Variable order panel clustering

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Abstract

We present a new version of the panel clustering method for a sparse representation of boundary integral equations. Instead of applying the algorithm separately for each matrix row (as in the classical version of the algorithm) we employ more general block partitionings. Furthermore, a variable order of approximation is used depending on the size of blocks.

We apply this algorithm to a second kind Fredholm integral equations and show that the complexity of the method only depends linearly on the number, say n, of unknowns. The complexity of the classical matrix oriented approach is $O\left(n^2\right)$ while, for the classical panel clustering algorithm, it is $O\left(n \log^7 n\right)$.

1 Introduction

Elliptic boundary value problems with constant coefficients can be transformed into integral equations on the boundary of the domain via the method of integral equations. From the numerical point of view, this approach is interesting especially for problems on unbounded domains where the direct discretization with finite element or finite differences is not straightforward.

Boundary integral equations are discretised in many engineering applications via the boundary element method by lifting conventional finite element spaces onto the surface of the domain. Due to the non-localness of the integral operators the arising system of equations is fully populated. Hence, the work for the classical matrix-oriented approach grows *quadratically* in the number (n) of unknowns.

In [5], [6], and [10], the panel clustering algorithm was introduced for collocation methods. By using polynomial approximations of the kernel function of the integral operator it was possible to split the dependence of the integration variable from the source points. The algorithm was applied for each matrix row separately. In [6], it was shown that the complexity of the algorithm is proportionally to $O\left(n \log^\kappa n\right)$ with moderate κ. In [15], [7], [12], [4], the panel clustering algorithm was introduced for the Galerkin discretization of boundary integral equations. The key role plays a symmetric factorization of the kernel function with respect to both variables. Again, the algorithm is applied to each matrix row separately.

In [11], the fast multipole method was introduced for the efficient evaluation of sums in multiple particle systems. Here, the algorithm was applied not pointwise but appropriate block partitionings are employed. The complexity of the algorithm is again proportionally to $O\left(n \log^k n\right)$. In [9], a block version of the panel clustering algorithm was introduced. The complexity is still $O\left(n \log^\kappa n\right)$ while the constants in the complexity estimates are smaller as for the classical approach.

In our paper, we introduce a variable order approximation on the clusters resulting in an algorithm with complexity $O\left(n\right)$. As a model problem we consider a Galerkin discretization of a second kind Fredholm integral equation. The fact that boundary integral equations can be realized (with full stability and consistency) in $O\left(n\right)$ operations whilst the classical matrix oriented approach has complexity $O\left(n^2\right)$ seems to be of interest. Generalizations of our approach to more general integral equations are the topic of future research.

Another way of a sparse approximation of boundary integral operators are wavelet discretizations. In the past decade they were intensively developed for boundary integral equations. There are versions for second kind integral equations by [18], [17] (complexity $O\left(n \log^\kappa n\right)$). The approach presented in [16] reduces the complexity to $O\left(n\right)$. However, the efficiency of wavelet methods depends on the number of (smooth) charts being employed for the representation of the surface. If the surface is rough and complicated the efficiency breaks down while the panel clustering method works especially well for complicated surfaces.

An algebraic approach to the data-sparse realization of non-local operators are $\mathcal{H}$-matrices (see [2], [3]). Matrix blocks are approximated by low rank matrices. The choice of the approximation system can be based on a singular value decomposition and is suited to approximate inverses of sparse matrices efficiently.

2 Setting

Let $\Gamma \subset \mathbb{R}^3$ denote an orientable, sufficiently smooth manifold ($\Gamma \in C^2$ is sufficient). On Γ, we consider the integral equation with the classical double layer potential in the weak form: For given $g \in L^2\left(\Gamma\right)$, find $u \in L^2\left(\Gamma\right)$ so that

$$2\pi \left(v, u\right)_{0,\Gamma} + \left(v, Ku\right)_{0,\Gamma} = \left(v, g\right)_{0,\Gamma}, \qquad \forall v \in L^2\left(\Gamma\right) \tag{1}$$

holds with

$$\begin{aligned} K\left[u\right]\left(x\right) &= \int_\Gamma k\left(x, y\right) u\left(y\right) d\Gamma_y, \\ k\left(x, y\right) &= \frac{\partial}{\partial n\left(y\right)} \frac{1}{\left\|x - y\right\|} \end{aligned} \tag{2}$$

Here, $L^2\left(\Gamma\right)$ denotes the set of all measurable functions on Γ which are square integrable and $\left(\cdot, \cdot\right)_{0,\Gamma}$ denotes the L^2-scalar product on Γ. The vector field $n\left(y\right)$ denotes the oriented normal vector field at a surface point $y \in \Gamma$.

The *Galerkin discretization* of (1) is given by replacing the infinite dimensional space $L^2(\Gamma)$ by a finite dimensional subspace $\mathcal{V}$. The Galerkin solution u_G is the solution of

$$2\pi (v, u_G)_{0,\Gamma} + (v, K u_G)_{0,\Gamma} = (v, g)_{0,\Gamma}, \qquad \forall v \in \mathcal{V}. \tag{3}$$

Our aim is to use finite element spaces lifted to the manifold Γ as the subspace $\mathcal{V}$. Finite element spaces are defined on finite element grids. We introduce the two-dimensional *master triangle* Q having the vertices $(0,0)^\intercal$, $(1,0)^\intercal$, $(1,1)^\intercal$.

Definition 1 *A finite element grid of Γ is a set $\mathcal{G} = \{\tau_1, \tau_2, \ldots, \tau_n\}$ consisting of disjoint, open surface pieces $\tau_i \subset \Gamma$ satisfying*

- $\overline{\Gamma} = \overline{\bigcup \mathcal{G}}$,
- *for all $\tau \in \mathcal{G}$, there exists a C^k-diffeomorphism $\chi_\tau : Q \to \tau$ (k sufficiently large) which can be extended to a C^k-diffeomorphism $\chi_\tau^\star : Q^\star \to \tau^\star$ for some open neighbourhoods $\tau^\star \supset \overline{\tau}$ and $Q^\star \supset \overline{Q}$.*

Notation 2 *The elements of a finite element grid are called "geometric finite elements". In the context of boundary element methods they are alternatively called "panels".*

In this paper, we restrict to piecewise constant approximations on triangulations.

Definition 3 *The space $S^{-1,0}$ is given by*

$$S^{-1,0} = \left\{ v \in L^2(\Gamma) : \forall \tau \in \mathcal{G} : v\,|_\tau = const \right\}.$$

A local basis of $S^{-1,0}$ is formed by the characteristic functions on the triangles:

$$\begin{aligned} \varphi_\tau &: \Gamma \to \mathbb{R}, \\ \forall \tau &\in \mathcal{G}: \qquad \varphi_\tau(x) = \begin{cases} 1 & x \in \tau, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By using the basis representation

$$u_G(x) = \sum_{\tau \in \mathcal{G}} u_G(\tau)\, \varphi_\tau(x), \tag{4}$$

the Galerkin discretization can be transformed into a system of linear equations:

$$(\mathbf{M} + \mathbf{K})\, \mathbf{u}_G = \mathbf{g}$$

where $\mathbf{M}, \mathbf{K} \in \mathbb{R}^{\mathcal{G}\times\mathcal{G}}$ and $\mathbf{u}_G$, $\mathbf{g} \in \mathbb{R}^{\mathcal{G}}$ are given, for all $\tau, t \in \mathcal{G}$, by

$$\begin{aligned} \mathbf{M}_{\tau,t} &= 2\pi (\varphi_\tau, \varphi_t)_{0,\Gamma}, & \mathbf{u}_G &= (u_G\,|_\tau)_{\tau\in\mathcal{G}}, \\ \mathbf{K}_{\tau,t} &= (\varphi_\tau, K[\varphi_t])_{0,\Gamma}, & \mathbf{g} &= \Big((\varphi_\tau, g)_{0,\Gamma}\Big)_{\tau\in\mathcal{G}}. \end{aligned}$$

The matrix $\mathbf{M}$ is diagonal while $\mathbf{K}$ is a fully populated $n \times n$-matrix. Hence, the classical matrix oriented approach costs (at least) $O(n^2)$ operations.

The idea of the panel clustering method is to use an alternative representation of the discrete integral operator which can be written in the form

$$\mathbf{K} \approx \mathbf{N} + \mathbf{B}^\intercal \mathbf{F} \mathbf{C}, \tag{5}$$

where the matrix $\mathbf{N}$ is sparse containing only $O(n)$ non-zero entries. Furthermore, $\mathbf{B}$, $\mathbf{C} \in \mathbb{C}^{m \times n}$ with $m \ll n$ and $\mathbf{F} \in \mathbb{C}^{m \times m}$. Note that, by using this representation, the matrix elements of $\mathbf{K}$ are not known, i.e., direct solvers cannot be applied to this system. However, for large n, iterative solvers are much more efficient than direct solvers and should be used instead. For iterative solvers, the matrix elements of $\mathbf{K}$ are not required. Matrix-vector multiplications appear as elementary operations which can be performed efficiently by using the splitting (5). The rest of the paper is concerned with the definition and analysation of an approximate factorization of the integral operator in (3).

First, we have to introduce some geometric notations.

Definition 4 *A cluster is the union of one or more panels.*

The geometric size of a cluster can be described via the Čebyšev radius of the cluster.

Definition 5 *For a subset $\omega \subset \mathbb{R}^d$, the Čebyšev ball B_ω is the ball with minimal radius containing ω. The Čebyšev centre M_ω it the midpoint of this ball and the Čebyšev radius ρ_ω its radius.*

Notation 6 *For a cluster c, the Čebyšev ball, Čebyšev centre, and Čebyšev radius are alternatively denoted by cluster ball, cluster centre, and cluster radius.*

For the efficiency of the algorithm, it is important to organize the clusters in a hierarchical tree. In this light, a set (set of sons) has to be associated with each cluster.

Definition 7 *A set of sons $\sigma(c)$ associated with a cluster c*

1. *is either the empty set,*
2. *or is the union of one or more disjoint clusters satisfying*

$$\overline{c} = \overline{\bigcup \sigma(c)}.$$

3. *If $\sigma(c) = \emptyset$ then $c \in \mathcal{G}$.*

A cluster c with $\sigma(c) = \emptyset$ is called a *leaf*.

Definition 8 *A cluster tree T corresponding to a grid $\mathcal{G}$ consists of clusters with associated sets of sons satisfying:*

1. $\Gamma \in T$,
2. *Any $c \in T$ with associated set of sons $\sigma(c)$ satisfies either*
   (a) $\sigma(c) = \emptyset$,
   (b) $\overline{c} = \overline{\bigcup \sigma(c)}$.

Remark 9 *We do not require that $\sharp\sigma(c) \neq 1$. For the later constructions, it will be convenient to allow $\sharp\sigma(c) = 1$ implying $\tilde{c} = c$ for $\tilde{c} \in \sigma(c)$.*

In the next step, we will associate to each cluster a *level* indicating the depth in the cluster tree. Since the largest cluster, i.e., the surface Γ, is subdivided recursively into smaller clusters, it is natural to use the depth of a cluster τ as an indication of the geometric size as well.

Definition 10 *The function* LEVEL: $T \to \mathbb{N}_0$ *is the recursive function*

$$\begin{aligned} \text{LEVEL}(\Gamma) &= 0, \\ \text{LEVEL}(\tilde{c}) &= \text{LEVEL}(c) + 1, \qquad \forall \tilde{c} \in \sigma(c), \qquad \forall c \in T \backslash \mathcal{G}. \end{aligned}$$

The ***depth*** *of the cluster tree is*

$$L = \max\{\text{LEVEL}(c) : c \in T\}$$

while the ***minimal depth*** *is given by*

$$L_{\min} := \min\{\text{LEVEL}(\tau) : \tau \in \mathcal{G}\}. \tag{6}$$

For $0 \leq \ell \leq L$, the ***tree level*** *$T(\ell)$ contains all clusters $c \in T$ with* LEVEL$(c) = \ell$.

The term $(v, Ku)_{0,\Gamma}$ in (3) contains an integral over $\Gamma \times \Gamma$:

$$(v, Ku)_{0,\Gamma} = \int_{\Gamma \times \Gamma} v(x)\, u(y)\, k(x,y)\, d\Gamma_y d\Gamma_x.$$

In the next step, the product $\Gamma \times \Gamma$ is partitioned into pairs of clusters defining a block partitioning of $\Gamma \times \Gamma$. A pair $\mathbf{c} = (c_1, c_2) \in T \times T$ is called a *block*.

Definition 11 *Let $\eta \in (0,1)$. A block $\mathbf{c} \in T \times T$ is η-admissible if*

$$\max\{\rho_{c_1}, \rho_{c_2}\} \leq \eta \operatorname{dist}(c_1, c_2) \tag{7}$$

holds with ρ_{c_1}, ρ_{c_2} as in Definition 5.

If there is no ambiguity we write "admissible" short for "η-admissible".

Definition 12 *Let* $\mathbf{c} = (c_1, c_2) \in T \times T$. *The set of sons of* $\mathbf{c}$ *is given by*

- $\sigma(\mathbf{c}) = \sigma(c_1) \times \sigma(c_2)$ *provided* $\sigma(c_1) \neq \emptyset$ *and* $\sigma(c_2) \neq \emptyset$,
- $\sigma(\mathbf{c}) = \sigma(c_1) \times \{c_2\}$ *provided* $\sigma(c_1) \neq \emptyset$ *and* $\sigma(c_2) = \emptyset$,
- $\sigma(\mathbf{c}) = \{c_1\} \times \sigma(c_2)$ *provided* $\sigma(c_1) = \emptyset$ *and* $\sigma(c_2) \neq \emptyset$,
- $\sigma(\mathbf{c}) = \emptyset$ *provided* $\sigma(c_1) = \sigma(c_2) = \emptyset$.

A block $\mathbf{c} \in T \times T$ is called a leaf if $\sigma(\mathbf{c}) = \emptyset$. The tree T induces a block cluster tree $T^{(2)}$ of $\Gamma \times \Gamma$.

