Transformation of hypersingular integrals and black-box cubature (extended version)

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Abstract

In this paper, we will consider hypersingular integrals as they arise by transforming elliptic boundary value problems into boundary integral equations. First, local representations of these integrals will be derived. These representations contain so-called finite-part integrals. In the second step, these integrals are reformulated as improper integrals. We will show that these integrals can be treated by cubature methods for weakly singular integrals as they exist in the literature.

1 Introduction

In this paper, we will consider Fredholm integral equations on two-dimensional surfaces in $\mathbb{R}^3$ which typically arise by applying the boundary element method to boundary value problems (see, e.g. [8], [27]).

With raising interest in the numerical solution of these integral equations the need of appropriate cubature[1] methods for computing the elements of the system matrix arises.

For weakly or Cauchy singular integrals, there exist appropriate cubature methods for approximating the elements of the system matrix (see [2], [25], [21], [9], [26], [4], [15], [17]). For many important problems as, e.g., mixed boundary value problems or transmission problems, the kernel functions are not integrable in the sense of Cauchy principal values. They are hypersingular and have to be regularised in the sense of Hadamard (see [8], [24]). For these kinds of integrals, cubature methods for Galerkin discretisations are missing in the literature. To overcome this difficulty a regularisation on the continuous level is often applied rendering the integrals weakly or Cauchy singular (see [20], [11], [8]). The drawback of this technique is that it has to be worked out for each kernel function separately, i.e., is not fully implicit. Here and in the following, the term *fully implicit* is used in the sense that the definition of the cubature method does not depend on the explicit form of the integrand but works as a black-box method for all kernel functions specified in Section 3. Only the

[1]In more than one dimension the term *quadrature* is replaced by *cubature*.

subroutine for evaluating the kernel function in pairs of cubature points has to be exchanged.

In our paper we present a direct approach for evaluating hypersingular integrals which are efficient in the sense that this *family* of cubature rules is

1. fully implicit,
2. exponentially convergent (with respect to the order of the rule),
3. uniformly stable (with respect to the order of the rule).

For collocation methods such techniques are described in [7], [6], [25], [13]. We will use these results to analyse the behaviour of the integrand for the *outer* integration appearing for the Galerkin method. For piecewise flat surfaces and the hypersingular kernel function corresponding to the Laplace operator, semi-analytic cubature techniques for the Galerkin method are worked out in [14].

The paper is organised as follows. In Section 2, we will specify the class of boundary integral equations which will be considered and formulate the Galerkin discretisation of the arising weak formulation.

In Section 3, properties of boundary integral equations and corresponding kernel functions are collected.

Then, in Section 4, it is explained how the arising finite-part integrals (over the whole surface) can be localized as finite-part integrals over pairs of panels.

In the next section, the local finite-part integrals are reformulated as a sum of weakly singular integrals by analysing the singular behaviour of the arising integrands.

Finally, in Section 6, families of cubature rules are defined for the approximation of the derived weakly singular integrals which converge *exponentially* with respect to the order.

2 The boundary element method

Let Γ be a piecewise analytic, orientable Lipschitz surface of a bounded domain $\Omega \subset \mathbb{R}^3$. The assumption on the analyticity of Γ is merely imposed for convenience. We expect that this condition can be replaced by "sufficiently smooth" in a similar fashion as worked out in [19]. However, the detailed extension of the theory below to that more general case is not worked out yet.

Let $L^2(\Gamma)$ denote the space of all measurable functions $u : \Gamma \to \mathbb{C}$ which are square integrable with respect to the surface measure $d\Gamma$. $H^1(\Gamma)$ is defined as usual by employing a Lipschitz atlas and a partition of unity. The intermediate spaces $H^s(\Gamma)$, $0 < s < 1$, are defined via interpolation while, for $-1 \leq s < 0$, $H^s(\Gamma)$ is the dual space of $H^{-s}(\Gamma)$ with respect to the L^2-scalar product.

We consider Fredholm integral equations in the variational form. For given

$$f \in H^{\tilde{s}}(\Gamma) \tag{1}$$

(for the definition of $\tilde{s}, s_1, s_2$, see (4), (5), and (6)), we are seeking $u \in H^{s_1}(\Gamma)$ such that

$$(v, \lambda_1 u)_0 + (v, \mathcal{K}_1 u)_0 = (v, \lambda_2, f)_0 + (v, \mathcal{K}_2 f)_0, \qquad \forall v \in H^{s_1}(\Gamma) \tag{2}$$

holds, where the $L^2(\Gamma)$-scalar product $(\cdot,\cdot)_0$ is identified with its extension to $H^{s_1}(\Gamma) \times H^{-s_1}(\Gamma)$ by means of Riesz' representation theorem. In (2), λ_1, λ_2 are analytic on smooth parts of the surface and the integral operators $\mathcal{K}_i$, $i = 1, 2$, are given by

$$\mathcal{K}_i[w](x) = p.f. \int_\Gamma k_i(x, y, y - x)\, w(y)\, d\Gamma_y. \tag{3}$$

If the *kernel function* contains non-integrable singularities, then, the integral (3) has to be understood in the regularised sense of Hadamard which will be explained in Section 3.2.

For $i = 1, 2$, we assume that the mapping

$$\lambda_i I + \mathcal{K}_i : H^{s_i} \to H^{-s_i} \tag{4}$$

is continuous and $\tilde{s}$ in (1) has to satisfy

$$\tilde{s} = 2s_2 - s_1. \tag{5}$$

Furthermore, we assume that the operator $\lambda_1 I + \mathcal{K}_1$ satisfies a Gårding inequality, i.e. there exist $\varepsilon > 0$ and constants c_1, c_2 such that

$$(u, (\lambda_1 I + \mathcal{K}_1)\, u)_0 \geq c_1 \|u\|_{s_1}^2 - c_2 \|u\|_{s_1 - \varepsilon}^2 \tag{6}$$

is satisfied for all $u \in H^{s_1}(\Gamma)$. The left-hand side of (2) defines the bilinear form $a : H^{s_1}(\Gamma) \times H^{s_1}(\Gamma) \to \mathbb{C}$ and the right-hand side the functional $F \in H^{-s_1}(\Gamma)$.

Throughout this paper we assume that

$$s_1, s_2 \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$$

holds (this requirement is satisfied for most practical applications in three dimensions). However, we hasten to say that our theory is by no means limited to this case and can be generalised to more general integral operators (see Remark 5).

The Galerkin discretisation of (2) is given by replacing the Sobolev space $H^{s_1}(\Gamma)$ by a finite dimensional subspace which will be constructed below.

Let $\tilde{\Gamma}$ be the (piecewise plane) surface of a polyhedron which interpolates Γ. Let $\tilde{\tau} := \left\{\tilde{K}_1, \tilde{K}_2, \ldots, \tilde{K}_N\right\}$ denote a grid on the surface $\tilde{\Gamma}$ consisting of plane (open) triangles and parallelograms satisfying

$$\begin{array}{l} \tilde{\Gamma} = \overline{\bigcup_{\tilde{K} \in \tilde{\tau}} \tilde{K}}, \\ \widetilde{K} \cap \widetilde{K'} = \emptyset, \quad \forall \widetilde{K}, \widetilde{K'} \in \tilde{\tau} \text{ with } \widetilde{K} \neq \widetilde{K'}. \end{array}$$

The following assumption links the true surface Γ with the auxiliary surface $\tilde{\Gamma}$. We assume that there exists a bi-Lipschitz mapping $\eta : \tilde{\Gamma} \to \Gamma$ having the property that, for all $K \in \tau$, the restriction $\eta \mid_{\tilde{K}}$ can be extended to an analytic mapping $\eta : \overline{\tilde{K}} \to \Gamma$ and the inverse η^{-1} has the analogue property.

The grid $\tilde{\tau}$ induces a grid on the true surface Γ by

$$\tau := \left\{ \eta\left(\tilde{K}\right) : \tilde{K} \in \tilde{\tau} \right\}.$$

The space of finite element functions on the surface Γ is defined as usual by lifting polynomial spaces on a reference element onto the true surface. For $r \in \mathbb{N}_0$, let

$$S_{\tau}^{r,p} := \{u \in C^r(\Gamma) \mid \forall K \in \tau : u \mid_K \circ \eta \circ \kappa_K \in P_K\} \tag{7}$$

where

$$\kappa_K : Q_K \to \tilde{K} \tag{8}$$

is an affine-linear mapping and

$$Q_K = \begin{cases} (0,1)^2 & \text{if } \tilde{K} \text{ is a parallelogram,} \\ \text{triangle with verices } \binom{0}{0}, \binom{1}{0}, \binom{0}{1} & \text{if } \tilde{K} \text{ is a triangle.} \end{cases}$$

We emphasize that throughout the paper the reference domains Q_K are considered to be *open* sets. For triangles, P_K is the space of bivariate polynomials of total degree p while, for quadrilaterals P_K, is the space of polynomials of degree p in each variable. For $r = -1$, the condition $u \in C^r(\Gamma)$ in (7) has to be replaced by $u \in L^\infty(\Gamma)$. For continuous finite elements, i.e. $r \geq 0$, we assume that, for all $K, K' \in \tau$, $K' \neq K$, the intersection $\overline{K} \cap \overline{K'}$ is either empty, a common vertex, or a common edge. In the following, we write V_τ short for $S_\tau^{r,p}$.

The Galerkin discretisation of (2) is given by finding $u_G \in V_\tau$ such that

$$a(u_G, v) = F(v), \qquad \forall v \in V_\tau. \tag{9}$$

This problem can be reformulated as a system of linear equations by introducing the basis representation of u_G :

$$u_G(x) = \sum_{i=1}^{n} \mathbf{u}_i \varphi_i(x)$$

where $n := \dim V_\tau$. Then, (9) is equivalent to

$$\mathbf{A}\mathbf{u} = \mathbf{F},$$

where the system matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ and the vector $\mathbf{F} \in \mathbb{C}^n$ are given by

$$\begin{aligned} \mathbf{A}_{i,j} &= a(\varphi_i, \varphi_j), \\ \mathbf{F}_i &= F(\varphi_i). \end{aligned}$$

To compute the matrix entries $\mathbf{A}$ and the right-hand side $\mathbf{F}$ fast cubature techniques are needed for the evaluation of

$$\begin{aligned} &\int_\Gamma \varphi_i(x)\,\varphi_j(x)\,\lambda(x)\,dx, \\ &\int_\Gamma \varphi_i(x)\,p.f.\int_\Gamma k(x,y,y-x)\,\varphi_j(y)\,d\Gamma_y d\Gamma_x, \end{aligned} \tag{10}$$

where λ is analytic on smooth parts of the surface and the kernel function k is either k_1 or k_2. The evaluation of the first integral is not problematic and we will discuss in the following only the second one. In this paper, we will focus on the definition of cubature rules for the numerical integration of (10) which approximate (10) to any required accuracy with a priori known convergence behaviour. The effect of replacing the true Galerkin matrix by a cubature approximation on the discretisation error is not the topic of this paper but is studied thoroughly in [22], [5].

3 Properties of boundary integral equations

3.1 The kernel function

The properties of an integral operator

$$\mathcal{K}[u](x) = p.f.\int_\Gamma k(x,y,y-x)\,u(y)\,d\Gamma_y \tag{11}$$

are determined by the kernel function $k : \Gamma \times \Gamma \to \mathbb{C}$. We assume that k has the following representation

$$k(x,y,z) = \|z\|^{-s}\sum_{i,j=0}^{b}\kappa_i(x)\,\rho_j(y)\,A_{i,j}\left(\|z\|,\frac{z}{\|z\|}\right), \qquad \forall x,y\in\Gamma,\ z=y-x,\ x\neq y, \tag{12}$$

where b is a finite number and $A_{i,j}(r,\xi)$ is analytic with respect to r in any compact neighbourhood of zero and analytic with respect to ξ in a neighbourhood of the sphere S_2. The functions κ_i,ρ_i are assumed to be in $L^\infty(\Gamma)$ and analytic on analytic parts of the boundary. To be more precise we assume that, for all $K\in\tau$, the restrictions $\kappa_i|_{\overline{K}}$ and $\rho_i|_{\overline{K}}$ are analytic. We state that practically all kernel functions arising by transforming elliptic boundary value problems into integral equations are of the form (12) (see [5] and the references therein). The kernel functions are associated with fundamental solutions to differential equations. The following examples illustrates that the fundamental solution of elliptic, scalar differential operators in $\mathbb{R}^3$ are of the form (12).

