Quadrature for hp-Galerkin BEM in $\mathbb{R}^3$

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Dedicated to Prof. Dr. I. Babuška on the occasion of his 70th birthday

Abstract

The Galerkin discretization of a Fredholm integral equation of the second kind on a closed, piecewise analytic surface $\Gamma \subset \mathbb{R}^3$ is analyzed. High order, hp-boundary elements on grids which are geometrically graded toward the edges and vertices of the surface give exponential convergence, similar to what is known in the hp Finite Element Method. A quadrature strategy is developed which gives rise to a fully discrete scheme preserving the exponential convergence of the hp-Boundary Element Method. The total work necessary for the consistent quadratures is shown to grow algebraically with the number of degrees of freedom. Numerical results on a curved polyhedron show exponential convergence with respect to the number of degrees of freedom as well as with respect to the CPU-time.

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1 Introduction

Elliptic boundary value problems in domains $\Omega \subset \mathbb{R}^3$ with piecewise smooth boundary $\Gamma = \partial\Omega$ can be reformulated as boundary integral equations if a fundamental solution is available. The approximate solution of such boundary integral equations by means of finite element spaces V^L on Γ gives rise to the so-called *boundary element method (BEM)*. Under the assumption of strong ellipticity of the boundary integral operators (valid in many cases), Galerkin-BEM are known to exhibit quasi-optimal asymptotic convergence [29], i.e., the rate of convergence is governed by the best approximation error of the boundary density from V^L.

If Γ and the boundary data are piecewise analytic (as in many cases of engineering interest), so are the unknown densities approximated by BEM [7]. It has been shown in [7], [9], [16] that proper design of V^L ensures *exponential convergence* of the best approximation in terms of $N = \dim V^L$ for piecewise analytic solutions. "Proper design" means here anisotropic meshes that are geometrically graded towards the edges of Γ have to be used possibly in conjunction with variable polynomial degree which increases linearly off the edges of Γ. Such subspaces V^L pose special challenges to the numerical evaluation of the BEM stiffness matrix. Quadrature of singular or near singular integrals over domains of arbitrary high aspect ratio has to be performed with an error that is exponentially decreasing in N in order to preserve the convergence rate of the scheme.

To develop and analyze such a quadrature strategy is the purpose of the present paper. We show how a numerically integrated stiffness matrix which preserves the $O\left(\exp\left(-bN^{0.25}\right)\right)$

convergence rate of the hp-Galerkin BEM can be computed in $O(N^{3.25})$ kernel evaluations. We prove the result for second kind integral equations but hasten to add that our quadrature error estimates are actually also applicable to weakly singular as well as hypersingular kernels (after proper regularization) on piecewise analytic surfaces. Likewise, our quadrature schemes also show how fully discrete h-type Galerkin BEM with anisotropic graded meshes can be realized computationally in optimal (up to logarithmic terms) complexity.

Our quadrature scheme will use tensor product Gaussian quadratures in the reference square and geometric subdivisions of the integration domains if necessary [24] (see also [2]). In addition, for the singular and also certain near singular integrals over edge-parallel, high aspect-ratio elements arising in the hp-BEM, this will be combined with certain regularizing coordinate transformations from [11, 20, 21]. The quadrature error analysis is nevertheless novel in several respects. In [11, 21], kernel expansions in local coordinates were used to reduce curvilinear panels to flat panels. Then, *regularizing coordinate transforms* were introduced in combination with semi-analytic techniques. For the h-version BEM on non-degenerate meshes, a satisfactory error analysis was presented. However, this expansion technique becomes inefficient and numerically unstable for elements with high aspect ratio and for high order approximation, especially from the viewpoint of practical implementations. The use of kernel expansions can be avoided by a fully implicit treatment of the kernel presented in [20] where also the problem of near singular integration over elements differing in size by orders of magnitude was solved. There, the order of the elements was fixed and the quadrature error estimates were h-asymptotic. Here, we prove exponential convergence for all quadratures, uniform in p, the degree of the shape functions and moreover, in the *aspect ratio* of the edge elements. In hp-BEM (and also in h-versions with mesh grading towards the edges) this aspect ratio must become arbitrarily large to ensure efficiency of approximation. Our quadrature strategy ensures exponential convergence *uniform* in the aspect ratio of the elements.

This allows to compute a numerically integrated stiffness matrix $\tilde{A}^L$ satisfying (42) with work W_L of order $O(N_L^a)$ for some $a > 0$, i.e., in *algebraic* complexity. Therefore the fully discrete scheme will exhibit exponential convergence also in terms of the work measure.

The outline of the paper is as follows. In Section 2, we formulate the boundary integral equation and the assumption on Γ. We define the hp-Galerkin scheme, present an exponential convergence result and a general framework for the analysis of consistency errors due to quadrature. Section 3 contains the quadrature error analysis and the main results. In Section 4 we report the results of numerical experiments which are in full agreement with our error and complexity estimates.

2 hp-Boundary Element Method

2.1 Problem formulation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a piecewise analytic, orientable Lipschitz boundary manifold $\Gamma = \partial\Omega$. We assume that there is a polyhedron $\tilde{\Omega}$ with surface $\tilde{\Gamma}$ consisting of open (plane) quadrangles and triangles $\tilde{K}_j$, $1 \leq j \leq M$ which have the property that

$$\tilde{\Gamma} = \overline{\bigcup_{1\leq j\leq M} \tilde{K}_j}, \tag{1}$$

$$\overline{\tilde{K}} \cap \overline{\tilde{K}'} \text{ is either empty, a vertex, an edge, or } \overline{\tilde{K}}.$$

These surface pieces form the *covering* $\tilde{\tau}_0 := \left\{\tilde{K}_j : 1 \leq j \leq M\right\}$. Furthermore, we assume that there exists a global bi-Lipschitz continuous mapping $\eta : \tilde{\Gamma} \to \Gamma$, i.e.,

$$C_1 \|\tilde{x} - \tilde{y}\| \leq \|\eta(\tilde{x}) - \eta(\tilde{y})\| \leq C_2 \|\tilde{x} - \tilde{y}\|, \qquad \forall \tilde{x}, \tilde{y} \in \tilde{\Gamma}$$

and, for all $\tilde{K} \in \tilde{\tau}_0$, there are plane extensions $\tilde{K}^{ext}$ of $\tilde{K}$, i.e., $\tilde{K} \subset\subset \tilde{K}^{ext}$ where $\eta\mid_{\tilde{K}}$ can be extended analytically.

$$(\eta\mid_{\tilde{K}})^{ext} \text{ is analytic on } \tilde{K}^{ext} \text{ for all } \tilde{K} \in \tilde{\tau}_0.$$

The mapping η defines a *covering* of Γ by $\tau_0 := \left\{\eta\left(\tilde{K}\right) : \tilde{K} \in \tilde{\tau}_0\right\}$.

Let K_0 be the reference element, either the unit triangle $T_0 := \{(\xi_1, \xi_2) : 0 < \xi_1 < 1, 0 < \xi_2 < 1 - \xi_1\}$ or the unit square $Q_0 := \{(\xi_1, \xi_2) : -1 < \xi_1 < 1, -1 < \xi_2 < 1\}$. Since K_0 can be transported onto $\tilde{K}$ by an affine (bi-) linear mapping $\tilde{\kappa}_{\tilde{K}} : K_0 \to \tilde{K}$, the composite mapping

$$\kappa_K := \eta \circ \kappa_{\tilde{K}} : K_0 \to K \tag{2}$$

can be extended analytically to a larger domain K_0^{ext} satisfying $K_0 \subset\subset K_0^{ext} \subset \mathbb{R}^2$. The situation is illustrated in Figure 1.

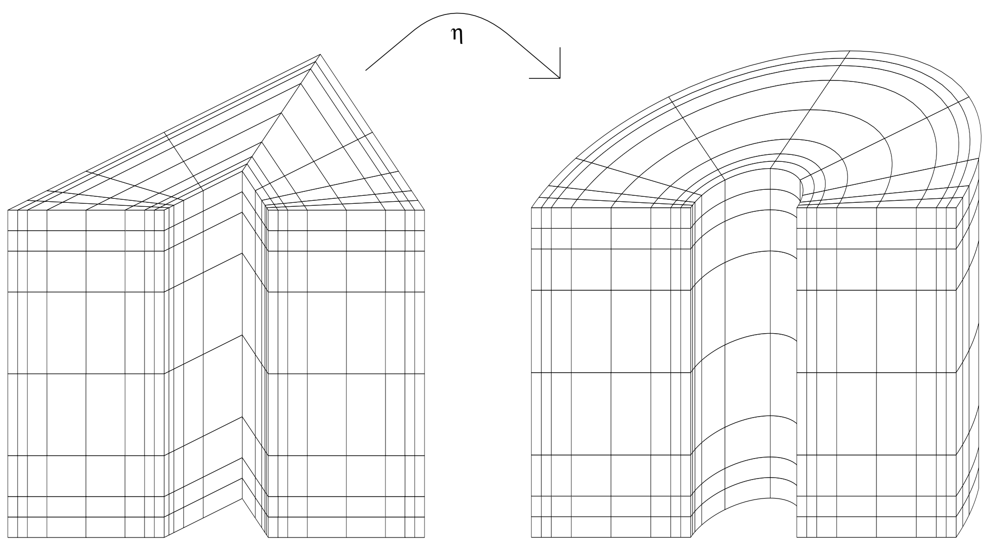

Figure 1: Surface of a halved tube and corresponding interpolating polyhedron. The inner radius is 1 while the outer radius and the height is 4. The mesh shown corresponds to the third refinement level.

By $d\Gamma$ we denote the surface measure defined almost everywhere on Γ. We consider the space $L^2(\Gamma)$ of functions $u : \Gamma \to \mathbb{C}^N$ which are square integrable with respect to $d\Gamma$. An inner product on $L^2(\Gamma)$ is given by

$$\langle u, v \rangle = \int_\Gamma u\bar{v}d\Gamma. \tag{3}$$

Another inner product $(\cdot, \cdot)$, equivalent to $\langle \cdot, \cdot \rangle$ (i.e., giving rise to equivalent norms) in $L^2(\Gamma)$, can then be defined by

$$(u, v) = \sum_{\tilde{K} \in \tilde{\tau}_0} \int_{K_0} (u \circ \kappa_K(\xi)) \overline{(v \circ \kappa_K(\xi))}\, d\xi. \tag{4}$$

Given a continuous operator $A : L^2(\Gamma) \to L^2(\Gamma)$, we are interested in the numerical solution of the equation

$$u \in L^2(\Gamma) \qquad \langle Au, v \rangle = \langle f, v \rangle \qquad \forall v \in L^2(\Gamma). \tag{5}$$

The operator A is a boundary integral operator which can be represented in the form

$$(Au)(x) = c(x)u(x) + p.v. \int_\Gamma k(x,y)u(y)d\Gamma_y \tag{6}$$

where $k(x,y) = \breve{k}(x,y;x-y)$ and the kernel $\breve{k}$ has the form

$$\breve{k}(x,y,z) = \sum_{|\alpha| \geq t} s_\alpha(x,y) z^\alpha \|z\|^{-2-t}, \qquad x,y \in \mathbb{R}^3, 0 \neq z \in \mathbb{R}^3. \tag{7}$$

We assume that for $K, K' \in \tau_0$, the coefficient functions $s_\alpha : \bar{K} \times \bar{K}' \to \mathbb{C}$ and $c : \bar{K} \to \mathbb{C}$ are analytic functions and that the series (7) is finite. In (7) α denotes a three-dimensional multi-index, $\alpha \in \mathbb{N}_0^3$, and t an odd integer (cf. [11, Assumption 1.1] and [21, p. 42]).

Proposition 1 *The assumption that t is an odd integer implies that*

$$k(x,y) + k(y,x) = \sum_{|\alpha| \geq t+1} \tilde{s}_\alpha(x,y)(y-x)^\alpha \|y-x\|^{-2-t}, \qquad x,y \in \mathbb{R}^3, y \neq x.$$

Proof. Expansion (7) implies

$$k(x,y) + k(y,x) = \sum_{|\alpha| \geq t} \left(s_\alpha(x,y) + (-1)^{|\alpha|} s_\alpha(y,x) \right) (y-x)^\alpha \|y-x\|^{-2-t}.$$

For $|\alpha| = t$ we obtain by using the analyticity of s_α

$$\begin{aligned} s_\alpha(x,y) - s_\alpha(y,x) &= s_\alpha(x,x) - s_\alpha(y,x) + (s_\alpha(x,y) - s_\alpha(x,x)) \\ &= \sum_{|\beta| \geq 1} \gamma_{\alpha,\beta} (y-x)^\beta \end{aligned}$$

and hence

$$k(x,y) + k(y,x) = \sum_{|\alpha| \geq t+1} \tilde{s}_\alpha(x,y)(y-x)^\alpha \|y-x\|^{-2-t}.$$

■

A typical example for (6), (7) is the classical double layer potential operator where the sum (7) is finite and

$$c(x) = 1, \quad t = 1, \quad s_\alpha(x,y) = \begin{cases} -\frac{1}{2\pi} n_\alpha(y) & \text{if } |\alpha| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Here $n_\alpha(y)$ denotes the exterior unit normal vector to Ω at $y \in \Gamma$.

The integral in (6) is in general to be understood in the Cauchy principal value sense, i.e.,

$$p.v. \int_\Gamma k(x,y)u(y)d\Gamma_y = \lim_{\varepsilon \to 0} \int_{\Gamma \setminus B_\varepsilon(x)} k(x,y)u(y)d\Gamma_y. \tag{9}$$

Here $B_\varepsilon(x) = \{y \in \mathbb{R}^3 : |x-y| < \varepsilon\}$ denotes the open ball of radius ε about the point x and the limit is assumed to exist for $x \in K \in \tau_0$.

We assume that, for the given data $f \in L^2(\Gamma)$, the problem $Au = f$ admits a unique solution $u \in L^2(\Gamma)$.

Approximate solutions to (5) are obtained by the Galerkin method: Given a dense sequence $\{V^L\}_{L=0}^{\infty}$ of finite dimensional subspaces of $L^2(\Gamma)$, we solve

$$u^L \in V^L \quad \left\langle Au^L, v \right\rangle = \langle f, v \rangle \qquad \forall v \in V^L. \tag{10}$$

We denote by P_L the orthogonal projection

$$P_L : L^2(\Gamma) \to V^L, \qquad ((v - P_L v), \phi) = 0 \quad \forall \phi \in V^L.$$

Proposition 2 *Assume that, for sufficiently large L, the approximate problem (10) is stable in the sense that*

$$\left\| P_L A u^L \right\|_0 \geq C_s \left\| u^L \right\|_0 \qquad \forall u^L \in V^L. \tag{11}$$

Then there exist unique solutions u^L of (10) which converge quasioptimally to the unique solution u of (5), i.e.,

$$\left\| u - u^L \right\|_0 \leq C \inf_{v \in V^L} \| u - v \|_0 . \tag{12}$$

The relation (12) states that the Galerkin approximations converge quasioptimally to the exact solution. The actual rate of convergence is therefore determined by the regularity of the exact solution and the selection of the spaces V^L. In order to achieve exponential convergence, we need to control derivatives of all orders of the exact solution u simultaneously. This is conveniently expressed by regularity statements in countably normed spaces $B_\varrho(\Gamma)$ of piecewise analytic functions which we now present.

2.2 Regularity

Let $\mathbf{V}(K_0)$ denote the set of vertices of the reference element K_0. For $X \in \mathbf{V}(K_0)$, let $B_R(X)$ denote the ball with radius R centered at X where R is chosen such that

$$U_X := B_R(X) \cap K_0$$

satisfies

$$K_0 = \bigcup_{X \in \mathbf{V}(K_0)} U_X, \text{ and } \qquad \mathbf{V}(K_0) \backslash X \cap U_X = \emptyset.$$

For a vertex $X \in \mathbf{V}(K_0)$, let us introduce polar coordinates (r_X, ϑ_X) on U_X centered at X such that $\vartheta_X \in (0, \alpha_X)$. For a parameter $\varrho \geq 0$, we can then define

$$\begin{aligned} B_\varrho(K_0) &= \{ v \in L^2(K_0) : \\ &\quad \left\| r_X^{k-\varrho} (\partial/\partial r_X)^k (\vartheta_X(\alpha_X - \vartheta_X))^{l-\varrho} (\partial/\partial \vartheta_X)^l v \right\|_{L^2(U_X)} \leq C d^{k+l+1} k! l!, \\ &\quad k, l \in \mathbb{N}, \quad X \in \mathbf{V}(K_0), \text{ and } C, d \text{ independent of } k, l \}, \\ B_\varrho(\Gamma) &= \{ v \in L^2(\Gamma) : v \circ \kappa_K \in B_\varrho(K_0), K \in \tau_0 \}. \end{aligned} \tag{13}$$

We assume the following regularity property of the operator A:

$$\text{if } u \in L^2(\Gamma) \text{ and } Au \in B_\varrho(\Gamma) \text{ for some } \varrho < 1/2, \text{ then } u \in B_\varrho(\Gamma). \tag{14}$$

This holds for example for the classical double layer potential operator on all convex (and also certain nonconvex) polyhedra, see [7, Theorem 3.1].

Remark 3 *If the right hand side f in (5) is analytic in $\overline{K}$, for all $K \in \tau_0$, then $f \in B_\varrho$ and, due to (14), $u \in B_\varrho(\Gamma)$.*

2.3 hp-Boundary Elements

We construct hp-subspaces $V^L \subset L^2(\Gamma)$ of dimension N_L such that for any $u \in B_\varrho(\Gamma)$

$$\inf_{v \in V^L} \|u - v\|_{L^2(\Gamma)} \leq C \exp(-b\sqrt[4]{N_L}) \tag{15}$$

with $C > 0$ and $b > 0$ independent of L. For the model double layer potential problem on a convex polyhedron in $\mathbb{R}^3$, these spaces also satisfy (11) and, due to (12), the Galerkin boundary element method converges at the exponential rate (15).

We begin with explaining how hp-subspaces V^L can be generated on the surface Γ in $\mathbb{R}^3$ described above. We will use the notation introduced in the previous section. Let $\mathfrak{E}$ denote the set of (possibly curved) edges of Γ while $\tilde{\mathfrak{E}} = \eta^{-1}(\mathfrak{E})$ is the pull-back on $\tilde{\Gamma}$.

2.3.1 Polyhedral surfaces and geometric quadrangulations

First, we will consider the case that Γ is the surface of a polyhedron. Then, w.l.o.g., the function η (cf. Section 2.1) may be taken as the identity. Let the initial mesh τ_0 be the covering defined in the previous section. It will turn out that, for the quadrature methods, the following assumption is very convenient and will be made throughout the paper.

Assumption 4 *We assume that for any $K \in \tau_0$ the following holds*

if K has an edge $e \subset \mathfrak{E}$ then the opposite edge is parallel to e.

The hierarchy of geometric meshes depends on the *grading parameter* $\sigma \in]0, 1/2]$ and is constructed recursively by the following procedure:

First we assume that τ_0 only contains quadrangles. In the following algorithm, the notation $i + 1$ stands for $(i + 1) \bmod 4$ and $i_2 := i \bmod 2$. Let us assume that τ_{L-1} was generated for $L \geq 1$.

Geometric Refinement:

for all $K \in \tau_{L-1}$ **do begin**
  let $\{X_i\}_{0 \leq i \leq 3}$ denote the set of vertices of K (counterclockwise ordering);
  let $e_i := \overline{X_i X_{i+1}}$ denote the set of edges;
  for $i = 0$ **to** 1 **do begin**
    if $e_i \cup e_{i+2} \subset \mathfrak{E}$ **then** connect the midpoints of e_{i+1} and e_{i+3};
  end;
  for $i = 0$ **to** 3 **do begin**
    if $e_i \subset \mathfrak{E}$ **and** $e_{i+2} \cap \mathfrak{E} = \emptyset$ **then** connect $X_{i+1} + \sigma(X_{i+2} - X_{i+1})$
      with $X_i + \sigma(X_{i+3} - X_i)$;
  end;
end;

The resulting geometric mesh is denoted by $\tau_L = \{K_1, K_2, \ldots, K_{M_L}\}$. Note that, for $L > 0$, each element of τ_L has a uniquely determined parent $\mathcal{P}(K) \in \tau_{L-1}$ characterized by $K \subset \mathcal{P}(K)$.

2.3.2 Polynomial degree distribution

The subspace V^L consists of piecewise polynomials (in local coordinates) of degrees $p^K = \left(p_1^K, p_2^K\right)$ on $K \in \tau_K$ which we combine in the *linear degree vector* $\delta p = \left\{p^K : K \in \tau_L\right\}$ with initial degree L_0 and slope $\mu > 0$. It is constructed recursively by the following algorithm. The notation $\lfloor m \rfloor$ denotes the largest integer smaller than or equal to m.

