Relative Frobenius algebras are groupoids

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Abstract

We functorially characterize groupoids as special dagger Frobenius algebras in the category of sets and relations. This is then generalized to a non-unital setting, by establishing an adjunction between H*-algebras in the category of sets and relations, and locally cancellative regular semigroupoids. Finally, we study a universal passage from the former setting to the latter.

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1. Introduction

Groupoids generalize groups in two ways. They can be regarded as groups with more than one object, leading to the definition as (small) categories in which every morphism is invertible. Alternatively, they can be regarded as groups whose multiplication is relaxed to a partial function. This article makes the connection between these two views precise by detailing isomorphisms between the appropriate categories. The latter view is made rigorous by so-called special dagger Frobenius algebras in the category of sets and relations. Thus we give a non-commutative and functorial generalization of [6] in Section 2.

These results are somewhat surprising from the point of view of quantum groups, another generalization of the concept of group. Quantum groups are usually formalized as some sort of Hopf algebra. However, this notion is at

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odds with that of Frobenius algebra: if a multiplication carries both Hopf and Frobenius structure, then it must be trivial [4, Proposition 2.4.10].

Reformulating groupoids as relative Frobenius algebras has two advantages. First, it yields several interesting new choices of morphisms between groupoids. Second, Frobenius algebras can be interpreted in many categories without limits, whereas the usual formulation of a groupoid as an internal category requires the ambient category to have finite limits.

For example, a commutative Frobenius algebra structure on a finite-dimensional Hilbert space corresponds to a choice of orthonormal basis for that space. For this correspondence to hold in arbitrary dimension, the Frobenius algebra structure must be relaxed to a so-called H*-algebra structure, basically dropping units [1]. Section 3 considers the relative version: it turns out that H*-algebras in the category of sets and relations correspond to so-called locally cancellative regular semigroupoids. The correspondence is functorial, but this time gives adjunctions instead of isomorphisms of categories. Finally, Section 4 considers a universal passage from H*-algebras to Frobenius algebras.

The generalization to H*-algebras is useful for an application to geometric quantization that will be presented in subsequent work, where one is forced to work with semigroupoids instead of groupoids. Rather than in the category of sets and relations, this plays out in the smooth setting of symplectic manifolds and canonical relations, corresponding to Lie groupoids. One could imagine similar applications in a topological or localic setting [7].

2. Relative Frobenius algebras and groupoids

A (small) category can be defined as a category internal to the category **Set** of sets and functions, see [5, Section XII.1]. This is a collection

$$\begin{tikzcd} G_0 \arrow[r, "e" description] & G_1 \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & G_1 \times_{G_0} G_1 \arrow[l, "m"'] \end{tikzcd}$$

of objects G_0 (objects) and G_1 (morphisms) and morphisms s (source), t (target), e (identity), and m (composition). These data have to satisfy familiar algebraic formulae, stating *e.g.* that composition m is associative. A functor then is a pair of functions $f_i\colon G_i \to G'_i$ that commute with the above structure. A category is a *groupoid* when there additionally is a morphism $i\colon G_1 \to G_1$ (inverse) satisfying $m \circ (1 \times i) \circ \Delta = e \circ s\colon G_1 \to G_1$. Notice that this formulation requires the monoidal structure $\times$ to have diagonals $\Delta\colon G_1 \to G_1 \times G_1$, and the category to have pullbacks.

This section proves that groupoids correspond precisely to so-called relative Frobenius algebras. To introduce the latter concept, we pass to the category **Rel** of sets and relations, where morphisms $X \to Y$ are relations $r \subseteq X \times Y$, and

$$s \circ r = \{(x,z) \mid \exists y\,.\,(x,y) \in r, (y,z) \in s\}.$$

It carries a contravariant identity-on-objects involution $\dagger\colon \mathbf{Rel}^{\mathrm{op}} \to \mathbf{Rel}$ given by relational converse. It also has compatible monoidal structure, namely Cartesian product of sets. This makes **Rel** into a so-called dagger symmetric monoidal category. To distinguish **Rel** from its subcategory **Set**, we write morphisms in the former category as $r\colon X \nrightarrow Y$, and morphisms in the latter as $f\colon X \to Y$.

Definition 1. For a morphism $m\colon X \times X \nrightarrow X$ in **Rel**, consider the following properties:

$$(1 \times m) \circ (m^\dagger \times 1) = m^\dagger \circ m = (m \times 1) \circ (1 \times m^\dagger), \tag{F}$$

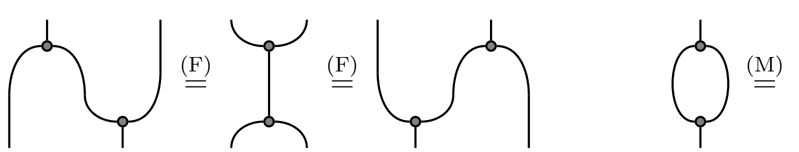
$$m \circ m^\dagger = 1, \tag{M}$$

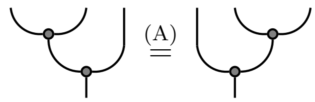
$$m \circ (1 \times m) = m \circ (m \times 1), \tag{A}$$

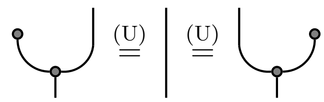
$$\text{there is } u\colon 1 \nrightarrow X \text{ with } m \circ (u \times 1) = 1 = m \circ (1 \times u). \tag{U}$$

If u exists then it is automatically unique. An object X together with such a morphism m is called a (unital) special dagger Frobenius algebra in **Rel**, or *relative Frobenius algebra* for short. Notice that this definition requires neither pullbacks nor diagonals, and makes sense in any dagger monoidal category.

The defining conditions of Frobenius algebras can also be presented graphically. Such string diagrams encode composition by drawing morphisms on top of each other, and the monoidal product by drawing morphisms next to each other. The dagger becomes a vertical reflection; we refer to [8] for more information. We depict $m\colon X \times X \nrightarrow X$ as , and $u\colon 1 \nrightarrow X$ as .







To prevent jumping back and forth between formalisms, we will mostly compute algebraically. But occasionally we will illustrate conditions graphically.

2.1. From relative Frobenius algebras to groupoids

For the rest of this subsection, fix a relative Frobenius algebra (X, m).

