ON L_∞-MORPHISMS OF CYCLIC CHAINS

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Abstract. Recently the first two authors [1] constructed an L_∞-morphism using the S^1-equivariant version of the Poisson Sigma Model (PSM). Its role in deformation quantization was not entirely clear. We give here a "good" interpretation and show that the resulting formality statement is equivalent to formality on cyclic chains as conjectured by Tsygan and proved recently by several authors [5], [10].

1. Introduction and Structure

We begin by drawing the big picture; precise definitions will be given below.

1.1. **Big picture on cochains.** Let M be a smooth d-dimensional manifold and $A = C^\infty(M)$ ($A_c = C^\infty_c(M)$) the commutative algebras of smooth (compactly supported) functions. We denote by $T^\bullet$ the dgla of multivector fields and by $C^\bullet(A)$ the multidifferential Hochschild complex. Kontsevich's famous Formality Theorem asserts that there is an L_∞-quasi-isomorphism of dglas

$$\mathcal{U}_K : T^\bullet \to C^\bullet(A).$$

Next, assume that M is orientable[1] and pick a volume form Ω. This endows $T^\bullet$ with an additional differential div_Ω, the divergence, that is compatible with the Schouten bracket on $T^\bullet$. We will denote the dgla $(T^\bullet[[u]], u\,\mathrm{div}_\Omega, [\cdot,\cdot]_S)$ shortly by $T^\bullet[[u]]$. Here u is a formal parameter of degree $+2$. There is a morphism of dglas

$$T^\bullet[[u]] \xrightarrow{u=0} T^\bullet.$$

We denote the composition of this morphism with $\mathcal{U}_K$ also by $\mathcal{U}_K$ for simplicity.

1.2. **Big picture on chains.** Let us turn to homology. Denote the negatively graded Hochschild (chain) complex by $C_\bullet(A) = C_\bullet(A, A)$. It is a mixed complex, with the Hochschild differential b of degree $+1$ and with the Rinehart (or Connes) differential B of degree -1. The cohomology $H_\bullet(A)$ of $C_\bullet(A)$ wrt. the differential b is the de Rham complex $(\Omega^{-\bullet}(M), \mathrm{d})$, which we view as a bicomplex with vanishing first differential.

$C_\bullet(A)$ also carries a compatible dgla module structure over the Hochschild cochains $C^\bullet(A)$. Pulling back this module structure along $\mathcal{U}_K$, we obtain an L_∞-module structure over multivector fields $T^\bullet$. The Hochschild Formality Theorem on chains [7, 4, 8] states that there is a quasi-isomorphism of L_∞-modules over $T^\bullet$

$$\mathcal{V} : C_\bullet(A) \to \Omega^{-\bullet}(M)$$

Actually, this morphism is compatible with the additional second differentials B and d on both sides. Hence we obtain an L_∞-quasi-isomorphism

$$\mathcal{V} : (C_\bullet(A)[[u]], b + uB) \to (\Omega^{-\bullet}(M)[[u]], u\mathrm{d})$$

This last statement is known as the Cyclic Formality Theorem on chains [10, 5, 8].

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[1] This is actually not necessary, but we will assume it for simplicity.

1.3. **Dual picture.** Recall that $A = C^\infty(M)$. The following statement is a particularly simple case of van den Bergh duality [9] (note the negative grading on the left)

$$H_\bullet(A, A) \cong H^{d+\bullet}(A, \Omega^d(M))$$

Concretely, the left hand side is $\Omega^{-\bullet}(M)$, and the right hand side is $VT^{d+\bullet} := T^{d+\bullet} \otimes \Omega^d(M)$. The isomorphism from right to left is by contraction. Note that we can pull back the de Rham differential along this isomorphism, obtaining a differential "div" on $VT^\bullet$. Note in particular that this differential div does not depend on a choice of volume, in contrast to the div_Ω defined before.

The dualized Hochschild formality theorem on chains states that there is a quasi-isomorphism of L_∞-modules

$$\mathcal{V}^* : VT^\bullet \to C^\bullet(A, \Omega^d)\,.$$

The dualized cyclic formality theorem states that this morphism is compatible with the additional differentials div on the left and the (adjoint of the) Connes differential B on the right.

We will only consider such morphisms that are differential operators in each argument. In this case there is a canonical way to obtain an adjoint morphism $\mathcal{V}^*$ from the "direct" one $\mathcal{V}$ and vice versa. Concretely, there is a pairing between $C^\bullet(A, \Omega^d)$ and $C_\bullet(A_c)$ given by

$$\langle \phi, a_0 \otimes \cdots \otimes a_n \rangle = \int_M a_0 \phi(a_1, \dots, a_n)$$

and a pairing between $VT^\bullet$ and $\Omega^\bullet(M)$ given by

$$\langle \gamma\Omega, \alpha \rangle = \int_M (\iota_\gamma \alpha)\Omega\,.$$

Here the insertion ι_γ is defined such that $\iota_{\gamma_1 \wedge \gamma_2} = \iota_{\gamma_1}\iota_{\gamma_2}$. One can see that to any direct multidifferential L_∞ morphism $\mathcal{V}$ there is a unique morphism $\mathcal{V}^*$ such that

$$\langle \gamma\Omega, \mathcal{V}(a_0 \otimes \cdots \otimes a_n) \rangle = \pm \langle \mathcal{V}^*(\gamma\Omega), a_0 \otimes \cdots \otimes a_n \rangle\,.$$

It follows that the direct and adjoint (multidifferential) formality statements are equivalent.

