COMPUTABLE ESTIMATES OF THE MODELING ERROR RELATED TO KIRCHHOFF-LOVE PLATE MODEL

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ABSTRACT. The Kirchhoff-Love plate model is a widely used in the analysis of thin elastic plates. It is well known that Kirchhoff-Love solutions can be viewed as certain limits of displacements and stresses for elastic plates where the thickness tends to zero. In this paper, we consider the problem from a different point of view and derive computable upper bounds of the difference between the exact three-dimensional solution and a solution computed by using the Kirchhoff-Love hypotheses. This estimate is valid for any value of the thickness parameter. In combination with a posteriori error estimates for approximation errors, this estimate allows the direct measurement of both, approximation and modeling errors, encompassed in a numerical solution of the Kirchhoff-Love model. We prove that the upper bound possess necessary asymptotic properties and, therefore, does not deteriorate as the thickness tends to zero.

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1. Introduction

In many practically important cases, an approximation of the solution $u^{(d)}$ of a d-dimensional problem (we call it Problem $\mathcal{P}^{(d)}$ and assume that u^d belongs to a Banach space V) is found by solving some simplified problem $\mathcal{P}^{(d-k)}$ (where k is a positive integer number). A solution of this problem we denote by $u^{(d-k)}$. An approximation of $u^{(d-k)}$ is usually obtained by projecting $\mathcal{P}^{(d-k)}$ onto a finite dimensional space and solving the corresponding discrete problem $\mathcal{P}_\tau^{(d-k)}$, where τ is a small parameter related to the respective mesh $\mathcal{T}_\tau$. Thus, instead of $u^{(d)}$ we compute $u_\tau^{(d-k)}$ (see Fig. 1).

The general purpose of $u_\tau^{(d-k)}$ is to present a reliable information on $u^{(d)}$. It should be outlined that the functions $u^{(d)}$, $u^{(d-k)}$ and $u_\tau^{(d-k)}$ belong to different spaces, so that to compare these functions we need a *dimension reconstruction* operator $R : V^{(d-k)} \to V$ that forms d-dimensional images of $(d-k)$-dimensional solutions. One can construct such an operator by different methods, but obviously it must satisfy two conditions: computational simplicity and boundedness. Additionally, we assume that R satisfies the

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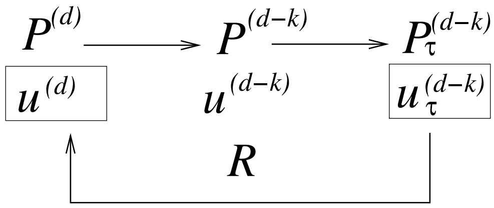

FIGURE 1. Dimension reduction and reconstruction

Lipshitz condition

$$\|Rw_1 - Rw_2\|_V \le \mathrm{C}_R \, \|w_1 - w_2\|_{V^{(d-k)}} \qquad \forall w_1, w_2 \in V^{(d-k)}, \tag{1.1}$$

where $\mathrm{C}_R > 0$ does not depend on w_1 and w_2.

Usually, the modeling error $\mathcal{E}_{mod} := \left\|u^{(d)} - Ru^{(d-k)}\right\|_V$ is assumed to be much smaller than the approximation error $\mathcal{E}_{app} := \|u^{(d-k)} - u_\tau^{(d-k)}\|_{V^{(d-k)}}$. However, in reliable computations this assumption must be verified.

In practice, we are interested to know how large is the difference between $u^{(d)}$ and the function $Ru_\tau^{(d-k)}$. By (1.1), we find that

$$\begin{aligned} \|u^d - Ru_\tau^{(d-k)}\|_V &\le \|u^d - Ru^{d-k}\|_V + \|Ru^{d-k} - Ru_\tau^{d-k}\|_V \\ &\le \|u^d - Ru^{d-k}\|_V + \mathrm{C}_R \, \|u^{d-k} - u_\tau^{(d-k)}\|_{V^{(d-k)}}. \end{aligned} \tag{1.2}$$

Hence

$$\|u^d - Ru_\tau^{(d-k)}\|_V \le \mathcal{E}_{mod} + \mathrm{C}_R \, \mathcal{E}_{app}. \tag{1.3}$$

This additive splitting into a modelling error term and an error term related to the numerical discretization gives insights how these two parts of the overall error are balanced.

Historically, the subject of error estimation in dimension reduction models was mainly focused on a priori asymptotic error estimates that evaluate the difference between original and reduced models in terms of small (geometric) parameters. In this context, models in the elasticity theory have been studied by different authors (see e.g., [1, 5, 6, 19] and the references therein). Among these models, the Kirchhoff-Love (KL) plate model (originally based on the heuristic "direct normal" hypothesis [8]) is one of the most known. KL solutions can be viewed as certain limits of 3D solutions of elastic plate-type bodies if the thickness parameter h tends to zero.

In this paper, we present computable estimates of $\mathcal{E}_{mod}$ associated with the KL plate model. The paper is organized as follows. In Sect.2, we outline some basic facts related to KL model of thin plates. In Sect. 3 we discuss estimates of the deviation from exact solutions for linear elasticity problems. Theorem 3.3 presents a new estimate, which provides guaranteed and fully computable upper bounds of modeling errors adapted to thin (plate type) elastic bodies. We emphasize that the estimate presented in Theorem 3.3 does not involve 3D constants (as the constant in the Korn's inequaliy associated with mixed boundary conditions) and instead contains only 2D constant in the Friedrichs inequality (on the middle surface), which is easy to estimate from above. This estimate is used in Sect. 4, where we consider the case of isotropic elasticity. We present error estimates for solutions obtained with the help of KL model (Theorems 4.1 and 5.1) and show that they are asymptotically equivalent to the modeling error.

2. Statement of the problem

We consider a bounded three-dimensional elastic body occupying the domain

$$\Omega := \left\{ (x_1, x_2) \in \omega, \quad x_3 \in \left] -\frac{h}{2}, \frac{h}{2} \right[\right\},$$

where ω is a bounded open domain in the x_1, x_2–plane with Lipschitz continuous boundary $\partial\omega$ and h is a positive constant (see Fig. 2). We assume that h is small with respect to the size of ω. This requirement can be formalized as follows: there exists a point $O \in \omega$ such that

$$h \leq \inf_{O' \in \partial\omega} |O'O|.$$

We define the *middle surface* by

$$S_0 := \{ x \in \Omega \mid x_3 = 0 \}.$$

Let $\mathbf{x} := (x_1, x_2, x_3)$ denotes the 3D coordinate vector and $\widehat{\mathbf{x}} := (x_1, x_2)$ stands for the vector associated with the plane part.

We define the *lower* and *upper* faces of the plate as follows:

$$S_\ominus := \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \left(\widehat{\mathbf{x}}, -\frac{h}{2} \right), \quad \widehat{\mathbf{x}} \in \omega \right\},$$
$$S_\oplus := \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \left(\widehat{\mathbf{x}}, +\frac{h}{2} \right), \quad \widehat{\mathbf{x}} \in \omega \right\},$$

and the lateral surface

$$\Gamma := \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = (\widehat{\mathbf{x}}, x_3), \ \widehat{\mathbf{x}} \in \partial\omega, \quad x_3 \in \left(-\frac{h}{2}, \frac{h}{2} \right) \right\}.$$

Henceforth, $S := S_\ominus \cup S_\oplus$, $\| \,.\, \|_\Omega$ and $\| \,.\, \|_S$ denote L^2-norms of a function (vector-function) associated with Ω and S, respectively.

On Γ, we impose the Dirichlet boundary condition

$$\mathbf{u}(\mathbf{x}) = 0 \tag{2.1}$$

for the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ (here and later, on vectors and tensors are denoted by bold letters). It is worth noting, that we consider homogeneous Dirichlet boundary conditions only for the sake of simplicity. Our analysis is applicable to nonhomogeneous Dirichlet and mixed boundary conditions.

