Reliability of Finite Element Methods for the numerical computation of waves

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1 Introduction

The numerical computation of stationary waves in exteriour and scattering problems is based on indefinite variational forms connected with the Helmholtz equation

$$\Delta u + k^2 u = 0$$

where k is a real parameter (scalar wavenumber). Unlike the case of linear elasticity, stable dependence of both analytical (if existing) and numerical solutions on the data is not straightforward. Stability estimates of the form $\|u\|_i \leq C_{ij}\|f\|_j$ do hold for various norms i, j but the constants C_{ij} depend in general on the parameter k. Hence also the quality of the discrete solution depends on k, as well as on the parameters of the numerical model (stepwidth h, degree of approximation p). For practical application it is essential to have reliable "rules of the thumb" for the choice of the numerical parameters as a function of physical parameters. It is well known from computations that, for the "classical" Galerkin FEM, the linear rule for mesh-design $kh =$ const. leads to reliable results only in the low frequency range. This leads to two questions:

1. How do error estimators for the Galerkin FEM depend on k, h?
2. Can the classical Galerkin approach be improved towards a linear rule?

These questions are addressed by analysis of one- and two-dimensional Helmholtz problems. We give error estimates in integral norms for the h-version of the Galerkin FEM with general degree of approximation (section 2). Unlike previous estimates – cf. Bayliss et al. [4], Aziz et al [1], i.e. constraining the magnitude of kh only. The results are discussed in the context of engineering dispersion analysis (section 3). We then turn to the investigation of generalized FEM for the Helmholtz equation. While it can be shown that there is no method that can eliminate the entire phase error in 2D, independently of the direction of the wave, one still can construct a generalized method with minimal pollution (equivalently, minimal phase error – section 4). Numerical examples of one- and two-dimensional computations and a summary of the results conclude the paper.

2 Error estimates for the Galerkin FEM

For analytical purpose, we consider the one-dimensional model problem: Let $\Omega = (0,1)$ and

$$\begin{aligned} -u''(x) - k^2 u(x) &= f(x) \quad , x \in \Omega && (2.1)\\ u(0) &= 0 && (2.2)\\ u'(1) - iku(1) &= 0. && (2.3) \end{aligned}$$

consisting of the ordinary Helmholtz equation with Dirichlet and (exactly absorbing) Robin condition at the boundaries. We will use the notation of Sobolev spaces $H^s(\Omega)$ in the usual way, denoting by $\|u\|_s$ and $|u|_s$ the norms and seminorms in these spaces, resp. If $s < 0$, the norm is computed in the dual (to H^{-s}) space. It is well known, see Douglas et al. [5], that for $f \in L^2(\Omega)$

$$|u|_s \leq C_s k^{s-1} \|f\|_o \tag{2.4}$$

holds for $s = 0, 1, 2$ with C_s independent of k. The inf-sup-condition for the problem holds with a constant $\gamma = Ck^{-1}$, hence the dual estimate $|u|_1 \leq Ck|f|_{-1}$ applies [6]. Using a Greens function approach on uniform mesh, the same stability conditions are shown for the discrete solution [6]. Using these stability results, one can prove (for piecewise linear approximation in the subspace $S_h \subset H^1$) the error estimate

Theorem 1 *Let $u \in H^2(\Omega)$; $u_h \in S_h(\Omega)$ be the exact and the finite element solutions to the BVP (2.1-2.3), resp. Then for $hk < 1$*

$$|u - u_h|_1 \leq (C_1 hk + C_2 h^2 k^3) \|f\| \tag{2.5}$$

holds with constants C_1, C_2 not depending on k, h.

If the exact solution has the form of a sinusiodal wave of wavelength k (or, more generally, if u is such that $|u|_s / |u|_t \leq Ck^{(s-t)}$ holds for some C independent of k) one easily derives

$$\tilde{e}_1 := \frac{|u - u_h|_1}{|u|_1} \leq C_1(kh) + C_2 k (kh)^2. \tag{2.6}$$

that generalizes the results of Bayliss [4], Douglas [5] and Aziz [1] who had shown that $\tilde{e}_1 \leq Ckh$ if k^2h is small.

An analysis of the h-p-version [7] shows that this estimate carries over to the case of piecewise polynomial approximation. On uniform $h-p$-mesh with approximation space S_h^p, the following theorem holds.