Remark 1 (on quantization). The cohomology $H^0(A, \Omega^d)$ is important because it classifies smooth traces on A_c, i.e., top degree differential forms Ω such that the functional $f \mapsto \int_M f\Omega$ is a trace on A_c. Of course, in the current commutative setting, these are just all top degree differential forms. However, due to dual Hochschild formality we can quantize. Let $A_\star$ be the algebra $C^\infty(M)[[\hbar]]$ with the Kontsevich star product [6] associated to a Poisson structure π. The relevant cohomology is then $H^0(A_\star, \Omega^d_\star) \cong \{\omega \in \Omega^d(M)[[\hbar]] \mid \mathrm{div}_\omega \pi = 0\}$. The quantized bimodule structure on $\Omega^d_\star = \Omega^d[[\hbar]]$ is defined such that for all $a, b \in A_c$, $\omega \in \Omega^d_\star$

$$\int_M a \cdot (L_b \omega) = \int_M (a \star b) \cdot \omega = \int_M b \cdot (R_a \omega)\,.$$

1.4. **Other module structures.** The cyclic chain formality morphisms above are quasi-isomorphisms of L_∞-modules over $(T^\bullet, 0, [\cdot,\cdot]_S)$. One may be tempted to replace this latter dgla by its "cyclic" counterpart $(T^\bullet[[u]], u\,\mathrm{div}_\Omega, [\cdot,\cdot]_S)$, and ask whether the above formality statements remain true. Of course, if we use the module structures obtained via pulling back along the dgla morphism

$$T^\bullet[[u]] \stackrel{u=0}{\to} T^\bullet$$

the new formality statements will be equivalent to the original ones. However, one may try to change the module structures. We will only consider changing the module structure on the classical (differential forms) side.[2] We show in section 2.3 that there is a whole family of dgla actions $L^{(t)}$ reducing to the original Lie derivative action for $t = 0$. However, all these module structures will be shown to be L_∞-quasi-isomorphic in Proposition 2.

[2]One can also "naturally" change the action on the Hochschild side. but we don't discuss it here.

1.5. **Meaning of the PSM morphism.** Using the S^1-equivariant version of the Poisson Sigma Model the first two authors [1] recently constructed an L_∞-morphism $\mathcal{V}_{PSM,orig}$, the "PSM morphism". This paper is devoted to clarifying the meaning of this morphism, which was not entirely clear. To do this, we will reinterpret $\mathcal{V}_{PSM,orig}$ slightly, yielding a morphism $\mathcal{V}^*_{PSM}$. Concretely, we introduce a new complex $E^\bullet$ which is quasi-isomorphic (as bicomplex and $C^\bullet(A)$-module) to $C^\bullet(A,\Omega^d)$. The morphism $\mathcal{V}^*_{PSM}$ can then be understood as an adjoint cyclic chain formality morphism on

$$\mathcal{V}^*_{PSM} : T^\bullet[[u]] \cong VT^\bullet[[u]] \to E^\bullet[[u]]\,.$$

Here the action of $T^\bullet[[u]]$ on the very left is the adjoint action, on the middle it is the (dual of the) action $L^{(1)}$, and on the right it is the action defined through pullback via $\mathcal{U}_K$. The isomorphism on the left is defined by choosing a volume form.

1.6. **Organisation of the paper.** The remainder of the paper is divided into two parts:

(1) In the first part we introduce the structures involved, i.e., the Hochschild and cyclic chain and cochain complexes. Here there are two novel aspects: (i) We introduce the natural "extended" complex $E^\bullet$ mentioned above that allows us to give a nice interpretation of the PSM morphism and (ii) we introduce the aforementioned family $L^{(t)}$ of $T^\bullet[[u]]$-actions on differential forms that was (to our knowledge) not studied before.
(2) In the second part we define $\mathcal{V}^*_{PSM}$ and prove the formality statement made above.

2. Part I: The objects of study

In this section we define the different complexes that will be related to each other through formality morphisms. Each complex can either constitute a differential graded Lie algebra (dgla) or serve as a module over one of the dglas. We will indicate the roles in the titles of each subsection. Of course, every dgla is also a module over itself.

2.1. **Multivector fields $T^\bullet$ (dgla).** The algebra of multivector fields on M, $T^\bullet$, is the algebra of smooth sections of $\wedge^\bullet TM$. There is a Lie bracket $[\cdot,\cdot]_S$ on $T^{\bullet+1}(M)$, the Schouten bracket, extending the Lie derivative and making $T^\bullet$ a Gerstenhaber algebra. More concretely,

$$[v_1\wedge\cdots\wedge v_m, w_1\wedge\cdots\wedge w_n]_S = \\ = \sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i+j}\,[v_i,w_j]\wedge v_1\wedge\cdots\wedge\hat{v}_i\wedge\cdots\wedge v_m\wedge w_1\wedge\cdots\wedge\hat{w}_j\wedge\cdots\wedge w_n\,.$$

Assume now that M is oriented, with volume form Ω. Contraction with Ω defines an isomorphism $T^\bullet\to\Omega^{d-\bullet}(M)$. The divergence operator div_Ω on $T^\bullet$ is defined as the pullback of the de Rham differential d on $\Omega^\bullet(M)$ under this isomorphism. Concretely

$$\iota_{\mathrm{div}_\Omega\gamma}\Omega = \mathrm{d}\iota_\gamma\Omega\,.$$

One can check that div_Ω is a derivation with respect to the Schouten bracket, i.e.,[3]

$$\mathrm{div}_\Omega\,[\gamma_1,\gamma_2]_S = [\mathrm{div}_\Omega\,\gamma_1,\gamma_2]_S + (-1)^{k_1-1}\,[\gamma_1,\mathrm{div}_\Omega\,\gamma_2]_S\ .$$

Introducing a new formal variable u of degree $+2$, the complex $T^{\bullet+1}(M)[[u]]$ is a dgla with differential $u\,\mathrm{div}_\Omega$ and bracket the u-linear extension of the Schouten bracket.

