STRIPS, TUBES, BOTTLES, CAPS, UMBRELLAS — A FEW EXAMPLES OF VISUALIZATIONS OF SURFACES

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1. Introduction

This article grew out of a talk given by the author at the Homi Bhabha Centre for Scientific Education (HBCSE) in Bombay in the framework of the seminar "Perspectives in Mathematics Education" on 16th December 2000. We aim to encourage teachers, future teachers and their teachers to use visualizations as a mean to bridge the gap between every day experience and (higher) mathematics.

There is a classical field of surface topology which is perfectly suited to our purpose: the theory of models of non-oriented closed surfaces. Indeed, this theory illustrates beautifully how "the invisible" (e.g. surfaces which cannot be embedded in 3-space) may be approached by "the visible" (e.g. models of such surfaces). Clearly, such a "theory of models" needs a mathematical notion of model: a model of a surface is a continuous map (subject to various suitable requirements) from this surface to 3-space. But in the spirit of the present paper one should not be ashame to understand the word of model in its everyday meaning and to think of an object made of plaster, glass, wood or paper, which can be visualized by photographs, computer graphics — or by hand made drawings.

Clearly, we can only treat some very limited topic in the mentionned field and so restrict ourselves to consider the most basic examples: the Klein bottle, the (real) projective plane and (in order to illustrate the concept of non-orientability) the Möbius strip. So, we shall speak on three well known objects (each occuring with 4000 - 5000 references in the internet) but usually not treated in regular courses.

We shall touch some of the historic landmarks of the subject, though not in chronologic order. "Prehistory" is presented by Steiner's Roman surface and the cross cap model (1844). The "foundational period" is symbolically touched with the Möbius strip (1858). The "classical period" shows up with the Klein bottle (1874) and the Boy surfaces (1901). Finally the occurence of Whitney umbrellas (on Steiner's surfaces) indicates a link to "the modern aera", which arises with Whitney's embedding theorems (1941).

Contrary to what we did in our talk, we now try to complete the "step from pre-mathematics to mathematics". So, we add here a few comments in which our heuristic considerations are brought to a strict mathematical form. In these comments we address ourselves to a reader with basic knowledge in calculus of several variables and in elementary set topology. In particular we do not use nor introduce the notion of topological or differentiable surface. Instead we use ad hoc descriptions of the occuring surfaces. Our background comments are set in smaller print and a reader who is in a hurry may skip them.

For readers, who wish to have more specific information on models of the projective plane (and the Klein bottle) a highly recommendable reference is [Ap]. Readers, who got attracted by the interplay of visible phenomena and mathematical concepts should consult the fascinating booklet [Fr] or also [Gr]. For those readers, who got motivated to learn more systematically about surfaces (and we hope there are some of them) we recommend [B-G], [DC] or [K]. For those readers, who wish to know more on the projective plane and on projective geometry, we suggest [Cox].

Finally, we express our gratitude to the HBCSE, the Indian Institute of Technology and the Tata Institute of Fundamental Research in Bombay for their hospitality offered during and after the Conference "Cohomological Methods in Commutative Algebra", December 2000.

2. The Möbius Strip: Making it by Paper and by Thought

We begin with a few more (see Figure 1) or less (see Figure 2) serious considerations about

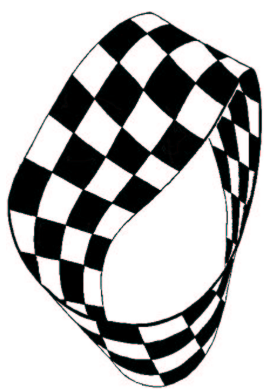

Figure 1

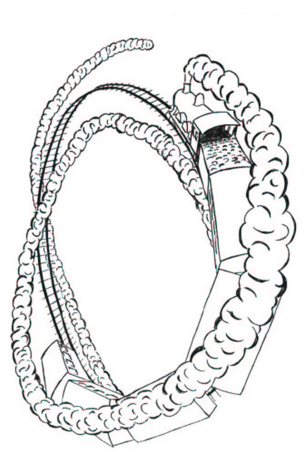

Figure 2

the well known *Möbius strip* or *Möbius band*, named after its "mathematical father", the German astronomer and mathematician A.F. Möbius (1790 - 1868). It seems rather evident, that this "one - (or no - ?) sided band" was known before Möbius (in 1858) made it an object of mathematical thought. So, following the path of history, let us start with the pre-mathematical aspect of the Möbius strip, by making it from a rectangular strip of paper, as indicated in Figure 3.

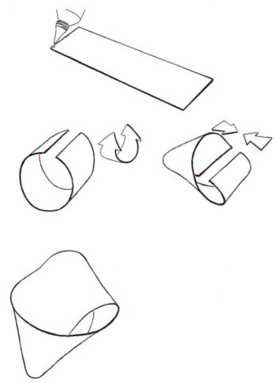

Figure 3

Here already some questions arise: For which rectangular strips of paper do we get a non-ceased band by the above procedure? Is there something like a "natural shape" of our "real" Möbius strip in this case?

But, instead of thinking of these difficult problems, let us cut our Möbius strip as indicated below. For the sake of intellectual challange, guess what comes out, before you start cutting!

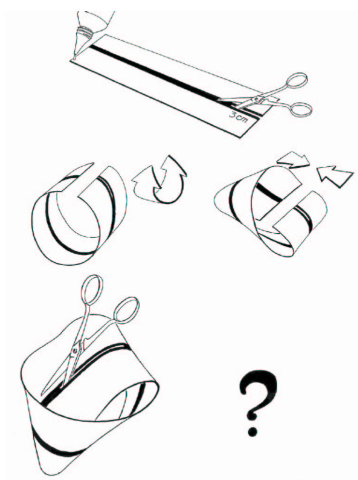

Figure 4 Figure 5

The most striking property of the Möbius strip is its "one-sidedness": if we try to paint its "front"- and "back"-sides (which do not exist actually) in different colours, we run into troubles (see Figure 6). Mainly by this property the Möbius strip attracts the mathematicians attention.

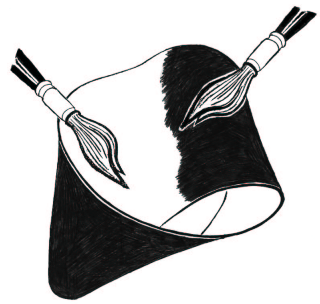

Figure 6

Now, let us do a first step from pre-mathematical thinking to mathematical thinking: we make a Möbius strip "by thought"!

Here is our task: Glue two opposite edges of a rectangle, reversing one of the edges before gluing it to its opponent. So, refering to Figure 7, we have to glue the shorter edges such that the black arrows they carry coincide.

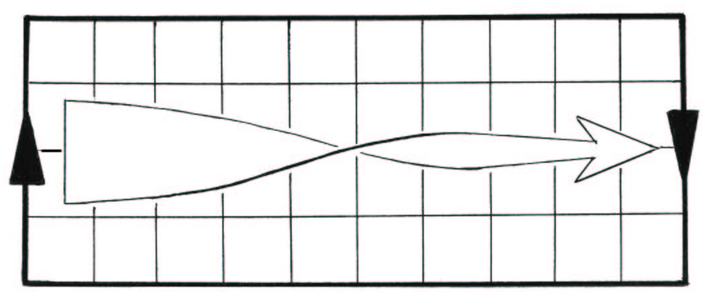

Figure 7

In Figure 8 three consecutive steps of a deformation and gluing process, which leads to our goal are visualized.

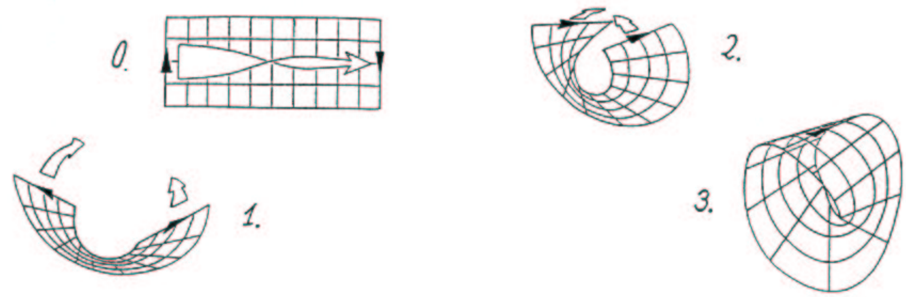

Figure 8

So, instead of a strip of paper, we have used geometric imagination to produce a Möbius strip. Now, the final step consists in replacing geometric imagination by a strict mathematical description of a Möbius strip, for example by a *parametrization*, which reflects our "gluing prescriptions".

