A refined Galerkin error and stability analysis for highly indefinite variational problems

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Abstract

Recently, a refined finite element analysis for highly indefinite Helmholtz problems was introduced by the second author. We generalise the analysis to the Galerkin method applied to an *abstract*, highly indefinite variational problem. In the refined analysis, the condition for stability and a quasi-optimal error estimate is expressed in terms of approximation properties $\mathcal{T}(S) \approx S$ and $\mathcal{T}(u + S) \approx S$. Here, u is the solution of the original variational problem, $\mathcal{T}$ is a certain continuous solution operator, and S is the finite dimensional test and trial space.

The abstract analysis can be applied to both finite and boundary element solutions of high-frequency Helmholtz problems. We apply the analysis to investigate the properties of the Brakhage-Werner boundary integral formulation of the Helmholtz problem, discretised by a standard Galerkin boundary element method. In the case of scattering by the unit sphere, we derive the explicit dependence of the error and of the stability condition on the wave number k. We show that $hk \lesssim 1$ is a sufficient condition for stability and a quasi-optimal error estimate. Further, we show that the constant of quasi-optimality is independent of k, which is an improvement over previously available results. Thus, the boundary element method does not suffer from the *pollution effect*.

1 Introduction

The numerical solution of high-frequency Helmholtz problems has attracted much interest in recent years; see for example [3, 4, 7, 10, 11, 12, 16, 27, 28]. The main aim of this paper is to develop a refined analysis for the error and the stability of the Galerkin discretisation of high-frequency Helmholtz problems. The analysis should be general enough to include both boundary and finite element methods and allow for discussion of standard and special finite/boundary elements such as the ones used in [22, 26, 28]. Most importantly, it should be possible to obtain optimal results on the dependence of the error bounds and the stability condition on the wave number k. The explicit dependence on k is rarely given in existing literature; for exceptions see [8, 11, 13].

It is well known that the Galerkin finite element method with standard piecewise polynomial basis functions suffers from the so called *pollution effect* [3]. If piecewise linear basis functions are used, the stability condition in the mesh width h is very strong: $hk^2 \lesssim 1$. In [3], a generalised finite element method was presented in one dimension, with the stability condition reduced to $hk \lesssim 1$; see also [16]. The proofs rely on the explicit knowledge of the Green's function and, hence, do not carry over to higher dimensions. Further, the general stability and convergence analysis given in [22] does not yield the improved stability condition. With this in mind, in [28] a refined finite element analysis was developed that gives improved stability and error estimates.

In this paper, we generalise the results of [28] to an abstract theory applicable to a general indefinite variational problem. We prove that the condition $\mathcal{T}(S) \approx S$, of approximate invariance of the test and trial space S under a certain continuous solution operator $\mathcal{T}$, is sufficient for stability.

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The quasi-optimal error estimate is proved under a similar condition $\mathcal{T}(u+S) \approx S$, where u is the solution of the continuous variational problem. This new concept is the crux of the abstract analysis we develop. We describe how the abstract analysis can be used to prove the results of [28] for the finite element method. As a further example of its use, we consider the boundary element method for the solution of high-frequency Helmholtz problems using the Brakhage-Werner boundary integral formulation. This problem has already been considered in [13] and recently in [11]. There, the stability condition $hk \lesssim 1$ and a quasi-optimal error estimate, with the constant of quasi-optimality proportional to $k^{1/3}$, was proved for the case of the unit sphere. In [11], the authors consider the problem of high-frequency scattering by a convex object in two dimensions. Known asymptotics of the scattered wave were used to reduce the problem to the computation of unknown amplitudes, which are less oscillatory than the original scattered wave. These were then computed using a Galerkin method for which the quasi-optimal error with constant of $\mathcal{O}(k^{1/3})$ was proved in the case of the unit disk and sphere.

We obtain a sharper error estimate, with the quasi-optimality constant independent of k. More importantly, our paper provides a framework in which to investigate the properties of boundary element methods with special basis elements such as plane waves [26]. For special finite element methods, it was already shown in [28] that the refined analysis obtains results outside the reach of standard analyses. We give reasons to expect the same to be true for boundary element methods. Further, the condition of the approximability of $\mathcal{T}(S)$ and $\mathcal{T}(u+S)$ by the boundary element space, can give guidelines for the construction of special boundary elements.

2 A highly indefinite variational problem

Let H and V be Hilbert spaces such that H is continuously imbedded in V, and hence, V' is continuously imbedded in H', where V' and H' are the dual spaces; see [31]. Denote by $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_V$ the respective inner products, and by $\|\cdot\|_H$ and $\|\cdot\|_V$ the induced norms.

We are interested in the following abstract variational problem: Given $f \in H'$, find $u \in H$ such that

$$a(u,v) = \langle f, v\rangle, \quad \text{for all } v \in H, \tag{1}$$

where $a(\cdot,\cdot) : H \times H \to \mathbb{C}$ and we have written $\langle f, v\rangle = f(v)$ for the value of the functional f at v.

Naturally, we need to place some conditions on the above problem.

Assumptions:

A1: $a(\cdot,\cdot) : H \times H \to \mathbb{C}$ is a bounded sesquilinear form. Thus, $a(u,v)$ is linear in u, conjugate linear in v, and

$$|a(u,v)| \le C_{\mathrm{c}} \|u\|_H \|v\|_H.$$

A2: There exist bounded sesquilinear forms $a_H(\cdot,\cdot) : H \times H \to \mathbb{C}$ and $a_V(\cdot,\cdot) : V \times V \to \mathbb{C}$ such that

$$a(u,v) = a_H(u,v) + a_V(u,v)$$

and

$$|a_H(u,u)| \ge \alpha_H \|u\|_H^2, \quad |a_V(u,v)| \le C_V \|u\|_V \|v\|_V, \text{ for any } u, v \in H.$$

A3: Problem (1) and its adjoint have a unique solution $u \in H$. Further,

$$\|u\|_H \le C_{\mathrm{reg}} \|f\|_{H'}.$$

The sesquilinear forms $a(\cdot,\cdot), a_H(\cdot,\cdot)$, and $a_V(\cdot,\cdot)$, define the corresponding bounded, linear operators:

$$A : H \to H', \quad A_H : H \to H', \quad \text{and } A_V : V \to V'. \tag{2}$$

In view of A3, the inverses of A and the adjoint A^* are also bounded linear operators:

$$A^{-1} : H' \to H \quad \text{and} \quad A^{*-1} : H' \to H. \tag{3}$$

We now investigate the properties of the Galerkin discretisation of (1).

2.1 Abstract stability and convergence analysis of the Galerkin method

Let $S \subset H$ be a finite dimensional subspace of H. We wish to consider the Galerkin discretisation of problem (1): Given $f \in H'$ find $u_S \in S$ such that

$$a(u_S, v) = \langle f, v \rangle, \quad \text{for all } v \in S. \tag{4}$$

We now derive a condition on S that guarantees the existence and uniqueness of u_S and a quasi-optimal error estimate.