Hence we have two dglas, $T^{\bullet+1}(M)$ and $T^{\bullet+1}(M)[[u]]$, related by a dgla morphism

$$T^{\bullet+1}(M)[[u]] \xrightarrow{u=0} T^{\bullet+1}(M).$$

This morphism in particular allows us to view any $T^{\bullet+1}(M)$-module also as $T^{\bullet+1}(M)[[u]]$-module.

[3]Actually div_Ω is a BV operator generating $[\cdot,\cdot]_S$ for any volume form Ω.

2.2. **Hochschild cochains $C^\bullet(A)$ (dgla).** The normalized multidifferential Hochschild complex $C^\bullet(A)$ is the complex of $\bullet$-differential operators, which vanish upon insertion of a constant function in any of its arguments. E.g., $C^1(M)$ are differential operators D such that $D1 = 0$. $C^{\bullet+1}(A)$ is a differential graded Lie algebra with the Gerstenhaber bracket

$$\begin{aligned}[\phi,\psi]_G(a_1,\dots,a_{p+q-1}) = \phi(&\psi(a_1,\dots,a_q),a_{q+1},\dots,a_{p+q-1})\\ &+(-1)^{q-1}\phi(a_1,\psi(a_2,\dots,a_{q+1}),a_{q+2},\dots,a_{p+q-1})\\ &\pm\dots\\ &+(-1)^{(p-1)(q-1)}\phi(a_1,\dots,\psi(a_p,\dots,a_{p+q-1}))\\ &-(-1)^{(p-1)(q-1)}(\phi\leftrightarrow\psi)\end{aligned}$$

for $\phi \in C^p(A), \psi \in C^q(A)$, and the Hochschild differential

$$b^H = [m_0, \cdot]_G\,.$$

Here $m_0 \in C^2(A)$ is the usual (commutative) multiplication of functions.

2.3. **The differential forms $\Omega^\bullet(M)$ (module).** Let $\Omega^\bullet = \Omega^\bullet(M)$ be the graded algebra of differential forms on M, with negative grading. Let $\mathrm{d} = \mathrm{d}_{dR}$ be the de Rham differential. Denote the insertion operators by ι_γ. They take a form and contract it with the multivector field γ. The signs are such that

$$\begin{aligned}\iota : T^\bullet &\to End(\Omega^\bullet)\\ \gamma &\mapsto \iota_\gamma\end{aligned}$$

is a morphism of graded algebras. For example, for a function f, ι_f is multiplication by f, for a vector field ξ, ι_ξ is a derivation of the dga $\Omega^\bullet$ and for any multivector fields γ, ν, $\iota_{\gamma\wedge\nu} = \iota_\gamma\iota_\nu$. The Lie derivative L is:

$$L_\gamma := [\mathrm{d}, \iota_\gamma]\ .$$

It satisfies the following relation, which can alternatively be taken as the definition of the Schouten bracket.

$$\iota_{[\gamma,\nu]_S} = [\iota_\gamma, L_\nu] = (-1)^{|\gamma|}\,[L_\gamma, \iota_\nu]$$

It follows that L forms a representation of the differential graded Lie algebra $T^{\bullet+1}$. Here and everywhere in the paper the degrees $|\gamma|$ are such that $\gamma \in T^{|\gamma|+1}$.

Next consider module structures on $(\Omega^\bullet[[u]], u\mathrm{d})$ over the dgla $(T^\bullet[[u]], [\cdot,\cdot]_S\,, u\,\mathrm{div}_\Omega)$. Let us introduce a family of actions $L^{(t)}_\gamma$ as follows. Let $S^{(t)}$ be the u-scaling operation on multivector fields given by

$$S^{(t)}\gamma = S^{(t)}\left(\sum_{j\geq 0} u^j\gamma_j\right) = \sum_{j\geq 0}(tu)^j\gamma_j\,.$$

Let further

$$\iota^{(t)}_\gamma = \iota_{S^{(t)}\gamma}\,.$$

The family of dgla actions is then given by

$$L^{(t)}_\gamma = (1/u)(\left[u\mathrm{d}, \iota^{(t)}_\gamma\right] + \iota^{(t)}_{u\,\mathrm{div}_\Omega\,\gamma}) = \sum_{j\geq 0}(ut)^j(L_{\gamma_j} + t\iota_{\mathrm{div}_\Omega\,\gamma_j})$$

where $\gamma = \sum_{j\geq 0} u^j\gamma_j \in T^\bullet[[u]]$.

Proposition 2. *For any $t \in \mathbb{C}$, $L^{(t)}_\gamma$ defines a dgla module structure on $\Omega^\bullet[[u]]$. Furthermore all these module structures are L_∞-isomorphic to each other.*

Proof. To show that the $L^{(t)}_\gamma$ are indeed dgla actions, compute

$$\begin{aligned}\left[u\mathrm{d}, L^{(t)}_\gamma\right] &= \sum_{j\geq 0}(ut)^j t\left[u\mathrm{d}, \iota_{\mathrm{div}_\Omega \gamma_j}\right] \\ &= \sum_{j\geq 0}(ut)^{j+1}L_{\mathrm{div}_\Omega \gamma_j} = L_{u\,\mathrm{div}_\Omega \gamma^{(t)}}\,.\end{aligned}$$

