A Posteriori Error Estimation for the Poisson Equation with Mixed Dirichlet/Neumann Boundary Conditions

Sergey Repin[a], Stefan Sauter[b] and Anton Smolianski[b]

[a] V.A. Steklov Institute of Mathematics
Russian Academy of Sciences
191011 St. Petersburg, Russia
E-mail: repin@pdmi.ras.ru

[b] Institute of Mathematics, University of Zurich
CH-8057 Zurich, Switzerland
E-mails: stas@amath.unizh.ch, antsmol@amath.unizh.ch

Abstract

The present work is devoted to the a posteriori error estimation for the Poisson equation with mixed Dirichlet/Neumann boundary conditions. Using the duality technique we derive a reliable and efficient a posteriori error estimator that measures the error in the energy norm. The estimator can be used in assessing the error of any approximate solution which belongs to the Sobolev space H^1, independently of the discretization method chosen. Only two global constants appear in the definition of the estimator; both constants depend solely on the domain geometry, and the estimator is quite non-sensitive to the error in the constants evaluation. It is also shown how accurately the estimator captures the local error distribution, thus, creating a base for a justified adaptivity of an approximation.

Key words and phrases: mixed Dirichlet/Neumann boundary conditions, a posteriori error estimator, reliability, efficiency, local error distribution

1 Introduction

A posteriori error estimation is known to be essential for reliable scientific computing, and many research efforts have been focused on this subject during the last decade (see, e.g., the monographs [1], [3], [14] and also [4], [8]). It is now well-understood that an adaptivity (e.g., an adaptive mesh refinement) is, in general, required to calculate the approximate solution accurately and efficiently. It is also clear that the discrete solution should be supplemented by a reliable estimate of the corresponding discretisation error, in order to provide the evidence that the problem has been solved with the prescribed accuracy. Both tasks can be successfully accomplished with the use of an a posteriori error estimator, if the latter correctly represents the local error distribution and serves as a guaranteed upper bound for the exact error. In this work, we derive an a posteriori error estimator possessing these properties and being, in addition, independent of any particular discretisation; it can, thus, be used in combination with finite element or finite difference or finite volume method, as well as for assessing the error in a post-processed solution.

In its original form, the estimator (like most existing a posteriori error estimators) is derived under the assumption that the approximate solution satisfies the Dirichlet boundary condition exactly. However, in many practically interesting cases, the essential boundary condition can be satisfied merely approximately either owing to complicated, e.g., non-polynomial Dirichlet data or because of accounting the boundary condition in a weak sense, like in fictitious domain methods (see, e.g., [6]). Thus, the approximate solution does not, in general, belong to the set of admissible functions of the original problem, i.e. presents a *non-conforming* approximation to the exact solution (see [13]). It will be shown here how the estimator can be modified to take into account the error in the approximation of the Dirichlet boundary condition. This issue has been recently addressed in [12] for the case of the Dirichlet boundary value problem; the present work can be considered as an extension of the results of [12] to the case of mixed Dirichlet/Neumann boundary conditions.

2 Preliminaries

Let Ω be a bounded domain with Lipschitz continuous boundary Γ. Suppose that Γ consists of two measurable parts Γ_D and Γ_N, and the area (the length in 2D) of Γ_D is non-zero. Consider the mixed boundary value problem: Find a function u such that

$$-\Delta u = f \qquad \text{in } \Omega\,, \tag{2.1}$$

$$u = u_0 \qquad \text{on } \Gamma_D\,, \tag{2.2}$$

$$\tfrac{\partial u}{\partial n} = g \qquad \text{on } \Gamma_N\,, \tag{2.3}$$

where the trace of the given function $u_0 \in H^1(\Omega)$ defines the boundary condition on Γ_D and n is the outward normal to Γ. Throughout this paper, we assume that $f \in L^2(\Omega),\ g \in L^2(\Gamma_N)$.

In the sequel, we will use the notation $|\cdot|$ for the standard Euclidean norm of a vector and $\|\cdot\|$ for the L^2-norm on Ω.

Let $V_0 := \{w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma_D\}$ and $V_0 + u_0 := \{v \in H^1(\Omega) \mid v = w + u_0,\ w \in V_0\}$. A weak formulation of the problem (2.1)–(2.3) is: Find $u \in V_0 + u_0$, such that

$$\int\limits_\Omega \nabla u \cdot \nabla w \, dx = \int\limits_\Omega f w \, dx + \int\limits_{\Gamma_N} g w \, ds \quad \forall w \in V_0\,. \tag{2.4}$$

It is well known that the solution to this problem exists and is unique. This solution can be characterised equivalently as the minimiser of the following variational problem:

Problem $\mathcal{P}$. Find $u \in V_0 + u_0$ such that $\quad J(u) = \inf\limits_{v \in V_0 + u_0} J(v)$,

where

$$J(v) = \frac{1}{2} \int\limits_\Omega |\nabla v|^2 \, dx - \int\limits_\Omega f v \, dx - \int\limits_{\Gamma_N} g v \, ds\,.$$

