East-West J. Numer. Math., Vol. 7, No. 1, pp. 61–77 (1999)

Extension operators and approximation on domains containing small geometric details

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Received September 8, 1998
Received in revised form December 15, 1998

Abstract — In [6] a new class of finite elements has been introduced for the coarse-scale discretization of partial differential equations on complicated domains. In contrast to standard finite element discretizations, the minimal dimension of these so-called *composite finite elements* is not linked to the number and size of the geometric details. The approximation property of these finite element spaces can be proved in an analogous fashion as for classical finite elements while the constant in these estimates depend on the norm of the minimal extension operator.

This motivates the study of the dependence of the norm of extension operators on geometric details as e.g. holes and rough boundaries. We also consider the case of so-called cuspoidal domains and prove the existence of extension operators in weighted Sobolev spaces.

Keywords: discretization of PDEs, composite finite elements, extension operator, geometric details.

AMS(MOS) subject classification: 46E35, 65N30.

In many practical applications as e.g. in environmental modelling or simulation of complicated engines, partial differential equations on complicated domains have to be solved numerically. Such problems are usually discretized via the finite element method since, in contrast to finite difference methods, the use of non-uniform and adaptive meshes is straightforward. However, the condition that a finite element mesh has to resolve the boundary of the domain links the minimal dimension of finite elements spaces directly to the number and size of geometric details contained in the domain. On the other hand, the efficiency of many fast solution techniques as e.g. multi-grid, extrapolation, wavelets, etc. depends on a multi-scale discretization of the problem: Features of the solution which are visible also on coarse scales should be computed cheaply on coarse scales.

In [4-7] *composite finite elements* have been introduced for coarse-level discretizations of PDEs on complicated domains. It was proved that the approximation property holds in an analogous way as for standard finite elements. The proof makes use of the existence of appropriate extension operators of functions in Sobolev spaces. Hence, a key problem is the investigation of the dependence of extension operators on e.g. small geometric details or the number of details. In [11], it was proved that, for domains containing arbitrary many holes, there exist extension operators for Sobolev spaces, where the operator norm is bounded independently of the size and number of the holes as long as the holes satisfy an appropriate separation condition. In this paper, we generalize these results to more general domains and study also the situation where the separation condition is violated.

Furthermore, we also consider domains containing cusps and prove the existence of extension operators for weighted Sobolev spaces.

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Our approach generalizes the technique developed in [10] for the extension of functions in the Sobolev space H^1 on domains with periodically distributed holes to more general domains and to function spaces H^m for arbitrary $m \in \mathbb{N}_0$.

The paper is organized as follows.

In Section 1 we introduce necessary notations and specify the class of domains we are dealing with. Furthermore, a simple finite element space is defined allowing coarse-scale discretizations of partial differential equations.

In Subsection 2.1 we introduce an extension operator for functions in H^m which has the property that the seminorms of the extended functions can be bounded by the seminorms of the original function.

Using these results we will prove in Subsection 2.2 that, for domains which consist of subdomains which are locally scaled images of "nice" domains and satisfy an appropriate separation condition, there exists an extension operator which is independent of the size and number of geometric details.

In Subsection 2.3 we consider the case that the separation condition is violated. We will obtain a quantitative estimate describing the growth of the norm of the minimal extension operator* with decreasing distance of two holes.

In Section 3 we will consider cuspoidal domains and prove that there exist extension operators for functions in weighted Sobolev spaces.

1. PRELIMINARIES

In this Section we will introduce necessary notations and collect well-known properties about classical extension operators. Furthermore, we define a simple finite element space for coarse-scale discretizations of boundary value problems on complicated domains.

1.1. Lipschitz continuous domains

Throughout the paper, $\Omega \subset \mathbb{R}^d$ denotes a bounded open set. Mostly, we will assume that Ω is a bounded domain or a Lipschitz domain. The precise definition is given below.

Definition 1.1. A domain is an open, connected subset of $\mathbb{R}^d$.

Definition 1.2. A subset $A \subset \mathbb{R}^d$ is a *special Lipschitz domain* if there exists an orthogonal mapping Φ and a function $\varphi : \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}$ satisfying the Lipschitz condition:

$$|\varphi(x) - \varphi(x')| \leq M |x - x'| \quad \text{for all } x, x' \in \mathbb{R}^{d-1}$$

and

$$A = \Phi\left(\left\{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : \; y > \varphi(x)\right\}\right).$$

For $x \in \mathbb{R}^m$, the m-dimensional ball with radius s about x is denoted by $B_s^m(x)$ or just by $B_s(x)$.

Definition 1.3. A bounded domain Ω is a *Lipschitz domain* if, for all $x \in \partial\Omega$, there is a neighborhood U_x of x and a special Lipschitz domain A_x such that

$$U_x \cap \Omega = U_x \cap A_x.$$

*Here and in the following the term "minimal extension operator" stands for "extension operator with minimal norm".

1.2. Sobolev spaces

Throughout this Subsection we assume that Ω is a bounded open set. Let $C^\infty(\Omega)$ denote the space of all functions having infinitely many derivatives. The spaces $C^\infty\left(\overline{\Omega}\right)$ and $C_0^\infty(\Omega)$ are defined by

$$C^\infty\left(\overline{\Omega}\right) := \left\{f|_\Omega : f \in C^\infty\left(\mathbb{R}^d\right)\right\}$$
$$C_0^\infty(\Omega) := \left\{f \in C^\infty\left(\mathbb{R}^d\right) : \operatorname{supp} f \subset \Omega \text{ is compact}\right\} .$$

The space of functions defined on Ω with bounded L^2-norm:

$$\|f\|_{L^2(\Omega)} := \left\{\int_\Omega |f(x)|^2 \,\mathrm{d}x\right\}^{1/2}$$

is denoted by $L^2(\Omega)$. For $k \in \mathbb{N}$, the Sobolev space $H^k(\Omega)$ is defined by:

$$H^k(\Omega) := \left\{f \in L^2(\Omega) : \forall \alpha \in \mathbb{N}_0^d, |\alpha| \le k : \partial^\alpha f \in L^2(\Omega)\right\}.$$

The space $H^k(\Omega)$ is a Hilbert space with scalar product

$$(f, g)_{H^k(\Omega)} := \sum_{|\alpha| \le k} \int_\Omega (\partial^\alpha f)(\partial^\alpha g)\,\mathrm{d}x$$

and norm

$$\|f\|_{H^k(\Omega)} := (f, f)_{H^k(\Omega)}^{1/2} . \tag{1.1}$$

For latter purpose, we define the bilinear form $(\cdot,\cdot)_{=k,\Omega}$ by

$$(f, g)_{=k,\Omega} := \sum_{|\alpha| = k} \int_\Omega (\partial^\alpha f)(\partial^\alpha g)\,\mathrm{d}x$$

and the seminorm $|\cdot|_{H^k(\Omega)}$ by

$$|f|_{H^k(\Omega)} := (f, f)_{=k,\Omega}^{1/2} .$$

For Lipschitz domains Ω the space $H^k(\Omega)$ can be defined in an equivalent way by using a simpler norm as in (1.1).

Definition 1.4. Let Ω be a bounded open set. For $k \in \mathbb{N}$, let

$$V^k(\Omega) := \left\{f \in L^2(\Omega) : \forall \alpha \in \mathbb{N}_0^d, |\alpha| = k : \partial^\alpha f \in L^2(\Omega)\right\} .$$

The norm on $V^k(\Omega)$ is defined by:

$$|||f|||_{H^k(\Omega)} = \left\{\|f\|_{L^2(\Omega)}^2 + |f|_{H^k(\Omega)}^2\right\}^{1/2} . \tag{1.2}$$

Theorem 1.1. *Let Ω be a bounded Lipschitz domain. Then, $H^k(\Omega) = V^k(\Omega)$ and the norms $\|\cdot\|_{H^k(\Omega)}$ and $|||\cdot|||_{H^k(\Omega)}$ are equivalent, i.e. there exists a constant $K > 0$ such that*

$$|||u|||_{H^k(\Omega)} \le \|u\|_{H^k(\Omega)} \le K|||u|||_{H^k(\Omega)} \quad \text{for all } u \in H^k(\Omega) .$$

Proof. See [8, 1.1.11]. □

Finally, we will introduce the Sobolev spaces which vanish on a certain part of the boundary. For this purpose, let Ω denote a bounded set and $\Gamma \subset \partial\Omega$ denote a closed subset with bounded $(d-1)$-dimensional measure. Then, we define :

$$C_{\Gamma,k}^\infty(\Omega) := \{f \in C^\infty(\Omega) : \forall |\alpha| < k : \partial^\alpha f = 0 \text{ in a neighborhood of } \Gamma\} .$$

The Sobolev space $H_\Gamma^k(\Omega)$ is then obtained as the closure of $C_{\Gamma,k}^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^k(\Omega)}$. If $\Gamma = \partial\Omega$, we write $H_0^k(\Omega)$ instead of $H_\Gamma^k(\Omega)$.