Example 1 *Let $G\in\mathbb{R}^{3\times 3}$ be a symmetric and positive definite matrix, let $\beta\in\mathbb{R}^3$ and $c\in\mathbb{R}$. Consider the differential operator*

$$Lu = -\operatorname{div}(G\operatorname{grad} u) + 2\langle\beta,\operatorname{grad} u\rangle + cu.$$

Let $B \in \mathbb{R}^{3\times 3}$ be a matrix satisfying $B^T B = G^{-1}$. Then, the fundamental solution of L (satisfying $LS = \delta_0$ with δ_0 denoting Dirac's functional centred at the origin) is given by

$$S(z) = \frac{|\det B|}{4\pi \|Bz\|} e^{\langle B\beta, Bz\rangle - \sqrt{c+\|B\beta\|^2}\|Bz\|}. \tag{13}$$

This function can be rewritten as

$$\|z\|^{-1} A\left(\|z\|, \frac{z}{\|z\|}\right).$$

with the function A defined by

$$A(r,\xi) = \frac{|\det B|}{4\pi \|B\xi\|} e^{r\left(\langle B\beta, B\xi\rangle - \sqrt{c+\|B\beta\|^2}\|B\xi\|\right)}.$$

Hence, $S(z)$ is of the form (12) and satisfies the analyticity properties due to the regularity of B.

Example 2 *The kernel of the classical double layer potential for Laplace's equation in 3-d is the normal derivative of (13) with $A = I$, $\beta = 0$, $c = 0$:*

$$k(x,y,z) = -\frac{\langle n(y), z\rangle}{4\pi \|z\|^3}.$$

This function can be rewritten as

$$k(x,y,z) = -\|z\|^{-2} \sum_{i=1}^{3} \left(\frac{n_i(y)}{4\pi}\right) \frac{z_i}{\|z\|}.$$

Since the components of the normal vector n are piecewise analytic the kernel function is of the form (12).

Finally, we remark that the kernel functions arising from the Lamé equation and the velocity part of kernel functions corresponding to the Stokes equation satisfy our general assumptions on k, too.

In our paper, we will concentrate on elliptic boundary value problems of second order. In [5], it is explained that for such problems the order of singularity s in (12) typically satisfies

$$s \leq 3.$$

For $s \leq 2$ (in combination with the so-called Giraud-Mikhlin condition, see [24, formula (11)] and [18, Chap. 9]), the finite-part integral reduces to a Cauchy principal value where transformation rules and cubature techniques already exist in the literature (see [8], [21], [9], [26], [4]). In this paper, we will assume throughout that

$$s = 3$$

holds. We state that all our statements remain valid also for $s \leq 3$ while some of the assumptions can be weakened and formulae simplified. In Remark 5, it is explained how our results can be extended to the case $s > 3$.

3.2 Finite-part integrals

We come now to the definition of the finite part integral involved in (11). For this, let $x \in \Gamma$ be a point inside a smooth part of the surface and, for $\varepsilon > 0$, let $B_\varepsilon(x)$ denote the (three-dimensional) ball with radius ε centred at x. Let $\gamma \subset \Gamma$ be a measurable subset of Γ satisfying $x \notin \partial\gamma$. We consider a function $u \in L^\infty(\gamma)$ being smooth in a neighbourhood of x (Hölder continuous with exponent $\lambda > 1$ is sufficient). Since the kernel function $k(x, y, y-x)$ is bounded for $y \neq x$ the following integral exists as a usual Riemann integral

$$I_{\varepsilon,\gamma}[u](x) := \int_{\gamma/B_\varepsilon(x)} k(x, y, y-x)\, u(y)\, d\Gamma_y.$$

In [24] and [12], it was shown that the functional $I_{\varepsilon,\gamma}$ admits an expansion as

$$I_{\varepsilon,\gamma}[u](x) = A_{-1,\gamma}[u](x)\,\varepsilon^{-1} + A_{\log,\gamma}[u](x)\log\varepsilon + A_{0,\gamma}[u](x) + R_{\varepsilon,\gamma}[u](x),$$

where $R_{\varepsilon,\gamma}[u](x) \to 0$ as $\varepsilon \to 0$. The finite part integral then is defined by

$$p.f. \int_\gamma k(x, y, y-x)\, u(y)\, d\Gamma_y := A_{0,\gamma}[u](x).$$

In [24] and [12], it was proved that the right-hand side above is finite.

The following general assumption on the integral operator $\mathcal{K}$ in (11) is assumed throughout the paper. $\mathcal{K}$ is a bounded operator from H^μ onto $H^{-\mu}$ with $\mu \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$. For the computation of the matrix elements, the integrals

$$(\varphi_q, \mathcal{K}[\varphi_r])_0 \qquad 1 \leq q, r \leq n \tag{14}$$

have to evaluated. For an arbitrary function $v \in H^\mu$, the image $\mathcal{K}[v]$ lies in $H^{-\mu}$ and, for $\mu = 1/2$, does not belong necessarily to L^2. This would complicate the development of cubature techniques for approximating the dual pairing $(w, \mathcal{K}[v])_0 = (w, \mathcal{K}[v])_{\mu\times-\mu}$ substantially. In particular, the splitting

$$(w, \mathcal{K}[v])_0 = \int_\Gamma w\mathcal{K}[v]\, dx = \sum_{K\in\tau} \int_K w\mathcal{K}[v]\, dx = \sum_{K\in\tau} (w, \mathcal{K}[v])_{L^2(K)}$$

is not valid for all functions $v, w \in H^\mu(\Gamma)$. However, in many cases the operator $\mathcal{K}$ satisfies a so-called shift property, i.e. $\mathcal{K}$ is a bounded operator from $H^{\mu+\sigma}$ into $H^{-\mu+\sigma}$ for a certain range of σ. For our purpose, it is sufficient to assume throughout the paper that

$$\mathcal{K} \;:\; H^\mu(\Gamma) \to H^{-\mu}(\Gamma) \tag{15}$$
$$\mathcal{K} \;:\; H^1(\Gamma) \to L^2(\Gamma) \tag{16}$$

is bounded.

Corollary 3 *Let (15) and (16) be satisfied. The definition (7) of the finite-dimensional spaces V_τ and $V_\tau \subset H^\mu$ implies that*

$$\mathcal{K}[u] \in L^2(\Gamma), \qquad \forall u \in V_\tau. \tag{17}$$

Proof. For $\mu \leq 0$, the assertion follows from (17) and $V_\tau \subset H^\mu$ via

$$\mathcal{K}[V_\tau] \subset \mathcal{K}[H^\mu] \subset H^{-\mu} \subset L^2(\Gamma).$$

For $\mu = 1/2$, all functions in V_τ are Lipschitz continuous and the result follows from (16) by using $V_\tau \subset C^0(\Gamma) \subset H^1(\Gamma)$. ■

A comment on the validity of (16) is given below.

Remark 1 *Assumption (16) is satisfied, e.g. for the hypersingular integral operators corresponding to elliptic boundary value problems of 2nd order with the Laplace operator as the principal part, discretised by $S_\tau^{r,p}$ for $r \geq 0$ (for a proof, see [3]).*

4 Local representation of hypersingular integrals

For the approximation of the integrals (14), it is important to localize the integrals over the whole surface Γ by splitting it into a sum over the panels and to transform these local integrals onto fixed reference panels. Then, it suffices to develop cubature rules on these reference elements. In view of the finite part integrals, this splitting and transformation is much more delicate as for weakly singular integrals where such transformations are straightforward. For simplicity, we abbreviate the integrand in (14) with

$$k^{new}(x, y, z) := \varphi_q(x)\, \varphi_r(y)\, k(x, y, z)$$

and skip the superscript *new* in the following. For $K_i, K_j \in \tau$, we define the function $H_{i,j} : K_i \to \mathbb{C}$ by

$$H_{i,j}(x) := p.f. \int_{K_j} k(x, y, y - x)\, d\Gamma_y, \qquad \forall x \in K_i.$$

For $t \in \{i, j\}$, let $Q_t := (\eta_t)^{-1} K_t$ denote the reference element (either the unit square or the unit triangle) where $\eta_t := \eta \circ \kappa_{K_t}$ (see (8)). For the following, it is important that the reference elements Q_t (like the surface elements K_t) are assumed to be *open* sets.

The local kernel function is defined by

$$k_{i,j}(\hat{x}, \hat{y}) := k(\eta_i(\hat{x}), \eta_j(\hat{y}), \eta_j(\hat{y}) - \eta_i(\hat{x}))\, g_i(\hat{x})\, g_j(\hat{y})$$

where, for $t \in \{i, j\}$, the function g_t denotes the surface area element corresponding to the chart η_t. The local version of $H_{i,j}$ is defined by

$$\begin{aligned} &\hat{H}_{i,j} : Q_i \to \mathbb{C} \\ &\hat{H}_{i,j}(\hat{x}) := p.f. \textstyle\int_{Q_j} k_{i,j}(\hat{x}, \hat{y})\, d\hat{y}, \qquad \forall \hat{x} \in Q_i. \end{aligned} \tag{18}$$

Note that, for the regularisation of the finite-part integral in (18), an ε-ball in the *parameter plane* has to be subtracted. It is *not* necessary to perform the limit with respect to the distorted ball $\eta_i^{-1}(B_\varepsilon(x) \cap K_i)$. This fact will simplify the treatment of the hypersingular integrals substantially.

The connection of $H_{i,j}$ and $\hat{H}_{i,j}$ is expressed by the formula

$$g_i(\hat{x})(H_{i,j} \circ \eta_i)(\hat{x}) = \hat{H}_{i,j}(\hat{x}), \qquad \forall \hat{x} \in Q_i$$

which is proved in [24, Theorem 5]. The sum

$$\sum_{K_j \in \tau} \hat{H}_{i,j}(\hat{x}) =: \hat{H}_i(\hat{x}), \qquad \forall \hat{x} \in Q_i$$

can be regarded as a local version of the integral operator $\mathcal{K}$ (up to a bounded factor):

$$g_i(\hat{x})\varphi_q(x)\mathcal{K}[\varphi_r](x) = \hat{H}_i(\hat{x})$$

for all $\hat{x} \in Q$ and $x = \eta_i(\hat{x})$.

The mapping property of $\mathcal{K}$ (see 17) implies $\hat{H}_i \in L^2(\Gamma)$. It follows that the integral (14) equals

$$\sum_{K_i \in \tau} \int_{Q_i} \sum_{K_j \in \tau} \hat{H}_{i,j}(\hat{x})\, d\hat{x}. \tag{19}$$

In Lemma 12, we will prove that $\hat{H}_{i,j}$ is possibly singular only if $\hat{x} \to \partial Q_i$. In this light, we define, for $\delta > 0$, the reduced element Q_i^δ,

1. for $Q_i = (0,1)^2$, by

$$Q_i^\delta := \{\hat{x} \in Q_i \mid \operatorname{dist}(\hat{x}, \partial Q_i) > \delta\}$$

2. for Q_i =unit simplex, by

$$Q_i^\delta := \left\{ \begin{array}{c} \delta < \hat{x}_1 < 1 - 2\delta \\ \delta < \hat{x}_2 < 1 - \delta - \hat{x}_1 \end{array} \right\}.$$

It follows from the weak singularity of $\hat{H}_i(\hat{x})$ that

$$\int_{Q_i} \hat{H}_i(\hat{x})\, d\hat{x} = \lim_{\delta \to 0} \int_{Q_i^\delta} \sum_{K_j \in \tau} \hat{H}_{i,j}(\hat{x})\, d\hat{x} < \infty. \tag{20}$$

Since $\hat{H}_{i,j}$ is possibly singular only if $\hat{x} \to \partial Q_i$ (see Lemma 12), the integrand is bounded on Q_i^δ and we may interchange the summation with the integration:

$$\int_{Q_i} \hat{H}_i(\hat{x})\, d\hat{x} = \lim_{\delta \to 0} \sum_{K_j \in \tau} \int_{Q_i^\delta} \hat{H}_{i,j}(\hat{x})\, d\hat{x}.$$

In Lemma 6(c), Lemma 8, and Lemma 11, we will prove that the integrals on the right-hand side above have an expansion of the form

$$\int_{Q_i^\delta} \hat{H}_{i,j}(\hat{x})\, d\hat{x} = I_{\log}^{i,j} \log \delta + I_0^{i,j} + I_1^{i,j}(\delta) \tag{21}$$

where $I_1^{i,j}(\delta) \to 0$ as $\delta \to 0$. From the boundedness of the integral (20), it follows that

$$\sum_{K_j \in \tau} I_{\log}^{i,j} = 0$$

holds. This motivates the definition of a further finite-part integral:

$$\underset{\delta}{p.f.} \int_{Q_i} \hat{H}_{i,j}(\hat{x})\, d\hat{x} := I_0^{i,j}.$$

Lemma 4 *The integral*

$$\int_\Gamma p.f. \int_\Gamma k(x, y, y - x)\, d\Gamma_y d\Gamma_x$$

has the local representation

$$\sum_{K_i \in \tau} \sum_{K_j \in \tau} \underset{\delta}{p.f.} \int_{Q_i} p.f. \int_{Q_j} k_{i,j}(\hat{x}, \hat{y})\, d\hat{y} d\hat{x}. \tag{22}$$

The inner finite-part integral reduces to the usual Riemann integral if $K_i \neq K_j$ holds. The outer finite-part integral reduces to an improper integral if $\overline{K_i}$, $\overline{K_j}$ share at most one point.