Concretely, (M) implies that m is single-valued. Therefore we may write $f = hg$ instead of $((h, g), f) \in m$. Notice that this notation implies that hg is defined, which fact we denote by $hg\downarrow$. We will use this *Kleene equality* throughout this article, by reading $x = y$ for $x, y \in X$ as follows: if either side is defined, so is the other, and they are equal. The concrete meaning of (M) is completed by

$$\forall f \in X\, \exists g, h \in X.\, f = hg.$$

Concretely, (F) means that for all $a, b, c, d \in X$

$$ab = cd \iff \exists e \in X.\, b = ed, c = ae \iff \exists e \in X.\, d = eb, a = ce.$$

The condition (A) comes down to $(fg)h = f(gh)$. Finally, identifying the morphism $u\colon 1 \mathrel{\mapstochar\!\!\rightarrow} X$ with a subset $U \subseteq X$, we find that (U) means

$$\begin{aligned}
&\forall f \in X\, \exists u \in U.\, fu = f,\\
&\forall f \in X\, \exists u \in U.\, uf = f,\\
&\forall f \in X\, \forall u \in U.\, fu\downarrow \implies fu = f,\\
&\forall f \in X\, \forall u \in U.\, uf\downarrow \implies uf = f.
\end{aligned}$$

Definition 2. Given a relative Frobenius algebra m, define the following objects and morphisms in **Rel**:

$$\begin{aligned}
G_1 &= X,\\
G_2 &= \{(g, f) \in X^2 \mid gf\downarrow\},\\
G_0 &= U,\\
e &= U \times U \colon G_0 \mathrel{\mapstochar\!\!\rightarrow} G_1,\\
s &= \{(f, x) \in G_1 \times G_0 \mid fx\downarrow\} \colon G_1 \mathrel{\mapstochar\!\!\rightarrow} G_0,\\
t &= \{(f, y) \in G_1 \times G_0 \mid yf\downarrow\} \colon G_1 \mathrel{\mapstochar\!\!\rightarrow} G_0,\\
i &= \{(g, f) \in G_2 \mid gf \in G_0, fg \in G_0\} \colon G_1 \mathrel{\mapstochar\!\!\rightarrow} G_1.
\end{aligned}$$

We will prove that the collection $\mathbf{G}$ of these data is a groupoid (in $\mathbf{Set}$). First, we show that the relations s, t, i are in fact (graphs of) functions, as is clearly the case for e.

Lemma 3. *For $f \in X$ and $u, v \in U$:*

1. *if $fu\downarrow$ then $u^2\downarrow$;*
2. *if $fu\downarrow$ and $fv\downarrow$ then $uv\downarrow$;*
3. *if $fu\downarrow$ and $fv\downarrow$ then $u = v$.*

Hence the relation s is (the graph of) a function. Similarly, t is (the graph of) a function.

Proof. If $fu\downarrow$, then $fu = f$ by (U), so that also $(fu)u = f$. By (A), this means in particular that $u^2\downarrow$, establishing (1). For part (2), assume that $fu\downarrow$ and $fv\downarrow$. Then $fu = f = fv$, and by (F) we have $u = ev$ for some $e \in X$, so that $uv = ev^2\downarrow$. Finally, for (3), notice that if $fu\downarrow$ and $fv\downarrow$ then $u = uv = v$ by (U). □

Lemma 4. *The pullback of s and t in* **Set** *is (isomorphic to) G_2.*

Proof. The pullback of s and t is given by $P = \{(g, f) \in X \mid s(g) = t(f)\}$. Now $s(g)$ is the unique $y \in U$ with $gy\downarrow$, and $t(f)$ is the unique $y' \in U$ with $y'f\downarrow$. So, if $(g, f) \in P$, then $y = y'$ so that $gyf\downarrow$, and by (U) also $gf\downarrow$ so $(g, f) \in G_2$. Conversely, if $(g, f) \in G_2$, then by (U) there exists $y \in U$ such that $gyf\downarrow$, and we have $s(g) = y = t(f)$. □

Lemma 5. *The following diagram in* **Rel** *commutes.*

$$\begin{tikzcd}
G_1 \arrow[rr, "s"] \arrow[d, "\Delta"'] & & G_0 \arrow[d, "e"] \\
G_1 \times G_1 \arrow[r, "1\times i"'] & G_1 \times G_1 \arrow[r, "m"'] & G_1
\end{tikzcd}$$

Here, Δ is (the graph of) the diagonal function $x \mapsto (x, x)$.

Proof. First we compute

$$e \circ s = \{(f, g) \in G_1 \times G_1 \mid \exists u \in U.\, g = u, fu\downarrow\} = \{(f, u) \in X \times U \mid fu\downarrow\},$$

and

$$\begin{aligned} m \circ (1 \times i) \circ \Delta &= \{(f, h) \in G_1^2 \mid \exists g \in G_1.\, fg \in U, gf \in U, h = gf\} \\ &= \{(f, gf) \in G_1^2 \mid g \in G_1,\ fg \in U \ni gf\}. \end{aligned}$$

Clearly $m \circ (1 \times i) \circ \Delta \subseteq e \circ s$. For the converse, suppose that $(f, u) \in e \circ s$. Since $fu\!\downarrow$ we then have $fu = f$ by (U). Therefore, again by (U), there exists $v \in U$ such that $fu = vf$. Now it follows from (F) that there exists g with $u = gf$ and $v = fg$. Thus $(f, u) = (f, gf) \in m \circ (1 \times i) \circ \Delta$. $\square$

Lemma 6. *The relation i is (the graph of) a function.*

Proof. We need to prove that to each $f \in X$ there is a unique $g \in X$ with $gf \in U \ni fg$. We already have existence of such a g by Lemma 5, so it suffices to prove unicity. Suppose that $gf \in U \ni fg$ and $g'f \in U \ni fg'$. Then in particular $fg\!\downarrow$ and $gf\!\downarrow$, so that by (A) also $fgf\!\downarrow$ and similarly $fg'f\!\downarrow$. Now (U) implies $fgf = f = fg'f$, so that by the previous conjecture $gf = g'f$. But then $g = gfg = g'fg = g'$. $\square$

Theorem 7. *If m is a relative Frobenius algebra,* $\mathbf{G}$ *is a groupoid (in* **Set***).*

Proof. The proof consists of routine verifications that the maps e, s, t, i indeed satisfy the axioms for a groupoid. The most interesting one has already been dealt with in Lemma 5. $\square$

2.2. From groupoids to relative Frobenius algebras

For the rest of this subsection, fix a groupoid

$$\mathbf{G} = \left(\begin{tikzcd} G_0 \arrow[r, "e" description] & G_1 \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] \arrow[loop above, "i"] & G_1 \times_{G_0} G_1 \arrow[l, "m"'] \end{tikzcd} \right).$$

Definition 8. For a groupoid $\mathbf{G}$, define $X = G_1$, and let $m \colon G_1 \times G_1 \twoheadrightarrow G_1$ be the graph of the function m.

We will prove that m is a relative Frobenius algebra. For starters, it follows directly from associativity of composition in the groupoid $\mathbf{G}$ that m satisfies (A).