2.1.1 Stability and convergence

For our analysis of the stability and convergence of (4), the following continuous dual problem will be crucial: Given $w \in H$, let $z \in H$ be such that

$$a(v, z) = -a_V(w, v), \quad \text{for all } v \in H.$$

From (A2) it follows that $a_V(w, \cdot)$ defines a bounded linear functional on V. Since H is continuously imbedded in V, i.e., the identity mapping $I : H \to V$ is continuous, $a_V(w, \cdot)$ defines also a bounded linear functional on H. Therefore, we can apply (A3) to obtain that the solution $z \in H$ of the above adjoint problem exists and is unique. Consequently, we can define a solution operator by $\mathcal{T}w := z$. Using again the fact that H is continuously imbedded in V and the properties of the operators in (2) and (3), we conclude that the solution operator $\mathcal{T} = -A^{*-1}A_V$ is a bounded linear operator mapping from H to H. Hence, there exists a constant $C_{\mathcal{T}}$ such that

$$\|\mathcal{T}u\|_H \leq C_{\mathcal{T}} \|u\|_H, \quad \text{for all } u \in H. \tag{5}$$

Remark 1. *In applications, the operator $\mathcal{T}$ will be a compact operator. Usually, it is also a smoothening operator; see Remark 4 and [28].*

Let us now define a measure of approximability in the space S. This measure depends on some subset $\widetilde{H} \subseteq H$, which satisfies $S \subset \widetilde{H}$ and $u + S \subset \widetilde{H}$, where u is the exact solution of (1). The measure is defined by

$$\eta(S) := \sup_{w \in \widetilde{H} \setminus \{0\}} \inf_{v \in S} \frac{\|\mathcal{T}w - v\|_H}{\|w\|_H}. \tag{6}$$

Remark 2.

1. *For a dense sequence $(S_l)_{l \geq 1}$ of spaces, i.e. $\overline{\cup_l S_l}^{\|\cdot\|_H} = H$, we have $\lim_{l \to \infty} \eta(S_l) = 0$.*
2. *We will prove stability of (4) and a quasi-optimal error estimate, under the condition that $\eta(S)$ is small enough.*
3. *Note that the choice $\widetilde{H} = H$ is always possible. However, a choice of a smaller set $\widetilde{H} \subsetneq H$ might result in a smaller value of $\eta(S)$ and a less restrictive stability condition.*

Theorem 1. *Let S be such that*

$$\eta(S) \leq \frac{\alpha_H}{2C_c}, \tag{7}$$

and let $u \in H$ be the solution of (1). Then there exists a unique solution $u_S \in S$ of the discrete problem (4). Moreover,

$$\|u - u_S\|_H \leq \frac{2C_c}{\alpha_H} \inf_{v \in S} \|u - v\|_H.$$

Proof. Since S is finite dimensional, it suffices to prove uniqueness. Given $w_S \in S$, let z_S be the best approximation to $z = \mathcal{T} w_S$ with respect to the H-norm. Then,

$$\begin{aligned} |a(w_S, w_S + z_S)| &= |a_H(w_S, w_S) - a(w_S, z - z_S)| \geq \alpha_H \|w_S\|_H^2 - C_c \|w_S\|_H \|z - z_S\|_H \\ &\geq \alpha_H \|w_S\|_H^2 - C_c \eta(S) \|w_S\|_H^2. \end{aligned}$$

From (5) we have that

$$\|z\|_H \leq C_{\mathcal{T}} \|w_S\|_H$$

and hence

$$\|w_S + z_S\|_H \leq \|w_S\|_H + \|z\|_H + \|z - z_S\|_H \leq (1 + C_{\mathcal{T}} + \eta(S)) \|w_S\|_H.$$

Using (7), we have that

$$|a(w_S, w_S + z_S)| \geq \frac{\alpha_H}{2} \|w_S\|_H^2 \geq \frac{\alpha_H}{2 + 2C_{\mathcal{T}} + 2\eta(S)} \|w_S\|_H \|w_S + z_S\|_H.$$

Hence, we have the discrete inf-sup condition:

$$\inf_{u \in S \setminus \{0\}} \sup_{v \in S \setminus \{0\}} \frac{|a(u, v)|}{\|u\|_H \|v\|_H} \geq \frac{\alpha_H}{2 + 2C_{\mathcal{T}} + 2\eta(S)} > 0,$$

and we have proved that the discrete solution u_S exists and is unique.

Next, let $z' = \mathcal{T} e$, where $e = u - u_S$, and again let z'_S be the best approximation to z' in the H-norm. Then,

$$|a_V(e, e)| = |a(e, z')| = |a(e, z' - z'_S)| \leq C_c \eta(S) \|e\|_H^2.$$

Hence, for any $v \in S$,

$$\begin{aligned} \alpha_H \|e\|_H^2 &\leq |a_H(e, e)| = |a(e, e) - a_V(e, e)| = |a(e, u - v) - a_V(e, e)| \\ &\leq C_c \|e\|_H \|u - v\|_H + C_c \eta(S) \|e\|_H^2. \end{aligned}$$

Therefore, using (7)

$$\|e\|_H \leq \frac{2C_c}{\alpha_H} \|u - v\|_H, \quad \text{for any } v \in S.$$

Thus, we have also proved the quasi-optimality of the Galerkin method. $\square$

Remark 3. *A result on the stability and convergence of the Galerkin finite element method applied to an indefinite PDE can be found in Theorem 5.7.6 of [6]. The same constant of quasi-optimality* $2C_c/\alpha_H$*, as above, is also given in [6]; this is an improvement over the usual estimate given by Céa's lemma, see Remark 5. The essential novelty of our concept, is that for stability and convergence it is sufficient to have* $\mathcal{T}(S) \approx S$ *and* $\mathcal{T}(u + S) \approx S$*. On the contrary, the approach taken in [6] requires that the adjoint problem has full regularity. Theorem 1 is a stronger result, that implies the result of [6]. In particular, the kind of condition given in [6] does not allow for improved stability estimates of [28]; for details see [28].*

2.1.2 Error estimate in the V-norm

By using the Aubin-Nitsche technique, we can bound the V-norm of the error by the H-norm of the error. Let $\psi \in H$ be such that

$$a(v, \psi) = (e, v)_V, \quad \text{for all } v \in H.$$

Let $\mathcal{S} : H \to H$ be the solution operator defined by $\mathcal{S} e := \psi$, and let

$$\mu(S) := \sup_{w \in \widetilde{H} \setminus \{0\}} \inf_{v \in S} \frac{\|\mathcal{S} w - v\|_H}{\|w\|_V}.$$

If ψ_S is the best approximation to ψ with respect to the H-norm, then

$$\|e\|_V^2 = a(e,\psi) = a(e,\psi-\psi_S) \le C_c\mu(S)\|e\|_H\|e\|_V.$$

Hence, we have an estimate of the V-norm of the error in terms of the H-norm of the error. We proceed now to obtain an alternative condition, to the one given in Theorem 1, for the existence of a quasi-optimal error estimate. For any $v \in H$,

$$\begin{aligned}\alpha_H\|e\|_H^2 &\le |a_H(e,e)| = |a(e,e) - a_V(e,e)| \le C_c\|e\|_H\|u-v\|_H + C_V\|e\|_V^2\\ &\le C_c\|e\|_H\|u-v\|_H + C_V\left(C_c\mu(S)\right)^2\|e\|_H^2.\end{aligned}$$

Hence, under the alternative condition

$$C_V\left(C_c\mu(S)\right)^2 < \alpha_H/2,$$

we have obtained the same quasi-optimal estimate as before. The results are collected in the following theorem.