To derive the dual variational formulation we employ the relation

$$J(u) = \inf_{v \in V_0 + u_0} \sup_{y^* \in L^2(\Omega, \mathbb{R}^n)} \left\{ \int\limits_\Omega \left(\nabla v \cdot y^* - \frac{1}{2} |y^*|^2 - f v \right) dx - \int\limits_{\Gamma_N} g v \, ds \right\}.$$

Then, using the representation $v = w + u_0$ with $w \in V_0$, we derive

Problem $\mathcal{P}^*$. Find $p^* \in Q^*_{f,g}$ such that $I^*(p^*) = \sup\limits_{q^* \in Q^*_{f,g}} I^*(q^*)$,

where

$$I^*(q^*) = \int\limits_\Omega \left(\nabla u_0 \cdot q^* - \frac{1}{2} \mid q^* \mid^2 - f u_0 \right) dx - \int\limits_{\Gamma_N} g u_0 \, ds$$

is the dual variational functional and

$$Q^*_{f,g} := \{ q^* \in L^2(\Omega, \mathbb{R}^n) \mid \operatorname{div} q^* = -f \text{ in } \Omega,\ q^* \cdot n = g \text{ on } \Gamma_N \} . \tag{2.5}$$

Both problems $\mathcal{P}$ and $\mathcal{P}^*$ have unique solutions u and p^*, which satisfy the duality relations (see, e.g., [7])

$$J(u) = I^*(p^*) , \quad \nabla u = p^* . \tag{2.6}$$

In view of (2.4), we have: $J(v) - J(u) = \frac{1}{2} \int\limits_\Omega \mid \nabla(v - u) \mid^2 \ dx$ for all $v \in V_0 + u_0$.

Using (2.6), one derives $\frac{1}{2} \|\nabla(v-u)\|^2 = \inf\limits_{q^* \in Q^*_{f,g}} \{ J(v) - I^*(q^*) \}$, and, since $J(v) - I^*(q^*) = \frac{1}{2} \int\limits_\Omega \mid \nabla v - q^* \mid^2 \ dx$ for all $v \in V_0 + u_0$ and $q^* \in Q^*_{f,g}$, we obtain:

$$\|\nabla(v-u)\|^2 = \inf_{q^* \in Q^*_{f,g}} \|\nabla v - q^*\|^2 \quad \forall v \in V_0 + u_0 . \tag{2.7}$$

From (2.7) we immediately see that $\nabla(v-u)$ in the L^2-norm (which may be viewed as the approximation error) is majorised by the L^2-norm of the difference $(\nabla v - q^*)$ with any $q^* \in Q^*_{f,g}$. However, if q^* does not belong to $Q^*_{f,g}$, the L^2-norm of the difference $(\nabla v - q^*)$ does not, in general, provide an upper bound for the error. This means that any numerical approximation of q^* should satisfy the constraint $q^* \in Q^*_{f,g}$ with very high accuracy in order to guarantee a reliability of the error estimate; thus, the estimate (2.7) is not very useful for practical application. Our further efforts will be focused on finding a computable upper bound for the right-hand side in (2.7).

3 Error majorant for conforming approximations

Functional-type a posteriori error majorants for conforming approximations have been derived by using general minimax theorems of convex analysis in

[10], [11]. For linear elliptic problems under consideration we will present in this section a much simplified way of deriving functional-type a posteriori estimates using a variant of the Helmholtz decomposition for the space $L^2(\Omega, \mathbb{R}^n)$.

Let y^* be any function from the space $\tilde{H}(\Omega, \mathrm{div}) := \{y^* \in L^2(\Omega, \mathbb{R}^n) \mid \mathrm{div}\, y^* \in L^2(\Omega)\,,\ y^* \cdot n \in L^2(\Gamma_N)\}$. Obviously, $\tilde{H}(\Omega, \mathrm{div})$ is a Banach space with the norm $\|y^*\|_{\tilde{H}(\Omega,\mathrm{div})} = \Big(\|y^*\|^2 + \|\mathrm{div}\, y^*\|^2 + \|y^* \cdot n\|^2_{L^2(\Gamma_N)}\Big)^{1/2}$.

For $y^* \in \tilde{H}(\Omega, \mathrm{div})$, define the auxiliary function w as the solution to the problem:

$$\Delta w = \mathrm{div}\, y^* + f \qquad \text{in } \Omega\,, \tag{3.1}$$

$$w = 0 \qquad \text{on } \Gamma_D\,, \tag{3.2}$$

$$\frac{\partial w}{\partial n} = y^* \cdot n - g \qquad \text{on } \Gamma_N\,. \tag{3.3}$$

It is evident that the problem (3.1)–(3.3) has a unique solution $w \in V_0$. Consider now the function $q^* := y^* - \nabla w$. It is clear that $q^* \in L^2(\Omega, \mathbb{R}^n)$; moreover, $\mathrm{div}\, q^* = \mathrm{div}\, y^* - \Delta w = -f$ in Ω and $q^* \cdot n = y^* \cdot n - \frac{\partial w}{\partial n} = g$ on Γ_N (see (3.1) and (3.3)). Thus, $q^* \in Q^*_{f,g}$. It is worth noting that the decomposition $y^* = q^* + \nabla w$ may be viewed as a non-orthogonal variant of the Helmholtz decomposition (see [9]).