Lemma 9. *The morphism m of* **Rel** *satisfies (U).*

Proof. Define a relation $u \colon 1 \twoheadrightarrow X$ by $u = \{(*, e(x)) \mid x \in G_0\}$. Then

$$\begin{aligned} m \circ (u \times 1) &= \{(f, e(x)f) \mid f \in G_1, x = t(f) \in G_0\} \\ &= \{(f, et(f)f) \mid f \in G_1\} = 1. \end{aligned}$$

The symmetric condition also holds, and so (U) is satisfied. $\square$

Lemma 10. *The morphism m of* **Rel** *satisfies (M).*

Proof. We have

$$m \circ m^\dagger = \{(f, f) \in G_1^2 \mid \exists g, h \in G_2.\, s(h) = t(g),\ f = hg\}.$$

Because we can always take $g = f$ and $h = e(t(f))$, this relation is equal to $\{(f, f) \in G_1^2 \mid f \in G_1\} = 1$. Thus (M) is satisfied. □

Lemma 11. *The morphism m of* **Rel** *satisfies (F).*

Proof. First compute

$$m^\dagger \circ m = \{((a, b), (c, d)) \in G_2^2 \mid ab = cd\},$$
$$(m \times 1) \circ (1 \times m^\dagger) = \{((a, b), (c, d)) \in G_2^2 \mid \exists e \in G_1.\, ed = b,\ ae = c\}.$$

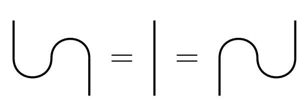
If $ed = b$ and $ae = c$, then $cd = aed = ab$. Hence $(m \times 1) \circ (1 \times m^\dagger) \subseteq m^\dagger \circ m$. Conversely, suppose that $((a, b), (c, d)) \in m^\dagger \circ m$. Taking $e = bd^{-1}$, we obtain $ed = bdd^{-1} = b$, and subsequently $ae = abd^{-1} = cdd^{-1} = c$. Therefore $m^\dagger \circ m \subseteq (1 \times m^\dagger) \circ (m \times 1)$. The symmetric condition is verified analogously. Thus (F) is satisfied. □

Theorem 12. *If* **G** *is a groupoid, then m is a relative Frobenius algebra.* □

2.3. Functoriality

Notice that the constructions $m \mapsto \mathbf{G}$ and $\mathbf{G} \mapsto m$ of the previous two sections are each other's inverse. This subsection proves that the assignments extend to an isomorphism of categories under various choices of morphisms: one that is natural for groupoids, one that is natural for relations, and one that is natural to Frobenius algebras. (See also [2].) We start by considering a choice of morphisms that is natural from the point of view of relations: namely, morphisms between groupoids are subgroupoids of the product.

The category **Rel** is *compact closed*, *i.e.* allows morphisms $\eta_X \colon 1 \to X \times X$ satisfying $(\eta^\dagger \times 1) \circ (1 \times \eta) = 1 = (1 \times \eta^\dagger) \circ (\eta \times 1)$. Drawing η as $\cup$, this property graphically becomes the following.



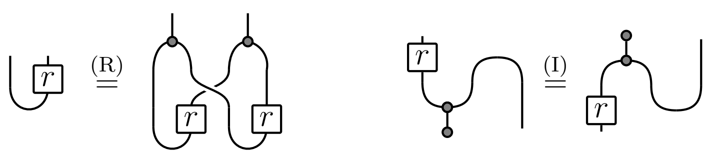
In fact, any relative Frobenius algebra induces such a compact structure on X, by $\eta = m^\dagger \circ u$. There is a canonical choice of compact structure, that we use from now on, namely $\eta = \{(*, (x,x)) \mid x \in X\}$. It is induced by the relative Frobenius algebra corresponding to the discrete groupoid on X, and therefore has a universal categorical characterization. It follows that a morphism $r\colon X \rightarrow\!\!\!\!\!\!+\; Y$ in **Rel** is completely determined by its transpose $\ulcorner r \urcorner = (1 \times r) \circ \eta = \{(*, (x,y)) \mid (x,y) \in r\}\colon 1 \rightarrow\!\!\!\!\!\!+\; X \times Y$. Furthermore, the dagger in **Rel** is compatible with the symmetric monoidal structure, giving a natural swap isomorphism $\sigma\colon X \times Y \to Y \times X$ with $\sigma^{-1} = \sigma^\dagger$.

Definition 13. The category $\mathbf{Frob}(\mathbf{Rel})^{\mathrm{rel}}$ has relative Frobenius algebras as objects. A morphism $(X, m_X) \to (Y, m_Y)$ is a morphism $r\colon X \rightarrow\!\!\!\!\!\!+\; Y$ in **Rel** satisfying

$$(m_X \times m_Y) \circ (1 \times \sigma \times 1) \circ (\ulcorner r \urcorner \times \ulcorner r \urcorner) = \ulcorner r \urcorner, \tag{R}$$

$$(r \times \eta^\dagger) \circ (m_X^\dagger \times 1) \circ (u_X \times 1) = (u_Y^\dagger \times 1) \circ (m_Y \times 1) \circ (r \times \eta) \tag{I}$$

These conditions translate into string diagrams as follows.



Proposition 14. $\mathbf{Frob}(\mathbf{Rel})^{\mathrm{rel}}$ *is a well-defined category.*

Proof. Clearly, identities $1_X = \{(x,x) \mid x \in X\}\colon X \rightarrow\!\!\!\!\!\!+\; X$ satisfy (R) and (I). Observe that the composition $(1 \times \eta^\dagger) \circ (m^\dagger \times 1) \circ (u \times 1)\colon X \rightarrow\!\!\!\!\!\!+\; X$ is the relation $\{(x, x^{-1}) \mid x \in X\}$, where x^{-1} is the inverse of x when regarding X as the set of morphisms of a groupoid as per Theorem 7. So (I) means that $(x,y) \in r$ if and only if $(x^{-1}, y^{-1}) \in r$. Now, if $r\colon X \rightarrow\!\!\!\!\!\!+\; Y$ and $s\colon Y \rightarrow\!\!\!\!\!\!+\; Z$ satisfy (R) and (I), then so does $s \circ r$:

$$\begin{aligned}
\ulcorner s \circ r \urcorner &= \{(*, x'', z'') \in 1 \times X \times Z \mid \exists y'' \in Y.\, (x'', y'') \in r, (y'', z'') \in s\} \\
&= \{(*, xx', zz') \mid x, x' \in X, z, z' \in Z, \exists y, y' \in Y. \\
&\qquad\qquad (x,y) \in r, (x', y') \in r, (y,z) \in s, (y', z') \in s\} \\
&= (m_X \times m_Z) \circ (1 \times \sigma \times 1) \circ (\ulcorner s \circ r \urcorner \times \ulcorner s \circ r \urcorner).
\end{aligned}$$

Let us justify the second equation. If $(x'', y'') \in r$, then $(x''^{-1}, y''^{-1}) \in r$ by (I), and hence $(1,1) \in r$ by (R). Hence we may take $x = x''$, $x' = 1$, $y = y''$ and $y' = 1$. $\square$

Definition 15. The category $\mathbf{Gpd}^{\mathrm{rel}}$ has groupoids as objects. Morphisms $\mathbf{G} \to \mathbf{H}$ are subgroupoids of $\mathbf{G} \times \mathbf{H}$.

That this is a well-defined category will follow from the following theorem. Identities are the diagonal subgroupoids, and composition of subgroupoids $\mathbf{R} \subseteq \mathbf{G} \times \mathbf{G}'$ and $\mathbf{S} \subseteq \mathbf{G}' \times \mathbf{G}''$ is the groupoid $S_1 \circ R_1 \rightrightarrows S_0 \circ R_0$.

