COHOMOLOGICAL PATTERNS OF COHERENT SHEAVES OVER PROJECTIVE SCHEMES

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ABSTRACT. We study the sets $P(X, \mathcal{F}) = \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid H^i(X, \mathcal{F}(n)) \neq 0\}$, where X is a projective scheme over a noetherian ring R_0 and where $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules. In particular we show that $P(X, \mathcal{F})$ is a so called tame combinatorial pattern if the base ring R_0 is semilocal and of dimension ≤ 1. If $X = \mathbb{P}^d_{R_0}$ is a projective space over such a base ring R_0, the possible sets $P(X, \mathcal{F})$ are shown to be precisely all tame combinatorial patterns of width $\leq d$. We also discuss the "tameness problem" for arbitrary noetherian base rings R_0 and prove some stability results for the R_0-associated primes of the R_0-modules $H^i(X, \mathcal{F}(n))$.

1. INTRODUCTION

Let $R = \oplus_{n \geq 0} R_n$ be a positively graded homogeneous noetherian ring and let $X = Proj(R)$.

1.1. Definition. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. We define the *cohomological pattern* of $\mathcal{F}$ as the set

$$P(X, \mathcal{F}) := P(\mathcal{F}) = \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid H^i(X, \mathcal{F}(n)) \neq 0\} . \qquad \bullet$$

The basic aim of this paper is to characterize those sets $P \subseteq \mathbb{N}_0 \times \mathbb{Z}$ which occur as cohomological patterns of coherent sheaves of $\mathcal{O}_X$-modules. In order to so, we introduce the following notion:

1.2. Definition. A) Let $w \in \mathbb{N}_0$. A set $P \subseteq \mathbb{N}_0 \times \mathbb{Z}$ is called a *combinatorial pattern of width w*, if it satisfies the following five conditions:

(π_1) $\exists m, n \in \mathbb{Z} : (0, m), (w, n) \in P$;

(π_2) $(i, n) \in P \Longrightarrow i \leq w$;

(π_3) $(i, n) \in P \Longrightarrow \exists\, j \leq i : (j, n + i - j + 1) \in P$;

(π_4) $(i, n) \in P \Longrightarrow \exists\, k \geq i : (k, n + i - k - 1) \in P$;

(π_5) $i > 0 \Longrightarrow (i, n) \notin P, \ \forall n \gg 0$.

B) A combinatorial pattern $P \subseteq \mathbb{N}_0 \times \mathbb{Z}$ is called *tame* if it satisfies the additional condition

$$(\pi_6) \qquad \forall i \in \mathbb{N} : \begin{cases} (i, n) \in P, \ \forall n \ll 0 \\ \text{or} \\ (i, n) \notin P, \forall n \ll 0 \end{cases}$$

$\bullet$

The basic aim of this paper is to establish the following results (s. (3.5), (4.3) and (4.7) respectively).

(1.3) *For each coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, the cohomological pattern $P(\mathcal{F})$ is a combinatorial pattern.*

(1.4) *If R_0 is semilocal and of dimension ≤ 1, the pattern $P(\mathcal{F})$ of (1.3) is tame.*

(1.5) *If one of the special fibers of the natural morphism $X \longrightarrow X_0 := Spec(R_0)$ is a projective space of dimension d, each tame combinatorial pattern of width $\leq d$ occurs as the cohomological pattern of a coherent sheaf of $\mathcal{O}_X$-modules.*

As a consequence of (1.4) and (1.5) we get (cf. (4.8))

(1.6) *If X is a projective d-space over a semilocal noetherian ring R_0 of dimension ≤ 1, the cohomological pattern of coherent sheaves of $\mathcal{O}_X$-modules are precisely the tame combinatorial patterns of width $\leq d$.*

Our conjecture is, that the restriction on the dimension on R_0 is not needed to guarantee that the pattern $P(\mathcal{F})$ of (1.3) is tame. We have not been able to show this. In the last section of our paper we consider some aspects of this tameness problem. We prove a partial result in the viewed direction, which gives the requested tameness "along the lowest non left bounded line" of $P(\mathcal{F})$. More precisely, we prove (in terms of local cohomology modules) the following result (cf. (5.6)):

(1.7) *Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $i \in \mathbb{N}_0$ be such that $H^j(X, \mathcal{F}(n)) = 0$ for all $j < i$ and all $n \ll 0$. Then, the sets*

$$Ass_{R_0}(H^i(X, \mathcal{F}(n))) \text{ are asymptotically stable for } n \longrightarrow -\infty.$$

Finally, we use an example of Singh [Si] to show that the asymptotic stability mentioned in (1.7) need not be true for arbitrary values of i (cf. (5.7)).