Polynomial Refinement:

if $L = 0$ **then** define $p^K = (L_0, L_0)$ for all $K \in \tau_0$
else begin
for all $K \in \tau_L$ **do begin**
  let $\{e_i\}_{0\le i\le 3}$ denote the edges of K and $\{E_i\}_{0\le i\le 3}$ the edges of $\mathcal{P}(K)$;
  for $i := 0$ **to** 3 **do begin**
    if there is E_j such that $e_i = E_j$ **then**

$$p_{i_2}^K := \min\left(L, \left\lfloor p_{j_2}^{\mathcal{P}(K)} + \mu\right\rfloor\right);$$

    else if there is E_j such that $e_i \subsetneq E_j$ **then** $p_{i_2}^K := L_0$
  end;
end;**end;**

Remark 5 *Note that the geometric refinement algorithm preserves Assumption 4 for the finer grids $\tilde{\tau}_L$, $L \geq 0$.*

2.3.3 *hp*-Finite Element Space

The *hp*-finite element spaces corresponding to the geometric meshes τ_L and the degree vector δp are defined by lifting tensor products of Legendre polynomials onto the surface elements $K \in \tau_L$. Let

$$\varphi_\alpha^0(\xi) := (\alpha + 1/2)^{1/2} L_\alpha(\xi), \qquad \alpha = 0, 1, 2, \ldots \tag{16}$$

with L_α denoting the Legendre polynomials of order α on the interval $(-1, 1)$ scaled such that $L_\alpha(1) = 1$ holds. This implies in particular that the φ_α^0 are orthonormal in $L^2(-1, 1)$, i.e.

$$\int_{-1}^{1} \varphi_\alpha \varphi_\beta d\xi = \delta_{\alpha\beta} \tag{17}$$

where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and 0 otherwise.

For $K \in \tau_L$, let χ_K denote an analytic chart mapping the unit cube $(-1, 1)^2$ onto K having the property that

$$\chi_K^{-1}(e_0) = \overline{\binom{-1}{-1}, \binom{1}{-1}}, \quad \chi_K^{-1}(e_1) = \overline{\binom{1}{-1}, \binom{1}{1}}, \tag{18}$$

where e_i denotes the *ith* edge of K. In the considered case where K is a quadrangle, χ_K is bi-linear. The basis functions on K are given by

$$\varphi_\alpha^0(\xi) := \varphi_{\alpha_1}^0(\xi_1)\, \varphi_{\alpha_2}^0(\xi_2) \tag{19}$$

$$\varphi_\alpha^K(x) = \begin{cases} \left(\varphi_\alpha^0 \circ \chi_K^{-1}\right)(x) / \sqrt{|K|} & \text{for } x \in K, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } 0 \leq \alpha_i \leq p_i^K, \quad i = 1, 2.$$

Here, and in the following $|K| := \|X_1 - X_2\| \cdot \|X_2 - X_3\|$. The resulting local *hp*-finite element space is given by

$$V^L(K) := \operatorname{span}\left\{\varphi_\alpha^K \mid 0 \le \alpha_i \le p_i^K, \quad 1 \le i \le 2.\right\}.$$

The global *hp*-finite element space is composed of the local ones: $V^L := V^L_{\sigma,\delta p} \subset L^2(\Gamma)$ is the set of all functions $u : \Gamma \to \mathbb{C}$ of the form

$$u(x) = \sum_{K \in \tau_L'} \sum_{\alpha=0}^{p^K} u_\alpha^K \varphi_\alpha^K(x) \tag{20}$$

with some coefficients $u_\alpha^K \in \mathbb{C}$ and the *reduced* mesh

$$\tau_L' := \left\{K \in \tau_L : \bar{K} \cap \mathfrak{E} = \emptyset\right\}.$$

Here and in the following the notation $\sum_{\alpha=0}^{p^K}$ stands for $\sum_{\alpha_1=0}^{p_1^K} \sum_{\alpha_2=0}^{p_2^K}$. In order to simplify the notation we introduce the set

$$\mathcal{I}_L := \left\{(K,\alpha) \mid K \in \tau_L', 0 \le \alpha_i \le p_i^K \text{ for } 1 \le i \le 2\right\}. \tag{21}$$

Notice that $u \in V^L$ vanishes in a neighborhood of the edges and vertices of Γ. This is necessary to prove stability (11) of the Galerkin scheme (10) by the finite section method [7]. Since this neighborhood of the edges is exponentially small in L, however, this does not lead to a deterioration in approximation.

The dimension N_L of V^L depends on the choice of δp. For the considered case of *linear* degree vectors, we obtain asymptotically

$$N_L = O((L+1)^4). \tag{22}$$

2.3.4 Curved surfaces and general meshes

In the following, we will explain how the construction given above for a polyhedron and quadrangles can be modified in order to treat more general situations.

Curved boundaries: Let Γ be the (possibly curved) surface of a 3-d domain Ω and $\tilde{\Gamma}, \eta$, and $\tilde{\tau}_0$ as explained in Subsection 2.1. We generate *hp*-meshes using $\tilde{\tau}_0$, $\tilde{\Gamma}$, and $\tilde{\mathfrak{E}}$ as explained above. The surface meshes are then given by

$$\tau_L := \left\{\eta\left(\tilde{K}\right) : \tilde{K} \in \tilde{\tau}_L\right\}.$$

The definition of the *hp*-finite element spaces is the same as in the polyhedral case. The charts (18) have to be replaced by $\eta \circ \chi_{\tilde{K}}$. For convenience, we replace the quantity $|K|$ by $|K| := \left|\tilde{K}\right|$.

Triangular elements: Triangular elements can be used in combination with quadrangles. We only have to guarantee that the initial mesh τ_0 has the property that

$$K \in \tau_0 \text{ is a triangle} \Leftrightarrow \ \bar{K} \cap \mathfrak{E} = \emptyset.$$

This condition implies that triangular elements will never be subdivided by geometric refinement. As a basis we use the Dubiner basis functions (see [5]) on the reference triangle and lift them onto the surface. Here, we do not go into the details but consider only quadrilateral meshes. The restriction to quadrangulations is not a severe restriction as can be seen in the following

Remark 6 *Any triangle can be split into three quadrilaterals by connecting the midpoints of the edges with the barycenter.*

2.4 Stability and convergence of hp-boundary element methods

The following theorem concerns the approximation properties of the subspace V^L for functions $u \in B_\varrho(\Gamma)$. We recall the definition of the slope μ and the parameter L_0 characterizing the polynomial grading function δp. For the following analysis we assume always that the mesh τ_L only consists of quadrilaterals.

Theorem 7 *Let $B_\varrho(\Gamma)$ be defined by (13) for some $0 \leq \varrho < 1/2$. For every $\sigma \in (0,1)$ there exists $\mu > 0, L_0 \geq 0$ (depending on σ, ϱ and d in (13)) such that, for any $u \in B_\varrho(\Gamma)$, there exists $v \in V^L_{\sigma,\delta p}$ such that*

$$\|u - v\|_{L^2(\Gamma)} \leq C(\sigma, d)(L+1)\sigma^{\varrho L} \tag{23}$$

is satisfied where C is a constant independent of L, but dependent on Γ, d and σ.

Remark 8 *The estimate (23) can also be expressed in terms of degrees of freedom:*

$$L\sigma^{\varrho L} \leq L e^{C\varrho \log\sigma \sqrt[4]{N_L}} \leq e^{C'\varrho \log\sigma \sqrt[4]{N_L}} = e^{-b\sqrt[4]{N_L}}$$

with $b := C'\varrho\,|\log\sigma|$. This is (15).

Results of this kind have first been proved by Babuska and Guo, [9]. The proof of Theorem 7 consists in a modification of their argument (somewhat simpler since we use discontinuous functions), see also [7] and [16]. In [7], Theorem 7 was proved for the case $L_0 = L$ and $\mu = 0$, i.e. uniform polynomial degree.

Let $u^L \in V^L$ denote the Galerkin solution defined in (10). The stability condition (11) ensures that (10) admits a unique solution u^L for sufficiently large L. Moreover, if $u \in B_\varrho(\Gamma)$ for some $0 < \varrho < 1/2$, Theorem 7 implies the error estimate

$$\left\|u - u^L\right\|_0 \leq C(L+1)\sigma^{\varrho L}. \tag{24}$$

The stability (11) of the Galerkin scheme based on V^L holds, for example, for the classical double layer potential operator on convex as well as certain nonconvex polyhedra and for a polynomial grading function δp characterized by $L_0 = L$ and $\mu = 0$, see [7]. The arguments there can be generalized to cover the case $L_0 = 1$, $\mu > 0$ sufficiently large as well [8].

For $u^L \in V^L$, let $\vec{u} = \{u_I\}_{I\in\mathcal{I}_L}$ denote the coefficients of the basis representation (20). The Galerkin equations (10) are then equivalent to finding $\vec{u}^L$ such that

$$A^L\vec{u} = \vec{f} \tag{25}$$

with the load vector $\vec{f} = \{\langle f, \varphi_I\rangle\}_{I\in\mathcal{I}_L}$ and the stiffness matrix $A^L = \{A^L_{II'}\}_{I,I'\in\mathcal{I}_L}$ given by

$$A^L_{II'} = \langle \varphi_{I'}, A\varphi_I\rangle\,, \quad I, I' \in \mathcal{I}_L.$$

Due to the way the φ_I are normalized, we have the following equivalence between the $L^2(\Gamma)$ norm and the discrete ℓ_2-norm of the coefficient vectors $\vec{u}$ of functions $u \in V^L$.

Lemma 9 *There exist constants $0 < C_1 \leq C_2 < \infty$ independent of L such that for every $u \in V^L$ there holds*

$$C_1 \|u\|^2_{L^2(\Gamma)} \leq \sum_{I \in \mathcal{I}_L} \left| u^L_I \right|^2 \leq C_2 \|u\|^2_{L^2(\Gamma)} . \tag{26}$$

Proof. Throughout the proof, $\sim$ denotes equivalence with constants independent of L.

Let $u \in V^L$. Then $u = \sum_{I \in \mathcal{I}_L} u_I \varphi_I$ and

$$\begin{aligned} \left\| \sum_{I \in \mathcal{I}_L} u_I \varphi_I \right\|^2_{L^2(\Gamma)} &= \left\| \sum_{K \in \tau'_L} \sum_{\alpha=0}^{p^K} u^K_\alpha \varphi^K_\alpha \right\|^2_{L^2(\Gamma)} = \sum_{K \in \tau'_L} \left\| \sum_{\alpha=0}^{p^K} u^K_\alpha \varphi^K_\alpha \right\|^2_{L^2(K)} \\ &\sim \sum_{K \in \tau'_L} \sum_{\alpha=0}^{p^K} \left| u^K_\alpha \right|^2 \left\| \varphi^0_\alpha \right\|^2_{L^2(K_0)} \end{aligned}$$

due to (19). By the normalization (17) of φ^0_α, the assertion follows since

$$\|u\|^2_{L^2(\Gamma)} \sim \sum_{I \in \mathcal{I}_L} |u_I|^2 .$$

■

The norm equivalence (26) and the stability (11) have the following consequence which is of interest for the iterative solution of the linear system (25).

Lemma 10 *There exists a constant C independent of L such that* $\mathrm{cond}_2(A^L) \leq C < \infty$.

2.5 Consistency analysis

In general, one has to use numerical quadrature to calculate approximate entries $\tilde{A}^L_{II'}$ of the stiffness matrix A^L, resulting in a perturbed matrix $\tilde{A}^L$. For the h-version of the Galerkin-BEM, this effect was thoroughly discussed in [22]. For the hp-BEM, the situation is different due to the following two points. The norms in which the consistency has to be measured are different and the required consistency changes from algebraic to exponential accuracy. The stiffness matrices A^L and $\tilde{A}^L$ define finite dimensional operators $\mathcal{A}^L$, $\tilde{\mathcal{A}}^L : V^L \to \left(V^L\right)'$ where $\left(V^L\right)'$ denotes the dual space of V^L (with respect to $L^2(\Gamma)$). We estimate the difference between $\mathcal{A}^L$ and $\tilde{\mathcal{A}}^L$.

Lemma 11 *Assume that the entries $\tilde{A}^L_{II'}$ of $\tilde{A}^L$ satisfy*

$$\left| E^L_{II'} \right| = \left| A^L_{II'} - \tilde{A}^L_{II'} \right| \leq \Phi(L). \tag{27}$$

Then there holds for every $u, \tilde{u} \in L^2(\Gamma)$

$$\left| \left\langle \left(\mathcal{A}^L - \tilde{\mathcal{A}}^L \right) P_L u, P_L \tilde{u} \right\rangle \right| \leq C N_L \Phi(L) \|u\|_0 \|\tilde{u}\|_0 . \tag{28}$$

Proof. Using Lemma 9, we have

$$\left\langle \left(\mathcal{A}^L - \tilde{\mathcal{A}}^L \right) P_L u, P_L \tilde{u} \right\rangle \leq \ C \|u\|_0 \|\tilde{u}\|_0 \left\| E^L \right\|_2 \quad u, \tilde{u} \in L^2(\Gamma) \tag{29}$$

with C independent of L and the matrix E^L given by $E^L_{II'} := A^L_{II'} - \tilde{A}^L_{II'}$. To estimate $\left\|E^L\right\|_2$, we use the Schur-Lemma (see, e.g., [17] page 269) with $\gamma_I = 1$. We estimate for every $I \in \mathcal{I}_L$ with (27)

$$\sum_{I' \in \mathcal{I}_L} \left|E^L_{II'}\right| \leq N_L \Phi(L)$$

and for every fixed $I' \in \mathcal{I}_L$ in the same way

$$\sum_{I \in \mathcal{I}_L} \left|E^L_{II'}\right| \leq N_L \Phi(L).$$

From the Schur-Lemma it follows then that $\left\|E^L\right\|_2 \leq N_L \Phi(L)$ and (29) imply the assertion. ■

Lemma 11 allows to estimate the impact of the consistency error (27) on the asymptotic convergence rate of the solution $\tilde{u}^L$ defined by

$$\tilde{\mathcal{A}}^L \tilde{u}^L = P_L f. \tag{30}$$

Theorem 12 *Assume that the Galerkin scheme (10) is stable, i.e. (11) holds, and that the approximate stiffness matrix $\tilde{A}^L$ used in the computation satisfies (27) with*

$$N_L \Phi(L) \to 0 \text{ as } L \to \infty. \tag{31}$$

Then (30) is stable, i.e. there exists $c > 0$ such that

$$\left\|\tilde{\mathcal{A}}^L v^L\right\|_0 \geq c \left\|v^L\right\|_0 \qquad \forall v^L \in V^L \tag{32}$$

for sufficiently large L.

Assume in addition that $u \in B_\varrho(\Gamma)$ for some $0 \leq \varrho < 1/2$. Then

$$\left\|u - \tilde{u}^L\right\|_0 \leq C_u L \sigma^{\varrho L}, \quad L \geq 1 \tag{33}$$

with $C > 0$ and $b > 0$ independent of L, provided (27) holds with

$$\Phi(L) = N_L^{-1} L \sigma^{\varrho L}, \quad L \geq 1. \tag{34}$$

Assume finally that, for every $g \in B_\varrho(\Gamma)$, $0 \leq \varrho < 1/2$, the solution φ of the adjoint equation

$$A^* \varphi = g \tag{35}$$

exists in $L^2(\Gamma)$ and belongs to $B_\varrho(\Gamma)$ as well. If the quadrature errors satisfy (27) with

$$\Phi(L) = N_L^{-1} L \sigma^{2\varrho L}, \quad L \geq 1 \tag{36}$$

then

$$\left|\langle g, u\rangle - \left\langle g, \tilde{u}^L\right\rangle\right| \leq C_u (L+1)^2 \sigma^{2\varrho L} \tag{37}$$

with a constant C_u depending only on σ and d.

Proof.

1. For sufficiently large L, the *discrete inf-sup condition* (11) can be written as

$$\left\| v^L \right\|_0 \leq c_s \left\| A v^L \right\|_{(V^L)'} \quad \forall v^L \in V^L. \tag{38}$$

   Using (28) with $v^L \in V^L$ we obtain

$$\left\| \tilde{\mathcal{A}}^L v^L \right\|_{(V^L)'} \geq \left\| A v^L \right\|_{(V^L)'} - \left\| (\tilde{\mathcal{A}}^L - A) v^L \right\|_{(V^L)'} \geq c_s^{-1} \left\| v^L \right\|_0 - C N_L \Phi(L) \left\| v^L \right\|_0 .$$

   Then (31) gives for sufficiently large L

$$\left\| v^L \right\|_0 \leq C \left\| \tilde{\mathcal{A}}^L v^L \right\|_{(V^L)'} \quad \forall v^L \in V^L. \tag{39}$$

2. We have

$$\left\| u - \tilde{u}^L \right\|_0 \leq \left\| u - P_L u \right\|_0 + \left\| P_L u - \tilde{u}^L \right\|_0 .$$

   Using (39) and $\left\langle \tilde{\mathcal{A}}^L \tilde{u}^L, v^L \right\rangle = \left\langle A u, v^L \right\rangle$ for $v^L \in V^L$ we obtain

$$\left\| P_L u - \tilde{u}^L \right\|_0 \leq C \left\| \tilde{\mathcal{A}}^L \left(P_L u - \tilde{u}^L \right) \right\|_{(V^L)'} = C \left\| \tilde{\mathcal{A}}^L P_L u - A u \right\|_{(V^L)'}$$

   yielding

$$\left\| u - \tilde{u}^L \right\|_0 \leq \left\| u - P_L u \right\|_0 + C \left\| A(u - P_L u) \right\|_{(V^L)'} + C \left\| (A - \tilde{\mathcal{A}}^L) P_L u \right\|_{(V^L)'} .$$

   The first two terms are estimated using the approximation property and the continuity of A. The estimate for the third term follows from (28) with (34) and $P^L v^L = v^L$:

$$\left| \left\langle (A - \tilde{\mathcal{A}}^L) P_L u, v^L \right\rangle \right| \leq C (L+1) \sigma^{\varrho L} \left\| u \right\|_0 \left\| v^L \right\|_0 .$$

3. Let $\phi^L := P_L \phi$ with ϕ denoting the solution of $A^\star \phi = g$. Then

$$\left| \left\langle u - \tilde{u}^L, g \right\rangle \right| = \left| \left\langle A \left(u - \tilde{u}^L \right), \phi \right\rangle \right| \leq \left| \left\langle A \left(u - \tilde{u}^L \right), \phi - \phi^L \right\rangle \right| + \left| \left\langle A \left(u - \tilde{u}^L \right), \phi^L \right\rangle \right|$$

   The first term can be estimated by $C \left\| u - \tilde{u}^L \right\|_0 \left\| \phi - P_L \phi \right\|_0$ which gives the desired bound using (33) and the regularity of ϕ. For the second term we have

$$\begin{aligned} \left\langle A \left(u - \tilde{u}^L \right), \phi^L \right\rangle &= \left\langle \left(\tilde{\mathcal{A}}^L - A \right) \tilde{u}^L, \phi^L \right\rangle \\ &= \left\langle \left(\tilde{\mathcal{A}}^L - A \right) \left(\tilde{u}^L - P_L u \right), P_L \phi \right\rangle + \left\langle \left(\tilde{\mathcal{A}}^L - A \right) P_L u, P_L \phi \right\rangle . \end{aligned}$$

   The second term on the right hand side can be estimated by (28) and (36) (here the higher quadrature accuracy is needed). Since $\tilde{u}^L - P_L u \in V^L$ we have for the first term using (28)

$$\begin{aligned} \left| \left\langle \left(\tilde{\mathcal{A}}^L - A \right) \left(\tilde{u}^L - P_L u \right), P_L \phi \right\rangle \right| &\leq C N_L \Phi(L) \left\| \tilde{u}^L - P_L u \right\|_0 \left\| \phi \right\|_0 \\ &\leq C (L+1) \sigma^{\varrho L} \left(\left\| \tilde{u}^L - u \right\|_0 + \left\| u - P_L u \right\|_0 \right) \left\| \phi \right\|_0 . \end{aligned}$$

■

Theorem 12 shows that the exponential convergence rates (33), (37) are preserved, provided the consistency error (27) for the matrix entries is controlled with $\Phi(L) = N_L^{-1} L \sigma^{2\varrho L}$. This rather tight bound on the quadrature error must be achieved with a work W_L of *algebraic* order in dependence on the problem size N_L since otherwise the convergence rate of the fully discrete Galerkin scheme will not be exponential in terms of the work W_L.

2.6 Solution of the linear system

The fully populated stiffness matrix $\tilde{A}^L$ has $O(N_L^2)$ nonvanishing entries and is nonsingular due to (32) for sufficiently large L under the assumptions of Theorem 12. Therefore Gaussian elimination will yield a solution in $O(N_L^3)$ operations.