To give such a parametrization, we choose $a, b, R \in \mathbb{R}$ with $0 < b < a, \frac{b}{2} < R$ and consider the rectangle

$$\mathbb{S} := \left\{ (u,v) \in \mathbb{R}^2 \mid 0 \le u \le a, -\frac{b}{2} < v < \frac{b}{2} \right\} \tag{2.1}$$

and the map

$$\varphi : \mathbb{S} \to \mathbb{R}^3;\ \varphi(u,v) := \begin{bmatrix} x(u,v) = \left(R + v \sin\left(\frac{\pi}{a}u\right)\right) \cos\left(\frac{2\pi}{a}u\right) \\ y(u,v) = \left(R + v \sin\left(\frac{\pi}{a}u\right)\right) \sin\left(\frac{2\pi}{a}u\right) \\ z(u,v) = v \cos\left(\frac{\pi}{a}u\right) \end{bmatrix} \tag{2.2}$$

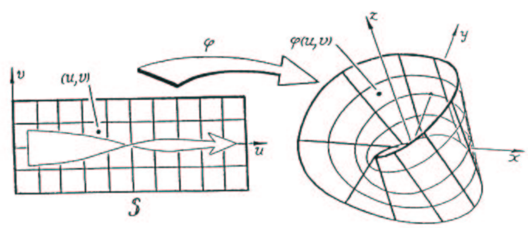

Figure 9

This map has the following properties:

(2.3) $$\varphi \text{ is injective on } \overset{\circ}{\mathbb{S}} := \{(u,v) \in \mathbb{S} \mid u \neq 0, a\}\,;$$

(2.4) $$\varphi^{-1}\left(\varphi(0,v)\right) = \{(0,v),(a,-v)\}\,, \left(-\frac{b}{2} < v < \frac{b}{2}\right);$$

φ is an immersion e.g. it is continuously differentiable and its Jacobinian

(2.5) $$\partial\varphi := \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \left[\tfrac{\partial}{\partial u}\varphi, \tfrac{\partial}{\partial v}\varphi\right] \text{ has constant rank 2.}$$

(2.6) $$\frac{\partial}{\partial u}\varphi(0,v) = \frac{\partial}{\partial u}\varphi(a,-v), \ \left(-\frac{b}{2} < v < \frac{b}{2}\right).$$

The conditions (2.3) and (2.4) reflect our "gluing prescription" (identification of $(0,v)$ and $(a,-v)$), whereas (2.5) and (2.6) say that φ parametrizes an "unceased" surface. So, we are justified to consider the set $\varphi(\mathbb{S}) \subseteq \mathbb{R}^3$ as a "mathematical model" of a Möbius strip.

This *ideal Möbius strip* differs in shape from our "real" one made of a rectangular strip of paper. Nevertheless, both types of bands share the property of being "one-sided".

More general, for any $n \geq 3$ we could define a set $\mathbb{M} \subseteq \mathbb{R}^n$ to be a *Möbius strip* if there is an immersive map $\varphi : \mathbb{S} \to \mathbb{R}^n$, satisfying the above conditions (2.3), (2.4), (2.6) and such that

(2.7) $$\mathbb{M} = \varphi(\mathbb{S}).$$

It is important to notice, that all Möbius strips are "topologically equivalent", more precisely:

(2.8) If $\mathbb{M} \subseteq \mathbb{R}^n, \mathbb{M}' \subseteq \mathbb{R}^{n'}$ are Möbius strips parametrized by $\varphi : \mathbb{S} \to \mathbb{R}$ and $\varphi' : \mathbb{S} \to \mathbb{R}^{n'}$ respectively, there is a unique homeomorphism $\varepsilon : \mathbb{M} \xrightarrow{\approx} \mathbb{M}'$ such that $\varepsilon \circ \varphi = \varphi'$.

In mathematical terms the "one-sidedness" of the Möbius strip means, that the Möbius strip is a *non-orientable surface*. Roughly speaking, there is a closed path on our surface such that (sufficiently small) geometric figures return with reversed orientation, if moved along this path (cf Figure 10).

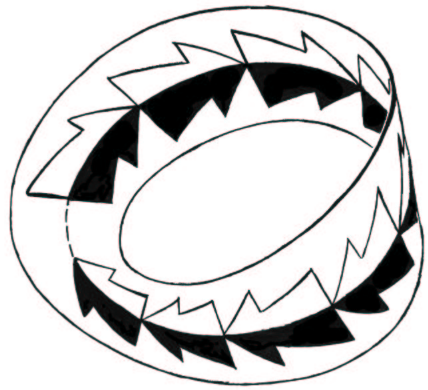

Figure 10

To explain this in mathematical terms, let $\mathbb{M} \subseteq \mathbb{R}^3$ the ideal Möbius strip parametrized by the map $\varphi : \mathbb{S} \to \mathbb{R}^3$ (cf (2.1), (2.2)). Then

(2.9) $$s : [0, a] \to \mathbb{R}^3;\ s(t) := \varphi(t, 0)$$

defines a closed path on $\mathbb{M}$. Moreover we can say:

(2.10) The two functions $s : [0, a] \to \mathbb{R}^3, t \mapsto \frac{\partial}{\partial u}\varphi(t, 0), t \mapsto \frac{\partial}{\partial v}\varphi(t, 0)$ are continuous .

(2.11) The two vectors $\frac{\partial}{\partial u}\varphi(t, 0), \frac{\partial}{\partial v}\varphi(t, 0) \in \mathbb{R}^3$ span the tangent plane $T_{s(t)}\mathbb{M}$ to $\mathbb{M}$ in the point $s(t) \in \mathbb{M}$, $(0 \le s \le a)$.

Finally we have

(2.12) $$\frac{\partial}{\partial u}\varphi(0, 0) = \frac{\partial}{\partial u}\varphi(a, 0);\ \frac{\partial}{\partial v}\varphi(0, 0) = -\frac{\partial}{\partial v}\varphi(a, 0).$$

Therefore:

(2.13) The "initial basis" $\left(\frac{\partial}{\partial u}\varphi(0, 0), \frac{\partial}{\partial v}\varphi(0, 0)\right)$ and the "final basis" $\left(\frac{\partial}{\partial u}\varphi(a, 0), \frac{\partial}{\partial v}\varphi(a, 0)\right)$ of $T_{s(0)}\mathbb{M} = T_{s(a)}\mathbb{M}$ have different orientation .

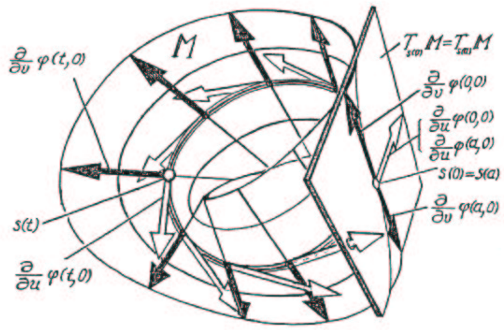

Figure 11

If $\mathbb{M} \subseteq \mathbb{R}^n$ is an arbitrary Möbius strip parametrized by $\varphi : \mathbb{S} \to \mathbb{R}^n$ (cf (2.8)) we get a closed path on $\mathbb{M}$ by the above definition (2.9) and for this path, the statements (2.10) - (2.13) hold as well.

For further reading on the Möbius strip, on A.F. Möbius, his mathematical work and his cultural and historic background, we recommend [Fa-Fl-W].

3. A Glance Beyond 3-Space

In the previous section we have produced a Möbius strip by gluing in an appropriate way *one pair of opposite edges* of a rectangle. Now, let us dare more: let us glue *two pairs of opposite edges* of a rectangle as indicated by Figure 12.

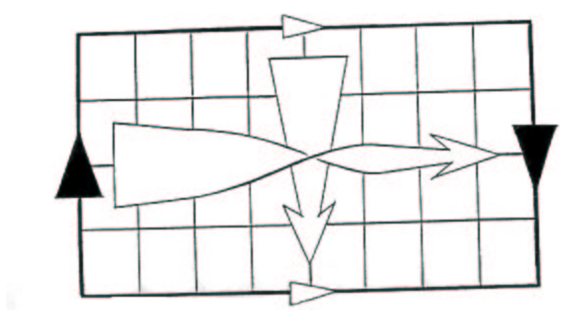

Figure 12

So, we want to glue the longer edges by identifying opposite points on both of them. Similar as in the case of the Möbius strip, we glue the short edges only after having reversed one of them. If we reach our goal, we shall have produced a kind of "closed Möbius strip".

We start our imaginative process by gluing the edges which carry the "white arrows". As shown below, we get a pipe in this way.

Figure 13

Then we start to bend the pipe with the goal to join its ends such that the black arrows become coincident.

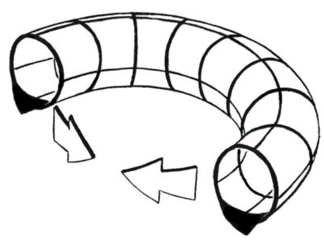

Figure 14

If one of the two black arrows would point in the opposite direction, we could go straight on with the bending and join the two ends in the requested way. Like this we would end up with a "swimming tube", a *torus*.

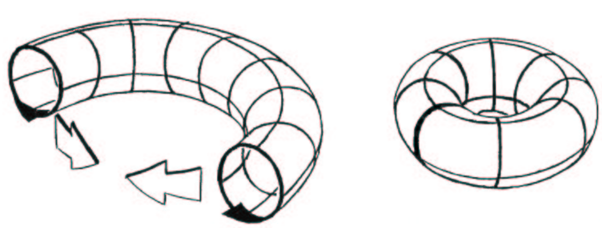

Figure 15

But this clearly does not solve our task, as we have reversed one of the black arrows!
Instead of the bending shown in Figure 14 let us try the following deformation.

Figure 16

Again, we run into trouble. It seems, that we cannot perform the requested process without producing a "self-penetration". If we go on joining the two ends of our pipe in the requested way and do not care about the occuring self-penetration, we get a surface of a "bottle-like" shape as shown in Figure 17.