Construction 13 $T^{(2)}$ *is a block cluster tree if*

- $(\Gamma, \Gamma) \in T^{(2)}$,
- *every* $\mathbf{c} \in T^{(2)}$ *satisfies one of the alternatives:*
  - $\mathbf{c}$ *is a leaf,*
  - $\overline{\mathbf{c}} = \overline{\bigcup \sigma(\mathbf{c})}$.

Note that the block cluster tree $T^{(2)}$ is fully determined by the cluster tree T.

Definition 14 *A subset* $P^{(2)} \subset T^{(2)}$ *is a block partitioning of* $\Gamma \times \Gamma$ *if the elements of* $P^{(2)}$ *are disjoint and*

$$\overline{\Gamma \times \Gamma} = \overline{\bigcup P^{(2)}}.$$

It is an η*-admissible block partitioning if every* $\mathbf{c} \in P^{(2)}$ *satisfies one of the alternatives*

- $\mathbf{c}$ *is a leaf,*
- $\mathbf{c}$ *is* η*-admissible.*

It is a minimal, η*-admissible block partitioning if there is no* η*-admissible block partitioning with less elements.*

Algorithm 15 *The minimal,* η*-admissible block partitioning of* $\Gamma \times \Gamma$ *is obtained as the result of the procedure* ***divide***$((\Gamma, \Gamma), \emptyset)$ *defined by (see [6])*

procedure divide$(\mathbf{c}, P)$;
begin
  if ($\mathbf{c}$ is a leaf) **then** $P := P \cup \{\mathbf{c}\}$
  else if ($\mathbf{c}$ is admissible) **then** $P := P \cup \{\mathbf{c}\}$
  else for all $\widetilde{\mathbf{c}} \in \sigma(\mathbf{c})$ **do divide**$(\widetilde{\mathbf{c}}, P)$;
end;

The partitioning $P_{\min}$ contains non-admissible leaves and admissible blocks. These subsets are denoted by N (nearfield) and F (farfield):

$$\begin{aligned} N &:= \{\mathbf{c} \in P_{\min} : \mathbf{c} \text{ is non-admissible}\}, \\ F &:= P_{\min} \backslash N. \end{aligned} \tag{8}$$

On blocks $\mathbf{c} \in F$, the kernel function will be replaced by an approximation of a certain order. The idea is that, on blocks consisting of clusters of similar size, the approximation order is the same and, in addition, the approximation order is smaller on smaller blocks.

Definition 16 *Let $L_{\min}$ be as in (6). The order distribution function $m : F \to \mathbb{N}_0$ depends on two constants $a, b \in \mathbb{N}_0$ and is given by*

$$m(\mathbf{c}) := a\,(L_{\min} - M_{\mathbf{c}})_+ + b \tag{9}$$

with

$$M_{\mathbf{c}} = \min\{\text{LEVEL}(c_1), \text{LEVEL}(c_2)\}$$

and

$$(\cdot)_+ = \max\{0, \cdot\}.$$

The order distribution is extended to a function $m : F \cup T \to \mathbb{N}_0$ by

$$m(c) = \max\{m(\mathbf{c}) : \mathbf{c} \in F \wedge c \in \{c_1, c_2\}\}, \qquad c \in T. \tag{10}$$

Remark 17 *One could generalise the function m by allowing $a, b \in \mathbb{R}_{\geq 0}$ and defining*

$$m(\mathbf{c}) := \lceil a\,(L_{\min} - M_{\mathbf{c}})_+ + b \rceil,$$

where $\lceil x \rceil$ denotes the smallest integer larger than x.

Remark 18 *Definition 16 implies that the approximation order on a block (c_1, c_2) is determined by the "larger cluster" $c = \operatorname{argmin}\{\text{LEVEL}(c_1), \text{LEVEL}(c_2)\}$. The approximation order is high on large clusters, e.g., $m(\Gamma, \Gamma) = aL_{\min} + b$ and small for small clusters as, e.g.,*

$$m(c_1, c_2) = b$$

for all c_1, c_2 satisfying LEVEL(c_1),LEVEL$(c_2) \geq L_{\min}$.

Remark 19 *In Subsection 3.1, a construction for the sets T, $P^{(2)}$ is presented which always guarantees that $(c_1, c_2) \in P^{(2)}$ implies that c_1 and c_2 belong to the same $T(\ell)$ for some ℓ. In this case, the order distribution m only depends on the level ℓ.*

3 The variable order panel clustering algorithm

In this section, we will define the panel clustering algorithm. In the previous section, we have defined a partitioning of $\Gamma \times \Gamma$ into a minimal, η-admissible block partitioning $P_{\min} = N \cup F$. On the portion $\bigcup N \subset \Gamma \times \Gamma$, the standard, matrix oriented approach is used while, on $\bigcup F$, the kernel function is approximated by suitable expansions. Let the kernel function k be as in (2).

Assumption 20 *There exist positive constants C_1, C_3, C_4, C_5, λ_1, λ_I, $\lambda_{II} < \infty$ and $C_2, \overline{\eta} \in (0,1)$ having the following properties. For all $\eta \in (0, \overline{\eta})$ and all η-admissible block partitionings $P^{(2)}$ of $\Gamma \times \Gamma$, for all $\mathbf{c} \in F$, there is a family of approximations $k_{\mathbf{c}}^{(m)}$, $m \in \mathbb{N}_0$, of the kernel function k satisfying*

$$\left| k(x,y) - k_{\mathbf{c}}^{(m)}(x,y) \right| \leq C_1 C_2^m \operatorname{dist}^{-1}(c_1, c_2), \qquad \forall (x,y) \in \mathbf{c}. \tag{11}$$

having the form

$$k_{\mathbf{c}}^{(m)}(x,y) = \sum_{(\nu,\mu) \in \mathbf{I}_m} \kappa_{\nu,\mu}^{(m)}(\mathbf{c}) \, \Phi_{c_1}^{(\nu)}(x) \, \Psi_{c_2}^{(\mu)}(y). \tag{12}$$

with index sets $\mathbf{I}_m$, $m \in \mathbb{N}_0$, satisfying

$$\begin{aligned}
\sharp \mathbf{I}_m &\leq C_3 (m+1)^{\lambda_1}, && (13)\\
\iota_m^I &:= \left\{ \nu \mid \exists \mu \in \mathbb{N}_0^3 : (\nu,\mu) \in \mathbf{I}_m \right\}, && (14)\\
\iota_m^{II} &:= \left\{ \mu \mid \exists \nu \in \mathbb{N}_0^3 : (\nu,\mu) \in \mathbf{I}_m \right\}, && (15)\\
\sharp \iota_m^s &\leq C_4 (m+1)^{\lambda_s}, \qquad s \in \{I, II\}, && (16)\\
|\nu| &\leq C_5 (m+1), \qquad \forall \nu \in I_m^s, \quad s \in \{I, II\}, && (17)\\
\iota_m^I &\subset \iota_M^{II}, \qquad \iota_m^{II} \subset \iota_M^{II}, \qquad \forall 0 \leq m \leq M. && (18)
\end{aligned}$$

The approximation of the kernel function is based on a modification of Taylor expansions. In this light, we begin with analysing the (true) Taylor approximation of the kernel function $k_{\mathbf{c}}^{(m)}$ and, then, explain the modification. We begin with introducing some notations.

For $\mathbf{c} \in F$, the difference domain $d(\mathbf{c})$ is given by

$$d(\mathbf{c}) = c_1 - c_2 = \left\{ z \in \mathbb{R}^3 \mid \exists (x,y) \in \mathbf{c} : z = x - y \right\}. \tag{19}$$

Put $z_{\mathbf{c}} = M_{c_1} - M_{c_2}$. One easily checks that, since $\mathbf{c}$ is admissible, $z_{\mathbf{c}} \neq 0$. The kernel function in relative coordinates defines the function $\tilde{k} : c_1 \times d(\mathbf{c}) \to \mathbb{R}$

$$\tilde{k}(y,z) = \langle n(y), z \rangle \, k_3(z)$$

with

$$k_3(z) := \frac{1}{\|z\|^3}. \tag{20}$$

Taylor expansion of k_3 about $z_{\mathbf{c}}$ yields (writing n short for $n(y)$):

$$\tilde{k}_{\mathbf{c}}^{(m)}(y,z) = \langle n, z\rangle \sum_{|\nu|<m} \frac{(z-z_{\mathbf{c}})^{\nu}}{\nu!} k_3^{(\nu)}(z_{\mathbf{c}}), \tag{21}$$

where we employed the usual multi-index notation for $\nu \in \mathbb{N}_0^3$. Re-substituting $z = x - y$, factorizing $(x - y - z_{\mathbf{c}})^{\nu}$ with respect to x and y, and rearranging the terms results in[1]

$$\check{k}_{\mathbf{c}}^{(m)}(x,y) := \tilde{k}_{\mathbf{c}}^{(m)}(y, x-y) = \sum_{i=1}^{3} \sum_{|\nu|+|\mu|\le m} (y - M_{c_2})^{\mu} n_i (x - M_{c_1})^{\nu} \kappa_{\nu,\mu,i}^{(m)}(\mathbf{c}) \tag{22}$$

with

$$\kappa_{\nu,\mu,i}^{(m)}(\mathbf{c}) = \frac{(-1)^{|\nu|}}{\mu!\nu!} \begin{cases} (\nu_i+\mu_i)\, k_3^{(\nu+\mu-e_i)}(z_{\mathbf{c}}) + (z_{\mathbf{c}})_i\, k_3^{(\nu+\mu)}(z_{\mathbf{c}}) & |\nu|+|\mu| < m, \\ (\nu_i+\mu_i)\, k_3^{(\nu+\mu-e_i)}(z_{\mathbf{c}}) & |\nu|+|\mu| = m. \end{cases}$$

Here, $\{e_i\}_{i=1}^3$ denotes the set of canonical unit vectors in $\mathbb{R}^3$. Introducing the seven-dimensional index set:

$$\widetilde{\mathbf{I}}_m = \left\{(\nu,\mu,i) \in \mathbb{N}_0^3 \times \mathbb{N}_0^3 \times \{1,2,3\} : |\nu| + |\mu| \le m\right\}.$$

and the function system

$$\check{\Phi}^{(\nu)}(x) = (x - M_{c_1})^{\nu}, \qquad \check{\Psi}^{(\mu,i)}(y) = (y - M_{c_2})^{\mu} n_i(y) \tag{23}$$

results in an expansion of the form (12). In order to reduce the number of indices the three-dimensional coefficients and functions $\overrightarrow{\kappa}$ and $\overrightarrow{\check{\Psi}}_c$ are introduced by

$$\overrightarrow{\kappa}_{\nu,\mu}^{(m)}(\mathbf{c}) = \left\{\kappa_{\nu,\mu,i}^{(m)}(\mathbf{c})\right\}_{i=1}^{3}, \qquad \overrightarrow{\check{\Psi}}^{(\mu)} = \left\{\check{\Psi}^{(\mu,i)}(y)\right\}_{i=1}^{3}.$$

The expansion (22) can be rewritten as

$$\check{k}_{\mathbf{c}}^{(m)}(x,y) = \sum_{(\nu,\mu)\in\mathbf{I}_m} \check{\Phi}_c^{(\nu)}\, \overrightarrow{\check{\Psi}}_c^{(\mu)}\, \overrightarrow{\kappa}_{\nu,\mu}^{(m)}(\mathbf{c})$$

with the six-dimensional index set

$$\mathbf{I}_m = \left\{(\nu,\mu) \in \mathbb{N}_0^3 \times \mathbb{N}_0^3 : |\nu| + |\mu| \le m\right\}$$

[1] For the *variable order* panel clustering algorithm, the functions $(\cdot - M_c)^{\nu}$ will be replaced by suitable approximations. This is the reason why we denote the function in (22) by $\check{k}_{\mathbf{c}}^{(m)}$ instead of $k_{\mathbf{c}}^{(m)}$.

and the convention

$$\overrightarrow{\check{\Psi}}_{c}^{(\mu)}\overrightarrow{\kappa}_{\nu,\mu}^{(m)}(\mathbf{c})=\sum_{i=1}^{3}\check{\Psi}_{c}^{(\mu,i)}\kappa_{\nu,\mu,i}^{(m)}(\mathbf{c}).$$

In [6, Appendix A], it was proved that there exists constants $\widetilde{C_1}$, $\widetilde{C_2}$, and $\widetilde{\eta_0}$ so that

$$\left|k(x,y)-\check{k}_{\mathbf{c}}^{(m)}(x,y)\right|\leq\widetilde{C_1}\left(\widetilde{C_2}\tilde{\eta}\right)^{m-1}|k(x,y)| \tag{24}$$

holds for all $(x,y)\in\mathbf{c}$ satisfying $\|x-y-z_{\mathbf{c}}\|\leq\tilde{\eta}\|z_{\mathbf{c}}\|$ and all $\tilde{\eta}\in(0,\widetilde{\eta_0})$.

Lemma 21 *Let $P^{(2)}$ denote an η-admissible block partitioning of $\Gamma\times\Gamma$ with $\eta\in(0,\overline{\eta})$ and $\overline{\eta}:=\min\{1/4,\widetilde{\eta_0}\}$. Then, Assumption 20 is satisfied.*

Proof. Since the block partitioning was assumed to be η-admissible we conclude:

$$\|x-y-z_{\mathbf{c}}\|\leq\|x-M_{c_1}\|+\|y-M_{c_2}\|\leq\rho_{c_1}+\rho_{c_2}\overset{(7)}{\leq}2\eta\,\mathrm{dist}(c_1,c_2).$$

The distance can be estimated by:

$$\mathrm{dist}(c_1,c_2)\leq\|M_{c_1}-M_{c_2}\|+\rho_{c_2}+\rho_{c_1}\leq\|M_{c_1}-M_{c_2}\|+2\eta\,\mathrm{dist}(c_1,c_2).$$

Using $\eta<\frac{1}{4}$ we get

$$\mathrm{dist}(c_1,c_2)\leq2\|M_{c_1}-M_{c_2}\|$$

and, finally,

$$\|x-y-z_{\mathbf{c}}\|\leq4\eta\|z_{\mathbf{c}}\|.$$

By choosing $\bar{\eta}=\widetilde{\eta_0}/4$ in Assumption 20 with $\widetilde{\eta_0}$ as in (24) results in $(4\eta)=:\tilde{\eta}\in(0,\widetilde{\eta_0})$ in (24). Hence, (11) holds with $C_2=4\widetilde{C_2}$. Let C_Γ denote the smallest constant so that, for all $\mathbf{c}\in P^{(2)}$ and all $(x,y)\in\mathbf{c}$:

$$|\langle n(y),x-y\rangle|\leq C_\Gamma\|x-y\|^2.$$

Then,

$$|k(x,y)|=\frac{\langle n(y),x-y\rangle}{\|x-y\|^3}\leq C_\Gamma\frac{1}{\|x-y\|}\leq C_\Gamma\,\mathrm{dist}^{-1}(c_1,c_2),\qquad\forall(x,y)\in\mathbf{c}$$

and (11) holds with $C_1=\widetilde{C_1}C_\Gamma$.