Furthermore

$$\begin{aligned}\left[L^{(t)}_\gamma, L^{(t)}_\nu\right] &= \sum_{j,k\geq 0}(ut)^{j+k}\left[L_{\gamma_j}+t\iota_{\mathrm{div}_\Omega \gamma_j}, L_{\nu_k}+t\iota_{\mathrm{div}_\Omega \nu_k}\right] \\ &= \sum_{j,k\geq 0}(ut)^{j+k}\left(L_{[\gamma_j,\nu_k]}+t(-1)^{|\gamma_j|}\iota_{[\gamma_j,\mathrm{div}_\Omega \nu_k]}+t\iota_{[\mathrm{div}_\Omega \gamma_j,\nu_k]}\right) \\ &= \sum_{j,k\geq 0}(ut)^{j+k}\left(L_{[\gamma_j,\nu_k]}+t\iota_{\mathrm{div}_\Omega[\gamma_j,\nu_k]}\right) \\ &= L^{(t)}_{[\gamma,\nu]}\,.\end{aligned}$$

Next we construct a family of L_∞ isomorphisms $H^{(t)}$ relating $L^{(t)}_\gamma$ and $L^{(0)}_\gamma$. These isomorphisms will be solutions of a differential equation

$$\dot{H}^{(t)} = h^{(t)}H^{(t)}$$

for some family of infinitesimal morphisms (L_∞-derivations) $h^{(t)}$. In fact, the $h^{(t)}$ will have vanishing zeroth Taylor component and will all commute, so that one can explicitly write down the solution

$$H^{(t)} = \exp\left(\int_0^t h^{(t)}dt\right).$$

The $h^{(t)}$ will have only a single non-vanishing Taylor coefficient of degree one, which we denote (admittedly slightly confusing) by

$$h^{(t)}_1(\gamma;\alpha) = -(-1)^{|\gamma|}h^{(t)}_\gamma\alpha\,.$$

One finds that the L_∞-derivation property is equivalent to the following two conditions for $h^{(t)}_\gamma$.

$$\begin{aligned}-\frac{d}{dt}L^{(t)}_\gamma &= \left[u\mathrm{d}, h^{(t)}_\gamma\right]+h^{(t)}_{u\,\mathrm{div}_\Omega \gamma} \\ h^{(t)}_{[\gamma,\nu]_S} &= \left[h^{(t)}_\gamma, L^{(t)}_\nu\right]+(-1)^{|\gamma|}\left[L^{(t)}_\gamma, h^{(t)}_\nu\right]\end{aligned}$$

All higher L_∞ relations are trivially satisfied.

We claim that

$$h^{(t)}_\gamma = -\frac{1}{u}\frac{d}{dt}\iota^{(t)}_\gamma$$

satisfies these equations.[4] Compute

$$\begin{aligned}\frac{d}{dt}L^{(t)}_\gamma &= (1/u)\left[u\mathrm{d}, \frac{d}{dt}\iota^{(t)}_\gamma\right]+(1/u)\frac{d}{dt}\iota^{(t)}_{u\,\mathrm{div}_\Omega \gamma} \\ &= -\left[u\mathrm{d}, h^{(t)}_\gamma\right]-h^{(t)}_{u\,\mathrm{div}_\Omega \gamma}\,.\end{aligned}$$

[4]Note that the expression on the right is well defined since $\frac{d}{dt}\iota^{(t)}_\gamma \sim O(u)$.

In second order

$$\begin{aligned}\left[h_\gamma^{(t)}, L_\nu^{(t)}\right] + (-1)^{|\gamma|}\left[L_\gamma^{(t)}, h_\nu^{(t)}\right] &= -\sum_{j,k}(tu)^{j+k}\left((j/t)\iota_{[\gamma_j,\nu_k]} + (k/t)\iota_{[\gamma_j,\nu_k]}\right)\\ &= -\frac{d}{dt}\sum_{j,k}(tu)^{j+k}\iota_{[\gamma_j,\nu_k]}\\ &= h_{[\gamma,\nu]}^{(t)}\,.\end{aligned}$$

□

In the special case $t = 0$ the action becomes

$$L_\gamma^{(0)}\alpha = L_{\gamma_0}\alpha$$

and in the case $t = 1$

$$L_\gamma^{(1)}\alpha = L_\gamma\alpha + \iota_{\mathrm{div}_\Omega\,\gamma}\alpha.$$

The quasi-isomorphism between these two structures is given by

$$H^{(1)} = e^{\int_0^1 h^{(t)}dt} = e^{-\iota^+/u}$$

where $\iota_\gamma^+ = \iota_\gamma^{(1)} - \iota_\gamma^{(0)} = \sum_{j\geq 1} u^j\iota_{\gamma_j}$. Concretely, the n-th Taylor component reads

$$H_n^{(1)}(\gamma_1,\dots,\gamma_n) = \pm\frac{1}{u^n}\iota_{\gamma_1}^+\cdots\iota_{\gamma_n}^+.$$

2.4. **Multivector field valued top forms $VT^\bullet$ (module).** We define the multivector field valued top forms

$$VT^\bullet := \Omega^d(M;\wedge^\bullet TM)\,.$$

There is a natural non-degenerate pairing

$$\begin{gathered}\langle\cdot,\cdot\rangle : VT^\bullet\otimes\Omega_c^\bullet(M)\to\mathbb{C}\\ \langle\nu\Omega,\alpha\rangle = \int_M \Omega(\iota_\nu\alpha)\,.\end{gathered}$$

Its obvious u-bilinear extension allows for dualizing the dgla-module structures L and $L^{(t)}$ on $\Omega^\bullet(M)[[u]]$ discussed above to dgla-module structures on $VT^\bullet[[u]]$. We denote these dual module structures also by $L^{(t)}$ and hope that no confusion arises. Concretely, in our sign conventions the differential, temporarily called δ, and action are defined such that

$$\begin{aligned}\langle\delta(\nu\Omega),\alpha\rangle &= -(-1)^{|\nu|}\langle\nu\Omega, u\mathrm{d}\alpha\rangle\\ \left\langle L_\gamma^{(t)}(\nu\Omega),\alpha\right\rangle &= -(-1)^{|\nu||\gamma|}\left\langle\nu\Omega, L_\gamma^{(t)}\alpha\right\rangle\,.\end{aligned}$$