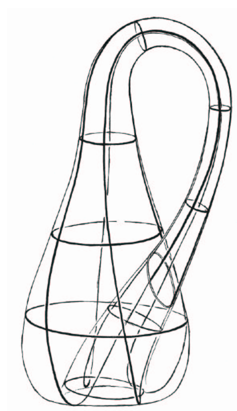

Figure 17

In fact in "4-space, one could join both ends of our tube in the requested way" such that now self-penetration occurs. The surface obtained like this in 4-space is called a *Klein bottle*, after the German mathematician F. Klein (1849 - 1924), who described this surface in 1874. Clearly, this surface cannot be visualized properly, as it is embedded into 4-space.

What is shown in Figure 17 is only a so called *model of the Klein bottle.* It is typical, that this model has a self-penetration. We can think of this model as the image of the actual Klein bottle under a projection from 4-space. In Figure 18 we visualize the relation between the Klein bottle and its model by a lower dimensional analogue. Instead of the Klein bottle — a surface in 4-space — we choose a knot — a curve in 3-space. Then, a "model" of the knot is the curve obtained by projecting our knot to a plane, in analogy to the model of the Klein bottle which is obtained by projecting the actual Klein bottle to 3-space. The occurence of self-penetrations in the model of the Klein bottle corresponds to the occurence of self-intersections on the projected knot.

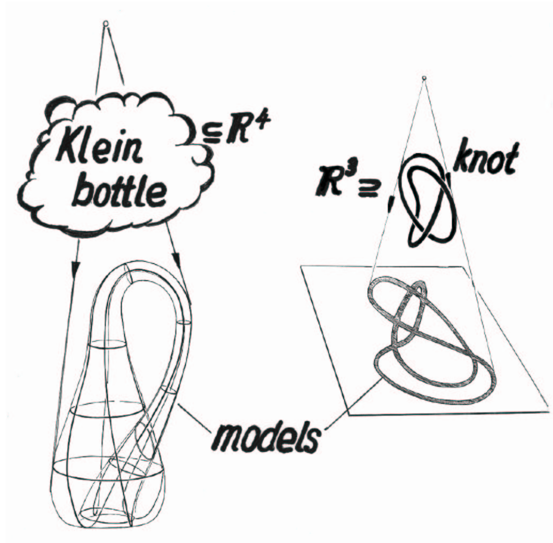

Figure 18

The reason, why the Klein bottle cannot be embedded into 3-space may be visualized at its model: the Klein bottle contains a Möbius strip (cf Figure 19)! This means that the Klein bottle is not orientable. On the other hand, the Klein bottle is a closed surface — as the torus for example. But a closed non-orientable surface cannot be embedded into 3-space. (The torus may be embedded in 3-space as it is an orientable surface, like the sphere for example.)

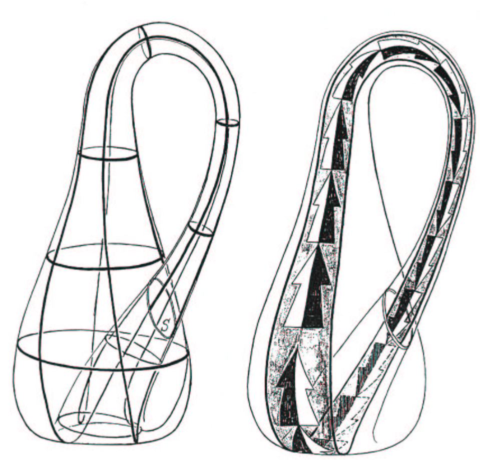

Figure 19

Clearly, our treatment of the Klein bottle was only of pre-mathematical nature. So, our gluing procedure must be described in strict mathematical terms. Notably our use of 4-space and the notion of model as used above need mathematical justification. As in the case of the Möbius strip we give a short outline of a mathematical presentation of the subject, leaving the verification of our statements to the reader.

We first give a way to parametrize a model of the Klein bottle similar to the surface presented in Figure 17.

Again, we choose $a, b \in \mathbb{R}$ with $0 < b < a$. Then, we consider the closed rectangle

$$\overline{\mathbb{S}} := \left\{ (u,v) \in \mathbb{R}^2 \mid 0 \le u \le a,\ -\frac{b}{2} \le v \le \frac{b}{2} \right\}. \tag{3.1}$$

Moreover we consider three twice continuously differentiable functions $y, z, \varrho : [0,a] \to \mathbb{R}$. Our idea is to use the path

$$\gamma : [0,a] \to \mathbb{R}^2;\ \gamma(t) := \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \tag{3.2}$$

as the "central curve" of our model and $\varrho(t)$ as the "radius of the normal cross section (!) of our model through the point $\gamma(t)$" (see Figure 20).

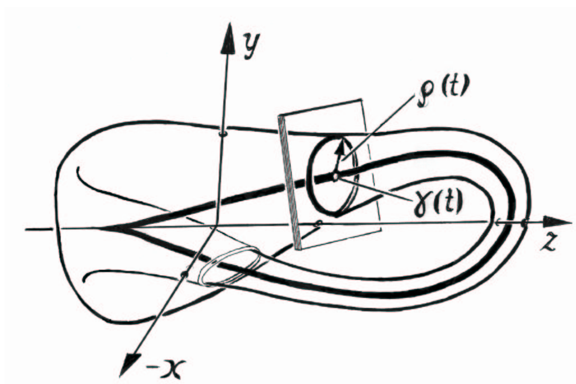

Figure 20

We choose our "*supporting functions*" $y, z : [0,a] \to \mathbb{R}$ such that

$$\begin{aligned} &\text{a)}\ \ y(t) = -y(a-t), z(t) = z(a-t) \text{ for all } t \in [0,a]; \\ &\text{b)}\ \ y(t) > 0 \text{ for all } t \in]0, \frac{a}{2}[; \\ &\text{c)}\ \ \dot{z}(t) > 0 \text{ for all } t \in]0, \frac{a}{2}[. \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\text{a)}\ \ \dot{\gamma}(t) := \begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} \ \begin{cases} = 0 \text{ for } t \in \{0,a\} \\ \neq 0 \text{ for } t \in]0,a[\end{cases}; \\ &\text{b)}\ \ \lim_{t \downarrow 0} \left\| \dot{\gamma}(t) \right\|^{-1} \dot{\gamma}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \lim_{t \uparrow a} \left\| \dot{\gamma}(t) \right\|^{-1} \dot{\gamma}(t) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \end{aligned} \tag{3.4}$$

So, $\gamma : [0,a] \to \mathbb{R}^2$ defines a simply closed path, symmetric with respect to the z-axis, with a cusp at $\gamma(0) = \gamma(a)$ and smooth in all other points.

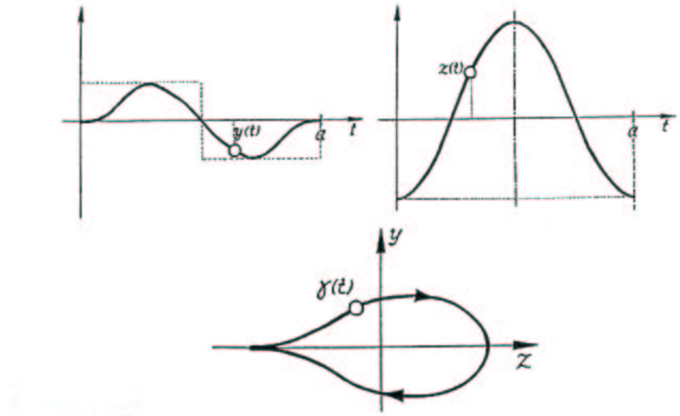

Figure 21

In particular, by (3.4) we have two continuously differentiable functions $\alpha, \beta : [0, a] \to \mathbb{R}$ such that

(3.5) $$\tau(t) := \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix}, \ (t \in [0, a])$$

is the "*unit tangent vector*" to the path γ in the point $\gamma(t)$:

(3.6) $$\begin{aligned} &\text{a)} \ \ \tau(t) = \|\dot\gamma(t)\|^{-1}\dot\gamma(t) \text{ for all } t \in]0, a[; \\ &\text{b)} \ \ \tau(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \tau(a) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \end{aligned}$$

Now, we choose the twice continuously differentiable "*shape function*" $\varrho : [0, a] \to \mathbb{R}$ such that

(3.7) $$\begin{aligned} &\text{a)} \ \ \varrho(t) > 0 \text{ for all } t \in [0, a]; \\ &\text{b)} \ \ \varrho(a) = \varrho(0) \text{ and } \varrho(a - t) \neq \varrho(t) \text{ for } t \in]0, \frac{a}{2}[; \\ &\text{c)} \ \ \text{If } t_0 \in]0, a[\text{ and } \dot y(t_0) = 0, \text{ then } \varrho(t_0) < |y(t_0)|. \end{aligned}$$

(3.8) $$\begin{aligned} &\text{a)} \ \ \dot\varrho(0) = \dot\varrho(a) > 0, \\ &\text{b)} \ \ \text{If } t_1 \in]0, a[\text{ with } \dot\varrho(t_1) = 0, \text{ then } \|\dot\gamma(t_1)\| > \varrho(t_1)\|\dot\tau(t_1)\|, \\ &\qquad (\text{e.g. } \varrho(t_1) < r_\gamma(t_1) := \text{ radius of curvature of } \gamma \text{ at } t_1). \end{aligned}$$