Theorem 2. *Let $u \in H$ be the solution of (1) and $u_S \in S$ be a solution of (4). Then*

$$\|u-u_S\|_V \le C_c\mu(S)\|u-u_S\|_H.$$

Further, if S is such that $C_V\left(C_c\mu(S)\right)^2 < \alpha_H/2$, then

$$\|u-u_S\|_H \le \frac{2C_c}{\alpha_H}\inf_{v\in S}\|u-v\|_H.$$

2.2 An example application in a finite element setting

The abstract analysis given here is a generalisation of the finite element analysis for highly indefinite Helmholtz problems introduced in [28]. The appropriate choice of spaces H and V for the finite element method in [28] is

$$H = H^1(\Omega), \quad V = L^2(\Omega),$$

where the space H is equipped with a weighted norm (cf. [22]):

$$\|u\|_{\mathcal{H}} := \left(|u|_{1,\Omega}^2 + k^2\|u\|_{0,\Omega}^2\right)^{1/2}.$$

With this choice of spaces, the assumptions A1–A3 are proved in [28]. Theorem 2.2 and Theorem 2.5 of [28] are then implied by Theorem 1 and Theorem 2, respectively. For details we refer the reader to [28].

We now turn to another case to which the abstract theory can be applied. Namely, we consider the solution of a Helmholtz problem by a Galerkin boundary element method.

3 A Helmholtz scattering problem

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2,3$, with a smooth boundary Γ. We consider the following exterior Helmholtz problem: Given $g \in H^{1/2}(\Gamma)$ find $u \in H^1_{\mathrm{loc}}(\Omega^c)$ such that

$$\begin{aligned}-\Delta u - k^2 u &= 0, && \text{in } \Omega^c,\\ u &= g, && \text{on } \Gamma,\\ \lim_{r\to\infty} r^{(d-1)/2}\left(\frac{\partial u}{\partial r} - iku\right) &= 0, && \text{where } r := \|x\|,\end{aligned} \tag{8}$$

is satisfied in a weak sense. The equation governs the process of acoustic scattering by a sound soft object; see [24].

Let $G_k(\cdot)$ be the fundamental solution of the Helmholtz equation:

$$\begin{aligned} G_k(r) &= \frac{i}{4} H_0(kr), \quad \text{for } d = 2, \\ G_k(r) &= \frac{1}{4\pi} \frac{e^{ikr}}{r}, \quad \text{for } d = 3, \end{aligned}$$

with $r > 0$. Throughout the paper H_ν is the Hankel function of the first kind of order ν defined by

$$H_\nu(x) := J_\nu(x) + i\, Y_\nu(x), \quad x > 0,$$

where J_ν and Y_ν are the Bessel functions of the first and second kind. Employing the fundamental solution, we define, respectively, the single layer and the double layer integral operators:

$$\left(S_k\varphi\right)(x) := \int_\Gamma G_k(\|x - y\|)\varphi(y) d\Gamma_y, \quad x \in \mathbb{R}^d \setminus \Gamma, \tag{9}$$

$$\left(D_k\varphi\right)(x) := \int_\Gamma \frac{\partial}{\partial n_y} G_k(\|x - y\|)\varphi(y) d\Gamma_y, \quad x \in \mathbb{R}^d \setminus \Gamma, \tag{10}$$

where n_y is the unit normal to the surface Γ at the point $y \in \Gamma$. The corresponding boundary integral operators are defined by

$$\left(V_k\varphi\right)(x) := \int_\Gamma G_k(\|x - y\|)\varphi(y) d\Gamma_y, \quad x \in \Gamma, \tag{11}$$

$$\left(K_k\varphi\right)(x) := \int_\Gamma \frac{\partial}{\partial n_y} G_k(\|x - y\|)\varphi(y) d\Gamma_y, \quad x \in \Gamma. \tag{12}$$

We state now the well-known mapping properties of the above operators; see [9, 29].

Proposition 3. *Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded domain with smooth boundary Γ. Then for any $s \in \mathbb{R}$ the following are bounded linear operators:*

(a) $V_k : H^s(\Gamma) \to H^{s+1}(\Gamma)$,

(b) $K_k : H^s(\Gamma) \to H^{s+1}(\Gamma)$.

It is well known that every solution $\varphi \in H^{-1/2}(\Gamma)$ of $V_k\varphi = g$, has the property that $u = S_k\varphi$ satisfies the exterior Helmholtz problem (8). However, for countable many wave numbers k the operator V_k is not injective. To avoid this problem Brakhage and Werner [5], Leis [21], and Panich [25], independently suggested to represent the solution as a combination of the single and double layer potentials:

$$u = D_k\varphi - i\alpha S_k\varphi, \tag{13}$$

for some coupling parameter $\alpha > 0$. The unknown density φ in (13) satisfies the boundary integral equation

$$g = \left(\frac{1}{2} I + K_k - i\alpha V_k\right)\varphi, \tag{14}$$

where I is the identity operator. We denote by $(\cdot,\cdot)_0$ the $L^2(\Gamma)$ inner product and by $\|\cdot\|_0$ the corresponding norm, and define

$$a(\varphi, v) := (R_k\varphi, v)_0, \text{ where } R_k := \frac{1}{2} I + K_k - i\alpha V_k. \tag{15}$$

To be able to apply the abstract theory developed in Section 2, we need to prove that the assumptions A1–A3 hold in this case. Proposition 3 implies that the condition A1 is satisfied with the choice $H = L^2(\Gamma)$. We can then define

$$a_H(\varphi, v) := \frac{1}{2}(I\varphi, v)_0 \text{ and } a_V(\varphi, v) := (\widetilde{R}_k\varphi, v)_0, \text{ where } \widetilde{R}_k := K_k - i\alpha V_k.$$

Therefore, $A := R_k$, $A_H := \frac{1}{2}I$, and $A_V := \widetilde{R}_k$. Again by Proposition 3, it follows that the condition A2 holds with the choice $V = L^2(\Gamma)$; trivially, V is then continuously imbedded in H. Furthermore, we can clearly set $\alpha_H = 1/2$. The following proposition deals with assumption A3.

Proposition 4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary Γ. Then, for any $g \in L^2(\Gamma)$ there exists a unique $\varphi \in L^2(\Gamma)$ such that*

$$a(\varphi, v) = (g, v)_0, \quad \textit{for all } v \in L^2(\Gamma), \tag{16}$$

and there exists a constant $C_{reg} > 0$, which depends on both k and Ω, such that

$$\|\varphi\|_0 \le C_{reg} \|g\|_0.$$

Moreover,

$$u = (D_k \varphi) - i\alpha\,(S_k \varphi)\,,$$

is the solution of the Helmholtz problem (8).