Theorem 16. *There is an isomorphism of categories* $\mathbf{Frob}(\mathbf{Rel})^{\mathrm{rel}} \cong \mathbf{Gpd}^{\mathrm{rel}}$.

Proof. Let (X, m_X) and (Y, m_y) be relative Frobenius algebras, inducing groupoids $\mathbf{G}$ and $\mathbf{H}$. First, notice that if $r \colon X \twoheadrightarrow Y$ satisfies (R), then

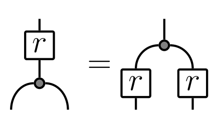
$$\begin{aligned} m_r &= (m_X \times m_Y) \circ (1 \times \sigma \times 1) \\ &= \{(((a,b),(c,d)),(ac,bd)) \mid a,b,c,d \in X\} \colon r \times r \twoheadrightarrow r \end{aligned}$$

is a well-defined morphism in **Rel**. In fact, since (X, m_X) and (Y, m_Y) are relative Frobenius algebras, so is (r, m_r): one readily verifies that it satisfies (M), (A), and (F). Also, (U) is satisfied by the intersection $1 \twoheadrightarrow R$ of r with $U_X \times U_Y$. Theorem 7 thus shows that r induces a groupoid $\mathbf{R}$. It is a subgroupoid of $\mathbf{G} \times \mathbf{H}$: we have $R_1 \subseteq (G \times H)_1$ by construction, and if $u \in U_R$, then $u = u^{-1}$, so $u \in (G \times H)_0$. The structure maps of $\mathbf{R}$ are easily seen to be restrictions of those of $\mathbf{G} \times \mathbf{H}$.

Conversely, if $\mathbf{R}$ is a subgroupoid of $\mathbf{G} \times \mathbf{H}$, then clearly $R_1 \subseteq X \times Y$ is a morphism in **Rel** satisfying (R) and (I). It now suffices to observe that these constructions are inverses. □

Next, we consider a choice of morphisms that is natural to groupoids, namely functors. This entails dealing with functions. Fortunately, functions can be characterized among relations purely categorically. The category **Rel** is in fact a 2-category, where there is a single 2-cell $r \Rightarrow s$ when $r \subseteq s$. Hence it makes sense to speak of adjoints of morphisms in **Rel**. A morphism has a right adjoint if and only if it is (the graph of) a function.

Definition 17. The category $\mathbf{Frob}(\mathbf{Rel})$ has relative Frobenius algebras as objects. Morphisms $(X, m_X) \to (Y, m_Y)$ are morphisms $r \colon X \twoheadrightarrow Y$ that satisfy (I) and preserve the multiplication: $r \circ m_X = m_Y \circ (r \times r)$.



We write $\mathbf{Frob}(\mathbf{Rel})^{\text{func}}$ for the subcategory of all morphisms r that have a right adjoint and allow a 2-cell $r \circ u_X \Rightarrow u_Y$.

Lemma 18. *Morphisms in* $\mathbf{Frob}(\mathbf{Rel})$ *satisfy (R). Hence we have inclusions*

$$\mathbf{Frob}(\mathbf{Rel})^{\text{func}} \hookrightarrow \mathbf{Frob}(\mathbf{Rel}) \hookrightarrow \mathbf{Frob}(\mathbf{Rel})^{\text{rel}}.$$

Proof. Let $r \colon X \twoheadrightarrow\!\!\!\!\!+\, Y$ be a morphism in $\mathbf{Frob}(\mathbf{Rel})$. Then

$$\begin{aligned}
&(m_X \times m_Y) \circ (1 \times \sigma \times 1) \circ (\ulcorner r \urcorner \times \ulcorner r \urcorner) \\
&= (m_X \times m_Y) \circ (1 \times r \times r) \circ (1 \times \sigma \times 1) \circ \{(*, (x, x, y, y)) \mid x, y \in X\} \\
&= (1 \times r) \circ (m_X \times m_X) \circ \{(*, (x, y, x, y)) \mid x, y \in X\} \\
&= (1 \times r) \circ \{(*, (xy, xy)) \mid xy\!\downarrow\} \\
&= (1 \times r) \circ \{(*, (z, z)) \mid z \in X\} \\
&= \ulcorner r \urcorner,
\end{aligned}$$

because we can always choose $x = z$ and $y = 1$. □

Write $\mathbf{Gpd}$ for the category of groupoids and functors.

Theorem 19. *There is an isomorphism of categories* $\mathbf{Frob}(\mathbf{Rel})^{\text{func}} \cong \mathbf{Gpd}$.

Proof. Let (X, m_X) and (Y, m_Y) be relative Frobenius algebras, inducing groupoids $\mathbf{G}$ and $\mathbf{H}$. Let $r \colon m_X \to m_Y$ be a morphism in $\mathbf{Frob}(\mathbf{Rel})^{\text{func}}$. The condition that r has a right adjoint means it is in fact a function $r \colon G_1 \to H_1$. Furthermore, the condition $r \circ u_X \subseteq u_Y$ means precisely that it sends G_0 to H_0. Finally, the condition that r preserve multiplication makes it functorial $\mathbf{G} \to \mathbf{H}$, because relational composition of graphs coincides with composition of functions. Conversely, it is easy to see that a functor between groupoids induces a morphism in $\mathbf{Frob}(\mathbf{Rel})^{\text{func}}$. Finally, these two constructions are inverse to each other. □

Corollary 20. *The category* $\mathbf{Gp}$ *of groups and homomorphisms is isomorphic to the full subcategory of* $\mathbf{Frob}(\mathbf{Rel})^{\text{func}}$ *consisting of those relative Frobenius algebras for which u has a right adjoint.*

Proof. The morphism $u \colon 1 \twoheadrightarrow\!\!\!\!\!+\, U$ has a right adjoint precisely when it is a function $u \colon 1 \to U$ and hence amounts to an element of U. That is, the corresponding groupoid has a single identity, *i.e.* is a group. □

Finally, we can consider a choice of morphisms that is natural from the point of view of Frobenius algebras, namely the category $\mathbf{Frob}(\mathbf{Rel})$. There is a category between $\mathbf{Gpd}$ and $\mathbf{Gpd}^{\mathrm{rel}}$, that corresponds to the middle category in the sequence $\mathbf{Frob}(\mathbf{Rel})^{\mathrm{func}} \hookrightarrow \mathbf{Frob}(\mathbf{Rel}) \hookrightarrow \mathbf{Frob}(\mathbf{Rel})^{\mathrm{rel}}$, as follows.

Definition 21. A *multi-valued functor* $\mathbf{G} \to \mathbf{H}$ between categories is a multi-valued function $F\colon G_1 \to H_1$ that preserves identities and composition:

$$\begin{array}{ll} \text{for } g, f \in G_1 \times_{G_0} G_1: & g \circ f \ni h \;\Rightarrow\; F(g) \circ F(f) \ni F(h), \\ \text{for } x \in G_0: & F(e(x)) \ni H_0. \end{array}$$

We denote the category of groupoids and multi-valued functors by $\mathbf{Gpd}^{\mathrm{mfunc}}$.