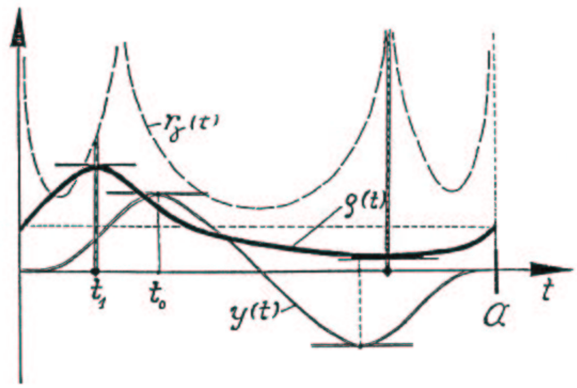

Figure 22

Now, following the idea expressed by Figure 20, we try to parametrize our model by the map

$$\overline{\psi} : \mathbb{S} \to \mathbb{R}^3 ; \overline{\psi}(u,v) := \begin{bmatrix} \varrho(u)\cos\left(\frac{2\pi}{b}v\right) \\ y(u) + \varrho(u)\beta(u)\sin\left(\frac{2\pi}{b}v\right) \\ z(u) - \varrho(u)\alpha(u)\sin\left(\frac{2\pi}{b}v\right) \end{bmatrix} . \tag{3.9}$$

We consider the *boundary* of $\overline{\mathbb{S}}$:

$$\partial\overline{\mathbb{S}} := \overline{\mathbb{S}} \setminus \mathring{\mathbb{S}} = \left\{ (u,v) \in \overline{\mathbb{S}} \mid u \in \{0,a\} \text{ or } v = \pm\frac{b}{2} \right\} . \tag{3.10}$$

Then, on use of statements (3.3) a), b), c), (3.6) b) and (3.7) a), b) we can check that our parametrization satisfies our "gluing prescriptions" along the boundary:

$$\begin{aligned} &\text{a)} \quad \overline{\psi}^{-1}\left(\overline{\psi}(0,\frac{b}{2})\right) \cap \partial\overline{\mathbb{S}} = \left\{ \left(0, \pm\frac{b}{2}\right), \left(a, \pm\frac{b}{2}\right) \right\} ; \\ &\text{b)} \quad \overline{\psi}^{-1}\left(\overline{\psi}(u,\frac{b}{2})\right) \cap \partial\overline{\mathbb{S}} = \left\{ \left(u, \pm\frac{b}{2}\right) \right\} \text{ for all } u \in]0,a[; \\ &\text{c)} \quad \overline{\psi}^{-1}\left(\overline{\psi}(0,v)\right) \cap \partial\overline{\mathbb{S}} = \{(0,v),(a,-v)\} \text{ for all } v \in]-\frac{b}{2},\frac{b}{2}[. \end{aligned} \tag{3.11}$$

Moreover, by (3.4) a), (3.6) a), b) and (3.8) a), b) and keeping in mind that $\tau \cdot \dot{\tau} \equiv 0$, one may prove

$$\overline{\psi} : \mathbb{S} \to \overline{\mathbb{R}}^3 \text{ is an immersion , (cf (2.5) for this notion) .} \tag{3.12}$$

Finally, by (3.4) a) and (3.8) a), we see that

$$\begin{aligned} &\text{a)} \quad \frac{\partial}{\partial u}\overline{\psi}(0,v) = \frac{\partial}{\partial u}\overline{\psi}(a,v) \text{ for all } v \in \left[-\frac{b}{2},\frac{b}{2}\right] ; \\ &\text{b)} \quad \frac{\partial}{\partial v}\overline{\psi}\left(u,\frac{b}{2}\right) = \frac{\partial}{\partial v}\overline{\psi}\left(u,-\frac{b}{2}\right) \text{ for all } u \in [0,a]. \end{aligned} \tag{3.13}$$

Statements (3.12) and (3.13) say that $\overline{\Psi} : \overline{\mathbb{S}} \to \mathbb{R}^3$ parametrizes an "unceased" surface. So, $\overline{\psi}(\overline{\mathbb{S}}) =: \overline{\mathbb{K}}$ may be viewed as a mathematical description of our model of the Klein bottle. What

we cannot achieve is that the map $\overline{\psi}\restriction_{\overset{\circ}{\mathbb{S}}}: \overset{\circ}{\mathbb{S}} \to \mathbb{R}^3$ is injective, so that the model $\overline{\psi}(\mathbb{S}) = \overline{\mathbb{K}}$ would have no self-penetration.
Now, to describe the "actual" Klein bottle instead of its model, we introduce the map

$$\psi : \overline{\mathbb{S}} \to \mathbb{R}^4;\ \psi(u,v) := \begin{bmatrix} \overline{\psi}(u,v) \\ \sin^2\left(\frac{\pi}{a}u\right) \end{bmatrix}. \tag{3.14}$$

For this map we have:

$$\text{If } (u,v) \in \overset{\circ}{\mathbb{S}}, (u',v') \in \overline{\mathbb{S}} \text{ and } \psi(u,v) = \psi(u',v'),\ \text{then } (u,v) = (u',v'). \tag{3.15}$$

The essential part in the proof of this statement is to show that $u' = u$. We give a few hints for this. Assume, that $u' \neq u$. Comparing fourth components of ψ, we obtain $u' = a - u, u \neq \frac{a}{2}$. In particular, we may assume $u \in]0, \frac{a}{2}[$. Assume first, that $\dot{y}(u) = 0$. Then (3.3) a) implies $\dot{y}(u') = \dot{y}(a-u) = 0$. Applying (3.7) c) with $t_0 = u$ resp. $t_0 = u'$ we get $\varrho(u) <\mid y(u) \mid, \varrho(u') < \big| y(u') \big|$, thus $\varrho(u) + \varrho(u') < 2 \mid y(u) \mid$ (see (3.3) a)). On the other hand, comparing second components of ψ and observing (3.3) a) we get $2y(u) = \varrho(u')\beta(u')\sin\left(\frac{2\pi}{b}v'\right) - \varrho(u)\beta(u)\sin\left(\frac{2\pi}{b}v\right)$. As $\mid \beta(u)\sin\left(\frac{2\pi}{b}v'\right) \mid, \mid \beta(u')\sin\left(\frac{2\pi}{b}v\right) \mid \leq 1$, this yields a contradiction.
Assume now, that $\dot{y}(u) \neq 0$, so that $\alpha(u) \neq 0$. As a consequence of (3.3) a) we have $z(a-u) = z(u), \alpha(a-u) = \alpha(u)$. So, comparing third components of ψ, we obtain $\varrho(a-u)\sin\left(\frac{2\pi}{b}v'\right) = \varrho(u)\sin\left(\frac{2\pi}{b}v\right)$. Comparing first components of ψ also gives $\varrho(a-u)\cos\left(\frac{2\pi}{b}v'\right) = \varrho(u)\cos\left(\frac{2\pi}{b}v\right)$. These two inequalities give $\varrho(a-u)^2 = \varrho(u)^2$. In view of (3.7) a), b), this yields again a contradiction.
Now, on use of (3.15) and (3.11) we obtain

$$\psi \text{ is injective on the interior } \overset{\circ}{\mathbb{S}} \text{ of } \overline{\mathbb{S}}. \tag{3.16}$$

$$\begin{aligned} &\text{a)}\quad \psi^{-1}\left(\psi(0,\frac{b}{2})\right) = \left\{\left(0, \pm\frac{b}{2}\right), \left(a, \pm\frac{b}{2}\right)\right\}; \\ &\text{b)}\quad \psi^{-1}\left(\psi(u,\frac{b}{2})\right) = \left\{\left(u, \pm\frac{b}{2}\right)\right\} \text{ for all } u \in]0,a[; \\ &\text{c)}\quad \psi^{-1}\left(\psi(0,v)\right) = \{(0,v),(a,-v)\} \text{ for all } v \in \left]-\frac{b}{2}, \frac{b}{2}\right[. \end{aligned} \tag{3.17}$$

Also,by (3.12) and (3.13) it follows easily

$$\psi : \overline{\mathbb{S}} \to \mathbb{R}^4 \text{ is an immersion .} \tag{3.18}$$

$$\begin{aligned} &\text{a)}\quad \frac{\partial}{\partial u}\psi(0,v) = \frac{\partial}{\partial u}\psi(a,v) \text{ for all } v \in \left[-\frac{b}{2}, \frac{b}{2}\right]; \\ &\text{b)}\quad \frac{\partial}{\partial v}\psi\left(u, \frac{b}{2}\right) = \frac{\partial}{\partial v}\psi\left(u, -\frac{b}{2}\right) \text{ for all } u \in [0,a]. \end{aligned} \tag{3.19}$$

But (3.16) and (3.17) say that $\psi : \overline{\mathbb{S}} \to \mathbb{R}^4$ parametrizes a surface $\mathbb{K} := \psi(\overline{\mathbb{S}}) \subseteq \mathbb{R}^4$ without self-penetration and such that our "gluing prescriptions" are respected by ψ. So, we may consider $\psi(\overline{\mathbb{S}}) = \mathbb{K} \subseteq \mathbb{R}^4$ as a mathematical description of the Klein bottle.

To understand the relation between the Klein bottle $\mathbb{K} = \psi(\overline{\mathbb{S}}) \subseteq \mathbb{R}^4$ and its model $\overline{\mathbb{K}} = \overline{\psi}(\overline{\mathbb{S}}) \subseteq \mathbb{R}^3$, we consider the projection

$$\pi : \mathbb{R}^4 \to \mathbb{R}^3; \quad \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{3.20}$$

Then, we have

$$\pi \circ \psi = \overline{\psi}, \text{ thus } \overline{\mathbb{K}} = \overline{\psi}(\overline{\mathbb{S}}) = \pi\left(\psi(\overline{\mathbb{S}})\right) = \pi(\mathbb{K}). \tag{3.21}$$

So $\overline{\mathbb{K}}$ is the image of $\mathbb{K}$ under the projection π, as indicated in Figure 18.