1.3. A simple finite element space

In this Subsection a simple finite element space for the discretization of PDEs on complicated domains will be introduced. This space can be regarded as a course scale generalization of classical finite element spaces where the condition that the finite element mesh has to resolve the boundary is relaxed. The relation to composite finite element spaces which have been introduced in [4-7] is explained at the end of this Subsection.

To avoid technicalities we restrict here to two-dimensional problems and triangulations while more general situations can be treated in the same way.

Definition 1.5. The union $\mathcal{G} = \{\tau_1, \tau_2, \ldots, \tau_n\}$ of triangles τ_i is a triangulation if, for all $\tau, t \in \mathcal{G}$, either $\tau = t$ holds or the intersection $\overline{\tau} \cap \overline{t}$ is either empty, a common vertex, or a common edge.

The domain covered by a finite element mesh is denoted by $\Omega_{\mathcal{G}} := \cup\mathcal{G}$ and the set of vertices of $\mathcal{G}$ by $\Theta_{\mathcal{G}}$. The step size $h_{\mathcal{G}}$ is given by $h_{\mathcal{G}} := \max_{\tau\in\mathcal{G}} h_\tau$ with $h_\tau := \operatorname{diam}(\tau)$.

For the approximation property of finite element spaces the quality of the mesh plays an important role.

Definition 1.6. Let $\mathcal{G}$ denote a triangulation. $\mathcal{G}$ has quality $C_r > 0$ if, for all $\tau \in \mathcal{G}$:

$$\rho_\tau / h_\tau \geq C_r$$

holds where ρ_τ denotes the diameter of the maximal ball contained in τ.

On triangulations, continuous, piecewise linear finite element spaces can be defined.

Definition 1.7. Let $\mathcal{G}$ be a finite element mesh. The space of continuous, piecewise linear finite elements is given by

$$S_{\mathcal{G}} := \left\{ u \in C^0(\Omega_{\mathcal{G}}) : \forall \tau \in \mathcal{G} : \ u|_\tau \text{ is affine} \right\}.$$

Note that, for the definition of $S_{\mathcal{G}}$, the domain Ω was never used. For classical finite elements, the condition

$$\Omega = \Omega_{\mathcal{G}}$$

usually appears. This condition links the minimal possible dimension of $S_{\mathcal{G}}$ to the number and size of geometric details. For the definition of the new finite element space for coarse-scale discretizations, this condition is replaced by an overlap condition.

Definition 1.8. Let Ω be a domain. Let $\mathcal{G}$ be a triangulation satisfying the overlap condition

$$\Omega \subset \Omega_{\mathcal{G}}.$$

Then, $S_{\mathcal{G}}^{\Omega}$ is given by

$$S_{\mathcal{G}}^{\Omega} := S_{\mathcal{G}}|_\Omega := \left\{ u \in C^0(\Omega) : \ \exists u^\star \in S_{\mathcal{G}} : \ u = u^\star|_\Omega \right\}.$$

Composite finite elements are defined in [6], [12], and [7]. They can be regarded as perturbations of the simple finite element space $S_{\mathcal{G}}^{\Omega}$. This modification is motivated by practical reasons: composite finite elements are defined on a hierarchy of (refined) finite element grids overlapping the domain. They have the property that the spaces are nested and an intergrid transfer operator can be defined which is stable in the H^1-norm. We do not go into these details here but turn now to the proof of the approximation property. We state that the proof of the approximation property of composite finite elements is based on the approximation property of $S_{\mathcal{G}}^{\Omega}$ and a perturbation argument which is independent of the physical domain Ω.

Theorem 1.2. *Let Ω be a domain, sufficiently smooth such that a continuous extension operator $\mathfrak{E}: H^2(\Omega) \to H^2(\mathbb{R}^d)$ exists. Let $\mathcal{G}$ denote a triangulation with quality C_τ. For $u \in H^2(\Omega)$, there exists $u_{\mathcal{G}} \in S_{\mathcal{G}}^{\Omega}$ so that, for $m = 0, 1$:*

$$\|u - u_{\mathcal{G}}\|_{H^m(\Omega)} \leq C h_{\mathcal{G}}^{2-m} \|u\|_{H^2(\Omega)}$$

is satisfied where C only depends on C_τ and the norm of the minimal extension operator:

$$C_{\mathfrak{e}} := \sup_{\substack{u \in H^2(\Omega) \\ \|u\|_{H^2(\Omega)} = 1}} \inf_{\substack{u^\star \in H^2(\mathbf{R}^2) \\ u^\star|_\Omega = u}} \|u^\star\|_{H^2(\mathbf{R}^2)} . \tag{1.3}$$

Proof. Let $u \in H^2(\Omega)$ and $u^\star := \mathfrak{E}u$. From the standard approximation property for finite element functions (see [3, Theorem 3.2.1]) it is well known that there exists a function $u_{\mathcal{G}}^\star \in S_{\mathcal{G}}$ satisfying

$$\left\|u^\star - u_{\mathcal{G}}^\star\right\|_{H^m(\Omega_{\mathcal{G}})} \leq C h_{\mathcal{G}}^{2-m} \|u^\star\|_{H^2(\Omega_{\mathcal{G}})} .$$

Let $u_{\mathcal{G}} := u_{\mathcal{G}}^\star|_\Omega \in S_{\mathcal{G}}^{\Omega}$ and $C_{\mathfrak{e}}$ as in (1.3). Then:

$$\|u - u_{\mathcal{G}}\|_{H^m(\Omega)} \leq \left\|u^\star - u_{\mathcal{G}}^\star\right\|_{H^m(\Omega_{\mathcal{G}})} \leq C h_{\mathcal{G}}^{2-m} \|u^\star\|_{H^2(\Omega_{\mathcal{G}})} \leq C C_{\mathfrak{e}} h_{\mathcal{G}}^{2-m} \|u\|_{H^2(\Omega)} .$$

□

From this proof it is clear that the dependence of the norm of the minimal extension operator on the geometric properties of the domain plays an important role for the approximation property of the space $S_{\mathcal{G}}^{\Omega}$ and also for composite finite element spaces. This motivates the study of the dependence of this norm on various parameters characterizing the domain Ω.

2. EXTENSION OPERATORS FOR DOMAINS CONTAINING SMALL GEOMETRIC DETAILS

In many practical applications (e.g. environmental modelling or composite materials) the physical domain contains geometric details. These details might be holes of various size as in porous media, or rough boundaries as e.g. the shore of a lake, or thin but long holes appearing when thin wires are imbedded in isolating materials. In this Section we will discuss the existence of extension operators for such domains and the dependence on various parameters.

Let Ω be a Lipschitz domain. In [13], for all $k \in \mathbb{N}$, an extension operator

$$\mathfrak{E}_0 : H^k(\Omega) \to H^k(\mathbb{R}^d) \tag{2.1}$$

is constructed explicitly satisfying

$$\|\mathfrak{E}_0 u\|_{H^k(\mathbb{R}^d)} \leq C_0 \|u\|_{H^k(\Omega)} \quad \text{for all } u \in H^k(\Omega)$$

where C_0 depends on k but is independent of u. However, it turns out that C_0 depends critically on various parameters describing the domain, e.g. the size of holes. Hence, we will modify this extension operator for different situations considered below.