On the upper and lower faces, we impose the Neumann type boundary conditions

$$\boldsymbol{\sigma} \cdot \boldsymbol{n}_\ominus = 0 \quad \text{on } S_\ominus \qquad \text{and} \qquad \boldsymbol{\sigma} \cdot \boldsymbol{n}_\oplus = \mathbf{F} \quad \text{on } S_\oplus, \tag{2.2}$$

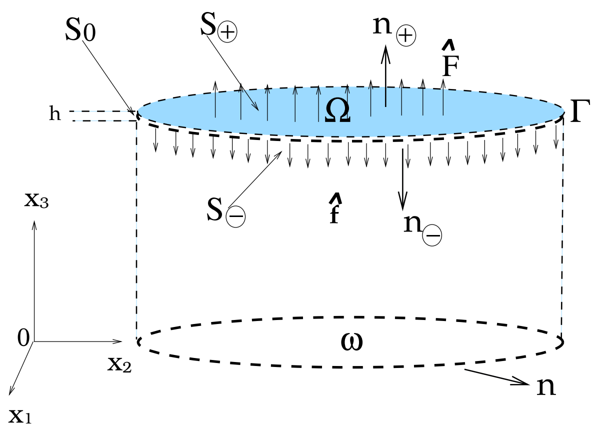

FIGURE 2. Plate type elastic body.

where $\mathbf{F} = (0, 0, \widehat{F})$. By $\boldsymbol{n}_{\oplus}(\widehat{\mathbf{x}})$ and $\boldsymbol{n}_{\ominus}(\widehat{\mathbf{x}})$ we denote the unit normal vectors. The body is subject to the action of a volume and surface loads $\mathbf{f} = \left(0, 0, \widehat{f}\,(\widehat{\mathbf{x}})\right)$ and $\widehat{F}$, respectively. We assume that $\widehat{f}$ and $\widehat{F}$ belong to $L^2(\omega)$. The exact solution of the 3D elasticity problem in question is presented by the displacement vector $\mathbf{u}$ and the stress tensor $\boldsymbol{\sigma}\,(\mathbf{x}) = (\boldsymbol{\sigma}_{ij}\,(\mathbf{x}))_{i,j=1}^{3}$ that satisfy the *equilibrium equation*

$$\mathrm{Div}\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \qquad \text{in } \Omega, \tag{2.3}$$

where the strain and stress fields are coupled linearly via Hooke's law

$$\boldsymbol{\sigma} = \mathbb{L}\boldsymbol{\varepsilon}\,(\mathbf{u}) \qquad \text{in } \Omega \tag{2.4}$$

with the tensor $\mathbb{L} = (L_{ijkl})$ of elastic constants. The strain tensor $\boldsymbol{\varepsilon}\,(\mathbf{u})$ is given by the relation (within the framework of small strains theory)

$$\boldsymbol{\varepsilon}\,(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\intercal}\right). \tag{2.5}$$

In the important case of isotropic media

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\,(\mathrm{tr}\boldsymbol{\varepsilon})\,\mathbb{I} = \frac{E}{1+\nu}\boldsymbol{\varepsilon} + \frac{\nu E}{(1+\nu)(1-2\nu)}\,(\mathrm{tr}\boldsymbol{\varepsilon})\,\mathbb{I}, \tag{2.6}$$

and

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\,(\mathrm{tr}\boldsymbol{\sigma})\,\mathbb{I}, \tag{2.7}$$

where λ and μ are the Lamé constants, E and ν are the Young modulus and the Poisson coefficient respectively, $\mathrm{tr}\,(\cdot)$ stands for the first invariant of a tensor and $\mathbb{I}$ is the unit tensor. In general, $\mathbb{L}$ must be subject to the conditions

$$c_1^2|\varepsilon|^2 \leq \mathbb{L}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \leq c_2^2|\varepsilon|^2, \qquad \forall\boldsymbol{\varepsilon} \in \mathbb{M}^{3\times3}_{sym}, \tag{2.8}$$

where $\mathbb{M}^{3\times 3}_{sym}$ is the space of symmetric real valued 3×3 tensors with square summable coefficients.

Also, we assume that the coefficients of the elasticity tensor are bounded and possess natural symmetry, i.e.,

$$\mathbb{L}_{ijkm} = \mathbb{L}_{jikm} = \mathbb{L}_{kmij}, \; i, j, k, m = 1, \ldots, d, \tag{2.9}$$

$$\mathbb{L}_{ijkm} \in L^\infty(\Omega). \tag{2.10}$$

A function $\mathbf{u} \in \mathbf{V}_0 := \{\mathbf{w} \in H^1(\Omega, \mathbb{R}^3) \mid \mathbf{w} = 0 \quad \text{on } \Gamma\}$ is a generalized solution of (2.1)–(2.4) if it satisfies the variational relation

$$\int\limits_\Omega \mathbb{L}\, \varepsilon\left(\mathbf{u}\right) : \varepsilon\left(\mathbf{w}\right)\, d\mathbf{x} = \int\limits_\Omega \widehat{f} w_3\left(\mathbf{x}\right)\, d\mathbf{x} + \int\limits_\omega \widehat{F} w_3\left(\widehat{\mathbf{x}}, \frac{h}{2}\right) d\widehat{\mathbf{x}} \tag{2.11}$$

for all $\mathbf{w} \in \mathbf{V}_0$. The corresponding stress tensor $\boldsymbol{\sigma} = \mathbb{L}\varepsilon(\mathbf{u})$ belongs to the space

$$\Sigma := L^2\left(\Omega, \mathbb{M}^{3\times 3}_{sym}\right).$$

The existence and uniqueness of $\mathbf{u}$ follow from the Korn's inequality which allows us to establish the coercivity of the energy norm

$$\|\!| \varepsilon(\mathbf{w}) \|\!|^2 := \int\limits_\Omega \mathbb{L}\varepsilon(\mathbf{w}) : \varepsilon(\mathbf{w})\, d\mathbf{x}$$

on the space $\mathbf{V_0}$. By $\|\!|\ .\ \|\!|_*$ we denote the energy norm associated with $\mathbb{L}^{-1}$, i.e.,

$$\|\!| \boldsymbol{\tau} \|\!|^2_* := \int\limits_\Omega \mathbb{L}^{-1}\boldsymbol{\tau} : \boldsymbol{\tau}\, d\mathbf{x}.$$

In the classical theory of Kirchhoff-Love plates, the above-described 3D model is replaced by a simplified one, in which displacements and stresses are found in accordance with the Kirchhoff-Love hypothesis. The first hypothesis states that the unit normal to the middle surface remains unstretched during the deformation of the plate. It means that the displacement vector is sought in the form

$$u_1\left(\mathbf{x}\right) = -x_3\widehat{w}_{,1}; \quad u_2\left(\mathbf{x}\right) = -x_3\widehat{w}_{,2}; \quad u_3\left(\mathbf{x}\right) = \widehat{w}\left(\widehat{\mathbf{x}}\right), \tag{2.12}$$

where $\widehat{w}$ is a scalar-valued function that represents deflections of S_0.

Another (static) hypothesis is that the components σ_{i3}, $i = 1, 2, 3$, are negligibly small compared to σ_{11}, σ_{12}, and σ_{22} so that they are set to zero. Thus, only the plane part of the stress tensor is considered. For the case of isotropic media, it is defined by the relations

$$\sigma_{11} = -\frac{Ex_3}{1-\nu^2}\left(\widehat{w}_{,11} + \nu\widehat{w}_{,22}\right) = -\frac{2\mu x_3}{1-\nu}\left(\widehat{w}_{,11} + \nu\widehat{w}_{,22}\right), \tag{2.13}$$

$$\sigma_{22} = -\frac{Ex_3}{1-\nu^2}\left(\nu\widehat{w}_{,11} + \widehat{w}_{,22}\right) = -\frac{2\mu x_3}{1-\nu}\left(\nu\widehat{w}_{,11} + \widehat{w}_{,22}\right), \tag{2.14}$$

$$\sigma_{12} = -\frac{Ex_3}{1+\nu}\widehat{w}_{,12} = -2\mu x_3\widehat{w}_{,12}. \tag{2.15}$$

In order to deduce the equation for $\widehat{w}$, we use (2.11) and accordingly define the test functions $\mathbf{w}$ in the form $\mathbf{w} = \left(-x_3\widehat{\varphi}_{,1}; -x_3\widehat{\varphi}_{,2}; \widehat{\varphi}\,(\widehat{\mathbf{x}})\right)$, where $\widehat{\varphi} \in H^2(\omega)$ is an arbitrary function vanishing on $\partial\omega$ together with its first derivatives. In view of the static hypothesis, the left-hand side of (2.11) contains only plane components and can be rewritten as follows:

$$\int_\Omega \sigma_{11}(\mathbf{u})\varepsilon_{11}(\mathbf{v})d\mathbf{x} =$$

$$= \int_\Omega \frac{Ex_3^2}{1-\nu^2}(\widehat{w}_{,11} + \nu\widehat{w}_{,22})\widehat{\varphi}_{,11}d\widehat{\mathbf{x}} = \int_\omega D(\widehat{w}_{,11} + \nu\widehat{w}_{,22})\widehat{\varphi}_{,11}d\widehat{\mathbf{x}},$$

where $D := \frac{Eh^3}{12(1-\nu^2)} = \frac{\mu h^3}{6(1-\nu)}$. Analogously,

$$\int_\Omega \sigma_{22}(\mathbf{u})\varepsilon_{22}(\mathbf{v})d\mathbf{x} = \int_\omega D(\nu\widehat{w}_{,11} + \nu\widehat{w}_{,22})\widehat{\varphi}_{,22}d\widehat{\mathbf{x}}$$

and

$$2\int_\Omega \sigma_{12}(\mathbf{u})\varepsilon_{12}(\mathbf{v})d\mathbf{x} = 2(1-\nu)\int_\omega D\widehat{w}_{,12}\widehat{\varphi}_{,12}d\widehat{\mathbf{x}}.$$