Proof. In the original paper of Brakhage and Werner [5], the existence and uniqueness have been proved for the classical formulation. To extend the proof to the variational formulation we proceed as in [13][1]. Since $\widetilde{R}_k$ is a continuous operator from $L^2(\Gamma)$ to $H^1(\Gamma)$, and $H^1(\Gamma)$ is compactly imbedded in $L^2(\Gamma)$, we have that $\widetilde{R}_k$ is a compact operator from $L^2(\Gamma)$ to $L^2(\Gamma)$. Therefore we can apply the Fredholm-Riesz-Schauder theory to the operator $R_k = I/2 + \widetilde{R}_k$, which implies that to prove invertibility it suffices to prove injectivity, i.e., it suffices to prove that $\operatorname{Ker} R_k = \{0\}$.

Let $R_k \varphi = 0$, then $\varphi = -2\widetilde{R}_k \varphi$. Applying the mapping property $\widetilde{R}_k : H^s(\Gamma) \to H^{s+1}(\Gamma)$ twice, we have that $\varphi \in H^2(\Gamma)$ and is hence continuous. For continuous functions the proof of uniqueness given in [5] is applicable, therefore $\varphi = 0$. □

To find an approximation to the solution φ numerically, we use the Galerkin discretisation. Let S be a finite dimensional subset of $L^2(\Gamma)$. Then, find a $\varphi_S \in S$ such that

$$a(\varphi_S, v) = (g, v)_0, \quad \text{for all } v \in S. \tag{17}$$

Since we have checked that all the assumptions of the abstract theory hold, from Theorem 1 we immediately obtain the following result.

Corollary 5. *Let S be such that $C_c \eta(S) \le 1/4$. Then (17) has a unique solution $\varphi_S \in L^2(\Gamma)$ and*

$$\|\varphi - \varphi_S\|_0 \le 4C_c \inf_{v \in S} \|\varphi - v\|_0,$$

where $\varphi \in L^2(\Gamma)$ is the solution of (16).

Remark 4. *Recall the definition of $\mathcal{T}$ from the previous section. Since $\mathcal{T} = R_k^{*-1} \widetilde{R}_k$, from Proposition 3 we have that $\mathcal{T} : L^2(\Gamma) \to H^1(\Gamma)$; therefore, $\mathcal{T}$ is a smoothening operator. To emphasise the dependence of $\mathcal{T}$ on k, for the rest of the paper we denote it by $\mathcal{T}_k := \mathcal{T}$.*

We will later show that for the case of $\Omega = \mathbb{S}^2$ and a particular choice of the coupling parameter α, the constant C_c is independent of k. The result of Theorem 2 brings little new in this setting, since $V = H$. For the finite element method of [28], Theorem 2 is of more interest.

So far we have made no specification for the set S except that it is a finite dimensional subspace of $L^2(\Gamma)$. Next, we consider the special case of the usual piecewise polynomial boundary elements.

[1] In [13] a weaker assumption is made on the smoothness of Γ but stronger on the spaces: $\Gamma \in C^{2,\lambda}$, $0 < \lambda < 1$, and $u, f \in H^{1/2}(\Gamma)$.

3.1 Piecewise polynomial boundary elements

Let $\mathcal{G}$ be a shape-regular triangulation of Γ. We assume that no approximation of the boundary occurs, namely:

$$\Gamma = \bigcup_{\tau \in \mathcal{G}} \tau.$$

The mesh width h is defined to be

$$h := \max\{h_\tau : \tau \in \mathcal{G}\}, \quad \text{where } h_\tau := \sup_{x,y \in \tau} \|x - y\|.$$

The set S is then defined to be a space of piecewise polynomial functions on the triangulation $\mathcal{G}$. In particular we are interested in the space $\mathcal{S}^{0,-1}_{\mathcal{G},h}$ of functions constant on each triangle $\tau \in \mathcal{G}$.

Next we give the well-known approximation property of the piecewise-constant finite element spaces.

Theorem 6. *Let $\varphi \in H^1(\Gamma)$ and $S = \mathcal{S}^{0,-1}_{\mathcal{G},h}$. There exists a constant C_A, that depends only on the minimal angle of the triangulation $\mathcal{G}$, such that*

$$\inf_{v \in S} \|\varphi - v\|_0 \leq C_A h \|\varphi\|_1.$$

We now proceed to investigate the dependence of the stability and the Galerkin error on the wave number. To do this, we make the assumption that the derivatives of the solution grow proportionally with the wave number k.

Definition 1. *For a given $\rho > 0$, the set $\mathcal{O}_{\rho,k,l}$ contains functions $\varphi \in H^l(\Gamma)$ such that*

$$\|\varphi\|_l \leq \rho k^l \|\varphi\|_0.$$

The conditions under which the solution of (16) belongs to a class $\mathcal{O}_{\rho,k,l}$ are discussed in [8].

Corollary 7. *Let $S = \mathcal{S}^{0,-1}_{\mathcal{G},h}$ and let $\varphi \in L^2(\Gamma)$ be the solution of (16). If*

$$C_c C_A h \|\mathcal{T}_k\|_{H^1(\Gamma) \leftarrow L^2(\Gamma)} < 1/4,$$

the discrete problem (17) has a unique solution $\varphi_S \in S$. If further $\varphi \in \mathcal{O}_{\rho,k,1}$ and $\varphi \neq 0$, then the relative error is bounded as

$$\frac{\|\varphi - \varphi_S\|_0}{\|\varphi\|_0} \leq 4 C_c C_A h k.$$

Proof. Using the approximation property of the piecewise-constant space and choosing $\widetilde{H} = H = L^2(\Gamma)$, we have that

$$\eta(S) = \sup_{\varphi \in L^2(\Gamma)\setminus\{0\}} \inf_{v \in S} \frac{\|\mathcal{T}_k \varphi - v\|_0}{\|\varphi\|_0} \leq C_A \sup_{\varphi \in L^2(\Gamma)\setminus\{0\}} \frac{h \|\mathcal{T}_k \varphi\|_1}{\|\varphi\|_0} \leq \frac{1}{4 C_c}.$$

Hence, by Corollary 5, we have the required stability condition.

Let us now assume that $\varphi \in \mathcal{O}_{\rho,k,l}$. Using Corollary 5 again,

$$\|\varphi - \varphi_S\|_0 \leq 4 C_c \inf_{v \in S} \|\varphi - v\|_0 \leq 4 C_c C_A h \|\varphi\|_1 \leq 4 C_c C_A h k \|\varphi\|_0.$$

$\square$

In the next section we investigate the dependence of C_c and of $\|\mathcal{T}_k\|_{H^1(\Gamma) \leftarrow L^2(\Gamma)}$ on the wave number k. Our goal is to state the dependence on k of all the constants in Corollary 7 in the case of the sphere.