Theorem 22. *There is an isomorphism of categories* $\mathbf{Frob}(\mathbf{Rel}) \cong \mathbf{Gpd}^{\mathrm{mfunc}}$.

Proof. Let m_X and m_Y be relative Frobenius algebras, inducing groupoids $\mathbf{G}$ and $\mathbf{H}$. Let $r\colon m_X \to m_Y$ be a morphism in $\mathbf{Frob}(\mathbf{Rel})$; by Theorem 16 it induces a subgroupoid of $\mathbf{G}\times\mathbf{H}$. By the argument of the proof of Theorem 19, r is a multi-valued function $G_1 \to H_1$. But then it is precisely a multi-valued functor. $\square$

In summary, we have the following commutative diagram of categories.

$$\begin{tikzcd} \mathbf{Frob}(\mathbf{Rel})^{\mathrm{func}} \arrow[r, hook] \arrow[d, "\cong"] & \mathbf{Frob}(\mathbf{Rel}) \arrow[r, hook] \arrow[d, "\cong"] & \mathbf{Frob}(\mathbf{Rel})^{\mathrm{rel}} \arrow[d, "\cong"] \\ \mathbf{Gpd} \arrow[r, hook] & \mathbf{Gpd}^{\mathrm{mfunc}} \arrow[r, hook] & \mathbf{Gpd}^{\mathrm{rel}} \end{tikzcd}$$

3. Relative H*-algebras and semigroupoids

This section establishes a non-unital generalization of the correspondence of the previous section. Frobenius algebras are relaxed to the following non-unital version.

Definition 23. A *relative H*-algebra* is a morphism $m\colon X \times X \nrightarrow X$ in $\mathbf{Rel}$ satisfying (M), (A), and

$$\begin{array}{c} \text{there is an involution } *\colon \mathbf{Rel}(1, X) \to \mathbf{Rel}(1, X) \text{ such that} \\ m \circ (1 \times x^*) = (1 \times x^\dagger) \circ m^\dagger \text{ and } m \circ (x^* \times 1) = (x^\dagger \times 1) \circ m^\dagger \\ \text{for all } x\colon 1 \nrightarrow X. \end{array} \tag{H}$$

In the presence of (U) and (A), condition (H) above is equivalent to (F) [1]. But in the absence of (U), it is stronger than (F), and means concretely that

$$\forall A \subseteq X\, \forall x, y \in X \left[\begin{array}{l} \exists a \in A.\, xa = y \iff \exists a^* \in A^*.\, ya^* = x, \\ \exists a \in A.\, ax = y \iff \exists a^* \in A^*.\, a^*y = x. \end{array} \right.$$

Equivalently,

$$\forall a \in A \subseteq X\, \forall x \in X \left[\begin{array}{l} xa\downarrow \implies \exists a^* \in A^*.\, xaa^* = x, \\ ax\downarrow \implies \exists a^* \in A^*.\, a^*ax = x. \end{array} \right.$$

We will prove that relative H*-algebras are precisely semigroupoids that are regular and locally cancellative. Recall that a semigroupoid is a 'category without identities', just like a semigroup is a 'non-unital monoid' [3]. More precisely, a *semigroupoid* consists of a diagram

$$G_0 \underset{t}{\overset{s}{\leftleftarrows}} G_1 \xleftarrow{\;m\;} G_1 \times_{G_0} G_1$$

(in the category **Set** of sets and functions) such that $m \circ (m \times 1) = m \circ (1 \times m)$. We may also assume that s and t are jointly epic, *i.e.* $G_0 = s(G_1) \cup t(G_1)$. A *semifunctor* is then a 'functor without preservation of identities', *i.e.* a pair of functions $f_i \colon G_i \to G_i'$ that commute with the above structure.

A *pseudoinverse* of $f \in G_1$ is an element $f^* \in G_1$ satisfying ($s(f) = t(f^*)$ and $t(f) = s(f^*)$ and) $f = ff^*f$ and $f^* = f^*ff^*$. A semigroupoid is *regular* when every $f \in G_1$ has a pseudoinverse. Finally, a semigroupoid is *locally cancellative* when $fhh^* = gh^*$ implies $fh = g$, and $h^*hf = h^*g$ implies $hf = g$, for any $f, g, h \in G_1$ and any pseudoinverse h^* of h. The following lemma shows that the asymmetry in the latter condition is only apparent.

Lemma 24. *A semigroupoid is locally cancellative if and only if $fh^*h = gh$ implies $fh^* = g$ for any $f, g, h \in G_1$ and any pseudoinverse h^* of h.*

Proof. If h^* is a pseudoinverse of h, then h is a pseudoinverse of h^*, too. $\square$

Examples of locally cancellative semigroupoids are semigroupoids in which every morphism is both monic and epic. Clearly, groupoids are examples of locally cancellative regular semigroupoids. The following lemma gives a converse in the presence of identities.

Lemma 25. *If a locally cancellative regular semigroupoids has identities, then it is a groupoid.*

Proof. Let $f \in G_1$. By regularity, there is a pseudoinverse f^*. In the definition of local cancellativity, take $h = f^*$ and $g = 1_{t(f)}$, and $h^* = f$. Then $fhh^* = ff^*f = f = 1f = gh^*$, and so $ff^* = 1$. By Lemma 24 similarly $f^*f = 1$. Thus f^* is an inverse of f, and therefore unique. $\square$

3.1. From relative H-algebras to semigroupoids*

For the rest of this subsection, fix a relative H*-algebra $m\colon X \times X \twoheadrightarrow X$.

Definition 26. Define $\mathbf{G}$ by

$$\begin{aligned} G_0 &= \{f \in X \mid m(f,f) = f\}, \\ G_1 &= X, \\ s &= \{(f, f^*f) \mid f^* \textit{ is a pseudoinverse of } f\}\colon G_1 \twoheadrightarrow G_0 \\ t &= \{(f, ff^*) \mid f^* \textit{ is a pseudoinverse of } f\}\colon G_1 \twoheadrightarrow G_0. \end{aligned}$$

Lemma 27. *Each element a in a relative H*-algebra has $a^* \in \{a\}^*$ satisfying $a^*aa^* = a^*$ and $aa^*a = a$.*

Proof. By (M), we have $\forall y \in X\, \exists a, x \in X.\, y = ax$. Applying (H) with $A = X$ gives $\forall a \in X\, \exists x \in X.\, xa\!\downarrow$. Now let $a \in X$. If we substitute $A = \{a\}$, then (H) becomes

$$\begin{aligned} &\forall x, y \in X\big[xa = y \iff \exists a^* \in \{a\}^*.\, ya^* = x\big] \\ &\forall x, y \in X\big[ax = y \iff \exists a^* \in \{a\}^*.\, a^*y = x\big] \end{aligned}$$

As above, there exists $x \in X$ with $xa\!\downarrow$. So by the first condition above, there is $a' \in \{a\}^*$ with $aa'\!\downarrow$. Hence, by the second condition, there is $a'' \in \{a\}^*$ with $a''aa' = a'$. Applying the first condition again, now with $x = a'$ and $y = a''a$, gives $a'a = a''a$. Therefore we have $a^* = a^*aa^*$ for $a^* = a' \in \{a\}^*$. Finally, applying the first condition again, this time with $x = aa^*$ and $y = a$, we find that also $aa^*a = a$. $\square$

Lemma 28. *The data $\mathbf{G}$ form a well-defined semigroupoid.*

Proof. By (A), the condition $m \circ (m \times 1) = m \circ (1 \times m)$ is clearly satisfied. It remains to prove that m, s and t are well-defined functions.