Also for any $c \in]0, b]$ and with $\mathbb{S}' := \left\{(u,v) \in \overline{\mathbb{S}} \mid v \in \left]-\frac{c}{2}, \frac{c}{2}\right[\right\}$ we see from (3.16) - (3.19):

$$\begin{aligned} &\mathbb{M}' := \psi(\mathbb{S}') \subseteq \psi(\overline{\mathbb{S}}) = \mathbb{K} \text{ is a Möbius strip in } \mathbb{R}^4 \text{ parametrized by} \\ &\psi \restriction_{\mathbb{S}'} : \mathbb{S}' \to \mathbb{R}^4 \ (\text{cf } (2.7)) \ . \end{aligned} \tag{3.22}$$

So, what has been shown heuristically in Figure 19 has found mathematical justification now.

Finally, for any $n \geq 4$ we may define a set $\mathbb{K} \subseteq \mathbb{R}^n$ to be a (embedded) *Klein bottle* if there is an immersive map $\psi : \overline{\mathbb{S}} \to \mathbb{R}^n$ satisfying (3.16), (3.17) and (3.19) such that $\mathbb{K} = \psi(\overline{\mathbb{S}})$. Then, similarly as in (2.8) any two Klein bottles are topologically equivalent.

For further reading on the Klein bottle we recommend [Ap], [Fr], [Gr] or [H-Co].

4. The Projective Plane

In the previous section we have produced a closed surface by gluing opposite edges of a rectangle, reversing just one of the edges before gluing it to its opposite. So, it seems natural, to try to glue opposite edges of a rectangle after having reversed two edges which are not opposite to each other. To describe the resulting imaginative process we rather like to start with a square instead of a general rectangle - just in view of the "symmetry of the situation". So, we wish to produce a surface by gluing opposite edges of a square as indicated by Figure 23.

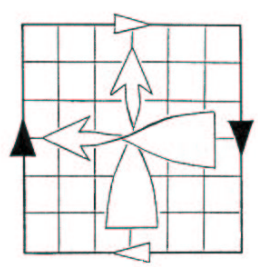

Figure 23

The Figures 24 - 27 show four consecutive steps of an imaginative deformation and gluing procedure which should produce a surface of the requested type. Having in mind our experience with the Klein bottle it is not surprising that again, we cannot solve our task without admitting self-penetrations on the resulting surface (cf Figure 26). If we accept to do so, we end up with a surface as shown in Figure 27.

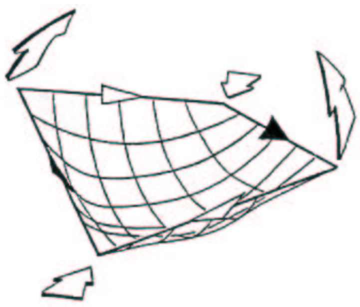

Figure 24

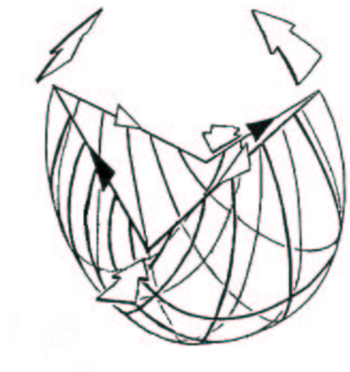

Figure 25

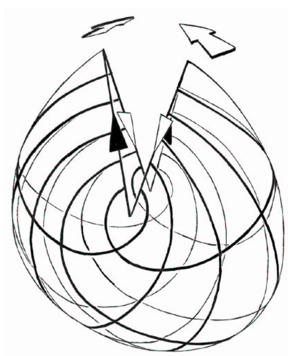

Figure 26

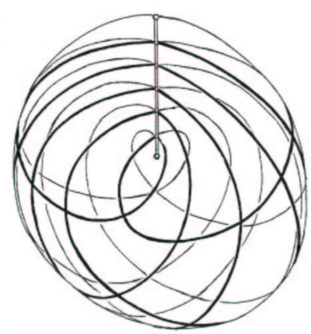

Figure 27

In Figure 28, another presentation of the obtained surface is given. This surface is a so called (closed) *cross cap*, a name which is made evident by Figure 29. This type of surface was discovered by the native Swiss mathematician J. Steiner (1796 - 1863), who was a professor in Berlin.

Figure 28

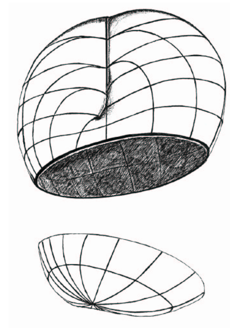

Figure 29

Comparing with the Klein bottle, we expect that our cross cap surface is again a "model" of some other surface which has no self-penetrations and is not contained in 3-space. This new "invisible" surface then would solve our gluing problem "properly". To identify this surface heuristically, we consider the interior of our original square as an appropriately shrinked plane. Then, the points on the boundary of the square correspond to "points at infinity" of our plane. According to our gluing prescription, the surface we are looking for is obtained by identifying two points at infinity if they appear in opposite directions (cf Figure 30).

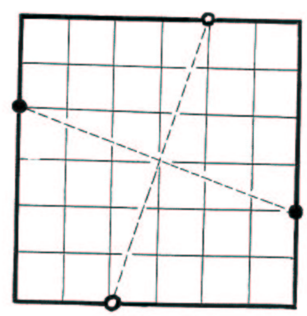

Figure 30

But this is just one way, in which the *projective plane* may be obtained from the "ordinary" (affine) plane. So, the surface produced by our gluing process is actually a projective plane $\mathbb{P}^2$. Therefore, the surface shown in Figure 28 is called the *cross cap model of the projective plane.*

If we delete the horizontal edges of our square, our gluing-prescriptions correspond precisely to those which furnish a Möbius strip (cf Section 2). So, the projective plane must contain a Möbius strip, and in fact this also shows up at the cross cap model (cf Figure 31).

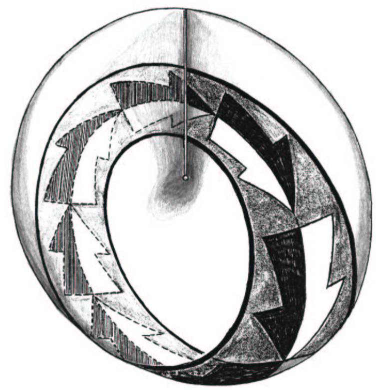

Figure 31

To be more formal, let $(x : y : z) \in \mathbb{P}^2$ denote the point with homogeneous coordinates x, y, z $\big((x, y, z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}\big)$, let us consider the standard sphere

$$S^2 := \big\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\big\} \tag{4.1}$$

and the natural map

$$\mathbb{P} : S^2 \to \mathbb{P}^2;\ (x, y, z) \mapsto (x : y : z). \tag{4.2}$$

Then clearly

$$\begin{aligned} &\text{a)}\ \ \mathbb{P} \text{ is surjective ;} \\ &\text{b)}\ \ \mathbb{P}^{-1}(\mathbb{P}(P)) = \{P, -P\},\ \text{ for all } P \in S^2. \end{aligned} \tag{4.3}$$

Moreover, $\mathbb{P}^2$ carries the *quotient topology* of S^2, so that a set $U \subseteq \mathbb{P}^2$ is open, if and only $\mathbb{P}^{-1}(U) \subseteq S^2$ is. But this means:

$$\mathbb{P} \text{ is continuous and open (e.g. maps open sets to open sets).} \tag{4.4}$$

Now, let $a > 0$, consider the square

$$\mathbb{T} := \{(u, v) \in \mathbb{R}^2 \mid |u|, |v| \leq \frac{a}{2}\}, \tag{4.5}$$

the unit disk

$$\mathbb{D} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \tag{4.6}$$

and the continuous map

$$\varepsilon : \mathbb{T} \to \mathbb{D}; \varepsilon(u, v) := \begin{cases} (0,0), & \text{if } u = v = 0 \\ \frac{2\max\{|u|,|v|\}}{a\sqrt{u^2+v^2}}(u, v), & \text{otherwise} \end{cases}, \tag{4.7}$$

which is in fact a homeomorphism. Also, we consider the continuous map

$$\delta : \mathbb{D} \to S^2;\ (x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2}) \tag{4.8}$$

and the composition (cf Figure 32)

$$\eta : \mathbb{T} \to \mathbb{P}^2,\ \eta := p \circ \delta \circ \varepsilon. \tag{4.9}$$

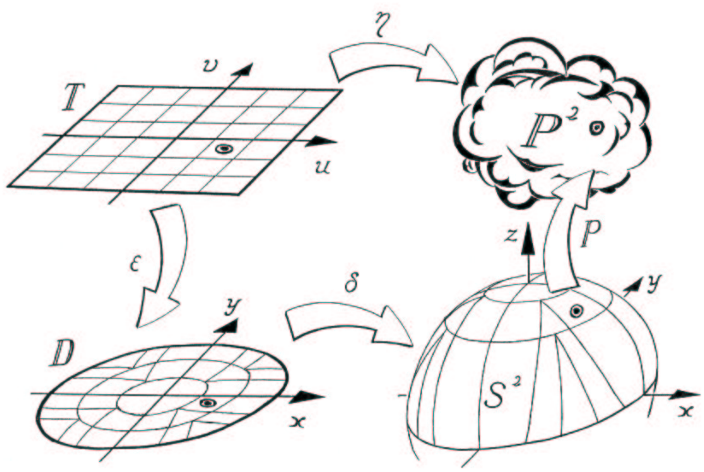

Figure 32

Then introducing the *interior* $\mathring{\mathbb{T}} := \{(u,v) \in \mathbb{R}^2 \mid |u|, |v| < \frac{a}{2}\}$ and the boundary $\partial\mathbb{T} := \mathbb{T} \backslash \mathring{\mathbb{T}}$ of $\mathbb{T}$ we obtain:

(4.10)

a) $\eta : \mathbb{T} \to \mathbb{P}^2$ is surjective and continuous ;

b) $\eta \restriction_{\mathring{\mathbb{T}}} : \mathring{\mathbb{T}} \to \mathbb{P}^2$ is injective and open ;

c) $\eta^{-1}\left(\eta(P)\right) = \{P, -P\}$ for all $P \in \partial\mathbb{T}$.