Lemma 3. *The dgla module structure $L^{(t)}$ on $VT^\bullet[[u]]$ is given explicitly by the following data: The differential is $\delta = u\,\mathrm{div}$ with*

$$\mathrm{div}(\nu\Omega) := (\mathrm{div}_\Omega\,\nu)\Omega.$$

The action is

$$L_\gamma^{(t)}(\nu\Omega) = \sum_{j\geq 0}(tu)^j\left([\gamma,\nu]_S\,\Omega + (-1)^{|\gamma_j|}(1-t)(\mathrm{div}_\Omega\,\gamma\wedge\nu)\Omega\right)$$

where $\gamma = \sum_{j\geq 0} u^j\gamma_j$.

Proof. Note first that

$$\int_M(\iota_\gamma\alpha)\Omega = \int_M \alpha\wedge\iota_\gamma\Omega\,.$$

It follows that

$$\begin{aligned}\langle\delta(\nu\Omega),\alpha\rangle &= -(-1)^{|\nu|}u\int_M(\iota_\nu \mathrm{d}\alpha)\Omega = -(-1)^{|\nu|}u\int_M(\mathrm{d}\alpha)\iota_\nu\Omega\\ &= (-1)^{|\nu|+|\alpha|}u\int_M \alpha \mathrm{d}\iota_\nu\Omega = u\int_M \alpha\iota_{\mathrm{div}_\Omega\,\nu}\Omega\\ &= u\int_M(\iota_{\mathrm{div}_\Omega\,\nu}\alpha)\Omega = \langle u\,\mathrm{div}(\nu\Omega),\alpha\rangle\ .\end{aligned}$$

In the fourth line we used that everything is zero unless $|\alpha| = |\gamma|$. Furthermore, note that by a small computation

$$\int_M (L_\gamma\alpha)\Omega = -\int_M(\iota_{\mathrm{div}_\Omega\,\gamma}\alpha)\Omega\,.$$

Hence we obtain

$$\begin{aligned}\left\langle L^{(t)}_{u^j\gamma_j}(\nu\Omega),\alpha\right\rangle &= -(-1)^{|\nu||\gamma_j|}u^j\int_M \iota_\nu\left(t^jL_{\gamma_j}\alpha + t^{j+1}\iota_{\mathrm{div}_\Omega\,\gamma_j}\alpha\right)\Omega\\ &= -(-1)^{|\nu||\gamma_j|}(tu)^j\int_M\Big(\iota_{[\nu,\gamma_j]_S} + (-1)^{(|\nu|+1)|\gamma_j|}L_{\gamma_j}\iota_\nu\alpha\\ &\qquad\qquad +(-1)^{(|\nu|+1)|\gamma_j|}t\iota_{\mathrm{div}_\Omega\,\gamma_j\wedge\nu}\alpha\Big)\Omega\\ &= -(-1)^{|\nu||\gamma_j|}(tu)^j\int_M\Big(\iota_{[\nu,\gamma_j]_S} + (-1)^{(|\nu|+1)|\gamma_j|}\iota_{\mathrm{div}_\Omega\,\gamma_j\wedge\nu}\alpha\\ &\qquad\qquad +(-1)^{(|\nu|+1)|\gamma_j|}t\iota_{\mathrm{div}_\Omega\,\gamma_j\wedge\nu}\alpha\Big)\Omega\\ &= -(-1)^{|\nu||\gamma_j|}(tu)^j\int_M\Big(-(-1)|\nu||\gamma_j|\iota_{[\gamma_j,\nu]_S} - (-1)^{(|\nu|+1)|\gamma_j|}\iota_{\mathrm{div}_\Omega\,\gamma_j\wedge\nu}\alpha\\ &\qquad\qquad +(-1)^{(|\nu|+1)|\gamma_j|}t\iota_{\mathrm{div}_\Omega\,\gamma_j\wedge\nu}\alpha\Big)\Omega\\ &= \left\langle(tu)^j([\gamma,\nu]_S + (-1)^{|\gamma_j|}(1-t)\,\mathrm{div}_\Omega\,\gamma_j\wedge\nu)\Omega,\alpha\right\rangle\ .\end{aligned}$$

□

In view of the PSM morphism, the most interesting case is $t = 1$. Here the action is the pushforward of the adjoint action along the isomorphism

$$\begin{aligned}T^\bullet[[u]] &\to VT^\bullet[[u]]\\ \gamma &\mapsto \gamma\otimes\Omega\,.\end{aligned}$$

2.5. **The Hochschild chains (module).** The (normalized) Hochschild chain complex of the algebra A is the complex

$$C_{-\bullet}(A) = A\otimes\bar{A}^{\otimes\bullet}$$

where $\bar{A} = A/\mathbb{C}\cdot 1$. It is equipped with differential b_H

$$b_H(a_0\otimes\cdots\otimes a_n) = a_0a_1\otimes a_2\otimes\cdots\otimes a_n \pm\cdots + (-1)^n a_na_0\otimes a_1\otimes\cdots\otimes a_{n-1}.$$

The normalized Hochschild cochain complex acts on the normalized chain complex through the (dgla) action

$$\text{(1)}\quad \begin{aligned}L_D(a_0\otimes\cdots\otimes a_n) = &\sum_{j=n-d+1}^{n}(-1)^{n(j+1)}D(a_{j+1},\ldots,a_0,\ldots)\otimes a_{d+j-n}\otimes\cdots\otimes a_j+\\ &+\sum_{i=0}^{n-d}(-1)^{(d-1)(i+1)}a_0\otimes\cdots\otimes a_i\otimes D(a_{i+1},\ldots,a_{i+d})\otimes\cdots\otimes a_n.\end{aligned}$$

In particular $b_H = L_{m_0}$.