2.1. A first extension operator

In this Subsection we will describe the basic extension operator which will be used on scaled domains later on. We study the local situation of one geometric detail.

Assumption 2.1. Let $k \in \mathbb{N}$. Let $\Omega, \omega, \Omega^\star \subset \mathbb{R}^d$ be bounded domains satisfying

- $\Omega, \omega \subset \Omega^\star$,
- $\Omega = \Omega^\star \setminus \overline{\omega}$ is a Lipschitz domain and
- $\operatorname{meas}(\Gamma) \neq 0$ for $\Gamma := \partial\omega \cap \Omega^\star$.

Theorem 2.1. *Let Ω and $\Omega^\star$ be as in Assumption rm2.1 and $k \in \mathbb{N}$. Then there exists an extension operator $\mathfrak{E}_\Omega : H^k(\Omega) \to H^k(\Omega^\star)$ (depending on k) satisfying*

$$\begin{aligned} \|\mathfrak{E}_\Omega u\|_{H^k(\Omega^\star)} &\leq C_1 \|u\|_{H^k(\Omega)} \qquad \text{for all } u \in H^k(\Omega) \\ |\mathfrak{E}_\Omega u|_{H^k(\Omega^\star)} &\leq C_2 |u|_{H^k(\Omega)} \qquad \text{for all } u \in H^k(\Omega) \end{aligned} \tag{2.2}$$

where C_1, C_2 depend on k but are independent of u.

Proof. Let $u \in H^k(\Omega)$. Since Ω is a Lipschitz domain $u_0 := \mathfrak{E}_0(u)|_{\Omega^\star}$ is well defined. From Friedrichs' inequality (see [9, Théorème 1.9]) and, by induction, one concludes that $(\cdot,\cdot)_{=k,\omega}$ is a scalar product on $H^k_\Gamma(\omega)$. Hence, by Riesz' representation theorem it follows that the problem: find $z \in H^k_\Gamma(\omega)$ such that

$$(z, v)_{=k,\omega} = (u_0, v)_{=k,\omega} \qquad \text{for all } v \in H^k_\Gamma(\omega) \tag{2.3}$$

is uniquely solvable. This defines the projection $P : H^k(\omega) \to H^k_\Gamma(\omega)$ by

$$P(u_0|_\omega) := z\,.$$

The extension operator $\mathfrak{E}_\Omega$ is defined by:

$$\mathfrak{E}_\Omega(u)(x) := \begin{cases} u(x), & x \in \Omega \\ u_0(x) - P(u_0|_\omega)(x), & x \in \omega\,. \end{cases}$$

In the first step we show that $\mathfrak{E}_\Omega$ is continuous. Let $u_\omega := u_0|_\omega$. Using Cauchy-Schwarz' inequality one derives:

$$\begin{aligned} |P(u_\omega)|^2_{H^k(\omega)} &= (P(u_\omega), P(u_\omega))_{=k,\omega} = (u_0, P(u_\omega))_{=k,\omega} \\ &= (\mathfrak{E}_0(u), P(u_\omega))_{=k,\omega} \leq |\mathfrak{E}_0(u)|_{H^k(\omega)} |P(u_\omega)|_{H^k(\omega)}\,. \end{aligned}$$

Using Friedrichs' inequality again, one obtains

$$\|P(u_\omega)\|_{H^k(\omega)} \leq \eta \|\mathfrak{E}_0(u)\|_{H^k(\omega)}$$

where η only depends on k and ω. Let $C_0 = C_0(\Omega)$ denote the norm of the extension operator $\mathfrak{E}_0$ as in (2.1). The continuity of $\mathfrak{E}_\Omega$ follows from

$$\begin{aligned} \|\mathfrak{E}_\Omega(u)\|^2_{H^k(\Omega^\star)} &= \|u\|^2_{H^k(\Omega)} + \|u_0 - P(u_\omega)\|^2_{H^k(\omega)} \\ &\leq \|u\|^2_{H^k(\Omega)} + \left(\|u_0\|_{H^k(\omega)} + \|P(u_\omega)\|_{H^k(\omega)}\right)^2 \\ &\leq \|u\|^2_{H^k(\Omega)} + \left(\|\mathfrak{E}_0(u)\|_{H^k(\Omega^\star)} + \eta\,\|\mathfrak{E}_0(u)\|_{H^k(\omega)}\right)^2 \\ &\leq \|u\|^2_{H^k(\Omega)} + (1+\eta)^2 C_0^2 \|u\|^2_{H^k(\Omega)} \leq C_1^2 \|u\|^2_{H^k(\Omega)} \end{aligned} \tag{2.4}$$

where C_1 depends on the constant of Friedrichs' inequality and the norm of the extension operator $\mathfrak{E}_0$.

Now we estimate the seminorm separately. Let $\mathbb{P}_{k-1}$ denote the space of all polynomials in d variables of total degree $\leq k-1$. From the generalized Poincaré inequality (see [9, Théorème 1.5]) one concludes that, for all $u \in H^k(\Omega)$, there exists a polynomial $p \in \mathbb{P}_{k-1}$ such that

$$\|u - p\|_{H^k(\Omega)} \leq C\, |u|_{H^k(\Omega)}\,. \tag{2.5}$$

In Lemma 2.1 it will be proved that:

$$\mathfrak{E}_\Omega(p) = p \quad \text{for all } p \in \mathbb{P}_{k-1}\,. \tag{2.6}$$

Hence, from (2.4), (2.5), and (2.6) it follows that:

$$\begin{aligned} |\mathfrak{E}_\Omega(u)|_{H^k(\Omega^\star)} &= |\mathfrak{E}_\Omega(u) - p|_{H^k(\Omega^\star)} = |\mathfrak{E}_\Omega(u-p)|_{H^k(\Omega^\star)} \\ &\leq \|\mathfrak{E}_\Omega(u-p)\|_{H^k(\Omega^\star)} \leq C_1 \|u-p\|_{H^k(\Omega)} \leq CC_1\, |u|_{H^k(\Omega)}\,. \end{aligned}$$

□

It remains to prove (2.6).

Lemma 2.1. *Let k be as in the previous theorem. Let $p \in \mathbb{P}_{k-1}$. Then $\mathfrak{E}_\Omega(p) = p$.*

Proof. For all $v \in H_\Gamma^k(\omega)$:

$$(p - \mathfrak{E}_0(p), v)_{=k,\omega} = (p, v)_{=k,\omega} - (\mathfrak{E}_0(p), v)_{=k,\omega} = -(\mathfrak{E}_0(p), v)_{=k,\omega}\,. \tag{2.7}$$

Formulae (2.3) and (2.7) imply

$$(P(p), v)_{=k,\omega} = (\mathfrak{E}_0(p), v)_{=k,\omega} = -(p - \mathfrak{E}_0(p), v)_{=k,\omega}\,.$$

From $p - \mathfrak{E}_0(p) = 0$ on Ω one derives $p - \mathfrak{E}_0(p) \in H_\Gamma^k(\omega)$ and, hence, $P(p) = \mathfrak{E}_0(p) - p$. From $\mathfrak{E}_\Omega(p) = \mathfrak{E}_0(p) - P(p) = p$ on ω the assertion follows. □

2.2. Locally scaled domains

A scaling argument implies that the extension operator $\mathfrak{E}_\Omega$ is independent of the size of the geometric detail ω. In this light we introduce, for bounded domains $a, A \subset \mathbb{R}^d$, the scaling $\chi_a : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\chi_a(x) = \frac{x}{\operatorname{diam}(a)}$$

and the scaled domain $A_a := \chi_a(A)$.