Hence, we arrive at the following problem.
Find $\widehat{w} \in V_{00}(\omega) := \left\{\widehat{\eta} \in H^2(\omega) \mid \widehat{\eta} = \widehat{\eta}_{,\boldsymbol{n}} = 0 \text{ on } \partial\omega\right\}$ such that

$$\int_\omega D\Big((\widehat{w}_{,11} + \nu\widehat{w}_{,22})\widehat{\varphi}_{,11} + (\nu\widehat{w}_{,11} + \nu\widehat{w}_{,22})\widehat{\varphi}_{,22} + 2(1-\nu)\widehat{w}_{,12}\widehat{\varphi}_{,12}\Big)d\widehat{\mathbf{x}}$$

$$= \int_\omega \widehat{g}\widehat{\varphi}\,d\widehat{\mathbf{x}} \qquad \forall\widehat{\varphi} \in V_{00}(\omega), \tag{2.16}$$

where $\widehat{g}(\widehat{\mathbf{x}}) = h\widehat{f} + \widehat{F}$.

If $\widehat{w}$ is sufficiently regular, then (2.16) implies the classical plate equation (see, e.g.,[7])

$$\widehat{w}_{,1111} + 2\widehat{w}_{,1122} + \widehat{w}_{,2222} = \frac{\widehat{g}}{D},$$

a weak form of which is

$$\int\limits_\omega D\widehat{\Delta}\ \widehat{w}\ \widehat{\Delta}\ \widehat{\varphi}\ d\widehat{\mathbf{x}} = \int\limits_\omega \widehat{f}\ \widehat{\varphi}\ d\widehat{\mathbf{x}} \qquad \widehat{\varphi} \in V_{00}\,(\omega)\,.$$

This simplified 2D model is often used for numerical analysis of plate-type elastic bodies (see, e.g., [2]).

Finally, we note that in general $\widehat{w}$ depends on h (so that it would be right to denote it by $\widehat{w}^h$). To exclude this dependence, we scale the external forces and set $\widehat{f} = h^2\widehat{f}_0$ and $\widehat{F} = h^3\widehat{F}_0$. In this case,

$$\widehat{g}_0 := \frac{\widehat{g}}{D} = \frac{6(1-\nu)}{\mu}\left(\widehat{f}_0 + \widehat{F}_0\right)$$

so that $\widehat{w}$ satisfies the equation

$$\widehat{w}_{,1111} + 2\widehat{w}_{,1122} + \widehat{w}_{,2222} = \widehat{g}_0 \tag{2.17}$$

and does not depend on h.

3. Errors of simplified models in linear elasticity

Let $\mathbf{v} \in \mathbf{V}_0$ denote an approximation of the exact solution $\mathbf{u}$ of (2.11) obtained by some suitable reconstruction of a plate model. In this section, we present different estimates of the modeling error generated by $\mathbf{v}$. For the sake of convenience, we hereafter use an additional notation, namely

$$r_i(\boldsymbol{\tau}) := \operatorname{div}\{\tau_{ij}\}_{j=1}^3 = \tau_{i1,1} + \tau_{i2,2} + \tau_{i3,3}, \quad i = 1, 2, 3.$$

3.1. **Prager-Synge estimate.**

Prager-Synge estimate [11] is the first mathematical tool that can be used to derive computable estimates of modeling errors generated by dimension reduction models in linear elasticity. In the context of our problem, it yields the following result

Theorem 3.1. *For any* $\mathbf{v} \in V_0$,

$$\| \varepsilon(\mathbf{u} - \mathbf{v}) \| + \| \boldsymbol{\sigma} - \boldsymbol{\tau} \|_* \leq M_1(\mathbf{v}, \boldsymbol{\tau}) := \| \boldsymbol{\tau} - \mathbb{L}\varepsilon(\mathbf{v}) \|_*, \tag{3.1}$$

where $\boldsymbol{\tau}$ *is an arbitrary tensor valued function in the set*

$$Q_f := \Big\{ \boldsymbol{\tau} \in \Sigma \mid r_1(\boldsymbol{\tau}) = r_2(\boldsymbol{\tau}) = 0, \quad r_3(\boldsymbol{\tau}) + h^2 \widehat{f}_0 = 0,$$
$$\tau_{13} = \tau_{23} = 0 \text{ on } S, \ \tau_{33} = h^3 \widehat{F}_0 \text{ on } S_\oplus, \ \tau_{33} = 0, \text{ on } S_\ominus \Big\}.$$

It is easy to see that for $\boldsymbol{\tau} = \boldsymbol{\sigma}$ the upper bound coincides with the true error. To obtain a sharp upper bound of the error we need $\boldsymbol{\tau} \in Q_f$ to be a good approximation of $\boldsymbol{\sigma}$. On the other hand, the functions in the set Q_f of admissible stresses must exactly satisfy the three differential equations appearing in the definition of this set. This is numerically a complicated task. In the following, we will modify the majorant so that the auxiliary stress $\boldsymbol{\tau}$ can be chosen from an essentially larger space.

3.2. **Functional type a posteriori estimate for linear elasticity problem.**

Define the sets

$$\Sigma_{\mathrm{Div}} := \Big\{ \boldsymbol{\tau} \in \Sigma \mid \operatorname{Div}\boldsymbol{\tau} \in L^2(\Omega, \mathbb{R}^3) \Big\}$$

and

$$\Sigma_+ := \Big\{ \boldsymbol{\tau} \in \Sigma \mid \boldsymbol{\tau} \cdot \boldsymbol{n}_\ominus \in L^2(S_\ominus, \mathbb{R}^3), \quad \boldsymbol{\tau} \cdot \boldsymbol{n}_\oplus \in L^2(S_\oplus, \mathbb{R}^3) \Big\}.$$

Theorem 3.2. *For any* $\boldsymbol{\tau} \in \Sigma_{\mathrm{Div}} \cap \Sigma_+$, *the following estimate holds:*

$$\begin{aligned} | \ \| \varepsilon(\mathbf{u} - \mathbf{v}) \| \leq M_2(\mathbf{v}, \boldsymbol{\tau}) &:= \| \boldsymbol{\tau} - \mathbb{L}\varepsilon(\mathbf{v}) \|_* + \\ &+ C_\Omega \Big(\|r_1(\boldsymbol{\tau})\|_\Omega^2 + \|r_2(\boldsymbol{\tau})\|_\Omega + \|r_3(\boldsymbol{\tau}) + h^2 \widehat{f}_0\|_\Omega^2 + \\ &+ \sum_{s,t=1}^{2} \left(\left\| \tau_{st}\left(\widehat{\mathbf{x}}, \frac{h}{2}\right) \right\|_\omega^2 + \left\| \tau_{st}\left(\widehat{\mathbf{x}}, -\frac{h}{2}\right) \right\|_\omega^2 \right) + \\ &+ \|\tau_{33}(\widehat{\mathbf{x}}, -\frac{h}{2})\|_\omega^2 + \|\tau_{33}(\widehat{\mathbf{x}}, \frac{h}{2}) - \widehat{F}\|_\omega \Big)^{1/2}, \end{aligned} \tag{3.2}$$

where

$$C_\Omega := \sup_{\mathbf{v}\in\mathbf{V_0(\Omega)}} \frac{\sqrt{\|\mathbf{v}\|^2_\Omega + \|\mathbf{v}\|^2_{S_\ominus} + \|\mathbf{v}\|^2_{S_\oplus}}}{\| \,\boldsymbol{\varepsilon}(\mathbf{v})\, \|_\Omega}.$$

This estimate is a particular form of the general a posteriori estimate for linear elasticity problem (see [12, 13, 14]). Obviously, it is valid for a much wider set of stress tensors $\boldsymbol{\tau}$.