Proof. Let $K_i \neq K_j$ and $x \in K_i$. The pull back $\hat{x} := \eta_i^{-1}(x)$ satisfies $\hat{x} \in Q_i$. For sufficiently small $\varepsilon > 0$, the ball $B_\varepsilon(\hat{x})$ has positive distance from ∂Q_i. Hence, the transformed ball $\eta_i(B_\varepsilon(\hat{x}))$ has positive distance from ∂K_i and, consequently, also positive distance from K_j. Therefore, the integrand is bounded and the integral converges to the usual Riemann integral as $\varepsilon \to 0$.

If $\overline{K_i}$ and $\overline{K_j}$ share at most one point the result follows from Lemma 6 (b), (c). ■

Remark 2 *Formula (22) is a local representation of hypersingular kernel functions. The elements Q_i, Q_j are fixed reference elements (either the unit square or the unit triangle). Thus, it is sufficient to develop cubature rules on these reference elements (see Section 6).*

Remark 3 *In [14], a splitting of (14) into local quantities is derived and worked out in the case of flat triangular panels. The approach in the cited paper is different from (22); the local quantities are of the form*

$$\underset{\varepsilon}{p.f.} \int_{K_i} \int_{K_j} \ldots dx dy \qquad \|x-y\| \geq \varepsilon$$

while additional line functionals (which can be evaluated efficiently) appear in the local representation.

5 Finite-part-free representation of hypersingular integrals

As mentioned before, the single terms $\hat{H}_{i,j}$ are not integrable in general and, hence, the inner sum in (19) may not be interchanged with the outer integral. In the following, we will work out the character of the singularity of $\hat{H}_{i,j}$ in detail. These results will play the key role in Section 6 for constructing appropriate variable transforms rendering the integrands analytic such that the integrals can be approximated efficiently by (tensor versions) of Gaussian quadrature rules.

In order to characterise the regularity of the function $\hat{H}_{i,j}$ we will distinguish the following three cases.

I. $\overline{K_i}$ and $\overline{K_j}$ share at most one common point.

II. $\overline{K_i}$ and $\overline{K_j}$ share exactly one edge.

III. $K_i = K_j$.

<u>**Case I:**</u>

From the analyticity of the charts η_i, η_j it follows that the functions g_i, g_j and the coefficients κ_i, ρ_j from (12) are analytic in local coordinates. The pull backs of the basis functions $\varphi_i \circ \eta_i$, $\varphi_j \circ \eta_j$ are analytic, too. Thus, the singular behaviour of the kernel function is characterized by the singular behaviour of the function[2] $\|z\|^{-3} A\left(\|z\|, \frac{z}{\|z\|}\right)$ in local coordinates. If $\overline{K_i} \cap \overline{K_j} = \emptyset$ the kernel function is analytic in local coordinates. Therefore, we assume for the following that the panels share exactly one point: $\overline{K_i} \cap \overline{K_j} = P$. In local coordinates, the difference $z = y - x$ takes the form

$$z = \eta_j(\hat{y}) - \eta_i(\hat{x}).$$

Without loss of generality we assume that $\eta_j\binom{0}{0} = \eta_i\binom{0}{0} = P$. Obviously, $z = 0$ if and only if $\hat{y} = \hat{x} = (0,0)^T$. Taylor expansion of η_j and η_i about the origin results in

$$z = \sum_{m=1}^{\infty} \frac{\left(\langle \hat{y}, \nabla \rangle^m \eta_j(0)\right) - \left(\langle \hat{x}, \nabla \rangle^m \eta_i(0)\right)}{m!}, \tag{23}$$

where the differential operator $\langle \hat{y}, \nabla \rangle^m$ is defined by

$$\langle \hat{y}, \nabla \rangle^m \eta_j = \sum_{k=0}^{m} \binom{m}{k} \hat{y}_1^k \hat{y}_2^{m-k} \left(\partial_1^k \partial_2^{m-k} \eta_j\right).$$

Let $\hat{z} = (\hat{x}, \hat{y})$. We introduce four-dimensional polar coordinates by

$$\hat{z} = r\xi \tag{24}$$

[2]For simplicity, we write A instead of $A_{i,j}$.

with $r = \|\hat{z}\|$ and $\xi = \hat{z}/\|\hat{z}\| \in S_3$. Then, (23) becomes

$$z = r \sum_{m=0}^{\infty} r^m l_m(\xi) =: r a_1(r, \xi) \tag{25}$$

with

$$l_m(\xi) := \frac{\left(\langle \xi_{34}, \nabla \rangle^{m+1} \eta_j(0)\right) - \left(\langle \xi_{12}, \nabla \rangle^{m+1} \eta_i(0)\right)}{(m+1)!}$$

and $\xi_{\theta t} = (\xi_\theta, \xi_t)^T$. The function $a_1(r, \xi)$ is analytic in any compact neighbourhood of $r = 0$ and is analytic in ξ in a suitable neighbourhood of S_3. As in [24, Lemma 1, Remark 7] one can show that $a_1(r, \xi)$ has no zero in a neighbourhood of $r = 0$ and $\xi \in S_3$. Consequently, $\|z\|^{-s}$ admits the local representation about $r = 0$:

$$\|z\|^{-s} = r^{-s} a_{2,s}(r, \xi) \tag{26}$$

where $a_{2,s}$ is analytic in a neighbourhood of $r = 0$ and $\xi \in S_3$. The ratio $\frac{z}{\|z\|}$ similarly can be expanded by multiplying (25) with (26) (choosing $s = 1$) resulting in

$$\frac{z}{\|z\|} = a_{2,1}(r, \xi) a_1(r, \xi) =: a_3(r, \xi),$$

where the function $a_3(r, \xi)$ is analytic with respect to r in a neighbourhood of $r = 0$ and with respect to ξ in a neighbourhood of S_3. Combining these expansions we have proven that, for sufficiently small r and $\xi \in S_3$, the kernel function $k_{i,j}(x, y)$ can be expressed in local coordinates by

$$k_{i,j}(\hat{x}, \hat{y}) = r^{-s} a_4(r, \xi). \tag{27}$$

where a_4 is analytic for $r \leq \delta$ with sufficiently small $\delta > 0$, and analytic in ξ in a neighbourhood of S_3. On the other hand, $r > \delta/2$ implies that $\|x - y\| \geq C\delta$ holds. Hence, in this case the kernel function is analytic, too. It follows that $a_4(r, \xi)$ is analytic for all $r = \|\hat{z}\|$, $\xi = \hat{z}/\|\hat{z}\|$ with $\hat{z} = (\hat{x}, \hat{y})$ and all $\hat{x} \in \overline{Q_i}$, $\hat{y} \in \overline{Q_j}$.

Proposition 5 *Let $\overline{K_i}$ and $\overline{K_j}$ share at most one common point (Case I). Then, the integral*

$$\int_{Q_i} \int_{Q_j} k_{i,j}(\hat{x}, \hat{y}) \, d\hat{y} d\hat{x} \tag{28}$$

exists as an improper integral.

Proof. Due to the analyticity of a_4, the integrand can be estimated by cr^{-s}. By introducing four-dimensional polar coordinates $(\hat{x}, \hat{y}) = r\psi(\alpha_1, \alpha_2, \alpha_3)$ with $r^2 = \|\hat{x}^2\| + \|\hat{y}\|^2$ and $\psi = \frac{(\hat{x}, \hat{y})^T}{r} \in S_3$ the integrand in the new variables can be estimated by the constant c from which the integrability follows. ■

Lemma 6 *(a) If $\overline{K_i}$ and $\overline{K_j}$ share at most one point then the function $\hat{H}_{i,j}$ has the representation*

$$\hat{H}_{i,j}(\hat{x}) = \int_{Q_j} k_{i,j}(\hat{x},\hat{y})\, d\hat{y},$$

where, for all $\hat{x} \in Q_i$, the integral exists as a usual Riemann integral.

(b) The function $\hat{H}_{i,j}(\hat{x})$ is weakly singular, i.e.

$$\int_{Q_i} \hat{H}_{i,j}(\hat{x})\, d\hat{x}$$

coincides with (28).

(c) Expansion (21) is valid with $I_{\log}^{i,j} = 0$:

$$\int_{Q_i^\delta} \hat{H}_{i,j}(\hat{x})\, d\hat{x} = \int_{Q_i} \hat{H}_{i,j}(\hat{x})\, d\hat{x} + R(\delta) \tag{29}$$

where $R(\delta) \to 0$ as $\delta \to 0$.

(d) The function $\hat{H}_{i,j}(\hat{x})$ is analytic in Q_i.

Proof. The first assertion follows from (12) since, for fixed $\hat{x} \in Q_i$, the integrand is bounded and the finite part integral coincides with the usual Riemann integral. The second assertion follows from Proposition 5 and Fubini's theorem.

Expansion (29) follows from statement (b) as in [8, Chap. 6.1.3].

The analyticity of $\hat{H}_{i,j}$ is a direct consequence of (27).

■

This result will later be the base for the construction of the cubature method.

Case II:

In the following, we will investigate the singular behaviour of $H_{i,j}$ in the case that $\overline{K_i}$ and $\overline{K_j}$ share exactly one edge. Without loss of generality we assume that the charts η_i, η_j mapping the reference elements Q_i, Q_j onto K_i, K_j satisfy

$$\eta_i\binom{t}{0} = \eta_j\binom{t}{0} \qquad \forall t \in [0,1]. \tag{30}$$

Hence, the difference

$$z = \eta_j(\hat{y}) - \eta_i(\hat{x})$$

is zero if and only if the three-dimensional relative coordinates

$$\hat{z} := \begin{pmatrix} \hat{y}_1 - \hat{x}_1 \\ \hat{y}_2 \\ \hat{x}_2 \end{pmatrix} \tag{31}$$

equals zero. The difference z then can be rewritten as

$$z = \eta_j\binom{\hat{x}_1+\hat{z}_1}{\hat{z}_2} - \eta_i\begin{pmatrix} \hat{x}_1 \\ \hat{z}_3 \end{pmatrix}.$$

Using the abbreviation $r = \|\hat{z}\|$ and $\xi = \hat{z}/r$ and expanding z about $r = 0$ yields the representation

$$z = \eta_j \binom{\hat{x}_1 + \hat{z}_1}{\hat{z}_2} - \eta_i \begin{pmatrix} \hat{x}_1 \\ \hat{z}_3 \end{pmatrix} = r \sum_{m=0}^{\infty} r^m \lambda_m \left(\hat{x}_1, \xi\right)$$

with

$$\lambda_m \left(\hat{x}_1, \xi\right) = \frac{\left\langle \xi_{12}, \nabla \right\rangle^{m+1} \eta_j \binom{\hat{x}_1}{0} - \left(\xi_3 \partial_2\right)^{m+1} \eta_i \binom{\hat{x}_1}{0}}{(m+1)!}.$$

Similarly as in Case I, the following expansion is derived

$$k_{i,j} \left(\hat{x}, \hat{y}\right) = r^{-s} b \left(\hat{x}_1, r, \xi\right), \tag{32}$$

where $b\left(\hat{x}_1, r, \xi\right)$ is analytic with respect to

1. $\hat{x}_1$ in a neighbourhood of $(0, 1)$,
2. r in a neighbourhood of $\left\{r \in \mathbb{R} \mid \exists \hat{x} \in Q_i, \hat{y} \in Q_j : r^2 = \hat{x}_2^2 + \hat{y}_2^2 + \left(\hat{y}_1 - \hat{x}_1\right)^2\right\}$,
3. ξ in a neighbourhood of S_2.