Let us mention a few further problems which arise with the concept of cohomological patterns:

(1.8) *Let K be a field and let $X = \mathbb{P}_K^w$. Is there a purely combinatorial characterization of those patterns, which may be realized as the pattern $P(\mathcal{E})$ of an indecomposable algebraic vector bundle $\mathcal{E}$ over X, (cf. (4.9) B))?*

(1.9) *Let X be as in (1.8) and let P be a minimal combinatorial pattern of width w. What is the minimal rank of an algebraic vector bundle $\mathcal{E}$ over X for which $P(\mathcal{E}) = P$, (cf. (4.9) A))?*

(1.10) *Is there a purely combinatorial characterization of those patterns, which occur as the pattern $P(\mathcal{O}_X)$ of the structure sheaf $\mathcal{O}_X$ of an irreducible projective variety X? (A modification of the non-rigidity result [Ev-G, (4.13)].*

For the unexplained terminology we refer to [E], [H] and [B-S].

2. COMBINATORIAL PATTERNS

2.1. Notation, Definition and Remark. A) By $\mathcal{P}$ we denote the set of all combinatorial patterns. If $P \in \mathcal{P}$ is of width w, we write $w(P) = w$. By $\mathcal{P}'$ we denote the set of tame combinatorial patterns. The following facts are easily verified.

$P, Q \in \mathcal{P} \Longrightarrow P \cup Q \in \mathcal{P}, \ w(P \cup Q) = \max\{w(P), \ w(Q)\};$

$P, Q \in \mathcal{P}' \Longrightarrow P \cup Q \in \mathcal{P}';$

$\{0\} \times \mathbb{Z} \in \mathcal{P}';$

$P \in \mathcal{P}, \ (0, n) \in P \Longrightarrow (0, n+1) \in P;$

$P \in \mathcal{P}, \ (w(P), n) \in P \Longrightarrow (w(P), n-1) \in P;$

$w(P) = 0 \Longleftrightarrow P = \{0\} \times \mathbb{Z}, \ (P \in \mathcal{P}).$

B) For $P \in \mathcal{P}$ and $t \in \mathbb{Z}$, we define the *t-th shift* of P by

$$P(t) := \{(i, n+t) \mid (i, n) \in P\} \in \mathcal{P}.$$

Obviously we have

$P \in \mathcal{P}, \ t \in \mathbb{Z} \Longrightarrow P(t) \in \mathcal{P}, \ w(P(t)) = w(P);$

$P, Q \in \mathcal{P}, \ t \in \mathbb{Z} \Longrightarrow (P \cup Q)(t) = P(t) \cup Q(t);$

$P \in \mathcal{P} \Longrightarrow P(0) = P, \ P(t+s) = P(t)(s) \ (\forall s, \ t \in \mathbb{Z});$

$P \in \mathcal{P}', \ t \in \mathbb{Z} \Longrightarrow P(t) \in \mathcal{P}'.$

C) Next, we consider the *diagonal projection*

$$\rho : \mathbb{N}_0 \times \mathbb{Z} \longrightarrow \mathbb{Z}; \quad (i, n) \mapsto i + n.$$

In view of the axioms (π_1), (π_3) and (π_4) we have

$$\text{If } P \in \mathcal{P}, \text{ then } \rho : P \longrightarrow \mathbb{Z} \quad \text{is surjective}.$$

D) Let $P \in \mathcal{P}$ and $(i,n) \in P$. By (π_3) and (π_4) we then may define the following non-negative integers

$$j_P(i,n) := \max\{j \leq i \mid (j, n+i-j+1) \in P\} \,;$$
$$k_P(i,n) := \min\{k \geq i \mid (k, n+i-k-1) \in P\} \,.$$

Observe that

$$j_P(i,n) \leq i \leq k_P(i,n) \quad \text{for all} \quad (i,n) \in P \,. \qquad \bullet$$

2.2. **Definition and Remark.** A) A *minimal combinatorial pattern* is a combinatorial pattern P such that there is no combinatorial pattern Q with $Q \subsetneq P$. The set of minimal combinatorial pattern is denoted by $\mathcal{P}_{\min}$.