It is easy to show that an upper bound of C_Ω can be expressed throughout the Friedrichs' constant $C_{F\omega}$ for ω and the constant $C_{K\Omega}$ in the Korn's inequality for Ω. However, the computation (or sharp estimates) of the constant $C_{K\Omega}$ for 3D elasticity problem with mixed boundary conditions is far from trivial, especially for complicated domains. The estimate derived below presents a certain compromise between M_1 and M_2. It depends only on $C_{F\omega}$ (instead of constants associated with three-dimensional problems) and reduces the number of terms in the majorant by imposing suitable conditions on τ. In our subsequent analysis, we show that the latter conditions can indeed be satisfied.

3.3. **Error bound for plate type elastic bodies.** Now, our goal is to derive estimates that do not contain $C_{K\Omega}$ and are valid for a much wider set of stresses than the set Q_f.

Theorem 3.3. *Assume that*

$$r_3(\boldsymbol{\tau}) + h^2 \widehat{f}_0 = 0 \quad \text{a.e. in } \Omega \tag{3.3}$$

and

$$\tau_{33}\left(\widehat{x}, \frac{h}{2}\right) = h^3 \widehat{F},\ \tau_{13}\left(\widehat{x}, \frac{h}{2}\right) = \tau_{23}\left(\widehat{x}, \frac{h}{2}\right) = 0,\ \tau_{i3}\left(\widehat{x}, -\frac{h}{2}\right) = 0 \tag{3.4}$$

for $i = 1, 2, 3$. Then,

$$\begin{aligned} \| \,\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v})\, \|^2 \leq \quad & M_3(\mathbf{v},\boldsymbol{\tau}) := (1+\beta)\, \|\, \mathbb{L}\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\tau}\, \|^2_\star + \\ & + \frac{1+\beta}{c_1^2\beta} 2C_\omega^2 \left(\|r_1(\boldsymbol{\tau})\|^2_\Omega + \|r_2(\boldsymbol{\tau})\|^2_\Omega \right), \end{aligned} \tag{3.5}$$

where $\beta > 0$ and C_ω is a constant in the Friedrichs inequality related to the plain domain ω.

Proof. We rewrite (2.11) in the form

$$\begin{aligned} \int_\Omega \mathbb{L}\, \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{w})\, d\mathbf{x} = & \int_\Omega (\mathbf{f} + \mathrm{Div}\boldsymbol{\tau}) \cdot \mathbf{w}\, d\mathbf{x} + \int_\Omega \boldsymbol{\tau} : \boldsymbol{\varepsilon}(\mathbf{w})\, d\mathbf{x} + \\ + \int_{S_\ominus} (\boldsymbol{\tau}\cdot\mathbf{n}_\ominus)\cdot \mathbf{w} d\widehat{\mathbf{x}} + & \int_{S_\oplus} (\boldsymbol{\tau}\cdot\mathbf{n}_\oplus - \mathbf{F})\cdot \mathbf{w} d\widehat{\mathbf{x}} - \int_\Omega \mathbb{L}\, \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{w})\, d\mathbf{x} \end{aligned} \tag{3.6}$$

which holds for any $\mathbf{w} \in \mathbf{V_0}$. In view of (3.3) the integrals related to upper and lower faces vanish. Using (3.2), we find that

$$\int_\Omega (\mathbf{f} + \mathrm{Div}\boldsymbol{\tau}) \cdot \mathbf{w} d\mathbf{x} = \int_{-h/2}^{h/2} (I_1(x_3) + I_2(x_3)) dx_3,$$

where

$$I_k(x_3) = \int_\omega (r_k(\boldsymbol{\tau})(\widehat{\mathbf{x}}, x_3) w_k(\widehat{\mathbf{x}}, x_3) d\widehat{\mathbf{x}}, \quad k = 1, 2.$$

To estimate this integral, we first assume that $\mathbf{w}$ is a smooth function (the result for functions in $\mathbf{V_0}$ then follows by density arguments).

Consider $w_k(\widehat{\mathbf{x}}, x_3)$ as a function of $\widehat{\mathbf{x}}$. For any $x_3 \in \left(-\frac{h}{2}, +\frac{h}{2}\right)$, we have the estimate

$$\|w_k(\cdot, x_3)\|_\omega \leq C_\omega \|\widehat{\nabla} w_k(\cdot, x_3)\|_\omega,$$

which follows from $2D$ Friedrichs' inequality. For this reason,

$$\int_{-h/2}^{h/2} I_k(x_3) dx_3 \leq C_\omega \int_{-h/2}^{h/2} \left[\|r_k(\cdot, x_3)\|_\omega \left(\int_\omega \left(w_{k,1}^2 + w_{k,2}^2\right) d\widehat{\mathbf{x}} \right)^{1/2} \right] dx_3$$

$$\leq C_\omega \left(\int_{-h/2}^{h/2} \|r_k(\cdot, x_3)\|_\omega^2 \, dx_3 \right)^{1/2} \left(\int_{-h/2}^{h/2} \int_\omega \left(w_{k,1}^2 + w_{k,2}^2\right) d\widehat{\mathbf{x}} dx_3 \right)^{1/2}$$

and

$$\int_{-h/2}^{h/2} (I_1(x_3) + I_2(x_3)) dx_3 \leq$$

$$\leq C_\omega \left(\|r_1(\boldsymbol{\tau})\|_\Omega^2 + \|r_2(\boldsymbol{\tau})\|_\Omega^2 \right)^{1/2} \left(\int_{-h/2}^{h/2} \int_\omega \left(w_{1,1}^2 + w_{1,2}^2 + w_{2,1}^2 + w_{2,2}^2\right) d\widehat{\mathbf{x}} dx_3 \right)^{1/2}.$$

Since

$$\int_\omega (w_{1,2}^2 + w_{2,1}^2) d\widehat{\mathbf{x}} = \int_\omega ((w_{1,2} + w_{2,1})^2 - 2w_{1,2}w_{2,1}) d\widehat{\mathbf{x}} =$$

$$= \int_\omega ((w_{1,2} + w_{2,1})^2 - 2w_{1,1}w_{2,2}) d\widehat{\mathbf{x}} \leq \int_\omega ((w_{1,2} + w_{2,1})^2 + w_{1,1}^2 + w_{2,2}^2) d\widehat{\mathbf{x}}$$

we find that

(3.7)
$$\int\limits_{-h/2}^{h/2} (I_1(x_3)+I_2(x_3))dx_3 \leq \sqrt{2}\frac{C_\omega}{c_1}\left(\|r_1(\boldsymbol{\tau})\|^2_\Omega+\|r_2(\boldsymbol{\tau})\|^2_\Omega\right)^{1/2} \|\!| \,\varepsilon(\mathbf{w})\, \|\!|_\Omega\,.$$

Finally, we note that

$$\int\limits_\Omega (\boldsymbol{\tau}-\mathbb{L}\varepsilon(\mathbf{v})):\varepsilon(\mathbf{w})d\mathbf{x} \leq \|\!|\, \boldsymbol{\tau}-\mathbb{L}\varepsilon(\mathbf{v})\, \|\!|_* \|\!|\, \varepsilon(\mathbf{w})\, \|\!|\,. \tag{3.8}$$

By (3.6) and (3.7), and (3.8) we conclude that

$$\int\limits_\Omega \mathbb{L}\varepsilon(\mathbf{u}-\mathbf{v}):\varepsilon(\mathbf{w})\,d\mathbf{x} \leq$$

$$\leq \left(\|\!|\, \mathbb{L}\varepsilon(\mathbf{v})-\boldsymbol{\tau}\, \|\!|_* + \sqrt{2}\frac{C_\omega}{c_1}\left(\|r_1(\boldsymbol{\tau})\|^2_\Omega+\|r_2(\boldsymbol{\tau})\|^2_\Omega\right)^{1/2}\right) \|\!|\, \varepsilon(\mathbf{w})\, \|\!|_\Omega\,. \tag{3.9}$$

Now, we use density of smooth functions in $\mathbf{V_0}$ and obtain the same inequality for $\mathbf{w}\in\mathbf{V_0}$.