3.2 The special case of the unit sphere

In this section we restrict to the case $\Gamma = \mathbb{S}^2$. This case was investigated by Giebermann in [13] and by Domínguez et al. in [11]. Our final result will be a slight improvement on the results of [13] and [11]. The improvement is in part due to the abstract theory developed at the start of the paper and in part due to some stronger bounds on the eigenvalues that we prove; the details are stated in Remark 5. In this section we draw heavily on results stated in [1] and [14]. We denote these two references by [AS] and [GR] for the rest of the paper.

The Fourier coefficients of a function $f \in L^2(\mathbb{S}^2)$ are defined by

$$f_n^m := \int_{\mathbb{S}^2} Y_n^m(\hat{x})\overline{f(\hat{x})}ds_x, \tag{18}$$

where Y_n^m are the spherical harmonics; see [AS]. Equivalent spaces to the usual Sobolev spaces on $\mathbb{S}^2$ can be defined through the Fourier coefficients.

Definition 2. *For any $s \geq 0$, let $\mathcal{H}^s(\mathbb{S}^2)$ be the space containing all functions $f \in L^2(\mathbb{S}^2)$ whose Fourier coefficients satisfy*

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} |f_n^m|^2(1+n^2)^s < \infty.$$

The inner product is defined by

$$\langle f, g\rangle_s := \sum_{n=0}^{\infty}(1+n^2)^s \sum_{m=-n}^{n} f_n^m \overline{g_n^m}.$$

For negative s, $\mathcal{H}^s(\mathbb{S}^2)$ is the dual space of $\mathcal{H}^{-s}(\mathbb{S}^2)$.

In the following j_n, y_n, and $h_n^{(1)}$ are spherical Bessel functions of the first, second, and third kind respectively; see [AS]. These can be defined through the Bessel functions:

$$\begin{aligned} j_n(x) &:= \sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x), \\ y_n(x) &:= \sqrt{\frac{\pi}{2x}}Y_{n+\frac{1}{2}}(x), \\ h_n^{(1)}(x) &:= j_n(x) + iy_n(x) = \sqrt{\frac{\pi}{2x}}H_{n+\frac{1}{2}}(x). \end{aligned} \tag{19}$$

Lemma 8.

(a) The space $\mathcal{H}^s(\mathbb{S}^2)$ is a Hilbert space and is equivalent to $H^s(\mathbb{S}^2)$. Namely, the norms induced by the inner products are equivalent and the sets $\mathcal{H}^s(\mathbb{S}^2)$ and $H^s(\mathbb{S}^2)$ coincide.

(b) The spherical harmonics form a complete orthogonal system in $\mathcal{H}^s(\mathbb{S}^2)$ and are the eigenfunctions of operators V_k, K_k, R_k, and $\mathcal{T}_k$. We have that

$$\begin{aligned} V_k Y_n^m &= \lambda_{n,k}^{(V)} Y_n^m, \quad \textit{with } \lambda_{n,k}^{(V)} := 2ikh_n^{(1)}(k)j_n(k), \\ K_k Y_n^m &= \lambda_{n,k}^{(K)} Y_n^m, \quad \textit{with } \lambda_{n,k}^{(K)} := -1/2 + ik^2 h_n^{(1)}(k)j_n'(k), \\ R_k Y_n^m &= \lambda_{n,k}^{(R)} Y_n^m, \quad \textit{with } \lambda_{n,k}^{(R)} := 1/2 + \lambda_{n,k}^{(K)} - i\alpha\lambda_{n,k}^{(V)} = ik^2 h_n^{(1)}(k)j_n'(k) + 2\alpha h_n^{(1)}(k)j_n(k). \\ \mathcal{T}_k Y_m^n &= R_k^{*-1}\widetilde{R}_k Y_m^n = \lambda_{n,k}^{(\mathcal{T})} Y_n^m, \quad \textit{with } \lambda_{n,k}^{(\mathcal{T})} := \left(\lambda_{n,k}^{(K)} - i\alpha\lambda_{n,k}^{(V)}\right) \Big/ \left(1/2 + \overline{\lambda_{n,k}^{(K)}} + i\alpha\overline{\lambda_{n,k}^{(V)}}\right). \end{aligned}$$

(c) For $s \geq 0$,

$$\|R_k\|_{\mathcal{H}^s(\mathbb{S}^2)\leftarrow\mathcal{H}^s(\mathbb{S}^2)} = \sup_{n\in\mathbb{N}_0} |\lambda_{n,k}^{(R)}|, \quad \|\mathcal{T}_k\|_{\mathcal{H}^{s+1}(\mathbb{S}^2)\leftarrow\mathcal{H}^s(\mathbb{S}^2)} = \sup_{n\in\mathbb{N}_0} \sqrt{1+n^2}|\lambda_{n,k}^{(\mathcal{T})}|.$$

Proof. For (a) see [8]. The eigenvalues of the operators V_k and K_k are given in [18]. From these it is easy to derive the eigenvalues of the remaining two operators. A proof of (c) can be found in [13]; see also [23]. $\square$

The above result justifies us writing $H^s(\mathbb{S}^2)$ for both $H^s(\mathbb{S}^2)$ and $\mathcal{H}^s(\mathbb{S}^2)$. We now prove some results on the Bessel functions that, in view of (19) and Lemma 8, have direct use in bounding eigenvalues $\lambda_{n,k}^{(R)}$. Recall that the Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ are real valued for $\nu \in \mathbb{R}$ and $x \geq 0$.

Lemma 9.

(a) $J_\nu(x), J_\nu'(x), Y_\nu'(x) > 0,\ Y_\nu(x) < 0$, *for* $0 < x < \nu$,

(b) $J_\nu(x)$ *and* $xJ_\nu'(x)$ *are positive increasing functions of* x, *for* $0 < x < \nu$,

(c) for $x > 0$ *the product* $x\left[J_\nu^2(x) + Y_\nu^2(x)\right]$, *as a function of* x, *decreases monotonically, if* $\nu > 1/2$, *and increases monotonically if* $\nu < 1/2$.

Proof. Parts *(a)* and *(b)* are proved in Watson [30, §15.3]. A proof of part *(c)* can found in Watson [30, §13.74]. $\square$

Proposition 10. *There exists a constant* $C > 0$, *such that for any* $x \geq 1$ *and* $\nu \in [1/2, \infty) \cup \{0\}$

(a) $|J_\nu(x)H_\nu(x)| \leq Cx^{-2/3}$,

(b) $|xJ_\nu'(x)H_\nu(x)| \leq C$.

Proof. A proof of part *(a)* for $\nu > 1/2$ is given in [13] and [11], where also a bound that is less sharp than what we prove here is given for part *(b)*.

In the proof we make use of the following asymptotic expansions [AS, (9.3.31)–(9.3.34)] :

$$\begin{aligned}
J_\nu(\nu) &= a\nu^{-1/3} + O(\nu^{-5/3}),\\
Y_\nu(\nu) &= -\sqrt{3}a\nu^{-1/3} + O(\nu^{-5/3}),\\
J_\nu'(\nu) &= b\nu^{-2/3} - c\nu^{-4/3} + O(\nu^{-8/3}),\\
Y_\nu'(\nu) &= \sqrt{3}\left(b\nu^{-2/3} + c\nu^{-4/3}\right) + O(\nu^{-8/3}),
\end{aligned} \tag{20}$$

where a, b, and c are certain positive constants.

