - XA_{1,2}X. \quad (1)$$

Let C_1, C_2, C_3, C_4 be matrices of size $p \times n, p \times m, n \times n$, and $n \times m$, and assume that the output measurements of the systems parameters are given through the Möbius transformation

$$Y(t) = (C_1 + C_2X(t))(C_3 + C_4X(t))^{-1}. \quad (2)$$

We say that the state parameters $X(t)$ are observable from the output measurements $Y(t)$, if for any $0 \leq t_1 < t_2$ it is possible to compute $X(t)$ from the observation

$$Y(t), \quad t_1 < t < t_2.$$

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Note that the “read-out map” (2) is not as artificial as it might appear, since we include in particular the situation

$$Y(t) = C_1 + C_2X(t)$$

where C_1 and C_2 are arbitrary but fixed matrices. Moreover, the linear observability problem $\dot{x} = A_{2,2}x, y = C_2x$ is included as well, and it is one goal of this note to provide an algebraic criterion of observability for (1) similar to the well-known observability criterion of Popov, Belevitch, and Hautus [3].

In the sequel we derive a sufficient and a necessary condition for observability. For this let

$$A := \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}, \quad C := \begin{bmatrix} C_3 & C_4 \\ C_1 & C_2 \end{bmatrix}.$$

Theorem 2.1: If for all (real and complex) eigenvalues

$$\lambda_1, \dots, \lambda_{n+1} \text{ of } A$$

one has

$$\text{rank} \begin{bmatrix} (A - \lambda_1 I) \cdots (A - \lambda_{n+1} I) \\ C \end{bmatrix} = m + n \quad (3)$$

then $X(t)$ can be observed from $Y(t)$.

The proof will be given in the next section.

Then give a necessary condition: For this, let e_i be the i th standard basis vector of \mathbb{K}^{m+n} . We call a set

$$S := \{\beta_1, \dots, \beta_{n+1}\}$$

a cyclic set of generalized eigenvectors if the vectors in S are linearly independent, and if S has the property that if $\beta \in S$ is a generalized eigenvector of A with corresponding eigenvalue λ then β is an eigenvector or $(A - \lambda I)\beta \in S$.

Theorem 2.2: If the matrix A has a cyclic set of generalized eigenvectors

$$\{\beta_1, \dots, \beta_{n+1}\}$$

such that

$$\text{span} \{\beta_1, \dots, \beta_{n+1}, e_{n+1}, \dots, e_{m+n}\} = \mathbb{K}^{m+n}$$

and

$$\{C\beta_1, \dots, C\beta_{n+1}\}$$

are linearly dependent, then the state $X(t)$ cannot be observed from the output measurements $Y(t)$ as defined in (2).

We will give the proof again in the next section.

So far we just presented a sufficient and a necessary condition, and it is not quite clear how those conditions are related. The following lemma establishes this connection. The lemma essentially states that if the sufficient condition of Theorem 2.1 is not satisfied, then under certain “generic” conditions the necessary condition of Theorem 2.2 is satisfied, i.e., (1) is not observable.

Lemma 2.3: If there are eigenvalues $\lambda_1, \dots, \lambda_{n+1}$ of A such that

$$\text{rank} \begin{bmatrix} (A - \lambda_1 I) \cdots (A - \lambda_{n+1} I) \\ C \end{bmatrix} < m + n \quad (4)$$

then there exists a cyclic set of generalized eigenvectors

$$\{\beta_1, \dots, \beta_{n+1}\}$$

such that the vectors

$$\{C\beta_1, \dots, C\beta_{n+1}\}$$

are linearly dependent.

The proof of this Lemma is a direct consequence of [2, Proposition 11].

III. PROOFS AND EXAMPLES

It has been observed by many authors (see, e.g., [4]–[6]) that it is possible to extend the domain of the RDE from the vector space \mathbb{K}^{mn} to the Grassmann manifold $\text{Grass}(n, \mathbb{K}^{m+n})$ which parameterizes the set of n -dimensional subspaces in \mathbb{K}^{m+n} . For this, consider the embedding

$$\begin{aligned} \psi: \mathbb{K}^{mn} &\rightarrow \text{Grass}(n, \mathbb{K}^{m+n}) \\ X &\mapsto \text{colspan} \begin{bmatrix} I \\ X \end{bmatrix}. \end{aligned} \quad (5)$$

Let X_1, X_2 be matrices of size $n \times n$, respectively, $m \times n$, and consider the linear differential equation

$$\frac{d}{dt} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}. \quad (6)$$

Equation (6) describes the dynamics of an m -dimensional plane

$$P(t) = \text{colspan} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$$

in \mathbb{K}^{m+n} , i.e., it describes a flow on a Grassmann manifold. Moreover one immediately verifies that

$$X(t) := X_2(t)X_1(t)^{-1}$$

satisfies (1) as long as $X_1(t)$ is invertible and every solution of (1) arises in this way.