B) The following statements are obvious from the definition

$$\{0\} \times \mathbb{Z} \in \mathcal{P}_{\min} \,; \quad P \in \mathcal{P}_{\min}, \; n \in \mathbb{Z} \Longrightarrow P(n) \in \mathcal{P}_{\min} \,.$$

Moreover, as $\rho : P \longrightarrow \mathbb{Z}$ is surjective for each $P \in \mathcal{P}$ (s. (2.1) C)), we can say

$$\rho : P \longrightarrow \mathbb{Z} \quad \text{bijective} \quad \Longrightarrow P \in \mathcal{P}_{\min}, \; (P \in \mathcal{P}) \,.$$

C) Now, fix $w \in \mathbb{N}_0$ and let $\mathbb{M}_w$ be the set of all monotonously decreasing functions $\mu : \mathbb{Z} \longrightarrow \{0, \ldots, w\}$ for which $0, w \in \mu(\mathbb{Z})$. For any function $\mu \in \mathbb{M}_w$ we define the "skew graph" of μ, e.g. the set

$$P[\mu] := \{(\mu(n), \; n - \mu(n)) \mid n \in \mathbb{Z}\} \,.$$

It is easy to verify (on use of the last observation of part B)), that

$$P[\mu] \in \mathcal{P}_{\min}, \quad w(P[\mu]) = w, \quad (\forall \; \mu \in \mathbb{M}_w) \,. \qquad \bullet$$

2.3. **Lemma.** *Let $P \in \mathcal{P}$ and let $(i,n) \in P$. Then, there is an integer $w \in \{0, \ldots, w(P)\}$ and a function $\mu \in \mathbb{M}_w$ such that $(i,n) \in P[\mu] \subseteq P$.*

Proof It is easy to verify (by ascending and descending induction beginning with $m = n + i$) that there is a function $\mu : \mathbb{Z} \longrightarrow \mathbb{N}_0$ such that

$$(\mu(m), \; m - \mu(m)) \in P \quad \text{for all} \quad m \in \mathbb{Z}$$

and such that

$$\mu(m) = \begin{cases} i, & \text{if } m = n+i; \\ j_P(\mu(m-1), m-1-\mu(m-1)), & \text{if } m > n+i; \\ k_P(\mu(m+1), m+1+\mu(m+1)), & \text{if } m < n+i. \end{cases}$$

In view of the inequalities given in 2.1 (D), we see that μ is monotonously decreasing. Moreover, $w := \max\{\mu(\mathbb{Z})\}$ exists and satisfies $w \leq w(P)$. As P satisfies the axiom (π_5), we have $\mu(m) = 0$, $\forall m \gg 0$. Therefore, $\mu \in \mathbb{M}_w$. Altogether we now have $(i,n) \in P[\mu] \subseteq P$. $\qquad \square$

2.4. **Remark.** A) Let $w \in \mathbb{N}_0$. Then, as an easy consequence of (2.3), we get a bijection

$$\mathbb{M}_w \overset{\approx}{\longrightarrow} \{P \in \mathcal{P}_{\min} \mid w(P) = w\}\,, \quad (\mu \mapsto P[\mu])\,.$$

B) It is obvious, that each combinatorial pattern $P[\mu]$ with $\mu \in \mathbb{M}_w$ is tame. So, in view of A) we have

$$\mathcal{P}_{\min} \subseteq \mathcal{P}'\,. \qquad\qquad\qquad\qquad\qquad \bullet$$

2.5. **Proposition.** *For a set $P \subseteq \mathbb{N}_0 \times \mathbb{Z}$ the following statements are equivalent:*

   (i): *P is a tame combinatorial pattern.*

   (ii): *P is the union of finitely many minimal combinatorial patterns.*

Proof "(ii)$\Longrightarrow$ (i)": Clear from the fact that minimal combinatorial patterns are tame (cf. (2.4) B)) and that unions of finitely many tame combinatorial pattern are again tame combinatorial patterns, (cf. (2.1) A)).