In contrast to the result of Proposition 5, the function $\hat{H}_{i,j}\left(\hat{x}\right)$ contains non-integrable singularities for $\hat{x}_2 \to 0$. In this light, we will investigate the integrals

$$\int_{Q_i^\delta} \hat{H}_{i,j} \left(\hat{x}\right) d\hat{x}$$

as $\delta \to 0$ (cf. (21)). In Lemma 7, we will prove that Q_i^δ can be replaced by the simpler domain $\tilde{Q}_i^\delta$ defined by

$$\tilde{Q}_i^\delta := \left\{\hat{x} \in Q_i \mid \hat{x}_2 > \delta\right\}. \tag{33}$$

Lemma 7 *The difference*

$$R(\delta) := \int_{Q_i^\delta} \hat{H}_{i,j}\left(\hat{x}\right) d\hat{x} - \int_{\tilde{Q}_i^\delta} \hat{H}_{i,j}\left(\hat{x}\right) d\hat{x}$$

converges to zero as $\delta \to 0$.

Proof. Let Q_i^I denote the triangle with vertices $(0,0)^T$, $(1,0)^T$, and $(1/4, 1/4)^T$. The complement is denoted by $Q_i^{II} := Q_i \backslash \overline{Q_i^I}$. It suffices to prove that $\hat{H}_{i,j}$ is weakly singular on Q_i^{II}. The assertion then follows from [8, Chap. 6.1.3]. From (32), it follows

$$\left|\hat{H}_{i,j}\left(\hat{x}\right)\right| \leq \left|\int_{Q_j} k_{i,j}\left(\hat{x}, \hat{y}\right) d\hat{x}\right| \leq \int_0^1 \int_0^1 \frac{c}{\left(\hat{x}_2^2 + \left(\hat{y}_1 - \hat{x}_1\right)^2 + \hat{y}_2^2\right)^{3/2}} d\hat{y}.$$

Introducing polar coordinates about $(\hat{x}_1, 0)^T$ results in

$$\left|\hat{H}_{i,j}(\hat{x})\right| \leq \int_0^{\pi} \int_0^{R(\alpha,\hat{x}_1)} \frac{cr}{(\hat{x}_2^2 + r^2)^{3/2}} dr d\alpha$$

where $R(\alpha, \hat{x}_1)$ denotes the upper limit of the r-integration. Performing the r-integration analytically yields

$$\left|\hat{H}_{i,j}(\hat{x})\right| \leq c \int_0^{\pi} \frac{1}{\hat{x}_2} - \frac{1}{\sqrt{\hat{x}_2^2 + R(\alpha, \hat{x}_1)^2}} d\alpha \leq \frac{\pi c}{\hat{x}_2}.$$

The weak singularity of $\hat{H}_{i,j}$ on Q_i^{II} follows from

$$\int_{Q_i^{II}} \left|\hat{H}_{i,j}(\hat{x})\right| d\hat{x} \leq \int_0^1 \int_{\hat{x}_1}^1 \frac{\pi c}{\hat{x}_2} d\hat{x} + \int_0^1 \int_{(1-\hat{x}_1)/3}^1 \frac{\pi c}{\hat{x}_2} d\hat{x} = \pi c (2 + \ln 3).$$

■

In view of (21), we have to show that

$$I(\delta) := \int_{\tilde{Q}_i^{\delta}} \int_{Q_j} k_{i,j}(\hat{x}, \hat{y}) \, d\hat{y} d\hat{x} \tag{34}$$

admits an expansion of the form

$$I(\delta) = I_{\log} \log \delta + I_0 + I_1(\delta) \tag{35}$$

where $I_1(\delta)$ converges to zero as $\delta \to 0$. Since the integrand in (34) is analytic the transformation rule of variable applies and the domain of integration can be split into appropriate subdomains. We introduce relative coordinates by

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} := M\hat{z} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{z}_3 \\ \hat{z}_4 \end{pmatrix}.$$

As an abbreviation we write $\hat{k}(\hat{z}) = k_{i,j}((M\hat{z})_{12}, (M\hat{z})_{34})$. The domain of integration is given by

$$D = \left\{ \begin{array}{c} \delta \leq \hat{z}_2 \leq 1 \\ 0 \leq \hat{z}_1 \leq q_i(\hat{z}_2) \\ -\hat{z}_1 \leq \hat{z}_3 \leq 1 - \hat{z}_1 \\ 0 \leq \hat{z}_4 \leq q_j(\hat{z}_3 + \hat{z}_1) \end{array} \right\}, \tag{36}$$

where, for $t \in \{i, j\}$,

$$q_t(w) := \begin{cases} 1 & \text{if } Q_t = (0,1)^2, \\ 1 - w & \text{if } Q_t \text{ is the unit triangle.} \end{cases} \tag{37}$$

The integrand $\hat{k}$ is singular only if $(\hat{z}_1, \hat{z}_2, \hat{z}_3)^T = 0$ (cf. (31)). Since $\hat{k}$ is smooth with respect to $\hat{z}_1$, we interchange the ordering of integration such that the $\hat{z}_1$ becomes the innermost integration (cf. [21] and [9]). In order to characterize D by a system of inequalities, where $\hat{z}_1$ stands at the last position, one has to split D into subdomains:

$$\begin{aligned} D &= \bigcup_{m=1}^{\nu} D_m, \\ I(\delta) &= \sum_{m=1}^{\nu} \int_{D_m} \hat{k}(\hat{z})\, d\hat{z}, \end{aligned}$$

where $\nu = 2$ if $Q_i = Q_j = (0,1)^2$, $\nu = 5$ if both reference elements are triangles and $\nu = 3$ otherwise. These subdomains are given explicitly below.

Case a: $Q_i = Q_j = (0,1)^2$:

$$D_1 = \left\{ \begin{array}{c} \delta \leq \hat{z}_2 \leq 1 \\ -1 \leq \hat{z}_3 \leq 0 \\ 0 \leq \hat{z}_4 \leq 1 \\ -\hat{z}_3 \leq \hat{z}_1 \leq 1 \end{array} \right\}, \qquad D_2 = \left\{ \begin{array}{c} \delta \leq \hat{z}_2 \leq 1 \\ 0 \leq \hat{z}_3 \leq 1 \\ 0 \leq \hat{z}_4 \leq 1 \\ 0 \leq \hat{z}_1 \leq 1 - \hat{z}_3 \end{array} \right\}.$$

Case b: $Q_i = (0,1)^2$, $Q_j =$unit triangle:

$$D_1 = \left\{ \begin{array}{c} \delta \leq \hat{z}_2 \leq 1 \\ -1 \leq \hat{z}_3 \leq 0 \\ 0 \leq \hat{z}_4 \leq -\hat{z}_3 \\ -\hat{z}_3 \leq \hat{z}_1 \leq 1 \end{array} \right\}, \qquad D_2 = \left\{ \begin{array}{c} \delta \leq \hat{z}_2 \leq 1 \\ -1 \leq \hat{z}_3 \leq 0 \\ -\hat{z}_3 \leq \hat{z}_4 \leq 1 \\ -\hat{z}_3 \leq \hat{z}_1 \leq 1 - \hat{z}_3 - \hat{z}_4 \end{array} \right\},$$

$$D_3 = \left\{ \begin{array}{c} \delta \leq \hat{z}_2 \leq 1 \\ 0 \leq \hat{z}_3 \leq 1 \\ 0 \leq \hat{z}_4 \leq 1 - \hat{z}_3 \\ 0 \leq \hat{z}_1 \leq 1 - \hat{z}_3 - \hat{z}_4 \end{array} \right\}.$$

Case c: $Q_i =$unit triangle, $Q_j = (0,1)^2$:

$$D_1 = \left\{ \begin{array}{c} \delta \leq \hat{z}_2 \leq 1 \\ \hat{z}_2 - 1 \leq \hat{z}_3 \leq 0 \\ 0 \leq \hat{z}_4 \leq 1 \\ -\hat{z}_3 \leq \hat{z}_1 \leq 1 - \hat{z}_2 \end{array} \right\}, \qquad D_2 = \left\{ \begin{array}{c} \delta \leq \hat{z}_2 \leq 1 \\ 0 \leq \hat{z}_3 \leq \hat{z}_2 \\ 0 \leq \hat{z}_4 \leq 1 \\ 0 \leq \hat{z}_1 \leq 1 - \hat{z}_2 \end{array} \right\},$$

$$D_3 = \left\{ \begin{array}{c} \delta \leq \hat{z}_2 \leq 1 \\ \hat{z}_2 \leq \hat{z}_3 \leq 1 \\ 0 \leq \hat{z}_4 \leq 1 \\ 0 \leq \hat{z}_1 \leq 1 - \hat{z}_3 \end{array} \right\}.$$

Case d: $Q_i = Q_j$ =unit triangle:

$$D_1 = \left\{ \begin{array}{c} \delta \le \hat{z}_2 \le 1 \\ \hat{z}_2 - 1 \le \hat{z}_3 \le 0 \\ 0 \le \hat{z}_4 \le \hat{z}_2 - \hat{z}_3 \\ -\hat{z}_3 \le \hat{z}_1 \le 1 - \hat{z}_2 \end{array} \right\}, \qquad D_2 = \left\{ \begin{array}{c} \delta \le \hat{z}_2 \le 1 \\ \hat{z}_2 - 1 \le \hat{z}_3 \le 0 \\ \hat{z}_2 - \hat{z}_3 \le \hat{z}_4 \le 1 \\ -\hat{z}_3 \le \hat{z}_1 \le 1 - \hat{z}_3 - \hat{z}_4 \end{array} \right\},$$

$$D_3 = \left\{ \begin{array}{c} \delta \le \hat{z}_2 \le 1 \\ 0 \le \hat{z}_3 \le \hat{z}_2 \\ 0 \le \hat{z}_4 \le \hat{z}_2 - \hat{z}_3 \\ 0 \le \hat{z}_1 \le 1 - \hat{z}_2 \end{array} \right\}, \qquad D_4 = \left\{ \begin{array}{c} \delta \le \hat{z}_2 \le 1 \\ 0 \le \hat{z}_3 \le \hat{z}_2 \\ \hat{z}_2 - \hat{z}_3 \le \hat{z}_4 \le 1 - \hat{z}_3 \\ 0 \le \hat{z}_1 \le 1 - \hat{z}_3 - \hat{z}_4 \end{array} \right\},$$

$$D_5 = \left\{ \begin{array}{c} \delta \le \hat{z}_2 \le 1 \\ \hat{z}_2 \le \hat{z}_3 \le 1 \\ 0 \le \hat{z}_4 \le 1 - \hat{z}_3 \\ 0 \le \hat{z}_1 \le 1 - \hat{z}_3 - \hat{z}_4 \end{array} \right\}.$$

By applying suitable four-dimensional rotations these subdomains can be mapped onto four-dimensional polyhedrons having the property that the origin is a corner point. We will need the following reference elements

$$R_1 = \left\{ \begin{array}{c} \delta \le v_1 \le 1 \\ 0 \le v_2 \le 1 \\ 0 \le v_3 \le 1 \\ v_3 \le v_4 \le 1 \end{array} \right\} \qquad R_2 = \left\{ \begin{array}{c} \delta \le v_1 \le 1 \\ 0 \le v_2 \le 1 \\ 0 \le v_3 \le v_2 \\ v_2 \le v_4 \le 1 \end{array} \right\} \qquad R_3 = \left\{ \begin{array}{c} 0 \le v_1 \le 1 \\ \delta \le v_2 \le 1 \\ 0 \le v_3 \le v_2 \\ v_2 \le v_4 \le 1 \end{array} \right\}$$

$$R_4 = \left\{ \begin{array}{c} 0 \le v_1 \le 1 \\ \delta \le v_2 \le 1 \\ \delta \le v_3 \le v_2 \\ v_2 \le v_4 \le 1 \end{array} \right\} \qquad R_5 = \left\{ \begin{array}{c} \delta \le v_1 \le 1 \\ \delta \le v_2 \le v_1 \\ \delta \le v_3 \le v_2 \\ v_1 \le v_4 \le 1 \end{array} \right\} \qquad R_6 = \left\{ \begin{array}{c} \delta \le v_1 \le 1 \\ \delta \le v_2 \le v_1 \\ 0 \le v_3 \le v_2 \\ v_1 \le v_4 \le 1 \end{array} \right\}$$

$$R_7 = \left\{ \begin{array}{c} \delta \le v_1 \le 1 \\ 0 \le v_2 \le v_1 \\ 0 \le v_3 \le v_2 \\ v_1 \le v_4 \le 1 \end{array} \right\} \qquad R_8 = \left\{ \begin{array}{c} \delta \le v_1 \le 1 \\ \delta \le v_2 \le v_1 \\ 0 \le v_3 \le v_1 \\ v_1 \le v_4 \le 1 \end{array} \right\}.$$