Due to the norm equivalence Lemma 9 and the stability (32) of the fully discrete scheme, the condition number of $\tilde{A}^L$ is uniformly bounded:

$$\mathrm{cond}_2(\tilde{A}^L) \le C < \infty. \tag{40}$$

Classical iterative methods, such as Richardson iteration, yield a sequence $\{\vec{u}^L(j)\}_{j=0}^{\infty}$ of coefficient vectors $\{\vec{u}^L(j)\}_{j=0}^{\infty}$ and corresponding approximate solutions $\{u^L(j)\}_{j=0}^{\infty}$ which satisfy

$$\left\| u^L(j) - \tilde{u}^L \right\|_{L^2(\Gamma)} \le C \left\| \vec{u}^L(j) - \tilde{u}^L \right\|_{\ell^2} \le Cq^j, \qquad j = 0, 1, \ldots \tag{41}$$

with $q < 1$ and $C > 0$ independent of L and j. The iterations are stopped when the error is of the order of the discretization error. Assuming that $u \in B_\varrho(\Gamma)$, this is the case if $q^j \le \exp\left(-bN_L^{1/4}\right)$. Hence, for $j \ge bN_L^{1/4} / |\ln q|$ iterations an approximate solution of the linear system with (33) can be obtained. Since each step requires one matrix-vector multiplication, the total work for the iterative solution of the linear system is $O(N_L^{2.25})$ operations. For the optimal convergence rate (37) of the postprocessed solution at an interior point, twice the number of iterations needed for optimal $L^2(\Gamma)$-convergence is necessary (since then, b is replaced by $2b$ in the above argument).

3 Quadrature error analysis

The purpose of this section is to develop and analyze a quadrature scheme such that the resulting numerically integrated stiffness matrix $\tilde{A}^L$ satisfies the consistency estimate

$$\left| A_{II'}^L - \tilde{A}_{II'}^L \right| \le N_L^{-1} L \sigma^{2\varrho L}. \tag{42}$$

By Theorem 12, this will ensure the exponential convergence rates (33), (37) for the fully discrete scheme, provided that the exact solution u belongs to $B_\varrho(\Gamma)$. The parameter ϱ defined in Theorem 7 will be explicit in the quadrature error estimates. By Theorem 12, (42) will ensure the exponential "energy" convergence (33) of the solution $\tilde{u^L}$ of the fully discrete problem. In order to achieve the optimal convergence rates (37) at an interior point $x \in \Omega$, one should replace ϱ by 2ϱ in our estimates for the quadrature points.

In the following we will work out the quadrature methods only for quadrilateral meshes. One possbility to treat triangles is to map the quadrilateral reference domain to the triangular one by a degenerate mapping. Then the techniques presented below can be applied also to this case while then the number of quadrature points has to be increased by one in each direction.

3.1 Some auxiliary results

We begin with a classical quadrature error estimate in one dimension. It goes back to Davis and Rabinowitz, see, e.g. [4, Eqn. (4.6.1.11)]. Throughout, we denote by $G_{(a,b)}^n$ the n-point Gaussian quadrature formula in (a, b). If no confusion is possible we skip the integration interval and write simply G^n. Let $\mathcal{E}^{\rho}_{a,b} \subset \mathbb{C}$ be the closed ellipse with foci at $z = a, b$, semimajor axis $\bar{a} > (b-a)/2$ and semiminor axis $\bar{b} > 0$. The semiaxis sum is $\rho = \bar{a} + \bar{b}$. For $a = -1$ and $b = 1$, we write also $\mathcal{E}^{\rho}$ instead of $\mathcal{E}^{\rho}_{-1,1}$. A classical estimate for the error in Gaussian quadrature is (see, e.g. [4])

Proposition 13 *Let $f(x)$ be analytic in $[-1,1]$ and admit an analytic continuation $f(z)$ into the ellipse $\mathcal{E}_\rho \subset \mathbb{C}$. Then*

$$|E^n f| = |If - G^n f| \leq C\rho^{-2n} \max_{z \in \partial\mathcal{E}_\rho} |f(z)|. \tag{43}$$

Higher dimensional analogs of (43) can be obtained by a tensor product construction. In tensor products Gaussian formulae we denote by $G^{n_i}_{a_i,b_i}$ the n-point quadrature formula scaled on (a_i, b_i) with respect to the *ith* variable.

Proposition 14 *For $a_i, b_i \in \mathbb{R}$, $b_i \geq a_i$ we define the cuboid $D = \bigotimes_{i=1}^d (a_i, b_i)$. Let $f(x) \in C^0(\overline{D})$. Then*

$$\left| \int_D f(x)dx - \left(\prod_{i=1}^d G^{n_i}_{a_i,b_i} \right) f \right| \leq C(d) \sum_{i=1}^d |D_i^c| \max_{x_i^c \in D_i^c} |(E_i f)(x_i^c)| \tag{44}$$

where

$$x_i^c = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_d), \quad D_i^c = \bigotimes_{\substack{j=1 \\ j \neq i}}^d (a_i, b_i), \quad |D_i^c| = \prod_{\substack{j=1 \\ j \neq i}}^d |b_j - a_j| \tag{45}$$

and

$$(E_i f)(x_i^c) = \int_{a_i}^{b_i} f(x)dx_i - G^{n_i}_{(a_i,b_i)}[f].$$

Proof. We only consider the case $d = 2$. The case $d = 1$ is trivial while, for $d > 2$, the result follows by induction. We use a classical tensor product argument. Let $\Omega = \Omega_1 \times \Omega_2$, $If = I_1 I_2 f$ where $I_i f = \int_{\Omega_i} f(x_i)dx_i$ and $Q_i f = \sum_{j=1}^{n_i} w_j^{(i)} f(x_j^{(i)})$ are quadrature formulas in Ω_i with positive weights $w_j^{(i)}$, $i = 1, 2$. Then

$$\begin{aligned} (I - Q) f &= (I_1 I_2 - Q_1 Q_2) f = (I_1 I_2 - I_1 Q_2 + I_1 Q_2 - Q_1 Q_2) f \\ &= I_1 [(I_2 - Q_2) f] + Q_2 (I_1 - Q_1) f \end{aligned}$$

and we estimate

$$\begin{aligned} |(I - Q) f| &\leq |\Omega_1| \max_{x_1 \in \bar{\Omega}_1} |(I_2 - Q_2) f(x_1, \cdot)| + \sum_{j=1}^{N_2} w_j^{(2)} \left| (I_1 - Q_1) f\left(\cdot, x_j^{(2)}\right) \right| \\ &\leq |\Omega_1| \max_{x_1 \in \bar{\Omega}_1} |(I_2 - Q_2) f(x_1, \cdot)| + |\Omega_2| \max_{x_2 \in \bar{\Omega}_2} |(I_1 - Q_1) f(\cdot, x_2)|. \end{aligned}$$

■

To apply the estimate in Proposition 13 to the transformed integrands, we will also require estimates on the growth of Legendre polynomials on $\partial\mathcal{E}_\rho$.

Proposition 15 *Let $\mathcal{E}_\rho \subset \mathbb{C}$ denote the ellipse with foci at ± 1 and semiaxis sum $\rho \geq 1$. Let further $L_n(x)$ denote the Legendre Polynomial of degree n on $(-1,1)$, normalized such that $L_n(1) = 1$, for $n \in \mathbb{N}_0$. Then, for $\rho \geq 1$,*

$$\max_{z \in \mathcal{E}_\rho} |L_n(z)| \leq \rho^n, \tag{46}$$

and

$$\max_{u \in \mathcal{E}_\rho} \max_{v \in \mathcal{E}_\rho} \left| \frac{L_n(v) - L_n(u)}{v - u} \right| \leq \frac{n(n+1)}{2} \rho^{n-1}. \tag{47}$$

Proof. The conformal map $z = (w + w^{-1})/2$, $|w| \geq 1$, maps $\mathbb{C} \setminus [-1,1]$ into the exterior of the unit circle. Circles of radius $\rho > 1$ in the w-plane correspond to $\mathcal{E}^\rho_{-1,1}$ in the z-plane. Moreover (see [28], (8.3.1))

$$L_n(z) = \sum_{m=0}^{n} g_m g_{n-m} w^{n-2m} = w^n \sum_{m=0}^{n} g_{n-m} g_m w^{-2m}$$

where the numbers g_m are defined by

$$g_m = 4^{-m} \binom{2m}{m}.$$

Inserting this into the representation formula, we obtain

$$\max_{z \in \mathcal{E}^\rho_{-1,1}} |L_n(z)| \leq \rho^n \sum_{m=0}^{n} g_{n-m} g_m = \rho^n 4^{-n} \sum_{m=0}^{n} \binom{2m}{m} \binom{2(n-m)}{n-m} = \rho^n.$$

Next, let $v = (t + t^{-1})/2$ and $u = (s + s^{-1})/2$. Then

$$\begin{aligned} \frac{L_n(v) - L_n(u)}{v - u} &= 2 \sum_{m=0}^{n} g_m g_{n-m} \frac{t^{n-2m} - s^{n-2m}}{t + t^{-1} - s - s^{-1}} \\ &= 2 \sum_{m=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} g_m g_{n-m} \frac{t^{n-2m} + t^{-n+2m} - s^{n-2m} s^{-n+2m}}{t + t^{-1} - s - s^{-1}}. \end{aligned}$$

For $r > 0$ we obtain

$$\frac{t^r + t^{-r} - s^r - s^{-r}}{t + t^{-1} - s - s^{-1}} = (ts)^{1-r} \frac{((ts)^r - 1)}{(ts - 1)} \frac{(t^r - s^r)}{t - s}.$$

From $1 \leq |s|, |t| \leq \rho$, it follows that

$$\left| \frac{t^r + t^{-r} - s^r - s^{-r}}{t + t^{-1} - s - s^{-1}} \right| \leq (ts)^{1-r} \sum_{j=0}^{r-1} (ts)^j \, r\rho^{r-1} = \sum_{j=0}^{r-1} (ts)^{1-r+j} \, r\rho^{r-1} \leq r^2 \rho^{r-1}.$$

holds. Consequently

$$\begin{aligned} \max_{(u,v) \in \mathcal{E}^\rho_{(-1,1)} \times \mathcal{E}^\rho_{(-1,1)}} \left| \frac{L_n(v) - L_n(u)}{v - u} \right| &\leq 2 \sum_{m=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} g_m g_{n-m} (n - 2m)^2 \rho^{n-2m-1} \\ &\leq \rho^{n-1} 2 \sum_{m=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} g_m g_{n-m} (n - 2m)^2 = \frac{n(n+1)}{2} \rho^{n-1}. \end{aligned}$$

■

For later purpose we define the scaling function $\tilde{\pi}_K := \left(p_1^K \left(p_1^K + 1\right) + p_2^K \left(p_2^K + 1\right) + 1\right)/2$ and in view of (16) the function $\pi_K = \left(p_1^K + 1/2\right)^{1/2} \left(p_2^K + 1/2\right)^{1/2}$ for $K \in \tau_L$.

3.2 Surface integrals in hp-BEM: The basic cases

Let τ_L denote the hp-mesh generated by the geometric refinement algorithm presented in Section 2.3. We recall the definition of the set $\mathcal{I}_L$ (see (21)) and use the notation of Subsection 2.1. In order to assemble the stiffness matrix A^L one has to compute integrals of the form

$$A^L_{I,I'} := \lim_{\varepsilon\to 0} \int_{\substack{K_x\times K_y \\ \|x-y\|\geq\varepsilon}} k(x,y)\,\varphi_I(x)\,\varphi_{I'}(y)\,dydx. \tag{48}$$

In the following we present quadrature methods for the approximation of $A^L_{I,I'}$. The strategies will depend on the singular or near singular behaviour of the kernel function. As already mentioned the arising kernel functions have a special, characteristic behaviour which can be (globally) expressed by (7). What is more important for the quadrature methods is the behaviour of the integrands in the local coordinates. We have to distinguish between the following three basic cases. From the assumption on the initial parametrization (1) and the algorithm for the geometric refinement, it follows that there exists a constant C_s depending only on $\Gamma, \tilde{\Gamma}$, on the function $\eta : \tilde{\Gamma} \to \Gamma$, and on $\tilde{\tau}_0$ such that for all pairs of panels $\tilde{K}_x \times \tilde{K}_y \in \tilde{\tau}_L$ one of the following conditions is satisfied

1. $\tilde{K}_x = \tilde{K}_y$, "case of identical panels".
2. Condition 1 is violated and $\operatorname{dist}\left(\tilde{K}_x, \tilde{K}_y\right) \leq C_s \max_{z\in\{x,y\}} \operatorname{diam} \tilde{K}_z$ holds. Furthermore, there exist two plane quadrangles $\tilde{K}^\star_x, \tilde{K}^\star_y \subset \tilde{\Gamma}$ which share exactly one common edge and have the property that, at least, three edges of $\tilde{K}_x$ belong to $\partial\tilde{K}^\star_x$ and, at least, three edges of $\tilde{K}_y$ belong to $\partial\tilde{K}^\star_y$. "edge-parallel case".
3. Conditions 1 and 2 are violated. "vertex-singular, near-singular, and regular farfield case".

The integral over the surface pieces K_x and K_y has to be pulled back onto suitable parameter domains $\hat{K}_x$ and $\hat{K}_y$ in $\mathbb{R}^2$. The transformations have to be chosen such that the geometric situation of the parameter panels $\hat{K}_{x,y}$ is the same as on the surface, e.g., $\hat{K}_x, \hat{K}_y$ share an edge if this is the case for K_x and K_y. In the following we will discuss the three cases above separately.

3.3 Identical panels

3.3.1 Regularizing coordinate transforms

In the first step we will apply certain coordinate transforms which render the integrand tractable for automatic quadrature methods.

Let $K_x = K_y =: K \in \tau_L$. The corresponding flat panel on $\tilde{\Gamma}$ is denoted by $\tilde{K} := \eta^{-1}(K) \in \tilde{\tau}_L$. We first have to transform the surface panel onto a suitable reference element in $\mathbb{R}^2$. Let $\{X_i\}_{1\leq i\leq 4}$ denote the vertices of $\tilde{K}$ (counterclockwise ordering) and $\varepsilon_i := \|X_{i+1} - X_i\|$ the side lengths. The reference domain is given by $\hat{K} := (0,\varepsilon_1)\times(0,\varepsilon_2)$. The mapping $\tilde{\kappa}_{\tilde{K}} : \hat{K} \to \tilde{K}$ is affine bi-linear:

$$\tilde{\kappa}_{\tilde{K}}(u) = X_1 + \frac{u_1}{\varepsilon_1}(X_2 - X_1) + \frac{u_2}{\varepsilon_2}(X_4 - X_1) + \frac{u_1 u_2}{\varepsilon_1\varepsilon_2}(X_1 - X_2 + X_3 - X_4)$$

and depends only on the angles of $\tilde{K}$ but not on the side lengths. The composite mapping $\kappa_K := \eta \circ \tilde{\kappa}_{\tilde{K}}$ transports $\hat{K}$ onto K. The kernel function in local coordinates on $\hat{K} \times \hat{K}$ is given by

$$k_{loc}(u,v) := k(\kappa_K(u), \kappa_K(v)).$$

The product of the basis functions in local coordinates takes the form

$$\begin{aligned} |K| \varphi_\alpha^K(\kappa_K(u)) \varphi_{\alpha'}^K(\kappa_K(y)) &= \left(\varphi_\alpha^0 \circ \chi_K^{-1} \circ \kappa_K\right)(u) \cdot \left(\varphi_{\alpha'}^0 \circ \chi_K^{-1} \circ \kappa_K\right)(v) \\ &=: \left(\varphi_\alpha^0 \circ \hat{\kappa}_K\right)(u) \cdot \left(\varphi_{\alpha'}^0 \circ \hat{\kappa}_K\right)(v). \end{aligned}$$

Since $\hat{K}$ is a rectangle and $Q_0 := (-1,1)^2$, the transformation $\hat{\kappa}_K := \chi_K^{-1} \circ \kappa_K$ mapping $\hat{K}$ onto Q_0 is given by $\hat{\kappa}_K(u) = \binom{-1}{-1} + 2\binom{u_1/\varepsilon_1}{u_2/\varepsilon_2}$. In local coordinates, the integral (48) takes the form

$$I_s := \lim_{\varepsilon \to 0} \int_{\substack{\hat{K} \times \hat{K} \\ \|u-v\| \geq \varepsilon}} k_{loc}(u,v) B(u,v)\, dv du \tag{49}$$

with

$$B(u,v) := B_{\alpha,\alpha'}(u,v) := g_K(u) g_K(v) \left(\varphi_\alpha^0 \circ \hat{\kappa}_K\right)(u) \cdot \left(\varphi_{\alpha'}^0 \circ \hat{\kappa}_K\right)(v) / |K|.$$

Here, $g_K(u) := \left|\det \left\{\left\langle \frac{\partial \kappa_K}{\partial u_i}, \frac{\partial \kappa_K}{\partial u_j}\right\rangle\right\}_{1 \leq i,j \leq 2}\right|^{1/2}$ can be extended analytically into a neighborhood of $\hat{K}$. Note that $g_K(u)$ does not depend on the side lengths of K but only on the angles. For $u \in \hat{K}$, define the shifted rectangle $\hat{K}_u$ by

$$\hat{K}_u = \left\{z \in \mathbb{R}^2 \mid \exists v \in \hat{K} : z = v - u\right\}. \tag{50}$$

Then

$$I_s = \lim_{\varepsilon \to 0} \int_{u \in \hat{K}} \int_{\substack{z \in \hat{K}_u \\ \|z\| \geq \varepsilon}} k_{loc}(u, u+z) B(u, u+z)\, dz du. \tag{51}$$

The domain of integration is given by the system of inequalities of the form

$$\begin{array}{cc} 0 \leq u_1 \leq \varepsilon_1, & 0 \leq u_2 \leq \varepsilon_2, \\ -u_1 \leq z_1 \leq \varepsilon_1 - u_1, & -u_2 \leq z_2 \leq \varepsilon_2 - u_2. \end{array}$$

This system can be reordered. An equivalent description of this domain is given by

$$\begin{aligned} -\varepsilon_1 &\leq z_1 \leq \varepsilon_1, \\ -\varepsilon_2 &\leq z_2 \leq \varepsilon_2, \\ \max(0, -z_1) &\leq u_1 \leq \min(\varepsilon_1, \varepsilon_1 - z_1), \\ \max(0, -z_2) &\leq u_2 \leq \min(\varepsilon_2, \varepsilon_2 - z_2). \end{aligned}$$

Splitting the domain in four sub-domains

$$D_1 = \left\{ \begin{array}{c} 0 \leq z_1 \leq \varepsilon_1 \\ 0 \leq z_2 \leq \varepsilon_2 \\ 0 \leq u_1 \leq \varepsilon_1 - z_1 \\ 0 \leq u_2 \leq \varepsilon_2 - z_2 \end{array} \right\} \qquad D_2 = \left\{ \begin{array}{c} 0 \leq z_1 \leq \varepsilon_1 \\ -\varepsilon_2 \leq z_2 \leq 0 \\ 0 \leq u_1 \leq \varepsilon_1 - z_1 \\ -z_2 \leq u_2 \leq \varepsilon_2 \end{array} \right\}$$

$$D_3 = \left\{ \begin{array}{c} -\varepsilon_1 \leq z_1 \leq 0 \\ 0 \leq z_2 \leq \varepsilon_2 \\ -z_1 \leq u_1 \leq \varepsilon_1 \\ 0 \leq u_2 \leq \varepsilon_2 - z_2 \end{array} \right\} \qquad D_4 = \left\{ \begin{array}{c} -\varepsilon_1 \leq z_1 \leq 0 \\ -\varepsilon_2 \leq z_2 \leq 0 \\ -z_1 \leq u_1 \leq \varepsilon_1 \\ -z_2 \leq u_2 \leq \varepsilon_2 \end{array} \right\}$$

we avoid the min / max -expressions and I_s takes the form

$$I_s = \lim_{\varepsilon \to 0} \sum_{j=1}^{4} \int_{\substack{D_j \\ \|z\| \geq \varepsilon}} k_{loc}\left(u, u+z\right) B\left(u, u+z\right) du dz.$$

We transform the integration domains D_j onto D_1 by the following transformations

$$D_1 : \begin{pmatrix} u_1^1 \\ u_2^1 \\ v_1^1 \\ v_2^1 \end{pmatrix} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_1 + \hat{z}_1 \\ \hat{u}_2 + \hat{z}_2 \end{pmatrix}, \qquad D_2 : \begin{pmatrix} u_1^2 \\ u_2^2 \\ v_1^2 \\ v_2^2 \end{pmatrix} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 + \hat{z}_2 \\ \hat{u}_1 + \hat{z}_1 \\ \hat{u}_2 \end{pmatrix},$$

$$D_3 : \begin{pmatrix} u_1^3 \\ u_2^3 \\ v_1^3 \\ v_2^3 \end{pmatrix} = \begin{pmatrix} \hat{u}_1 + \hat{z}_1 \\ \hat{u}_2 \\ \hat{u}_1 \\ \hat{u}_2 + \hat{z}_2 \end{pmatrix}, \qquad D_4 : \begin{pmatrix} u_1^4 \\ u_2^4 \\ v_1^4 \\ v_2^4 \end{pmatrix} = \begin{pmatrix} \hat{u}_1 + \hat{z}_1 \\ \hat{u}_2 + \hat{z}_2 \\ \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}.$$

The integral takes the form

$$I_s = \lim_{\varepsilon \to 0} \int_{\substack{D_1 \\ \|\hat{z}\| \geq \varepsilon}} \sum_{i=1}^{4} k_{loc}\left(u^i, v^i\right) B\left(u^i, v^i\right) d\hat{u} d\hat{z} \tag{52}$$

with the functions $v^i = v^i\left(\hat{z}, \hat{u}\right)$ and $v^i = u^i\left(\hat{z}, \hat{u}\right)$ defined above. The integrand in (52) defines the function $H\left(\hat{u}, \hat{z}\right)$.