Some combinatorial manipulations yield

$$C_3=3,\qquad\lambda_1=6.$$

Obviously:

$$\iota_m^I=\left\{\nu\in\mathbb{N}_0^3:|\nu|\leq m\right\} \tag{25}$$

$$\tilde{\iota}_m^{II}=\left\{\nu\in\mathbb{N}_0^3:|\nu|\leq m\right\}\times\{1,2,3\}. \tag{26}$$

Again, some combinatorial manipulations lead to

$$C_4 = 3, \qquad \lambda_1^I = \lambda_2^{II} = 3.$$

Finally, $C_5 = 1$ is trivial. ■

In [15], [7], it was proved that all kernel functions corresponding to elliptic boundary value problems admit an approximation satisfying Assumption 20.

Remark 22 *For the variable order panel clustering algorithm, the Taylor-based expansion derived in the previous example will be modified by replacing the expansion functions (23) by approximations having more hierarchical structure with respect to the order m.*

Remark 23 *The panel clustering method is by no means linked to Taylor based expansions. Other expansions as, e.g., expansions in spherical harmonics could be preferable for special applications.*

The panel clustering approximation of $(v, Ku)_{0,\Gamma}$ is given by

$$\begin{aligned}(v, Ku)_{0,\Gamma} \approx & \sum_{\mathbf{c}\in N} \int_{\mathbf{c}} v(x)\, u(y)\, k(x,y) \\ & + \sum_{\mathbf{c}\in F} \sum_{(\nu,\mu)\in \mathbf{I}_{m(\mathbf{c})}} \overrightarrow{\kappa}_{\nu,\mu}^{(m)}(\mathbf{c}) \int_{c_1} \Phi_{c_1}^{(\nu)}(x)\, v(x)\, dx \int_{c_2} \overrightarrow{\Psi}_{c_2}^{(\mu)}(y)\, u(y)\, d\Gamma_y.\end{aligned} \tag{27}$$

The function $m(\mathbf{c})$ determines the order of approximation on blocks $\mathbf{c} \in F(\ell)$. It was defined in Definition 16 while the constants $a, b \in \mathbb{N}_0$ will be fixed in Definition 54. Assumption 20 implies

$$\mathbf{I}_{m(\mathbf{c})} \subset \iota_{m(\mathbf{c})}^I \times \iota_{m(\mathbf{c})}^{II}.$$

Let $\mathbf{c} = (c_1, c_2) \in F$. Then, $m(c_i) \geq m(\mathbf{c})$ for $i = 1, 2$ (cf. (10)), resulting in

$$\mathbf{I}_{m(\mathbf{c})} \subset \iota_{m(c_1)}^I \times \iota_{m(c_2)}^{II}. \tag{28}$$

Property (28) will allow to decompose the computations related to the index set $\mathbf{I}_{m(\mathbf{c})}$ into separate computations on the index sets $\iota_{m(c_1)}^I$, $\iota_{m(c_2)}^{II}$.

For the evaluation of a matrix-vector multiplication, expression (27) has to be evaluated for all basis functions $v = b_\tau$, $\tau \in \mathcal{G}$.

The variable order panel clustering algorithm

The variable order panel clustering algorithm depends on various parameters:

- η: The constant appearing in the definition of η-admissibility.
- The choice of the constants a, b in the definition of $m = m(\mathbf{c})$ (as in (9)) in (11). The precise choice of a and b is given in Definition 54.

Setup phase:

1. For a given mesh $\mathcal{G}$, build up the cluster tree T and compute all cluster radii and cluster centres.
2. Compute $P_{\min}$ by using the procedure **divide** of Algorithm 15.
3. For all $\mathbf{c} \in F$, $(\nu, \mu) \in \mathbf{I}_{m(\mathbf{c})}$: compute the coefficients $\overrightarrow{\kappa}^{m(\mathbf{c})}_{\nu,\mu}(\mathbf{c})$.
4. Compute the *nearfield matrix* entries:
$$\mathbf{N}_{\tau,t} = \int_{\tau\times t} k(x,y)\, d\Gamma_y d\Gamma_x, \qquad \forall (\tau,t) \in N.$$
5. For all $\tau \in \mathcal{G}$: compute the basis farfield coefficients:
$$\begin{aligned} J^I_{\tau,\nu} &= \int_\tau \Phi^{(\nu)}_\tau(x)\, d\Gamma_x, \qquad \forall \nu \in \iota^I_{m(\tau)}, \\ \overrightarrow{J}^{II}_{\tau,\nu} &= \int_\tau \overrightarrow{\Psi}^{(\nu)}_\tau(x)\, d\Gamma_x, \qquad \forall \nu \in \iota^{II}_{m(\tau)}. \end{aligned}$$

Evaluation phase:

Let $u \in S^{-1,0}$ and $\mathbf{u} \in \mathbb{R}^{\mathcal{G}}$ so that $u = \sum_{\tau\in\mathcal{G}} \mathbf{u}(\tau)\varphi_\tau$ as in (4).

1. Compute the farfield coefficients: For all $c \in T$:
$$\overrightarrow{J}^{II}_{c,\mu}[u] := \int_c \overrightarrow{\Psi}^{(\mu)}_c(x)\, u(x)\, d\Gamma_x, \qquad \forall \mu \in \iota^{II}_{m(c)}.$$
2. For all $\mathbf{c} = (c_1, c_2) \in P^{(2)}(\ell)$, $\nu \in \iota^I_{m(\mathbf{c})}$:
$$A^{(\nu)}_{\mathbf{c}}[u] := \sum_{\mu:(\nu,\mu)\in\mathbf{I}_{m(\mathbf{c})}} \overrightarrow{\kappa}^{m(\mathbf{c})}_{\nu,\mu}(\mathbf{c})\, \overrightarrow{J}^{II}_{c_2,\mu}[u].$$
3. Approximate a matrix vector multiplication by
$$\sum_{t\in\mathcal{G}} \mathbf{N}_{\tau,t}\mathbf{u}(t) + \sum_{\mathbf{c}\in F}\sum_{\nu\in\iota^I_{m(\mathbf{c})}} \int_{c_1} A^{(\nu)}_{\mathbf{c}}[u]\, \Phi^{(\nu)}_{c_1}(x)\, b_\tau(x)\, dx, \qquad \forall \tau \in \mathcal{G}. \tag{29}$$

Remark 24 *Let $\mathbf{c} = (c_1, c_2) \in F$ and $m_i = m(c_i)$, $i = 1, 2$. For the realization of the algorithm, it is essential that*
$$\mathbf{I}_{m(\mathbf{c})} \subset \iota^I_{m_1} \times \iota^{II}_{m_2}$$
holds. This condition is guaranteed since, in view of (10), we have $m_i \geq m(\mathbf{c})$ and
$$\mathbf{I}_{m(\mathbf{c})} \subset \iota^I_{m(\mathbf{c})} \times \iota^{II}_{m(\mathbf{c})} \subset \iota^I_{m_1} \times \iota^{II}_{m_2}.$$

In the sequel, we will comment on the realization of the single steps in the algorithm which is essential for both, the practical implementation and the complexity analysis. Some further approximations and relaxations will occur.

3.1 Construction of the cluster tree

Let $\mathcal{G}$ denote the given mesh of Γ. In a first step, one has to compute the centre and radius (cf. Definition 5) of each panel $\tau \in \mathcal{G}$. The smallest radius defines the quantity

$$\underline{\rho} := \min_{\tau \in \mathcal{G}} \rho_\tau \tag{30}$$

while the "step size" h of $\mathcal{G}$ is given by

$$h = \max_{\tau \in \mathcal{G}} \operatorname{diam} \tau \tag{31}$$

We give a construction based on an auxiliary uniform grid with a uniform partitioning. This grid is not needed in the true computations but inherits a simple logical structure to the true grid $\mathcal{G}$. Let Q denote the smallest cube containing Γ with edges parallel to the coordinate axes. Without loss of generality we may assume that $Q = (0,1)^3$. We introduce a sequence of physically and logically nested grids on Q.

For $\ell \in \mathbb{N}_0$, let $h_\ell = 2^{-\ell}$ and $n_\ell = 2^\ell$. The interval $\pi_{i,\ell}$ is defined, for $1 \leq i \leq n_\ell$, by $\pi_{i,\ell} := ((i-1)\, h_\ell, i h_\ell)$. For $\mu \in (\mathbb{N}_{\leq n_\ell})^3$, a cell $q_{\mu,\ell}$ is given by

$$q_{\mu,\ell} = \pi_{\mu_1,\ell} \times \pi_{\mu_2,\ell} \times \pi_{\mu_3,\ell}.$$

Lemma 25 *For $\mu \in (\mathbb{N}_{\leq n_\ell})^3$ and $\ell \in \mathbb{N}_0$, the centre and radius of $q_{\mu,\ell}$ are given by*

$$\begin{aligned} \rho_{\mu,\ell} &:= \rho_{\tau_{\mu,\ell}} = \sqrt{3}\frac{h_\ell}{2}, \\ M_{\mu,\ell} &:= M_{\tau_{\mu,\ell}} = h_\ell \left(\mu - 2^{-1}\,(1,1,1)^\intercal\right). \end{aligned} \tag{32}$$

The reference grid $\mathcal{Q}_\ell$ is defined by

$$\mathcal{Q}_\ell := \left\{ q_{\mu,\ell} : \mu \in (\mathbb{N}_{\leq n_\ell})^3 \right\}. \tag{33}$$

Obviously, each element $q \in \mathcal{Q}_\ell$ has exactly eight sons in $\mathcal{Q}_{\ell+1}$ satisfying

$$\overline{q} = \overline{\bigcup \sigma(q)}.$$

In other words, $\{\mathcal{Q}_\ell\}_{\ell \in \mathbb{N}_0}$ is an oct-tree. This tree will be associated to $\mathcal{G}$. Let L denote the smallest number so that

$$\sqrt{3}\frac{h_L}{2} \leq \underline{\rho} \tag{34}$$

holds with $\underline{\rho}$ as in (30). Hence, a cluster tree for the auxiliary grid $\mathcal{Q}_L$ is given by $\mathcal{Q} = \{\mathcal{Q}_\ell\}_{0 \leq \ell \leq L}$.

Any element $\tau \in \mathcal{G}$ is associated to that element $q \in \mathcal{Q}_L$ containing the centre of τ. (If there are multiple possibilities, choose one of them). This defines a mapping REF: $\mathcal{G} \to \mathcal{Q}_L$. Define INVREF: $\mathcal{Q}_L \to \mathcal{G} \cup \{\emptyset\}$ via

$$\text{INVREF}\,(q) = \begin{cases} \tau & \text{if } q = \text{REF}\,(\tau) \\ \emptyset & \text{otherwise.} \end{cases}$$

The following procedure builds up the cluster tree along with the tree levels. Before we present the formal description of the algorithm we explain the underlying ideas. Our aim is to generate a balanced tree with the additional properties that

1. the number of sons of any cluster is different from one,
2. the geometric size of a cluster on level $T(\ell)$ is of order $2^{-\ell}$, i.e., there exists $C_7 \geq 1$ so that, for all $c \in T(\ell)$:

$$C_7^{-1} 2^{-\ell} \leq \rho_c \leq C_7 2^{-\ell}. \tag{35}$$

The cluster ball, centre, and radius are approximated as follows. A *box* is a rectangular parallelepiped with axes parallel to the coordinate system. For a cluster, it is quite simple to determine the minimal box $b(c)$ containing c. The approximate cluster ball, centre, and radius are defined as the Čebysev ball, centre, and radius of $b(c)$ and are denoted by $\tilde{B}(c)$, $\tilde{M}_c$, and $\tilde{\rho}_c$. By this construction it is guaranteed that

$$c \subset \tilde{B}(c), \qquad s_c \leq \rho_c \leq \tilde{\rho}_c,$$

where s_c denotes the maximal side length of $b(c)$.

The clusters (corresponding to a reference cube $q \in \mathcal{Q}_\ell$) are built recursively by collecting the clusters $\bigcup_{\tilde{q} \in \sigma(q)} \{\text{INVREF}\,(\tilde{q})\}$. However, if a cluster contains only one son or the maximal side length s_c is so small that (35) is violated, this cluster is absorbed in the neighbouring cluster. The choice of the neighbouring cluster involves the definition of layers about a set ω.

Definition 26 *Let $\mathcal{Q}_\ell$ be as in (33). For $\omega \in \mathbb{R}^3$, the layers $\mathcal{L}_k^i$ about ω are given by $\mathcal{L}_k^0(\omega) := \omega$ and, for $0 \leq k \leq L$, $i \in \mathbb{N}_0$, by the recursion:*

$$\begin{aligned} \mathcal{L}_k^1(\omega) &:= \bigcup \{q \in \mathcal{Q}_k \mid \overline{q} \cap \overline{\omega} \neq \emptyset\}, \\ \mathcal{L}_k^{i+1}(\omega) &:= \mathcal{L}_k^1\left(\mathcal{L}_k^i(\omega)\right). \end{aligned}$$

If a cluster $c \in T(\ell)$ has only one son or is too small it will be "absorbed" in a "neighbouring" cluster $\tilde{c} \in T(\ell)$ (with reference cluster $\tilde{q} :=$REF$(\tilde{c})$) satisfying

1. $c \subset \mathcal{L}_{\ell+1}^1(\tilde{q})$, i.e., $\tilde{c}$ is "close" to c

2. $s_{\tilde{c}} \geq c_{\min} 2^{-\ell}$, i.e., $\tilde{c}$ is "sufficiently big".