Set $\mathbf{w}=\mathbf{v}-\mathbf{u}\in\mathbf{V_0}$. We obtain

$$\|\!|\, \varepsilon(\mathbf{u}-\mathbf{v})\, \|\!| \leq \|\!|\, \mathbb{L}\varepsilon(\mathbf{v})-\boldsymbol{\tau}\, \|\!|_* + \sqrt{2}\frac{C_\omega}{c_1}\left(\|r_1(\boldsymbol{\tau})\|^2_\Omega+\|r_2(\boldsymbol{\tau})\|^2_\Omega\right)^{1/2} \tag{3.10}$$

and (3.4) follows by Young's inequality. □

Remark 3.4. *It is easy to see that there exists $\boldsymbol{\tau}$ such that*

$$M_s(\mathbf{v},\boldsymbol{\tau}) = \|\!|\, \varepsilon(\mathbf{u}-\mathbf{v})\, \|\!|, \quad s=1,2,3.$$

Indeed, let $\boldsymbol{\tau}=\boldsymbol{\sigma}:=\mathbb{L}\varepsilon(\mathbf{u})$. In this case, the equilibrium equations and the boundary conditions on $S_{\ominus,\oplus}$ are satisfied. Since

$$\|\!|\, \mathbb{L}\varepsilon(\mathbf{v})-\boldsymbol{\tau}\, \|\!|_* = \|\!|\, \mathbb{L}\varepsilon(\mathbf{v})-\mathbb{L}\varepsilon(\mathbf{u})\, \|\!|_* = \|\!|\, \varepsilon(\mathbf{u}-\mathbf{v})\, \|\!|\,.$$

we see that all the estimates show the value of the modeling error provided that $\boldsymbol{\tau}$ is properly selected.

In the next section, we will use the majorant M_3 in order to evaluate the accuracy of KL model.

4. Error bound for KL model

Let $\widehat{w}=\widehat{w}(x_1,x_2)$ be a solution of the Kirchhoff-Love problem (for the sake of simplicity we consider the case where the plate is under the action of only volume forces and $\widehat{F}=0$). To measure the corresponding modeling error we need to reconstruct 3D displacements and stresses. For this purpose, we define reconstruction operators $R_{\mathbf{v}}$ (for displacements) and $R_{\boldsymbol{\tau}}$ (for stresses).

4.1. **Reconstruction of 3D displacements.** In the simplest case, the first two components of the reconstructed displacement vector are first order polynomials with respect to x_3 and the third one does not depend on x_3 (cf. (2.12)). Henceforth, we call it (110)- reconstruction and write

$$\mathbf{v}^{110} := R^{110}_{\mathbf{v}}(\widehat{w}) = (-x_3\widehat{w}_{,1}, -x_3\widehat{w}_{,2}, \widehat{w})^{\mathsf{T}} . \tag{4.1}$$

Then, the corresponding strain tensor has the form

$$\varepsilon(\mathbf{v}^{110}) \quad := x_3 \begin{bmatrix} -\widehat{w}_{,11} & -\widehat{w}_{,12} & 0 \\ -\widehat{w}_{,12} & -\widehat{w}_{,22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.2}$$

with only plane nonzero components.

In a more advanced (112)-reconstruction (see [1, 9, 3, 4, 18]), we set

$$\mathbf{v}^{112} := R^{112}_{\mathbf{v}}(\widehat{w}) = \Big(-x_3\widehat{w}_{,1}, -x_3\widehat{w}_{,2}, \widehat{w} + x_3^2\widehat{W}(\widehat{x})\Big)^{\mathsf{T}} , \tag{4.3}$$

where $\widehat{W}\,(\widehat{\mathbf{x}}) \in H^1_0\,(\omega)$ is a specially selected function. (112)-reconstruction generates the strain tensor

$$\varepsilon(\mathbf{v}^{112}) := x_3 \begin{bmatrix} -\widehat{w}_{,11} & -\widehat{w}_{,12} & \frac{1}{2}x_3\widehat{W}_{,1} \\ -\widehat{w}_{,12} & -\widehat{w}_{,22} & \frac{1}{2}x_3\widehat{W}_{,2} \\ \frac{1}{2}x_3\widehat{W}_{,1} & \frac{1}{2}x_3\widehat{W}_{,2} & 2\widehat{W} \end{bmatrix} \in \Sigma. \tag{4.4}$$

It is easy to see that $\operatorname{tr}\varepsilon(\mathbf{v}^{112}) = x_3(2\widehat{W} - \widehat{\Delta}\widehat{w})$.

4.2. **Reconstruction of 3D stresses.** In our analysis, we use two reconstructions of 3D stresses based on $\widehat{w}$. The first one reconstructs stresses in accordance with 3D elasticity relations, i.e.,

$$R^{\mathbb{L}}_{\boldsymbol{\tau}}(\mathbf{v}) = \mathbb{L}\varepsilon(\mathbf{v}) = 2\mu\varepsilon(\mathbf{v}) + \frac{2\mu\nu}{1-2\nu}\operatorname{tr}\varepsilon(\mathbf{v})\mathbb{I}.$$

Here, instead of $\boldsymbol{\tau}^{110} := R^{\mathbb{L}}_{\boldsymbol{\tau}}(\mathbf{v}^{110}(\widehat{w}))$ (which may not provide a sufficiently good approximation of $\boldsymbol{\sigma}$), we consider the stress tensor generated by (112) model:

$$\boldsymbol{\tau}^{112} := R^{\mathbb{L}}_{\boldsymbol{\tau}}(\mathbf{v}^{112}(\widehat{w})) = 2\mu \begin{bmatrix} -x_3\widehat{w}_{,11} & -x_3\widehat{w}_{,12} & \frac{1}{2}x_3^2\widehat{W}_{,1} \\ -x_3\widehat{w}_{,12} & -x_3\widehat{w}_{,22} & \frac{1}{2}x_3^2\widehat{W}_{,2} \\ \frac{1}{2}x_3^2\widehat{W}_{,1} & \frac{1}{2}x_3^2\widehat{W}_{,2} & 2x_3\widehat{W} \end{bmatrix} + \\ +\frac{2\mu\nu}{1-2\nu}x_3(2\widehat{W} - \widehat{\Delta}\widehat{w})\mathbb{I}.$$

Another possible reconstruction of the stress tensor uses the KL relations (2.13)–(2.15) for the components τ_{sk} $s, k = 1, 2$, while the components τ_{i3} $i = 1, 2, 3$ are not zero (as in the classical KL theory) and are defined with the help of correction functions $\theta, \widehat{\boldsymbol{q}}$, and ψ. This *improved* reconstruction has the form

$$\boldsymbol{\tau}_{KL}^{im} := \begin{bmatrix} -2\mu x_3\left(\widehat{w}_{,11}+\frac{\nu}{1-\nu}\widehat{\Delta}\widehat{w}\right) & -2\mu x_3\widehat{w}_{,12} & \theta(x_3)\widehat{q}_1 \\ -2\mu x_3\widehat{w}_{,12} & -2\mu x_3\left(\widehat{w}_{,22}+\frac{\nu}{1-\nu}\widehat{\Delta}\widehat{w}\right) & \theta(x_3)\widehat{q}_2 \\ \theta(x_3)\widehat{q}_1 & \theta(x_3)\widehat{q}_2 & \psi(x_3)h^2\widehat{f}_0 \end{bmatrix}.$$

We choose $\theta(x_3)$, $\widehat{q}_1$, $\widehat{q}_2$, and $\psi(x_3)$ in such a way that (3.3) and (3.4) are satisfied. For this purpose, we require that θ and ψ are continuous functions vanishing at $x_3 = \pm\frac{h}{2}$. Besides, we assume that

$$\widehat{\boldsymbol{q}} \in \mathbf{Q}_{c\widehat{f}_0} := \left\{\widehat{\boldsymbol{q}}(\widehat{\mathbf{x}}) \in H(\omega, \mathrm{div}) \mid \mathrm{div}\widehat{\boldsymbol{q}} + c\,\widehat{f}_0 = 0, \quad \text{for a.e.}\, \widehat{x} \in \omega\right\},$$

where c is a constant, which we define in (4.6) . Now (3.3) leads to

$$h^2\psi' + h^2 = c\,\theta, \quad \text{for a.e.}\, x_3 \in \left[-\frac{h}{2}, \frac{h}{2}\right], \tag{4.5}$$

where

$$c = h^3\left(\int\limits_{-h/2}^{h/2}\theta dx_3\right)^{-1}. \tag{4.6}$$

From (4.5), the form of ψ follows:

$$\psi\left(x_3\right) = \frac{c}{h^2}\int\limits_{-h/2}^{x_3}\theta dx_3 - x_3 - \frac{1}{2}h. \tag{4.7}$$

Now, it is easy to see that on upper and lower faces we have

$$\boldsymbol{\tau}_{KL}^{im}\cdot\boldsymbol{n}\mid_{\pm\frac{h}{2}} = 0.$$

Since

$$r_3(\boldsymbol{\tau}_{KL}^{im}) + h^2\widehat{f}_0 = \tau_{31,1}^{im} + \tau_{32,2}^{im} + \tau_{33,3}^{im} = \theta\left(q_{1,1}+q_{2,2}\right) + (h^2\psi' + h^2)\widehat{f}_0 = 0$$

we also see that the conditions of Theorem 3.3 are satisfied.