The integral over D can be rewritten as

$$\int_D \hat{k}\,(\hat{z})\, d\hat{z} = \sum_{m=1}^{\mu} \int_{\hat{D}_m} \hat{k}_m\,(v)\, dv, \tag{38}$$

where $\hat{D}_m$ is one of the above reference elements. More precisely

- if $Q_i = Q_j = (0,1)^2$ then

$$\begin{aligned} &\mu = 1 \\ &\hat{D}_1 = R_1, \\ &\hat{k}_1\,(v) = \hat{k}\,(M_1 v) + \hat{k}\,(M_2 v) \end{aligned}$$

with

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

- if $Q_i = (0,1)^2$, Q_j =unit triangle then

$$\begin{aligned} &\mu = 1 \\ &\hat{D}_1 = R_2 \\ &\hat{k}_1(v) = \hat{k}(M_1 v) + \hat{k}(M_2 v) + \hat{k}(M_3 v) \end{aligned}$$

with

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix},$$

- if Q_i =unit triangle, $Q_j = (0,1)^2$ then

$$\begin{aligned} &\mu = 2 \\ &\hat{D}_1 = R_4, && \hat{D}_2 = R_3, \\ &\hat{k}_1(v) = \hat{k}(M_1 v) + \hat{k}(M_2 v), && \hat{k}_2(v) = \hat{k}(M_3 v) \end{aligned}$$

with

$$M_1 = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

- if $Q_i = Q_j$ =unit triangle then

$$\begin{aligned} &\mu = 4, \\ &\hat{D}_1 = R_5, && \hat{D}_m = R_{m+4}, && m = 2,3,4 \\ &\hat{k}_1(v) = \hat{k}(M_1 v) + \hat{k}(M_2 v), && \hat{k}_m(v) = \hat{k}(M_{m+1} v), \end{aligned}$$

with

$$M_1 = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \qquad M_5 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The linearity of the transformations M_m implies that the kernel function $\hat{k}_m$ is singular if and only if $(v_1, v_2, v_3)^T = 0$ (cf. (31)). Furthermore, the smoothness behaviour of $\hat{k}_m$ can be derived from the smoothness behaviour of the function $k_{i,j}$ of (32). In particular, $\hat{k}_m$ is analytic in the variable v_4 (which corresponds to the behaviour of $k_{i,j}(\hat{x}, \hat{y})$ with respect to $\hat{x}_1$). Hence, the innermost integration (with respect to v_4) in (38) defines a function $\hat{\kappa}_m(v_1, v_2, v_3)$ which has the same smoothness behaviour as $k_{i,j}$ as a function of $\hat{z}$ (see (31)). Let $\hat{D}_m^-$ denote the domain $\hat{D}_m$ reduced by the last variable:

$$\hat{D}_m^- := \left\{ w \in \mathbb{R}^3 \mid \exists v \in \hat{D}_m, \forall i = 1, 2, 3 : w_i = v_i \right\}$$

resulting in

$$\int_{\hat{D}_m} \hat{k}_m(v)\, dv = \int_{\hat{D}_m^-} \hat{\kappa}_m(w)\, dw. \tag{39}$$

For the further examination of the last integral we introduce spherical coordinates. Let

$$\psi(\alpha, \beta) = \begin{pmatrix} \cos\alpha \cos\beta \\ \sin\alpha \cos\beta \\ \sin\beta \end{pmatrix}$$

and the permutation operator $\Pi_{i,j,k} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\Pi_{i,j,k} = (\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k),$$

where $(\mathbf{e}_m)_n = \delta_{m,n}$ and $\delta_{m,n}$ denotes the Kronecker delta. In order to describe the domains D_m^- in spherical coordinates (r, α, β) we will employ the following six three-dimensional parameter domains $P_{\theta,m}(\delta)$, $1 \le \theta \le 3$, $1 \le m \le 2$. Let

$$\begin{array}{ll} a_1 = 0, & a_2 = a_3 = \arctan\delta, \\ b_1(\alpha) = b_2(\alpha) = 0, & b_3(\alpha) := \arctan(\delta\cos\alpha), \\ B_1(\alpha) := \arctan\cos\alpha, & B_2(\alpha) := \arctan\sin\alpha. \end{array} \tag{40}$$

Then,

$$P_{\theta,m} = \left\{ \begin{array}{c} \alpha_\theta \le \alpha \le \frac{\pi}{4} \\ b_\theta(\alpha) \le \beta \le B_m(\alpha) \\ \frac{\delta}{\psi_\theta(\alpha,\beta)} \le r \le \frac{1}{\cos\alpha\cos\beta} \end{array} \right\}.$$

We do not indicate explicitly the dependence of $P_{\theta,m}$ on δ, while, for $\delta = 0$, we write $P_{\theta,m}^0$ instead. Then, the integral (38) can be written in the following form

$$\int_D \hat{k}(v)\, dv = \sum_{m=1}^{\lambda} \int_{\check{D}_m^-} \check{\kappa}_m(r\psi)\, r^2 \cos\beta\, dr\, d\beta\, d\alpha. \tag{41}$$

More precisely

- if $Q_i = Q_j = (0,1)^2$ then

$$\begin{array}{ll} \lambda = 2 & \\ \check{D}_1^- = P_{1,1}, & \check{D}_2^- = P_{2,1}, \\ \check{\kappa}_1(w) = \hat{\kappa}_1(w), & \check{\kappa}_2(w) = \hat{\kappa}_1(\Pi_{2,1,3}w) + \hat{\kappa}_1(\Pi_{3,1,2}), \end{array}$$

- if $Q_i = (0,1)^2$, Q_j =unit triangle then

$$\begin{array}{ll} \lambda = 2 & \\ \check{D}_1^- = P_{1,2}, & \check{D}_2^- = P_{2,1}, \\ \check{\kappa}_1(w) = \hat{\kappa}_1(w), & \check{\kappa}_2(w) = \hat{\kappa}_1(\Pi_{2,1,3}w), \end{array}$$

- if Q_i =unit triangle, $Q_j = (0,1)^2$ then

$$\begin{array}{llll} \lambda = 4 & & & \\ \check{D}_1^- = P_{2,1}, & \check{D}_2^- = P_{3,2}, & \check{D}_3 = P_{1,1}, & \check{D}_4 = P_{2,2}, \\ \check{\kappa}_1(w) = \hat{\kappa}_1(\Pi_{2,3,1}w), & \check{\kappa}_2(w) = \hat{\kappa}_1(w), & \check{\kappa}_3(w) = \hat{\kappa}_2(\Pi_{2,1,3}w), & \check{\kappa}_4(w) = \hat{\kappa}_2(w), \end{array}$$

- if $Q_i = Q_j$ =unit triangle then

$$\begin{array}{llll} \lambda = 4 & & & \\ \check{D}_1^- = P_{3,2}, & \check{D}_2^- = P_{2,2}, & \check{D}_3^- = P_{1,2}, & \check{D}_4^- = P_{2,1}, \\ \check{\kappa}_1(w) = \hat{\kappa}_1(w), & \check{\kappa}_2(w) = \hat{\kappa}_2(w) & \check{\kappa}_2(w) = \hat{\kappa}_3(w) & \check{\kappa}_2(w) = \hat{\kappa}_4(w) \end{array}.$$

Since the transformation of the spherical coordinates appearing in (41) onto the coordinates (32) is analytic, the smoothness and singular behaviour of the function $\hat{\kappa}_m(r\psi)$ carries over from the behaviour of $k_{i,j}$ (see (32) and the remarks thereafter). This means that $r^2\check{\kappa}_m(r\psi)\cos\beta$ can be represented by

$$r^2\check{\kappa}_m(r\psi)\cos\beta = \frac{\check{\kappa}_m^0(\psi)}{r} + \check{\kappa}_m^1(r\psi),$$

where

$$\check{\kappa}_m^0(\psi) \quad : \quad = \lim_{r\to 0}\left(r^3\check{\kappa}_m(r\psi)\cos\beta\right), \tag{42}$$

$$\check{\kappa}_m^1(r\psi) \quad : \quad = \check{\kappa}_m(r\psi) - \check{\kappa}_m^0(\psi) \tag{43}$$

are analytic with respect to r, α, β. The domain $\check{D}_m^-$ depends on δ. Since κ_m^1 is analytic the coefficient $I_{\log}$ in (35) depends only on the integral

$$\int_{\check{D}_m^-} \frac{\check{\kappa}_m^0(\psi)}{r} dr d\alpha d\beta.$$

The r-integration can be carried out analytically. We obtain

$$\begin{aligned} \int_{\check{D}_m^-} \frac{\check{\kappa}_m^0(\psi)}{r} dr d\alpha d\beta &= -\log\delta \int_{a_\theta}^{\pi/4}\int_{b_\theta(\alpha)}^{B_t(\alpha)} \check{\kappa}_m^0(\psi)\, d\alpha d\beta \\ &\quad + \int_{a_\theta}^{\pi/4}\int_{b_\theta(\alpha)}^{B_t(\alpha)} \check{\kappa}_m^0(\psi) \log\frac{\psi_\theta(\alpha,\beta)}{\cos\alpha\cos\beta} d\alpha d\beta, \end{aligned}$$

where a_θ, b_θ, B_t are as in (40) and (θ, t) is determined by

$$\check{D}_m^- = P_{\theta,t}. \tag{44}$$

The first integral defines the coefficient $I_{\log,m}$ while the second integrand is weakly singular. From [8, Chap. 6.1.3], it follows that

$$\int_{\check{D}_m^-} \frac{\check{\kappa}_m^0(\psi)}{r} dr d\alpha d\beta = I_{\log,m}\log\delta + \int_0^{\pi/4}\int_0^{B_t(\alpha)} \check{\kappa}_m^0(\psi)\log\frac{\psi_\theta(\alpha,\beta)}{\cos\alpha\cos\beta} d\alpha d\beta + I_{1,m}(\delta) \tag{45}$$

holds, where $I_{1,m}(\delta) \to 0$ as $\delta \to 0$. The considerations above are summarized in the next lemma.

Lemma 8 *Let $K_i, K_j \in \tau$ share exactly one edge. Then*

$$\int_{Q_i^\delta} \hat{H}_{i,j}(\hat{x})\, d\hat{x} = I_{\log}\log\delta + I_0 + I_1(\delta)$$

holds, where $I_1(\delta) \to 0$ as $\delta \to 0$ and

$$\begin{aligned} I_{\log} &= \textstyle\sum_{m=1}^{\lambda} I_{\log,m}, \\ I_0 &= \textstyle\sum_{m=1}^{\lambda} \displaystyle\int_0^{\pi/4}\int_0^{B_t(\alpha)} \check{\kappa}_m^0(\psi)\log\tfrac{\psi_\theta(\alpha,\beta)}{\cos\alpha\cos\beta} d\alpha d\beta + \int_{\check{D}_m^0} \check{\kappa}_m^1(r\psi)\, dr d\beta d\alpha. \end{aligned}$$

The function B_t and indices (θ, t) are defined as in (44) and $\check{D}_m^0$ is obtained by setting $\delta = 0$ in the definition of $P_{\theta,t}$.

Proof. The proof follows from the representation (45). ■

Cubature rules for computing the integrals appearing in the definition of I_0 will be presented in Section 6.

Case III:

It remains to discuss the case of coinciding panels $K_i = K_j$. The function $\hat{H}_{i,j}$ is defined as a finite-part integral

$$\hat{H}_{i,i}(\hat{x}) := \underset{\delta}{p.f.} \int_{Q_i} k_{i,i}(\hat{x}, \hat{y})\, d\hat{y}.$$

The integrand is singular only if $y = x$. The difference

$$z = \eta_i(\hat{y}) - \eta_i(\hat{x})$$

is zero only if $\hat{y} = \hat{x}$. In this light, we introduce the following two-dimensional relative coordinates (cf. [21], [9])

$$\hat{z} = \hat{y} - \hat{x}$$

resulting in

$$z = \eta_i (\hat{x} + \hat{z}) - \eta_i (\hat{x}) .$$

Using the abbreviation $r = \|\hat{z}\|$ and $\xi = \hat{z}/r$ and expanding z about $r = 0$ yields the representation

$$z = \eta_i (\hat{x} + r\xi) - \eta_i (\hat{x}) = r \sum_{m=0}^{\infty} r^m \chi_m (\hat{x}, \xi) ,$$

where

$$\chi_m (\hat{x}, \xi) = \frac{\langle \xi, \nabla \rangle^{m+1} \eta_i (\hat{x})}{(m+1)!}.$$

Similarly as in Case I, the following expansion is derived

$$k_{i,i} (\hat{x}, \hat{y}) = r^{-s} c (\hat{x}, r, \xi) , \tag{46}$$

where $c (\hat{x}, r, \xi)$ is analytic with respect to

1. $\hat{x}$ in a neighbourhood of Q_i,
2. r in a neighbourhood of $\{r \in \mathbb{R} \mid \hat{x}, \hat{y} \in Q_i : r = \|\hat{y} - \hat{x}\|\}$,
3. ξ in a neighbourhood of S_1.