Remark 2.1. Let Ω, ω and $\Omega^\star$ be as in Assumption 2.1. Then

$$\mathfrak{E}_\Omega(u) = \left(\mathfrak{E}_{\Omega_\omega}\left(u \circ \chi_\omega^{-1}\right)\right) \circ \chi_\omega\,.$$

Lemma 2.2. *Let C_1 denote the operator norm of $\mathfrak{E}_{\Omega_\omega}$ and C_2 the bound of the H^k-seminorm estimate for $\mathfrak{E}_{\Omega_\omega}$ (cf. Theorem 2.1). Then $\mathfrak{E}_\Omega$ satisfies:*

$$|||\mathfrak{E}_\Omega(u)|||_{H^k(\Omega^\star)} \leq \left(C_2 + \left(1 + \operatorname{diam}(\omega)^k\right) C_1\right) \|u\|_{H^k(\Omega)} \quad \textit{for all } u \in H^k(\Omega)\,.$$

Remark 2.2. The norm $|||\cdot|||$ was defined in (1.2). The result of Lemma 2.2 has an auxiliary character. It is used in Theorem 2.2 to show that the "usual" operator norm of an extension operator is independent of the size and number of geometric details.

Remark 2.3. The result of Lemma 2.2 is interesting especially for small geometric details: $\operatorname{diam}(\omega) \leq \varepsilon_0$. Then, the operator norm of $\mathfrak{E}_\Omega$ can be bounded by a constant depending only on ε_0 and the norm of the operator $\mathfrak{E}_{\Omega_\omega}$ on the scaled domain.

Proof of Lemma 2.2. Let $m \leq k$ be an integer, $u \in H^k(\Omega)$ and $\varepsilon := \operatorname{diam}(\omega)$. The transformation rule for integrals implies

$$|\mathfrak{E}_\Omega(u)|_{H^m(\Omega^\star)} = \varepsilon^{d/2-m} \left|\mathfrak{E}_{\Omega_\omega}\left(u \circ \chi_\omega^{-1}\right)\right|_{H^m(\Omega_\omega^\star)}$$

and, vice versa,

$$\left|u \circ \chi_\omega^{-1}\right|_{H^m(\Omega_\omega)} = \varepsilon^{m-d/2} |u|_{H^m(\Omega)} .$$

Using the assumption that the norm of $\mathfrak{E}_{\Omega_\omega}$ is bounded by C_1, the L^2-norm can be estimated by:

$$\begin{aligned}\|\mathfrak{E}_\Omega u\|^2_{L^2(\Omega^\star)} &= \varepsilon^d \|\mathfrak{E}_{\Omega_\omega}(u \circ \chi_\omega^{-1})\|^2_{L^2(\Omega_\omega^\star)} \leq \varepsilon^d \|\mathfrak{E}_{\Omega_\omega}(u \circ \chi_\omega^{-1})\|^2_{H^k(\Omega_\omega^\star)} \\ &\leq \varepsilon^d C_1^2 \|u \circ \chi_\omega^{-1}\|^2_{H^k(\Omega_\omega)} = \varepsilon^d C_1^2 \textstyle\sum_{m=0}^k |u \circ \chi_\omega^{-1}|^2_{H^m(\Omega_\omega)} \\ &\leq C_1^2 \left(1+\varepsilon^k\right)^2 \textstyle\sum_{m=0}^k |u|^2_{H^m(\Omega)} = C_1^2\left(1+\varepsilon^k\right)^2 \|u\|^2_{H^k(\Omega)} .\end{aligned}$$

The H^k-seminorm is estimated by:

$$|\mathfrak{E}_\Omega u|^2_{H^k(\Omega^\star)} = \varepsilon^{d-2k} \left|\mathfrak{E}_{\Omega_\omega}\left(u \circ \chi_\omega^{-1}\right)\right|^2_{H^k(\Omega_\omega^\star)} \leq \varepsilon^{d-2k} C_2^2 \left|u \circ \chi_\omega^{-1}\right|^2_{H^k(\Omega_\omega)} = C_2^2 |u|^2_{H^k(\Omega)} .$$

□

Using the results of Theorem 2.1 and Lemma 2.2, the norm of extension operators on domains containing arbitrary many geometric details of various size can be estimated.

Assumption 2.2. Let $k \in \mathbb{N}$ and $N \subset \mathbb{N}$. $\Omega, \Omega^\star \subset \mathbb{R}^d$ are bounded domains and $(\omega_i)_{i\in N}$ is a family of domains (geometric details) satisfying $\omega_i \subset \Omega^\star$ and $\overline{\omega}_i \cap \overline{\omega}_j = \emptyset$ for all $i, j \in N$, $i \neq j$, such that

$$\Omega = \Omega^\star \setminus \bigcup_{i\in N} \overline{\omega}_i .$$

Furthermore, there exist constants $C_1^\star, C_2^\star > 0$ and a family of disjoint domains (neighborhoods) $(\Omega_i^\star)_{i\in N}$ satisfying, for all $i \in N$, the following conditions (see Fig. 1).

1. $\omega_i \subset \Omega_i^\star \subset \Omega^\star$.
2. $\Omega_i := \Omega_i^\star \setminus \overline{\omega}_i$ is a Lipschitz domain.
3. $\operatorname{meas}(\Gamma_i) \neq 0$ with $\Gamma_i := \partial\omega_i \cap \Omega_i^\star$.
4. For $i \in N$, the norm $C_{1,i}$ of the extension operator of Theorem 2.1 applied to the scaled domain $\Omega_{i,\omega_i} = \chi_{\omega_i}(\Omega_i)$:

$$\mathfrak{E}_{\Omega_{i,\omega_i}} : H^k\left(\Omega_{i,\omega_i}\right) \to H^k\left(\Omega^\star_{i,\omega_i}\right)$$

is bounded by $C_1^\star$ while the constant in the estimate for the H^k-seminorm for $\mathfrak{E}_{\Omega_{i,\omega_i}}$ (cf. (2.2)) is bounded by $C_2^\star$.

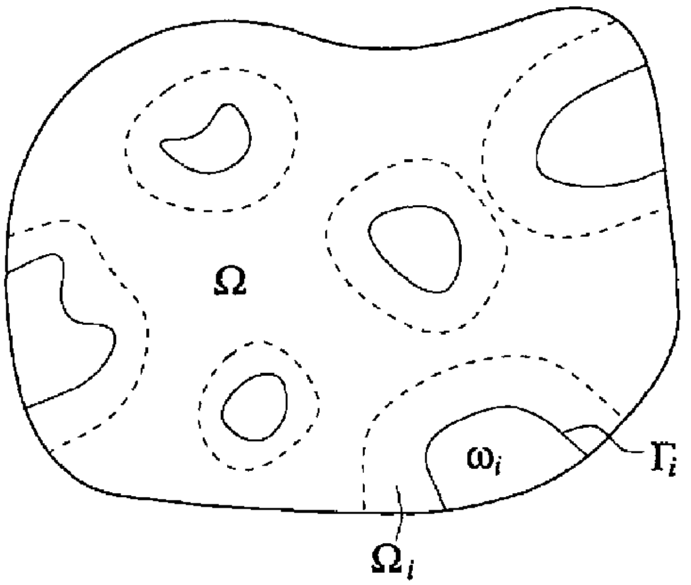

Figure 1. The notation.

Under these assumptions one can proof the existence of an extension operator $\mathfrak{E}_1 : H^k(\Omega) \to H^k(\Omega^\star)$.