Because (6) extends the phase space of the RDE to the whole Grassmann manifold, we will call (6) the extended Riccati differential equation (ERDE).

As it is possible to extend the RDE to the Grassmann manifold, it is also possible to extend the observation map (2) to the Grassmann manifold. For this, consider the linear transformation

$$\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} C_3 & C_4 \\ C_1 & C_2 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = C \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}. \quad (7)$$

Restricting (7) to matrices X_1, X_2 having the property that $\text{rank} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = n$, we can view (7) as a map

$$\varphi: \text{Grass}(n, \mathbb{K}^{m+n}) - B \rightarrow \text{Grass}(n, \mathbb{K}^{n+p})$$

where the base locus B describes all n -planes in \mathbb{K}^{m+n} which are projected under C to a lower dimensional plane in \mathbb{K}^{n+p} . From the identity

$$\begin{aligned} Y_2 Y_1^{-1} &= (C_1 X_1 + C_2 X_2)(C_3 X_1 + C_4 X_2)^{-1} \\ &= (C_1 + C_2 X_2 X_1^{-1})(C_3 X_1 + C_4 X_2 X_1^{-1})^{-1} \end{aligned}$$

it immediately follows that

$$X(t) := X_2(t)X_1(t)^{-1}, \quad Y(t) := Y_2(t)Y_1(t)^{-1}$$

satisfy (2) whenever $X_1(t), Y_1(t)$ are both invertible and $X_1(t), X_2(t), Y_1(t), Y_2(t)$ satisfy (7).

After those preliminaries, we can now connect to the main result in [2]. For this, consider an n -dimensional plane $P(t) \in \text{Grass}(n, \mathbb{K}^{m+n})$ and matrices A, C as introduced earlier. We say $P(t)$ can be observed from the linear system

$$\frac{d}{dt} P(t) = AP(t), \quad Q(t) = CP(t) \quad (8)$$

if for any $0 \leq t_1 < t_2$, it is possible to calculate $P(0)$ from the observation of the ‘‘moving plane’’

$$Q(t) = CP(t) = Ce^{At}P(0)$$

in \mathbb{K}^{n+p} , $t_1 \leq t \leq t_2$. The main result which we need for our purposes was derived in [2] and reads as follows.

Theorem 3.1: System (8) observes any n -dimensional affine subspace $P(0)$ through the measurements $Q(t)$ in \mathbb{K}^{n+p} if for any set of eigenvalues $\lambda_1, \dots, \lambda_{n+1}$ of A one has

$$\text{rank} \begin{bmatrix} (A - \lambda_1 I) \cdots (A - \lambda_{n+1} I) \\ C \end{bmatrix} = m + n. \quad (9)$$

Moreover, this condition is also necessary if $n = 0$ or if all eigenvalues of the matrix A are in \mathbb{K} .

Combining the stated results we have:

Proof: Proof of Theorem 2.1: If (3) is satisfied, then the extended system defined through (6) and the observation map of (7) is observable by Theorem 3.1. In particular, the original system defined through (1) and (2) is observable as well. ■

Proof: Proof of Theorem 2.2: Let $P_0 := \text{span} \{\beta_1, \dots, \beta_{n+1}\}$, and let $P(t) := e^{At}P_0$ be the unique $n + 1$ -dimensional plane satisfying

$$\frac{d}{dt} P(t) = AP(t), \quad P(0) = P_0 \quad \forall t.$$

Since $\{C\beta_1, \dots, C\beta_{n+1}\}$ are linearly dependent, it follows that $CP_0 \subset \mathbb{K}^{n+p}$ has a dimension at most n . Without loss of generality we can assume that the dimension is n , and we leave the small additional argument needed for the case when the dimension is strictly less than n to the reader.

It therefore follows that almost any two n -dimensional subspaces $R_1, R_2 \subset P_0 \subset \mathbb{K}^{m+n}$ have the property that

$$Ce^{At}R_1 = Ce^{At}R_2 = Ce^{At}P_0 \stackrel{\perp}{=} CP_0$$

almost everywhere. System (6) with the extended read-out map of (7) is therefore not observable. Since

$$\text{span} \{P_0, e_{n+1}, \dots, e_{m+n}\} = \mathbb{K}^{m+n}$$

it follows that (1) cannot be observed from the output measurements of (2) either. ■

We illustrate the results through an example whose numbers have also been used in [2].

Example 3.2: Let X be the 2×2 matrix $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, and consider the RDE

$$\begin{aligned} \dot{X} &= \begin{bmatrix} 62 & 43 \\ 203 & 138 \end{bmatrix} + \begin{bmatrix} -46 & 9 \\ -149 & 31 \end{bmatrix} X \\ &\quad - X \begin{bmatrix} -81 & -56 \\ 146 & 102 \end{bmatrix} - X \begin{bmatrix} 57 & -11 \\ -106 & 20 \end{bmatrix} X. \end{aligned}$$

In the following, we will consider two seemingly similar output observation maps. As it turns out, only one of the maps has the observability property.