To apply tensor product Gaussian quadrature, we transform the region of integration in (52) to a 4-dimensional cuboid by the substitutions

$$\eta_i(\varepsilon_i - \hat{z}_i) = u_i, \quad i = 1, 2. \tag{53}$$

Thus, integral (52) takes the form

$$I_s = \lim_{\varepsilon \to 0} \int_{\substack{\hat{K} \\ \|\hat{z}\| \geq \varepsilon}} \int_0^1 \int_0^1 H\left(\binom{\eta_1(\varepsilon_1 - \hat{z}_1)}{\eta_2(\varepsilon_2 - \hat{z}_2)}, \hat{z}\right)(\varepsilon_1 - \hat{z}_1)(\varepsilon_2 - \hat{z}_2) d\eta d\hat{z}. \tag{54}$$

The singular behaviour of the integrand is analyzed in

Proposition 16 *The function $\hat{H}\left(\eta, \hat{z}\right)$ defined by the integrand of (54) is weakly singular at $z = 0$ and, for any $\hat{z} \neq 0$, analytic in η and, for any η, analytic in $\hat{z} \neq 0$.*

Proof. It is sufficient to show that $\left|H\left(\hat{u}, \hat{z}\right)\right| \leq C \left\|\hat{z}\right\|^{-1}$ for $\left|\hat{z}\right|$ sufficiently small. Consider in (52) the sum of the terms with $i = 1, 4$:

$$\begin{aligned} h_{1,4}\left(\hat{u}, \hat{z}\right) \quad : &= k_{loc}\left(u^1, u^1 + z^1\right) B\left(u^1, u^1 + z^1\right) + k_{loc}\left(u^4, u^4 + z^4\right) B\left(u^4, u^4 + z^4\right) \\ &= k_{loc}\left(\hat{u}, \hat{u} + \hat{z}\right) B\left(\hat{u}, \hat{u} + \hat{z}\right) + k_{loc}\left(\hat{u} + \hat{z}, \hat{u}\right) B\left(\hat{u} + \hat{z}, \hat{u}\right) \\ &= k_0\left(\hat{u}, \hat{u} + \hat{z}\right) B\left(\hat{u}, \hat{u}\right) + k_1\left(\hat{u}, \hat{u} + \hat{z}\right) B_{diff}^{I}\left(\hat{u}, \hat{u} + \hat{z}\right) \\ &\quad + k_1\left(\hat{u} + \hat{z}, \hat{u}\right) B_{diff}^{II}\left(\hat{u} + \hat{z}, \hat{u}\right) \end{aligned}$$

with

$$\begin{aligned}
k_0\left(\hat{u},\hat{u}+\hat{z}\right) &: = k_{loc}\left(\hat{u},\hat{u}+\hat{z}\right)+k_{loc}\left(\hat{u}+\hat{z},\hat{u}\right),\\
k_1\left(\hat{u},\hat{u}+\hat{z}\right) &: = \left\|z\right\| k_{loc}\left(\hat{u},\hat{u}+\hat{z}\right),\\
B^I_{diff}\left(\hat{u},\hat{v}\right) &: = \left(B\left(\hat{u},\hat{v}\right)-B\left(\hat{u},\hat{u}\right)\right)/\left\|\hat{v}-\hat{u}\right\|,\\
B^{II}_{diff}\left(\hat{u},\hat{v}\right) &: = \left(B\left(\hat{u},\hat{v}\right)-B\left(\hat{v},\hat{v}\right)\right)/\left\|\hat{v}-\hat{u}\right\|.
\end{aligned}$$

Inserting the local parametrization κ_K into the global representation it follows with Proposition 2.1 that

$$\left|k_0\left(\hat{u},\hat{u}+\hat{z}\right)\right|+\left|k_1\left(\hat{u}+\hat{z},\hat{u}\right)\right|+\left|k_1\left(\hat{u},\hat{u}+\hat{z}\right)\right|\leq\frac{C}{\left\|\hat{z}\right\|},$$

holds (see also [20, Lemma 4.2]). Due to the analyticity of $B\left(u,v\right)$ the assertion follows from

$$\left|B\left(\hat{u},\hat{u}+\hat{z}\right)-B\left(\hat{u},\hat{u}\right)\right|+\left|B\left(\hat{u}+\hat{z},\hat{u}\right)-B\left(\hat{u},\hat{u}\right)\right|\leq C\left\|\hat{z}\right\|.$$

The remaining terms in the sum corresponding to $i=2,3$ defines the function $h_{2,3}$ which can be treated analogously. ■

Proposition 16 implies that

$$I_s=\int_0^{\varepsilon_1}\int_0^{\varepsilon_2}\int_0^1\int_0^1\hat{H}\left(\eta,\hat{z}\right)d\eta d\hat{z} \tag{55}$$

exists as an improper integral.

Remark 17 Before proceeding in deriving a representation of the integral I_s which is appropriate for numerical quadrature, let us first motivate our strategy. At $\hat{z}=0$, the integrand $\hat{H}(\eta,\hat{z})$ has a weak singularity. The common strategy for such kind of integrals is to split $(0,\varepsilon_1)\times(0,\varepsilon_2)$ into two triangles and to apply the so-called Duffy transformation which removes the singularity and renders the integrand analytic (see [26] and [1]). Gaussian quadratures would then yield exponential convergence. The problem with this strategy, however, is that, due to the possibly high aspect ratio $\varepsilon_1/\varepsilon_2$ of the domain of integration, the size of the region of analyticity of the transformed integrand will not be uniform in ε_i. More precisely, the exponential rates of convergence of Gaussian quadrature applied to the transformed integrand $\hat{H}(\eta,\hat{z})$ deteriorate for high element aspect ratio. We show next that a *variable order composite quadrature* [24] can achieve exponential convergence with algebraic work independently of the elemental aspect ratio. The key to our strategy is, as in [24], an appropriate splitting of the domain of integration such that for each subdomain the distance to the singularity versus the diameter of the subdomain is bounded uniformly from above and below. This ensures uniform domains of analyticity for the integrands and, by Proposition 13, uniform exponential convergence of Gaussian Quadrature. The situation is illustrated in Figure 2.

Without loss of generality, we assume that $\varepsilon_2\leq\varepsilon_1$ and define $j_0=\left\lfloor\left|\log_2\frac{\varepsilon_2}{\varepsilon_1}\right|\right\rfloor$. Due to the geometric subdivision algorithm in Section 2.3.1, we know that $\varepsilon_2\geq\sigma^L\varepsilon_1$ holds implying $j_0\leq L\left|\log\sigma\right|$. Then

$$I_s=\sum_{j=0}^{j_0-1}\int_{\hat{Q}_j}\hat{H}\left(\eta,\hat{z}\right)d\eta d\hat{z}+\int_0^{\varepsilon_{1,j_0}}\int_0^{\varepsilon_2}\int_0^1\int_0^1\hat{H}\left(\eta,\hat{z}\right)d\eta d\hat{z} \tag{56}$$

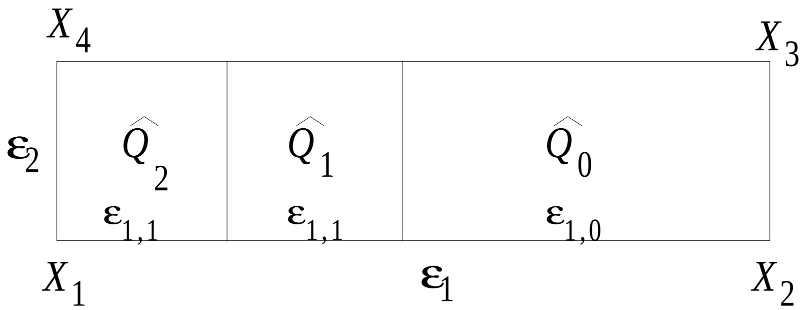

Figure 2: Subdivision of the domain $\hat{K} = (0, \varepsilon_1) \times (0, \varepsilon_2)$ into Q_i. Singular Vertex is X_1.

where the domains $\hat{Q}_j$ are given by

$$\hat{Q}_j := (\varepsilon_{1,j+1}, \varepsilon_{1,j}) \times (0, \varepsilon_2) \times (0,1)^2, \quad 0 \leq j \leq j_0 - 1. \tag{57}$$

with $\varepsilon_{1,j} := 2^{-j}\varepsilon_1$. Note that the aspect ratio of the last integration domain in (56) (where $\hat{H}(\eta, \hat{z})$ is singular) is now bounded and that $\hat{H}(\eta, \hat{z})$ is analytic over $\overline{\hat{Q}_j}$ with size of the domain of analyticity proportional to that of $\hat{Q}_j$. In order to apply Duffy coordinates for the singular integral we split $(0, \varepsilon_{1,j_0}) \times (0, \varepsilon_2)$ into two triangles

$$\begin{aligned}\int_0^{\varepsilon_{1,j_0}} \int_0^{\varepsilon_2} \int_0^1 \int_0^1 \hat{H}(\eta, \hat{z})\, d\eta d\hat{z} &= \int_0^{\varepsilon_{1,j_0}} \int_0^{\frac{\varepsilon_2}{\varepsilon_{1,j_0}} z_1} \int_0^1 \int_0^1 \hat{H}(\eta, \hat{z})\, d\eta d\hat{z} \\ &+ \int_0^{\varepsilon_{1,j_0}} \int_{\frac{\varepsilon_2}{\varepsilon_{1,j_0}} z_1}^{\varepsilon_2} \int_0^1 \int_0^1 \hat{H}(\eta, \hat{z})\, d\eta d\hat{z}.\end{aligned}$$

For the first integral we substitute

$$\begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ \frac{\varepsilon_2}{\varepsilon_{1,j_0}} z_1 z_2 \end{pmatrix}$$

while for the second one we put

$$\begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon_{1,j_0}}{\varepsilon_2} z_1 z_2 \\ z_2 \end{pmatrix}$$

and obtain

$$\begin{aligned}\int_0^{\varepsilon_{1,j_0}} \int_0^{\varepsilon_2} \int_0^1 \int_0^1 \hat{H}(\eta, \hat{z})\, d\eta d\hat{z} &= \int_0^{\varepsilon_{1,j_0}} \int_0^1 \int_0^1 \int_0^1 \frac{\varepsilon_2}{\varepsilon_{1,j_0}} z_1 \hat{H}(\eta, \hat{z}(z))\, d\eta dz \\ &+ \int_0^1 \int_0^{\varepsilon_2} \int_0^1 \int_0^1 \frac{\varepsilon_{1,j_0}}{\varepsilon_2} z_2 \hat{H}(\eta, \hat{z}(z))\, d\eta dz.\end{aligned} \tag{58}$$

Summarizing the above transformations we have shown that

$$\begin{aligned}I_s &= \sum_{j=0}^{j_0-1} \int_{\hat{Q}_j} \hat{H}(\eta, \hat{z})\, d\eta d\hat{z} \\ &+ \int_0^{\varepsilon_{1,j_0}} \int_0^1 \int_0^1 \int_0^1 \frac{\varepsilon_2}{\varepsilon_{1,j_0}} z_1 \hat{H}(\eta, \hat{z}(z))\, d\eta dz + \int_0^1 \int_0^{\varepsilon_2} \int_0^1 \int_0^1 \frac{\varepsilon_{1,j_0}}{\varepsilon_2} z_2 \hat{H}(\eta, \hat{z}(z))\, d\eta dz\end{aligned}$$

holds with analytic integrands as we will show below. This representation of the integral I_s is now well-suited for numerical approximation (cf. Remark 17).

3.3.2 Quadrature and error analysis

We approximate the integrals above by properly scaled tensor Gaussian rules of possibly non-uniform order for the different variables. First, we consider the integrals over $\hat{Q}_j$. The quadrature error is given by

$$E_j := \left| \int_{\hat{Q}_j} \hat{H}(\eta, \hat{z}) d\eta d\hat{z} - G^{n_1}_{\eta_1} G^{n_2}_{\eta_2} G^{n_3}_{\hat{z}_1} G^{n_4}_{\hat{z}_2} \hat{H} \right|, \quad j = 0, ..., j_0 - 1, \tag{59}$$

where G^n_x denotes a (properly scaled) n-point Gaussian quadrature in the variable x and $n_l = n_l(j), l = 1, .., 4,\ 0 \leq j < j_0$. Since $\hat{Q}_j$ is a tensor product domain, we use Proposition 14 to bound the quadrature error as follows:

$$E_j \leq C \sum_{l=1}^{4} \hat{E}_{l,j}, \quad \sum_{j=0}^{j_0-1} E_j \leq C \sum_{j=0}^{j_0-1} \sum_{l=1}^{4} \hat{E}_{l,j} \tag{60}$$

where $\hat{E}_{l,j}$ are one-dimensional quadrature errors to be estimated. The details are in the following

Theorem 18 *For all l and j, the quadrature errors $\hat{E}_{l,j}$ in (60) can be estimated by*

$$\begin{aligned} \hat{E}_{l,j} &\leq C \pi_K^2 \tilde{\pi}_K \varepsilon_2 \left(1 + \lambda_l \gamma\right)^{2\left(p_1^K - n_l\right) + \nu_l}, \quad l = 1, 3 \\ \hat{E}_{l,j} &\leq C \pi_K^2 \tilde{\pi}_K \varepsilon_2 \left(1 + \lambda_l \gamma\right)^{2\left(p_2^K - n_l\right) + \nu_l}, \quad l = 2, 4 \end{aligned} \tag{61}$$

with constants C and γ depending only on the mapping η and the angles of $\tilde{K}$. The numbers λ_l and ν_l are given by

$$\lambda_l := \begin{cases} 1/\varepsilon_l & l = 1, 2, \\ 1 & l = 3, \\ \varepsilon_{1,j+1}/\varepsilon_2 & l = 4, \end{cases} \qquad \nu_l := \begin{cases} -1 & l = 1, 2, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let us consider the error of the $\hat{z}_1$-integration. We use the splitting of the proof of Proposition 16, $H = h_{1,4} + h_{2,3}$, inducing an analogous splitting of $\hat{H} = \hat{h}_{1,4} + \hat{h}_{2,3}$. We first consider the function $\hat{h}_{1,4}$. Let $\hat{u}_i := \hat{u}_i(\eta, \hat{z}) := (\varepsilon_i - \hat{z}_i)\, \eta_i$. Using the notation of the proof of Proposition 16 we obtain

$$\begin{aligned} \hat{h}_{1,4}(\eta, \hat{z}) \quad : \quad &= (\varepsilon_1 - z_1)(\varepsilon_2 - z_2) \left\{ k_0(\hat{u}, \hat{u} + \hat{z})\, B(\hat{u}, \hat{u}) \right. \\ &\left. + k_1(\hat{u}, \hat{u} + \hat{z})\, B^I_{diff}(\hat{u}, \hat{u} + \hat{z}) + k_1(\hat{u} + \hat{z}, \hat{u})\, B^{II}_{diff}(\hat{u} + \hat{z}, \hat{u}) \right\}. \end{aligned} \tag{62}$$

We scale the $\hat{z}_1$-integration to $(-1, 1)$ by $\hat{z}_1 := \hat{z}_1(t) = \varepsilon_{1,j+1}(t + 3)/2$. The quadrature error is then given by

$$\hat{E}_{3,j} \leq \frac{\varepsilon_{1,j+1}\varepsilon_2}{2} \max_{\eta \in (0,1)^2} \max_{z_2 \in (0, \varepsilon_2)} \left| \int_{-1}^{1} \hat{h}_{1,4}(\eta, \hat{z})\, dt - G^{n_3}_{t,(-1,1)} \hat{h}_{1,4}\left(\eta, \binom{\hat{z}_1(t)}{\hat{z}_2}\right) \right|.$$

From [20, Lemma 4.2] we know that there exists $\gamma > 0$ depending only on the global kernel function, the initial grid $\tilde{\tau}_0$ and the mapping η such that $\hat{h}_{1,4}$ can be extended analytically onto

$$D_1^{\rho} := \mathcal{E}^{\rho}_{(-1,1)} \times (0, \varepsilon_2) \times (0, 1)^2$$

with $\rho := \left(1+\gamma \operatorname{dist}\left(\hat{Q}_j, 0\right)/\varepsilon_{1,j+1}\right) = 1+\gamma$. We will estimate the terms in (62) separately. In [20, Lemma 4.2] it was shown that

$$\max_{(t,\hat{z}_2,\eta)\in D_1^\rho} \left|k_1\left(\hat{u}\left(\eta,\hat{z}\right),\hat{z}\right)\right| \leq \frac{M}{(1+\gamma)\,\varepsilon_{1,j+1}}$$

where M is independent of the discretization parameters $\varepsilon_{1,2}$, L, and K. The proof of

$$\max_{(t,\hat{z}_2,\eta)\in D_1^\rho} \left|k_0\left(\hat{u}\left(\eta,\hat{z}\right),\hat{z}\right)\right| \leq \frac{M}{(1+\gamma)\,\varepsilon_{1,j+1}}$$

is completely analogous.

We turn now to the estimates of the products of basis functions, i.e., the function B. From the definition of B it follows that

$$\max_{(t,\hat{z}_2,\eta)\in D_1^\rho} \left|B\left(\hat{u},\hat{u}\right)\right| \leq C\frac{\pi_K^2}{\varepsilon_1\varepsilon_2} \max_{(t,\hat{z}_2,\eta)\in D_1^\rho} \left|L_{\alpha_1}\circ\hat{\kappa}_{K,1}\left(\hat{u}_1\right)\right| \left|L_{\alpha_1'}\circ\hat{\kappa}_{K,1}\left(\hat{u}_1\right)\right|.$$

Tracing back the coordinate transform it follows that $\hat{\kappa}_{K,1}\left(\hat{u}_1\right) = -1+2\eta_1\left(1-2^{-j}\left(t+3\right)/4\right)$ is contained in the ellipse $\mathcal{E}^\rho_{(-1,1)}$. Using Proposition 15 we obtain

$$\max_{(t,\hat{z}_2,\eta)\in D_1^\rho} \left|B\left(\hat{u},\hat{u}\right)\right| \leq C\frac{\pi_K^2}{\varepsilon_1\varepsilon_2}\left(1+\gamma\right)^{2p_1^K}.$$

Let $\varphi_{\alpha_i}\left(\hat{u}_i\right) := \left(L_{\alpha_i}\circ\hat{\kappa}_{K,i}\right)\left(\hat{u}_i\right)$. In order to estimate the difference quotients we write

$$\begin{aligned}
|K|\,\|\hat{z}\|\,B^I_{diff}\left(\hat{u},\hat{u}+\hat{z}\right) &= |K|\left(B\left(\hat{u},\hat{u}+\hat{z}\right)-B\left(\hat{u},\hat{u}\right)\right)\\
&= \left(p_1^K+1/2\right)\left(p_2^K+1/2\right)g_K\left(\hat{u}\right)\varphi_{\alpha_1}\left(\hat{u}_1\right)\varphi_{\alpha_2}\left(\hat{u}_2\right)\cdot\\
&\quad \left\{g_K\left(\hat{u}+\hat{z}\right)\varphi_{\alpha_1'}\left(\hat{u}_1+\hat{z}_1\right)\varphi_{\alpha_2'}\left(\hat{u}_2+\hat{z}_2\right)-g_K\left(\hat{u}\right)\varphi_{\alpha_1'}\left(\hat{u}_2\right)\varphi_{\alpha_2'}\left(\hat{u}_2\right)\right\}.
\end{aligned}$$

The expression in the parenthesis can be rewritten in the form

$$\begin{aligned}
\{\ldots\} &= \left(g_K\left(\hat{u}+\hat{z}\right)-g_K\left(\hat{u}\right)\right)\varphi_{\alpha_1'}\left(\hat{u}_1+\hat{z}_1\right)\varphi_{\alpha_2'}\left(\hat{u}_2+\hat{z}_2\right)\\
&\quad +g_K\left(\hat{u}\right)\left\{\varphi_{\alpha_1'}\left(\hat{u}_1+\hat{z}_1\right)-\varphi_{\alpha_1'}\left(\hat{u}_1\right)\right\}\varphi_{\alpha_2'}\left(\hat{u}_2+\hat{z}_2\right)\\
&\quad +g_K\left(\hat{u}\right)\varphi_{\alpha_1'}\left(\hat{u}_1\right)\left(\varphi_{\alpha_2'}\left(\hat{u}_2+\hat{z}_2\right)-\varphi_{\alpha_2'}\left(\hat{u}_2\right)\right).
\end{aligned}$$

Using the analyticity of g_K, the estimates of Proposition 15 and the boundedness of $\frac{\hat{z}_j}{\|\hat{z}\|}$ on $D_1^{\rho_1}$ we obtain

$$\max_{(t,\hat{z}_2,\eta)\in D_1^\rho} \left|B^I_{diff}\left(\hat{u},\hat{u}+\hat{z}\right)\right| \leq C\pi_K^2\tilde{\pi}_K\frac{\left(1+\gamma\right)^{2p_1^K-1}}{\varepsilon_1\varepsilon_2}$$

The estimate for $B^{II}_{diff}\left(\hat{u}+\hat{z},\hat{u}\right)$ follows in the same fashion. Estimating the leading factor $\left|\left(\varepsilon_1-\hat{z}_1\right)\left(\varepsilon_2-\hat{z}_2\right)\right|$ by $\varepsilon_1\varepsilon_2\left(1+\gamma\right)$ we obtain the error bound

$$\begin{aligned}
\hat{E}_{3,j} &\leq C\left(1+\gamma\right)^{-2n}\varepsilon_1\varepsilon_2\frac{\varepsilon_{1,j+1}\varepsilon_2}{2}\frac{\left(1+\gamma\right)^{2p_1^K}}{2\varepsilon_{1,j+1}\varepsilon_1\varepsilon_2}\pi_K^2\tilde{\pi}_K\\
&= C\pi_K^2\tilde{\pi}_K\varepsilon_2\left(1+\gamma\right)^{2\left(p_1^K-n_3\right)}.
\end{aligned}$$

The estimates of the quadrature errors corresponding to the remaining variables are just a repetition of the arguments. Note however that due to the scaling $\hat{u}_i := \eta_i\,(\varepsilon_i - z_i)$ the semiaxes sum ρ for the η_i-integration error can be chosen as $\rho = 1 + \gamma/\varepsilon_i$ and for the z_2-integration as $\rho = 1 + \gamma\varepsilon_{1,j+1}/\varepsilon_2$. Furthermore, for the η-integration, the leading factor $(\varepsilon_1 - \hat{z}_1)\,(\varepsilon_2 - \hat{z}_2)$ can be estimated by $\varepsilon_1 \cdot \varepsilon_2$ yielding the different values of ν_l in the assertion. ■

To achieve (42), it is sufficient that

$$\sum_{j=0}^{j_0-1} \hat{E}_{lj} \leq N_L^{-1} L \sigma^{\varrho L}$$

with ϱ from Theorem 7. This estimate is guaranteed for quadrature orders $n_{l,j}$ satisfying

$$\hat{E}_{lj} \leq N_L^{-1} L \sigma^{\varrho L} / j_0 \tag{63}$$

with $\hat{E}_{l,j}$ from (61).