The algorithm depends on the parameter $c_{\min} < 1$ controlling the relative smallness of a cluster. The precise choice of $c_{\min}$ (Assumption 50) is a result of the convergence analysis while we expect that, in practical applications, a larger value of $c_{\min}$ might be preferable. Numerical experiments in this direction will be presented in a forthcoming paper.

The recursion starts on the panel level and we put $T(L) = \mathcal{G}$ and $T(L-1) = \emptyset$. On the panel level, we assume that the cluster centres, balls, and radii are computed exactly. Then, the procedure **build_cluster_tree** generates a coarser level from the finer level recursively and is called by

$\ell := L - 1;$
while $T_{\ell+1} \neq \emptyset$ **do begin**
  build_cluster_tree$(T_{\ell+1}, T_\ell, \ell)\,; \ell := \ell - 1;$
end;

while the procedure **build_cluster_tree** is defined by

procedure build_cluster_tree$(T_{\ell+1}, T_\ell, \ell)\,;$
begin
  for all $q \in \mathcal{Q}_\ell$ **do begin** $c := \emptyset; \sigma(c) := \emptyset;$INVREF$(q) := \emptyset;$
Comment: The sons of the reference cluster q will be collected;
    for all $\tilde{q} \in \sigma(q)$ **do begin**
      $\tilde{c} :=$INVREF$(\tilde{q})\,; \quad c := c \cup \tilde{c};$
      if $\tilde{c} \neq \emptyset$ **then** $\sigma(c) := \sigma(c) \cup \{\tilde{c}\};$
    end;
    if $c \neq \emptyset$ **then begin**
      $T_\ell := T_\ell \cup \{c\}\,;$ REF$(c) = q;$ INVREF$(q) := c;$ LEVEL$(c) = \ell;$
    end;
  end;
Comment: Clusters having only one son or too small radius are absorbed in a neighbouring cluster;
  for all $c \in T_\ell$ **do begin**
    compute $b(c)$ as the minimal box containing $\bigcup_{\tilde{c} \in \sigma(c)} b(\tilde{c})$,
    the approximate cluster centre $\tilde{M}_c$;
    the approximate cluster radius $\tilde{\rho}_c$;
    and the minimal side length s_c of $b(c)$;
    if $\sharp\sigma(c) = 1$ **or** $s_c \leq c_{\min} 2^{-\ell}$ **then begin**[2]

$$\mathcal{N}(c) := \left\{ \tilde{c} \in T_\ell \mid c \subset \mathcal{L}^1_{\ell+1}(q_{\tilde{c}}) \right\}; \tag{36}$$

[2]We employ the notation, that, for a cluster c, the reference cube is denoted by $q_c :=$REF(c) and, for a cube $q \in \mathcal{Q}_\ell$, the pullback by $c_q :=$INVREF(c).

if $\mathcal{N}(c) \neq \emptyset$ **then** determine $\tilde{c} \in \mathcal{N}(c)$ so that

$$s_{\tilde{c}} \geq s_{c'}, \qquad \forall c' \in \mathcal{N}(c);$$

$$T_\ell := T_\ell \setminus \{c\}; \qquad \tilde{c} := \tilde{c} \cup c; \qquad \sigma(\tilde{c}) := \sigma(\tilde{c}) \cup \sigma(c);$$
$$\text{INVREF}^{-1} q_c = \emptyset; \textbf{ update } b(\tilde{c}),\ \tilde{M}(\tilde{c}),\ \tilde{\rho}_{\tilde{c}}, \text{ and } s_{\tilde{c}};$$

end;end;end;end;

The approximations of the cluster radii and cluster centres will be employed to check whether a pair of clusters is η-admissible. A sufficient condition is given in the next lemma.

Lemma 27 *Let the approximate cluster centre, radius and ball be as in the procedure* ***build_cluster_tree****. Let $c_1, c_2 \in T$ and put, for $i = 1, 2$, $\tilde{\rho}_i := \tilde{\rho}_{c_i}$ and $\tilde{B}_i := \tilde{B}(c_i)$. Then, the condition*

$$\max\{\tilde{\rho}_1, \tilde{\rho}_2\} \leq \eta \operatorname{dist}\left(\tilde{B}_1, \tilde{B}_2\right), \tag{37}$$

implies that the block (c_1, c_2) is η-admissible.

Proof. Let $i \in \{1, 2\}$. Our construction directly implies that the minimal ball B_i containing c_i is contained in $\tilde{B}_i$. Hence, the true cluster radii ρ_1, ρ_2 can be estimated by

$$\max\{\rho_1, \rho_2\} \leq \max\{\tilde{\rho}_1, \tilde{\rho}_2\}.$$

Since c_i is contained in $\tilde{B}_i$ evidently

$$\operatorname{dist}\left(\tilde{B}_1, \tilde{B}_2\right) \leq \operatorname{dist}(c_1, c_2).$$

We have proved that condition (37) implies

$$\max\{\rho_1, \rho_2\} \leq \max\{\tilde{\rho}_1, \tilde{\rho}_2\} \leq \eta \operatorname{dist}\left(\tilde{B}_1, \tilde{B}_2\right) \leq \eta \operatorname{dist}(c_1, c_2)$$

and (c_1, c_2) is η-admissible. ■

The lemma above motivates the definition of strong η-admissibility. We employ the same notation as in that lemma.

Definition 28 *A block $\mathbf{c} = (c_1, c_2) \in T^{(2)}$ is strongly η-admissible iff*

$$\max\{\tilde{\rho}_1, \tilde{\rho}_2\} \leq \eta \operatorname{dist}\left(\tilde{B}_1, \tilde{B}_2\right).$$

In order to check the strong η-admissibility the approximate centre and radius of the clusters have to be stored. The computation of an η-admissible, block partitioning $P^{(2)}$ of $\Gamma \times \Gamma$ is performed by using Algorithm 15, where the check of η-admissibility is replaced by checking the strong η-admissibility.

Remark 29 *If the clusters along with the associated set of sons are constructed by the algorithm* ***build_cluster_tree****, then,* T *is a cluster tree.*

Remark 30 *For* $0 \leq \ell \leq L$*, the construction of the cluster tree implies that the tree levels*

$$T(\ell) = \{c \in T \mid \text{LEVEL}(c) = \ell\}. \tag{38}$$

satisfy

$$\begin{aligned} \Gamma &= \bigcup T(\ell), \qquad \forall 0 \leq \ell \leq L, \\ \mathcal{G} &= T(L). \end{aligned}$$

The block-cluster tree $T^{(2)}$ is determined from T via Construction 13.

Remark 31 *All blocks* $(c_1, c_2) \in T^{(2)}$ *consists of clusters of the same level:*

$$\text{LEVEL}(c_1) = \text{LEVEL}(c_2).$$

3.2 Computation of the expansion coefficients, nearfield matrix and basis farfield coefficients

The coefficients $\overrightarrow{\kappa}_{\nu,\mu}^{(m)}(\mathbf{c})$ in (12) have to be computed. We will not use the *expansion system* (23) but approximations to it. Nevertheless, the *expansion coefficients* $\overrightarrow{\kappa}_{\nu,\mu}^{(m)}(\mathbf{c})$ of (22) are used. Efficient algorithms for computing $\overrightarrow{\kappa}_{\nu,\mu}^{(m)}(\mathbf{c})$ are developed, for collocation discretizations, in [6] and [14] and, for Galerkin discretizations, in [7], [15], [8]. We do not recall here the details of the algorithms.

It will turn out from the error analysis that the nearfield matrix $\mathbf{N}$ can be replaced by the zero-matrix. No work at all is needed for this step.

It remains to compute the basis farfield coefficients. We consider here only the more involved case II:

$$\overrightarrow{J}_{\tau,\nu}^{II} = \int_{\tau} \overrightarrow{\Psi}_{\tau}^{(\nu)}(y)\, d\Gamma_y, \qquad \forall \nu \in \iota_{m(\tau)}^{II}, \qquad \forall \tau \in \mathcal{G}.$$

It will turn out that on the *panel level*, we restrict to polynomial expansions, i.e.,

$$\Phi_{\tau}^{(\nu)}(x) = (x - M_\tau)^{\nu}, \qquad \overrightarrow{\Psi}_{\tau}^{(\nu)} = (y - M_\tau)^{(\nu)}\, n(y).$$

In the case of *flat panels*, the normal vector n is constant on τ and the integration can be performed analytically (cf. [14]). For more general parametrisations, the integrals have to be evaluated numerically. Transforming τ onto the master element Q (cf. Definition 1) yields:

$$\overrightarrow{J}_{\tau,\nu}^{II} = \int_0^1 \int_0^{\xi_1} g_\tau(\xi)\, \overrightarrow{\Psi}_{\tau}^{(\nu)} \circ \chi_\tau(\xi)\, d\xi,$$

where g_τ denotes the surface element. Since χ_τ and n are smooth the integrand is smooth as well. Due to (17) we know

$$|\nu| \leq C_5 \left(m\left(L\right)+1\right) \overset{(9)}{=} C_5 \left(b+1\right).$$

Hence, standard quadrature formulae as, e.g., conical Gauß rules, could be applied. For given α, the number of quadrature points for conical Gauß rules approximating $J^{II}_{\tau,\nu}$ with an accuracy of $O\left(\operatorname{diam}^\alpha \tau\right)$ is independent of $\operatorname{diam}\tau$.

3.3 Computation of the farfield coefficients

In this subsection, we will define precisely the kind of approximation we will use for the variable order panel clustering method. Before we present the formal algorithm, we start with some motivations. For the efficiency of the algorithm, it is essential that the expansion system is organized hierarchically. The idea is to compose the functions $\overrightarrow{\Psi}^{(\nu)}_c, \Phi^{(\nu)}_c$ as linear combinations of the functions $\overrightarrow{\Psi}^{(\nu)}_{\tilde{c}}, \Phi^{(\nu)}_{\tilde{c}}$ where $\tilde{c} \in \sigma\left(c\right)$. We consider first the functions $\Phi^{(\nu)}_c$and make the ansatz:

$$\Phi^{(\nu)}_c\left(x\right) |_{\tilde{c}} := \sum_{\widetilde{\nu} \in \iota_{\widetilde{m}}} \gamma_{\nu,\widetilde{\nu},\tilde{c}} \Phi^{(\nu)}_{\tilde{c}}\left(x\right)$$

where $\iota_{\widetilde{m}}$ is short for $\iota^I_{m(\tilde{c})}$ and ι_m abbreviates $\iota^I_{m(c)}$. Since $\Phi^{(\nu)}_c$ should approximate the functions

$$\check{\Phi}^{(\nu)}_c\left(x\right) := \left(x - M_c\right)^\nu$$

we determine $\gamma_{\nu,\widetilde{\nu},\tilde{c}}$ by the equation

$$\check{\Phi}^{(\nu)}_c\left(x\right) |_{\tilde{c}} = \sum_{\widetilde{\nu} \in \iota_m} \gamma_{\nu,\widetilde{\nu},\tilde{c}} \check{\Phi}^{(\nu)}_{\tilde{c}}\left(x\right). \tag{39}$$

This leads to

$$\left(x - M_c\right)^\nu = \sum_{\widetilde{\nu} \leq \nu} \frac{1}{\widetilde{\nu}!} \partial^{\widetilde{\nu}} \left[\check{\Phi}^{(\nu)}_c\right]\left(M_{\tilde{c}}\right) \left(x - M_{\tilde{c}}\right)^{\widetilde{\nu}}.$$

This motivates the definition

$$\gamma_{\nu,\widetilde{\nu},\tilde{c}} := \begin{cases} \binom{\nu}{\widetilde{\nu}} \left(M_{\tilde{c}} - M_c\right)^{\nu-\widetilde{\nu}} & \widetilde{\nu} \leq \nu \\ 0 & \text{otherwise} \end{cases} \tag{40}$$

for all $\nu \in \iota_m$ and $\widetilde{\nu} \in \iota_{\widetilde{m}}$.