"(i) $\Longrightarrow$ (ii)": Let $P \in \mathcal{P}'$ and let $w = w(P)$. Let

$$\mathbb{I} := \{i \in \mathbb{N}_0 \mid \#\{n \mid (i,n) \in P\} = \infty\}\,.$$

Clearly, $\mathbb{I} \subseteq \{0,\ldots,w\}$. As P is tame (and in view of the last but third observation made in (2.1) A)), there is some $r \in \mathbb{Z}$ such that

$$(i,n) \in P \quad \text{for all } n \leq r - i \text{ and all } i \in \mathbb{I}\,;$$
$$(j,n) \notin P \quad \text{for all } n \leq r - j \text{ and all } j \notin \mathbb{I}\,.$$

Now, fix $i \in \mathbb{I}$. Then, by (2.3) there is some $w_i \in \{0,\ldots,w\}$ and some $\mu_i \in \mathbb{M}_{w_i}$ such that $(i, r-i) \in P[\mu_i] \subseteq P$. As $\mu_i(r) = i$, we have $i \leq w_i$. Next, define a function $\overline{\mu}_i : \mathbb{Z} \longrightarrow \mathbb{N}_0$ by

$$\overline{\mu}_i(m) := \begin{cases} \mu_i(m), & \text{for } m \geq r; \\ i, & \text{for } m < r. \end{cases}$$

Clearly $\overline{\mu}_i \in \mathbb{M}_i$. Moreover, by our choice of r

$$(\{i\} \times \mathbb{Z}_{\leq r-i}) \subseteq P[\overline{\mu}_i] \subseteq P\,.$$

As P satisfies the axiom (π_5), there is some $s \geq r$ such that $\rho^{-1}(\mathbb{Z}_{\geq s}) \cap P = \{0\} \times \mathbb{Z}_{\leq s} \subseteq P[\overline{\mu}_i]$ for all $i \in \mathbb{I}$. So, the set $Q := P \setminus \bigcup_{i \in \mathbb{I}} P[\overline{\mu}_i]$ is contained in $\bigcup_{j=0}^{w} \{j\} \times \{r - j + 1, r - j + 2, \ldots, s - j - 1\}$ and hence is finite.

Now, for each $q \in Q$ there is some $P_q \in \mathcal{P}_{\min}$ with $q \in P_q \subseteq P$, (s. (2.3)). So, we see that indeed

$$P = \bigcup_{i \in \mathbb{I}} P[\overline{\mu}_i] \cup \bigcup_{q \in Q} P_q\,. \qquad\qquad\qquad \square$$

2.6. Remark. A) For each $w \in \mathbb{N}_0$ and each $\mu \in \mathbb{M}_w$, the projection $\rho : P[\mu] \longrightarrow \mathbb{Z}$ is bijective. So, in view of (2.3) and the last observation of (2.2) B) we can say

$$\rho : P \longrightarrow \mathbb{Z} \quad \text{bijective} \quad \Longleftrightarrow P \in \mathcal{P}_{\min}, \quad (P \in \mathcal{P}).$$

B) Let $P \in \mathcal{P}'$. If we define r, s as in the proof of (2.5) it follows from that proof and from the previous observation:

To cover P, at least $\max\{\#\rho^{-1}(n) \cap P \mid n \in \mathbb{Z}\} =: t$ minimal combinatorial patterns are needed. Moreover, P can be written as the union of $t + (s - r - 1)w$ minimal combinatorial patterns. •

For later use, we want to give another description of minimal combinatorial patterns.

2.7. Notation and Remark. A) For $u, v \in \mathbb{Z} \cup \{-\infty, \infty\}$ we write $]u, v[:= \{n \in \mathbb{Z} \mid u < n < v\}$. Now, fix some $w \in \mathbb{N}$ and set

$$\mathcal{S}_w := \{(r_1, \ldots, r_w) \in \mathbb{Z}^w \mid r_1 < r_2 < \cdots < r_w\}.$$

Choose $\underline{r} = (r_1, \ldots, r_w) \in \mathcal{S}_w$. We set $r_0 := -\infty$, $r_{w+1} := \infty$ and define the following set

$$P\langle r_1, \ldots, r_w \rangle := P\langle \underline{r} \rangle := \bigcup_{i=0}^{w} \Big(\{i\} \times] - r_{i+1} - 1, -r_i - 1[\Big) \subseteq \mathbb{N}_0 \times \mathbb{Z}.$$

B) Observe that $\mathbb{Z} = \bigcup_{0 \leq i \leq w}] - r_{i+1} - 1 + i, r_i - 1 + i[$. So we may define a function $\mu_{\underline{r}} : \mathbb{Z} \longrightarrow \mathbb{N}_0$ by

$$\mu_{\underline{r}}(n) := i, \quad \text{if } n \in] - r_{i+1} - 1 + i, -r_i - 1 + i[.$$