Remark 19 *We have seen that only the z_2-integration depends on the index of the block Q_j which is expressed by $\lambda_4 = 2^{-j-1}\varepsilon_1/\varepsilon_2$. On the other hand, we recommend* ***not*** *to use a variable order quadrature with respect to j in an implementation of the hp-quadrature but rather to employ the estimate $\lambda_4 \leq 1$ since, for practical problems, the administration overhead for this additional case dominates the asymptotic gain. This is done in the implementation discussed in Section 4.*

In any implementation of *hp*-boundary element methods one should choose quadrature order n directly from relation (61) and (63). For a bound on the *asymptotic* complexity we further simplify this relation.

Proposition 20 *For large L, condition (63) is guaranteed for*

$$n_l\,(j) = O\left(\varrho L \left|\log \sigma\right|\right). \tag{64}$$

Proof. For L large enough we put $N_L = L^4$, $j_0 = L\left|\log\sigma\right|$, $p_i^K = L$, $\varepsilon_2 = 1$ and obtain, for all l and j

$$2 n_l \log\left(1+\gamma\right) = \varrho L \left|\log\sigma\right| + \log\left|\log\sigma\right| + 2L\log\left(1+\gamma\right).$$

Neglecting the second term and the constants $\log\left(1+\gamma\right)$ we obtain

$$n_l = O\left(\varrho L \left|\log\sigma\right|\right).$$

■

We consider now the first integral of the right hand side of (58) approximated by Gauß-Legendre tensor formulae.

$$E := \frac{\varepsilon_2}{\varepsilon_{1,j_0}} \left(\int_0^{\varepsilon_{1,j_0}} \int_0^1 \int_0^1 \int_0^1 z_1 \hat{H}\left(\eta, \hat{z}\left(z\right)\right) d\eta dz - G_{\eta_1}^{n_1} G_{\eta_2}^{n_2} G_{z_1}^{n_3} G_{z_2}^{n_4} z_1 \hat{H}\left(\eta, \hat{z}\left(z\right)\right) \right)$$

Since ε_{1,j_0} is of the same order as ε_2 we may assume for the quadrature error analysis that $\varepsilon_{1,j_0} = \varepsilon_2$. The quadrature errors are estimated in the following

Theorem 21 *The one-dimensional quadrature errors* E_l *corresponding to the* η_1, η_2, z_1, z_2*-integration can be estimated by*

$$E_l \leq C\pi_K^2 \tilde{\pi}_K \varepsilon_2 \left(1 + \lambda_l \gamma\right)^{2\left(p_1^K + p_2^K - n_l\right) + \nu_l}$$

with constants C *and* γ *depending only on the mapping* η *and the angles of* $\tilde{K}$. *The numbers* λ_l, ν_l *are given by*

$$\lambda_l = \begin{cases} 1/\varepsilon_i & l = 1,2 \\ 1/\varepsilon_2 & \text{for } l = 3, \\ 1 & \text{otherwise}, \end{cases} \qquad \nu_l = \begin{cases} 0 & \text{for } l = 1,2, \\ 2 & \text{for } l = 3, \\ 1 & \text{for } l = 4. \end{cases}$$

Proof. Let us first consider the error corresponding to the $\hat{z}_1$-integration. We scale the interval to $(-1,1)$ by $\hat{z}_1(t) = \varepsilon_2 (t+1)/2$ resulting in

$$E_3 := \frac{\varepsilon_2}{2} \max_{(\hat{z}_2,\eta) \in (0,1)^3} \left| \int_{-1}^{1} \hat{z}_1(t) \hat{H}\left(\eta, \left(\begin{smallmatrix} \hat{z}_1(t) \\ \hat{z}_1(t) z_2 \end{smallmatrix}\right)\right) d\hat{z}_1 - G_{t,(-1,1)}^{n_3}\left[\hat{z}_1(t) \hat{H}\left(\eta, \left(\begin{smallmatrix} \hat{z}_1(t) \\ \hat{z}_1(t) z_2 \end{smallmatrix}\right)\right)\right]\right|.$$

In [26] it was shown that the integrand above can be extended analytically with respect to t onto an ellipse $\mathcal{E}_{(-1,1)}^{\rho}$ with $\rho = 1 + \gamma/\varepsilon_2$ while

$$\max_{(t,\hat{z}_2,\eta) \in D_1^{\rho}} \left|\hat{z}_1 k_0\left(\hat{u}(\eta,\hat{z}), \hat{z}\right)\right| \leq M.$$

and

$$\max_{(t,\hat{z}_2,\eta) \in D_1^{\rho}} \|\hat{z}\| \left|\hat{z}_1 k_0\left(\hat{u}(\eta,\hat{z}), \hat{z}\right)\right| \leq M$$

holds with $\hat{z}_1 = \hat{z}_1(t)$ and $D_1^{\rho} := \mathcal{E}_{(-1,1)}^{\rho} \times (0,1)^3$. Again, the constant M depends only on the global kernel function, the initial mesh $\tilde{\tau}_0$ and the mapping η. The corresponding combinations of basis functions can be estimated as follows. Let $\varphi_{\alpha_i}(\hat{u}_i) := \left(L_{\alpha_i} \circ \hat{\kappa}_{K,i}\right)(\hat{u}_i)$. Then

$$\max_{(t,\hat{z}_2,\eta) \in D_1^{\rho_1}} \left|B(\hat{u}, \hat{u})\right| \leq C \frac{\pi_K^2}{\varepsilon_1 \varepsilon_2} \max_{(t,\hat{z}_2,\eta) \in D_1^{\rho}} \left|\varphi_{\alpha_1}(\hat{u}_1) \varphi_{\alpha_2}(\hat{u}_2) \varphi_{\alpha_1'}(\hat{u}_1) \varphi_{\alpha_1'}(\hat{u}_2)\right|.$$

Tracing back the coordinate transform it follows that

$$\begin{aligned} \hat{\kappa}_{K,1}(\hat{u}_1) &= -1 + 2\eta_1 - \eta_1 \varepsilon_2/\varepsilon_1 (t+1) \\ \hat{\kappa}_{K,2}(\hat{u}_2) &= -1 + 2\eta_2 - z_2 \eta_2 (t+1) \end{aligned}$$

are contained in the ellipse $\mathcal{E}_{(-1,1)}^{\rho}$. As in the proof of Theorem 18, it follows that

$$\max_{(t,\hat{z}_2,\eta) \in D_1^{\rho_1}} \left|B(\hat{u}, \hat{u})\right| \leq C \frac{\pi_K^2}{\varepsilon_1 \varepsilon_2} \left(1 + \frac{\gamma}{\varepsilon_2}\right)^{2\left(p_1^K + p_2^K\right)}.$$

and

$$\max_{(t,\hat{z}_2,\eta) \in D_1^{\rho_1}} \left|B_{diff}^I(\hat{u}, \hat{u} + \hat{z})\right| \leq C\pi_K^2 \tilde{\pi}_K \frac{\left(1 + \frac{\gamma}{\varepsilon_2}\right)^{2\left(p_1^K + p_2^K\right) - 1}}{\varepsilon_1 \varepsilon_2}.$$

The leading factor $(\varepsilon_1 - z_1(t))(\varepsilon_2 - z_1(t) z_2)$ can be estimated by $\varepsilon_1\varepsilon_2(1+\gamma)^2$. Altogether, we obtain

$$E_3 \leq C\varepsilon_2 \left(1 + \frac{\gamma}{\varepsilon_2}\right)^{2\left(p_1^K + p_2^K - n_3 + 1\right)} \pi_K^2 \tilde{\pi}_K.$$

The estimates for the remaining variables are just a repetition of the arguments. However, the size of the ellipses changes due to the different scaling of the variables. For the z_2-integration the interval is $(0,1)$ and hence, $\rho_4 = 1+\gamma$. For the η_i integration, we may choose $\rho_l = 1+\gamma/\varepsilon_i$ due to the scaling $\hat{u}_i = \eta_i(\varepsilon_i - z_i)$. The different values of ν_l stem from the fact that $(\varepsilon_1 - z_1(t))(\varepsilon_2 - z_2(t) z_2)$ can be estimated for the η_1 and η_2 integration by $\varepsilon_1\varepsilon_2$ while for the z_2 integration we have $\varepsilon_1\varepsilon_2(1+\gamma)$. ■

Corollary 22 *The estimates for the second integral of the right hand side of (58) are the same as in the previous theorem but the roles of z_1 and z_2 have to be interchanged.*

To ensure (42) the quadrature orders have to be chosen such that $E_l \leq N_L^{-1} L\sigma^{\varrho L}$. We strictly recommend to use this condition in any computer realization of the *hp*-BEM instead of the following asymptotic consideration, analogous to Proposition 19.

Proposition 23 *Asymptotically, i.e., for sufficiently large L, the quadrature orders have to be chosen according to*

$$n_l = O(\varrho L |\log \sigma|) \tag{65}$$

where $O(\cdot)$ is uniform in L and σ and ϱ is as in (14).

We sum up the foregoing considerations in

Proposition 24 *If the singular integrals (49) are transformed as in (54), the integration domain is subdivided according to (56), and Gaussian Quadrature is applied to the resulting integrals with orders (64),(65), the consistency estimate (42) holds. The total work for the consistent quadrature of all singular integrals is bounded by $W_L \leq CL^4(\varrho|\log\sigma| L)^5 = C(\varrho|\log\sigma|)^5 N_L^{2.25}$ kernel evaluations.*

3.4 Singular and near singular, edge-parallel case

The singular and near singular, edge-parallel case is characterized by the following condition. Let K_x, K_y be two non-identical panels of τ_L. The pull-backs on $\tilde{\tau}_L$ are given by $\tilde{K}_x := \eta^{-1}(K_x)$ and $\tilde{K}_y := \eta^{-1}(K_y)$. We assume that there are plane quadrangles with disjoint interior $\tilde{K}_x^\star$, $\tilde{K}_y^\star \subset \tilde{\Gamma}$ which share exactly one edge and have the property that, at least, three edges of $\tilde{K}_x$ belongs to $\partial\tilde{K}_x^\star$ and the same holds for $\tilde{K}_y$ and $\tilde{K}_y^\star$. We use the following conventions and notations. For $z \in \{x,y\}$, let $\{P_i^z\}_{1\leq i\leq 4}$ denote the vertices of $\tilde{K}_z^\star$ with $P_1^x = P_1^y$ and $P_2^x = P_4^y$. Let $\varepsilon_1 := \|P_2^x - P_1^x\|$, $d_x := \|P_4^x - P_1^x\|$, and $d_y := \|P_2^y - P_1^y\|$. Furthermore, let X_i, Y_i denote the vertices of K_x and K_y with $X_3 = P_3^x$, $X_4 = P_4^x$ and $Y_2 = P_2^y$, $Y_3 = P_3^y$. If $\overline{\tilde{K}_x} \cap \overline{\tilde{K}_x} \neq \emptyset$ we assume that $\tilde{K}_x = \tilde{K}_x^\star$ and $\tilde{K}_y = \tilde{K}_y^\star$. Figure 3 illustrates the situation.

We have to design the quadrature formula such that they are robust with respect to any possible aspect ratio of ε_1, d_2^x, and d_2^y. Let the extended reference domains be defined by

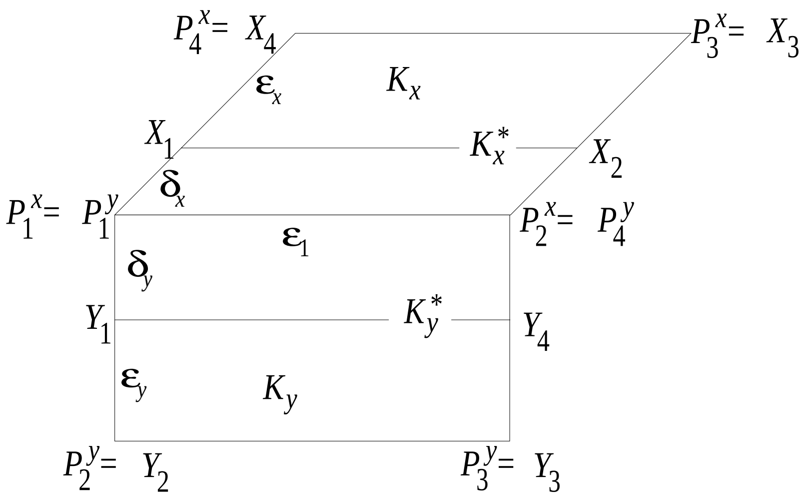

Figure 3: Near-singular, edge-parallel case.

$\hat{K}_x^\star := (0, \varepsilon_1) \times (0, d_x)$ and $\hat{K}_y^\star := (0, \varepsilon_1) \times (-d_y, 0)$. The affine bilinear mappings

$$\begin{aligned}
\tilde{\kappa}_x (u) \quad &: \quad = P_1^x + \frac{u_1}{\varepsilon_1} (P_2^x - P_1^x) + \frac{u_2}{d_x} (P_4^x - P_1^x) + \frac{u_1 u_2}{\varepsilon_1 d_x} (P_1^x - P_2^x + P_3^x - P_4^x) \\
\tilde{\kappa}_y (v) \quad &: \quad = P_1^x + \frac{v_1}{\varepsilon_1} (P_2^x - P_1^x) - \frac{v_2}{d_y} (P_2^y - P_1^x) - \frac{v_1 v_2}{\varepsilon_1 d_y} (P_1^x - P_2^y + P_3^y - P_2^x)
\end{aligned}$$

transporting $\hat{K}_{x,y}^\star$ onto $\tilde{K}_{x,y}^\star$ define the reference domains $\hat{K}_{x,y} := \kappa_{x,y}^{-1} \left(\tilde{K}_{x,y} \right)$. Due to Assumption 4 we may assume for the following that

$$\hat{K}_x = (0, \varepsilon_1) \times (\delta_x, \delta_x + \varepsilon_x), \quad \hat{K}_x = (0, \varepsilon_1) \times (-\delta_y - \varepsilon_y, -\delta_y)$$

with $\varepsilon_x := \|X_4 - X_1\|$, $\varepsilon_y := \|Y_2 - Y_1\|$ and $\delta_{x,y} := d_{x,y} - \varepsilon_{x,y}$. The composite mappings $\kappa_{x,y} := \eta \circ \tilde{\kappa}_{x,y}$ map $\hat{K}_{x,y}$ onto the surface elements. Note that, due to the chosen scaling, $\kappa_{x,y}$ does not depend on the side lengths of $\tilde{K}_{x,y}$ but only on the angles of $\tilde{K}_{x,y}$ and the mapping η. The kernel in local coordinates is given by

$$k_{loc} (u, v) := k (\kappa_x (u), \kappa_y (v)) \tag{66}$$

and the combination of the basis functions defines

$$B (u, v) := B_{\alpha, \alpha'} (u, v) := g_x (u) \, g_y (v) \left(\varphi_\alpha^0 \circ \hat{\kappa}_{K_x} \right) (u) \left(\varphi_\alpha^0 \circ \hat{\kappa}_{K_y} \right) (v) / \sqrt{\left| \tilde{K}_x \right| \left| \tilde{K}_y \right|}$$

with the affine linear mapping $\hat{\kappa}$ defined as in the previous section. The integral in parameter coordinates takes the form

$$I_e = \int_{u \in \hat{K}_x} \int_{v \in \hat{K}_y} k_{loc} (u, v) \, B (u, v) \, dv du. \tag{67}$$

3.4.1 Regularizing coordinate transformations

As in the previous section, we will first transform the integral into a sum of integrals over domains Q_j having a proper distance from the singularity and a further domain where the kernel function is singular in a vertex such that simplex coordinates render the integrand analytic. Due to the choice of the local coordinate system and the assumption that η is bi-Lipschitz continuous it follows that the kernel function is singular if and only if $u = v$, i.e., $v_1 - u_1 = u_2 = v_2 = 0$. Hence, we employ here one-dimensional relative coordinates and write

$$z_1 = v_1 - u_1, \quad z_2 = v_2, \quad z_3 = u_2. \tag{68}$$

The domain of integration is described by the following system of inequalities

$$\begin{array}{ll} 0 \leq u_1 \leq \varepsilon_1, & \delta_x \leq z_3 \leq \delta_x + \varepsilon_x \\ -u_1 \leq z_1 \leq \varepsilon_1 - u_1, & -\delta_y - \varepsilon_y \leq z_2 \leq -\delta_y. \end{array}$$

We exchange the order of integrations in u_1 and z_1 (this is justified by Fubini's theorem; observe that k_{loc} in (66) belongs to $L^1(\hat{K}_x \times \hat{K}_y)$ for $\delta_x + \delta_y \geq 0$, see [21]). An equivalent description of the parameter domain is given by $D_1 \cup D_2$

$$D_1 = \begin{cases} -\varepsilon_1 \leq z_1 \leq 0, \\ -\delta_y - \varepsilon_y \leq z_2 \leq -\delta_y, \\ \delta_x \leq z_3 \leq \delta_x + \varepsilon_x, \\ -z_1 \leq u_1 \leq \varepsilon_1, \end{cases} \qquad D_2 = \begin{cases} 0 \leq z_1 \leq \varepsilon_1, \\ -\delta_y - \varepsilon_y \leq z_2 \leq -\delta_y, \\ \delta_x \leq z_3 \leq \delta_x + \varepsilon_x, \\ 0 \leq u_1 \leq \varepsilon_1 - z_1. \end{cases} \tag{69}$$

The integral takes the form

$$I_e = \sum_{j=1}^{2} \int_{D_i} k_{loc}\left(\binom{u_1}{z_3}, \binom{z_1+u_1}{z_2}\right) B\left(\binom{u_1}{z_3}, \binom{z_1}{z_2}\right) du_1 dz$$

The integrand above defines the function $H(u_1, z)$. Replacing z_1 by $-\hat{z}_1$ and u_1 by $\hat{u}_1 + \hat{z}_1$ in the first integral maps D_1 onto D_2. We obtain

$$I_e = \int_{D_2} \left(H(\hat{u}_1 + \hat{z}_1, \hat{u}_1, \hat{z}_2, \hat{z}_3) + H(\hat{u}_1, -\hat{z}_1, \hat{z}_2, \hat{z}_3)\right) d\hat{u}_1 d\hat{z}$$

We henceforth omit the hat from the variables u and z and denote D_2. In order to obtain a four-dimensional tensor product domain we replace u_1 by $u_1(\eta, z_1) := \eta(\varepsilon_1 - z_1)$ and obtain

$$\begin{aligned} I_e \quad = \quad & \int_0^{\varepsilon_1} \int_{-\delta_y-\varepsilon_y}^{-\delta_y} \int_{\delta_x}^{\delta_x+\varepsilon_x} \int_0^1 (\varepsilon_1 - z_1)\left\{H(u_1(\eta, z_1) + z_1, -z_1, z_2, z_3)\right. \\ & \left. + H(u_1(\eta, z_1), -z_1, z_2, z_3)\right\} d\eta d\hat{z}. \end{aligned} \tag{70}$$

The integrand above defines the function $\tilde{H}(\eta, z)$. The region Q of the $z-$integration is depicted in Figure 4. We remark that

$$\operatorname{dist}(Q, 0) \geq \sqrt{\delta_x^2 + \delta_y^2} =: \delta. \tag{71}$$

The integrand $\tilde{H}$ in (70) is analytic on $\overline{Q \times (0,1)}$ but has a (near) singularity at $z = 0$ (see Figure 4). This and the problem of high aspect ratio are overcome by a judicious subdivision of Q which we describe next (see also Figure 4 and Remark 17). Our aim is to split Q into subdomains Q_j such that $\operatorname{diam} Q_j \sim \operatorname{dist}(Q_j, 0)$ and, for $\delta = 0$, a cube $Q_\star$ which contains the singularity at a vertex and has sides of comparable length, uniformly in L.