So $\eta : \mathbb{T} \to \mathbb{P}^2$ is a (topological) parametrization of $\mathbb{P}^2$ such that our gluing prescriptions are satisfied. Hence, in strict mathematical terms $\mathbb{P}^2$ is a (topological) solution of our gluing problem in which self-penetrations are avoided.

One could use this parametrization to describe the cross cap model. But we prefer another way to do so, which is much more comfortable. Let us begin with a few preliminary remarks.

A map $\mu : S^2 \to \mathbb{L}$ from the sphere S^2 to some set $\mathbb{L}$ is said to be *even*, if $\mu(-P) = \mu(P)$ for all $P \in S^2$. According to (4.3) an even map $\mu : S^2 \to \mathbb{L}$ induces a unique map $\mu^* : \mathbb{P}^2 \to \mathbb{L}$ such that $\mu^*\left(p(P)\right) = \mu(P)$ for all $P \in S^2$. In these notations we get by (4.3) and (4.4)

(4.11)

a) The assignment $\mu \mapsto \mu^*$ gives a one to one correspondence between all even maps $\mu : S^2 \to \mathbb{L}$ and all maps $\mu^* : \mathbb{P}^2 \to \mathbb{L}$.

b) If $\mathbb{L}$ is a topological space, then μ is continuous (resp. open) if μ^* is .

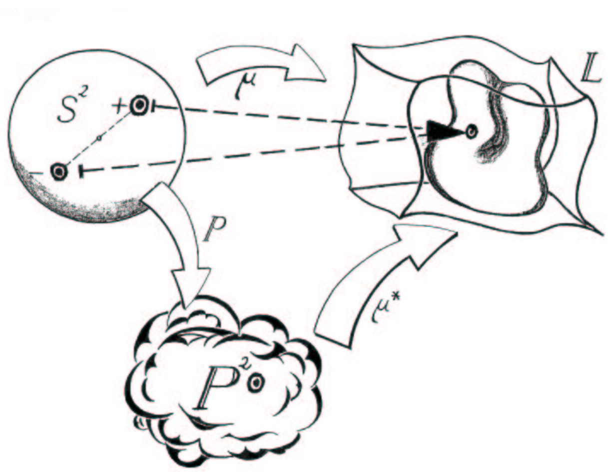

Figure 33

Now, consider the map

$$\overline{\nu} : \mathbb{R}^3 \to \mathbb{R}^3; \overline{\nu}(u,v,w) := \begin{bmatrix} x(u,v,w) := & vw \\ y(u,v,w) := & 2uv \\ z(u,v,w) := & u^2 - v^2 \end{bmatrix}. \tag{4.12}$$

Then, clearly $\overline{\nu} \restriction_{S^2} : S^2 \to \mathbb{R}^3$ is an even continuous map and hence induces a continuous map $\overline{\mu} := (\overline{\nu} \restriction_{S^2})^* : \mathbb{P}^2 \to \mathbb{R}^3$ given by

$$\overline{\mu}(u : v : w) = \frac{\overline{\nu}(u,v,w)}{u^2 + v^2 + w^2}. \tag{4.13}$$

Now, $\overline{\mu}(\mathbb{P}^2) = \overline{\nu}(S^2)$ has the shape of our original cross cap model (cf Figure 34)

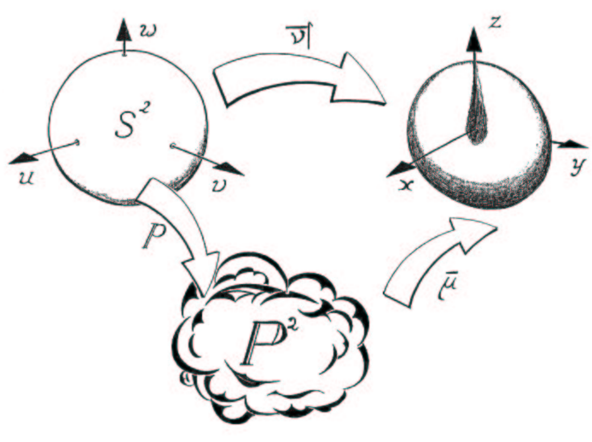

Figure 34

It is easy to verify that for each point $(u, v, w) \in S^2$ we have

$$(\overline{\nu} \restriction_{S^2})^{-1} (\overline{\nu}(u, v, w)) = \begin{cases} \{(\varepsilon u, \varepsilon v, \varepsilon w) \mid \varepsilon \in \{\pm 1\}\}, & \text{if } v \neq 0 \\ \{(\varepsilon u, 0, \delta w) \mid \varepsilon, \delta \in \{\pm 1\}\}, & \text{if } v = 0 \end{cases}. \tag{4.14}$$

In particular we see that for each point $(x, y, z) \in \overline{\mu}(\mathbb{P}^2)$

$$\sharp \overline{\mu}^{-1}(x, y, z) = \begin{cases} 2, & \text{if } x = y = 0,\ 0 < z < 1 \\ 1, & \text{otherwise}. \end{cases} \tag{4.15}$$

This shows in strict mathematical terms that the cross cap model $\overline{\mu}(\mathbb{P}^2)$ has a self-penetration along the set $\{(0, 0, z) \mid 0 < z < 1\}$. In particular $\overline{\mu} : \mathbb{P}^2 \twoheadrightarrow \overline{\mu}(\mathbb{P}^2)$ is not injective. This is in perfect accordance with the fact that $\mathbb{P}^2$ is a "closed non-orientable" surface as indicated by our gluing prescriptions. But, on use of the map (4.12) it is easy to embed $\mathbb{P}^2$ topologically into $\mathbb{R}^4$. Namely, consider the map

$$\nu : \mathbb{R}^3 \to \mathbb{R}^4;\ \nu(u, v, w) := \begin{bmatrix} \overline{\nu}(u, v, w) \\ uw \end{bmatrix}. \tag{4.16}$$

Then, clearly $\nu \restriction_{S^2} : S^2 \to \mathbb{R}^4$ is an even continuous map. Moreover, from (4.14) we see

$$(\nu \restriction_{S^2})^{-1} (\nu(u, v, w)) = \{(\varepsilon u, \varepsilon v, \varepsilon w) \mid \varepsilon \in \{\pm 1\}\} \text{ for all } (u, v, w) \in S^2. \tag{4.17}$$

Therefore, there is an induced map $\mu := (\nu \restriction_{S^2})^* : \mathbb{P}^2 \to \mathbb{R}^4$ and we can say:

$$\mu : \mathbb{P}^2 \to \mathbb{R}^4 \text{ is given by } \mu(u; v; w) = \frac{\nu(u, v, w)}{u^2 + v^2 + w^2}, \text{ continuous, open and injective}. \tag{4.18}$$

Thus μ embeds $\mathbb{P}^2$ into $\mathbb{R}^4$ and $\mu(\mathbb{P}^2) \subseteq \mathbb{R}^4$ may be viewed as an embedded copy of $\mathbb{P}^2$. Finally, if we consider the projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ of (3.20) we have

$$\pi \circ \mu = \overline{\mu},\ \text{thus } \overline{\mu}(\mathbb{P}^2) = \pi\left(\mu(\mathbb{P}^2)\right). \tag{4.19}$$

So, the cross cap $\overline{\mu}(\mathbb{P}^2)$ is obtained by projecting the embedded copy $\mu(\mathbb{P}^2) \subseteq \mathbb{R}^4$ of the projective plane to the space $\mathbb{R}^3$ - a situation similar to what we met for the Klein bottle. On the other hand - as we shall see later, there is also an important difference between the two situations.

If we cut out an appropriate piece of the cross cap model around its self-intersection, we get a surface of the type shown in Figure 35 and - with a Möbius strip on it - in Figure 36.

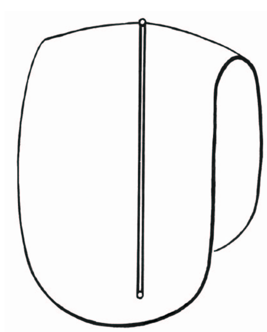

Figure 35

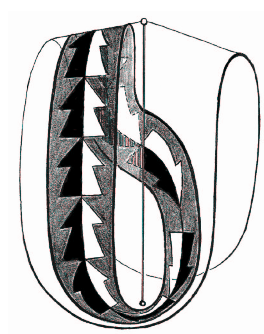

Figure 36

This surface sometimes is called a *Plücker conoid*, after the German mathematician J. Plücker (1801 - 1868). We rather like to refer to the above surface as a *double Whitney umbrella*: if we cut our surface horizontally, we get two so called *Whitney umbrellas*, as shown in Figure 37. This latter type of surface is named after the American mathematician H. Whitney (1907 - 1989) and it contains a typical singularity, sometimes called a *pinch point* (see Figure 38).