We divide the proof into two cases:

Case 1: $\nu > x \geq 0$

Using the identity $J_\nu(x)Y_\nu'(x) - J_\nu'(x)Y_\nu(x) = 2/(\pi x)$ [AS, (9.1.16)] we have that

$$0 \overset{\text{Lemma 9a}}{\leq} J_\nu(x)Y_\nu'(x) \overset{\text{[AS, (9.1.16)]}}{=} \frac{2}{\pi x} + J_\nu'(x)Y_\nu(x).$$

Therefore,

$$|xJ_\nu'(x)Y_\nu(x)| \overset{\text{Lemma 9a}}{=} -xJ_\nu'(x)Y_\nu(x) \leq \frac{2}{\pi}.$$

Also,

$$|xJ_\nu'(x)J_\nu(x)| = xJ_\nu'(x)J_\nu(x) \overset{\text{Lemma 9b}}{\leq} \nu J_\nu'(\nu)J_\nu(\nu) \overset{(20)}{\leq} C,$$

where C is independent of x and ν. Combining the last two results we have that

$$|xJ_\nu'(x)H_\nu(x)| \leq |xJ_\nu'(x)J_\nu(x)| + |xJ_\nu'(x)Y_\nu(x)| \leq C + \frac{2}{\pi}, \text{ for } x < \nu. \tag{21}$$

Case 2: $1/2 < \nu \leq x$

We use the following definitions:

$$M_\nu(x) := |H_\nu(x)| \text{ and } N_\nu(x) := |H'_\nu(x)|.$$

We have that

$$x^2|J'_\nu(x)H_\nu(x)|^2 \leq x^2N^2_\nu(x)M^2_\nu(x) \overset{\text{[AS, (9.2.22)]}}{=} x^2{M'_\nu}^2(x)M^2_\nu(x) + \frac{4}{\pi}. \tag{22}$$

Next,

$$\begin{aligned} x\frac{d}{dx}\{-xM'_\nu(x)\} &\overset{\text{[AS, (9.2.25)]}}{=} (x^2-\nu^2)M_\nu(x) - \frac{4}{\pi^2}\frac{1}{M^3_\nu(x)} = M_\nu(x)(x^2-\nu^2-\frac{4}{\pi^2}M_\nu^{-4}(x)) \\ &\overset{\text{[GR,(8.479)]}}{\leq} M_\nu(x)(x^2-\nu^2-x^2) \leq 0. \end{aligned}$$

Hence, $-xM'_\nu(x)$ is a monotonically decreasing function. From Lemma 9c we have that, for $\nu > 1/2$, $xM^2_\nu(x)$ is monotonically decreasing, and hence $M'_\nu(x) \leq 0$. It is now not difficult to see that $x{M'_\nu}^2(x)$ is also a monotonically decreasing function. Therefore,

$$x{M'_\nu}^2(x)xM^2_\nu(x) \leq \nu^2{M'_\nu}^2(\nu)M_\nu(\nu)^2 \overset{(20)}{\leq} C, \text{ for } x \geq \nu > \frac{1}{2}. \tag{23}$$

Combining this last result with (21) and (22) gives the required bound for $\nu > 1/2$. The result for $\nu = 1/2$ is obtained by the continuity of Bessel functions in the argument ν.

Finally we prove *(a)* and *(b)* for $\nu = 0$.

$$|J_0(k)H_0(k)| \leq \frac{1}{k}kM^2_0(k) \overset{\text{Lemma 9c}}{\leq} \frac{1}{k}\lim_{k\to\infty} kM^2_0(k) \overset{\text{[AS, (9.2.3)]}}{\leq} C\frac{1}{k} \leq Ck^{-2/3}.$$

Similarly,

$$k|J'_0(k)H_0(k)| = k|J_1(k)H_0(k)| \leq \sqrt{k}M_1(k)\sqrt{k}M_0(k) \overset{\text{Lemma 9c}}{\leq} M_1(1)\lim_{k\to\infty}\sqrt{k}M_0(k) \overset{\text{[AS, (9.2.3)]}}{\leq} C.$$

□

Corollary 11. *Let $R_k : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ be the operator defined, as in (15), by*

$$R_k = I/2 + K_k - i\alpha V_k.$$

Then R_k is bounded and there exists a constant $C > 0$ independent of k such that

$$\|R_k\|_{L^2(\mathbb{S}^2)\leftarrow L^2(\mathbb{S}^2)} \leq C(1+\alpha k^{-2/3}).$$

Proof. In view of Lemma 8, to prove the statement we need to find bounds on the eigenvalues of the operator R_k. Using the definition of spherical Bessel functions (19) and Proposition 10 we have that

$$\left|\lambda^{(V)}_{n,k}\right| = \left|2kh^{(1)}_n(k)j_n(k)\right| = \left|\pi H_{n+\frac{1}{2}}(k)J_{n+\frac{1}{2}}(k)\right| \leq Ck^{-2/3},$$

and

$$\begin{aligned} \left|\frac{1}{2} + \lambda^{(K)}_{n,k}\right| &= \left|k^2h^{(1)}_n(k)j'_n(k)\right| = \left|\frac{\pi}{2}kH_{n+\frac{1}{2}}(k)\left(J'_{n+\frac{1}{2}}(k) + \frac{1}{2k}J_{n+\frac{1}{2}}(k)\right)\right| \\ &\leq \left|\frac{\pi}{2}kH_{n+\frac{1}{2}}(k)J'_{n+\frac{1}{2}}(k)\right| + \left|\frac{\pi}{4}H_{n+\frac{1}{2}}(k)J_{n+\frac{1}{2}}(k)\right| \leq C(1+k^{-2/3}). \end{aligned}$$

The result now follows from the identity

$$\|R_k\|_{\mathcal{H}^s(\mathbb{S}^2)\leftarrow\mathcal{H}^s(\mathbb{S}^2)} = \sup_{n\in\mathbb{N}_0}\left|\lambda^{(R)}_{n,k}\right| = \sup_{n\in\mathbb{N}_0}\left|1/2 + \lambda^{(K)}_{n,k} - i\alpha\lambda^{(V)}_{n,k}\right|.$$

□

Note that for $\alpha \le k^{2/3}$, $\|R_k\|_{L^2(\mathbb{S}^2)\leftarrow L^2(\mathbb{S}^2)}$ is bounded by a constant independent of k. Numerical experiments suggest $C_c = \|R_k\|_{L^2(\mathbb{S}^2)\leftarrow L^2(\mathbb{S}^2)} \le 1.76$, for $\alpha = k^{2/3}$.