Without loss of generality we assume that $\varepsilon_1 \geq \varepsilon_x \geq \varepsilon_y$. Otherwise, one has to permute the indices in the formulae below. Let $M_1 := \max(\delta, \varepsilon_x)$ and $M_2 := \max(\delta, \varepsilon_y)$, $i_0 := \left\lfloor \log_2 \frac{\varepsilon_1}{M_1} \right\rfloor - 1$

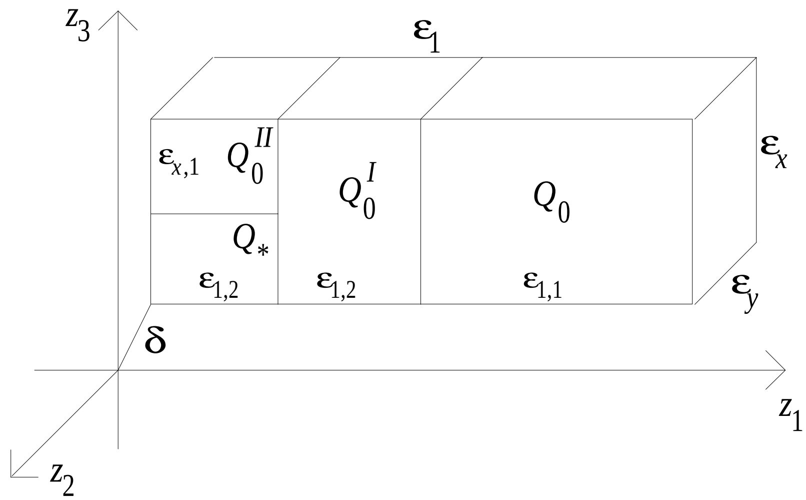

Figure 4: Subdivison of the domain Q into Q_i, $Q_j^{I,II}$ and $Q_\star$.

and $j_0 := \left\lfloor \log_2 \frac{\varepsilon_x}{M_2} \right\rfloor - 1$. For $z \in \{1, x, y\}$, we put $\varepsilon_{z,i} := 2^{-i}\varepsilon_z$. Define a sequence of domains for $i = 0, 1, 2, \ldots, i_0$ and $j = 0, 1, 2, \ldots, j_0$.

$$\begin{aligned}
Q_i \quad &: \quad = (\varepsilon_{1,i+1}, \varepsilon_{1,i}) \times (-\delta_y - \varepsilon_y, \delta_y) \times (\delta_x, \delta_x + \varepsilon_x)\,, \\
Q_j^I \quad &: \quad = (\varepsilon_{1,j+i_0+2}, \varepsilon_{1,j+i_0+1}) \times (-\delta_y - \varepsilon_y, \delta_y) \times (\delta_x, \delta_x + \varepsilon_{x,j}) \\
Q_j^{II} \quad &: \quad = (0, \varepsilon_{1,j+i_0+2}) \times (-\delta_y - \varepsilon_y, \delta_y - \varepsilon_y) \times (\delta_x + \varepsilon_{x,j+1}, \delta_x + \varepsilon_{x,j}) \\
Q_\star \quad &: \quad = (0, \varepsilon_{1,j_0+i_0+2}) \times (-\delta_y - \varepsilon_y, \delta_y) \times (\delta_x, \delta_x + \varepsilon_{x,j_0+1})
\end{aligned}$$

If $i_0 < 0$ and/or $j_0 < 0$, no subdivisions in the respective direction are performed. One verifies that

$$\frac{\operatorname{dist}(D, 0)}{\operatorname{diam} D} \geq C > 0, \qquad \forall D \in \left\{ Q_i, Q_j^I, Q_j^{II} \right\}. \tag{72}$$

for $0 \leq i < i_0$, $0 \leq j \leq j_0$ where C does not depend on ε_i and δ. Due to the geometric refinement algorithm we know that (72) holds also for $D = Q_\star$ if $\delta > 0$. The integral takes the following form

$$\begin{aligned}
I_e \quad = \quad & \sum_{i=0}^{i_0} \int_{Q_i} \int_0^1 \tilde{H}(\eta, z)\, d\eta dz + \sum_{j=0}^{j_0} \left(\int_{Q_j^I} \int_0^1 \tilde{H}(\eta, z)\, d\eta dz + \int_{Q_j^{II}} \int_0^1 \tilde{H}(\eta, z)\, d\eta dz \right) \\
& + \int_{Q_\star} \int_0^1 \tilde{H}(\eta, z)\, d\eta dz. \qquad (73)
\end{aligned}$$

We remark that some of the sums in (73) may vanish, depending on the size of δ. For $\delta = 0$ the integrand of the last integral of (73) is singular. We have to apply additional coordinate transform in that case. This is discussed in the sequel.

As mentioned above the side lengths of $Q_\star$ are of the same magnitude, namely, ε_x. Hence,

$$\hat{H}(\eta, z) := \frac{\varepsilon_{1,j_0+i_0+2}\varepsilon_{x,j_0+1}}{\varepsilon_y^2} \tilde{H}\left(\eta, \frac{\varepsilon_{1,j_0+i_0+2}}{\varepsilon_y} z_1, z_2, \frac{\varepsilon_{x,j_0+1}}{\varepsilon_y} z_3 \right) \tag{74}$$

has the same behaviour as $\tilde{H}$. We split $Q_\star$ furthermore according to

$$\int_{Q_\star}\int_0^1 \tilde{H}(\eta,z)\,d\eta dz = \int_0^{\varepsilon_y}\int_0^{\varepsilon_y}\int_0^{\varepsilon_y}\int_0^1 \hat{H}(\eta,z)\,d\eta dz$$
$$= \int_0^{\varepsilon_y}\int_0^{z_1}\int_0^{z_1}\int_0^1 \hat{H}(\eta,z)\,d\eta dz + \int_0^{\varepsilon_y}\int_0^{z_2}\int_0^{z_2}\int_0^1 \hat{H}(\eta,z)\,d\eta dz_1 dz_3 dz_2$$
$$+\int_0^{\varepsilon_y}\int_0^{z_3}\int_0^{z_3}\int_0^1 \hat{H}(\eta,z)\,d\eta dz_2 dz_1 dz_3.$$

The simplicial integration domain for the z-integration contains 0 as a vertex. Hence, we apply simplex coordinates

$$\begin{aligned} z^{(1)}(\hat{z}) &:= \hat{z}_1(1,\hat{z}_2,\hat{z}_3)^T && \text{for the first integral,}\\ z^{(2)}(\hat{z}) &:= \hat{z}_2(\hat{z}_1,1,\hat{z}_3)^T && \text{for the second integral,}\\ z^{(3)}(\hat{z}) &:= \hat{z}_3(\hat{z}_1,\hat{z}_2,1)^T && \text{for the last integral}\end{aligned}$$

and obtain

$$\int_{Q_\star}\int_0^1 \tilde{H}(\eta,z)\,d\eta dz = \int_0^{\varepsilon_y}\int_{(0,1)^3}\hat{z}_1^2\hat{H}\left(\eta,z^{(1)}(\hat{z})\right)d\eta d\hat{z} \tag{75}$$
$$+\int_0^{\varepsilon_y}\int_{(0,1)^3}\hat{z}_2^2\hat{H}\left(\eta,z^{(2)}(\hat{z})\right)d\eta d\hat{z}_1 d\hat{z}_3 d\hat{z}_2 + \int_0^{\varepsilon_y}\int_{(0,1)^3}\hat{z}_3^2\hat{H}\left(\eta,z^{(3)}(\hat{z})\right)d\eta d\hat{z}_2 d\hat{z}_1 d\hat{z}_3.$$

It will turn out that the integrands above are analytic and can be approximated by Gauß-Legendre formulae. Summarizing the transformations above we have shown that I_e can be written in the form (73) with analytic integrands for the integration over Q_i, Q_j^I, Q_j^{II} and either analytic integrand for the integration over $Q_\star$ or, in the case of $\delta = 0$, analytic integrands in the representation (75). Up to now no numerical approximation has been applied but only regularizing coordinate transforms. The splitting (73) and (75) is an *exact representation* of the initial integral. In the next subsection we will discuss numerical quadrature along with the corresponding error analysis.

3.4.2 Numerical quadrature and error analysis

For the approximation of each integral we employ tensor product Gauß-Legendre formulas $G_{z_1}^{n_1}G_{z_2}^{n_2}G_{z_3}^{n_3}G_\eta^{n_4}$ of orders n_l possibly different for different variables and different integration domains. For $i = 0, 1, \dots i_0$, we define the quadrature error $E_{i,1}$ by

$$E_{i,1} := |Q_1^c| \max_{(z_1^c,\eta)\in Q_1^c\times(0,1)}\left|\int_{\varepsilon_{1,i+1}}^{\varepsilon_{1,i}}\tilde{H}(\eta,z)\,dz_1 - G^{n_1}_{z_1,(\varepsilon_{1,i+1},\varepsilon_{1,i})}\tilde{H}(\eta,z)\right|,$$

with $Q_1^c = (-\delta_y - \varepsilon_y, \times -\delta_y)\times(\delta_x, \delta_x+\varepsilon_x)$, while the remaining errors, $E_{i,l}$, $E_{j,l}^I$, $E_{j,l}^{II}$, $E_l^\star$, $l \in \{1,2,3,4\}$, are defined analogously. The total quadrature error E will be estimated, using Proposition 14 as in (60), by a sum of 1-dimensional quadrature errors:

$$\left|I_e - \tilde{I}_e\right| \le C\sum_{l=1}^4\left\{\left(\sum_{i=0}^{i_0-1}E_{i,l}\right) + \left(\sum_{j=0}^{j_0-1}\sum_{R\in\{I,II\}}E_{j,l}^R\right) + E_l^\star\right\}. \tag{76}$$

In the singular case, i.e. $\delta = 0$, the error $E_l^\star$ has to be split into $E_l^\star = \sum_{k=1}^3 E_{k,l}^\star$ where, for $k \in \{1,2,3\}$, $E_{k,l}^\star$ corresponds to the kth integral of (75). We estimate these errors with the aid of Propositions 13 and 15.

Theorem 25 *Let $\delta > 0$. This implies in particular that $\delta \geq C\varepsilon_y$. Then the quadrature errors can be estimated by*

$$E_{i,l} \leq C\pi_{K_x}\pi_{K_y}\eta\sqrt{\frac{\varepsilon_y}{\varepsilon_x}}\,(1+\lambda_l\gamma)^{q_l-2n_l}$$

where the constants γ and C depend only on η, $\tilde{\tau}_0$ and the angles of $\tilde{K}_x$, $\tilde{K}_y$. The numbers η, q_l and λ_l are given by $\eta = \varepsilon_x/\varepsilon_{1,i}$ and

$$q_l = \begin{cases} p_1^{K_x} + p_1^{K_y} + 1 & z_1\text{-integration}, \\ p_2^{K_y} & z_2\text{-integration}, \\ p_1^{K_x} & z_3\text{-integration}, \\ p_1^{K_x} + p_1^{K_y} & \eta\text{-integration}, \end{cases} \qquad \lambda_l = \begin{cases} 1 & l=1, \\ \varepsilon_{1,i+1}/\varepsilon_y & l=2, \\ \varepsilon_{1,i+1}/\varepsilon_x & l=3, \\ 1/\varepsilon_1 & l=4. \end{cases}$$

The quadrature errors corresponding to the domains $Q_j^{I,II}$ are given by replacing $\varepsilon_{1,i+1}$ and ε_x by $\varepsilon_{x,j+1}$. The quadrature errors for the domain $Q_\star$ are given by replacing $\varepsilon_{1,i+1}$ and ε_x by ε_y. In all those cases, η equals 1.

Proof. Let us first consider the error $E_{i,1}$ of the z_1-integration for the domain Q_i, i.e.,$E_{i,1}$. In order to apply Proposition 13 we have to transport the integral to the unit interval $(-1,1)$ by $z_1 = z_1(t) = \frac{\varepsilon_{1,j+1}}{2}(t+3)$ yielding

$$E_{i,1} := \frac{\varepsilon_{1,i+1}\varepsilon_x\varepsilon_y}{2} \max_{\left(z_1^c,\eta\right)\in Q_1^c\times(0,1)} \left|\int_{-1}^{1} \tilde{H}(\eta,z)\,dt - G^{n_1}_{t,(-1,1)}\tilde{H}(\eta,z)\right|.$$

The integrands above were defined in (70) by

$$\tilde{H}(\eta,z) = (\varepsilon_1 - z_1)\left\{H(u_1+z_1,u_1,z_2,z_3) + H(u_1,-z_1,z_2,z_3)\right\}$$

with $u_1 = \eta(\varepsilon_1 - z_1)$. In the following we consider only the first summand above since the second one has the same behaviour. Tracing back the coordinate transform we obtain

$$\tilde{H}_1(\eta,z) = (\varepsilon_1 - z_1)\,k_{loc}\left(\binom{u_1+z_1}{z_3},\binom{u_1}{z_2}\right) B\left(\binom{u_1+z_1}{z_3},\binom{u_1}{z_2}\right). \tag{77}$$

In [20, Lemma 4.2] it was shown that $\tilde{H}_1(\eta,z)$ can be extended analytically to $D^{\rho_1} := \mathcal{E}^{\rho_1}_{-1,1} \times Q_1^c \times (0,1)$ with $\rho_1 = 1 + \gamma\frac{\operatorname{dist}(Q_i,0)}{\varepsilon_{1,i+1}} = 1+\gamma$, while k_{loc} can be estimated by

$$\max_{\left(t,z_1^c,\eta\right)\in D^{\rho_1}} \left|k_{loc}\left(\binom{u_1+z_1}{z_3},\binom{u_1}{z_2}\right)\right| \leq \frac{C}{\varepsilon^2_{1,i+1}}.$$

The estimate

$$\max_{\left(t,z_1^c,\eta\right)\in D^{\rho_1}} \left|B\left(\binom{u_1+z_1}{z_3},\binom{u_1}{z_2}\right)\right| \leq C\frac{\pi_{K_x}\pi_{K_y}}{\varepsilon_1\sqrt{\varepsilon_x\varepsilon_y}}\,(1+\gamma)^{p_1^{K_x}+p_1^{K_y}} \tag{78}$$

follows as in the proof of Theorem 18. The leading factor $(\varepsilon_1 - z_1)$ is estimated by $\varepsilon_1(1+\gamma)$. Summarizing we have shown that

$$E_{i,1} := C\pi_{K_x}\pi_{K_y}\frac{\sqrt{\varepsilon_x\varepsilon_y}}{2\varepsilon_{1,i+1}}\,(1+\gamma)^{p_1^{K_x}+p_1^{K_y}+1-2n_1}.$$

The proof for the remaining variables is the same. However, due to the possibly different side lengths of Q_i as, e.g. $\varepsilon_x \ll \varepsilon_1$, the sum of the semiaxes might be larger. We have $\rho_2 = 1 + \gamma \operatorname{dist}(Q_i, 0)/\varepsilon_y = 1 + \gamma\varepsilon_{1,i+1}/\varepsilon_y$ and analogously, $\rho_3 = 1 + \gamma\varepsilon_{1,i+1}/\varepsilon_x$, $\rho_4 = 1 + \gamma/\varepsilon_1$. The leading factor $(\varepsilon_1 - z_1)$ in these cases can always be estimated by ε_1. The different powers q_l of the assertion correspond to different exponents in (78) for the different variables.

The estimates for the integrals over $Q_j^{I,II}$ are just a repetition of the arguments taking the modified side lengths and distances into account. ■

In order to adapt the quadrature error to the required consistency one has to chose the orders of the Gauß formulae such that $E \leq CN_L^{-1}L\sigma^{\varrho L}$. In view of the sum (76) this is guaranteed if

$$E_{i,l} \leq CN_L^{-1}L\sigma^{\varrho L}/i_0, \quad E_{j,l}^{I,II} \leq CN_L^{-1}L\sigma^{\varrho L}/j_0, \quad E_\star \leq CN_L^{-1}L\sigma^{\varrho L}. \tag{79}$$

Together with the above error bounds, this gives rules for the minimum number of Gauss points to be used. We recommend to use these conditions together with the error estimates above to determine the precise quadrature order. However, for an investigation of the asymptotic complexity, we simplify the bounds on the quadrature orders as follows.

Proposition 26 *For L large enough, the orders of the integration have to be chosen according to*

$$n = O\left(\varrho \left|\log\sigma\right| L\right).$$

Proof. The assertion follows by the same arguments as in the proof of Proposition 20. ■

We come now to the integral over $Q_\star$ in the case that $\delta = 0$. As mentioned before, this integral is approximated by replacing the integrals in (75) by Gauß-Legendre formulae.

Theorem 27 *Let $\delta = 0$ and consider the approximation of the integrals (75) by Gauß-Legendre formulae $G_{z_1}^{n_1}G_{z_2}^{n_2}G_{z_3}^{n_3}G_\eta^{n_4}$. The corresponding errors are denoted by $E_{k,l}^\star$, $l \in \{1,2,3,4\}$ and $k \in \{1,2,3\}$. Then the estimate*

$$E_{1,l}^\star \leq C\pi_{K_x}\pi_{K_y}\sqrt{\frac{\varepsilon_y}{\varepsilon_x}}\left(1 + \lambda_l\gamma\right)^{q_l - 2n_1}$$

holds with

$$\lambda_l = \begin{cases} 1/\varepsilon_y & z_1\text{-integration,} \\ 1 & z_{2,3}\text{-integration,} \\ 1/\varepsilon_1 & \eta\text{-integration,} \end{cases} \qquad q_l = \begin{cases} \left|p^{K_x} + p^{K_y}\right| + 1 & l = 1, \\ p_2^{K_y} & l = 2, \\ p_2^{K_x} & l = 3, \\ p_1^{K_x} + p_1^{K_y} & l = 4. \end{cases}$$

The estimates of $E_{k,l}^\star$ are the same as above but the indices of $\hat{z}_i$ have to be interchanged appropriately.