The former means that $(g, f) \in G_1 \times_{G_0} G_1$ implies $gf\downarrow$. Now Assume $s(g) = t(f)$, *i.e.* $g^*g = ff^*$ for some pseudoinverses g^* and f^* of g and f. Because g^*g is idempotent, we have $g^*gff^* = g^*gg^*g = g^*g\downarrow$, and therefore also $gf\downarrow$. Hence m is well-defined.

As for t, suppose that f^* and f' are both pseudoinverses of f, so that $(f, ff^*) \in s$ and $(f, ff') \in s$. Then $ff^*f = f = ff'f$. Set $A = \{f^*\}$, $a = f^*$, $x = f$, and $y = ff'$. By Lemma 27, we obtain $f \in A^*$, and so $ya^* = x$ for $a^* = f$. Now it follows from (H) that $ff^* = xa = y = ff'$. Similarly, s is a well-defined function. □

Theorem 29. *If m is a relative H^*-algebra, then $\mathbf{G}$ is a locally cancellative regular semigroupoid.*

Proof. Regularity is precisely Lemma 27. Suppose that $fhh^* = gh^*$ for a pseudoinverse h^* of h. Applying (H) to $A = \{h\}$, $x = fhh^*$, $y = g$, $a = h$ and $a^* = h^*$ yields $fh = fhh^*h = xa^* = y = g$. Hence $\mathbf{G}$ is locally cancellative. □

3.2. From semigroupoids to relative H^*-algebras

For the rest of this subsection, fix a semigroupoid $\mathbf{G}$.

Definition 30. Define

$$\begin{aligned} X &= G_1, \\ m &= \{(g, f, gf) \mid s(g) = t(f)\} \colon G_1 \times G_1 \twoheadrightarrow G_1, \\ A^* &= \{a^* \in X \mid a^*aa^* = a^* \text{ and } aa^*a = a \text{ for all } a \in A\}. \end{aligned}$$

Theorem 31. *If $\mathbf{G}$ is a locally cancellative regular semigroupoid, then m is a relative H^*-algebra.*

Proof. Clearly, (A) is satisfied. Because

$$m^\dagger \circ m = \{(f, f) \in G_1^2 \mid \exists (g, h) \in G_2 \,.\, f = hg\}$$

we have $m^\dagger \circ m \subseteq 1$. Conversely, if $f \in G_1$, setting $g = f$ and $h = f^*f$ for some pseudoinverse f^* of f, then $f = gh$. Hence (M) is satisfied.

Finally, we verify (H). Let $A \subseteq X$ be given, let $a \in A$ and $x \in X$, and suppose that $xa\downarrow$. That means that $s(f) = t(a)$. By regularity, a has a pseudoinverse $a^* \in A^*$, and we have $xa = xaa^*a$. Setting $f = xa$, $g = x$, $h = a$ and $h^* = a^*$ in the definition of local cancellativity yields $xaa^* = x$. The symmetric condition is verified similarly. Hence (H) is satisfied. □

3.3. Functoriality

This subsection proves that the assignments $m \mapsto \mathbf{G}$ and $\mathbf{G} \mapsto m$ extend functorially to an adjunction. The following definitions give well-defined categories, just as in Subsection 2.3. Condition (I) has to be adapted to the non-unital setting of H*-algebras, and becomes the following.

$$y \circ r \circ x = y^* \circ r \circ x^* \text{ for all } x\colon 1 \twoheadrightarrow X, y\colon 1 \twoheadrightarrow Y \tag{I'}$$

Concretely, this means that $(x, y) \in r$ if and only if $(x^*, y^*) \in r$ for any pseudoinverses x^* of x and y^* of y.

Definition 32. Relative H*-algebras can be made into the objects of several categories. A morphism $(X, m_X) \to (Y, m_Y)$ in $\mathbf{Hstar}(\mathbf{Rel})^{\mathrm{rel}}$ is a morphism $r\colon X \twoheadrightarrow Y$ in **Rel** satisfying (R) and (I'). A morphism in $\mathbf{Hstar}(\mathbf{Rel})$ is a morphism $r\colon X \twoheadrightarrow Y$ in **Rel** that satisfies (I') and preserves multiplication: $r \circ m_X = m_Y \circ (r \times r)$. A morphism in $\mathbf{Hstar}(\mathbf{Rel})^{\mathrm{func}}$ is a morphism in $\mathbf{Hstar}(\mathbf{Rel})$ that additionally has a right adjoint.

Definition 33. Locally cancellative regular semigroupoids can be made into the objects of several categories. Morphisms in **LRSgpd** are semifunctors. Morphisms in $\mathbf{G} \to \mathbf{H}$ in $\mathbf{LRSgpd}^{\mathrm{mfunc}}$ are multi-valued semifunctors, *i.e.* multi-valued functions $G_i \to H_i$ satisfying only the first condition of Definition 21. Morphisms in $\mathbf{LRSgpd}^{\mathrm{rel}}$ are locally cancellative regular subsemigroupoids of $\mathbf{G} \times \mathbf{H}$.

Proposition 34. *The assignments $m \mapsto \mathbf{G}$ and $\mathbf{G} \mapsto m$ extend to functors*

$$\begin{aligned} \mathbf{Hstar}(\mathbf{Rel})^{\mathrm{rel}} &\leftrightarrows \mathbf{LRSgpd}^{\mathrm{rel}}, \\ \mathbf{Hstar}(\mathbf{Rel}) &\leftrightarrows \mathbf{LRSgpd}^{\mathrm{mfunc}}, \\ \mathbf{Hstar}(\mathbf{Rel})^{\mathrm{func}} &\leftrightarrows \mathbf{LRSgpd}. \end{aligned}$$

Proof. We prove the case $\mathbf{Hstar}(\mathbf{Rel})^{\mathrm{rel}} \leftrightarrows \mathbf{LRSgpd}^{\mathrm{rel}}$. Let (X, m_X) and (Y, m_Y) be relative H*-algebras, inducing locally cancellative regular semigroupoids **G** and **H**. Given $r\colon m_X \to m_Y$, define $m_r\colon r \times r \twoheadrightarrow r$ as in the proof of Theorem 16; it satisfies (A) and (M). It also satisfies (H), as we now verify. For $A \subseteq r$, take $A^* = \{(x^*, y^*) \mid (x, y) \in A, x^* \in \{x\}^*, y^* \in \{y\}^*\}$.