Due to the analyticity of c we can define the following functions

$$c_0 (\hat{x}, \xi) = \lim_{r \to 0} c (\hat{x}, r, \xi) \tag{47}$$

$$c_1 (\hat{x}, \xi) = \lim_{r \to 0} \partial_r c (\hat{x}, r, \xi) \tag{48}$$

$$k_{reg} (\hat{x}, \hat{x} + r\xi) = \frac{c (\hat{x}, r, \xi) - c_0 (\hat{x}, \xi) - r c_1 (\hat{x}, \xi)}{r^s}. \tag{49}$$

In view of the analyticity of the coefficients c, c_0, c_1, and of the kernel function, $k_{reg} (\hat{x}, \hat{x} + r\xi)$ can be written as

$$k_{reg} (\hat{x}, \hat{x} + r\xi) = \frac{\tilde{c} (\hat{x}, r, \xi)}{r}, \tag{50}$$

where $\tilde{c}$ has the same analyticity behaviour as the function c from (46). Similarly as worked out in Proposition 5 and Lemma 6 one proves that

$$\hat{H}_{i,i}^{reg} (\hat{x}) := \int_{Q_i} k_{reg} (\hat{x}, \hat{y}) \, d\hat{y}$$

exists as an improper integral and is weakly singular with respect to $\hat{x}$. Altogether, the representation

$$\begin{aligned}\hat{H}_{i,i}(\hat{x}) &= \underset{\delta}{p.f.}\int_{\mathcal{P}_i}\frac{c_0(\hat{x},\psi(\alpha))}{r^2}drd\alpha + \underset{\delta}{p.f.}\int_{\mathcal{P}_i}\frac{c_1(\hat{x},\psi(\alpha))}{r}drd\alpha + \hat{H}^{reg}_{i,i}(\hat{x}) \\ &=: \hat{H}^0_{i,i}(\hat{x}) + \hat{H}^1_{i,i}(\hat{x}) + \hat{H}^{reg}_{i,i}(\hat{x}) \end{aligned} \tag{51}$$

is proved, where

$$\psi(\alpha) := \begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}$$

and $\mathcal{P}_i$ denotes the domain $(Q_i - \hat{x})$ in polar coordinates:

$$\mathcal{P}_i = \left\{\begin{array}{c} 0 \le \alpha \le 2\pi \\ \varepsilon \le r \le R(\alpha,\hat{x})\end{array}\right\}.$$

The function $R(\alpha,\hat{x})$ is defined as follows. Let $\nu \in \{3,4\}$ denote the numbers of corners of Q_i, $\{P_m\}_{m=1}^{\nu}$ the corner points of Q_i (counterclockwise ordering and $P_1 = (0,0)^T$), and $e_m = P_{m+1} - P_m$ the edges of Q_i. The distance of $\hat{x} \in Q_i$ from the *mth* edge is given by

$$d_m(\hat{x}) := \inf_{t\in(0,1)} \|P_m + te_m - \hat{x}\|$$

and the auxiliary functions $a_m(\hat{x}), p_m(\alpha)$ by

	$Q_i = (0,1)^2$		Q_i =unit triangle	
m	$a_m(\hat{x}) =$	$p_m(\alpha) =$	$a_m(\hat{x}) =$	$p_m(\alpha) =$
0	$-\pi + \arctan\frac{\hat{x}_2}{\hat{x}_1}$		$-\pi + \arctan\frac{\hat{x}_2}{\hat{x}_1}$	
1	$-\arctan\frac{\hat{x}_2}{1-\hat{x}_1}$	$-\sin\alpha$	$-\arctan\frac{\hat{x}_2}{1-\hat{x}_1}$	$-\sin\alpha$
2	$\arctan\frac{1-\hat{x}_2}{1-\hat{x}_1}$	$\cos\alpha$	$\pi - \arctan\frac{1-\hat{x}_2}{\hat{x}_1}$	$\frac{\cos\alpha+\sin\alpha}{\sqrt{2}}$
3	$\pi - \arctan\frac{1-\hat{x}_2}{\hat{x}_1}$	$\sin\alpha$	$\pi + \arctan\frac{\hat{x}_2}{\hat{x}_1}$	$-\cos\alpha$
4	$\pi + \arctan\frac{\hat{x}_2}{\hat{x}_1}$	$-\cos\alpha$		

Then, $R(\alpha,\hat{x})$ is given by

$$R(\alpha,\hat{x}) = \frac{d_m(\hat{x})}{p_m(\alpha)}, \qquad \forall\alpha \in (a_{m-1}(\hat{x}), a_m(\hat{x})).$$

The r-integration in the definition of $\hat{H}^0_{i,i}$ and $\hat{H}^1_{i,i}$ can be performed analytically and the finite-part process as well:

$$\hat{H}^0_{i,i}(\hat{x}) = \underset{\delta}{p.f.}\int_0^{2\pi}\int_{\varepsilon}^{R(\hat{x},\alpha)}\frac{c_0(\hat{x},\psi(\alpha))}{r^2}drd\alpha = -\int_0^{2\pi}\frac{c_0(\hat{x},\psi(\alpha))}{R(\hat{x},\alpha)}d\alpha$$

$$= -\sum_{m=1}^{\nu} \frac{1}{d_m(\hat{x})} \int_{a_{m-1}(\hat{x})}^{a_m(\hat{x})} p_m(\alpha)\, c_0(\hat{x}, \psi(\alpha))\, d\alpha =: \sum_{m=1}^{\nu} \frac{C_m(\hat{x})}{d_m(\hat{x})} \quad (52)$$

$$\hat{H}_{i,i}^1(\hat{x}) = \sum_{m=1}^{\nu} \int_{a_{m-1}(\hat{x})}^{a_m(\hat{x})} c_1(\hat{x}, \psi(\alpha)) \log \frac{d_m(\hat{x})}{p_m(\alpha)} d\alpha. \quad (53)$$

The properties of the functions $\hat{H}_{i,i}^0$ and $\hat{H}_{i,i}^1$ are collected in the following Lemma.

Lemma 9 *The integrands in (52) and (53) are weakly singular (with respect to α).*

The function $\hat{H}_{i,i}^1$ is weakly singular while the function $\hat{H}_{i,i}^0$ contains non-integrable singularities as $x \to \partial Q_i$. The function C_m in (52) is analytic. More precisely, the function $\frac{C_m}{d_m}$ is strongly singular only if $\hat{x} \to \overline{P_m P_{m+1}}$ and analytic in Q_i.

Proof. These statements follows directly from (52) and the analyticity of p_m and c_0. ■

Next, we have to prove that

$$\int_{Q_i^\delta} \hat{H}_{i,i}(\hat{x})\, d\hat{x} = I_{\log} \log \delta + I_0 + I_1(\delta) \quad (54)$$

holds with $I_1(\delta) \to 0$ as $\delta \to 0$. Similarly as in the case of a common edge (cf. Lemma 7), this problem can be simplified. Let $\tilde{Q}_i^\delta$ be defined as in (33) and $W_m : Q_i \to Q_i$ denote the affine-linear, orientation preserving function which maps the edge $(0,1) \times \{0\}$ onto the *mth* edge $\overline{P_m P_{m+1}}$ of Q_i and

$$g_m := \begin{cases} \sqrt{2} & \text{if } Q_i \text{ is a triangle and } m = 2, \\ 1 & \text{otherwise.} \end{cases} \quad (55)$$

Lemma 10 *The difference*

$$R(\delta) := \int_{Q_i^\delta} \hat{H}_{i,i}(\hat{x})\, d\hat{x} - \left\{ \int_{Q_i} \hat{H}_{i,i}^{reg}(\hat{x}) + \hat{H}_{i,i}^1(\hat{x})\, d\hat{x} + \sum_{m=1}^{\nu} g_m \int_{\tilde{Q}_i^\delta} \frac{C_m(W_m \hat{x})}{\hat{x}_2} d\hat{x} \right\}$$

converges to zero as $\delta \to 0$.

Proof. We have to show that

$$\tilde{R}(\delta) := g_m \int_{\tilde{Q}_i^\delta} \frac{C_m(W_m \hat{x})}{\hat{x}_2} d\hat{x} - \int_{Q_i^\delta} \frac{C_m(\hat{x})}{d_m(\hat{x})} d\hat{x}$$

converges to zero as $\delta \to 0$. We observe that W_m maps Q_i^δ onto Q_i^δ. Since $d_m(W_m \hat{x}) = \hat{x}_2 / g_m$, it suffices to prove that

$$\int_{\tilde{Q}_i^\delta} \frac{C_m(W_m \hat{x})}{\hat{x}_2} d\hat{x} - \int_{Q_i^\delta} \frac{C_m(W_m \hat{x})}{\hat{x}_2} d\hat{x}$$

converges to zero as $\delta \to 0$. This can be proved as Lemma 7. ■

Now we can prove that (54) holds.

Lemma 11 *The integral* $\int_{Q_i^\delta} \hat{H}_{i,i}(\hat{x})\,d\hat{x}$ *can be written as*

$$\int_{Q_i^\delta} \hat{H}_{i,i}(\hat{x})\,d\hat{x} = I_{\log}\log\delta + I_0 + I_1(\delta)$$

where $I_1(\delta) \to 0$ *as* $\delta \to 0$.

Proof. Due to the analyticity of C_m the functions

$$C_m^0(\hat{x}_1) \quad : \quad = \lim_{\hat{x}_2 \to 0} g_m C_m(W_m \hat{x}), \tag{56}$$

$$C_m^1(\hat{x}) \quad : \quad = \frac{g_m C_m(W_m \hat{x}) - C_m^0(\hat{x}_1)}{\hat{x}_2} \tag{57}$$

are analytic, too. Therefore

$$\begin{aligned} g_m \int_{\tilde{Q}_i^\delta} \frac{C_m(W_m \hat{x})}{\hat{x}_2} d\hat{x} \quad = \quad & -\log\delta \int_0^1 C_m^0(\hat{x}_1)\,d\hat{x}_1 + \int_0^1 C_m^0(\hat{x}_1)\log q_i(\hat{x}_1)\,d\hat{x}_1 \\ & + \int_{Q_i} C_m^1(\hat{x})\,d\hat{x} + R(\delta), \end{aligned} \tag{58}$$

where q_i is as in (37) and $R(\delta) \to 0$ as $\delta \to 0$. ■

Remark 4 *The second term in (58) vanishes if* Q_i *is the unit square due to* $\log q_i = \log 1 = 0$.

As a side result we obtain a statement on the analyticity of $\hat{H}_{i,j}$.

Lemma 12 *The function* $\hat{H}_{i,j}(\hat{x})$ *is analytic in* Q_i *and is possibly singular as* $\hat{x} \to \partial Q_i$.

Proof. Let $\overline{B} \subset Q_i$. For $K_i \neq K_j$, the function $k_{i,j} : B \times Q_j \to \mathbb{C}$ is analytic since $\operatorname{dist}(\eta_i(B), K_j) > 0$. Hence, $H_{i,j}(\hat{x}) = \int_{Q_j} k_{i,j}(\hat{x},\hat{y})\,d\hat{y}$ is analytic in B.

Now, let $K_i = K_j$. We employ the splitting (51). The term $\hat{H}_{i,i}^{reg}$ was defined by

$$\hat{H}_{i,i}^{reg}(\hat{x}) := \int_0^{2\pi} \int_0^{R(\alpha,x)} \tilde{c}\left(\hat{x}, r, \binom{\cos\alpha}{\sin\alpha}\right) dr\,d\alpha$$

with $\tilde{c}$ as in (50). The integrand is analytic in all variables and the upper bound $R(\alpha,\hat{x})$ for the r-integration is also analytic in Q_i. Hence, the same holds for $\hat{H}_{i,i}^{reg}$.