Substituting $q^* = y^* - \nabla w$ into (2.7) and using Young's inequality, we obtain the estimate

$$\|\nabla(v-u)\|^2 \le (1+\beta)\|\nabla v - y^*\|^2 + (1+\frac{1}{\beta})\,\|\nabla w\|^2 \quad \forall \beta > 0\,,\ \forall v \in V_0 + u_0\,, \tag{3.4}$$

which is valid for any $y^* \in \tilde{H}(\Omega, \mathrm{div})$ and w defined by (3.1)–(3.3).

Since $w \in V_0$ and $\Delta w \in L^2(\Omega)$, the second term on the right-hand side of the last inequality can be estimated by

$$\|\nabla w\|^2 = \int\limits_{\Gamma_N} \frac{\partial w}{\partial n} w\, ds - \int\limits_{\Omega} (\Delta w) w\, dx\,,$$

$$\|\nabla w\|^2 \le \left\|\frac{\partial w}{\partial n}\right\|_{L^2(\Gamma_N)} C_{\Gamma_N}(1 + C_\Omega^2)^{1/2}\|\nabla w\| + C_\Omega\|\Delta w\|\,\|\nabla w\|\,,$$

that is

$$\|\nabla w\| \le C_{\Gamma_N}(1 + C_\Omega^2)^{1/2}\left\|\frac{\partial w}{\partial n}\right\|_{L^2(\Gamma_N)} + C_\Omega\|\Delta w\|\,, \tag{3.5}$$

where C_Ω is the constant in Friedrichs' inequality $\left(C_\Omega^2 := \sup\limits_{w \in V_0 \setminus \{0\}} \frac{\|w\|^2}{\|\nabla w\|^2} \right)$ and C_{Γ_N} is the constant in the trace inequality $\left(C_{\Gamma_N}^2 := \sup\limits_{w \in V_0 \setminus \{0\}} \frac{\|w\|^2_{L^2(\Gamma_N)}}{\|w\|^2_{H^1(\Omega)}} \right)$.

Using (3.5), (3.1), (3.3) and Young's inequality we deduce from (3.4):

$$\|\nabla(v-u)\|^2 \leq (1+\beta)\|\nabla v - y^*\|^2 + (1+\frac{1}{\beta})(1+\frac{1}{\gamma})\, C_{\Gamma_N}^2 (1+C_\Omega^2)\|y^* \cdot n - g\|^2_{L^2(\Gamma_N)}$$
$$+ (1+\frac{1}{\beta})(1+\gamma)\, C_\Omega^2 \|\mathrm{div}\, y^* + f\|^2 \quad \forall v \in V_0 + u_0\,,\ \forall y^* \in \tilde{H}(\Omega, \mathrm{div})\,. \quad (3.6)$$

Here β and γ are arbitrary positive numbers stemming from Young's inequality. Minimising the right-hand side of (3.6) with respect to the scalar parameters β and γ, we obtain the a posteriori error estimate for any approximate solution $v \in V_0 + u_0$:

$$\|\nabla(v-u)\| \leq \|\nabla v - y^*\| + C_{\Gamma_N}(1+C_\Omega^2)^{1/2}\|y^* \cdot n - g\|_{L^2(\Gamma_N)}$$
$$+ C_\Omega \|\mathrm{div}\, y^* + f\| \quad \forall y^* \in \tilde{H}(\Omega, \mathrm{div})\,. \qquad (3.7)$$

Remark 3.1 Denote the right-hand side of (3.7) by $M(v; y^*)$. The error majorant $M(v; y^*)$ has the following properties:
1) $M(v; y^*)$ is always *reliable*, i.e. provides an upper bound for the exact error, as long as $y^* \in \tilde{H}(\Omega, \mathrm{div})$.
2) $M(v; y^*)$ is *asymptotically exact* in the sense that, if $y^* \to p^* = \nabla u$ in $\tilde{H}(\Omega, \mathrm{div})$, $M(v; y^*) \to \|\nabla(v-u)\|$.
3) Since $M(v; y^*) \leq \|\nabla(v-u)\| + \|y^* - p^*\| + C_{\Gamma_N}(1+C_\Omega^2)^{1/2}\|(y^*-p^*) \cdot n\|_{L^2(\Gamma_N)} + C_\Omega\|\mathrm{div}\,(y^*-p^*)\|$, $M(v; y^*)$ is also *efficient* (i.e. provides a reasonable lower bound for the exact error) if $y^* \to p^*$ in $\tilde{H}(\Omega, \mathrm{div})$.
4) Evidently, the majorant's first term $m_d(v; y^*) := \|\nabla v - y^*\|$ computed over any subdomain $\omega \subset \Omega$ is close to the exact error $\|\nabla(v-u)\|$ on ω if y^* is close to p^* in $L^2(\Omega, \mathbb{R}^n)$.

The following Proposition shows how to find a suitable $y^* \in \tilde{H}(\Omega, \mathrm{div})$ (the proof is very similar to the one for the analogous Proposition in [12]).