4.3. **Error estimate.** We apply (3.5) and find that

$$\begin{aligned} &|\ \|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v}^{112})\|^2 \leq \\ &\leq (1+\beta)\,\|\boldsymbol{\kappa}\|_*^2 + \frac{1+\beta}{c_1^2\beta}2C_\omega^2\left(\|r_1(\boldsymbol{\tau}_{KL}^{im})\|_\Omega^2 + \|r_2(\boldsymbol{\tau}_{KL}^{im})\|_\Omega^2\right), \end{aligned} \tag{4.8}$$

where $\boldsymbol{\kappa} := \boldsymbol{\tau}^{112} - \boldsymbol{\tau}_{KL}^{im}$.

4.3.1. *The quantity* $\|\boldsymbol{\kappa}\|_*^2$. We have

$$\boldsymbol{\kappa}_{11} = \boldsymbol{\kappa}_{22} = \frac{2\mu\nu}{1-2\nu} x_3 \rho(\widehat{w}, \widehat{W}), \quad \boldsymbol{\kappa}_{12} = 0,$$
$$\boldsymbol{\kappa}_{13} = \mu x_3^2 \widehat{W}_{,1} - \theta \widehat{q}_1, \quad \boldsymbol{\kappa}_{23} = \mu x_3^2 \widehat{W}_{,2} - \theta \widehat{q}_2,$$
$$\boldsymbol{\kappa}_{33} = 4\mu x_3 \widehat{W} + \frac{2\mu\nu}{1-2\nu} x_3 (2\widehat{W} - \widehat{\Delta}\widehat{w}) - \psi h^2 \widehat{f}_0 = \frac{2\mu x_3(1-\nu)}{1-2\nu} \rho(\widehat{w}, \widehat{W}) - \psi h^2 \widehat{f}_0,$$

where

$$\rho(\widehat{w}, \widehat{W}) := 2\widehat{W} - \frac{\nu}{1-\nu} \widehat{\Delta}\widehat{w}.$$

Since

$$\operatorname{tr} \boldsymbol{\kappa} = \frac{2\mu x_3(1+\nu)}{1-2\nu} \rho(\widehat{w}, \widehat{W}) - \psi h^2 \widehat{f}_0$$

we find that

$$\begin{aligned}
|||\boldsymbol{\kappa}|||_*^2 &= \int_\Omega \left(\frac{1}{2\mu} \boldsymbol{\kappa} : \boldsymbol{\kappa} - \frac{\nu}{2\mu(1+\nu)} (\operatorname{tr} \boldsymbol{\kappa})^2 \right) d\mathbf{x} \\
&= \frac{1}{2\mu} \int_\Omega 2 \left(|\mu x_3^2 \widehat{W}_{,1} - \theta \widehat{q}_1|^2 + |\mu x_3^2 \widehat{W}_{,2} - \theta \widehat{q}_2|^2 \right) d\mathbf{x} \\
&\quad + \frac{4\nu^2 \mu}{(1-2\nu)^2} \int_\Omega x_3^2 \rho^2(\widehat{w}, \widehat{W}) d\mathbf{x} \\
&\quad + \int_\Omega \frac{1}{2\mu} \left(\frac{2\mu x_3(1-\nu)}{1-2\nu} \rho(\widehat{w}, \widehat{W}) - \psi h^2 \widehat{f}_0 \right)^2 \\
&\quad - \int_\Omega \frac{\nu}{2\mu(1+\nu)} \left(\frac{2\mu x_3(1+\nu)}{1-2\nu} \rho(\widehat{w}, \widehat{W}) - \psi h^2 \widehat{f}_0 \right)^2 d\mathbf{x} \\
&= \frac{1}{2\mu} \int_\Omega 2 \left(|\mu x_3^2 \widehat{W}_{,1} - \theta \widehat{q}_1|^2 + |\mu x_3^2 \widehat{W}_{,2} - \theta \widehat{q}_2|^2 \right) d\mathbf{x} \\
&\quad + \int_\Omega \frac{2\mu(1-\nu)}{1-2\nu} x_3^2 \rho^2(\widehat{w}, \widehat{W}) d\mathbf{x} \\
&\quad -2 \int_\Omega x_3 \rho(\widehat{w}, \widehat{W}) \psi h^2 \widehat{f}_0 + \frac{1}{2\mu(1+\nu)} \psi^2 h^4 \widehat{f}_0^2 d\mathbf{x}.
\end{aligned} \tag{4.9}$$

Define the following quantities

$$\iota_1 = \int\limits_{-h/2}^{h/2} x_3^2 \theta(x_3) dx_3, \quad \iota_2 = \int\limits_{-h/2}^{h/2} \theta^2(x_3) dx_3, \tag{4.10}$$

$$\iota_3 = \int\limits_{-h/2}^{h/2} x_3 \psi(x_3) dx_3, \quad \iota_4 = \int\limits_{-h/2}^{h/2} \psi^2(x_3) dx_3. \tag{4.11}$$

Then

$$\begin{aligned} \| \boldsymbol{\kappa} \|_*^2 \;&=\; \int\limits_\omega \left(\frac{\mu h^5}{80} |\widehat{\nabla} \widehat{W}|^2 - 2\iota_1 (\widehat{W}_{,1} \widehat{q}_1 + \widehat{W}_{,2} \widehat{q}_2) + \frac{\iota_2}{\mu} |\widehat{\boldsymbol{q}}|^2 \right) d\widehat{\mathbf{x}} + \\ &\quad + \int\limits_\omega \left(\frac{\mu(1-\nu)h^3}{6(1-2\nu)} \rho^2(\widehat{w}, \widehat{W}) - 2\iota_3 \rho(\widehat{w}, \widehat{W}) h^2 \widehat{f}_0 + \frac{\iota_4 h^4 \widehat{f}_0^2}{2\mu(1+\nu)} \right) d\widehat{\mathbf{x}}. \end{aligned} \tag{4.12}$$

Henceforth, we assume that $\widehat{\Delta} \widehat{w}$ possesses square summable generalized derivatives (this assumption holds true, e.g., for domains with smooth boundaries or for convex polygonal domains). Also, we now choose the simplest quadratic form of the function θ, namely

$$\theta(x_3) = \frac{1}{2} \left(x_3 - \frac{h}{2} \right) \left(x_3 + \frac{h}{2} \right). \tag{4.13}$$

It is worth noting that this choice leads to the form of $\boldsymbol{\tau}_{KL}^{im}$ suggested by Morgenstern in [9].

In this case, the constant c in (4.6) is equal to -12,

$$\psi = -\frac{4x_3}{h^2} \theta(x_3), \; \iota_1 = -\frac{h^5}{240}, \; \iota_2 = -2\iota_1, \; \iota_3 = \frac{h^3}{60}, \; \iota_4 = \frac{h^3}{210},$$

r_1 and r_2 can be explicitly defined as follows:

$$\begin{aligned} r_1(\boldsymbol{\tau}_{KL}^{im}) &= \tau_{11,1}^{im} + \tau_{12,2}^{im} + \tau_{13,3}^{im} = -2\mu x_3 \left(\widehat{w}_{,111} + \frac{\nu}{1-\nu} \widehat{\Delta} \widehat{w}_{,1} \right) - 2\mu x_3 \widehat{w}_{,122} + x_3 \widehat{q}_1 \\ &= x_3 \left(-\frac{2\mu}{1-\nu} \widehat{\Delta} \widehat{w}_{,1} + \widehat{q}_1 \right), \\ r_2(\boldsymbol{\tau}_{KL}^{im}) &= \tau_{21,1}^{im} + \tau_{22,2}^{im} + \tau_{23,3}^{im} = x_3 \left(-\frac{2\mu}{1-\nu} \widehat{\Delta} \widehat{w}_{,2} + \widehat{q}_2 \right) \end{aligned}$$

and we have

$$\int\limits_\Omega r_s^2 \left(\boldsymbol{\tau}_{KL}^{im} \right) d\mathbf{x} = \frac{h^3}{12} \int\limits_\omega \left| \widehat{q}_s - \frac{2\mu}{1-\nu} \widehat{\Delta} \widehat{w}_{,s} \right|^2 d\widehat{\mathbf{x}} \quad s = 1, 2. \tag{4.14}$$

By (4.8), (4.12)–(4.14) we arrive at the following result.