2.6. **The cyclic chains (module).** The normalized Hochschild chain complex is equipped with an additional differential B of degree -1 discovered by Rinehart and rediscovered by Connes.

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n} (-1)^{jn} 1 \otimes a_j \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{j-1}$$

One can check that this differential (graded) commutes with the action (1) above, and hence anticommutes with b_H. Introducing an additional formal variable u of degree $+2$, one defines the negative cyclic chain complex as

$$(C_\bullet(A)[[u]], b_H + uB).$$

Its homology is called the negative cyclic homology. Other cyclic homology theories can be obtained from the negative cyclic complex by tensoring with an appropriate $\mathbb{C}[u]$-module and will not receive specialized treatment in this paper.

2.7. **Hochschild complex – sheaf version $E^\bullet$ (module).** Consider the sheaf $D^\bullet(M)$ of $\bullet$-differential operators. E.g., $D^1(M)$ is the sheaf of differential operators. It is a complex with the Hochschild differential[5]

$$(b\Phi)(a_0, \dots, a_n) = \pm \left(\Phi(a_0 a_1, a_2, \dots, a_n) - \Phi(a_0, a_1 a_2, \dots, a_n) \pm \dots \right.$$
$$\left. -(-1)^n \Phi(a_0, a_1, \dots, a_{n-1} a_n) + (-1)^n \Phi(a_n a_0, a_1, \dots, a_{n-1}) \right) .$$

Also, note that there is an action of the cyclic group(oid) on $D^\bullet(M)$ generated by

$$(\sigma\Phi)(a_0, \dots, a_n) = (-1)^n \Phi(a_1, a_2, \dots, a_n, a_0).$$

There is a canonical flat connection ∇ on $D^\bullet(M)$, compatible with the differential and the cyclic action. It is given by the de Rham differential:

$$(\nabla\Phi)(a_0, \dots, a_n) = \mathrm{d}(\Phi(a_0, \dots, a_n)).$$

Definition 4. The extended Hochschild cochain complex is the total complex

$$E^\bullet = \oplus_{p+q-d=\bullet} (\Gamma(D^{p+1}(M) \otimes_{C^\infty(M)} \Omega^q(M)), b + \nabla) .$$

The normalized extended Hochschild complex $E^\bullet_{norm}$ is the subcomplex of multidifferential operators Φ such that

$$\Phi(a_0, \dots, a_{j-1}, 1, a_{j+1}, \dots, a_n) = 0$$

for all $a_0, \dots a_n$ and all $j = 1, .., n$.

There is an action on $E^\bullet$ of the multidifferential operators, now considered as a sheaf of dg Lie algebras with differential d_H, by the formula dual to (1), i.e.,

$$(L_D\Phi)(a_0, \dots, a_n) =$$
$$= -(-1)^{|D||\Phi|} \left(\sum_{j=n-d+1}^{n} (-1)^{n(j+1)} \Phi(D(a_{j+1}, \dots, a_0, \dots), a_{d+j-n}, \dots, a_j) + \right.$$
$$\left. + \sum_{i=0}^{n-d} (-1)^{(d-1)(i+1)} \Phi(a_0, \dots, a_i, D(a_{i+1}, \dots, a_{i+d}), \dots, a_n) \right) .$$

In terms of this action, the differential can be written as $b = L_m$ where m is the multiplication cochain.

The complex $E^\bullet$ is just another complex computing Hochschild cohomology with values in $\Omega^d(M)$, as the following proposition shows.

[5] Note that this is not the b_H from above, there is no $a_0\Phi(a_1, a_2, \dots, a_n)$-term.

Proposition 5. *The embedding $C^\bullet(A, \Omega^d(M)) \to E^\bullet$ given by*
$$\Phi \mapsto ((a_0, \dots, a_n) \mapsto a_0 \Phi(a_1, ..., a_n))$$
is a quasi-isomorphism.

We will benefit from the following elementary result.

Lemma 6. *Let $(K^{p,q})_{0 \le p \le n, q \in \mathbb{Z}}$ be a double complex with differential $d_1 + d_2$, where*
$$d_1 : K^{p,q} \to K^{p+1,q} \qquad d_2 : K^{p,q} \to K^{p,q+1}.$$
Then the following holds:

(1) *If the d_1-cohomology is concentrated in bottom degree $p = 0$, then the inclusion of the d_1-closed, p-degree 0 elements*
$$\{k \in K^{0,\bullet} \mid d_1 k = 0\} \hookrightarrow K^{\bullet,\bullet}$$
is a quasi-isomorphism.

(2) *If the d_1-cohomology is concentrated in top degree $p = n$, then the projection onto the top p degree elements modulo exact elements*
$$K^{\bullet,\bullet} \twoheadrightarrow K^{n,\bullet}/d_1 K^{n-1,\bullet}$$
is a quasi-isomorphism.

Proof. At least the first statement is probably familiar to the reader. The proof of the second statement is essentially dual to the proof of the first. □

Proof of Proposition 5. It is more or less obvious that the above map is a map of complexes. It remains to be shown that it is a quasi-isomorphism.