Proposition 3.1 *Let $v \in V_0 + u_0$ and β, γ be some positive numbers. Let $y^*_{\beta\gamma} \in \tilde{H}(\Omega, \mathrm{div})$ be the minimiser of $M^2(v; y^*, \beta, \gamma)$, where $M^2(v; y^*, \beta, \gamma)$ denotes the right-hand side of inequality (3.6).*

Then, for any $\gamma > 0$, $y^*_{\beta\gamma}$ converges to $p^* = \nabla u$ in $\tilde{H}(\Omega, \mathrm{div})$ as $\beta \to 0$, and, moreover,

$$\|y^*_{\beta\gamma} - p^*\| \leq C\beta^{1/2}\,,$$

$$\|(y^* - p^*)\cdot n\|_{L^2(\Gamma_N)} \leq C\beta\frac{1}{1+\gamma}\left(\frac{\gamma}{\min(\frac{1}{\gamma}C_1, C_2)}\right)^{1/2}\,,$$

$$\|\mathrm{div}\,(y^*_\beta - p^*)\| \leq C\beta\frac{1}{1+\gamma}\left(\frac{1}{\min(\frac{1}{\gamma}C_1, C_2)}\right)^{1/2}\,,$$

where $C_1 = C^2_{\Gamma_N}(1 + C^2_\Omega)$, $C_2 = C^2_\Omega$ and the constant C depends only on f, v, g, C_Ω, C_{Γ_N}.

Remark 3.2
1) Obviously, if $\gamma \to 0$:

$$\|(y^* - p^*)\cdot n\|_{L^2(\Gamma_N)} = \mathcal{O}(\beta\gamma^{1/2})\,,\ \|\mathrm{div}\,(y^*_\beta - p^*)\| = \mathcal{O}(\beta)\,.$$

The Proposition also implies that $\|y^*_{\beta\gamma} - p^*\|_{\tilde{H}(\Omega,\mathrm{div})} = \mathcal{O}(\beta^{1/2})$ when $\beta \to 0$.
2) Proposition 3.1 suggests that, taking a small value β_0 of the parameter β, we can obtain a reasonably good $y^*_{\beta_0\gamma}$ to be used in the error majorant M. It is important to note that, unlike in the penalty method for solving the dual problem, we do not have to set β_0 very small as it is not necessary to strongly enforce the constraint $\mathrm{div}\, y^* = -f$ in $L^2(\Omega)$ for obtaining a good estimator.
3) Taking a small γ we can make the term $\|y^*\cdot n - g\|_{L^2(\Gamma_N)}$ be of higher order in the majorant as compared to other terms.
4) The constants C_Ω and C_{Γ_N} appear in the majorant $M(v; y^*)$ in front of the terms which are of the order $\mathcal{O}(\beta)$ and $\mathcal{O}(\beta\gamma^{1/2})$ respectively (for small β and γ). Thus, the majorant is generally not sensitive to possible errors in the constants evaluation.

Remark 3.3
1) The Friedrichs constant C_Ω can be easily estimated owing to the fact that C_Ω^{-2} is the smallest eigenvalue of the Laplace operator in Ω equipped with the homogeneous mixed boundary conditions (the Dirichlet condition on Γ_D and the Neumann on Γ_N). It is important that C_Ω must be evaluated only once

for each particular domain Ω as the constant depends only on the domain geometry.
2) For polygonal domains in 2D, the trace constant C_{Γ_N} can be estimated as follows. Suppose $\Gamma_N = \cup_{i=1}^{K} \Gamma_N^{(i)}$, where each $\Gamma_N^{(i)}$ is a line segment. Then, for each $\Gamma_N^{(i)}$ there exists a parallelogram $\omega_i \subset \Omega$ having $\Gamma_N^{(i)}$ as one of its sides. It is proved in [5] that $\|w\|^2_{L^2(\Gamma_N^{(i)})} \leq 2\left(\frac{|\Gamma_N^{(i)}|}{|\omega_i|}\|w\|^2_{L^2(\omega_i)} + \frac{|\omega_i|}{|\Gamma_N^{(i)}|}\|\nabla w\|^2_{L^2(\omega_i)}\right)$ for any $w \in H^1(\omega_i)$. Thus, for all $w \in H^1(\Omega)$, $\|w\|^2_{L^2(\Gamma_N)} \leq 2\, C_{\max} \sum_{i=1}^{K} (\|w\|^2_{L^2(\omega_i)} + \|\nabla w\|^2_{L^2(\omega_i)})$, where $C_{\max} := \max_{1\leq i\leq K} \max\left\{\frac{|\Gamma_N^{(i)}|}{|\omega_i|}, \frac{|\omega_i|}{|\Gamma_N^{(i)}|}\right\}$. Denote $C_{\text{overlap}} := \max_{x\in\Omega} L(x)$, where $L(x)$ is the number of parallelograms ω_i containing the point x. Then, $\|w\|^2_{L^2(\Gamma_N)} \leq 2\, C_{\max} C_{\text{overlap}} \|w\|^2_{H^1(\Omega)}$, and we obtain the estimate $C^2_{\Gamma_N} \leq 2\, C_{\max} C_{\text{overlap}}$ which is easy to use. For the 3D case and a polyhedral domain, C_{Γ_N} can be estimated analogously.