Theorem 2.2. *Let Assumption 2.2 be satisfied. Then, there exists an extension operator $\mathfrak{E}_1 : H^k(\Omega) \to H^k(\Omega^\star)$ where the operator norm is bounded by a constant depending only on the domain $\Omega^\star$ and the scaled domains Ω_{i,ω_i}, more precisely, on $C_1^\star$ and $C_2^\star$ and the maximal diameter of the holes ω_i.*

Proof. Let $u \in H^k(\Omega)$. Applying $\mathfrak{E}_{\Omega_i} : H^k(\Omega_i) \to H^k(\Omega_i^\star)$ to all subdomains Ω_i results in an extension operator $\mathfrak{E}_1 : H^k(\Omega) \to H^k(\Omega^\star)$. By using Lemma 2.2 its norm can be estimated by:

$$\begin{aligned} |||\mathfrak{E}_1(u)|||^2_{H^k(\Omega^\star)} &\le |||u|||^2_{H^k(\Omega)} + \sum_{i\in N} |||\mathfrak{E}_{\Omega_i}(u)|||^2_{H^k(\Omega_i^\star)} \\ &\le \|u\|^2_{H^k(\Omega)} + \sup_{i\in N}\left(C_2^\star + \left(1+\operatorname{diam}(\omega_i)^k\right) C_1^\star\right)^2 \sum_{i\in N} \|u\|^2_{H^k(\Omega_i)} \\ &\le C\,\|u\|^2_{H^k(\Omega)} \end{aligned}$$

where C depends only on $C_1^\star$, $C_2^\star$ and the maximal diameter of the holes ω_i. From Theorem 1.1 it follows that the norm $|||\cdot|||_{H^k(\Omega^\star)}$ is equivalent to the usual norm $\|\cdot\|_{H^k(\Omega^\star)}$, where the equivalence constants only depend on $\Omega^\star$ and are independent of the size and number of the geometric details. □

We will illustrate Theorem 2.2 by two examples. First, a domain containing holes is considered.

Example 2.1. *Let $\Omega^\star = (0,1)^d$ and $(B_{\varepsilon_i}(x_i))_{i\in N}$ a family of balls with $\overline{B_{\varepsilon_i}(x_i)} \subset \Omega$ satisfying the following separation condition. There exists a constant $C_{sep} > 0$ such that, for $i, j \in N$, $i \ne j$:*

$$\begin{aligned} \operatorname{dist}\left(B_{\varepsilon_i}(x_i), B_{\varepsilon_j}(x_j)\right) &\ge C_{sep}\max(\varepsilon_i, \varepsilon_j) \\ \operatorname{dist}\left(B_{\varepsilon_i}(x_i), \partial\Omega^\star\right) &\ge C_{sep}\,\varepsilon_i\,. \end{aligned} \tag{2.8}$$

Then, the extension operator $\mathfrak{E}_1$ applied to the domain $\Omega := \Omega^\star \setminus \bigcup_{i\in N} \overline{B_{\varepsilon_i}(x_i)}$ is independent of $\#N$ and ε_i, $i \in N$. Note that even $\#N = \infty$ is allowed.

Proof. Let $\omega_i := B_{\varepsilon_i}(x_i)$. From $\overline{\omega}_i \subset \Omega$ it follows that $\sup_{i\in N} \varepsilon_i \leq 1$. The domains $\Omega_i^\star$ can be chosen as the balls $B_{\rho\varepsilon_i}(x_i)$ with $\rho := (1 + C_{sep}/3)$. The scaled domains are given by $\Omega_{i,\omega_i}^\star = B_{\rho/2}(x_i)$ and $\chi_{\omega_i}(B_{\varepsilon_i}(x_i)) = B_{1/2}(x_i)$. Hence, Assumption 2.2 is satisfied and Theorem 2.2 can be applied with a constant being independent of ε_i and $\#N$. □

The following example shows that the norm of the *minimal* extension operator blows up if the separation condition (2.8) is violated.

Example 2.2. *Consider the domain (see Fig. 2):*

$$\Omega_\delta := \Omega_\delta^\star \setminus \overline{\omega} \quad \textit{with} \quad \Omega_\delta^\star := (-1-\delta, 1+\delta)^d \ \textit{and}\ \omega := (-1,1)^d .$$

Let $\mathfrak{E}_{\min} : H^1(\Omega_\delta) \to H^1(\Omega_\delta^\star)$ *denote the extension operator with minimal* H^1*-norm. Then:*

$$\|\mathfrak{E}_{\min}\|_{H^1(\Omega_\delta^\star)\leftarrow H^1(\Omega_\delta)} \geq C\,\delta^{-1/2} .$$

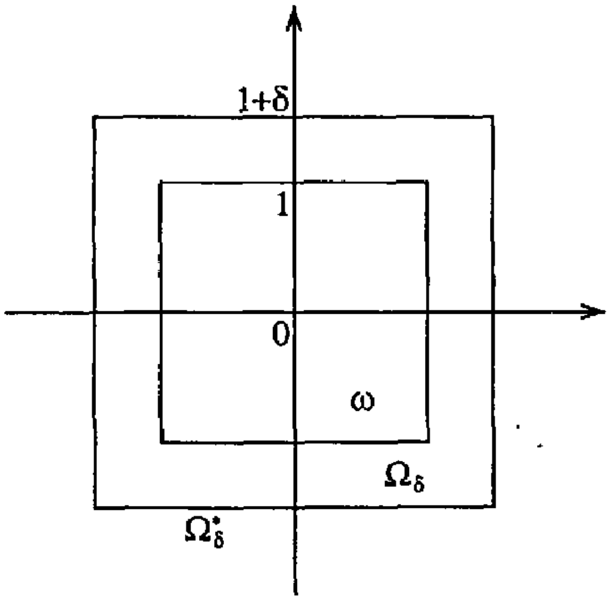

Figure 2. The domains.

Proof. It suffices to construct a function $u \in H^1(\Omega_\delta)$ such that

$$\|\mathfrak{E}_{\min} u\|_{H^1(\Omega_\delta^\star)} \geq C\,\delta^{-1/2} \|u\|_{H^1(\Omega_\delta)} .$$

This function $u : \Omega_\delta \to \mathbb{R}$ is given by

$$u(x) = \sinh(x_1) .$$

By using arguments as in the proof of Theorem 2.1 one can show that the extension $\mathfrak{E}_{\min}(u)$ is characterized by a suitable projection operator. Let $u^\star \in H^1(\Omega_\delta^\star)$ be any extension of u. Let $z \in H_0^1(\omega)$ be defined as the solution of

$$(z, v)_{H^1(\omega)} = (u^\star, v)_{H^1(\omega)} \quad \text{for all } v \in H_0^1(\omega) .$$

Then:

$$\mathfrak{E}_{\min}(u)(x) := \begin{cases} u(x), & x \in \Omega_\delta \\ u^\star(x) - z(x), & x \in \omega . \end{cases}$$

For our example we choose $u^*(x) = \sinh(x_1)$. Using Green's formula we obtain by taking into account $-\Delta u^* + u^* = 0$ on ω and $v = 0$ on $\partial\omega$:

$$(u^*, v)_{H^1(\omega)} = \int_\omega (\langle \nabla u^*, \nabla v\rangle + u^* v)\, \mathrm{d}x = \int_\omega (-\Delta u^* + u^*)\, v\, \mathrm{d}x + \int_{\partial\omega} \frac{\partial u^*}{\partial n} v\, \mathrm{d}x = 0\,.$$

Hence, $z = 0$ and $\mathfrak{E}_{\min}(u)\,(x) = \sinh(x_1)$.

Somewhat tedious calculations show that:

$$\frac{\|\mathfrak{E}_{\min}(u)\|^2_{H^1(\Omega^*_\delta)}}{\|u\|^2_{H^1(\Omega_\delta)}} = \frac{\|\sinh(x_1)\|^2_{H^1(\Omega^*_\delta)}}{\|\sinh(x_1)\|^2_{H^1(\Omega_\delta)}}$$
$$= \frac{(1+\delta)\sinh(1+\delta)\cosh(1+\delta)}{\delta\sinh(1+\delta)\cosh(1+\delta) + (\sinh(1+\delta)\cosh(1+\delta) - \sinh(1)\cosh(1))}$$
$$\geq \frac{1}{K_1\delta + K_2\delta} = K_3\,\frac{1}{\delta}\,.$$

□

2.3. Further estimates and examples

In this Subsection we will define extension operators for long and thin holes. As an example consider the domain (see Fig. 3):

$$\Omega := \Omega^* \setminus \overline{\omega} \quad \text{with } \Omega^* := (-1,1)^3 \text{ and } \omega := B^2_\varepsilon(0) \times (-1,1). \tag{2.9}$$

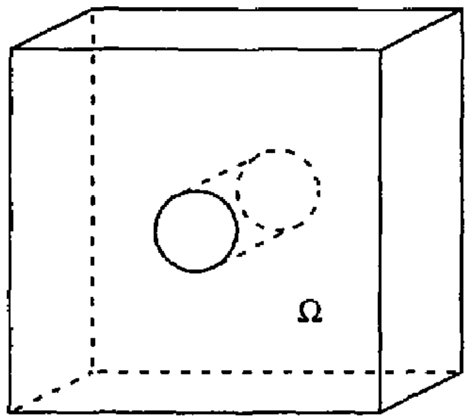

Figure 3. Domain with thin long hole.