Let us compute the cohomology of $E^\bullet$ wrt. ∇, i.e., the first term in the spectral sequence associated $E^\bullet$. We claim that it is concentrated in the top form-degree $d = \dim M$, and every class has exactly one representative in the image of the above quasi-isomorphism. To show this, consider the spectral sequence associated to the following filtration:
$$\mathcal{F}_p E = \{\Gamma(\Phi \in D^\bullet(M) \otimes_{C^\infty(M)} \Omega^k(M)) \mid k = 0, 1, \dots \text{ and } ord_0 \Phi \le p + k\}$$
where $ord_0 \Phi$ is the order of Φ as a differential operator in the first "slot" (i.e., the slot in which a_0 is inserted). One can check that $\nabla \mathcal{F}_p E \subset \mathcal{F}_p E$. The first term in the spectral series is the associated graded, i.e., multidifferential operators with values in $\wedge^\bullet T^*M \otimes S^\bullet TM$. The differential d_0 is, in local coordinates, the operator $d_0 = \sum_i (dx_i \wedge) \otimes (\partial_i \cdot)$, multiplying the $\wedge^\bullet T^*M$-part by dx^i and the $S^\bullet TM$-part by ∂_i. The cohomology is concentrated in form degree d and operator degree 0. Probably the quickest way to see this is to note that the complex $\wedge^\bullet T^*M \otimes S^\bullet TM$ with the above differential is isomorphic to the Koszul complex of $S^\bullet TM$, the isomorphism being given by contracting the first factor with a section of $\wedge^d TM$. The spectral sequence degenerates at this point by (form-)degree reasons. This means that any ∇-cohomology class has exactly one representative of form degree d and of differential operator degree 0 in the first slot. This proves the above claim, and hence the proposition. □

2.8. **Cyclic Cochains – sheaf version (module).**

Definition 7. The extended cyclic complex is the complex $(E^\bullet)^\sigma$ of invariants under the cyclic action. The extended cyclic (b, B)-complex is the complex $E^\bullet_{norm}[[u]]$ with differential $b + uB$, where B is Connes' B.

For an orientable manifold, this complex computes the cyclic cohomology.

Proposition 8. *For M orientable, the cohomology of the extended cyclic complexes $(E^\bullet)^\sigma$ and $E^\bullet_{norm}[[u]]$ is the cyclic cohomology of $C^\infty(M)$.*

Proof. Consider again the spectral sequence and compute the ∇-cohomology of the two complexes. As in the last proof, the first term of the spectral sequence for $E^\bullet_{norm}[[u]]$ is, as a vector space, isomorphic to $D^\bullet_{norm}[[u]]$, the isomorphism being given in the last proposition. One can see more or less by the definitions that the differentials b, B are mapped to b_H, B under this isomorphism.

For the case of $(E^\bullet)^\sigma$, note that ∇ commutes with the action of the cyclic group. It follows that taking the ∇-cohomology commutes with taking cyclic invariants. The result then follows as in the proof of the last proposition. □

3. Part II: The meaning of the PSM morphism

3.1. **The original PSM morphism.** Let M be orientable and choose a volume form Ω. The original PSM morphism $\mathcal{V}_{PSM,orig}$ is an L_∞-morphism of modules over the dg Lie algebra $(T^\bullet[[u]], u\,\mathrm{div}_\Omega, [\cdot,\cdot]_S)$, constructed by the first two authors in [1] using essentially an equivariant version of the Poisson sigma model. The two modules it relates are the cyclic chains and the multivector fields.

$$\mathcal{V}_{PSM,orig} : (C_\bullet(A,A)[[u]], b + uB) \to (T^\bullet[[u]], u\,\mathrm{div}_\Omega).$$

The module structure on the left is given by pulling back the $C^\bullet(A)$-action along $\mathcal{U}_K$. The module structure on the right is the trivial module structure (!). We copy the following proposition from [1]

Proposition 9. *The morphism $\mathcal{V}_{PSM,orig}$ is a morphism of L_∞-modules (but not a quasi-isomorphism).*

3.2. **The (reinterpreted) PSM morphism $\mathcal{V}^*_{PSM}$.** Here we give a new interpretation of the above morphism The (reinterpreted) PSM morphism $\mathcal{V}^*_{PSM}$ is a quasi-isomorphism of L_∞-modules over the dgla $(T^\bullet[[u]], u\,\mathrm{div}_\Omega, [\cdot,\cdot]_S)$. However, the two modules are the multivector-field-valued top forms, which can be identified with $T^\bullet[[u]]$ using the volume form, and the extended cyclic complex $E^\bullet_{norm}[[u]]$.

$$\mathcal{V}^*_{PSM} : (T^\bullet[[u]], u\,\mathrm{div}_\Omega) \cong (VT^\bullet[[u]], ud) \to (E^\bullet_{norm}[[u]], \nabla + b + uB).$$

The dgla module structure on the very left is the adjoint one, in contrast to the trivial one above, and on the middle $L^{(1)}$. The L_∞-module structure on the right is defined via pullback of the dgla action of $C^\bullet(A)$ via the (Kontsevich) L_∞-morphism $\mathcal{U}^{(0)}$.