Definition 32 *The index sets ι^I_m, ι^{II}_m, $\mathbf{I}_m$ are given by*

$$\begin{aligned} \iota_m &: = \iota^I_m := \iota^{II}_m := \left\{\nu \in \mathbb{N}^3_0 \mid |\nu| \leq m\right\}, \\ \mathbf{I}_m &: = \left\{(\nu,\mu) \in \mathbb{N}^3_0 \times \mathbb{N}^3_0 : |\nu| + |\mu| \leq m\right\} \end{aligned}$$

and the expansion functions $\Phi^{(\nu)}_c$, $\overrightarrow{\Psi}^{(\nu)}_c$ by the recursion:

- *for the panels* $\tau \in \mathcal{G}$:

$$\begin{array}{ll} \Phi_{\tau}^{(\nu)}(x) = (x - M_{\tau})^{\nu} & \forall \nu \in \iota_{m(\tau)}, \\ \overrightarrow{\Psi}_{\tau}^{(\nu)} = (y - M_{\tau})^{\nu} n(y) & \forall \nu \in \iota_{m(\tau)}, \end{array} \tag{41}$$

- *for the clusters* $c \in T \backslash \mathcal{G}$*:*

$$\begin{array}{lll} \Phi_{c}^{(\nu)} |_{\widetilde{c}} = \sum_{\widetilde{\nu} \in \iota_{m(\tilde{c})}} \gamma_{\nu,\widetilde{\nu},\widetilde{c}} \Phi_{\widetilde{c}}^{(\widetilde{\nu})} & \forall \nu \in \iota_{m(c)} & \forall \widetilde{c} \in \sigma(c), \\ \overrightarrow{\Psi}_{c}^{(\nu)} |_{\widetilde{c}} = \sum_{\widetilde{\nu} \in \iota_{m(\tilde{c})}} \gamma_{\nu,\widetilde{\nu},\widetilde{c}} \overrightarrow{\Psi}_{\widetilde{c}}^{(\widetilde{\nu})} & \forall \nu \in \iota_{m(c)} & \forall \widetilde{c} \in \sigma(c). \end{array} \tag{42}$$

For the computation of the farfield coefficients, the hierarchical definition of the expansion functions is used. The initial step is performed on the panel level $\mathcal{G}$. Compute and store, for all $\tau \in \mathcal{G}$:

$$\overrightarrow{J}_{\tau,\nu}^{II}[u] := u(\tau) \overrightarrow{J}_{\tau,\nu}^{II}, \qquad \forall \nu \in \iota_{m(\tau)}.$$

Assume inductively that all coefficients $\overrightarrow{J}_{\tilde{c},\nu}^{II}[u]$ are computed for all $\tilde{c} \in \sigma(c)$ and $c \in T \backslash \mathcal{G}$. For simplicity, we write ι_m short for $\iota_{m(c)}$ and $\iota_{\widetilde{m}}$ short for $\iota_{m(\tilde{c})}$. Then

$$\overrightarrow{J}_{c,\nu}^{II}[u] = \int_{c} \overrightarrow{\Psi}_{c}^{(\nu)}(y) u(y) d\Gamma = \sum_{\widetilde{c} \in \sigma(c)} \int_{\widetilde{c}} \overrightarrow{\Psi}_{c}^{(\nu)}(y) u(y) d\Gamma, \qquad \forall \nu \in \iota_{m}.$$

By using (42) we obtain

$$\overrightarrow{J}_{c,\nu}^{II}[u] = \sum_{\widetilde{c} \in \sigma(c)} \sum_{\widetilde{\nu} \in \iota_{\widetilde{m}}} \gamma_{\nu,\widetilde{\nu},\widetilde{c}} \overrightarrow{J}_{\widetilde{c},\widetilde{\nu}}^{II}[u], \qquad \forall \nu \in \iota_{m}. \tag{43}$$

3.4 Evaluation of a matrix vector multiplication

The computation of the quantities $A_{\mathbf{c},\nu}[u]$ is straightforward. It will turn out that it is preferable to compute and store directly the quantities[3]:

$$B_{c_1}^{(\nu)}[u] := \sum_{c_2:(c_1,c_2) \in F} \tilde{A}_{\mathbf{c}}^{(\nu)}[u], \qquad \forall c_1 \in T, \qquad \forall \nu \in \iota_{m(c_1)}^{I} \tag{44}$$

with

$$\tilde{A}_{\mathbf{c}}^{(\nu)}[u] := \begin{cases} A_{\mathbf{c}}^{(\nu)}[u], & \text{if } \nu \in \iota_{m(\mathbf{c})}^{I}, \\ 0 & \text{if } \nu \in \iota_{m(c_1)}^{I} \backslash \iota_{m(\mathbf{c})}^{I}, \end{cases} \qquad \forall \mathbf{c} = (c_1, c_2) \in F.$$

We turn to the evaluation of the sum in (29). Since we replaced the nearfield matrix $\mathbf{N}$ by the zero, the sum in (29) consists only of the farfield evaluation:

$$B_{\tau} = \sum_{\mathbf{c} \in F} \sum_{\nu \in \iota_{m(\mathbf{c})}^{I}} \int_{c_1} A_{\mathbf{c}}^{(\nu)}[u] \Phi_{c_1}^{(\nu)}(x) b_{\tau}(x) dx.$$

[3]Recall that, for $\mathbf{c} = (c_1, c_2) \in F$, we have $m(c_1) \geq m(\mathbf{c})$ (cf. (34), (18)) implying $\iota_{m(\mathbf{c})}^{I} \subset \iota_{m(c_1)}^{I}$)

In view of (44), it is advantageous to rewrite this formula as

$$B_\tau = \sum_{c_1 \in T} \sum_{\nu \in \iota^I_{m(c_1)}} \int_{c_1} B^{(\nu)}_{c_1}[u]\, \Phi^{(\nu)}_{c_1}(x)\, b_\tau(x)\, dx. \tag{45}$$

In the next step, we will derive a hierarchical representation of this formula. The summation over $c_1 \in T\backslash\mathcal{G}$ in (45) can be split into a sum over the sons $\sigma(c_1)$. Hence, setting $\iota_m = \iota^I_{m(c_1)}$, we obtain

$$\sum_{\nu \in \iota_m} \int_{c_1} B^{(\nu)}_{c_1}[u]\, \Phi^{(\nu)}_{c_1}(x)\, b_\tau(x)\, dx = \sum_{\widetilde{c_1} \in \sigma(c_1)} \sum_{\nu \in \iota_m} \int_{\widetilde{c_1}} B^{(\nu)}_{c_1}[u]\, \Phi^{(\nu)}_{c_1}(x)\, b_\tau(x)\, dx. \tag{46}$$

On the other hand, the summation in (45) contains a partial sum over $\sigma(c_1)$ of the form:

$$\sum_{\widetilde{c_1} \in \sigma(c_1)} \sum_{\tilde{\nu} \in \iota_{\tilde{m}}} \int_{\widetilde{c_1}} B^{(\tilde{\nu})}_{\widetilde{c_1}}[u]\, \Phi^{(\tilde{\nu})}_{\widetilde{c_1}}(x)\, b_\tau(x)\, dx \tag{47}$$

with $\iota_{\tilde{m}} = \iota^I_{m(\widetilde{c_1})}$. In the next step, the right-hand side in (46) will be added to (47). Plugging in (42) into (46) and re-organizing the terms shows that the sum in (46) equals

$$\sum_{\widetilde{c_1} \in \sigma(c_1)} \sum_{\widetilde{\nu} \in \iota_{\tilde{m}}} \int_{\widetilde{c_1}} R^{(\widetilde{\nu})}_{\widetilde{c_1}}[u]\, \Phi^{(\widetilde{\nu})}_{\widetilde{c_1}}(x)\, b_\tau(x)\, dx$$

with

$$R^{(\widetilde{\nu})}_{\widetilde{c_1}}[u] = \sum_{\nu \in \iota_m} \gamma_{\nu,\widetilde{\nu},\widetilde{c_1}} B^{(\nu)}_{c_1}[u].$$

Hence, (46) and (47) can be added resulting in

$$\sum_{\widetilde{c_1} \in \sigma(c_1)} \sum_{\tilde{\nu} \in \iota_{\tilde{m}}} \int_{\widetilde{c_1}} \left(R^{(\tilde{\nu})}_{\widetilde{c_1}}[u] + B^{(\tilde{\nu})}_{\widetilde{c_1}}[u] \right) \Phi^{(\tilde{\nu})}_{\widetilde{c_1}}(x)\, b_\tau(x)\, dx.$$

Iterating this algorithms over the hierarchical structure of T leads to Algorithm 33 for the evaluation of B_τ. The tree levels $T(\ell)$ are as in Definition 10 and $B^{(\nu)}_c[u]$ be as in (44).

Algorithm 33 ***procedure evaluate_sum;begin***
for all $\nu \in \iota_{m(\Gamma)}$ ***do*** $\mathcal{R}^{(\nu)}_\Gamma[u] = B^{(\nu)}_\Gamma[u]$;
for all $\ell = 0$ ***to*** $L-1$ ***do***
for all $c \in T(\ell)$ ***do***
for all $\widetilde{c} \in \sigma(c)$ ***do begin for all*** $\tilde{\nu} \in \iota_{m(\tilde{c})}$
$\mathcal{R}^{(\tilde{\nu})}_{\widetilde{c}}[u] = B^{(\tilde{\nu})}_{\tilde{c}}[u] + \sum_{\nu \in \iota_{m(c)}} \gamma_{\nu,\tilde{\nu},\widetilde{c}} \mathcal{R}^{(\nu)}_c[u]$;
end;

for all $\tau \in \mathcal{G}$ ***do***

$$B_\tau = \sum_{\nu \in \iota_{m(\tau)}} \mathcal{R}_\tau^{(\nu)} [u] \, J_{\tau,\nu}^I;$$

end;

4 Error analysis

4.1 Abstract error estimates

We have presented a variable order panel clustering algorithm based on block partitionings of $\Gamma \times \Gamma$ for the discretization of the second kind integral equation in (1). The discretization is based on piecewise constant finite element spaces. It is well known in the theory of boundary elements that the Galerkin solution to this problem converges as

$$\|u - u_G\|_{0,\Gamma} \leq Ch \, \|g\|_{1,\Gamma} \tag{48}$$

provided $g \in H^1(\Gamma)$ where $\|\cdot\|_{1,\Gamma}$ denotes the H^1-norm. Let $\widetilde{u_G} \in S^{-1,0}$ denotes the solution if the integral operator in (3) is replaced by the panel clustering approximation. In this section, we will prove that, under the abstract Assumption 20, the solution $\widetilde{u_G}$ exists and satisfies the error estimate (48), too, with a possibly larger constant C. In this section, the error estimates will be derived from abstract assumptions while, in Section 4.2 and [13], it is shown that this assumptions are satisfied for shape regular, quasi-uniform meshes.

Definition 34 *The uniformity of a mesh $\mathcal{G}$ is characterized by the smallest constant C_u satisfying*

$$h \leq C_u h_\tau, \qquad \forall \tau \in \mathcal{G}$$

where h is as in (31) and

$$h_\tau := \operatorname{diam} \tau.$$

Definition 35 *The quality of panels is characterized by the smallest constant C_q satisfying*

$$h_\tau^2 \leq C_q \, |\tau| \,, \qquad \forall \tau \in \mathcal{G}.$$

Remark 36 *Since $\mathcal{G}$ only contains finitely many panels, the constants C_u, C_q are always bounded. However, it will turn out that the constants in the estimates below behaves critically with increasing values of C_q, C_u and we assume here that C_q and C_u are of moderate size.*

Assumption 37 *The tree T is balanced in the sense that all panels $\tau \in \mathcal{G}$ have the same depth in the tree:*

$$\textsc{level}\,(\tau) = L, \qquad \forall \tau \in \mathcal{G}.$$

Remark 38 *By using the construction of Subsection 3.1, Assumption 37 is always guaranteed.*

Remark 39 *Construction 13 and Definition 14 implies that all blocks* $\mathbf{c} = (c_1, c_2) \in P^{(2)}$ *consist of clusters of the same level:*

$$\text{LEVEL}\,(c_1) = \text{LEVEL}\,(c_2)\,.$$

For $0 \leq \ell \leq L$*, we introduce the farfield levels* $F(\ell)$ *by*

$$F(\ell) = \{(c_1, c_2) \in F : \text{LEVEL}\,(c_1) = \text{LEVEL}\,(c_2) = \ell\}\,.$$

Then, the function $m : F \to \mathbb{N}_0$ *as in Definition 16 only depends on the level* ℓ*. For* $\mathbf{c} \in F(\ell)$*, we have*

$$m(\mathbf{c}) = a(L - \ell) + b. \tag{49}$$

The right-hand side in (49) defines a function $\tilde{m} : \mathbb{N}_0 \to \mathbb{N}_0$*. If there is no ambiguity we write again* m *instead of* $\tilde{m}$*.*

Assumption 40 *There exist constants* $C_6 < \infty$ *and* $1 < C_7 < \infty$ *so that, for all* $0 \leq \ell \leq L$ *and any* $c \in T(\ell)$*:*

$$\begin{aligned} C_7^{-1} 2^{-\ell} &\leq \rho_c \leq C_7 2^{-\ell}, \\ \operatorname{diam} c &\leq C_6 h 2^{L-\ell}. \end{aligned}$$

Assumption 41 *The constants* a *in (9) is chosen so that* $a > 1$ *and* $2C_2^a =: C_8 < 1$ *holds with* C_2 *as in Assumption 20.*

We need an assumption estimating, for $c_1 \in T(\ell)$, the number of clusters c_2 forming a block (c_1, c_2) in $F(\ell)$.

Assumption 42 *There exist positive constants* C_9^I*,* $C_9^{II} < \infty$ *so that, for all* $0 \leq \ell \leq L$ *and all* $c \in T(\ell)$*:*

$$\begin{aligned} \sharp\{\mathbf{c} \in F(\ell) : c_1 = c\} &\leq C_9^I, \\ \sharp\{\mathbf{c} \in F(\ell) : c_2 = c\} &\leq C_9^{II}. \end{aligned}$$

The nearfield matrix is replaced by zero. In order to estimate the arising error we need an assumption concerning the number of nearfield matrix entries.

Assumption 43 *There exist a positive constants* $C_{10}^I, C_{10}^{II} < \infty$ *so that, for all* $0 \leq \ell \leq L$ *and all* $\tau \in \mathcal{G}$*:*

$$\begin{aligned} \sharp\{t \in \mathcal{G} : (\tau, t) \in N\} &\leq C_{10}^I, \\ \sharp\{t \in \mathcal{G} : (t, \tau) \in N\} &\leq C_{10}^{II} \end{aligned}$$

with N *as in (8).*

The error estimate of the Galerkin discretization including panel clustering is based on the second Strang lemma [1]. For $u, v \in S^{-1,0}$, let

$$E := \left| \left(v, K[u] - \widetilde{K}[u] \right) \right|,$$

where $\tilde{K}$ denotes the panel clustering approximation to K. In order to estimate E, we need an auxiliary result.