Proof. Due to the chosen scaling we may assume for the quadrature error analysis that, in view of (74), $\varepsilon_y = \varepsilon_{1,i_0+j_0+2} = \varepsilon_{x,j_0+1}$ holds. Let $k = 1$ and consider the $\hat{z}_1$-integration. We scale the domain of integration onto $(-1,1)$ by $\hat{z}_1(t) := \frac{\varepsilon_y}{2}(t+1)$. Analogously as in the case of identical panels, one proves that the determinant of the simplex coordinates, i.e. $\hat{z}_1^2$, cancels the $\|z\|^{-2}$ singularity of the kernel function rendering the integrand analytic in any compact neighborhood of the integration interval. Hence, we choose the domain of analyticity by $D^{\rho_1} := \mathcal{E}_{-1,1}^{\rho_1} \times (0,1)^3$ with $\rho_1 = 1 + \gamma/\varepsilon_y$. As in the previous proof it sufficient to consider the function $\hat{z}_1^2 \tilde{H}_1 \left(\eta, z^{(1)}(\hat{z})\right)$ of (77). The factor $k_{loc}(\cdot,\cdot)$ is bounded on D^{ρ_1} by a constant independent of $\varepsilon_1, \varepsilon_x$, and ε_y. We have to investigate the arguments of the basis functions. All of them lie in the ellipse $\mathcal{E}_{-1,1}^{\rho_1}$ yielding with Proposition 15

$$\max_{\left(t,\hat{z}_1^c,\eta\right)\in D^{\rho_1}} \left|B\left(\begin{pmatrix}\eta(\varepsilon_1 - z_1)+z_1 \\ z_1 z_3\end{pmatrix}, \begin{pmatrix}\eta(\varepsilon_1 - z_1) \\ z_2 z_1\end{pmatrix}\right)\right| \leq C\frac{\pi_{K_x}\pi_{K_y}}{\varepsilon_1\sqrt{\varepsilon_x\varepsilon_y}}\left(1+\gamma/\varepsilon_y\right)^{p^{K_x}+p^{K_y}}. \tag{80}$$

The factor $(\varepsilon_1 - \hat{z}_1)$ can be estimated by $\varepsilon_1(1+\gamma/\varepsilon_y)$ yielding

$$E_{1,1}^{\star} \leq C\pi_{K_x}\pi_{K_y}\sqrt{\frac{\varepsilon_y}{\varepsilon_x}}\left(1+\gamma/\varepsilon_y\right)^{p^{K_x}+p^{K_y}+1-2n_1}.$$

The estimate for the other variables is just a repetition of the arguments above. However, due to the scaling of the $\hat{z}_2$ and $\hat{z}_3$-integration, the sum of the semi-axes is only $\rho_{2,3} = 1+\gamma$, while for the η-integration we have $\rho_4 = 1 + \gamma/\varepsilon_1$. On the other hand, the powers of ρ_l in (80) are reduced. We obtain

$$\max_{\left(t,\hat{z}_l^c,\eta\right)\in D^{\rho_l}} \left|B\left(\begin{pmatrix}\eta(\varepsilon_1 - \hat{z}_1)+\hat{z}_1 \\ \hat{z}_1 \hat{z}_3\end{pmatrix}, \begin{pmatrix}\eta(\varepsilon_1 - z_1) \\ \hat{z}_2 \hat{z}_1\end{pmatrix}\right)\right| \leq C\frac{\pi_{K_x}\pi_{K_y}}{\varepsilon_1\sqrt{\varepsilon_x\varepsilon_y}}\left(1+\lambda_l\gamma\right)^{q_l}, l = 1,2,3,4,$$

with $q_2 = p_2^{K_y}$, $q_3 = p_2^{K_x}$, $q_4 = p_1^{K_x} + p_1^{K_y}$. The leading factor $(\varepsilon_1 - \hat{z}_1)$ can be estimated by ε_1.

The estimates for the error $E_{k,l}$, $k \in \{2,3\}$ can be derived by a cyclic permutation of the indices of the $\hat{z}_i$-integration. ■

As already mentioned the quadrature orders have to be chosen such that $E_{k,l}^{\star} \leq CN_L^{-1}L\sigma^{\varrho L}$ holds. Repeating the arguments of Proposition 20 it follows that, for L large enough, the quadrature orders n behave like $n = O(\varrho L|\log\sigma|)$.

In summary, we have shown

Proposition 28 *All singular and near singular integrals $A_{II'}$ in (67) can be computed with variable order, composite quadratures based on the subdivision (73) and (75) with quadrature orders given, for the non-singular cases, by Theorem 25 in combination with (79) and for the singular case by Theorem 27 to the accuracy (42) with work $W \leq CL^8(\varrho L|\log\sigma|)^5 = C(\varrho|\log\sigma|)^5 N_L^{3.25}$ kernel evaluations. Here C depends on σ and ϱ, but is independent of L.*

3.5 Singular, near-singular and regular farfield case

3.5.1 Regularizing coordinate transforms

Let $K_x, K_y \in \tau_L$ be two panels which are neither identical nor belong to the edge parallel singular or near singular case discussed before. This implies in particular that $\overline{K_x} \cap \overline{K_y}$ is either empty or a vertex. In this section we will consider the approximation of the integrals

$$I_p := \int_{K_x\times K_y} k(x,y)\,\varphi_\alpha^{K_x}(x)\,\varphi_{\alpha'}^{K_y}(y)\,dydx.$$

Let the pullbacks on $\tilde{\Gamma}$ be denoted by $\tilde{K}_{x,y} = \eta^{-1}(K_{x,y})$ while $\{X_i\}_{1\leq i\leq 4}$, $\{Y_i\}_{1\leq i\leq 4}$ are the vertices of $\tilde{K}_{x,y}$ (counterclockwise ordering). The following definitions are illustrated in Figure 5.

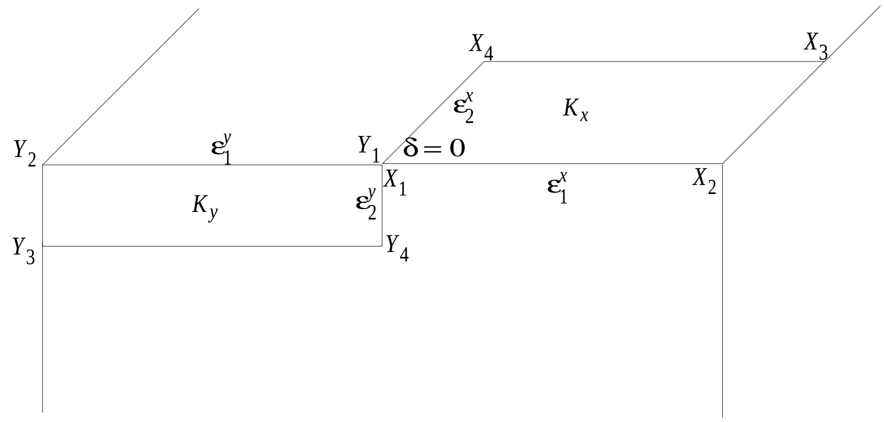

Figure 5: Singular farfield case.

Without loss of generality we assume that $\delta := \operatorname{dist}\left(\tilde{K}_x, \tilde{K}_y\right) = \|X_1 - Y_1\|$. Let $\varepsilon_1^x := \|X_2 - X_1\|$, $\varepsilon_2^x := \|X_4 - X_1\|$, and $\varepsilon_1^y := \|Y_2 - Y_1\|$, $\varepsilon_2^y := \|Y_4 - Y_1\|$. The parameter domains in the plane are given by $\hat{K}_x = (0, \varepsilon_1^x) \times (0, \varepsilon_2^x)$ and $\hat{K}_y := (-\delta - \varepsilon_1^y, -\delta) \times (0, -\varepsilon_2^y)$. The affine bilinear mappings $\tilde{\kappa}_{x,y} : \hat{K}_{x,y} \to \tilde{K}_{x,y}$ are given by

$$\begin{aligned} \tilde{\kappa}_x(u) &= X_1 + \frac{u_1}{\varepsilon_1^x}(X_2 - X_1) + \frac{u_2}{\varepsilon_2^x}(X_4 - X_1) + \frac{u_1 u_2}{\varepsilon_1^x \varepsilon_2^x}(X_1 - X_2 + X_3 - X_4), \\ \tilde{\kappa}_y(u) &= Y_1 - \frac{(v_1 + \delta)}{\varepsilon_1^y}(Y_2 - Y_1) - \frac{v_2}{\varepsilon_2^y}(X_4 - X_1) + \frac{(v_1 + \delta) v_2}{\varepsilon_1^y \varepsilon_2^y}(X_1 - X_2 + X_3 - X_4). \end{aligned}$$

The transformation onto the surface panels is then given by $\kappa_{x,y} := \eta \circ \tilde{\kappa}_{x,y}$ and is independent of the side lengths of $K_{x,y}$. It depends only on the angles of $\tilde{K}_{x,y}$ and the mapping η. The kernel in local coordinates is defined by

$$k_{loc}(u, v) := k(\kappa_x(u), \kappa_y(v)).$$

The local kernel is analytic if $\delta > 0$. For $\delta = 0, k_{loc}$ is singular if and only if $u = v$, i.e., $u = v = 0$. Hence the relative coordinates in this case reduces to a renaming of the variables. For $l = 1, 2$ we set

$$z_l = u_l, \quad z_{2+l} = v_l.$$

The combination of the basis functions with the determinants of the Jacobi matrices defines the function $B(u, v)$ as in the previous section. In order to simplify the notation we write $k_{loc}(z)$ instead of $k_{loc}\left(\binom{z_1}{z_2}, \binom{z_2}{z_3}\right)$ and $B(z)$ is defined analogously. In local coordinates the integral I_p takes the form

$$I_p := \int_{\hat{K}_x \times \hat{K}_y} k_{loc}(z)\, B(z)\, dz.$$

We first have to split the integration domain into subdomains Q_i such that either $(\operatorname{diam} Q_i) \sim \operatorname{dist}(Q_i, 0)$ or 0 is a vertex of Q_i. Without loss of generality we assume that $\varepsilon_1^x \geq \varepsilon_1^y \geq \varepsilon_2^x \geq \varepsilon_2^y$. Let $M_1 := \max(\delta, \varepsilon_1^y)$, $M_2 := (\delta, \varepsilon_2^x)$, $M_3 := (\delta, \varepsilon_2^y)$ and define

$$i_0 := \left\lfloor \log_2 \tfrac{\varepsilon_1^x}{M_1} \right\rfloor - 1, \qquad j_0 := \left\lfloor \log_2 \tfrac{\varepsilon_1^y}{M_2} \right\rfloor - 1, \qquad k_0 := \left\lfloor \log_2 \tfrac{\varepsilon_2^x}{M_3} \right\rfloor - 1,$$

and define the domains $\{Q_i\}_{0\leq i\leq i_0}$, $\left\{Q_j^{I,II}\right\}_{0\leq j\leq j_0}$, $\left\{Q_j^{III,IV,V}\right\}_{0\leq k\leq k_0}$, $Q_\star$ by

$$\begin{aligned}
Q_i &:= \left(\varepsilon^x_{1,i+1},\varepsilon^x_{1,i}\right)\times(0,\varepsilon^x_2)\times(-\delta-\varepsilon^y_1,-\delta)\times(-\varepsilon^y_2,0),\\
Q_j^I &:= \left(0,\varepsilon^x_{j+i_0+1}\right)\times(0,\varepsilon^x_2)\times\left(-\delta-\varepsilon^y_{1,j},-\delta-\varepsilon^y_{1,j+1}\right)\times(-\varepsilon^y_2,0),\\
Q_j^{II} &:= \left(\varepsilon^x_{j+i_0+2},\varepsilon^x_{j+i_0+1}\right)\times(0,\varepsilon^x_2)\times\left(-\delta-\varepsilon^y_{1,j+1},-\delta\right)\times(-\varepsilon^y_2,0),\\
Q_j^{III} &:= \left(0,\varepsilon^x_{k+j_0+i_0+2}\right)\times\left(\varepsilon^x_{2,k+1},\varepsilon^x_{2,k}\right)\times\left(-\delta-\varepsilon^y_{1,k+j_0+1},-\delta\right)\times(-\varepsilon^y_2,0),\\
Q_j^{IV} &:= \left(0,\varepsilon^x_{k+j_0+i_0+2}\right)\times\left(0,\varepsilon^x_{2,k+1}\right)\times\left(-\delta-\varepsilon^y_{1,k+j_0+1},-\delta-\varepsilon^y_{1,k+j_0+2}\right)\times(-\varepsilon^y_2,0),\\
Q_j^{V} &:= \left(\varepsilon^x_{k+j_0+i_0+3},\varepsilon^x_{k+j_0+i_0+2}\right)\times\left(0,\varepsilon^x_{2,k+1}\right)\times\left(-\delta-\varepsilon^y_{1,k+j_0+2},-\delta\right)\times(-\varepsilon^y_2,0),\\
Q_\star &:= \left(0,\varepsilon^x_{k_0+j_0+i_0+3}\right)\times\left(0,\varepsilon^x_{2,k_0+1}\right)\times\left(-\delta-\varepsilon^y_{1,k_0+j_0+2},-\delta\right)\times(-\varepsilon^y_2,0),
\end{aligned}$$

with $\varepsilon^z_{k,m}:=2^{-m}\varepsilon^z_k$ for $z\in\{x,y\}$ and $k\in\{1,2\}$. The integral I_p takes the following form

$$\begin{aligned}
I_p &:= \sum_{i=0}^{i_0}\int_{Q_i}k_{loc}(z)\,B(z)\,dz+\sum_{j=0}^{j_0}\sum_{R=I}^{II}\int_{Q_j^R}k_{loc}(z)\,B(z)\,dz\\
&+\sum_{k=0}^{k_0}\sum_{R=III}^{V}\int_{Q_k^R}k_{loc}(z)\,B(z)\,dz+\int_{Q_\star}k_{loc}(z)\,B(z)\,dz.
\end{aligned}\tag{81}$$

By our splitting strategy we have guaranteed that

$$\frac{\operatorname{diam}Q}{\operatorname{dist}(Q,0)}\leq C,\qquad\forall Q\in\left\{Q_i,Q_j^{I,II},Q_k^{III,IV,V}\right\}$$

where i,j,k range as explained above. This estimate also holds for $Q_\star$ for $\delta>0$. If $\delta=0$, then, all side lengths of $Q_\star$ are of order ε^y_2. Note that, if the distance δ of the panels is large compared to the side lengths, i.e. $\delta\geq\varepsilon^x_1$, the splitting (81) reduces to the integral over $Q_\star=\hat{K}_x\times\hat{K}_y$. All integrands in (81) are analytic except the last one if $\delta=0$. The integrand is singular in $Q_\star$ if and only if $\delta=0$. In that case we have to split the integration furthermore.

For the following, we assume that $\delta=0$. Since the side lengths of $Q_\star$ are of equal magnitude, the function

$$\begin{aligned}
\hat{H}(\hat{z}) &:= \frac{\varepsilon^x_{1,i_0+j_0+k_0+3}\varepsilon^x_{2,k_0+1}\varepsilon^y_{1,i_0+j_0+2}}{\left(\varepsilon^y_2\right)^3}k_{loc}(z(\hat{z}))\,B(z(\hat{z}))\\
z(\hat{z}) &= \left(\frac{\varepsilon^x_{1,i_0+j_0+k_0+3}}{\varepsilon^y_2}\hat{z}_1,\frac{\varepsilon^x_{2,k_0+1}}{\varepsilon^y_2}\hat{z}_2,\frac{\varepsilon^y_{1,i_0+j_0+2}}{\varepsilon^y_2}\hat{z}_3,\hat{z}_1,\right)^T
\end{aligned}$$

has the same (singular) behaviour as the integrand of $Q_\star$. The integration domain is split according to

$$\int_{Q_\star}k_{loc}(z)\,B(z)\,dz=\int_{\left(0,\varepsilon^y_2\right)^4}\hat{H}(\hat{z})\,d\hat{z}$$

$$=\int_0^{\varepsilon^y_2}\int_0^{z_1}\int_0^{z_1}\int_0^{z_1}\hat{H}(\hat{z})\,d\hat{z}+\int_0^{\varepsilon^y_2}\int_0^{z_2}\int_0^{z_2}\int_0^{z_2}\hat{H}(\hat{z})\,d\hat{z}_1d\hat{z}_4d\hat{z}_3d\hat{z}_2$$

$$+\int_0^{\varepsilon^y_2}\int_0^{z_3}\int_0^{z_3}\int_0^{z_3}\hat{H}(\hat{z})\,d\hat{z}_2d\hat{z}_1d\hat{z}_4d\hat{z}_3+\int_0^{\varepsilon^y_2}\int_0^{z_4}\int_0^{z_4}\int_0^{z_4}\hat{H}(\hat{z})\,d\hat{z}_3d\hat{z}_2d\hat{z}_1d\hat{z}_4.$$

For the kth integral, $k \in \{1,2,3,4\}$, we introduce 4-dimensional simplicial coordinates by

$$\hat{z}_i^{(k)}(\xi) = \begin{cases} \xi_i & \text{for } i = k, \\ \xi_k \xi_i & \text{otherwise}, \end{cases} \qquad \text{for } 1 \leq i \leq 4.$$

This leads us to the representation

$$\int_{Q_\star} k_{loc}(z) B(z) dz = \sum_{k=1}^{4} \int_0^{\varepsilon_2^y} \int_{(0,1)^3} \xi_k^3 \hat{H}\left(\hat{z}^{(k)}(\xi)\right) d\xi_k^c d\xi_k \tag{82}$$

with ξ_k^c defined by (45). We will see that the integrands on the right hands side are analytic and, hence, Gauß-Legendre formulae will converge exponentially. Summarizing the transformations above we conclude that the initial integral I_p can be split into (81) where either all integrands are analytic or, in the case of $\delta = 0$, after replacing the integral over $Q_\star$ by (82) all integrands are analytic. Again, we emphasize that, up to this point, no numerical approximation of the integral has been applied, but only regularizing splittings and coordinate transforms.

3.5.2 Numerical quadrature and error estimates

All integrals in (81) except the last one are approximated with tensor Gauß-Legendre quadrature with possibly different orders for different variables and integration domains. The integral over $Q_\star$ is replaced by Gauß-Legendre quadrature, too, if $\delta > 0$. Otherwise the representation (82) is employed and the four integrals are replaced by Gauß-Legendre quadrature. The convergence of the formulae are considered in the following. For $i = 0, 1, \ldots i_0$, we define the quadrature error $E_{i,1}$ by

$$E_{i,1} := |Q_1^c| \max_{z_1^x \in Q_1^c} \left| \int_{\varepsilon_{1,i+1}^x}^{\varepsilon_{1,i}^x} H(z) \, dz_1 - G_{z_1,\left(\varepsilon_{1,i+1}^x, \varepsilon_{1,i}^x\right)}^{n_1} H(z) \right|,$$

with $Q_1^c = (0, \varepsilon_2^x) \times (-\delta - \varepsilon_1^y, \times -\delta) \times (-\varepsilon_2^y, 0)$ and $H(z) := k_{loc}(z) B(z)$. The remaining errors, $E_{i,l}$, $E_{j,l}^{I,II}$, $E_{k,l}^{III,IV,V}$, $E_l^\star$, $l \in \{1,2,3,4\}$, are defined analogously. The total quadrature error E will be estimated, using Proposition 14 as in (60), by a sum of 1-dimensional quadrature errors:

$$\left| I_p - \tilde{I}_p \right| \leq C \sum_{l=1}^{4} \left\{ \left(\sum_{i=0}^{i_0-1} E_{i,l} \right) + \left(\sum_{j=0}^{j_0-1} \sum_{R=I}^{II} E_{j,l}^R \right) + \left(\sum_{k=0}^{k_0-1} \sum_{R=III}^{V} E_{k,l}^R \right) + E_l^\star \right\}. \tag{83}$$

In the singular case, i.e. $\delta = 0$, $E_l^\star$ has to be split into $E_l^\star \leq \sum_{k=1}^4 E_{k,l}^\star$ where, for $k \in \{1,2,3,4\}$, $E_{k,l}^\star$ corresponds to the kth integral of (82). We estimate these errors with the aid of Propositions 13 and 15. The details are in the following

Theorem 29 *Let $\delta > 0$. Then the quadrature errors can be estimated by*

$$E_{i,l} \leq C \pi_{K_x} \pi_{K_y} \eta \left(1 + \lambda_l \gamma\right)^{q_l - 2n} \tag{84}$$

where the constants C and γ depend only on τ_0, η, and the angles of $\tilde{K}_x$, $\tilde{K}_y$. The numbers η, q_l, and λ_l are defined by

$$\eta = \frac{|Q_i|}{\sqrt{\left|\tilde{K}_x\right| \left|\tilde{K}_y\right| \operatorname{dist}^2(Q_i, 0)}}, \qquad q_l := \begin{cases} p_l^{K_x} & l = 1,2, \\ p_{l-2}^{K_y} & l = 3,4, \end{cases} \qquad \lambda_l = \frac{\operatorname{dist}(Q_i, 0)}{D_l}$$

where D_l denotes the length of the integration interval of z_l. The estimates of the remaining errors are given by just replacing Q_i and D_l in the formulae above by the corresponding integration domains and interval lengths.

Proof. The proof is the same as the proof of Theorem 25 by taking into account the arising scales of the sides of the cube Q_i and the arguments of the basis functions. Hence, we skip the details. ■

Remark 30 *From definition of the domains Q_i, $Q_j^{I,II}$, and $Q_k^{III,IV,V}$, it follows directly that the constant η in (84) always can be bounded from above by $\sqrt{\varepsilon_2^x \varepsilon_2^y / (\varepsilon_1^x \varepsilon_1^y)}$. Furthermore, due to the assumption on the side lengths $\varepsilon_{1,2}^{x,y}$, we get the (possibly rough) estimate $\eta \leq 1$.*

For $\delta = 0$, it remains to consider the quadrature error for (82).

Theorem 31 *Let $\delta = 0$ and $E_{k,l}^\star$ as defined above. Then the error corresponding to the z_l integration can be estimated by*

$$E_{1,l}^\star \leq C \frac{\pi_{K_x} \pi_{K_y} \varepsilon_2^y}{\sqrt{\left|\tilde{K}_x\right| \left|\tilde{K}_y\right|}} (1 + \lambda_l \gamma)^{q_l - 2n}$$

with $\lambda_1 = 1/\varepsilon_2^y$ and $\lambda_l = 1$ for $l > 1$. The powers q_l are given by

$$q_l = \begin{cases} \left|p^{K_x} + p^{K_y}\right| & l = 1, \\ p_2^{K_x} & l = 2, \\ p_1^{K_y} & l = 3, \\ p_2^{K_y} & l = 4. \end{cases}$$

The estimates of the remaining errors $E_{k,l}^\star$ are given by a cyclic permutation of the indices.