4 A posteriori error estimate for functions that do not exactly satisfy the Dirichlet boundary condition

Consider the problem (2.1)–(2.3) and assume that v is any function from $H^1(\Omega)$ that satisfies the boundary condition (2.2) only approximately. Then, our aim is to control this additional error by an a posteriori error estimate.

In view of Young's inequality, we have

$$\|\nabla(v-u)\|^2 \leq (1+\frac{1}{\alpha_1})\|\nabla(v-\tilde{v})\|^2 + (1+\alpha_1)\|\nabla(\tilde{v}-u)\|^2 ,$$

where α_1 is any positive number and $\tilde{v} \in V_0 + u_0$. Applying the estimate (3.6) to the second term on the right-hand side and using once again Young's inequality one can obtain (see also [12])

$$\|\nabla(v-u)\| \leq 2m_0(v) + m_d(v,y^*) + C_{\Gamma_N}(1+C_\Omega^2)^{1/2} m_g(y^*) + C_\Omega m_f(y^*) , \quad (4.1)$$

where y^* is any function from $\tilde{H}(\Omega, \text{div})$ and the following notation is used:

$$m_0(v) := \inf_{\tilde{v}\in V_0+u_0} \|\nabla(v-\tilde{v})\| , \qquad m_d(v,y^*) := \|\nabla v - y^*\| ,$$
$$m_g(y^*) := \|y^* \cdot n - g\|_{L^2(\Gamma_N)} , \qquad m_f(y^*) := \|\text{div}\, y^* + f\| .$$

The quantity $m_0(v)$ is non-negative and vanishes if and only if v exactly satisfies boundary condition (2.2). Thus, it is a measure of the error in the Dirichlet boundary condition. The quantity $m_d(v, y^*)$ is a measure of the error in the duality relation for the pair (v, y^*), $m_g(y^*)$ is a measure of the error in the normal component of the dual variable on the Neumann part of the boundary, and, finally, $m_f(y^*)$ measures the error in the equation for the dual variable ("equilibrium equation").

Remark 4.1 The quantity $m_0^2(v)$ cannot, in general, be computed directly, but one can estimate it from above, for example, by choosing the function $\tilde{v} \in V_0 + u_0$ as $\tilde{v}(x) = \frac{\Phi(x)}{\Phi_0} v(x) + (1 - \frac{\Phi(x)}{\Phi_0}) u_0(x)$ in the domain $\{x \in \Omega \mid \Phi(x) \leq \Phi_0\}$ and $\tilde{v} \equiv v$ on the rest of Ω, where $\Phi(x)$ is the distance from x to Γ_D and $\Phi_0 > 0$ is some fixed number.

5 Numerical examples

5.1 The Dirichlet-Neumann singularity

Here we consider the problem (2.1)–(2.3) in the unit square $\Omega = (0,1) \times (0,1)$ with $\Gamma_N = \{(x; y) \in \mathbb{R}^2 \mid x \in (0, 0.5)\ ,\ y = 1\}$, $\Gamma_D = \partial\Omega \setminus \Gamma_N$. We set the homogeneous boundary conditions and $f \equiv 1$ in Ω. This is the standard model problem with the solution exhibiting a singularity at the point $(0.5; 1)$ of the change in the type of boundary conditions.

The exact solution behaves like $r^{1/2} \sin(\frac{\theta}{2})$ (in polar coordinates (r, θ)) near the point $(0.5; 1)$, hence we can expect the convergence rate $\mathcal{O}(h^{1/2})$ in the energy norm if a uniform mesh refinement is used. This can really be observed in Figure 1 (left), where both the exact energy error $\|\nabla e\| := \|\nabla(v - u)\|$ and the majorant M are plotted in logarithmic scale (the convergence rate is shown in dependence on the total number of unknowns N, thus, the slope should be 1/4). We computed with P_1 finite elements for both the primal v and the dual y^* variables; as the exact solution the finite element approximation on a very fine grid was taken. Figure 2 demonstrates the local error distribution delivered by the elementwise values of the exact error and the m_d-term of the majorant.

The adaptive mesh refinement based on the elementwise values of the m_d-term drastically improves the convergence rate for both the exact error and the majorant, as clearly seen in Figure 1 (right). The convergence rate is recovered to optimal $\mathcal{O}(N^{1/2})$; Figure 3 illustrates the process of the adaptive mesh refinement (we have used the Delaunay refinement algorithm based on the addition of new nodes and the subsequent Delaunay triangulation of the resulting point set).

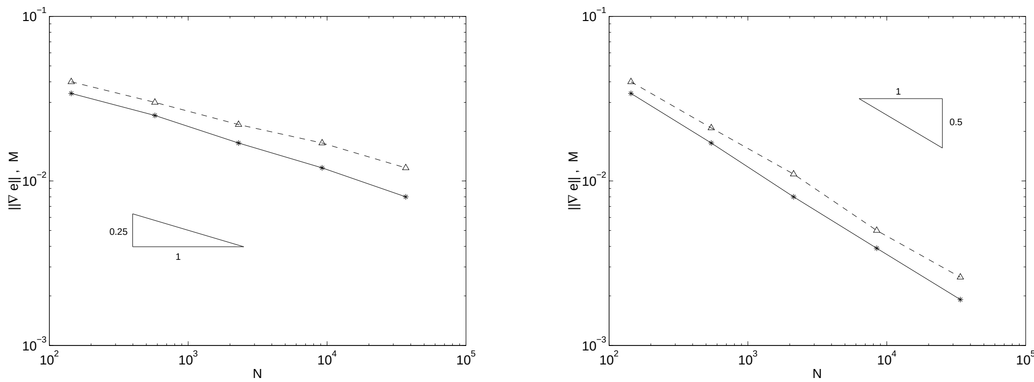

Figure 1: The convergence rates of the exact energy error ($*$) and the majorant ($\triangle$): (left) uniform mesh refinement, (right) adaptive mesh refinement.