$$\begin{aligned} &(1 \times A) \circ m_r^\dagger \\ &= \{((x, y), (a, b)) \in r \times r \mid \exists (c, d) \in A \,.\, y = bd, x = ac\} \\ &\overset{(\mathrm{H})}{=} \{((x, y), (a, b)) \in r \times r \mid \exists (c, d) \in A, c^* \in \{c\}^*, d^* \in \{d\}^* \,.\, a = xc^*, b = yd^*\} \\ &= m_r \circ (1 \times A^*). \end{aligned}$$

Theorem 29 now shows that m_r induces a subsemigroupoid of $\mathbf{G} \times \mathbf{H}$. Conversely, if $\mathbf{R}$ is a subsemigroupoid of $\mathbf{G} \times \mathbf{H}$, then $R_1 \colon G_1 \twoheadrightarrow H_1$ clearly satisfies (R) and (I'). Finally, the identity relation $r \colon m_X \twoheadrightarrow m_Y$ corresponds to the diagonal subsemigroupoid, which is indeed regular and locally cancellative. These constructions clearly restrict to the subcategories of the statement. $\square$

Theorem 35. *The functors from the previous proposition form adjunctions.*

$$\begin{tikzcd} \mathbf{LRSgpd}^{\mathrm{rel}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Hstar}(\mathbf{Rel})^{\mathrm{rel}} \arrow[l, shift left] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{LRSgpd}^{\mathrm{mfunc}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Hstar}(\mathbf{Rel}) \arrow[l, shift left] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{LRSgpd} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Hstar}(\mathbf{Rel})^{\mathrm{func}} \arrow[l, shift left] \end{tikzcd}$$

Proof. Starting with a relative H*-algebra $m \colon X \times X \twoheadrightarrow X$, we end up with

$$\{(g, f, gf) \mid \exists g^* \in \{g\}^* \exists f^* \in \{f\}^* \,.\, g^*g = ff^*\} \colon X \times X \twoheadrightarrow X.$$

This is a subrelation of m, and the inclusion forms the adjunction's unit.

Starting with a locally cancellative regular semigroupoid $\mathbf{G}$, we end with

$$\begin{tikzcd} \{f \in G_1 \mid f^2 = f\} & G_1 \arrow[l, shift right, "s'"'] \arrow[l, shift left, "t'"] & G_1 \times_{s',t'} G_1 \arrow[l, "m"'] \end{tikzcd}$$

where $s'(f) = f^*f$ and $t'(f) = ff^*$. Clearly, the original $\mathbf{G}$ maps into this, giving the counit of the adjunction. Naturality and the triangle equations are easily checked for $\mathbf{LRSgpd}^{\mathrm{rel}} \leftrightarrows \mathbf{Hstar}(\mathbf{Rel})^{\mathrm{rel}}$. Because the unit and counit are functions, the statement also holds for the other subcategories. $\square$

Proposition 36. *The largest subcategories making the adjunctions of Theorem 35 into equivalences are* $\mathbf{Gpd}^{\mathrm{rel}}$ *and* $\mathbf{Frob}(\mathbf{Rel})^{\mathrm{rel}}$*, and their variations. In that case, the equivalences are in fact isomorphisms.*

Proof. Consider the counit of the proof of Theorem 35. It is an isomorphism precisely when G_0 coincides with the idempotents of G_1. But then the unique idempotent on $x \in G_0$ is an identity, and $\mathbf{G}$ is a groupoid by Lemma 25. In other words, $\mathbf{Gpd}$ and its variations are the largest subcategories of $\mathbf{LRSgpd}$ and its variations turning the adjunctions into reflections.

Next consider the unit of the adjunctions. It is an isomorphism when $gf\downarrow$ implies $g^*g = ff^*$ for some pseudoinverses $f^* \in \{f\}^*$ and $g^* \in \{g\}^*$. In that case we can define a unit for the H*-algebra (X, m) by $\{u \in X \mid u = u^*u\}$, for if $uf\downarrow$ and $u = u^*u$ then $uf = u^*uf = ff^*f = f$. But recall that unital relative H*-algebras are relative Frobenius algebras. In other words, $\mathbf{Frob}(\mathbf{Rel})$ and its variations are the largest subcategories of $\mathbf{Hstar}(\mathbf{Rel})$ and its variations turning the above adjunctions into reflections. $\square$

4. Groupoids and semigroupoids

The forgetful functor $\mathbf{Gpd} \to \mathbf{Cat}$ has a left adjoint, that freely adds inverses. Similarly, the forgetful functor $\mathbf{Cat} \to \mathbf{Sgpd}$ to the category of semigroupoids and semifunctors has a left adjoint, that freely adds identities. The image of the latter left adjoint consists precisely of those categories in which the only isomorphisms are identities. Hence there is a functor $\mathbf{Sgpd} \to \mathbf{Gpd}$ giving the free groupoid on a semigroupoid. Restricting it gives a functor that turns a locally cancellative regular semigroupoid into a groupoid.

$$\begin{tikzcd} \mathbf{LRSgpd} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Gpd} \arrow[l, hook', shift left] \end{tikzcd}$$

The morphisms in these categories are (semi)functors. This section establishes right adjoints to the inclusion $\mathbf{Gpd}^{\mathrm{rel}} \hookrightarrow \mathbf{LRSgpd}^{\mathrm{rel}}$ and its variations with other choices of morphisms. This is then applied to obtain adjunctions between $\mathbf{HStar}(\mathbf{Rel})^{\mathrm{rel}}$ and $\mathbf{Frob}(\mathbf{Rel})^{\mathrm{rel}}$ (and their variations). The idea in building the right adjoint is to identify all idempotents: a group is a regular semigroup with a single idempotent.

Definition 37. For a semigroupoid $\mathbf{G}$, define $\sim$ as the congruence (see [5, Section II.8]) generated by $f \sim g$ when $s(f) = s(g)$ and $f^2 = f$ and $g^2 = g$. Set

$$\begin{aligned} G'_0 &= G_0, & s'([f]) &= s(f), & m'([g],[f]) &= [m(g,f)], \\ G'_1 &= G_1 / {\sim}, & t'([f]) &= t(f). \end{aligned}$$

Lemma 38. *If* $\mathbf{G}$ *is a (locally cancellative regular) semigroupoid, then*

$$\mathbf{G}' = \left(\begin{tikzcd} G'_0 & G'_1 \arrow[l, shift right, "s'"'] \arrow[l, shift left, "t'"] & G'_1 \times_{G'_0} G'_1 \arrow[l, "m'"'] \end{tikzcd} \right)$$

is again a well-defined (locally cancellative regular) semigroupoid.