Definition 3. *Let $\alpha := k^{2/3}$ in the definition of R_k; see (15).*

Remark 5. *The choice $\alpha \propto k$ is prevalent in the literature; see [2, 11, 13, 20]. In [2] and [20] the choice was made to minimise the condition number of the matrices arising from the discretisation of boundary integral operators in the case of the unit sphere and the unit disk. The same choice maximises the inf-sup constant and hence optimises the error estimate given by Céa's lemma; see [13]. The error estimate in Corollary 5 is not affected by the inf-sup constant and with the choice $\alpha = k^{2/3}$ the constant of quasi-optimality C_c is independent of k. Céa's lemma gives a more pessimistic bound, with quasi-optimality constant growing as $k^{1/3}$; see [11, 13].*

It remains now to find the dependence on k of the continuity constant of the operator $\mathcal{T}_k = R_k^{*-1}\widetilde{R}$. From Lemma 8 we have that

$$\|\mathcal{T}_k\|_{H^1(\mathbb{S}^2)\leftarrow L^2(\mathbb{S}^2)} = \sup_n \sqrt{1+n^2}|\lambda^{\mathcal{T}}_{n,k}| = \sup_n \sqrt{1+n^2}\left|\frac{\lambda^{(K)}_{n,k} - i\alpha\lambda^{(V)}_{n,k}}{1/2+\overline{\lambda^{(K)}_{n,k}} + i\alpha\overline{\lambda^{(V)}_{n,k}}}\right|.$$

By taking into account the properties of the zeros of Bessel functions, see [AS, (9.5)], it can be seen that the denominator in the above expression is never zero, however a proof of a useful upper bound for the whole expression is beyond the scope of this paper. Instead, we consider the three asymptotic cases: k fixed and $n \to \infty$, $n \approx k$, and n fixed and $k \to \infty$.

Proposition 12. *(a) For fixed ν, and $k \to \infty$ we have, for $\alpha \le k$,*

$$|\lambda^{\mathcal{T}}_{\nu,k}| = \left|1 - \frac{1}{2e^{i\chi}\left(-\frac{2\alpha}{k}\cos\chi + i\sin\chi\right)} + O(k^{-1})\right|,$$

where $\chi = k - \nu\pi/2 - \pi/2$.

(b) For $\nu + 1/2 = k$ and $\alpha \le k^{4/3}$ we have

$$|\lambda^{\mathcal{T}}_{\nu,k}| = 1 + \left|i\pi ab(1+\sqrt{3}i) + 2\pi a^2(1-\sqrt{3}i)\alpha k^{-2/3} + O(k^{-2/3})\right|^{-1},$$

where a and b are constants from the asymptotic expansions (20).

(c) For fixed k and $\nu \to \infty$ we have

$$\lambda^{\mathcal{T}}_{\nu,k} = O(\nu^{-1}).$$

Proof. Part *(a)*: We first use the definition of spherical functions to write the eigenvalues in terms of Bessel functions and then make use of asymptotic expansions given in [AS, (9.2)]. From (19), as in proof of Corollary 11, we have for ν fixed and $k \to \infty$, that

$$\begin{aligned}
|\lambda^{\mathcal{T}}_{\nu,k}| &= \left|\frac{-1/2 + i\frac{\pi}{2}kH_{\nu+\frac12}(k)J'_{\nu+\frac12}(k) - \frac{\pi}{2}(i/2-2\alpha)H_{\nu+\frac12}(k)J_{\nu+\frac12}(k)}{i\frac{\pi}{2}kH_{\nu+\frac12}(k)J'_{\nu+\frac12}(k) - \frac{\pi}{2}(i/2-2\alpha)H_{\nu+\frac12}(k)J_{\nu+\frac12}(k)}\right| \\
&= \left|1 - \frac{1}{i\pi kH_{\nu+\frac12}(k)J'_{\nu+\frac12}(k) - \pi(i/2-2\alpha)H_{\nu+\frac12}(k)J_{\nu+\frac12}(k)}\right| \\
&\overset{\text{[AS, (9.2)]}}{=} \left|1 - \frac{1}{2e^{i\chi}\left(-\frac{2\alpha}{k}\cos\chi + i\sin\chi\right) - \frac{\alpha}{k}O(k^{-1}) + O(k^{-1})}\right|,
\end{aligned}$$

where $\chi = k - (\nu + 1/2)\pi/2 - \pi/4 = k - \nu\pi/2 - \pi/2$. The result now follows from the assumption $\alpha \leq k$.
Part *(b)*: Using the asymptotic expansions (20) we obtain that

$$\begin{aligned}
\lambda^{\mathcal{T}}_{\nu,k} &= 1 + \left| i\pi kab\left((1+\sqrt{3}i)k^{-1} + O(k^{-5/3})\right) - \pi a^2(i/2 - 2\alpha)\left((1-\sqrt{3}i)k^{-2/3} + O(k^{-2})\right)\right|^{-1} \\
&= 1 + \left| i\pi ab(1+\sqrt{3}i) + 2\pi a^2(1-\sqrt{3}i)\alpha k^{-2/3} + O(k^{-2/3}) + \alpha O(k^{-2})\right|^{-1}.
\end{aligned}$$

Part *(c)*. For the proof, we use the asymptotic expansions given in [AS, (9.3)].

$$J_\nu(k)H_\nu(k) \overset{\text{[AS, (9.3.1)]}}{\sim} \frac{1}{2\pi\nu}\left(\frac{ek}{2\nu}\right)^{2\nu} - i\frac{1}{\pi\nu} = O(\nu^{-1}). \tag{24}$$

We also make use of Stirling's approximation to the Gamma function [AS, (6.1.39)]:

$$\begin{aligned}
J'_\nu(k) &\overset{\text{[AS, (9.1.10)]}}{=} \nu\frac{(\frac{1}{2}k)^\nu}{\Gamma(\nu+1)}\left(\frac{1}{k} - \frac{2+\nu}{2\nu}\left(\frac{1}{2}k\right)\frac{1}{\nu+1} + \ldots\right) \\
&\overset{\text{[AS, (6.1.39)]}}{\sim} \sqrt{\frac{\nu}{2\pi}}\left(\frac{ke}{2\nu}\right)^\nu\left(\frac{1}{k} + O(\nu^{-1})\right).
\end{aligned}$$

Hence,

$$J'_\nu(k)H_\nu(k) \overset{\text{[AS, (9.3.1)]}}{\sim} -i\frac{1}{\pi k} + O(\nu^{-1}). \tag{25}$$

Finally,

$$\lambda^{\mathcal{T}}_{\nu,k} \overset{(24),(25)}{\sim} \frac{-1/2 + 1/2 + O(\nu^{-1})}{1/2 + O(\nu^{-1})} = O(\nu^{-1}).$$

□

Part (c), in the above proposition merely confirms that $\mathcal{T}_k$ is a pseudo-differential operator of order -1. From part (b) we conclude that for $n + 1/2 = k$

$$\sqrt{1+n^2}|\lambda^{\mathcal{T}}_{n,k}| \sim O(k). \tag{26}$$

The denominator in the expression of part (a) is clearly never 0, however it becomes arbitrarily close to zero for certain, large enough values of k and for $\alpha < k$. Nevertheless, note that $|-\frac{2\alpha}{k}\cos\chi + i\sin\chi| \geq 2\alpha/k$, for $k > 2\alpha$. So that,

$$|\lambda^{\mathcal{T}}_{\nu,k}| = O(k/\alpha), \quad \text{for } k > 2\alpha.$$

Since $\alpha = k^{2/3}$, the condition $k > 2\alpha$ is equivalent to $k > 8$.