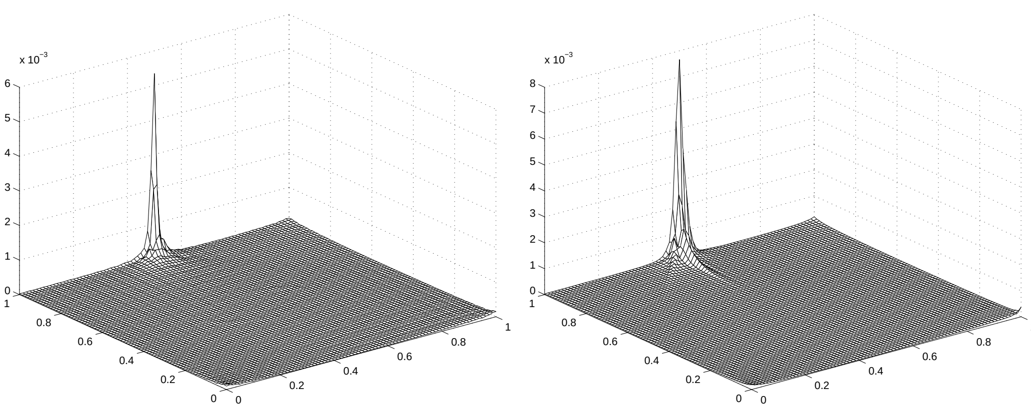

Figure 2: The local error distribution: (left) exact energy error, (right) m_d-term of the majorant.

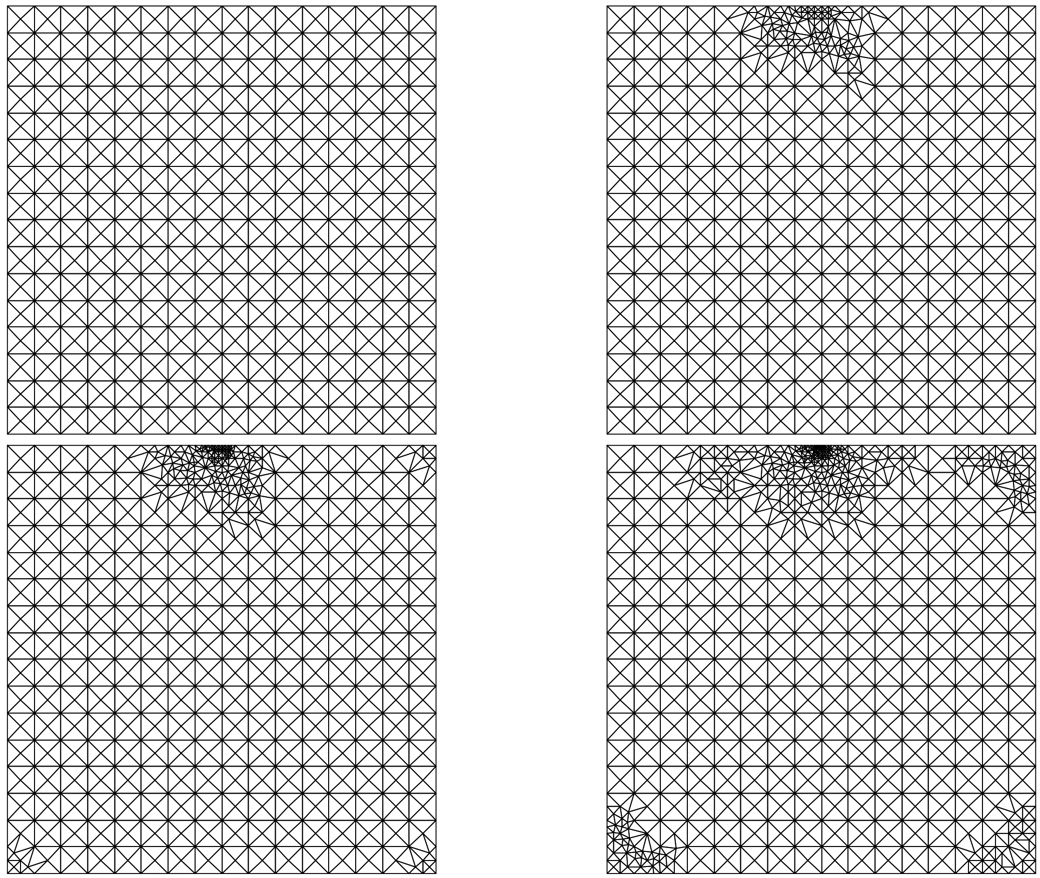

Figure 3: Adaptive mesh refinement based on the elementwise values of the m_d-term of the majorant.