Theorem 4.1. *Let θ, $\psi(x_3)$, and c be defined in accordance with (4.13), (4.6), and (4.7), respectively. Then, for any $\widehat{W} \in H^1_0(\Omega)$ and $\widehat{\mathbf{q}} \in \mathbf{Q}_{c\widehat{f}_0}(\omega)$ we have*

$$\begin{aligned}\Vert \varepsilon(\mathbf{u}-\mathbf{v}^{112}) \Vert^2 \leq \quad & M_4(\widehat{w}, \widehat{W}, \widehat{\boldsymbol{q}}) := (1+\beta)(\mathcal{M}_1(\widehat{W}, \widehat{\boldsymbol{q}})+ \\ & +\mathcal{M}_2(\widehat{w}, \widehat{W})) + \frac{1+\beta}{\beta}\mathcal{M}_3(\widehat{w}, \widehat{\boldsymbol{q}}),\end{aligned} \tag{4.15}$$

where

$$\mathcal{M}_1(\widehat{W}, \widehat{\boldsymbol{q}}) := 2\int\limits_\omega \left(\frac{\mu h^5}{160}|\widehat{\nabla}\widehat{W}|^2 - \iota_1(\widehat{W}_{,1}\widehat{q}_1 + \widehat{W}_{,2}\widehat{q}_2) + \frac{\iota_2}{2\mu}|\widehat{\boldsymbol{q}}|^2\right) d\widehat{\mathbf{x}},$$

$$\mathcal{M}_2(\widehat{w}, \widehat{W}) := \int\limits_\omega \left(\frac{\mu(1-\nu)h^3}{6(1-2\nu)}\rho^2(\widehat{w}, \widehat{W}) - 2\iota_3\rho(\widehat{w}, \widehat{W})h^2\widehat{f}_0 + \frac{\iota_4 h^4 \widehat{f}_0^2}{2\mu(1+\nu)}\right) d\widehat{\mathbf{x}},$$

$$\mathcal{M}_3(\widehat{w}, \widehat{\boldsymbol{q}}) := \frac{C_\omega^2 h^3}{6c_1^2}\int\limits_\omega \left(|\widehat{q}_1 - \frac{2\mu}{1-\nu}\widehat{\Delta}\widehat{w}_{,1}|^2 + |\widehat{q}_2 - \frac{2\mu}{1-\nu}\widehat{\Delta}\widehat{w}_{,2}|^2\right) d\widehat{\mathbf{x}},$$

and ι_k for $k = 1, 2, 3, 4$ are defined by (4.11) and (4.10).

Corollary 4.2. *Let $\omega \in \Pi$, where Π is a rectangle with sides a and b. Then*

$$C_\omega \leq C_\Pi = \frac{1}{\pi}\frac{ab}{\sqrt{a^2+b^2}}.$$

Thus, $M_4\left(\widehat{w}, \widehat{W}, \widehat{\mathbf{q}}\right)$ contains only known functions and constants. If $\widehat{w}$ is known, then the majorant is directly computable. By selecting $\widehat{\boldsymbol{q}}$ and $\widehat{W}$, we can minimize the value of the majorant. For this purpose, we can represent $\widehat{\boldsymbol{q}}$ and $\widehat{W}$ as series (using some trial functions) and minimize the majorant by a direct minimization method. The number obtained presents a guaranteed *upper bound of the modeling error encompassed in $\widehat{w}$. From Theorem 4.1 it follows that the sharpest error bound is given by the estimate*

$$\Vert \varepsilon(\mathbf{u}-\mathbf{v}^{112}) \Vert \leq \inf_{\substack{\widehat{W} \in H^1_0(\omega) \\ \boldsymbol{q} \in \mathbf{Q}_{c\widehat{f}_0}(\omega)}} \left\{ \left(\mathcal{M}_1(\widehat{W}, \widehat{\boldsymbol{q}}) + \mathcal{M}_2(\widehat{w}, \widehat{W})\right)^{1/2} + \mathcal{M}_3^{1/2}(\widehat{w}, \widehat{\boldsymbol{q}}) \right\}.$$

5. Asymptotic behavior of the error majorant

First, we use Hölder inequality and find that

$$|\iota_1| = \int\limits_{-h/2}^{h/2} x_2^3\theta(x_3)dx_3 \leq \left(\frac{h^5}{80}\right)^{1/2}\iota_2^{1/2}, \; |\iota_3| = \int\limits_{-h/2}^{h/2} x_3\psi(x_3)dx_3 \leq \left(\frac{h^3}{12}\right)^{1/2}\iota_4^{1/2}.$$

Now, we represent the estimate in a simpler form by noting that

$$\left|\int\limits_\omega (\iota_1(\widehat{W}_{,1}\widehat{q}_1+\widehat{W}_{,2}\widehat{q}_2))d\widehat{\mathbf{x}}\right| \leq |\iota_1| \|\nabla\widehat{W}\|_\omega \|\widehat{\mathbf{q}}\|_\omega \leq \left(\frac{h^5}{80}\right)^{1/2} \iota_2^{1/2}\|\nabla\widehat{W}\|_\omega \|\widehat{\mathbf{q}}\|_\omega \leq$$

$$\leq \mu\gamma\frac{h^5}{160}\|\nabla\widehat{W}\|^2_\omega + \frac{\iota_2}{2\mu\gamma}\|\widehat{\mathbf{q}}\|^2_\omega$$

and

$$2\left|\iota_3\int\limits_\omega \rho\widehat{f}d\widehat{\mathbf{x}}\right| \leq 2\left(\frac{h^3}{12}\right)^{1/2}\iota_4^{1/2}\|\rho\|_\omega\|\widehat{f}\|_\omega \leq \frac{\lambda\mu(1-\nu)}{(1-2\nu)}\frac{h^3}{12}\|\rho\|^2_\omega+\frac{\iota_4(1-2\nu)}{\lambda\mu(1-\nu)}h^4\|\widehat{f}_0\|^2_\omega.$$

In view of these relations,

$$\mathcal{M}_1(\widehat{W},\widehat{\boldsymbol{q}}) \leq m_{11}h^5\|\widehat{\nabla}\widehat{W}\|^2_\omega + m_{12}\iota_2\|\widehat{\boldsymbol{q}}\|^2_\omega,$$
$$\mathcal{M}_2(\widehat{w},\widehat{W}) \leq m_{21}h^3\|\rho(\widehat{w},\widehat{W})\|^2_\omega + m_{22}\iota_4 h^4\|\widehat{f}_0\|^2_\omega,$$

where the coefficients m_{ij} depend on material parameters and positive numbers λ and γ. They are defined by the relations

$$m_{11} = \tfrac{\mu(1+\gamma)}{80}, \qquad m_{12} = \frac{1+\gamma}{\mu\gamma},$$
$$m_{21} = \tfrac{\mu(1-\nu)(1+\lambda)}{6(1-2\nu)}, \quad m_{22} = \frac{1}{\mu}\left(\frac{1}{2(1+\nu)} + \frac{(1-2\nu)}{\lambda(1-\nu)}\right).$$

and we have derived the following form of the error bound.

Theorem 5.1. *Let the conditions of Theorem 4.1 be satisfied. Then*

$$\begin{aligned}
\|\!| \, \varepsilon(\mathbf{u}-\mathbf{v}^{112}) \, |\!\|^2 \quad &\leq \quad M_5\left(\widehat{w},\widehat{W},\widehat{\mathbf{q}}\right) := \\
&= \quad (1+\beta)\Big(m_{11}h^5\|\widehat{\nabla}\widehat{W}\|^2_\omega + m_{21}h^3\|\rho(\widehat{w},\widehat{W})\|^2_\omega + \\
&\qquad + m_{12}\iota_2\|\widehat{\mathbf{q}}\|^2_\omega + m_{22}\iota_4 h^4\|\widehat{f}_0\|^2_\omega + \frac{1}{\beta}\mathcal{M}_3(\widehat{w},\widehat{\mathbf{q}})\Big),
\end{aligned} \tag{5.1}$$

where the coefficients m depend only on elasticity coefficients and arbitrary positive numbers γ and λ.

We note that the $M_5(\widehat{w},\widehat{W},\widehat{\mathbf{q}})$ suggests a suitable form of the correction function $\widehat{W}$. Indeed, it is natural to define $\widehat{W}$ such that the value of M_5 be minimal, what leads to a *singularly perturbed variational problem*

$$\inf_{\widehat{W}\in H^1_0(\omega)}\left\{h^2\|\widehat{\nabla}\widehat{W}\|^2_\omega + \frac{m_{21}}{m_{11}}\|\rho(\widehat{w},\widehat{W})\|^2_\omega\right\}, \tag{5.2}$$

where $\frac{m_{21}}{m_{11}} = \frac{40(1-\nu)}{3(1-2\nu)}\frac{1+\lambda}{1+\gamma}$.