It is easy to verify, that

$$\mu_{\underline{r}} \in \mathbb{M}_w, \quad P[\mu_{\underline{r}}] = P\langle \underline{r} \rangle.$$

In particular, $P\langle \underline{r} \rangle$ is a minimal combinatorial pattern of width w:

$$\forall \underline{r} \in \mathcal{S}_w : P\langle \underline{r} \rangle \in \mathcal{P}_{\min}, \quad w(P\langle \underline{r} \rangle) = w.$$

C) Finally, if $\mu \in \mathbb{M}_w$ with $w \in \mathbb{N}$, we set $r_0 := -\infty$ and define integers $r_1 < r_2 < \cdots < r_w$ recursively by

$$r_i := \begin{cases} r_{i-1} + 1, & \text{if } i - 1 \notin \mu(\mathbb{Z}); \\ -\min\{n \mid \mu(n + i - 1) = i - 1\}, & \text{if } i - 1 \in \mu(\mathbb{Z}). \end{cases}$$

Writing $\underline{r}(\mu) = \underline{r} = (r_1, \ldots, r_w)$ we then have in the notation of part B)

$$\mu = \mu_{\underline{r}(\mu)}, \ (\forall \mu \in \mathbb{M}_w); \quad \underline{r}(\mu_{\underline{r}}) = \underline{r}, \ (\forall \underline{r} \in \mathcal{S}_w).$$

So, we get a bijection

$$\mathcal{S}_w \overset{\approx}{\longrightarrow} \mathbb{M}_w, \quad (\underline{r} \mapsto \mu_{\underline{r}});$$

and hence a bijection (see (2.4) A).

$$\mathcal{S}_w \xrightarrow[\approx]{} \{P \in \mathcal{P}_{\min} \mid w(P) = w\}, \quad (\underline{r} \mapsto P\langle \underline{r}\rangle).$$ •

3. Cohomological Patterns

We keep all notations and hypotheses of the introduction and give some further notation.

3.1. **Notation and Remark.** A) Let $x_0 \in X_0 := Spec(R_0)$ correspond to the prime ideal $\mathfrak{p}_0 \subseteq R_0$ of our base ring R_0. We write $\kappa(x_0)$ for the function field $(R_0/\mathfrak{p}_0)_{\mathfrak{p}_0}$ of X_0 at x_0. By $X(x_0)$ we denote the fibre of x_0 under the canonical morphism $X \longrightarrow X_0$, thus $X(x_0) := Proj(\kappa(x_0) \otimes_{R_0} R)$. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, we denote by $\mathcal{F}(x_0)$ the sheaf of $\mathcal{O}_{X(x_0)}$-modules induced by $\mathcal{F}$, thus $\mathcal{F}(x_0) := \kappa(x_0) \otimes_{\mathcal{O}_{X_0}} \mathcal{F}$. So, if M is a graded R-module and if $\widetilde{\bullet}$ is used to denote induced sheaves, we may write

$$\tilde{M}(x_0) = (\kappa(x_0) \otimes_{R_0} M)^\sim.$$

B) If the sheaf $\mathcal{F}$ is coherent, then $\mathcal{F}(x_0)$ is a coherent sheaf of $\mathcal{O}_{X(x_0)}$-modules for each $x_0 \in X_0$. Therefore, for any coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules we may introduce the invariant

$$d(\mathcal{F}) := \sup\{\dim_{X(x_0)}(\mathcal{F}(x_0)) \mid x_0 \in X_0\},$$

where the supremum is formed in $\mathbb{Z} \cup \{\pm\infty\}$ and where $\dim_y(\mathcal{G})$ denotes the dimension of the support of the coherent sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-modules over the noetherian scheme Y. (We use the convention that the Krull dimension of the empty set is $-\infty$.) In these notations we clearly have

$$d(\mathcal{F}) \leq d(\mathcal{O}_X) < \mu < \infty,$$

where μ denotes the minimal number of generators of the R_0-module R_1.