5.2 Error estimation for non-conforming approximate solution

Consider the problem

$$-\Delta u = f \qquad \text{in } \Omega\,, \tag{5.1}$$

$$u = u_0 \qquad \text{on } \Gamma_D^1\,, \tag{5.2}$$

$$u = 0 \qquad \text{on } \Gamma_D^2\,, \tag{5.3}$$

$$\tfrac{\partial u}{\partial n} = g \qquad \text{on } \Gamma_N\,, \tag{5.4}$$

where $\Omega = (0,1) \times (0,1) \setminus \bar{\omega}$, $\omega = \{(x;y) \in \mathbb{R}^2 \mid \frac{(x-0.5)^2}{(0.2)^2} + \frac{(y-0.5)^2}{(0.1)^2} < 1\}$, $\Gamma_D^2 = \partial\omega$, $\Gamma_N = \{(x;y) \in \mathbb{R}^2 \mid x = 1\,,\ y \in (0.25, 0.75)\}$, Γ_D^1 is the rest of the boundary of the square $(0,1) \times (0,1)$ (see Figure 4), having the exact solution

$$u_{\text{exact}}(x,y) = x\left(1 - e^{-0.5\left(1-\frac{(x-0.5)^2}{(0.2)^2}-\frac{(y-0.5)^2}{(0.1)^2}\right)^2}\right).$$

The solution is shown in Figure 4, its trace on the boundary piece Γ_D^1 and its normal derivative on Γ_N define the remaining part of the boundary conditions on $\partial\Omega$.

To approximate the problem (5.1)–(5.4) we make use of the so-called penalty/fictitious-domain method which is rather popular in the computational fluid dynamics community (see, e.g. [2] and the references therein). The penalised problem reads as follows:

$$-\Delta u_\varepsilon + \tfrac{1}{\varepsilon}\chi_\omega\, u_\varepsilon = \tilde{f} \qquad \text{in } (0,1) \times (0,1)\,, \tag{5.5}$$

$$u_\varepsilon = u_0 \qquad \text{on } \Gamma_D^1\,, \tag{5.6}$$

$$\tfrac{\partial u_\varepsilon}{\partial n} = g \qquad \text{on } \Gamma_N\,, \tag{5.7}$$

where χ_ω is the characteristic function of the domain ω, and $\tilde{f}$ is the original right-hand side f extended into ω (for example, by some constant value; we set $\tilde{f} = 30.0$ in ω). The main peculiarity of the method is that the mesh does not have to be adjusted to the curvilinear parts of the domain boundary. However, this convenience is paid off by the unavoidable error in the approximation of the Dirichlet boundary condition on the curvilinear parts, where the condition is satisfied in a weak sense only. It is also worth noting that the finite element solution of the penalised problem does not, in general, possesses the Galerkin orthogonality property with respect to the original problem (which is a requirement of most existing error estimators).

We start with $\varepsilon = 10^{-7}$, which is small enough to make the "boundary error" (m_0) almost negligible as compared to the "domain error" ($m_d + C_\Omega m_f$), and observe the convergence rate of our majorant with respect to the mesh width in Figure 5. Obviously, the majorant decreases with optimal rate for both P_1 and P_2 finite elements (we used the same approximation for the primal and dual variables and a uniform mesh refinement).

Next, we investigate the influence of the boundary error by taking sufficiently rough penalty parameter, namely $\varepsilon = 10^{-1}$. We can obtain detailed information on the local distribution of the error near the boundary Γ_D^2 by computing the term m_0 elementwise in the narrow band around Γ_D^2 (see Remark 4.1). The evaluated local distribution of the boundary error is depicted in Figure 6 together with the exact error distribution obtained via the $H^{1/2}$-norm of the approximate solution on Γ_D^2 (the exact solution is zero on Γ_D^2). It is clearly seen that the qualitative behaviour of the elementwise error along the boundary Γ_D^2 is captured well.

The detailed information delivered by the term m_0 can be used to adaptively improve the approximation near Γ_D^2 until the boundary error will become smaller than the error in the domain.

In Figure 7 we demonstrate the convergence of the majorant with respect to the penalty parameter ε, when the mesh width is sufficiently small ($h = 1/240$). The convergence rate is slightly better than theoretically predicted $\mathcal{O}(\varepsilon^{1/4})$ (see [2]). The most important fact is that the majorant obviously provides a reliable upper bound for the exact error even when the error in the approximation of boundary conditions is very large.