Such a domain does not satisfy Assumption 2.2 in the sense that $C_{1,i}$ (see Condition 4) is independent of ε. However, we will prove that, also for such domains, there exists an extension operator with norm bounded independently of ε. First, we will prove analogues of Lemma 2.2 and Theorem 2.2 where the norm $\|\cdot\|_{H^k(\Omega)}$ is replaced by the simpler norm $|||\cdot|||_{H^k(\Omega)}$ (see (1.2)).

Lemma 2.3. *Let C_1 and C_2 be as in Lemma 2.2 and let K be as in Theorem 1.1 applied to the scaled domain Ω_ω. Then the operator $\mathfrak{E}_\Omega$ satisfies*

$$|||\mathfrak{E}_\Omega(u)|||_{H^k(\Omega^*)} \leq \left(C_2 + \left(1 + \operatorname{diam}(\omega)^k\right) C_1 K\right) |||u|||_{H^k(\Omega)} \quad \text{for all } u \in H^k(\Omega)\,.$$

Proof. The proof is similar to the one of Lemma 2.2. Let $u \in H^k(\Omega)$ and $\varepsilon := \operatorname{diam}(\omega)$. Then:

$$\begin{aligned}
\|\mathfrak{E}_\Omega u\|^2_{L^2(\Omega^*)} &= \varepsilon^d \|\mathfrak{E}_{\Omega_\omega}(u \circ \chi_\omega^{-1})\|^2_{L^2(\Omega_\omega^*)} \leq \varepsilon^d \|\mathfrak{E}_{\Omega_\omega}(u \circ \chi_\omega^{-1})\|^2_{H^k(\Omega_\omega^*)} \\
&\leq \varepsilon^d C_1^2 \|u \circ \chi_\omega^{-1}\|^2_{H^k(\Omega_\omega)} \leq \varepsilon^d C_1^2 K^2 |||u \circ \chi_\omega^{-1}|||^2_{H^k(\Omega_\omega)} \\
&= \varepsilon^d C_1^2 K^2 \textstyle\sum_{m=0,k} |u \circ \chi_\omega^{-1}|^2_{H^m(\Omega_\omega)} \leq C_1^2 K^2 \left(1 + \varepsilon^k\right)^2 \sum_{m=0,k} |u|^2_{H^m(\Omega)} \\
&= C_1^2 K^2 \left(1 + \varepsilon^k\right)^2 |||u|||^2_{H^k(\Omega)} .
\end{aligned}$$

□

Assumption 2.3. Let Assumption 2.2 be satisfied and, for $i \in N$, let K_i denote the constant of Theorem 1.1 applied to the scaled domain Ω_{i,ω_i}. Let

$$K^* := \sup_{i \in N} K_i < \infty .$$

Theorem 2.3. *Let Assumption 2.3 be satisfied and $\mathfrak{E}_1$ as in Theorem 2.2. Then*

$$|||\mathfrak{E}_1 u|||_{H^k(\Omega^*)} \leq C \, |||u|||_{H^k(\Omega)} \quad \text{for all } u \in H^k(\Omega)$$

where

$$C = 1 + \sup_{i \in N} \left(C_2^* + (1 + \operatorname{diam}(\omega_i)^k) C_1^* K^*\right) .$$

Proof. The proof is just a repetition of the proof of Theorem 2.2 while Lemma 2.2 has to be replaced by Lemma 2.3. □

The following example is an application of the previous theorem. There, the extension operator is defined in two steps.

Example 2.3. *Let $k \in \mathbb{N}$ and $\varepsilon \in (0, \frac{1}{2})$ such that $\frac{1}{2}(1 + \frac{1}{\varepsilon}) \in \mathbb{N}$. Let Ω and Ω^* as in (2.9) (see Fig. 3). Then there exists an extension operator $\mathfrak{E}_{\text{hole}} : H^k(\Omega) \to H^k(\Omega^*)$ bounded by a constant independent of ε.*

Remark 2.4. The assumption $\frac{1}{2}(1 + \frac{1}{\varepsilon}) \in \mathbb{N}$ is only made for avoiding technicalities.

Proof. Let $m := \frac{1}{\varepsilon} \in 2\mathbb{N} - 1$ and $N := \mathbb{N}_{\leq m}$. For $i \in N$, we set:

$$\Omega_i^* := B^2_{2\varepsilon}(0) \times \left(\frac{i-1}{m}, \frac{i}{m}\right), \quad \omega_i := B^2_\varepsilon(0) \times \left(\frac{i-1}{m}, \frac{i}{m}\right) \quad \text{and } \Omega_i := \Omega_i^* \setminus \overline{\omega}_i .$$

Let $N_{\text{odd}} := \{i \in N : i \text{ is odd}\}$ and $N_{\text{even}} := \{i \in N : i \text{ is even}\}$. For intermediate use, let

$$\Delta := \Omega^* \setminus \bigcup_{i \in N_{\text{even}}} \overline{\omega}_i .$$

Let $\mathfrak{E}_{\text{odd}} : H^k(\Omega) \to H^k(\Delta)$ be the extension operator of Theorem 2.2 with N of Assumption 2.2 given by N_{odd} and $\mathfrak{E}_{\text{even}} : H^k(\Delta) \to H^k(\Omega^*)$ be the one with N given by N_{even}. For $u \in H^k(\Omega)$, we set

$$\mathfrak{E}_{\text{hole}}(u) := \mathfrak{E}_{\text{even}}\left(\mathfrak{E}_{\text{odd}}(u)\right) .$$

Since the domains ω_i are images of one reference domain via translation Assumptions 2.2 and 2.3 are satisfied. Thus, we may estimate $\mathfrak{E}_{\text{odd}}$ with Theorem 2.2 and $\mathfrak{E}_{\text{even}}$ with Theorem 2.3. It follows:

$$|||\mathfrak{E}_{\text{hole}}(u)|||^2_{H^k(\Omega^*)} \leq C_1 |||\mathfrak{E}_{\text{odd}}(u)|||^2_{H^k(\Delta)} \leq C_2 \|u\|^2_{H^k(\Omega)} .$$

From Theorem 1.1 one concludes that the norm $|||\cdot|||_{H^k(\Omega^*)}$ is equivalent to the usual norm $\|\cdot\|_{H^k(\Omega^*)}$, the equivalence constants only depend on Ω^* and are independent of ε. □

Many further domains with different kinds of geometric details as e.g. domains with rough boundaries, may be treated with this technique as well. We refer to [14] where further parameter studies are worked out.

3. CUSPS

A domain which contains two holes touching each other in one point, say z, is in general not a Lipschitz domain. The boundary contains a so-called *cusp* at z.

In this Section we will study the existence and behaviour of extension operators in the presence of cusps. As a model problem, we consider the domain with outward cusp (see Fig. 4):

$$\Omega := \{(x_1, x_2) \in (0,1)^2 : x_2 > x_1^\gamma\}$$

where $\gamma \in (0, 1)$.

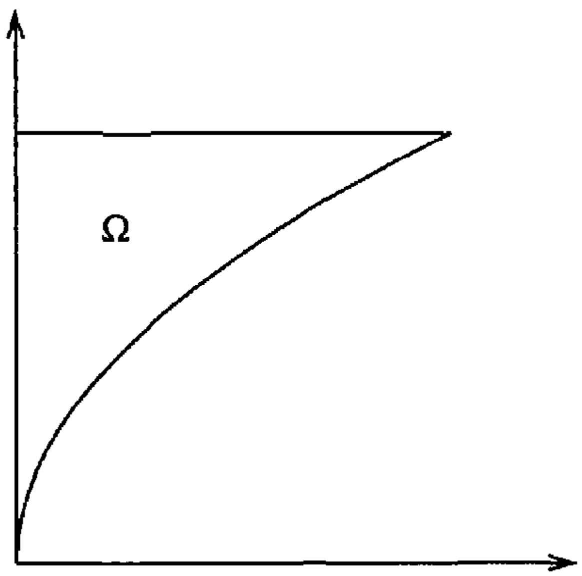

Figure 4. The cuspoidal domain Ω.