The term $\hat{H}_{i,i}^0$ can be written in the form $\sum_{m=1}^{\nu} \frac{C_m(\hat{x})}{d_m(\hat{x})}$ with analytic C_m and d_m denoting the distance to the mth edge. Hence, $\hat{H}_{i,i}^0$ has also the asserted analyticity properties.

Taylor expansion of the function $c_0(\hat{x}, \psi(\alpha))$ of (53) with respect to $\hat{x}$ and integrating the coefficients with respect to α implies that $\hat{H}_{i,i}^1$ has the asserted analyticity properties, too. ■

The following remark concerns the extension of the presented analysis to the case that the kernel function contains stronger singularities: $s > 3$. However, we emphasize that, in most applications in $\mathbb{R}^3$, $s \leq 3$ holds and the modifications below are irrelevant.

Remark 5 *Let the kernel function be of the form (12) with $s > 3$. If $\overline{K_i}$, $\overline{K_j}$ share exactly one common point, the kernel function in (27) is no longer integrable. One has to subtract a Taylor expansion of $a_4(r, \xi)$ about $r = 0$ (of order $s - 3$), i.e. introduce a further finite part process similarly as in the case of a common edge and $s = 3$.*

In the case of a common edge or for identical panels, the orders of the Taylor expansions appearing in (43), (49), and (57) have to be increased. Furthermore, the replacement of the domain Q_i^δ by the simplified domains $\tilde{Q}_i^\delta$ are no longer possible due to the arising stronger corner singularities. Instead one has to carry out the integration over Q_i^δ explicitly and introduce appropriate regularisations for both, edge and corner singularities.

6 Cubature techniques

In this section, we will define families of cubature rules for the approximation of the local, regularised integrals

$$\underset{\delta}{p.f.} \int_{Q_i} p.f. \int_{Q_j} k_{i,j}(\hat{x}, \hat{y})\, d\hat{y} d\hat{x} \tag{59}$$

appearing in the sum (22). We distinguish the following four cases:

1. $\overline{K_i} \cap \overline{K_j} = \emptyset$,
2. $\overline{K_i}, \overline{K_j}$ share exactly one point,
3. $\overline{K_i}, \overline{K_j}$ share exactly one edge,
4. $K_i = K_j$.

Case 1:

In the first case, the integrand is analytic and both finite part integrals in (59) reduce to usual Riemann integrals. Thus, four-dimensional tensor versions of properly scaled Gauß-Legendre quadrature rules are converging *exponentially* towards the true integral.

Case 2:

The integrand in this case is weakly singular and we can write

$$\int_{Q_i \times Q_j} k_{i,j}(\hat{x}, \hat{y})\, d\hat{y} d\hat{x}. \tag{60}$$

We have to discuss the different cases of Q_i separately.

Case 2a: $Q_i = Q_j$ =unit triangle.

The domain of integration is given by

$$\begin{aligned} 0 &\leq \hat{x}_1 \leq 1, \\ 0 &\leq \hat{x}_2 \leq 1 - \hat{x}_1, \\ 0 &\leq \hat{y}_1 \leq 1, \\ 0 &\leq \hat{y}_2 \leq 1 - \hat{y}_1. \end{aligned}$$

As explained in [4], [21] the transformations

$$\chi^I(\omega, \eta) := \begin{pmatrix} \eta_1 \omega \\ \omega(1 - \eta_1) \\ \eta_2 \eta_3 \omega \\ \eta_2 \omega (1 - \eta_3) \end{pmatrix}, \qquad \chi^{II}(\omega, \eta) := \begin{pmatrix} \eta_2 \eta_3 \omega \\ \eta_2 \omega (1 - \eta_3) \\ \eta_1 \omega \\ \omega(1 - \eta_1) \end{pmatrix}$$

map the four-dimensional unit cube onto two disjoint domains D_1, D_2 satisfying $Q_i \times Q_j = D_1 \cup D_2$. The Jacobian of both mappings is given by $\omega^3 \eta_2$. The integral in (60) becomes

$$\int_{(0,1)^4} \left\{ k_{i,j}\left(\chi_{12}^I, \chi_{13}^I\right) + k_{i,j}\left(\chi_{12}^{II}, \chi_{13}^{II}\right) \right\} \omega^3 \eta_2 d\omega d\eta. \tag{61}$$

Since the transform of these coordinates onto the polar coordinates (24) is analytic the integrand in (61) is analytic, too (cf. [8, Remark 9.4.2]). Thus, four-dimensional Gauß-Legendre formulae defines an *exponentially* convergent family of cubature rules for approximating (60).

Case 2b: $Q_i = Q_j = (0,1)^2$.

The mappings

$$\chi^I(\omega, \eta) := \begin{pmatrix} \omega \\ \eta_1 \omega \\ \eta_2 \omega \\ \eta_3 \omega \end{pmatrix}, \ \chi^{II}(\omega, \eta) := \begin{pmatrix} \eta_3 \omega \\ \omega \\ \eta_1 \omega \\ \eta_2 \omega \end{pmatrix}, \ \chi^{III}(\omega, \eta) := \begin{pmatrix} \eta_2 \omega \\ \eta_3 \omega \\ \omega \\ \eta_1 \omega \end{pmatrix}, \ \chi^{IV}(\omega, \eta) := \begin{pmatrix} \eta_1 \omega \\ \eta_2 \omega \\ \eta_3 \omega \\ \omega \end{pmatrix}$$

map the four dimensional unit cube onto domains $D_I, D_{II}, D_{III}, D_{IV}$ which are mutually disjoint and satisfy

$$Q_i \times Q_j = D_I \cup D_{II} \cup D_{III} \cup D_{IV}.$$

The Jacobian in all cases is given by ω^3. As before the integrand in

$$\int_{(0,1)^4} \omega^3 \sum_{R=I}^{IV} k_{i,j}\left(\chi_{12}^R, \chi_{34}^R\right) d\omega d\eta$$

is analytic and can be approximated by tensor version of Gauß-Legendre rules. This family of rules is again *exponentially* convergent with respect to the order.

Case 2c: Q_i =unit triangle, $Q_j = (0,1)^2$.

The mappings

$$\chi^I(\omega,\eta) := \begin{pmatrix} \eta_3\omega \\ \omega(1-\eta_3) \\ \eta_1\omega \\ \eta_2\omega \end{pmatrix}, \quad \chi^{II}(\omega,\eta) := \begin{pmatrix} \eta_3\eta_2\omega \\ \eta_2\omega(1-\eta_3) \\ \omega \\ \eta_1\omega \end{pmatrix}, \quad \chi^{III}(\omega,\eta) := \begin{pmatrix} \eta_2\eta_3\omega \\ \eta_2\omega(1-\eta_3) \\ \eta_1\omega \\ \omega \end{pmatrix}$$

map the four dimensional unit cube onto domains D_I, D_{II}, D_{III} which are mutually disjoint and satisfy

$$Q_i \times Q_j = D_I \cup D_{II} \cup D_{III}.$$

The Jacobian in the first case is given by ω^3 and, for the 2nd and 3rd case, by $\omega^3\eta_2$. As before the integrand in

$$\int_{(0,1)^4} \omega^3 \left\{ k_{i,j}\left(\chi^I_{12}, \chi^I_{34}\right) + \eta_2 k_{i,j}\left(\chi^{II}_{12}, \chi^{II}_{34}\right) + \eta_2 k_{i,j}\left(\chi^{III}_{12}, \chi^{III}_{34}\right) \right\} d\omega d\eta$$

is analytic and can be approximated by tensor versions of Gauß-Legendre rules. This family of rules is again *exponentially* convergent with respect to the order.

Case 2d: $Q_i = (0,1)^2$, Q_j =unit triangle.

This case can be obtained directly from the previous one by interchanging the role of the variables $\hat{x}$ and $\hat{y}$, i.e. applying the transformation to the function $\tilde{k}_{i,j}$ defined by $\tilde{k}_{i,j}(\hat{x},\hat{y}) := k_{i,j}(\hat{y},\hat{x})$.

We expect that the integral (60) can be treated with extrapolation techniques as well (cf. [16], [15], [17], [25]) while the corresponding error expansions are not worked out yet.

Case 3:

Now, we consider the case where $\overline{K_i}$ and $\overline{K_j}$ share exactly one edge. The following representation was proved in Lemma 8 where the quantities λ, B_m, $\check{\kappa}^{0,1}_m$, ψ, $\check{D}^0_m$ are also defined (for (θ,t), see (44)):

$$\underset{\delta}{p.f.} \int_{Q_i} p.f. \int_{Q_j} k_{i,j}(\hat{x},\hat{y})\, d\hat{y} d\hat{x} = \sum_{m=1}^{\lambda} \int_0^{\pi/4} \int_0^{B_t(\alpha)} \check{\kappa}^0_m(\psi) \log \frac{\psi_\theta(\alpha,\beta)}{\psi_1(\alpha,\beta)} d\alpha d\beta + \int_{\check{D}^0_m} \check{\kappa}^1_m(r\psi)\, dr d\beta d\alpha. \tag{62}$$

It was proved that the functions $\check{\kappa}^{0,1}_m$ are analytic. The domain of integration $\check{D}^0_m$ is given by

$$\begin{aligned} 0 &\leq \alpha \leq \frac{\pi}{4}, \\ 0 &\leq \beta \leq B_t(\alpha), \\ 0 &\leq r \leq \frac{1}{\cos\alpha\cos\beta}. \end{aligned}$$

The integration bounds depend analytically on the parameters. Hence, the second integral on the right-hand side above can be approximated by tensor versions of properly scaled Gauß-Legendre quadrature rules. This family of quadrature methods again is exponentially convergent with respect to the order.

For the first integral on the right-hand side of (62), we observe that,

- for $\theta = 1$, the integral vanishes,
- for $\theta = 2$, the integral can be rewritten as $\int_0^{\pi/4} \log\tan\alpha \int_0^{B_t(\alpha)} \check{\kappa}_m^0(\psi)\, d\alpha d\beta$
- for $\theta = 3$, the integral takes the form

$$\int_0^{\pi/4}\int_0^{B_t(\alpha)} \check{\kappa}_m^0(\psi)\log\frac{\tan\beta}{\cos\alpha} d\alpha d\beta. \tag{63}$$

For $\theta = 2$, the α-integration can be approximated by Gauß-like formula with logarithmic weight (substituting $\tan\alpha = s$) as explained, e.g., in [1], while the integrand is analytic with respect to β. Hence, integration with respect to β can be approximated with Gauß-Legendre quadrature rules.

For $\theta = 3$, we substitute the variable β by

$$\beta(\alpha,\xi) = \begin{cases} \arctan(\xi\cos\alpha) & \text{if } t = 1, \\ \arctan(\xi\sin\alpha) & \text{if } t = 2. \end{cases}$$

Let the auxiliary function $\rho^{(t)}$ and $\psi^{(t)}$ be defined by

$$\begin{array}{ll} \rho^{(1)}(\alpha,\xi) := \frac{\cos\alpha}{1+\xi^2\cos^2\alpha}, & \rho^{(2)}(\alpha,\xi) := \frac{\sin\alpha}{1+\xi^2\sin^2\alpha}, \\ \psi^{(1)}(\alpha,\xi) := \frac{(\cos\alpha,\sin\alpha,\xi\cos\alpha)^T}{\sqrt{1+\xi^2\cos^2\alpha}}, & \psi^{(2)}(\alpha,\xi) := \frac{(\cos\alpha,\sin\alpha,\xi\sin\alpha)^T}{\sqrt{1+\xi^2\sin^2\alpha}}. \end{array}$$

Then, (63) can be rewritten as

$$t = 1: \quad \int_0^1 \log\xi \int_0^{\pi/4} \rho^{(1)}\check{\kappa}_m^0\left(\psi^{(1)}\right) d\alpha d\xi, \tag{64}$$

$$t = 2: \quad \int_0^1 \log\xi \int_0^{\pi/4} \rho^{(2)}\check{\kappa}_m^0\left(\psi^{(2)}\right) d\alpha d\xi + \int_0^{\pi/4} (\log\tan\alpha) \int_0^1 \rho^{(2)}\check{\kappa}_m^0\left(\psi^{(2)}\right) d\xi d\alpha.$$

Hence, Gauß-Legendre rules can be used with respect to those variables where the integrand is analytic and Gauß-like rules with logarithmic weight with respect to the remaining variables.