Lemma 44 *Let Assumption 20, 37, and 40 be satisfied. There exist a constant $C_{11} < \infty$ so that, for all $0 \leq \ell \leq L$ and every $\mathbf{c} \in F(\ell)$*

$$\sqrt{|c_1|\,|c_2|} C_2^{m(\ell)} \operatorname{dist}^{-1}(c_1, c_2) \leq C_{11} h C_8^{L-\ell}. \tag{50}$$

Proof. Recall that $0 \leq C_2 < 1$. Let $\mathbf{c} \in F(\ell)$. Without loss of generality we assume that

$$\rho_{c_1} = \max\{\rho_{c_1}, \rho_{c_2}\}.$$

Hence,

$$\sqrt{|c_1|\,|c_2|} \leq C \rho_{c_1}^2,$$

where C depends only on (the curvature of) the surface Γ. Using (7), Assumption 40 and 37 we obtain

$$\sqrt{|c_1|\,|c_2|} C_2^{m(\ell)} \operatorname{dist}^{-1}(c_1, c_2) \leq C\eta(\rho_{c_1}) C_2^{m(\ell)} \leq C C_6 h 2^{L-\ell} C_2^{m(\ell)}$$

and, taking into account (58)

$$\sqrt{|c_1|\,|c_2|} C_2^{m(\ell)} \operatorname{dist}^{-1}(c_1, c_2) \leq C C_6 h (2C_2^a)^{L-\ell} = C C_6 h C_8^{L-\ell}.$$

■

Lemma 45 *Let Assumptions 20, 37, 40, 42, and 43 be satisfied. There exists a constant C so that, for all $u, v \in S^{-1,0}$:*

$$E \leq Ch \|u\|_{0,\Gamma} \|v\|_{0,\Gamma}.$$

Proof. We employ the splitting $E = E_1 + E_2$ with

$$\begin{aligned} E_1 &= \sum_{\ell=0}^{L} \sum_{\mathbf{c} \in F(\ell)} \int_{\mathbf{c}} u(x) v(y) \left(k(x,y) - k_{\mathbf{c}}^{m(\ell)}(x,y) \right) d\Gamma_y d\Gamma_x \\ E_2 &= \sum_{(t,\tau) \in N} \int_{t \times \tau} u(x) v(y) k(x,y) \, d\Gamma_y d\Gamma_x \end{aligned}$$

and estimate E_1, E_2 separately.

$$\begin{aligned}
E_1 &\leq \sum_{\ell=0}^{L}\sum_{\mathbf{c}\in F(\ell)}\int_{\mathbf{c}} |u(x)|\,|v(y)|\left|k(x,y)-k_{\mathbf{c}}^{m(\ell)}(x,y)\right| d\Gamma_y d\Gamma_x \\
&\overset{(11)}{\leq} \sum_{\ell=0}^{L}\sum_{\mathbf{c}\in F(\ell)} \sqrt{|c_1|\,|c_2|}C_1 C_2^{m(\ell)}\,\mathrm{dist}^{-1}(c_1,c_2)\,\|u\|_{0,c_1}\,\|v\|_{0,c_2} \\
&\overset{(50)}{\leq} C_{11}C_1 h\sum_{\ell=0}^{L} C_8^{L-\ell}\sum_{\mathbf{c}\in F(\ell)}\|u\|_{0,c_1}\,\|v\|_{0,c_2} \\
&\leq C_{11}C_1 h\sum_{\ell=0}^{L} C_8^{L-\ell}\left\{\sum_{\mathbf{c}\in F(\ell)}\|u\|_{0,c_1}^2\right\}^{1/2}\left\{\sum_{\mathbf{c}\in F(\ell)}\|v\|_{0,c_2}^2\right\}^{1/2} \\
&\leq C_{11}C_1 h\sum_{\ell=0}^{L} C_8^{L-\ell}\left\{\sum_{c_1\in T(\ell)}\|u\|_{0,c_1}^2\sum_{c_2:\mathbf{c}\in F(\ell)}1\right\}^{1/2}\left\{\sum_{c_2\in T(\ell)}\|v\|_{0,c_2}^2\sum_{c_1:\mathbf{c}\in F(\ell)}1\right\}^{1/2} \\
&\leq C_{11}C_1\sqrt{C_9^I C_9^{II}}h\,\|u\|_{0,\Gamma}\,\|v\|_{0,\Gamma}\sum_{\ell=0}^{L}C_8^{L-\ell} \leq \frac{C_{11}C_1\sqrt{C_9^I C_9^{II}}}{1-C_8}h\,\|u\|_{0,\Gamma}\,\|v\|_{0,\Gamma}.
\end{aligned}$$

For the estimate of E_2 we begin with considering a single pair of panels $(t,\tau)\in N$:

$$\begin{aligned}
\left|\int_t v(x)\int_\tau u(y)\,k(x,y)\right| &\leq C_\Gamma\int_t |v(x)|\,dx\int_\tau |u(y)|\,\|x-y\|^{-1}\,d\Gamma_y d\Gamma_x \\
&\leq C_\Gamma\,\|v\|_{\infty,t}\,\|u\|_{\infty,\tau}\int_{t\times\tau}\|x-y\|^{-1}\,d\Gamma_y d\Gamma_x. \qquad (51)
\end{aligned}$$

Since u is constant on t and v on τ, we get

$$\|v\|_{\infty,t}\,\|u\|_{\infty,\tau} = \frac{\|v\|_{0,t}\,\|u\|_{0,\tau}}{\sqrt{|t|\,|\tau|}}.$$

We turn to the integral in (51). We distinguish two cases:

(a) $\mathrm{dist}(\tau,t)>0$. The shape regularity and the quasi-uniformity of the meshes imply:

$$\mathrm{dist}(\tau,t)\geq Ch.$$

Hence,

$$\|v\|_{\infty,t}\,\|u\|_{\infty,\tau}\int_{t\times\tau}\|x-y\|^{-1}\,d\Gamma_y d\Gamma_x \leq C^{-1}h^{-1}\sqrt{|t|\,|\tau|}\,\|v\|_{0,t}\,\|u\|_{0,\tau} \leq C^{-1}h\,\|v\|_{0,t}\,\|u\|_{0,\tau}.$$

(b) $\mathrm{dist}(\tau,t)=0$. There exists a mapping $\chi:\mathbb{R}^2\to\mathbb{R}^3$ which is sufficiently smooth, independent of h along with a subset $\mathcal{U}\subset\mathbb{R}^2$ with $\chi(\mathcal{U})=t\cup\tau$. Furthermore,

me may assume that there exists a constant C independent of h so that $\mathcal{U}$ is contained in a ball B about the origin with radius Ch. Then,

$$\int_{t\times\tau} \|x-y\|^{-1} d\Gamma_y d\Gamma_x \leq C \int_{B\times B} \|\chi(\xi)-\chi(\theta)\|^{-1} d\theta d\xi.$$

We introduce polar coordinates about ξ:

$$\theta = \xi + r\psi(\alpha)$$

with $\psi(\alpha) = (\cos\alpha, \sin\alpha)^\intercal$. Hence,

$$\int_{t\times\tau} \|x-y\|^{-1} d\Gamma_y d\Gamma_x \leq C \int_B \int_0^{Ch} \int_0^{2\pi} r \|\chi(\xi)-\chi(\xi+r\psi)\|^{-1} d\alpha dr d\xi.$$

The quotient

$$\frac{r}{\|\chi(\xi)-\chi(\xi+r\psi)\|}$$

stays bounded as $r\to 0$ as a consequence of the regularity of Γ. Thus,

$$\int_{t\times\tau} \|x-y\|^{-1} d\Gamma_y d\Gamma_x \leq C \int_B \int_0^{Ch} \int_0^{2\pi} 1 d\alpha dr d\xi \leq Ch^3,$$

$$\|v\|_{\infty,t} \|u\|_{\infty,\tau} \int_{t\times\tau} \|x-y\|^{-1} d\Gamma_y d\Gamma_x \leq Ch \|v\|_{0,t} \|u\|_{0,\tau}.$$

Summing all nearfield entries yields:

$$\begin{aligned} E_2 &\leq Ch \sum_{(t,\tau)\in N} \|v\|_{0,t} \|u\|_{0,\tau} \leq Ch \left\{ \sum_{(t,\tau)\in N} \|v\|_{0,t}^2 \right\}^{1/2} \left\{ \sum_{(t,\tau)\in N} \|u\|_{0,\tau}^2 \right\}^{1/2} \\ &\leq Ch \left\{ \sum_{t\in\mathcal{G}} \|v\|_{0,t}^2 \sum_{\tau:(t,\tau)\in N} 1 \right\}^{1/2} \left\{ \sum_{\tau\in\mathcal{G}} \|u\|_{0,\tau}^2 \sum_{t:(t,\tau)\in N} 1 \right\}^{1/2} \leq C\sqrt{C_{10}^I C_{10}^{II}}\, h \|v\|_{0,\Gamma} \|u\|_{0,\Gamma}. \end{aligned}$$

■

Theorem 46 *Let the assumptions of Lemma 45 be satisfied. Then, the solution $\widetilde{u_G}$ to (3) with K replaced by the panel clustering approximation exists for any $g\in L^2(\Gamma)$. If $g\in H^1(\Gamma)$ the error estimate*

$$\|u-\widetilde{u_G}\|_{0,\Gamma} \leq Ch \|g\|_{1,\Gamma}$$

holds.

Proof. In view of Lemma 45 the result follows from [1, Lemma 4.1.1]. ■

In the remaining part of this paper, we will show that the hierarchical approximation system $\Phi_c^{(\nu)}$ and $\Psi_c^{(\nu)}$ (cf. Definition 32) satisfy the approximation property and that the panel clustering algorithm has linear complexity. We do not work out here the proof that Assumptions 37, 40, 42, and 43 hold for quasi-uniform meshes provided the cluster tree is assembled according to procedure **build_cluster_tree** and the block partitioning according to procedure **divide** but refer to [13].

4.2 Verifying the approximation property of the expansion system

In this subsection, we will prove that the expansion system defined in Definition 32 satisfies Assumption 20 with $m(\ell)$ as in (54). In order to reduce the technicalities in the proofs below we impose a weak assumption (Assumption 52) on the sizes of the sons of a cluster and two further assumptions on the geometry of the surface Γ: The first one is satisfied for all reasonable surfaces and the second one is imposed to reduce technicalities.

Notation 47 *The three dimensional ball (with respect to the maximum norm) about $x \in \mathbb{R}^3$ with radius r is denoted by $B_r^\infty(x)$. For $r > 0$, the r-neighbourhood of Γ is*

$$U_r(\Gamma) := \left\{ x \in \mathbb{R}^3 \mid \exists y \in \Gamma : \|x - y\| \leq r \right\}.$$

Assumption 48 *There exist positive constants c_Γ, C_Γ so that, for all $x \in \Gamma$ and all $0 < r \leq \operatorname{diam}\Gamma$*

$$\begin{aligned} |B_r^\infty(x) \cap \Gamma| &\geq c_\Gamma r^2, \\ |U_r(\Gamma)| &\leq C_\Gamma r. \end{aligned}$$

For all subsets $\gamma \subset \Gamma$, the diameter $\operatorname{diam}\gamma$ can be estimated by

$$\operatorname{diam}\gamma \geq c_\Gamma \sqrt{|\gamma|},$$

where $|\gamma|$ denotes the two-dimensional surface measure of γ.

Assumption 49 *Γ is a closed, simple connected surface.*

By using these assumptions the precise condition on the parameter $c_{\min}$ controlling the procedure **build_cluster_tree** can be formulated. However, for practical problems, we expect that the optimal value of $c_{\min}$ is larger and should be determined by numerical experiments. This will be the topic of a forthcoming paper.

Assumption 50 *The constant $c_{\min}$ in the procedure **build_cluster_tree** is chosen so that*

$$c_{\min} \leq \min\left\{4, \sqrt{c_\Gamma^3/2}\right\}/32$$

holds with c_Γ as in Assumption 48.

Definition 51 *Let $c \in T(i)$ and $\tilde{c} \in T(j)$ with $j > i$ and $\tilde{c} \subset c$. The chain*

$$\mathcal{K}_{\tilde{c},c} = (c_j, c_{j-1}, c_{j-2}, \ldots, c_i)$$

is given by the recursion:

$$\begin{aligned} c_j &= \tilde{c}, \\ c_{k-1} &: \sigma(c_{k-1}) \ni c_k, \qquad k = j, j-1, \ldots, i+1. \end{aligned}$$

Assumption 52 *There exist positive constants* $c_{12}, \bar{\rho} < 1$ *so that, for all clusters* $c \in T$ *and all sons* $\tilde{c} \in \sigma(c)$*, either* $\tilde{c} = c$ *or the ratio of the cluster radii satisfies:*

$$c_{12} \leq \rho_{\tilde{c}}/\rho_c \leq \bar{\rho}$$

For all clusters $c \in T$ *and all panels* $\tilde{c} \in \mathcal{G}$ *with* $\tilde{c} \subset c$*, the number of repeated clusters in the chain* $\mathcal{K}_{\tilde{c},c}$ *is bounded by :*

$$\sup_{c \in T} \sup_{\substack{\tilde{c} \in \mathcal{G} \\ \tilde{c} \subset c}} n_{\tilde{c},c} \leq \overline{n} \tag{52}$$

with

$$n_{\tilde{c},c} := \sharp \left\{ \check{c} \in \mathcal{K}_{\tilde{c},c} \mid \sharp\sigma(\check{c}) = 1 \right\}. \tag{53}$$

Definition 53 *Let* $\omega \in \mathbb{R}^d$ *with centre* M_ω*. The Taylor operator* $T_\omega^{(m)}$ *is given formally by*

$$T_\omega^{(m)}[f](x) = \sum_{|\nu| \leq m} \gamma_{\nu,\omega}[f] \check{\Phi}_\omega^{(\nu)}(x)$$

with

$$\gamma_{\nu,\omega}[f] = \frac{1}{\nu!} f^{(\nu)}(M_\omega)$$

and

$$\check{\Phi}_\omega^{(\nu)}(x) = (x - M_\omega)^\nu \tag{54}$$

The auxiliary functions $\overrightarrow{\check{\Psi}}_\omega^{(\nu)}$ are defined by

$$\overrightarrow{\check{\Psi}}_\omega^{(\nu)}(y) := \check{\Phi}_\omega^{(\nu)}(y)\, n(y).$$

The expansion functions $\Phi_c^{(\nu)}$ and $\overrightarrow{\Psi}_c^{(\nu)}$ (cf. Definition 32) can be regarded as approximations to the functions $\check{\Phi}_\omega^{(\nu)}$ and $\overrightarrow{\check{\Psi}}_\omega^{(\nu)}$. The precision is concerned in Lemma 55. The normal derivatives of the Taylor polynomials are denoted by

$$\check{N}_c^{(\nu)} := \sum_{i=1}^{3} n_i(y) \check{\Phi}_c^{(\nu+e_i)}(y) = \langle n, y - M_c \rangle \check{\Phi}_c^{(\nu)} \tag{55}$$

while an analogous quantity for the true expansion system is defined by

$$N_c^{(\nu)} := \sum_{i=1}^{3} n_i(y) \Phi_c^{(\nu+e_i)}(y). \tag{56}$$

It remains to define the function $m(\ell) = a(L - \ell) + b$ determining the degree of approximation on a block.