In [8, 1.5.1] inward cusps are considered and extension operators from Sobolev spaces into weighted Sobolev space are presented.

The following example, which is taken from [13], shows that there exists no extension operator

$$\epsilon : H^1(\Omega) \to H^1(\Omega^*)$$

for any Lipschitz domain Ω^* with $\Omega \subset \Omega^*$.

Example 3.1. *Consider the cuspoidal domain Ω with $\gamma = \frac{1}{2}$ and let Ω^* be a Lipschitz domain satisfying $\Omega \subset \Omega^*$. On Ω, consider the function:*

$$u(x) = x_2^{-1/4}.$$

Then $u \in H^1(\Omega)$ but u cannot be extended to Ω^*.

Proof by contradiction. Assume that an extension $u_1 = \mathfrak{E}u \in H^1(\Omega^*)$ exists. Then, u_1 can be extended onto $\mathbb{R}^2$ by Stein's extension operator (cf. (2.1)) $u_2 = \mathfrak{E}_0 u_1 \in H^1(\mathbb{R}^2)$. Thus, Sobolev's theorem (see [13]) implies:

$$u = u_2|_\Omega \in L^q(\Omega) \quad \text{for all } 1 \leq q < \infty . \tag{3.1}$$

Explicit computation yields that $u \notin L^{15}(\Omega)$ being a contradiction to (3.1). □

Remark 3.1. Example 3.1 shows that an extension operator can only be defined on an appropriate subspace of $H^1(\Omega)$. It turns out that it is possible to define extension operators in weighted Sobolev spaces.

For $p \in [1, \infty)$ and an open set $A \subset \mathbb{R}^d$, let the Sobolev space $W^{1,p}(A)$ be defined as usual, for example, see [1].

Definition 3.1. For $p \in [1, \infty)$, let $\sigma(x) := \gamma^p x_1^{\gamma - 1}$ be a weight function and let

$$\|u\|_{W^{1,p}(\Omega;\sigma)} = \left(\int_\Omega (|f(x)|^p + |\partial_2 f(x)|^p)\sigma(x) + |\partial_1 f(x)|^p \, dx \right)^{1/p}$$

which is a norm on

$$W^{1,p}(\Omega; \sigma) = \{u \in L^1_{loc}(\Omega) : \|u\|_{W^{1,p}(\Omega;\sigma)} < \infty\} .$$

Before we will give the definition of the extension operator, we shall introduce some notation.

Notation 3.1. Let

$$\Omega_L := \{(x_1, x_2) \in (0,1)^2 : x_2 > x_1\}, \quad \Omega_L^- := \{(x_1, x_2) : (-x_1, x_2) \in \Omega_L\}$$

and (see Fig. 5)

$$\Omega^* := \operatorname{int}(\overline{\Omega \cup \Omega_L^-}) .$$

Note that Ω^* is not a cuspoidal domain but a Lipschitz domain.

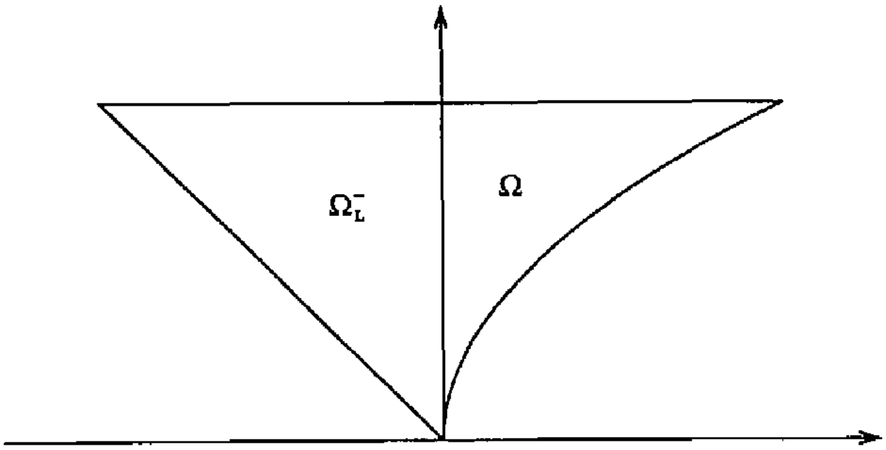

Figure 5. The Lipschitz domain Ω^*.

Let S denote the reflection operator $S : \Omega_L \to \Omega_L^-; (x_1, x_2) \mapsto (-x_1, x_2)$ and let $a : \Omega \to \Omega_L; (x_1, x_2) \mapsto (x_1^\gamma, x_2)$ be a transformation of Ω on Ω_L.

Theorem 3.1. *Let $p \in [1, \infty)$. The operator*

$$\mathfrak{E}: \ W^{1,p}(\Omega;\sigma) \to W^{1,p}(\Omega^*);$$
$$u \mapsto \mathfrak{E}u := \begin{cases} u & \text{on } \Omega, \\ u \circ a^{-1} \circ S & \text{on } \Omega_L^- \end{cases}$$

is an extension operator with norm bounded by $(\gamma^{-p} + \gamma^{1-p})^{1/p}$.

For the proof of Theorem 3.1 the following chain rule is required. In the Lemma the chain rule from [8, 1.1.7] is generalized to a chain rule for transformations with unbounded derivatives.

Lemma 3.1. *Suppose $p \in [1, \infty)$ and $u \in W^{1,p}(\Omega;\sigma)$, then $u \circ a^{-1} \in W^{1,p}(\Omega_L)$ and the weak derivatives can be computed according to the classical chain rule.*

Proof. Let $k \in \{1, 2\}$ and define $v(x) := (\partial_k u)(a^{-1}(x))\, \partial_k a_k^{-1}(x)$. Since a and a^{-1} map sets of measure zero to sets of measure zero, the definition of v does not depend on any special representative of u. We will show that $v = \partial_k(u \circ a^{-1})$. Let $\varphi \in C_0^\infty(\Omega_L)$. Then there is an $\varepsilon > 0$ such that

$$\operatorname{supp}\varphi \subset \Omega_\varepsilon := \Omega_L \cap ((\varepsilon, 1) \times (\varepsilon, 1)).$$

On Ω_ε, the transformation a is quasi-isometric and of the class $C^{0,1}(\overline{\Omega}_\varepsilon)$. Hence, the formula of the chain rule may be applied (cf. [8, 1.1.7]). We have:

$$\int_{\Omega_L} v(x)\varphi(x)\,\mathrm{d}x = \int_{\Omega_\varepsilon} v(x)\varphi(x)\,\mathrm{d}x = \int_{\Omega_\varepsilon} (\partial_k u)(a^{-1}(x))\, \partial_k a_k^{-1}(x)\varphi(x)\,\mathrm{d}x$$
$$= \int_{\Omega_\varepsilon} \partial_k(u \circ a^{-1})(x)\varphi(x)\,\mathrm{d}x = -\int_{\Omega_\varepsilon} (u \circ a^{-1})(x)\partial_k\varphi(x)\,\mathrm{d}x = -\int_{\Omega_L} (u \circ a^{-1})(x)\partial_k\varphi(x)\,\mathrm{d}x\,.$$