C) For $i \in \mathbb{Z}$ let $\mathcal{R}^i D_{R_+}$ denote the i-th right derived functor of the ideal transform functor

$$D_{R_+} = D_{R_+}(\bullet) = \varinjlim_n \operatorname{Hom}_R((R_+)^n, \bullet)$$

over R with respect to the irrelevant ideal $R_+ := \oplus_{n>0} R_n$ of R. Then, if M is a graded R-module, the modules $\mathcal{R}^i D_{R_+}(M)$ are naturally graded and there are natural isomorphisms of R_0-modules

$$H^i(X, \tilde{M}(n)) \cong \mathcal{R}^i D_{R_+}(M)_n, \quad (\forall i, n \in \mathbb{Z}),$$

where T_n denotes the n-th homogeneous part of a graded R-module T (cf. [B–S, (20.4.4)]). •

Next, we shall give some observations concerning the case in which the base ring R_0 is local.

3.2. **Remark.** A) Assume that the base ring $(R_0, \mathfrak{m}_0)$ is local. Let M be a finitely generated and graded R-module. Then $\dim_R(M/\mathfrak{m}_0 M) \leq 0$ if and only if $M_n/\mathfrak{m}_0 M_n = 0$ for all $n \gg 0$ hence, by Nakayama, if and only if $M_n = 0$ for all $n \gg 0$. If we use Γ_{R_+} to denote R_+-torsion, we thus have for a finitely generated graded R-module M:

$$\dim_R(M/\mathfrak{m}_0 M) \leq 0 \iff M = \Gamma_{R_+}(M) \,.$$

As $\Gamma_{R_+}(M)$ is concentrated in finitely many degrees, the modules

$$M/\mathfrak{m}_0 M \text{ and } (M/\Gamma_{R_+}(M))/\mathfrak{m}_0 \,(M/\Gamma_{R_+}(M))$$

differ only in finitely many degrees so that

$$\dim_R((M/\Gamma_{R_+}(M))/\mathfrak{m}_0(M/\Gamma_{R_+}(M))) = \dim_R(M/\mathfrak{m}_0 M) \,,$$

provided $\dim_R(M/\mathfrak{m}_0 M) > 0$.

B) Keep the hypothesis that $(R_0, \mathfrak{m}_0)$ is local and let $(R'_0, \mathfrak{m}'_0)$ be a faithfully flat noetherian local R_0-algebra. Then $R' := R'_0 \otimes_{R_0} R = \oplus_{n \geq 0} R'_0 \otimes_{R_0} R_n$ is a positively graded homogeneous ring, faithfully flat as an R-algebra and with $R'_+ = R_+ R'$. Let M be a finitely generated and graded R-module. Then $M' := R' \otimes_R M = R'_0 \otimes_{R_0} M = \oplus_{n \in \mathbb{Z}} R'_0 \otimes_{R_0} M_n$ is a finitely generated graded R'-module. Moreover, for each $n \in \mathbb{Z}$ we have an isomorphism of $R'_0/\mathfrak{m}'_0$-vector spaces

$$(M'/\mathfrak{m}'_0 M')_n \cong (R'_0/\mathfrak{m}'_0) \otimes_{R_0/\mathfrak{m}_0} (M/\mathfrak{m}_0 M)_n \,.$$

This shows that the graded $R/\mathfrak{m}_0 R$-module $M/\mathfrak{m}_0 M$ has the same Hilbert function as the graded $R'/\mathfrak{m}'_0 R'$-module $M'/\mathfrak{m}'_0 M'$. In particular, we have

$$\dim_R(M/\mathfrak{m}_0 M) = \dim_{R'}(M'/\mathfrak{m}'_0 M') \,.$$

C) Keep the hypotheses and notations of part B). Then, by the graded flat base change property of ideal transforms and their right derived functors, we get isomorphisms of R'_0-modules (s. [B–S, (15.2.2) (vi)])

$$\mathcal{R}^i D_{R_+}(M)_n \otimes_R R'_0 \cong \mathcal{R}^i D_{R'_+}(M')_n \,; \quad (\forall i, n \in \mathbb{Z}) \,.$$

So, as R'_0 is faithfully flat over R_0, we can say

$$\mathcal{R}^i D_{R_+}(M)_n \neq 0 \iff \mathcal{R}^i D_{R'_+}(M')_n \neq 0 \,; \quad (i, n \in \mathbb{Z}) \,.$$