The reinterpreted morphism is constructed from the original one as follows:

$$\mathcal{V}^*_{PSM}(\gamma_1,..,\gamma_m)(\gamma)(a_0,..,a_n) = \iota_{\mathcal{V}_{PSM,orig}(\gamma_1,..,\gamma_m,u\gamma;a_0,..,a_n)}\Omega.$$

Theorem 10. *The morphism $\mathcal{V}^*_{PSM}$ is a quasi-isomorphism of L_∞-modules.*

Proof. The fact that it is an L_∞-morphism is an easy consequence of Proposition 9 and the previous observation that for any multivector field ν

$$\iota_{\mathrm{div}\,\nu}\Omega = d\iota_\nu\Omega.$$

It remains to be shown that the zero-th Taylor component is an isomorphism on cohomology. In view of Lemma 6 it is sufficient to show that the composition with the projection onto the top form degree part modulo the image of ∇ is a quasi-isomorphism. Explicit computation yields that the 0-th Taylor component is

$$\gamma \mapsto \pm((a_0,..,a_k) \mapsto a_0\gamma(a_1,..,a_k)\Omega) + (\text{lower form degree}).$$

The first part is the HKR morphism, known to be a quasi-isomorphism, and the remainder does not matter due to the projection onto top form degree components. □

The statement of Theorem 10 can be seen as a dualized version of B. Tsygan's negative cyclic formality conjecture. The more precise relation is as follows. Note that the complex $E^\bullet$ is bigger than the dual (in an appropriate sense) of $C_\bullet(A)$. Concretely, it also contains non-top-degree differential forms. These forms do not show up in cohomology, but are needed to interpret the "white vertices", see [1], occuring in the original PSM morphism. The "true" dual of $C_\bullet(A)$ occurs after projecting $E^\bullet$ to top forms, modulo the image of ∇. This projection in particular kills all white vertices occuring in $\mathcal{V}^*_{PSM}$, but leaves it a quasi-isomorphism due to the proof of Proposition 8. By the remarks in Section 1.3, one can dualize this quasi-isomorphism again and obtain another solution of B. Tsygans formality conjecture.

Appendix A. Our signs conventions

There are many signs involved in the discussions above. Since sign computations are typically lengthy and boring, we did not explain them all. However, we list here the underlying conventions for the reader who believes $1 \neq -1$ and wants to check.

Let $\mathfrak{g}^\bullet$ be a graded vector space. An L_∞-algebra structure on $\mathfrak{g}^\bullet$ is a degree 1 coderivation Q on the cofree (graded) cocommutative coalgebra without counit cogenerated by $\mathfrak{g}^{\bullet+1}$, i.e. $S^+\mathfrak{g}^{\bullet+1}$, satisfying $Q^2 = 0$. Any such coderivation is determined by its Taylor coefficients

$$Q_n(x_1, \dots, x_n) = \pi Q(x_1, \dots, x_n)$$

where π is the projection on $\mathfrak{g}^{\bullet+1} \subset S^+\mathfrak{g}^{\bullet+1}$. If $\mathfrak{g}$ carries the structure $(d, [\cdot, \cdot])$ of a dgla, we associate to it an L_∞-structure by the following convention (others are possible)

$$Q_1(x) = dx \qquad\qquad Q_2(x_1, x_2) = -(-1)^{|x_1|}\,[x_1, x_2]\ .$$

An L_∞-module structure on the graded vector space $M^\bullet$ is a coderivation $\tilde{Q}$ lifting Q on the cofree comodule $S\mathfrak{g}^{\bullet+1} \otimes M^\bullet$. Again, it is determined by its Taylor coefficients $\pi_M \circ \tilde{Q}$. We identify (by convention) a dgla module $(M^\bullet, \delta, L)$ over the dgla $\mathfrak{g}$ with the L_∞-module

$$\tilde{Q}_0(m) = \delta m \qquad\qquad \tilde{Q}_1(x; m) = -(-1)^{|x_1|} L_x m\,.$$

Next let $\hat{M}^\bullet$ be another graded vector space and $\langle \cdot, \cdot \rangle$ be a nondegenerate pairing between $\hat{M}^\bullet$ and $M^\bullet$. This allows us to endow $\hat{M}^\bullet$ with an L_∞-structure $\tilde{Q}^*$ defined by

$$\left\langle \tilde{Q}^*_n(x_1, \dots, x_n; \hat{m}), m \right\rangle = -(-1)^{|\hat{m}|(n+1+\sum_j |x_j|)} \left\langle \hat{m}, \tilde{Q}_n(x_1, \dots, x_n; m) \right\rangle\ .$$

Let $M^\bullet, N^\bullet$ be L_∞-modules. A morphism ϕ between them is a degree zero morphism of the comodules intertwining the coderivations. It is also determined by the Taylor coefficients $\pi_N \phi$. Let $\hat{N}^\bullet, \hat{M}^\bullet$ be L_∞-modules, with the module structure determined by nondegenerate pairings as above. Then one can define an adjoint morphism $\phi^\star$ from $\hat{N}$ to $\hat{M}$ by the formula

$$\langle \phi^*_n(x_1, \dots, x_n; \hat{n}), m \rangle = (-1)^{|\hat{m}|(n+\sum_j |x_j|)} \langle \hat{n}, \phi_n(x_1, \dots, x_n; m) \rangle\ .$$

Finally, let us describe the signs involved in section 3. Let Q be the coderivation determining the L_∞-algebra structure on $T^\bullet[[u]]$. Then the (adjoint) L_∞-module structure on $T^\bullet[[u]]$ is simply given by

$$\tilde{Q}_n(x_1, \dots, x_n; x) = Q_{n+1}(x_1, \dots, x_n, x).$$

Let $\tilde{P}$ determine the L_∞ module structure on $C_\bullet(A, A)[[u]]$. Then the module structure on $E^\bullet_{norm}[[u]]$ is determined by the coderivation $\tilde{O}$, defined such that for a map $\lambda : C_\bullet(A, A)[[u]] \to T^\bullet[[u]]$:

$$\tilde{O}_n(x_1, \dots, x_n; \iota_{\lambda(\cdot)}\Omega) = -(-1)^{|\lambda|(n+1+\sum_j |x_j|)} \iota_{\lambda(\tilde{P}_n(x_1, \dots, x_n; \cdot))}\Omega + \delta_{n,0} \nabla \iota_{\lambda(\cdot)}\Omega\,.$$

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