Proof. Because $\sim$ is a congruence, m' is associative [5, Proposition II.8.1]). Because s and t are jointly epic, so are s' and t'. Hence $\mathbf{G}'$ is a semigroupoid. If $\mathbf{G}$ is regular, then $[f^*]$ is a pseudoinverse for $[f] \in G'_1$, where f^* is any pseudoinverse of f in G_1, and so $\mathbf{G}'$ is regular. Finally, it is easy to see that $\mathbf{G}'$ inherits local cancellativity from $\mathbf{G}$ using that $\sim$ is a congruence. $\square$

Lemma 39. *If the semigroup* $\mathbf{G}$ *is locally cancellative and regular, then*

$$F(\mathbf{G}) = \left(\begin{tikzcd} G_0' \arrow[r, shift right=2, leftarrow, "t'"'] \arrow[r, "e'"] \arrow[r, shift left=2, leftarrow, "s'"] & G_1' \arrow[loop above, "i'"] & G_1' \times_{G_0'} G_1' \arrow[l, "m'"'] \end{tikzcd} \right)$$

is a well-defined groupoid, where

$$e'(s(f)) = [f^*f], \qquad e'(t(f)) = [ff^*], \qquad i'([f]) = [f^*].$$

Proof. Because $\mathbf{G}$ is assumed regular, it makes sense to speak about f^*. Because $G_0' = \mathrm{Im}(s') \cup \mathrm{Im}(t')$, the above prescription completely defines e'. Finally, e' is well-defined, for if $s(f) = s(g)$, then $f^*f \sim g^*g$ because both are idempotent. Similarly, if $s(f) = t(g)$, then $f^*f \sim gg^*$.

We show that i is a well-defined function. If g and h are pseudo-inverses of f, then gf and hf are idempotent. Also $s(gf) = s(hf) = t(gf) = t(hf)$, so $[gf] = [hf]$. But then $[g] = [h]$ by local cancellativity of $\mathbf{G}'$.

By construction, these data makes $F(\mathbf{G})$ into a groupoid. $\square$

Proposition 40. *The assignment* $\mathbf{G} \mapsto F(\mathbf{G})$ *of the previous lemma extends to functors* $F^{\mathrm{rel}} \colon \mathbf{LRSgpd}^{\mathrm{rel}} \to \mathbf{Gpd}^{\mathrm{rel}}$*,* $F^{\mathrm{mfunc}} \colon \mathbf{LRSgpd}^{\mathrm{mfunc}} \to \mathbf{Gpd}^{\mathrm{mfunc}}$*, and* $F \colon \mathbf{LRSgpd} \to \mathbf{Gpd}$*.*

Proof. Let $\mathbf{R}$ be a morphism $\mathbf{G} \to \mathbf{H}$ in $\mathbf{LRSgpd}^{\mathrm{rel}}$. Then it is subsemigroupoid of $\mathbf{G} \times \mathbf{H}$, and hence a semigroupoid in its own right. Hence we can define $F^{\mathrm{rel}}(\mathbf{R})$ as in the previous lemma. It is easy to see that $F^{\mathrm{rel}}(\mathbf{R})$ is a subsemigroupoid of $F^{\mathrm{rel}}(\mathbf{G}) \times F^{\mathrm{rel}}(\mathbf{H})$. Finally, it clear that F^{rel} preserves identities and composition, and restricts to give functors F^{mfunc} and F. $\square$

Theorem 41. *The inclusion* $\mathbf{Gpd} \hookrightarrow \mathbf{LRSgpd}$ *has* F *as a right adjoint.*
The inclusion $\mathbf{Gpd}^{\mathrm{mfunc}} \hookrightarrow \mathbf{LRSgpd}^{\mathrm{mfunc}}$ *has* F^{mfunc} *as a right adjoint.*
The inclusion $\mathbf{Gpd}^{\mathrm{rel}} \hookrightarrow \mathbf{LRSgpd}^{\mathrm{rel}}$ *has* F^{rel} *as a right adjoint..*

Proof. We start by exhibiting the unit of the adjunctions. Let $\mathbf{G}$ be a locally cancellative regular semigroupoid. Then $G_0 = F(\mathbf{G})_0$, and there is a projection function $G_1 \twoheadrightarrow (G_1 / \sim) = F(\mathbf{G})_1$. By construction of s', t' and m', this induces a semifunctor $\mathbf{G} \to F(\mathbf{G})$, and hence a subsemigroupoid of $\mathbf{G} \times F(\mathbf{G})$. Because $\mathbf{G}$ is locally cancellative and regular itself, this subsemigroupoid is a well-defined morphism $\mathbf{G} \to F(\mathbf{G})$ in $\mathbf{LRSgpd}^{\mathrm{rel}}$ as well as in $\mathbf{LRSgpd}^{\mathrm{mfunc}}$ and $\mathbf{LRSgpd}$. It is easy to see that this is natural in $\mathbf{G}$.

As for the counit, notice that if $\mathbf{G}$ is a groupoid, then $G_1 \cong (G_1 / \sim)$. So the subsemigroupoid of $\mathbf{G} \times F(\mathbf{G})$ is in fact a groupoid, and hence gives a well-defined morphism $F(\mathbf{G}) \to \mathbf{G}$ in $\mathbf{Gpd}^{\mathrm{rel}}$, $\mathbf{Gpd}^{\mathrm{mfunc}}$ and $\mathbf{Gpd}$, that is natural in $\mathbf{G}$. One readily verifies that this unit and counit satisfy the triangle equations. □

Thus the functor F provides a universal way to pass from locally regular semigroupoids to groupoids. Restriction to the one-object case shows that collapsing all idempotents turns a locally cancellative regular semigroup into a group.

Corollary 42. *There are adjunctions*

$$\begin{tikzcd}
\mathbf{Hstar}(\mathbf{Rel})^{\mathrm{func}} \arrow[r, hook] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] & \mathbf{Hstar}(\mathbf{Rel}) \arrow[r, hook] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] & \mathbf{Hstar}(\mathbf{Rel})^{\mathrm{rel}} \arrow[d, shift left] \arrow[d, phantom, "\dashv"] \\
\mathbf{Frob}(\mathbf{Rel})^{\mathrm{func}} \arrow[r, hook] \arrow[u, shift left] & \mathbf{Frob}(\mathbf{Rel}) \arrow[r, hook] \arrow[u, shift left] & \mathbf{Frob}(\mathbf{Rel})^{\mathrm{rel}} \arrow[u, shift left]
\end{tikzcd}$$

Explicitly, the right adjoints map a relative H-algebra (X, m_X) to $(X / \sim, m')$, where $\sim$ is the equivalence relation generated by $f \sim g$ if $f^2 = f$ and $g^2 = g$ and $gf \downarrow$, and $m'([g],[f]) = [m(g,f)]$.*

Proof. Simply compose the following adjunctions, and similarly for the other choices of morphisms.

$$\begin{tikzcd}
\mathbf{Frob}(\mathbf{Rel})^{\mathrm{rel}} \arrow[r, shift left] \arrow[r, phantom, "\cong"] & \mathbf{Gpd}^{\mathrm{rel}} \arrow[l, shift left] \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathbf{LRSgpd}^{\mathrm{rel}} \arrow[l, shift left, "F^{\mathrm{rel}}"] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Hstar}(\mathbf{Rel})^{\mathrm{rel}} \arrow[l, shift left]
\end{tikzcd}$$

Applying Definitions 26, 37, and 8 in order to a given relative H*-algebra results in the relative Frobenius algebra of the statement. □

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