Definition 54 *Let Assumption 20 be satisfied. For $0 \leq \ell \leq L$, the function $m(\ell)$ determining the variable order of approximation is given by*

$$m(\ell) = a(L - \ell) + b, \tag{57}$$

with $a, b \in \mathbb{N}_0$ chosen so that

$$a > 1 \quad \textit{and} \quad 2C_2^a =: C_8 < 1 \tag{58}$$

(cf. Remark 59) and

$$b \geq \max\left\{ \frac{\left|\log \frac{1-\bar{\rho}}{2\bar{\rho}}\right|}{|\log \bar{\rho}|}, 1 + \frac{a \log \lambda_4}{\left|\log \frac{1+\bar{\rho}}{2}\right|}, 2\left(a\frac{\log \lambda_4}{\log 2} - 1\right), \frac{\log 2}{|\log \bar{\rho}|}, \frac{\log \frac{\lambda_4^a + C_\Gamma - 1}{2\left(1-\lambda_4^{-a}\right)}}{\log \frac{4}{3}} \right\} \tag{59}$$

with $\lambda_4 > 8$.

Note that the conditions on a and b stem from the proof of the approximation property which we expect are by far too restrictive. In a forthcoming paper, the results of numerical experiments will be presented dealing with the optimal choice of $a, b, \bar{\eta}$ for practical problems.

Lemma 55 *Let Assumptions 20, 40, 52, 48, 49, and 50 be satisfied. For all $\lambda_4 \geq c_{12}^{-1}$,*

$$\omega > \max\left\{ \frac{(a+3)^{a+1}}{2(a-1)!}, \frac{2}{1-\bar{\rho}}, e\lambda_4 \right\}, \qquad \omega_2 > C_7(1+\omega) \max\left\{ \sqrt{2C_\Gamma \frac{(a+3)^{a+2}}{(a-1)!}}, e\lambda_4 \right\} \tag{60}$$

and, for all $0 \leq \ell \leq L$ and all $c \in T(\ell)$, the estimates

$$\begin{aligned} \left\| \check{\Phi}_c^{(\nu)} - \Phi_c^{(\nu)} \right\|_{L^\infty(c)} &\leq \lambda_4^{a\bar{n} - m_\ell} (\omega \rho_\ell)^{|\nu|} && \forall \nu : |\nu| \leq m(\ell), \\ \left\| \check{N}_c^{(\nu)} - N_c^{(\nu)} \right\|_{L^\infty(c)} &\leq \frac{C_\Gamma \lambda_4^{a\overline{n}}}{1 - \lambda_4^{-a}} \lambda_4^{-m_\ell} \left(\omega_2 2^{-\ell}\right)^{|\nu|+2} && \forall \nu : |\nu| \leq m(\ell) - 1 \end{aligned} \tag{61}$$

hold with $\bar{n}$ as in (52).

Since the proof of this Lemma is rather technical we refer to the extended version of this paper (cf. [13]) for the detailed proof. The approximation of the kernel function on a block $\mathbf{c} = (c_1, c_2) \in P^{(2)}(\ell)$ is given by

$$k(x,y) \approx k_{\mathbf{c}}^{(m)}(x,y) := \sum_{(\nu,\mu) \in \mathbf{I}_m} \overrightarrow{\kappa}_{\nu,\mu}^{(m)}(\mathbf{c}) \, \Phi_{c_1}^{(\nu)}(x) \, \overrightarrow{\Psi}_{c_2}^{(\mu)}(y) \tag{62}$$

with $m = m(\ell)$ and $\overrightarrow{\kappa}_{\nu,\mu}^{(m)}$ as in (22) and $\Phi_{c_1}^{(\nu)}$, $\overrightarrow{\Psi}_{c_2}^{(\mu)}$ as in Definition 32. The error analysis consists of a consistency and stability part.

For the error analysis, it is preferably to write the Taylor approximation according to (22) in a different form (with $n = n(y)$)[4]

$$\begin{aligned} \check{k}_{\mathbf{c}}^{(m)}(x,y) &= \sum_{|\nu|+|\mu|\leq m-1} \left(\sum_{i=1}^{3} n_i \check{\Phi}_{c_1}^{(\nu+e_i)}(x) \right) \check{\Phi}_{c_2}^{(\mu)}(y)\, \kappa_{\nu,\mu}^{(m)}(\mathbf{c}) \qquad (63) \\ &\quad - \sum_{|\nu|+|\mu|\leq m-1} \check{\Phi}_{c_1}^{(\nu)}(x) \left(\sum_{i=1}^{3} n_i \check{\Phi}_{c_2}^{(\mu+e_i)}(y) \right) \kappa_{\nu,\mu}^{(m)}(\mathbf{c}) \\ &\quad + \langle n, z_{\mathbf{c}} \rangle \sum_{|\nu|+|\mu|\leq m-1} \check{\Phi}_{c_2}^{(\mu)}(y)\, \check{\Phi}_{c_1}^{(\nu)}(x)\, \kappa_{\nu,\mu}^{(m)}(\mathbf{c}) \\ &=: \sum_{s\in\{I,II,III\}} \check{k}_{\mathbf{c}}^{s,(m)}(x,y) \qquad (64) \end{aligned}$$

with

$$\kappa_{\nu,\mu}^{(m)}(\mathbf{c}) := \frac{1}{\nu!\mu!}(-1)^{|\mu|} k_3^{(\nu+\mu)}(z_{\mathbf{c}}) \qquad (65)$$

and $\check{\Phi}$, $\check{\Psi}$ as in Definition 53.

Proposition 56 *The approximation $k_{\mathbf{c}}^{(m)}$ can be written in the form (63) by replacing the Taylor polynomials $\check{\Phi}_c^{(\nu)}$ by the hierarchical approximations $\Phi_c^{(\nu)}$.*

Proof. By definition, $k_{\mathbf{c}}^{(m)}$ has the representation

$$\begin{aligned} k_{\mathbf{c}}^{(m)}(x,y) &:= \sum_{i=1}^{3} \sum_{|\nu|+|\mu|\leq m} \Phi_{c_2}^{(\mu)}(y)\, \Phi_{c_1}^{(\nu)}(x)\, n_i \nu_i \frac{(-1)^{|\nu|}}{\mu!\nu!} k_3^{(\nu+\mu-e_i)}(z_{\mathbf{c}}) \\ &\quad + \sum_{i=1}^{3} \sum_{|\nu|+|\mu|\leq m} \Phi_{c_2}^{(\mu)}(y)\, \Phi_{c_1}^{(\nu)}(x)\, n_i \mu_i \frac{(-1)^{|\nu|}}{\mu!\nu!} k_3^{(\nu+\mu-e_i)}(z_{\mathbf{c}}) \\ &\quad + \langle n, z_{\mathbf{c}} \rangle \sum_{|\nu|+|\mu|\leq m} \Phi_{c_2}^{(\mu)}(y)\, \Phi_{c_1}^{(\nu)}(x) \frac{(-1)^{|\nu|}}{\mu!\nu!} k_3^{(\nu+\mu)}(z_{\mathbf{c}}) \\ &=: \sum_{s\in\{I,II,III\}} k_{\mathbf{c}}^{s,(m)}(x,y). \qquad (66) \end{aligned}$$

Performing the same index manipulations as for the derivation of (22) yields the assertion. ■

For the estimate of the approximation error, we employ the splitting:

$$e_{\mathbf{c}}(x,y) := k(x,y) - k_{\mathbf{c}}^{(m)}(x,y) = \underbrace{k(x,y) - \check{k}_{\mathbf{c}}^{(m)}(x,y)}_{=:e_{\mathbf{c}}^{I}(x,y)} + \underbrace{\check{k}_{\mathbf{c}}^{(m)}(x,y) - k_{\mathbf{c}}^{(m)}(x,y)}_{=:e_{\mathbf{c}}^{II}(x,y)}. \qquad (67)$$

[4]This expansion is derived by re-substituting $z - z_{\mathbf{c}} = x - M_{c_1} - (y - M_{c_2})$ in (21), writing $\langle n, z\rangle = \langle n, x - M_{c_1}\rangle - \langle n, y - M_{c_2}\rangle + \langle n, z_{\mathbf{c}}\rangle$, and re-organizing the sums and products

The estimate of $e^I_{\mathbf{c}}(x,y)$ directly follows from Lemma 21 and we proceed with considering $e^{II}_{\mathbf{c}}(x,y)$. By employing (63), (64), and (66) the difference $\check{k}^{(m)}_{\mathbf{c}} - k^{(m)}_{\mathbf{c}}$ can be split into three parts:

$$e^{II}_{\mathbf{c}} = \check{k}^{(m)}_{\mathbf{c}} - k^{(m)}_{\mathbf{c}} = \sum_{s\in\{III,IV,V\}} \check{k}^{s,(m)}_{\mathbf{c}} - k^{s,(m)}_{\mathbf{c}} =: \sum_{s\in\{III,IV,V\}} e^{s}_{\mathbf{c}}. \tag{68}$$

We work out the details only for the case $e^{III}_{\mathbf{c}}$ while the estimate of the errors $e^{IV,V}_{\mathbf{c}}$ is just a repetition of the arguments:

$$\begin{aligned} e^{III}_{\mathbf{c}}(x,y) &= \sum_{|\nu|+|\mu|<m} \left(\check{\Phi}^{(\nu)}_{c_1}(x) - \Phi^{(\nu)}_{c_1}(x)\right)\left(\sum_{i=1}^{3} n_i \check{\Phi}^{\mu+e_i}_{c_2}(y)\right)\kappa^{(m)}_{\nu,\mu}(\mathbf{c}) \\ &\quad + \sum_{|\nu|+|\mu|<m} \Phi^{(\nu)}_{c_1}(x)\left(\sum_{i=1}^{3} n_i\left(\check{\Phi}^{\mu+e_i}_{c_2}(y) - \Phi^{\mu+e_i}_{c_2}\right)\right)\kappa^{(m)}_{\nu,\mu}(\mathbf{c}) \\ &= : e^{VI}_{\mathbf{c}}(x,y) + e^{VII}_{\mathbf{c}}(x,y). \end{aligned} \tag{69}$$

In order to estimate $e^{VI}_{\mathbf{c}}, e^{VII}_{\mathbf{c}}$ we need an auxiliary result estimating the size of $\kappa^{(m)}_{\nu,\mu}$.

Lemma 57 *Let $\mathbf{c} = (c_1, c_2)$ be η-admissible. Then,*

$$\kappa^{(m)}_{\nu,\mu}(\mathbf{c}) = 27\frac{(\nu+\mu)!}{\nu!\mu!}\left(\frac{4}{\operatorname{dist}(c_1,c_2)}\right)^{|\nu+\mu|+3}.$$

Proof. We start with estimating the derivatives of the function $k_3 : d_{\mathbf{c}} \to \mathbb{R}$ as in (20). Note that all $z \in d_{\mathbf{c}}$ satisfy $\|z\| \geq \operatorname{dist}(c_1,c_2)$. For any $w \in \mathbb{C}^3$ with $\|w\|_\infty \leq \|z_{\mathbf{c}}\| / \left(2\sqrt{3}\right)$, we have

$$\|z_{\mathbf{c}} + v\| \geq \|z_{\mathbf{c}}\| - \|w\| \geq \|z_c\| - \sqrt{3}\,\|w\|_\infty \geq \|z_{\mathbf{c}}\|/2 \geq \tfrac{1}{2}\operatorname{dist}(c_1,c_2).$$

Hence, the function

$$g_3(w) := \|z_{\mathbf{c}} + w\|^{-3}$$

is holomorphic with respect to each component in $B_{r_1}(0)$, i.e., in the ball in complex plane about the origin with radius $r_1 := \|z_{\mathbf{c}}\| / \left(2\sqrt{3}\right)$. Applying Cauchy's integral formula in each component results in

$$\frac{1}{\mu!} g^{(\mu)}_3(w) = \frac{1}{(2\pi i)^3}\oint_{|v_1|=r}\oint_{|v_2|=r}\oint_{|v_3|=r} \frac{g_3(v)}{(v-w)^{\mu+\mathbf{1}}} dv$$

with $\mathbf{1} = (1,1,1)^\intercal$ and $r = 3\|z_{\mathbf{c}}\|/4$. The function $g_3(v)$ can be estimated by

$$|g_3(v)| \leq \left(\frac{1}{4}\|z_{\mathbf{c}}\|\right)^{-3}.$$

The denominator satisfies

$$\left|(v-w)^{\mu+\mathbf{1}}\right| \geq \prod_{i=1}^{3}\left(|v_i|-|w_i|\right)^{\mu_i+1} \geq \prod_{i=1}^{3}\left(\frac{1}{4}\left\|z_{\mathbf{c}}\right\|\right)^{\mu_i+1} = \left(\tfrac{1}{4}\left\|z_{\mathbf{c}}\right\|\right)^{|\mu|+3}$$

and the length of a single integral path $2\pi r = 3\pi\left\|z_{\mathbf{c}}\right\|/2$. Hence,

$$\left|\frac{1}{\mu!}g_3^{(\mu)}(w)\right| \leq \frac{1}{8\pi^3}\left(3\pi\left\|z_{\mathbf{c}}\right\|/2\right)^3\left(\tfrac{1}{4}\left\|z_{\mathbf{c}}\right\|\right)^{-3}\frac{1}{\left(\frac{1}{4}\left\|z_{\mathbf{c}}\right\|\right)^{|\mu|+3}} = 27\left(\frac{4}{\left\|z_{\mathbf{c}}\right\|}\right)^{|\mu|+3}.$$

The connection between g_3 and k_3 is given by $k_3^{(\mu)}(z_{\mathbf{c}}) = g_3^{(\mu)}(0)$ and, hence,

$$\left|k_3^{(\mu)}(z_{\mathbf{c}})\right| = \left|g_3^{(\mu)}(0)\right| \leq 27\mu!\left(\frac{4}{\left\|z_{\mathbf{c}}\right\|}\right)^{|\mu|+3}.$$