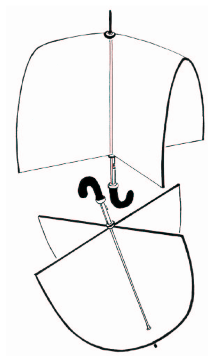

Figure 37

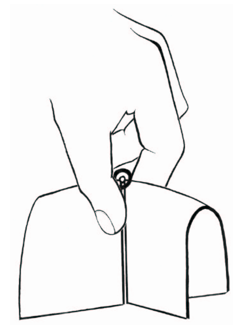

Figure 38

As shown in Figure 39, a double Whitney umbrella may be produced easily from a piece of paper. Using transparent paper, prepared as shown in Figure 40, one may produce the version shown in Figure 36.

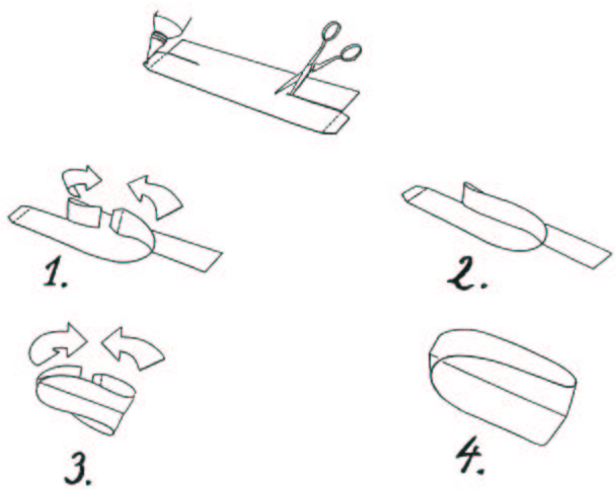

Figure 39

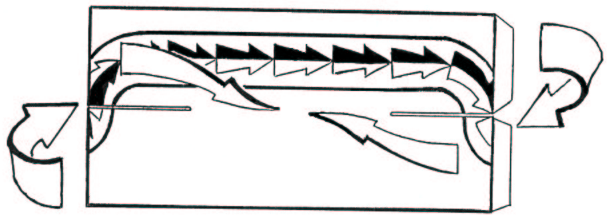

Figure 40

So, at least for a moment, we are back to our original activity: the making of surfaces by paper. Also, we do not forget to take the scissors and to cut a double Whitney umbrella as indicated in Figure 41.

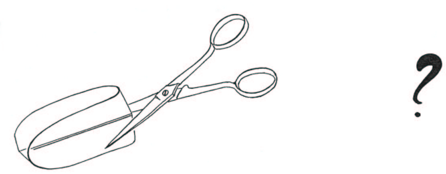

Figure 41

As the cross cap model has two pinch points (arising from the double Whitney umbrella) it is "ceased" in these two points: in a pinch point, a surface admits no tangent planes. So, the cross cap model is "not immersive" — in contrast to the immersive model of the Klein bottle we met earlier.

In the next paragraph, we try to make this more precise.

We consider the maps $\overline{\nu} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\nu : \mathbb{R}^3 \to \mathbb{R}^4$ of (4.12) resp. (4.16). Both maps are continuously differentiable and their Jacobians $\partial\overline{\nu}$ resp. $\partial\nu$ have the properties

$$\text{(4.20)} \quad \text{a)}\ \ Ker\,(\partial\overline{\nu}(u,v,w)) \begin{cases} \subseteq \mathbb{R}\begin{bmatrix} u \\ v \\ w \end{bmatrix}, & \text{for all } (u,v,w) \in S^2 \backslash \{(0,0,\pm 1),(\pm 1,0,0)\} \\ \nsubseteq \mathbb{R}\begin{bmatrix} u \\ v \\ w \end{bmatrix}, & \text{for } (u,v,w) \in \{(0,0,\pm 1),(\pm 1,0,0)\} \end{cases};$$

$$\text{b)}\ \ Ker\,(\partial\nu(u,v,w)) \subseteq \mathbb{R}\begin{bmatrix} u \\ v \\ w \end{bmatrix}, \text{ for all } (u,v,w) \in S^2.$$

Statement (4.20) b) says that for each point $P \in S^2$, the differential $d_P\nu = \partial\nu(P) : \mathbb{R}^3 \to \mathbb{R}^4$ is injective on the tangent vector space $T_PS^2 \subseteq \mathbb{R}^3$ of the sphere S^2 in P. This means (or can be taken as an ad hoc definition of the property) that the map $\nu\restriction_{S^2}: S^2 \to \mathbb{R}^4$ and the induced map $\mu : \mathbb{P}^2 \to \mathbb{R}^4$ (cf (4.13)) are immersive.

On the other hand (4.20) a) says that the differential $d_P\overline{\nu} = \partial\overline{\nu}(P) : \mathbb{R}^3 \to \mathbb{R}^3$ is injective on T_PS^2 if $P \in S^2 \setminus \{(0,0,\pm1),(\pm1,0,0)\}$ and non-injective if P is one of the four points $(0,0,\pm1),(\pm1,0,0) \in S^2$. This means that the map $\overline{\nu}\restriction_{S^2}: S^2 \to \mathbb{R}^3$ is singular at these four points and that the induced map $\overline{\mu} : \mathbb{P}^2 \to \mathbb{P}^3$ is singular at the two points $(0:0:1),(1:0:0) \in \mathbb{P}^2$. Indeed, these singularities appear on the cross cap model: their images $(0,0,0) = \overline{\nu}(0,0,\pm1) = \overline{\mu}(0:0:1)$ and $(1,0,0) = \overline{\nu}(\pm1,0,0) = \overline{\mu}(1:0:0)$ are precisely the two pinch points which occur on the cross cap, (Figure 34).

It is an everyday's experience, that things may appear rather different if considered from different points of view (cf Figure 42).

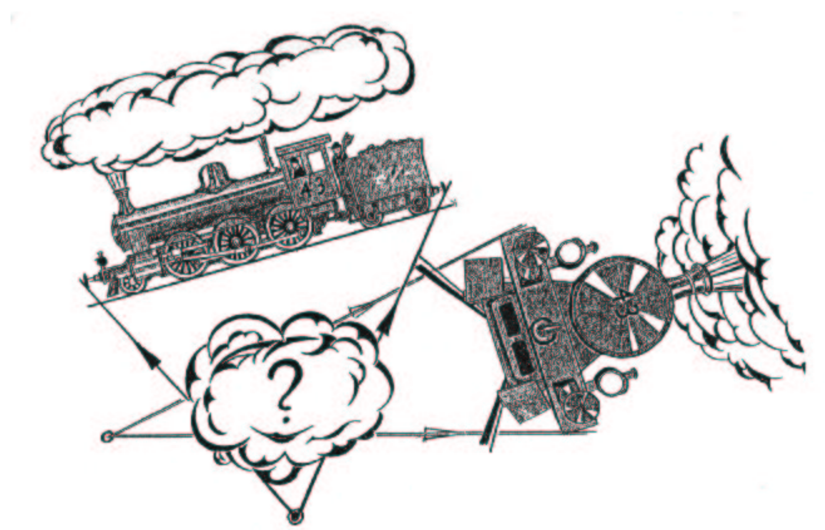

Figure 42

So, we expect that we also get rather different surfaces in 3-space, if "we look at (an embedded copy of) the projective plane from different directions". More precisely, we expect that there are models of the projective plane which look rather different from the cross cap model. Such a new model is shown in Figure 43: the so called *Roman surface*, also "discovered" by J. Steiner in 1844, on the occasion of a journey to Rome. This model has self-intersections along three pairwise orthogonal line segments which meet in their centers. Moreover the Roman surface contains three double Whitney umbrellas and hence 6 pinch points.

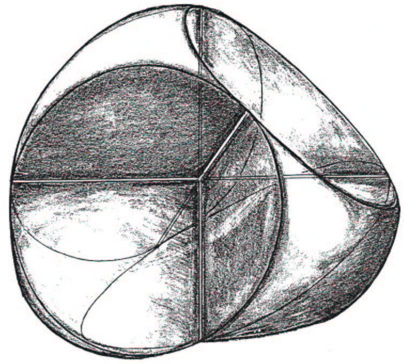

Figure 43

So, baring in mind what is shown in Figure 42 one is tempted to imagine a situation as shown in Figure 44.

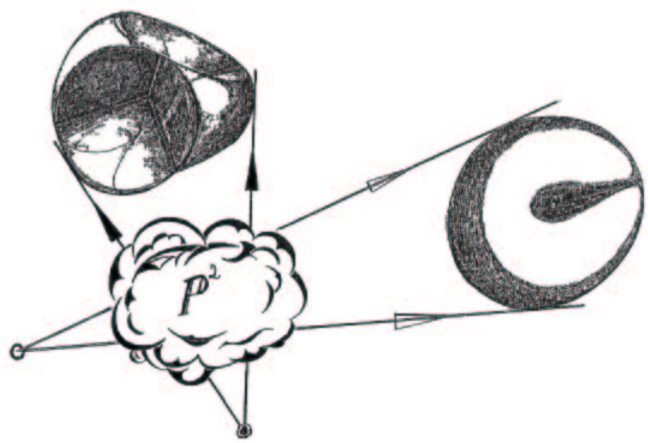

Figure 44

But is the situation really as simple as this? In order to answer this question in a satisfactory way, we have to describe the Roman surface in strict mathematical terms.