Proof. Again, the proof is just a repetition of the arguments in the proof of Theorem 27. Hence, we skip the details. ■

In order to satisfy the consistency requirements the orders have to be chosen such that

$$\begin{gathered} E_{i,l} \leq N_L^{-1} L \sigma^{\varrho L} / i_0, \quad E_{j,l}^{I,II} \leq N_L^{-1} L \sigma^{\varrho L} / j_0, \quad E_{k,l}^{III,IV,V} \leq N_L^{-1} L \sigma^{\varrho L} / k_0, \\ E_l^\star \leq N_L^{-1} L \sigma^{\varrho L} \text{ for } \delta > 0, \quad E_{k,l}^\star \leq N_L^{-1} L \sigma^{\varrho L} \text{ for } \delta = 0. \end{gathered}$$

The asymptotic behaviour is considered in the following

Proposition 32 *Asymptotically, i.e., for L large enough, the quadrature orders satisfies*

$$n = O\left(\varrho L \left|\log \sigma\right|\right)$$

while the total work for all singular, near-singular and regular farfield integrals is bounded by

$$W \leq C L^8 \left(\varrho L \left|\log \sigma\right|\right)^5 \leq C \left(\varrho \left|\log \sigma\right|\right)^5 N_L^{3.25}.$$

4 Numerical Experiments

In this section we present the numerical results of an implementation of the fully discrete *hp*-Galerkin BEM. The domain Ω was chosen as the half tube depicted in Figure 1. The initial grid consists of 10 panels. The geometric grading parameter was chosen as $\sigma = 0.5$ (this will be justified ahead). The polynomial degree vector was determined by the choice $\mu = 1$ and $L_0 = 0$ resulting in a "relatively" small dimension of the space $V^L := V^L_{\sigma,\delta p}$. We have used the procedure "geometric refinement" of Section 2.3.1 for the mesh refinement and the procedure "polynomial refinement" of Section 2.3.2 for setting up δp, the polynomial degree distribution. The following table lists the number of elements, the maximal polynomial degree p_{max}, and the number of unknowns N_L (i.e. the dimension of V^L), the number of iterations used by the solver and the overall CPU-time (i.e. the time for quadrature and linear system solution) used for this example.

Level	# of panels	p_{max}	$N_L = \dim V^L$	# of iterations	CPU[sec]
1	10	0	10	22	5.4e-2
2	40	0	40	26	1.135
3	128	1	160	29	21.48
4	264	2	576	30	430.2
5	448	3	1624	30	4264
6	600	4	3784	29	29778

We have used the precise panel-wise quadrature error bounds together with the consistency requirements derived in the previous sections to determine the optimal quadrature orders. We avoided, however, the use of variable order quadrature on subdivided panels in order to reduce the number of case statements in the code. Instead, we used slightly pessimistic estimates as outlined in Remark 19 and Remark 30.

To verify the sharpness of our estimates, we have considered the boundary value problem

$$\Delta u = 0 \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega$$

with $\varphi(x) = x_1$. Clearly, the exact solution is given by $u(x) = x_1$. We have employed the double layer ansatz

$$u(x) = -\frac{1}{4\pi}\int_\Gamma \frac{\langle n(y), y - x\rangle}{\|x - y\|^3} f(y)\, dy \tag{85}$$

leading to the following integral equation for the density f

$$-2\varphi(x) - \frac{1}{2\pi}\int_\Gamma \frac{\langle n(y), y - x\rangle}{\|x - y\|^3} f(y)\, dy = f(x) \qquad x \in \Gamma. \tag{86}$$

Note that although the solution $u(x)$ is smooth in $\overline{\Omega}$, the density f exhibits singularities due to the nonsmoothness of the domain making an accurate solution of (86) nontrivial. We have solved this integral equation with the fully discrete Galerkin BEM. We know that under Assumption 14 the Galerkin solution converges exponentially in interior points $x \in \overset{\circ}{\Omega}$. To avoid cancellation errors due to symmetry effects we have chosen the points $P_1 = (1.06, 1.9, 4.95)^T$ and $P_2 = (1.06, 1.75, 0.5)^T$ for the evaluation of the potential u and the approximation u_L which was obtained by inserting the fully discrete Galerkin solution f_L of (86) corresponding to the subspace V^L into (85). Then, for $i = 1, 2$, we put

$$e_L^i := |u(P_i) - u_L(P_i)|.$$

The discrete system was solved by an iterative solver of *generalized conjugate residual type*. The precise definition of the algorithms can be found in [14]. We emphasize that the time for solving the linear system is completely negligible compared to the CPU-time for generating the system of linear equations. As predicted by our theoretical results, the number of iterations for the solution process does *not* increase for increasing problem sizes since the condition number is bounded independently of L. About 30 iterations in each level were needed to get the residual in the Euclidean norm (and, by Lemma 9, also in the $\|\circ\|_{L^2(\Gamma)}$-norm) smaller than 1.0e-13.

The following plots show the convergence history of our method. We expect a convergence behaviour with respect to the refinement level L as

$$e_L^i \approx L^2 \sigma^{2\varrho L}$$

and, hence, a plot of $\log e_L^i$ versus L should be approximately a straight line as shown in Figure 6.

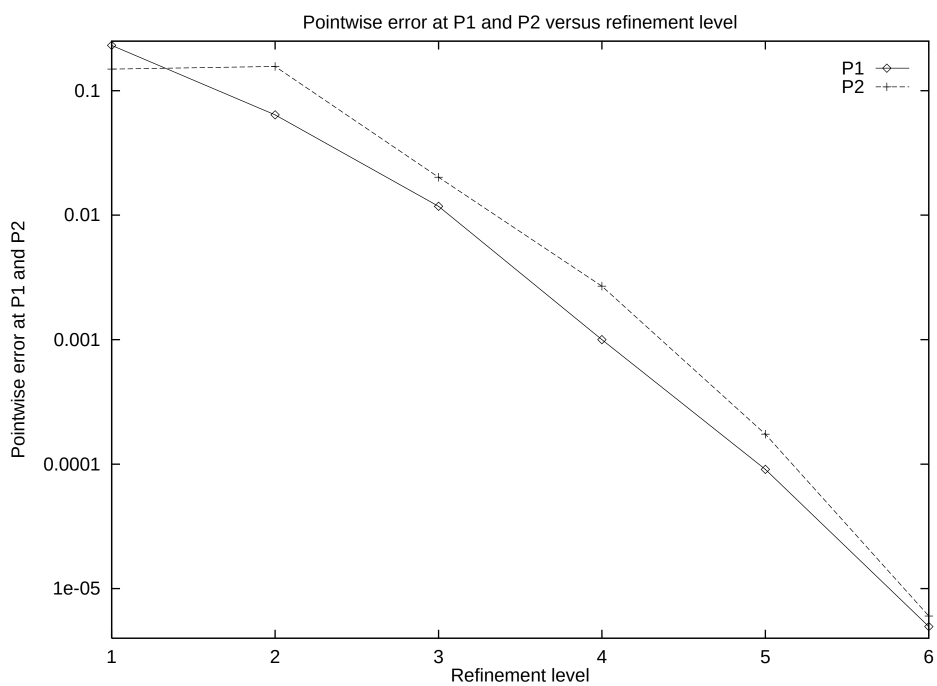

Figure 6: Pointwise error versus refinement level.

By Theorem 7, the error as a function of the work should behave like

$$e_N^i \approx \sqrt{N} \sigma^{2\varrho \sqrt[4]{N}}$$

and hence a plot of $\log e_L^i$ versus the fourth root of the degrees of freedom should be approximately a straight line. The corresponding graph is depicted in Figure 7.

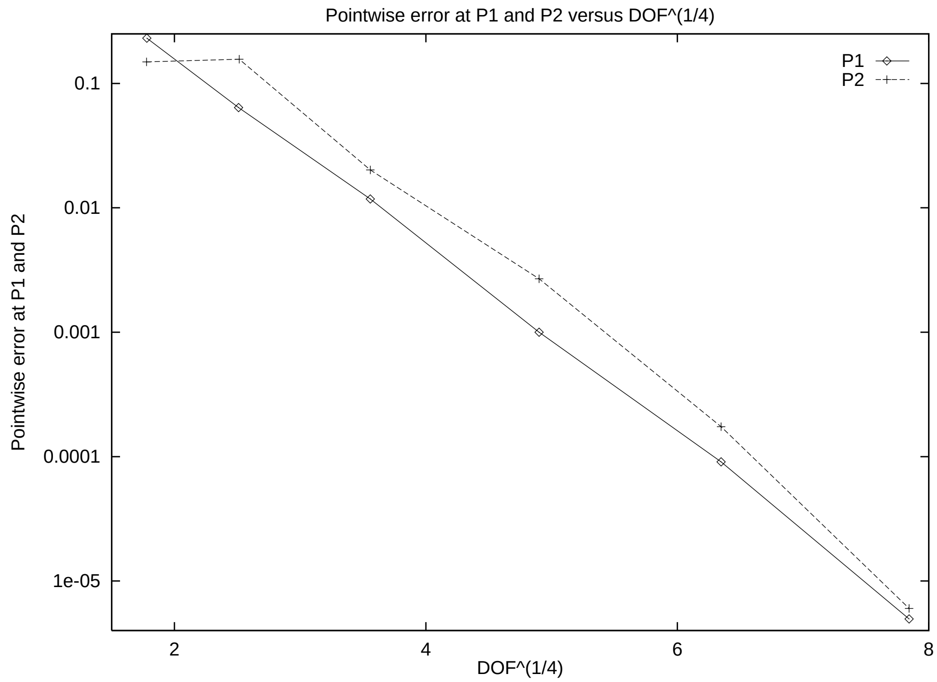

Figure 7: Pointwise error versus $N^{1/4}$.

The main result of our investigation, however, is the error versus the CPU-time. The convergence with respect to the CPU-time should be exponential as well. We have shown that the CPU-time is bounded by L^{13} resp. $N_L^{3.25}$. To verify the sharpness of this estimate, we plot $\log e^i_{work}$ versus the 13th root of the CPU-time in Figure 8.

In order to show the superiority over algebraic convergence behaviour, we have added a plot of e^i_{work} versus $work$ in a log-log scale in Figure 9. The exponential convergence is clearly visible.

We emphasize that a comparison with other codes as, e.g., the results reported in [21] for the h-version Galerkin BEM, show that the hp-method is a fast method also for moderate problem sizes and moderate accuracies. We further point out that, due to the high convergence rate of the method, the size of the stiffness matrix is moderate and its storage is not as severe a problem as in the h-version of the BEM.

We close with a comment on the sharpness of the work estimate given in Proposition 31. Figure 10 shows that the upper bound of $CN^{3.25}$ given in Proposition 31 is actually sharp and already attained for a moderate number of degrees of freedom N, as could be expected from Figure 8. This allows heuristically to give an optimal selection of the grading factor σ. We have from Theorem 7 (omitting terms algebraic in L) that $error \leq C\sigma^{\varrho L}$ holds and from Proposition 31 that W $\sim L^{13}(\varrho|\log\sigma|)^5$. Ignoring constants (which depend weakly on σ), the work to achieve a certain (small) tolerance tol can be determined in terms of tol, ϱ, σ to be $W \sim |\ln tol|^{13}/|\varrho \ln \sigma|^7$. This clearly indicates that in order to optimize error versus work, it is advantageous to select $\sigma = 0.5$ rather than $\sigma = 0.15$, as suggested by approximation theoretic considerations alone [9]. This was also clearly visible in our numerical experiments.

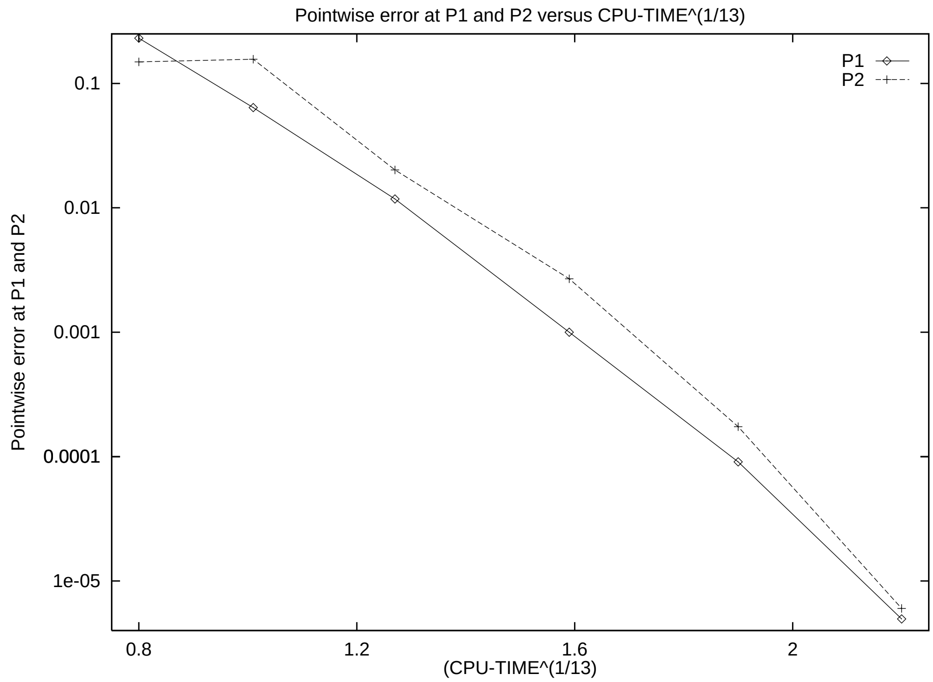

Figure 8: Pointwise error versus $(CPU - TIME)^{1/13}$.

5 Concluding remarks

In summary, we have presented quadrature methods for all types of integrals arising in *hp*-Galerkin BEM in 3-d. They were based on relative coordinates, an geometric splitting of the integration domain, regularizing coordinate transforms, and tensor product Gaussian quadrature. We showed how to compute exponentially convergent approximations of the system matrix satisfying (42) with work growing *algebraically* with the degrees of freedom. The quadrature methods are fully automatic, i.e., independent of the explicit form of the kernel function, the parametrization and the shape function. We have presented the double layer potential in Section 1 merely as an example of a kernel which satisfies our abstract requirements on the kernel function. It follows that an *integrator* based on our strategy will integrate a much broader class of integral equations by just replacing the subroutine which evaluates the kernel function at certain surface points. This class includes, for example, the kernel functions of the Helmholtz equation, the Lamé equation and the Stokes equation and in particular all weakly singular kernels for second order elliptic problems in $\mathbb{R}^3$.

We analyzed here in detail the impact of the quadrature errors on the convergence rate of the Galerkin boundary element discretization for second kind integral equations or more generally, integral operators of order zero, under the assumptions of *stability* (11) and *regularity* $u \in B_\rho(\Gamma)$ of the exact solution. Such regularity results appear to hold for a wide class of integral operators on piecewise analytic surfaces Γ [7, 16]. The stability of the Galerkin scheme based on V^L is trivial for first kind equations, but has been established for the second kind equations discretized here essentially only for convex polyhedra [7]. It likely holds, however, also for polyhedra with curved, analytic sides and convex edge and vertex angles [8].

Our quadrature error estimates apply, however, with minor modifications to weakly and hypersingular integrals. In the hypersingular case, the regularization of the integrals has to

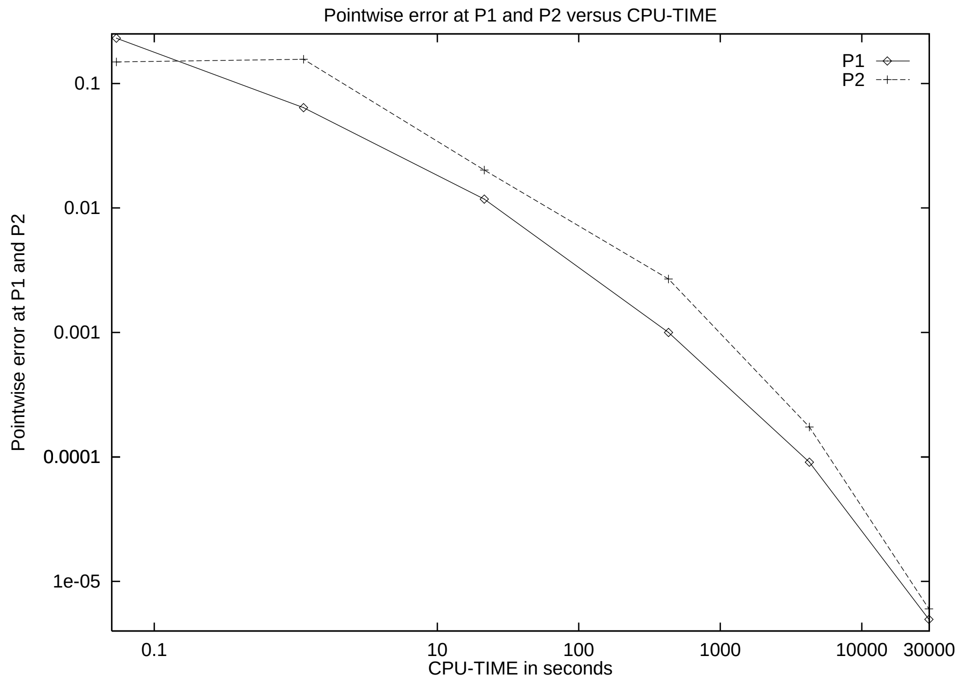

Figure 9: Pointwise error versus work in a $\log - \log$ plot.

be done on the *continuous level*, i.e. prior to discretization (cf. [10, Section 8.3], [12], [15], [19]). These regularizations render the integrand Cauchy-singular and hence, the techniques presented above can be applied directly.

The implementation of the coordinate transformations can be checked for simple test kernels as, e.g. polynomials, and should then work for all kernels which satisfy our assumptions. The selection rules for the number of Gauss points based on our quadrature error estimates are somewhat complicated at first sight. However, we found it essential that the lowest possible number of quadrature points sufficient to ensure the consistency is used, since simplified (upper) bounds for them result in substantially larger CPU-times at essentially no improvement in accuracy.

The estimates for the asymptotic complexity of the quadrature orders are rather rough. The effect of increasing distance from the singularity has been neglected and the different orders for the different variables as well. In practical implementations, however, it is *essential* to choose the quadrature orders directly from the error representations to obtain a method that is competitive also for practical problem sizes.

The numerical experiments fully confirmed our error and complexity estimates and indicated strongly that they are sharp. In order to achieve a given tolerance *tol* with the fully discrete method in minimal work, it appears best to utilize geometric meshes with grading factor $\sigma \sim 0.5$ rather than $\sigma = 0.15$. This is due to the strong dependence of the quadrature work on σ and confirmed by our work estimates as well as by numerical experience. This is important since geometric meshes with grading factor 0.5 are typically generated by adaptive mesh-refinement algorithms.

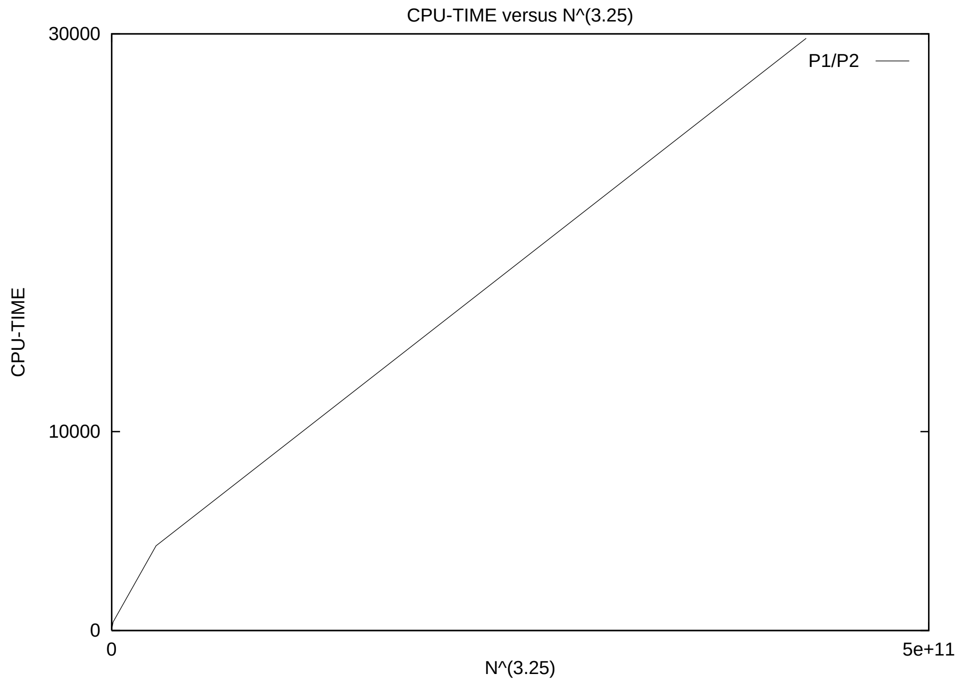

Figure 10: CPU-TIME versus $N^{3.25}$

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