We conclude this case by discussing how the function $\check{\kappa}_m^0$ (defined in (42)):

$$\check{\kappa}_m^0(\psi) := \lim_{r\to 0}\left(r^3\check{\kappa}_m(r\psi)\cos\beta\right) \tag{65}$$

can be evaluated in quadrature points where $\breve{\kappa}$ was defined by (41). Since the expression in the brackets in (65) is analytic in r the function κ_m^0 can be approximated for sufficiently small $\varepsilon > 0$ by

$$\widetilde{\breve{\kappa}}_m^0(\psi) := \varepsilon^3 \breve{\kappa}_m(\varepsilon\psi)\cos\beta$$

where the arising error is proportionally to ε.

Case 4:

In the case of identical panels, we have proved that

$$\begin{aligned} \underset{\delta}{p.f.}\int_{Q_i} p.f.\int_{Q_i} k_{i,i}(\hat{x},\hat{y})\,d\hat{y}d\hat{x} &= \int_{Q_i\times Q_i} k_{reg}(\hat{x},\hat{y})\,d\hat{y}d\hat{x} \\ &+ \sum_{m=1}^{\nu}\left\{\int_{Q_i}\int_{a_{m-1}}^{a_m} c_1(\hat{x},\psi(\alpha))\log\tfrac{d_m(\hat{x})}{p_m(\alpha)}d\alpha d\hat{x}\right. \\ &\left.+\int_{Q_i} C_m^1(\hat{x})\,d\hat{x} + \int_0^1 C_m^0(\hat{x}_1)\log q_i(\hat{x}_1)\,d\hat{x}_1\right\} \\ &=: I_{reg} + \sum_{m=1}^{\nu}\left\{I_m^{(1)} + I_m^{(2)} + I_m^{(3)}\right\} \end{aligned}$$

holds, where all quantities are defined in Section 4, Case III. For the approximation of I_{reg}, we employ the transformations and cubature techniques as described in [4] and [23, p. 203] which goes back to [21].

The functions C_m^1 are analytic on Q_i and, hence, $I_m^{(2)}$ can also be approximated by tensor versions of properly scaled Gauß-Legendre rules. The last integral $I_m^{(3)}$ vanishes if $Q_i = (0,1)^2$. If Q_i is the unit triangle, this integral can efficiently be approximated by using Gauß-like formulas with weight $\log(1-\hat{x}_1)$.

It remains to consider the approximation of the integral $I_m^{(1)}$. We employ the transformation W_m which was used in Lemma 10 and observe that $d_m(W_m\hat{x}) = \hat{x}_2/g_m$ (cf. (55)) holds. This means that

$$\sum_{m=1}^{\nu} I_m^{(1)} = \sum_{m=1}^{\nu}\int_{Q_i}\int_{a_{m-1}(W_m\hat{x})}^{a_m(W_m\hat{x})} c_1(W_m\hat{x},\psi(\alpha))\log\frac{\hat{x}_2}{g_m p_m(\alpha)}d\alpha d\hat{x}$$

holds, where the functions a_m are as in (53). In the next step, this integral will be transformed onto a standard domain such that the singular behaviour of the integrand is simplified. Let the constant γ_m be defined,

1. for $Q_i = (0,1)^2$, by

$$\gamma_m = (m-2)\frac{\pi}{2}, \qquad m = 1,2,3,4,$$

2. for Q_i =unit-triangle, by

$$\gamma_1 = -\frac{\pi}{2}, \qquad \gamma_2 = \frac{\pi}{4}, \qquad \gamma_3 = \pi.$$

Substituting $\alpha \leftarrow \gamma_m + \arctan \frac{\hat{z}}{\hat{x}_2}$, we obtain

$$I_m^{(1)} = \int_{Q_i} \int_{\rho_m^0(\hat{x})}^{\rho_m^1(\hat{x})} \check{c}_m(\hat{x}, \hat{z})\, \check{\omega}_m(\hat{x}, \hat{z})\, d\hat{z} d\hat{x} \tag{66}$$

where the quantities ρ_m, $\check{c}_m$, and $\check{\omega}_m$ are defined,

1. for $Q_i = (0,1)^2$, by

$$\begin{aligned} \rho_m^1 &= 1 - \hat{x}_1, \\ \rho_m^0 &= -\hat{x}_1, \end{aligned} \qquad m = 1, 2, 3, 4,$$

2. for Q_i =unit triangle, by

$$\begin{aligned} \rho_1^0 &= -\hat{x}_1, & \rho_1^1 &= 1 - \hat{x}_1, \\ \rho_2^0 &= -2\hat{x}_1 - \hat{x}_2, & \rho_2^1 &= 2 - 2\hat{x}_1 - \hat{x}_2, \\ \rho_3^0 &= -\hat{x}_1 - \hat{x}_2, & \rho_3^1 &= 1 - \hat{x}_1 - \hat{x}_2 \end{aligned}$$

and

$$\begin{aligned} \check{c}_m(\hat{x}, \hat{z}) \ &: = c_1\left(W_m \hat{x}, \psi\left(\gamma_m + \arctan \frac{\hat{z}}{\hat{x}_2}\right)\right), \\ \check{\omega}_m(\hat{x}, \hat{z}) \ &: = \frac{\hat{x}_2}{\hat{x}_2^2 + \hat{z}^2} \log \frac{\sqrt{\hat{z}^2 + \hat{x}_2^2}}{g_m}. \end{aligned}$$

Since the integrand in (66) is smooth with respect to $\hat{x}_1$, we interchange the ordering of integration such that $\hat{x}_1$ is the innermost variable. Again the integration domain has to be split into subdomains. Then, the following reference domains

$$R_1 := \left\{ \begin{array}{c} 0 \le v_1 \le 1 \\ 0 \le v_2 \le v_1 \\ 0 \le v_3 \le 1 - v_1 \end{array} \right\}, \qquad R_2 := \left\{ \begin{array}{c} 0 \le v_1 \le 1 \\ 0 \le v_2 \le v_1 \\ 0 \le v_3 \le 1 - v_2 \end{array} \right\}$$

are mapped onto these subdomains. Let transformations $\chi_{t,\mu}$, numbers μ_t, and domains D_t be defined

1. for $Q_i = (0,1)^2$, by:

$$\chi_{11}(v) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \chi_{12}(v) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$
$$\chi_{21}(v) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \chi_{22}(v) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

and

$$\mu_1 = 2, \qquad \mu_2 = 2, \qquad D_1 = R_1, \qquad D_2 = R_2,$$

2. for Q_i =unit triangle, by

$$\mu_1 = 3, \qquad \mu_2 = 0, \qquad D_1 = R_1,$$

and

- for $m = 1$, by:

$$\chi_{11}(v) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \chi_{12}(v) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$
$$\chi_{13}(v) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

- for $m = 2$, by:

$$\chi_{11}(v) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \chi_{12}(v) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$
$$\chi_{21}(v) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ -2 & 1 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

- for $m = 3$, by:

$$\chi_{11}(v) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \chi_{12}(v) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$
$$\chi_{13}(v) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Now, integral (66) can be written in the form

$$\sum_{t=1}^{2} g_m^2 \sum_{\mu=1}^{\mu_t} \int_{D_t} (\breve{\omega}_m \circ \chi_{t\mu})(v)\,(\check{c}_m \circ \chi_{t\mu})(v)\,dv.$$

The transform $v_2 = \xi v_1$ leads to

$$\sum_{t=1}^{2} \int_0^1 \int_0^1 \int_0^{b(v_1,\xi)} g_m^2 \sum_{\mu=1}^{\mu_t} v_1 (\breve{\omega}_m \circ \chi_{t\mu})(v_1, \xi v_1, v_3)\,(\check{c}_m \circ \chi_{t\mu})(v_1, \xi v_1, v_3)\,dv_3 d\xi dv_1. \tag{67}$$

Now, these integrals can be integrated by standard formulae. Exemplarily, we study only the case of the unit square and $\mu = t = 1$, where

$$v_1 \left(\check{\omega}_m \circ \chi_{t\mu}\right)(v_1, \xi v_1, v_3) = \left(\frac{\xi}{1+\xi^2} \log v_1\right) + \left(\frac{\xi \log \sqrt{1+\xi^2}}{1+\xi^2}\right)$$

holds. Hence, integral (67) can be decomposed into two integrals, where one integrand is analytic in all variables and the other is analytic in ξ and v_3 and contains a logarithmic singularity of the form $\log v_1$. Thus, properly scaled Gauß-Legendre formulae with respect to the smooth variables and Gauß-like formulae with logarithmic weight for the integral containing $\log v_1$ converge exponentially.

As in the previous chapter we will finish this section by explaining how the functions c_0 and c_1 appearing in (47) and (48) can be evaluated. Let a pair of points $\hat{x}, \hat{y} \in Q_i$ be given. First, the quantities $r = \|\hat{y} - \hat{x}\|$, $\xi = (\hat{y} - \hat{x})/r$, $y = \eta_i(\hat{y})$, $x = \eta_i(\hat{x})$, and $z = y - x$ have to be computed. As in (46) we define

$$c(\hat{x}, r, \xi) = r^s k_{i,i}(\hat{x}, r\xi + \hat{x}).$$

Then, c_0 is given by

$$c_0(\hat{x}, \xi) = \lim_{r \to 0} c(\hat{x}, r, \xi).$$

We have proven that the function c is analytic with respect to all variables. Hence, for small $\varepsilon > 0$, the function $c_0(\hat{x}, \xi)$ can be approximated by

$$c_0(\hat{x}, \xi) = \frac{1}{2}\left(c(\hat{x}, \varepsilon, \xi) + c(\hat{x}, -\varepsilon, \xi)\right) + O\left(\varepsilon^2\right).$$

For the evaluation of $c_1(\hat{x}, \xi)$ we use the formula

$$c_1(\hat{x}, \xi) = \lim_{r \to 0} \partial_r \left(c(\hat{x}, r, \xi)\right).$$

For sufficiently small value of ε, this quantity can be approximated by

$$\tilde{c}_1(\hat{x}, \xi) = \frac{c(\hat{x}, \varepsilon, \xi) - c(\hat{x}, -\varepsilon, \xi)}{2\varepsilon} + O\left(\varepsilon^2\right). \tag{68}$$

Note that these formulae are fully implicit; the derivatives of the kernel function or special expansions are never used in the cubature rules. However, if the three-dimensional derivatives of the kernel function $k(x, y, z)$ are available and all 2nd order derivatives of the chart η_i as well, then, $\partial_r(c(\hat{x}, r, \xi))$ can be expressed explicitly while the numerical evaluation then might behave more robustly with respect to cancellation errors.

It remains to explain the approximation of the function $C_m^0(\hat{x}_1)$ as defined in (56). The limit

$$C_m^0(\hat{x}_1) = g_m \lim_{\hat{x}_2 \to 0} \int_{a_{m-1}(\hat{x})}^{a_m(\hat{x})} p_m(\alpha)\, c_0(\hat{x}, \psi(\alpha))\, d\alpha$$

has to be computed. The integration bounds $a_m(\hat{x})$ can be evaluated at $\hat{x}_2 = 0$. The same holds for the function c_0. Again, the integrand is analytic and can be evaluated by properly scaled Gauß-Legendre rules.

Summarizing, we have developed cubature formulae for all integrals appearing in the context of hypersingular integral operators. First, appropriate variable transforms are applied rendering the integrand either analytic or analytic with a logarithmic weight. Such integrals finally can be treated by Gauß-like formulae. The algorithm is fully implicit, we never made use of the explicit form of the kernel function and/or the surface parametrisation. The transformations of the Gaussian points on true surface points is easy to implement and to debug since these transformations are given either by 4×4 or 3×3 matrices.

Concerning the work for assembling the system matrix we emphasize that the number of singular cases, i.e. $\overline{K_i} \cap \overline{K_j} \neq \emptyset$, is proportional to $O(N)$ where N denotes the number of panels. The integrand is regular in $O(N^2)$ cases. Hence, the overall complexity is dominated by the regular or so-called farfield integrals. For the singular cases, it is important to have robust cubature rules along with a proper convergence analysis to control the perturbation error arising by replacing the true Galerkin matrix by the cubature approximation (see [5]).

Alternative regularisation techniques like partial integration or global regularisation (i.e. subtracting functions lying in the null space of the operator) have the draw back that the evaluation of the integrand for the *farfield integrals* is possibly more costly (cf. [10], [11]) compared to the evaluation of the original kernel function. Hence, the work for approximating the *farfield integrals* could be substantially larger compared to using the true kernel function.

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