1) First assume that the output is given through

$$Y(t) = (0, 1)X(t) = (x_3(t), x_4(t)).$$

In this case

$$\begin{aligned} A &:= \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \\ &= \begin{bmatrix} -81 & -56 & 57 & -11 \\ 146 & 102 & -106 & 20 \\ 62 & 43 & -46 & 9 \\ 203 & 138 & -149 & 31 \end{bmatrix} \end{aligned}$$

and

$$C := \begin{bmatrix} C_3 & C_4 \\ C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

One immediately verifies that the eigenvalues of A are 0, 1, 2, 3, and that for all set of eigenvalues $\lambda_1, \lambda_2, \lambda_3$ one has

$$\text{rank} \begin{bmatrix} (A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I) \\ C \end{bmatrix} = 4.$$

It therefore follows from Theorem 3.1 that the extended RDE is observable, and by Theorem 2.1 it is also possible to compute $X(t)$ from the measurements $x_3(t), x_4(t)$.

2) Now assume that the output is given through

$$Y(t) = (1, 0)X(t) = (x_1(t), x_2(t)).$$

In this case, A is still the same matrix and

$$C := \begin{bmatrix} C_3 & C_4 \\ C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Since

$$A(A - I)(A - 2I) = \begin{bmatrix} -3 & -3 & 3 & 0 \\ 8 & 8 & -8 & 0 \\ -1 & -1 & 1 & 0 \\ -23 & -23 & 23 & 0 \end{bmatrix}$$

it is clear that (3) introduced in Theorem 2.1 is not satisfied for $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$. By Lemma 2.3, there exists a cyclic set of (generalized) eigenvectors $\{\beta_1, \beta_2, \beta_3\}$ having the property that $\{C\beta_1, C\beta_2, C\beta_3\}$ are linearly dependent. In our situation we can choose $\beta_1, \beta_2, \beta_3$ as eigenvectors of A with corresponding eigenvalues 0, 1, 2. Indeed, define

$$P_0 := \text{span} \{ \beta_1, \beta_2, \beta_3 \} \\ := \text{span} \left\{ \begin{bmatrix} -12 \\ 20 \\ 8 \\ 28 \end{bmatrix}, \begin{bmatrix} 35 \\ -60 \\ -25 \\ -85 \end{bmatrix}, \begin{bmatrix} -23 \\ 40 \\ 17 \\ 58 \end{bmatrix} \right\}$$

and let

$$R_1 := \text{colspan} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad R_2 := \text{colspan} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & \frac{13}{5} \end{bmatrix}.$$

One immediately verifies that $R_1, R_2 \subset P_0 \subset \mathbb{K}^4$ and that

$$C e^{At} R_1 = C e^{At} R_2 = C e^{At} P_0 = C P_0 \\ = \text{colspan} \begin{bmatrix} -12 & 35 \\ 20 & -60 \\ 8 & -25 \end{bmatrix}.$$

We conclude that the two different initial conditions

$$X(0) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \tilde{X}(0) = \begin{bmatrix} 1 & 1 \\ 2 & \frac{13}{5} \end{bmatrix}$$

result in the same output measurements $x_1(t), x_2(t)$ for all time t where the trajectory $X(t)$ is defined.

IV. CONCLUSION

This paper has studied the observability question of systems governed by RDE's. By extending the phase space of the differential equation to the Grassmann manifold, the author arrived at a necessary and at a sufficient observability criterion.

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Affine Parameter-Dependent Lyapunov Functions and Real Parametric Uncertainty

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Abstract—This paper presents new tests to analyze the robust stability and/or performance of linear systems with uncertain real parameters. These tests are extensions of the notions of quadratic stability and performance where the fixed quadratic Lyapunov function is replaced by a Lyapunov function with affine dependence on the uncertain parameters. Admittedly with some conservatism, the construction of such parameter-dependent Lyapunov functions can be reduced to a linear matrix inequality (LMI) problem and hence is numerically tractable.

These LMI-based tests are applicable to constant or time-varying uncertain parameters and are less conservative than quadratic stability in the case of slow parametric variations. They also avoid the frequency sweep needed in real- μ analysis, and numerical experiments indicate that they often compare favorably with μ analysis for time-invariant parameter uncertainty.

I. INTRODUCTION

When designing control systems, it is often desirable to obtain guarantees of stability and performance against uncertainty on the physical parameters of the system. Examples of physical parameters include stiffness, inertia, or viscosity coefficients in mechanical systems, aerodynamical coefficients in flight control, the values of

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