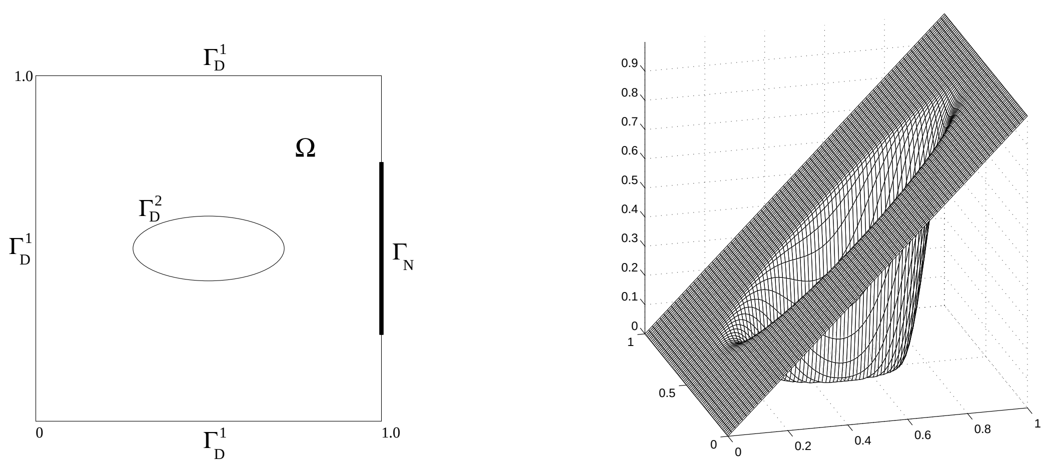

Figure 4: The domain geometry (left); the exact solution (right).

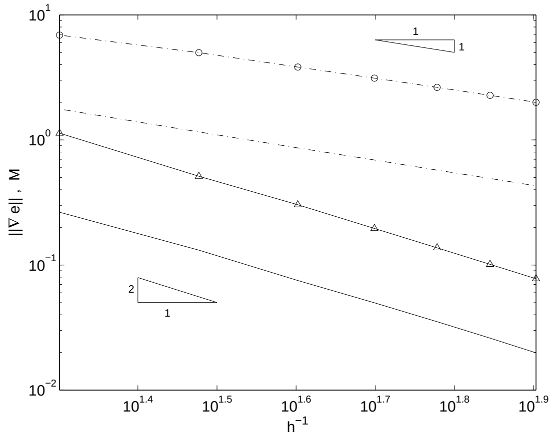

Figure 5: Decrease rates of the exact error and of the majorant ($\varepsilon = 10^{-7}$). The dash-dotted lines correspond to P_1 and the solid lines to P_2 elements; "$\circ$" and "$\triangle$" correspond to the majorant.

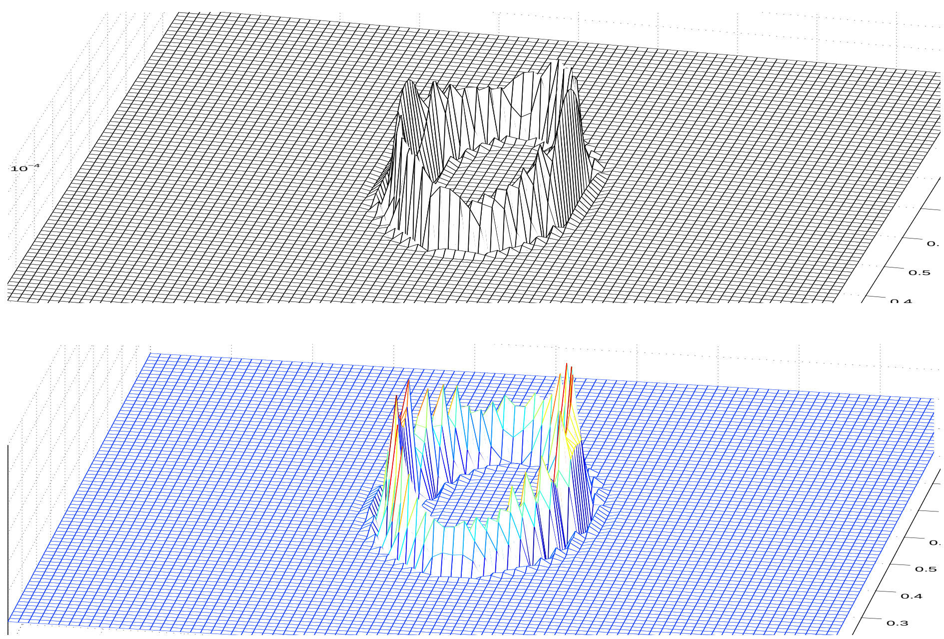

Figure 6: Local boundary-error distribution: exact error (top) and the m_0 term of the majorant (bottom); $\varepsilon = 10^{-1}$, P_1 elements, $h = 1/80$.

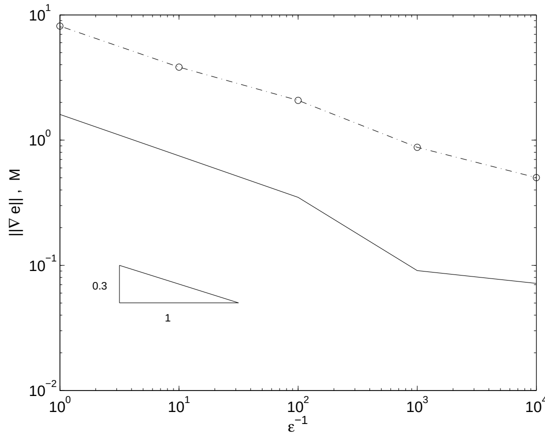

Figure 7: The convergence of the majorant (dash-dotted line) and the exact error (solid line) with respect to ε (P_1 elements, $h = 1/240$).

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