Theorem 2 *Let $u \in H^{p+1}(\Omega)$, $u_h \in S_h^p(\Omega)$ be the exact and the h-p-FEM solution to model problem (2.1-2.3). Assume that u is oscillating with frequency k and a constraint $hk < \pi$ is given for the stepwidth h. Then the relative error in H_1-seminorm is bounded by*

$$\tilde{e}_1 \leq C_1 \left(\frac{hk}{2p}\right)^p + C_2 k \left(\frac{hk}{2p}\right)^{2p} . \tag{2.7}$$

Here, the constants C_1 and C_2 grow moderately with p, see [7] for details. Again, continuous and discrete stability statements are the principal provision for the proof of the $h-p$-estimates. Numerical computations show that the estimates given in Theorems 1,2 are sharp.

3 Dispersion analysis and generalization of the Galerkin FEM

The first term on the right hand side of the estimate (2.7) is the error of approximation (interpolation error). This error is under control if a linear rule $hk = \alpha$ is used for the choice of the meshsize. The second term can be interpreted as numerical pollution caused by the indefiniteness of the variational form. Despite the constraint $hk = \alpha$, the error may grow infinitely with k. The error behaviour is different from the well known convergence pattern of h-p-extensions in definite problems, e.g. in linear elasticity. We show this exemplarily in Fig. 1 for $p = 1$. The relative errors of the Galerkin FE-solutions are plotted both for low and high frequency. The errors of the best approximation are displayed for comparison. Since the best approximation is computed from a positive definite projection problem, its error shows the expected pattern of convergence: the range of predicted asymptotic rate of convergence is preceded by a preasymptotic range (in the plot visible for $k = 100$) where the rate is suboptimal. In the case considered ($p = 1$), the relative projection error in the suboptimal range is 100% as long as the stepsize of the linear elements exceeds the size of one half-wave. The interpolation error is stable w.r. to k, the magnitude of the error depends on hk only. The behaviour of the Galerkin FEM-error is different. For both frequencies displayed, one observes an asymptotic range where the FE-error converges with optimal rate of convergence. Taking out hk in estimate (2.6) we see that quasioptimality is ensured, independently of k, `(fig1_h.ps)` was not found

Figure 1: Errors of H^1-projection versus error of finite element solution for Dirichlet problem; error in H^1-seminorm; wavenumbers $k = 10$, $k = 50$ and $k = 100$.

if hk^2 is constrained. In the preasymptotic range, the error first is above 100% before it begins to converge with superoptimal rate. The pollution error dominates the error, and the range of dominance of pollution in the convergence behaviour grows with k, as predicted by the estimates.

It is well known that discrete approximations of propagating solutions to the Helmholtz equation display, in general, a phase lead to the exact solution. On uniform mesh, one can assign a "discrete" wavenumber k' to the Galerkin FE-solution by discrete Fourier analysis. For regular solutions the pollution term is exactly of the order of the phase lag [7]:

Theorem 3 *Consider approximation of the ordinary Helmholtz equation by a Galerkin FEM in the approximation space S_h^p. Then, if $hk < 1$,*

$$|k' - k| \leq kC(p)\left(\frac{hk}{2p}\right)^{2p} . \tag{3.1}$$

This theorem generalizes previous results of dispersion analysis for wave computation using Galerkin FEM (see Thompson and Pinsky [10]). Modifying the classical Galerkin approach one can reduce the phase error of the FE solutions. For one-dimensional problems, the phase error can be entirely eliminated by suitable generalization of the variational form (cf. Harari and Hughes [9]; see also [3], Theorem 4). More generally, one can show [2] for any generalized FEM (GFEM) whos discrete matrix has certain natural properties that either $k' = k$ or

$$C_1 k(kh)^{s_o} \leq |k - k'| \leq C_2 k(kh)^{s_o}$$

for some even $s_o \geq 2$. Finally, the following statement is shown in [2].

(GFEM*) with nonvanishing phase difference* $k - k' \neq 0$. *Assume that* kh *and* $k(kh)^{s_o}$ *are bounded. Then, for sufficiently large* k,

$$C_1|k-k'| \leq \tilde{e}_o \leq C_2|k-k'| + C_3(kh)^2. \tag{3.2}$$

Hence the pollution term of the error and the phase lead are equivalent measures for the reliability of the discrete solution.