In view of (65), we obtain the assertion:

$$\left|\kappa_{\nu,\mu}^{(m)}(\mathbf{c})\right| = 27\frac{(\nu+\mu)!}{\nu!\mu!}\left(\frac{4}{\left\|z_{\mathbf{c}}\right\|}\right)^{|\nu+\mu|+3}.$$

■

Theorem 58 *Let $\mathbf{I}_m$, ι_m, and the expansion systems $\Phi_c^{(\nu)}$, $\Psi_c^{(\nu)}$ be chosen as in Definition 32 and the distribution of the expansion order as in Definition 54. Let the Assumptions of Lemma 55 be satisfied. Then, there exists $\overline{\eta}$ depending only on C_6, C_7, λ_3, a, b, c_{12}, $\bar{\rho}$ so that the expansion (62) satisfies Assumption 20.*

Proof. In view of (67) along with Lemma 21 we may restrict to the estimate of $e_{\mathbf{c}}^{II}$. As before, we work out the proof only for the partial error $e_{\mathbf{c}}^{III}$ in (68) while the estimate of $e_{\mathbf{c}}^{IV,V}$ is just a repetition of the arguments. Hence, it is sufficient to estimate the errors $e_{\mathbf{c}}^{VI}$ and $e_{\mathbf{c}}^{VII}$ (see (69)). Using Lemma 55 and 57 along with

$$\left|\sum_{i=1}^{3} n_i\check{\Phi}_c^{\mu+e_i}(y)\right| = \left|\langle n(y), y-M_c\rangle(y-M_c)^{\mu}\right| \leq C_\Gamma\rho_\ell^{|\mu|+2}, \qquad \forall x\in c\in T(\ell),$$

we obtain (putting $m = m(\ell)$):

$$\begin{aligned}\left|e_{\mathbf{c}}^{VI}(x,y)\right| &\leq 27\sum_{|\nu|+|\mu|<m_\ell}\lambda_4^{a\bar{n}-m_\ell}\left(\omega\rho_\ell\right)^{|\nu|}\left(C_\Gamma\rho_\ell^{|\mu|+2}\right)\frac{(\nu+\mu)!}{\nu!\mu!}\left(\frac{4}{\operatorname{dist}(c_1,c_2)}\right)^{|\nu+\mu|+3}\\ &\leq \tilde{C}\operatorname{dist}^{-1}(c_1,c_2)\,\lambda_4^{-m_\ell}\sum_{|\nu|+|\mu|<m_\ell}\binom{\nu+\mu}{\nu}\left(\frac{4\omega\rho_\ell}{\operatorname{dist}(c_1,c_2)}\right)^{|\nu+\mu|+2}\end{aligned}$$

with $\tilde{C} = 4 \times 27 C_\Gamma \lambda_4^{a\bar{n}}$. By using $\rho_\ell / \operatorname{dist}(c_1, c_2) \leq \eta$ and choosing $\bar{\eta} < (4\omega)^{-1}$ we get

$$\sum_{|\nu|+|\mu|<m_\ell} \binom{\nu+\mu}{\nu} \left(\frac{4\omega\rho_\ell}{\operatorname{dist}(c_1,c_2)} \right)^{|\nu+\mu|+2} \leq \sum_{|\nu|+|\mu|<m_\ell} \binom{\nu+\mu}{\nu} \leq 2^{3m_\ell} = 8^{m_\ell}.$$

Thus,

$$\left| e_{\mathbf{c}}^{VI}(x,y) \right| \leq \tilde{C} \operatorname{dist}^{-1}(c_1,c_2) \lambda_4^{-m_\ell} 8^{m_\ell}. \tag{70}$$

By choosing $\lambda_4 > 8$, we have proven an estimate of the form (11).

It remains to estimate $e_{\mathbf{c}}^{VII}$. The norm of the expansion functions $\Phi_{c_1}^{(\nu)}$ can be estimated by

$$\begin{aligned} \left\| \Phi_{c_1}^{(\nu)} \right\|_{L^\infty(c_1)} &= \left\| \check{\Phi}_{c_1}^{(\nu)} \right\|_{L^\infty(c_1)} + \left\| \Phi_{c_1}^{(\nu)} - \check{\Phi}_{c_1}^{(\nu)} \right\|_{L^\infty(c_1)} \\ &\leq \rho_\ell^{|\nu|} + \lambda_4^{a\bar{n}-m_\ell} (\omega\rho_\ell)^{|\nu|} \leq \rho_\ell^{|\nu|} \left(1 + \lambda_4^{a\bar{n}-m_\ell} \omega^{|\nu|}\right) \\ &\leq \rho_\ell^{|\nu|} (1 + \lambda_4^{a\bar{n}} \omega)^{|\nu|} =: (\tilde{\omega}\rho_\ell)^{|\nu|} \end{aligned} \tag{71}$$

with suitable $\tilde{\omega}$. Hence, by using (69) in combination with (71), (61), Lemma 57, and Assumption 40 we get

$$\begin{aligned} \left| e_{\mathbf{c}}^{VII}(x,y) \right| &\leq \frac{C_\Gamma \lambda_4^{a\bar{n}}}{1-\lambda_4^{-a}} \sum_{|\nu|+|\mu|<m} (\tilde{\omega}\rho_\ell)^{|\nu|} \lambda_4^{-m_\ell} \left(\omega_2 2^{-\ell}\right)^{|\mu|+2} 27 \frac{(\nu+\mu)!}{\nu!\mu!} \left(\frac{4}{\operatorname{dist}(c_1,c_2)} \right)^{|\nu+\mu|+3} \\ &\leq \check{C} \lambda_4^{-m_\ell} \operatorname{dist}^{-1}(c_1,c_2) \sum_{|\nu|+|\mu|<m} \frac{(\nu+\mu)!}{\nu!\mu!} \left(\frac{4M_\omega 2^{-\ell}}{\operatorname{dist}(c_1,c_2)} \right)^{|\nu+\mu|+2} \end{aligned}$$

where $M_\omega = \max\{C_7\tilde{\omega}, \omega_2\}$. Assumption 40 implies $2^{-\ell} \leq C_7\rho_\ell$ and, hence, the rest of the estimate is just a repetition of the arguments used for proving (70): For sufficiently small $\bar{\eta}$, an estimate of the form

$$\left| e_{\mathbf{c}}^{III}(x,y) \right| \leq \tilde{C} C_{2,III}^{m_\ell} \operatorname{dist}^{-1}(c_1,c_2)$$

holds with $C_{2,III} < 1$. Similarly, the error contributions $e_{\mathbf{c}}^{IV}$ and $e_{\mathbf{c}}^{V}$ (cf. (68)) can be estimated. ■

Remark 59 *For the approximation $k_{\mathbf{c}}^{(m)}(x,y)$ as in (62), the constant C_2 in Assumption 20 is* ***independent*** *of a in (49) (cf. (70)). Hence, a could be chosen so that*

$$\lambda_2 C_2^a =: C_8 < 1.$$

5 Complexity analysis

In this section, we will prove that, for quasi-uniform and shape regulars meshes, the storage amount and complexity of the variable order panel clustering method depend only linearly on the number of unknowns without any logarithmic terms. The key role in these proofs is played by sharp estimates on the number of blocks contained in the farfield levels $F(\ell)$ and in the nearfield N. Let n denote the number of panels, i.e., $n = \sharp\mathcal{G} = \dim S^{-1,0}$.

Lemma 60 *Let Assumption 48 be satisfied. There exist positive constants C_{13}, C_{14} so that, for all $0 \leq \ell \leq L$, the number of nearfield and farfield blocks is bounded by*

$$\begin{aligned} \sharp F(\ell) &\leq C_{13} 4^{\ell}, && (72) \\ \sharp N &\leq C_{14} n. && (73) \end{aligned}$$

Proof. First, we prove (72). By construction (cf. procedure **build_cluster_tree**) the cluster tree T is balanced implying $F(\ell) \subset T(\ell) \times T(\ell)$. In view of [13, Lemma 53, 54], it is sufficient to proof that there exists a constant C_{15} so that, for all $0 \leq \ell \leq L$,

$$\sharp T(\ell) \leq C_{15} 4^{\ell}.$$

All cluster centres M_τ of panels $\tau \in \mathcal{G}$ are contained in an h-neighbourhood $U_h(\Gamma)$ of Γ which was already introduced in Notation 47. The number of clusters contained in $T(\ell)$ is bounded from above by the number of cubes $q \in \mathcal{Q}_\ell$ satisfying

$$U_h(\Gamma) \cap q \neq \emptyset. \tag{74}$$

All cubes with this property are contained in $U_{h+d}(\Gamma)$ with $d = \sqrt{3} \times 2^{-\ell}$. Due to the quasi-uniformity of the grid, there exists $C < \infty$ so that $h + d \leq C 2^{-\ell}$. Hence, all cubes with property (74) are contained in $U_{C2^{-\ell}}(\Gamma)$. Due to Assumption 48 we have

$$|U_{C2^{-\ell}}| \leq C_\Gamma C 2^{-\ell}.$$

The volume of q is $2^{-3\ell}$ and, hence, the number of such cubes are bounded from above by

$$\frac{|U_{C2^{-\ell}}|}{2^{-3\ell}} \leq C_\Gamma C 4^{\ell}.$$

It remains to prove (73). The estimate follows directly from $\sharp\mathcal{G} = n$ and Lemma **??**. ■

The depth of the cluster tree is concerned in the next lemma.

Lemma 61 *Let Assumption 48, 49, and 50 be satisfied and the cluster tree constructed by the procedure **build_cluster_tree**. Then,*

$$4^L \leq \frac{12 C_u^2}{|\Gamma|} n.$$

Proof. Condition (34) implies $\sqrt{3}h_L \geq \underline{\rho}$. Taking into account Definition 34 and 35 along with

$$|\Gamma| = \sum_{\tau \in \mathcal{G}} |\tau| \leq nh^2$$

we obtain

$$4^L \leq 3\underline{\rho}^{-2} \leq \frac{12C_u^2}{h^2} \leq \frac{12C_u^2}{|\Gamma|} n$$

■

The following lemma estimates the amount of work per tree and farfield level. It has auxiliary character and will be used in the complexity estimates below.

Lemma 62 *Let $a, b, s \geq 0$. Then,*

$$\sum_{\ell=0}^{L} \left(a\left(L-\ell\right)+b\right)^s 4^\ell \leq 2\left(\frac{s\left(a+b\right)}{\ln 2}\right)^s 4^L.$$

Proof. Simple analysis yields:

$$\begin{aligned}\sum_{\ell=0}^{L} \left(a\left(L-\ell\right)+b\right)^s 4^\ell &\leq (a+b)^s \sum_{\ell=0}^{L} \left(L-\ell+1\right)^s 4^\ell = (a+b)^s 4^L \sum_{\ell=0}^{L} \left(\ell+1\right)^s 4^{-\ell} \\ &\leq (a+b)^s 4^L \left(\frac{s}{\ln 2}\right)^s \sum_{\ell=0}^{L} 2^\ell 4^{-\ell} = 2\left(\frac{s\left(a+b\right)}{\ln 2}\right)^s 4^L.\end{aligned}$$

■

In the sequel, we will estimate the number of operations in the single steps of the variable order panel clustering algorithm.

(a) Procedure **build_cluster_tree**.

Clearly, the complexity of the procedure is proportional to the number of elements in the cluster tree:

$$\sharp T = \sum_{\ell=0}^{L} 4^\ell \leq \frac{4}{3} 4^L \leq \frac{16C_u^2}{|\Gamma|} n.$$

(b) Computation of the expansion coefficients.

In [7], [15], [8], algorithms are presented where the computation of

$$\overrightarrow{\kappa}^m_{\nu,\mu}\left(\mathbf{c}\right), \qquad \forall \nu, \mu \in \mathbf{I}_m$$

can be performed in $O\left(m^7\right)$ operation. Hence, the computation for all coefficients and all farfield blocks costs (cf. (72) and Lemma 62)

$$\sum_{\ell=0}^{L} m^7\left(\ell\right) \times \sharp F\left(\ell\right) \leq C_{13} \sum_{\ell=0}^{L} \left(a\left(L-\ell\right)+b\right)^7 4^\ell \leq Cn \tag{75}$$

where C only depends on a, b, C_{13}, C_u, and $|\Gamma|$.

(c) Computation of the farfield coefficients.

The computation of all farfield coefficients $J^I_{\tau,\nu}$ and $J^{II}_{\tau,\mu}$ for the *panels* $\tau \in \mathcal{G}$ and $\nu \in \iota^I_{m(L)}$, $\mu \in \iota^{II}_{m(L)}$ is proportionally to n due $m(L) = b$.

The procedure **build_cluster_tree** implies that the number of sons of a cluster is bounded from above by 64. Thus, the evaluation of the recursion (43) costs $O(1)$ operation per farfield coefficients. Similar computations as in (75) yields that the number of operations is proportional to n.

(d) Computation of the recursion coefficients $\gamma_{\nu,\tilde{\nu},\tilde{c}}$ in (42). From (40), it follows directly, that the amount of computational work per coefficients is $O(1)$ while the total number of coefficients is bounded by $O(n)$.

(e) Evaluation of a matrix vector multiplication.

By similar considerations, one obtains that the evaluation of a matrix vector multiplication, i.e., Algorithm 33, costs $O(n)$ operations.

(f) Storage amount.

By using the same technique as for the computational complexity one can prove that the amount of memory for storing the quantities $\mathcal{G}$, T, F, $\kappa^m_{\nu,\mu}(\mathbf{c})$, $J^I_{c,\nu}$, $J^{II}_{c,\mu}$, and $\gamma_{\nu,\tilde{\nu},\tilde{c}}$ (as in (40)) is proportionally to n.

Theorem 63 *Let Assumption 48, 49, and 50 be satisfied and the cluster tree constructed by the procedure* ***build_cluster_tree****. The variable order panel clustering algorithm has linear complexity with respect to the computing time and the memory consumption.*

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