Remark 5.2. *If we set $\lambda = 1+2\gamma$, then (5.2) has the form*

$$\inf_{\widehat{W}\in H^1_0(\omega)}\left\{h^2\|\widehat{\nabla}\widehat{W}\|^2_\omega + \frac{80(1-\nu)}{3(1-2\nu)}\|\rho(\widehat{w},\widehat{W})\|^2_\omega\right\}, \tag{5.3}$$

which was used in [3]*. Hence, our analysis shows that this singularly perturbed problem follows from the functional a posteriori estimate if we define the correction function $\widehat{W}$ as the function that minimizes the majorant and select γ and λ in a special form. We note that other values of γ and λ lead to the same asymptotic rate (with respect to h), so that from the viewpoint of qualitative analysis the choice of γ and λ is not important. However, if the problem is considered in the quantitative context and it is necessary to find a sharp upper bound of the modeling error related to a concrete plate type body, then γ and λ should be selected such that the value of the majorant be minimal.*

Let us select $\widehat{\boldsymbol{q}}$ in a special form, namely

$$\widehat{q}_1 = \frac{2\mu}{1-\nu}\widehat{\Delta}\widehat{w}_{,1}, \quad \widehat{q}_2 = \frac{2\mu}{1-\nu}\widehat{\Delta}\widehat{w}_{,2}.$$

Now $\widehat{\mathbf{q}}$ depends on $\widehat{w}$. To outline this fact we henceforth write $\widehat{\mathbf{q}}(\widehat{w})$. It is easy to see that (cf. 2.17)

$$\mathrm{div}\widehat{\mathbf{q}}(\widehat{w}) = \frac{2\mu}{1-\nu}\widehat{\Delta}\widehat{\Delta}\widehat{w} = \widehat{f}_0\frac{6(1-\nu)}{\mu}\frac{2\mu}{1-\nu} = 12\widehat{f}_0.$$

Since we have set $c = -12$, the above relation means that

$$\widehat{\mathbf{q}}(\widehat{w}) \in \mathbf{Q}_{c\widehat{f}_0}.$$

Hence, we note that

$$\|\widehat{\mathbf{q}}(\widehat{w})\|^2_\omega = \frac{4\mu^2}{(1-\nu)^2}\int_\omega |\widehat{\nabla}\widehat{\Delta}\widehat{w}|^2 d\widehat{\mathbf{x}},$$

apply Theorem 5.1, and obtain

Theorem 5.3. *Let the conditions of Theorem 4.1 be satisfied. Then*

$$\begin{aligned} \interleave \varepsilon(\mathbf{u}-\mathbf{v}^{112}) \interleave^2 \quad &\leq \quad M_6(\widehat{w},\widehat{W},\widehat{\mathbf{q}}(\widehat{w})) := \\ &= \quad m_{11}h^5\|\widehat{\nabla}\widehat{W}\|^2_\omega + m_{21}h^3\|\rho(\widehat{w},\widehat{W})\|^2_\omega + h^5\mathcal{R}(\widehat{w},\widehat{f}_0), \end{aligned} \tag{5.4}$$

where γ, and λ are arbitrary positive numbers and

$$\mathcal{R}(\widehat{w},\widehat{f}_0) = m_{12}\frac{\mu^2}{30(1-\nu)^2}\|\widehat{\nabla}\widehat{\Delta}\widehat{w}\|^2_\omega + m_{22}\frac{h^2}{210}\|\widehat{f}_0\|^2_\omega.$$

From (4.4) it follows that

$$\interleave \varepsilon(\mathbf{v}^{112}) \interleave^2 \geq c_1^2\|\varepsilon(\mathbf{v}^{112})\|^2_\Omega \geq \frac{c_1^2h^3}{12}\|\mathbf{H}(\widehat{w})\|^2_\omega, \tag{5.5}$$

where $\mathbf{H}(\widehat{w})$ is the 2×2 matrix of second derivatives of $\widehat{w}$ (which is independent of h). In other words, the energy norm of the solution to (112)-model decreases not faster than $h^{3/2}$. Our goal is to show that majorant decreases with a faster rate. Our analysis of asymptotic properties is based upon the following result (see [1, 3]).

Lemma 5.4. *Let $\phi \in H^1(\omega)$ and $\widehat{W}_*$ be a minimizer of the functional*

$$\inf_{\widehat{W}\in H^1(\omega)} \left\{ h^2 \|\widehat{\nabla}\widehat{W}\|^2_\omega + \|\widehat{W} - \phi\|^2_\omega \right\}, \tag{5.6}$$

where h is a small positive number. Then

$$h^2\|\widehat{\nabla}\widehat{W}_*\|^2_\omega + \|\widehat{W}_* - \phi\|^2_\omega \le \widetilde{C}\left(h\|\phi\|_{\partial\omega} + h^2\|\phi\|_{1,\omega}\right). \tag{5.7}$$

By this Lemma and trace theorems, we derive (cf. [3])

$$h\|\widehat{\nabla}\widehat{W}_*\|_\omega + \|\rho(\widehat{w}, \widehat{W}_*)\|_\omega \le Ch^{1/2}\|\widehat{\Delta}\widehat{w}\|_\omega.$$

Therefore,

$$M_6(\widehat{w}, \widehat{W}_*, \widehat{\mathbf{q}}(\widehat{w})) \le Ch^4\left(m_{11} + m_{21}\right)\|\widehat{\Delta}\widehat{w}\|^2_\omega + h^5\mathcal{R}(\widehat{w}, \widehat{f}_0). \tag{5.8}$$

Now, we use (5.5) and Young's inequality and find that

$$\begin{aligned} \|\!| \boldsymbol{\varepsilon}(\mathbf{u}) \|\!|^2 &\ge \frac{1}{2} \|\!| \boldsymbol{\varepsilon}(\mathbf{v}^{112}) \|\!|^2 - M_6(\widehat{w}, \widehat{W}_*, \widehat{\mathbf{q}}(\widehat{w})) \ge \\ &\ge \frac{h^3 c_1^2}{24}\|H(\widehat{w})\|^2_\omega - Ch^4\left(m_{11} + m_{21}\right)\|\widehat{\Delta}\widehat{w}\|^2_\omega - h^5\mathcal{R}(\widehat{w}, \widehat{f}_0). \end{aligned} \tag{5.9}$$

By combining (5.8) and (5.5) for the first and (5.9) for the second estimate below we end up with

$$\frac{M_6(\widehat{w}, \widehat{W}_*, \widehat{\mathbf{q}}(\widehat{w}))}{\|\!| \boldsymbol{\varepsilon}(\mathbf{v}^{112}) \|\!|^2} \le ch \quad \text{and} \quad \frac{M_6(\widehat{w}, \widehat{W}_*, \widehat{\mathbf{q}}(\widehat{w}))}{\|\!| \boldsymbol{\varepsilon}(\mathbf{u}) \|\!|^2} \le ch. \tag{5.10}$$

Remark 5.5. *Typically, the solution $\widehat{w}$ of KL problem is known only approximately. However, using (1.3) we can estimate the distance to $\mathbf{u}$ provided that the reconstruction operator satisfies (1.1).*

It is not difficult to show that $R^{110}_{\mathbf{v}}$ satisfies this condition.

$$\|R^{110}_{\mathbf{v}}\widehat{w}_1 - R^{110}_{\mathbf{v}}\widehat{w}_2\|_{V_0} \le C^{110}(\omega)\frac{h^{3/2}}{12}\|\widehat{w}_1 - \widehat{w}_2\|_{H^2(\omega)},$$

which means that (1.1) holds.

Remark 5.6. *In* [3]*, generalizations of Lemma 5.4 have been established. They are related to cases where ϕ has lower regularity. In particular, if $\phi \in H^s(\omega)$, $0 < s < 1/2$, then we have the estimate*

$$h^2\|\widehat{\nabla}\widehat{W}_*\|^2_\omega + \|\widehat{W}_* - \phi\|^2_\omega \le C(s,\omega)h^{2s}\|\phi\|_{s,\omega}, \tag{5.11}$$

which can be used if $\Delta\widehat{w}$ does not belong to $H^1(\omega)$. It is easy to see that (5.11) yields convergence of the quotients in (5.10) but with a rate lesser than 1.

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