We consider the map

$$\varpi : \mathbb{R}^3 \to \mathbb{R}^3;\ \varpi(u,v,w) := \begin{bmatrix} vw \\ uv \\ uw \end{bmatrix}. \tag{4.21}$$

Then, clearly $\varpi \restriction_{S^2}: S^2 \to \mathbb{R}^3$ is an even continuous map and hence induces a continuous map $\overline{\sigma} := (\varpi \restriction_{S^2})^* : \mathbb{P}^2 \to \mathbb{R}^3$ given by

$$\overline{\sigma}(u:v:w) = \frac{\varpi(u,v,w)}{u^2+v^2+w^2}. \tag{4.22}$$

Now, $\overline{\sigma}(\mathbb{P}^2) = \varpi(S^2)$ has in fact the shape of the Roman surface (cf Figure 45).

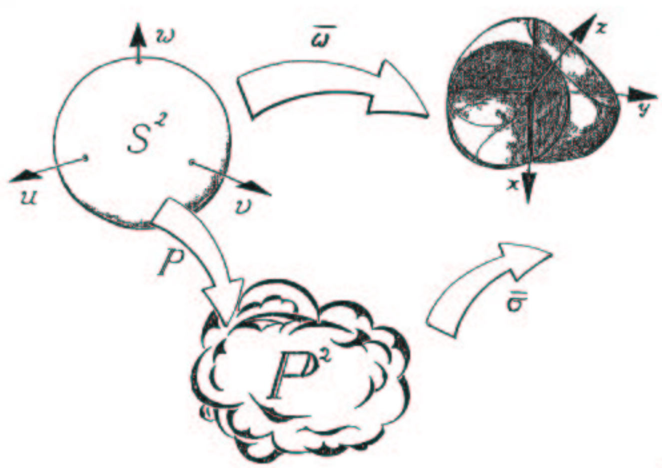

Figure 45

Now, consider the projection resp. the dilatation

$$\text{(4.23)}\qquad \begin{aligned} &\text{a)}\quad \zeta : \mathbb{R}^4 \to \mathbb{R}^3; \quad \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ t \end{bmatrix}; \\ &\text{b)}\quad \xi : \mathbb{R}^3 \to \mathbb{R}^3; \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ 2y \\ z \end{bmatrix}. \end{aligned}$$

Then, we have the commutative diagram

$$\text{(4.24)}\qquad \begin{array}{ccccc} \mathbb{P}^2 & = & \mathbb{P}^2 & \xrightarrow{\overline{\sigma}} & \mathbb{R}^3 \\ \overline{\mu}\big\downarrow & & \big\downarrow\mu & & \big\downarrow\xi \\ \mathbb{R}^3 & \xleftarrow{\pi} & \mathbb{R}^4 & \xrightarrow{\zeta} & \mathbb{R}^3 \end{array}$$

which shows that

$$\text{(4.25)}\qquad \overline{\mu}(\mathbb{P}^2) = \pi\left(\mu(\mathbb{P}^2)\right), \; \xi\left(\overline{\sigma}(\mathbb{P}^2)\right) = \zeta\left(\mu(\mathbb{P}^2)\right).$$

So (up to a dilatation by means of ξ) the Roman surface $\overline{\sigma}(\mathbb{P}^2)$ is obtained by projecting the embedded copy $\mu(\mathbb{P}^2) \subseteq \mathbb{R}^4$ of $\mathbb{P}^2$ to $\mathbb{R}^3$ by ζ, whereas the cross cap model $\overline{\mu}(\mathbb{P}^2)$ is obtained by projecting $\mu(\mathbb{P}^2)$ to $\mathbb{R}^3$ by π.

So, up to a dilatation, the Roman surface and the cross cap model are obtained by projecting the same embedded copy $\mu(\mathbb{P}^2)$ of $\mathbb{P}^2$ in $\mathbb{R}^4$ in two different directions. This comes close to the situation shown in Figure 44.

In spite of their rather different appearance, the cross cap model and the Roman surface have something striking in common: both contain (double) Whitney umbrellas and hence pinch points (cf Figure 46).

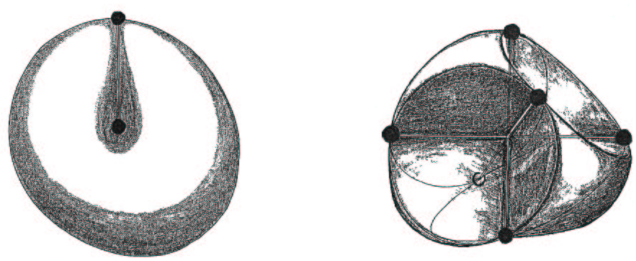

Figure 46

This makes arise the question, whether there are models of $\mathbb{P}^2$ in 3-space having no pinch points or — better — no singularities at all. Such "immersed models" of $\mathbb{P}^2$ in $\mathbb{R}^3$ actually exist. They are called *Boy surfaces*, after W. Boy (a student of D. Hilbert), who discovered such a surface in 1901. In Figure 47 two such surfaces are shown.

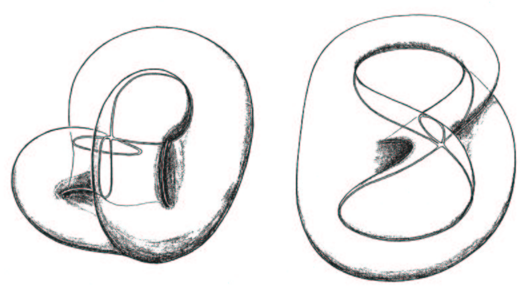

Figure 47

Given an even continuously differentiable map $\overline{\beta} : \mathbb{R}^3 \to \mathbb{R}^3$ with the property that

$$Ker\left(\partial\overline{\beta}(u,v,w)\right) \subseteq \mathbb{R}\begin{bmatrix} u \\ v \\ w \end{bmatrix} \text{ for all } (u,v,w) \in S^2 \tag{4.26}$$

(cf (4.20)), one gets a Boy surface $\mathbb{B}$ by setting $\mathbb{B} := \overline{\beta}(S^2)$. Moreover, each Boy surface may be obtained in this way. Obviously one also would subject $\overline{\beta}$ to further restrictions in order to keep the self-intersections of $\mathbb{B}$ as simple as possible: often, a Boy surface is understood also to satisfy the additional condition of having *transversal self-intersections of a specific type* (s [Ap]).

It is astonishingly complicated to find a map $\overline{\beta}$ which satisfies the above requirements. Also, it is known, that the components of such a map cannot be homogeneous polynomials of degree 2, (s

[Ap, pg 61]). Observe that the underlying maps $\overline{\nu}$ and $\overline{\omega}$ of the cross cap and the Roman surface are given by homogeneous polynomials of degree 2.

To conclude our considerations, let us recall three fundamental results of H. Whitney (cf [Wh]), which explain the phenomena we observed in the course of this paper from a general point of view:

($*$) *Any closed differentiable surface embeds immersively in* $\mathbb{R}^4$.

($**$) *Any closed smooth surface in* $\mathbb{R}^4$ *can be projected to a surface in* $\mathbb{R}^3$ *whose singularities all arise as pinch points on Whitney umbrellas.*

($***$) *There is no immersion* $\mathbb{R}^2 \to \mathbb{R}^3$ *which is obtained by embedding* $\mathbb{P}^2$ *into* $\mathbb{R}^4$ *and then projecting to* $\mathbb{R}^3$.

Our experience with the Klein bottle and the projective plane are in perfect accordance with ($*$). Obviously, we must admit that we have only used ad hoc descriptions of these two surfaces and have not made explicit their differentiable structures.

The cross cap model and the Roman surface are (essentially) both obtained by embedding $\mathbb{P}^2$ immersively into $\mathbb{R}^4$ and then projecting to $\mathbb{R}^3$. On both surfaces, all singularities arise on Whitney umbrellas. So, these two models illustrate what is said in statement ($**$).

Finally, the cross cap model and the Roman surface are singular, e.g. not obtained by immersions $\mathbb{P}^2 \to \mathbb{R}^3$. As both surfaces are obtained by projecting an immersively embedded copy of $\mathbb{P}^2$ in $\mathbb{R}^4$, this is in perfect accordance with statement ($***$).

For further reading on the projective plane and its models we recommend [Ap] and also [Br-K], [Gr] and [H-Co].

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A FEW INTERNET ADRESSES CONCERENING OUR SUBJECT...

http://mathworld.wolfram.com/MoebiusStrip.html
http://mathworld.wolfram.com/RealProjectivePlane.html
http://www.scidiv.bcc.ctc.edu/Math/Mobius.html
http://www.cut-the-knot.com/content.html
http://astronomy.swin.edu.au/pbourke/geometry/mobius/
http://astronomy.swin.edu.au/pbourke/geometry/crosscap/
http://astronomy.swin.edu.au/pbourke/geometry/klein/
http://astronomy.swin.edu.au/pbourke/
http://web.meson.org/topology/mobius.html
http://web.meson.org/topology/projective.html
http://web.meson.org/klein.html
http://vizwiz.gmd.de/ nitkin/vismat_html/node2.html
http://www.esat.kuleuven.ac.be/ pollefey/tutorial/tutorialECCV.html

Recommendation: Try to do your own search to find more !

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