Since $u \in W^{1,p}(\Omega;\sigma)$, the $W^{1,p}$-norm of $u \circ a^{-1}$ is finite:

$$\int_{\Omega_L} \left|u \circ a^{-1}(x)\right|^p \mathrm{d}x = \int_{\Omega} |u(x)|^p\, |\det(Da(x))|\,\mathrm{d}x = \int_{\Omega} |u(x)|^p\, \gamma x_1^{\gamma-1}\,\mathrm{d}x\,;$$

$$\int_{\Omega_L} |\partial_1(u \circ a^{-1})(x)|^p\,\mathrm{d}x = \int_{\Omega_L} |(\partial_1 u)(a^{-1}(x))|^p \left(\frac{1}{\gamma} x_1^{\frac{1}{\gamma}-1}\right)^p \mathrm{d}x$$
$$= \int_{\Omega} |\partial_1 u(x)|^p \left(\frac{1}{\gamma} x_1^{1-\gamma}\right)^p \gamma x_1^{\gamma-1}\,\mathrm{d}x \quad \le \gamma^{1-p} \int_{\Omega} |\partial_1 u(x)|^p\,\mathrm{d}x\,;$$

$$\int_{\Omega_L} |\partial_2(u \circ a^{-1})(x)|^p\,\mathrm{d}x = \int_{\Omega_L} |(\partial_2 u)(a^{-1}(x))|^p\,\mathrm{d}x = \int_{\Omega} |\partial_2 u(x)|^p\, \gamma x_1^{\gamma-1}\,\mathrm{d}x\,.$$

Thus:

$$\|u \circ a^{-1}\|^p_{W^{1,p}(\Omega_L)} \le \gamma^{1-p}\, \|u\|^p_{W^{1,p}(\Omega;\sigma)} < \infty\,. \tag{3.2}$$

□

Proof of Theorem 3.1. Obviously, $\mathfrak{E}$ is linear. Let $u \in W^{1,p}(\Omega;\sigma)$ and let

$$u^* := \begin{cases} u & \text{on } \Omega \\ u \circ a^{-1} \circ S & \text{on } \Omega_L^-\,. \end{cases}$$

Lemma 3.1 implies $u \circ a^{-1} \in W^{1,p}(\Omega_L)$. This is equivalent to $u \circ a^{-1} \circ S \in W^{1,p}(\Omega_L^-)$. Therefore, $u^* \in L^p(\Omega)$.

First, we will prove the weak differentiability of u^*. Since $C^\infty(\overline{\Omega}_L)$ is dense in $W^{1,1}(\Omega_L)$, we may approximate $u \circ a^{-1}$ by $v_\varepsilon \in C^\infty(\overline{\Omega}_L)$. We will show that $v_\varepsilon \circ a$ approximates u in $W^{1,1}(\Omega)$:

$$\int_\Omega |(u - v_\varepsilon \circ a)(x)|\,\mathrm{d}x = \int_{\Omega_L} |(u \circ a^{-1} - v_\varepsilon)(x)|\,|\det(Da^{-1}(x))|\,\mathrm{d}x$$

$$= \int_{\Omega_L} |(u \circ a^{-1} - v_\varepsilon)(x)|\,\frac{1}{\gamma} x_1^{\frac{1}{\gamma}-1}\,\mathrm{d}x \le \frac{1}{\gamma}\int_{\Omega_L} |(u \circ a^{-1} - v_\varepsilon)(x)|\,\mathrm{d}x\,;$$

$$\int_\Omega |\partial_1(u - v_\varepsilon \circ a)(x)|\,\mathrm{d}x = \int_\Omega |\partial_1 u(x) - (\partial_1 v_\varepsilon)(a(x))\,\gamma x_1^{\gamma-1}|\,\mathrm{d}x$$

$$= \int_{\Omega_L} |(\partial_1 u)(a^{-1}(x)) - \partial_1 v_\varepsilon(x)\,\gamma x_1^{1-\frac{1}{\gamma}}|\,\frac{1}{\gamma} x_1^{\frac{1}{\gamma}-1}\,\mathrm{d}x = \int_{\Omega_L} |\partial_1(u \circ a^{-1})(x) - \partial_1 v_\varepsilon(x)|\,\mathrm{d}x\,;$$

$$\int_\Omega |\partial_2(u - v_\varepsilon \circ a)(x)|\,\mathrm{d}x = \int_\Omega |\partial_2 u(x) - (\partial_2 v_\varepsilon)(a(x))|\,\mathrm{d}x$$

$$= \int_{\Omega_L} |(\partial_2 u)(a^{-1}(x)) - \partial_2 v_\varepsilon(x)|\,\frac{1}{\gamma} x_1^{\frac{1}{\gamma}-1}\,\mathrm{d}x = \int_{\Omega_L} |\partial_2(u \circ a^{-1}) - \partial_2 v_\varepsilon(x)|\,\frac{1}{\gamma} x_1^{\frac{1}{\gamma}-1}\,\mathrm{d}x$$

$$\le \frac{1}{\gamma}\int_{\Omega_L} |\partial_2(u \circ a^{-1} - v_\varepsilon)(x)|\,\mathrm{d}x\,.$$

Summarizing, we have:

$$\|u - v_\varepsilon \circ a\|_{W^{1,1}(\Omega)} \le \frac{1}{\gamma}\|u \circ a^{-1} - v_\varepsilon\|_{W^{1,1}(\Omega_L)} \to 0 \quad \text{for } \varepsilon \to 0\,. \tag{3.3}$$

To prove the weak differentiability of u^*, we approximate $u^*|_{\Omega_L^-}$ by v_ε in $W^{1,1}(\Omega_L^-)$. Thus, by (3.3), $w_\varepsilon := v_\varepsilon \circ a \circ S$ approximates $u^*|_\Omega$ in $W^{1,1}(\Omega)$. Let

$$u_\varepsilon^* := \begin{cases} w_\varepsilon & \text{on } \Omega \\ v_\varepsilon & \text{on } \Omega_L^-\,. \end{cases}$$

From $a(0, x_2) = (0, x_2)$ it follows that v_ε and w_ε equal for $x_1 = 0$. Hence, integration by parts for the smooth functions gives:

$$\int_{\Omega^*} \partial_1 u_\varepsilon^*(x)\,\varphi(x)\,\mathrm{d}x = \int_{x_2}\int_{x_1<0} \partial_1 v_\varepsilon(x)\,\varphi(x)\,\mathrm{d}x + \int_{x_2}\int_{x_1>0} \partial_1 w_\varepsilon(x)\,\varphi(x)\,\mathrm{d}x$$

$$= \int_{x_2} -v_\varepsilon(0, x_2)\,\varphi(0, x_2)\,\mathrm{d}x_2 - \int_{x_2}\int_{x_1<0} v_\varepsilon(x)\,\partial_1\varphi(x)\,\mathrm{d}x$$

$$+ \int_{x_2} w_\varepsilon(0, x_2)\,\varphi(0, x_2)\,\mathrm{d}x_2 - \int_{x_2}\int_{x_1>0} w_\varepsilon(x)\,\partial_1\varphi(x)\,\mathrm{d}x = -\int_{\Omega^*} u_\varepsilon^*(x)\,\partial_1\varphi(x)\,\mathrm{d}x\,.$$

The result for u^* follows if we let ε tend to zero. For ∂_2 it follows similarly, with the simplification that it is not necessary to split the domain in the parts $x_1 < 0$ and $x_1 > 0$.

It remains to show that the $W^{1,p}$-norm of $u^\star$ is finite. Using (3.2), we obtain:

$$\|u^\star\|^p_{W^{1,p}(\Omega^\star)} = \|u\|^p_{W^{1,p}(\Omega)} + \|u \circ a^{-1} \circ S\|^p_{W^{1,p}(\Omega_L^-)} = \|u\|^p_{W^{1,p}(\Omega)} + \|u \circ a^{-1}\|^p_{W^{1,p}(\Omega_L)}$$
$$\leq \|u\|^p_{W^{1,p}(\Omega)} + \gamma^{1-p}\|u\|^p_{W^{1,p}(\Omega;\sigma)} \leq (\gamma^{-p} + \gamma^{1-p})\|u\|^p_{W^{1,p}(\Omega;\sigma)}.$$

Thus, $\mathfrak{E}$ is bounded by $(\gamma^{-p} + \gamma^{1-p})^{1/p}$. □

Acknowledgments. The authors wish to thank Prof. W. Dahmen and Prof. S. Nepomnyaschikh for fruitful and stimulating discussions.

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