4 Error behaviour and quality improvement in two-dimensional Helmholtz problems

So far, no error estimates are proven for higher-dimensional Helmholtz problems. Computational experiments show, however, that the numerical effects predicted by one-dimensional analysis occur also in the 2D results. We consider the homogeneous Helmholtz equation

$$\Delta u(x_1,x_2) + k^2 u(x_1,x_2) = 0 \tag{4.1}$$

on the unit square $\Omega = (0,1)\times(0,1)$ with nonhomogeneous boundary conditions

$$iku + \frac{\partial u}{\partial \mathbf{n}} = g_s \quad \text{on} \quad \Gamma_s \quad , \quad s = 1,2,3,4 \tag{4.2}$$

where g is chosen such that the exact solution is

$$u = \exp(i\mathbf{kx})$$

with vector wavenumber $\mathbf{k} = (k_1,k_2)$ and $|\mathbf{k}| = k$. Bilinear shape-functions are used for approximation on uniform mesh [8, 2]. The error norms are computed in H^1- and L^2-norms and compared to the projection errors in these norms. One observes the same pollution effects as in the one-dimensional case [2, 8] - see the FEM-lines in Fig. 2.

It is possible to reduce this effect, and thus to raise reliability of the FEM on moderately refined mesh, by appropriate modification of the discrete model. However, unlike in the 1D-case it is not possible to eliminate pollution entirely. This is shown comparing the Fourier symbols of the differential operator and the difference operator resulting from any GFEM on square mesh. Denoting by $\mathbf{d}_{GFEM}(k,h)$ the distance (for some generalized FE-model) between the symbols in the Fourier image and by $\|\cdot\|_{-}$ a weighted L^2-norm, we have [3]

Theorem 5 *Let* u *and* u_{fe} *be the exact and GFEM solution, resp., to a well defined Helmholtz problem on a square domain* $\Omega = (-L,L)\times(-L,L)$. *a) For any GFEM, there exists a domain measure* L *and a data set such that*

$$\|u - u_{fe}\|_{-} \geq C_1\sqrt{\frac{\mathbf{d}}{h}}.$$

Fig. 6.2. from bips

Figure 2: Relative errors for 2D Helmholtz problem: finite element solution vs. best approximation for $k = 30$ and $k = 100$

b) The distance $\mathbf{d}$ *can be expanded as*

$$\mathbf{d} = r_{l_o}(kh)^{2l_o+1} + \mathcal{O}(kh)^{2l_o+3}$$

with $r_{l_o} \neq 0; \;\; 1 \leq l_o < \infty$*.*
c) There exists a function u_{opt} *in the finite element subspace satisfying*

$$\|u - u_{opt}\|_{-} \leq C_2(kh)^2.$$

One can show that for the standard Galerkin FEM $l_o = 1; \; r_1 = 1/24$. On the other hand, in [2] we construct a modified FEM with $l_o = 3; \; r_3 \approx 10^{-6}$. In Fig. 2 we show the error of this modified method and the error of the Galerkin method. For comparison, we also show the error of the Galerkin Least Squares Method proposed by Thompson and Pinsky [10].

The dependence of the errors of the direction *theta* is shown in Fig. 3.

5 Summary of conclusions

1. New error estimates are presented for wave computations using the Galerkin FEM. The estimates show that on normalized mesh with $kh \leq \alpha$, the energy norm of the error contains a pollution term of order $k(kh)^{2p}$, where p is the order of approximation. For reliability

Figs. 6.6.-6.8. from bips, one plot

Figure 3: Dependency of the H^1-error on the angle θ for $kh = 1.5$ and $k = 30, 100$.

of the FEM results, it is hence necessary *and* sufficient to constrain this term when choosing the meshwidth h for given wavenumber k.

2. Taking into account the pollution term it is shown that dispersion analysis of the phase lag of numerical solutions is equivalent to numerical analysis of the error in integral (H^1- or L^2-) norm. Hence modified FE methods proposed for phase error reduction equivalently lead to error reduction in integral norm.

3. Galerkin FE solutions to two-dimensional Helmholtz problems show the same error behaviour as one-dimensional solutions. However, unlike the 1D-case, it is not possible to eliminate the phase error by any generalized FEM in 2D. Still significant error reduction is possible by suitable modification of the classical approach. For any method, however, there exists a dispersive lower bound for the numerical error in integral norm.

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