To see how relevant these asymptotic cases are for estimating the continuity constant, we plot $\sqrt{1+n^2}|\lambda^{\mathcal{T}}_{n,k}|$ for different values of k and n in Figure 1. The picture suggests that the maximum occurs for $n + 1/2 \approx k$. Hence, in view of (26), we are lead to the following heuristic:

$$\|\mathcal{T}_k\|_{H^1(\mathbb{S}^2)\leftarrow L^2(\mathbb{S}^2)} \leq C_X k, \tag{27}$$

for some constant $C_X > 0$ independent of k. Numerical experiments suggest that $C_X \leq 1.7$. In [11], it is proved that, in two dimensions with the coupling parameter $\alpha = k$ and large enough k, the eigenvalues of R_k are bounded below by 1/2. This supports further our claim (27).

Now we are in a position to give estimates on the dependence on k of the stability and the accuracy of the boundary element method.

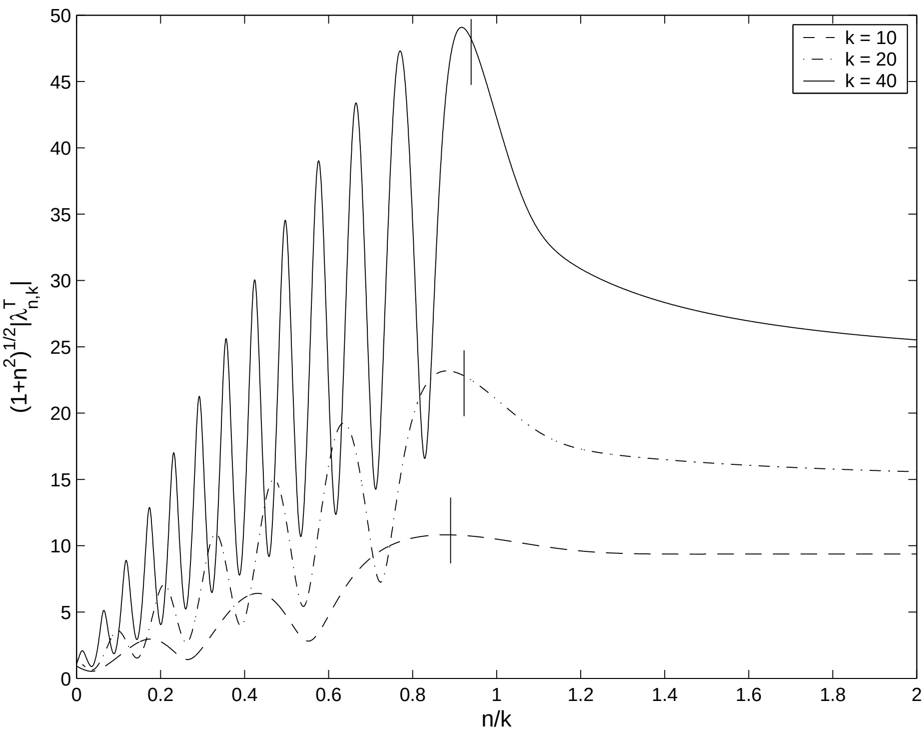

Figure 1: Plot of $\sqrt{1+n^2}|\lambda^{\mathcal{T}}_{n,k}|$ for different values of n and k. The vertical lines denote the positions at which $n+1/2=k$.

3.2.1 Piecewise-constant Galerkin boundary element method

Proposition 13. *Let $\Gamma=\mathbb{S}^2$, $S=\mathcal{S}^{0,-1}_{\mathcal{G},h}$, $\varphi\in L^2(\Gamma)$ be the solution of (16), and let (27) hold. There exists a constant c independent of k such that if $hk<c$, the discrete problem (17) has a unique solution $\varphi_S\in S$. If further $\varphi\in\mathcal{O}_{\rho,k,1}$, then there exists a constant C independent of k such that*

$$\|\varphi-\varphi_S\|_0\le Chk\|\varphi\|_0.$$

Therefore, the boundary element method does not suffer from the pollution effect, and a condition $hk\lesssim 1$ is sufficient to guarantee stability and a quasi-optimal error estimate.

Remark 6. *Let us consider the two dimensional case, $\Gamma=\{x\in\mathbb{R}^2:\|x\|=1\}$. The Sobolev space $H^s(\Gamma)$ can be identified with the space $H^s([0,2\pi])$ of 2π periodic distributions; see [2, 19]. Periodic functions, $e^{\pm in\theta}$, $n\in\mathbb{N}_0$, are then the eigenfunctions of the operators V_k and K_k with eigenvalues given by*

$$\lambda^{(V)}_{n,k}=\frac{i\pi}{2}J_n(k)H_n(k),\quad \lambda^{(K)}_{n,k}=-\frac{1}{2}+\frac{i\pi}{2}kJ'_n(k)H_n(k).$$

Comparing these with the case of the sphere it is clear that the analogous analysis of this section holds for the two dimensional case as well. In particular, the statement of Proposition 13 also holds for the case of the unit ball in two dimensions.

3.2.2 The h-p version of the Galerkin method

Just as in the finite element method [16, 17], the use of higher order polynomials improves the stability condition of the boundary element method. Let $S=\mathcal{S}^{p,1}_{\mathcal{G},h}$ be the usual boundary element space of continuous, piecewise polynomial functions of order p. Using the approximation properties of such spaces proved in [15, 16, 17] we proceed as in the case of piecewise-constant basis functions.

Assuming that $\widetilde{H} = \mathcal{O}_{\rho,k,l}$, where $1 \le l \le p$, we obtain the estimate

$$\begin{aligned}
\eta(S) &= \sup_{\psi\in\widetilde{H}\setminus\{0\}} \inf_{v\in S} \frac{\|\mathcal{T}_k\psi - v\|_0}{\|\psi\|_0} \overset{[15,\,16]}{\le} C_A(l) \sup_{\psi\in\widetilde{H}\setminus\{0\}} \frac{\|\mathcal{T}_k\psi\|_{l+1}}{\|\psi\|_0} \left(\frac{h}{2p}\right)^{l+1} \\
&\le C_A(l)C_X k \sup_{\psi\in\widetilde{H}\setminus\{0\}} \frac{\|\psi\|_l}{\|\psi\|_0} \left(\frac{h}{2p}\right)^{l+1} \\
&\le \rho C_A(l)C_X \left(\frac{kh}{2p}\right)^{l+1},
\end{aligned}$$

where $C(l)$ is a constant depending only on l. Therefore, the condition for stability and the quasi-optimal error estimate reduces to $hk \lesssim 2p$. Thus, higher order elements allow for a coarser mesh, and the following error estimate:

$$\|\varphi - \varphi_S\|_0 \le C \left(\frac{kh}{2p}\right)^{l+1} \|\varphi\|_0.$$

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