D) We shall apply what is said in parts B) and C) in the special case where $\mathbf{x}$ is an indeterminate and where R'_0 is the faithfully flat local R_0-algebra $R_0[\mathbf{x}]_{\mathfrak{m}_0 R[\mathbf{x}]}$ with maximal ideal $\mathfrak{m}'_0 = \mathfrak{m}_0 R'_0$. As $R'_0/\mathfrak{m}'_0 \cong (R_0/\mathfrak{m}_0)(\mathbf{x})$ is an infinite field we shall do this in order to be able to assume that $R_0/\mathfrak{m}_0$ is infinite – just by replacing R and M respectively by R' and M' if necessary. $\bullet$

Next we prove two auxiliary results concerning the case where the base ring R_0 is local. Observe that in view of the natural isomorphisms of 3.1 C), statement b) of Lemma 3.3 is nothing else than a module theoretic version of a result of Mumford (cf [Mu, pg 99]).

3.3. Lemma. *Assume that the base ring R_0 is local and let M be a finitely generated and graded R-module. Let $i \in \mathbb{N}_0$, $n \in \mathbb{Z}$.*

a) If $\mathcal{R}^j D_{R_+}(M)_{n-j} = 0$ for all $j \leq i$, then $\mathcal{R}^j D_{R_+}(M)_{m-j} = 0$ for all $j \leq i$ and all $m \leq n$.

b) If $\mathcal{R}^k D_{R_+}(M)_{n-k} = 0$ for all $k \geq i$, then $\mathcal{R}^k D_{R_+}(M)_{m-k} = 0$ for all $k \geq i$ and all $m \geq n$.

Proof Let $\mathfrak{m}_0$ be the maximal ideal of R_0. In view of (3.2) D) we may assume that $R_0/\mathfrak{m}_0$ is infinite. Now, we prove both statements by induction on $d := \dim_R (M/\mathfrak{m}_0 M)$. If $d \leq 0$, we have $M = \Gamma_{R_+}(M)$ (see (3.2) A)) and hence $\mathcal{R}^l D_{R_+}(M) = 0$ for all $l \in \mathbb{Z}$ (cf. [B-S, (2.2.8) (i), (2.2.4) (ii), (2.1.7) (i)]). So, both statements are clear in this case.

So, let $d > 0$. By (3.2) A) and in view of the natural isomorphisms $\mathcal{R}^l D_{R_+}(M) \xrightarrow{\cong} \mathcal{R}^l D_{R_+}(M/\Gamma_{R_+}(M))$ (see [B-S, (2.2.8) (ii), (2.2.4) (ii), (2.1.7) (iii)]) we may replace M by $M/\Gamma_{R_+}(M)$ and hence assume that $Ass_R(M) \cap Var(R_+) = \emptyset$. As $d > 0$, no minimal prime divisor of the annihilator $(\mathfrak{m}_0 M : M)$ of $M/\mathfrak{m}_0 M$ belongs to $Var(R_+)$. As R is homogeneous and as $R_0/\mathfrak{m}_0$ is infinite, we thus find some $x \in R_1$ which avoids all members of $Ass_R(M)$ and all minimal primes of $\mathfrak{m}_0 M_R : M$. Hence, x is M-regular and $\dim((M/xM)/\mathfrak{m}_0(M/xM)) = d-1$. In particular, we get exact sequences

$$
(1) \quad \begin{aligned}
&\mathcal{R}^{l-1} D_{R_+}(M)_t \longrightarrow \mathcal{R}^{l-1} D_{R_+}(M)_{t+1} \longrightarrow \mathcal{R}^{l-1} D_{R_+}(M/xM)_{t+1} \\
&\longrightarrow \mathcal{R}^l D_{R_+}(M)_t \longrightarrow \mathcal{R}^l D_{R_+}(M)_{t+1} \quad \text{for all } l, t \in \mathbb{Z}.
\end{aligned}
$$

To prove a), we assume that $\mathcal{R}^j D_{R_+}(M)_{n-j} = 0$ for all $j \leq i$. If we apply (1) with $t = n - l$ for all $l \leq i$ we see that $\mathcal{R}^{l-1} D_{R_+}(M/xM)_{n-(l-1)} = 0$ for all $l \leq i$. By induction we thus have $\mathcal{R}^{l-1} D_{R_+}(M/xM)_{m-(l-1)} = 0$ for all $l \leq i$ and all $m \leq n$. Another use of (1) with $t = m - l$ for all $m < n$ and all $l \leq i$ gives claim a).

To prove statement b), assume that $\mathcal{R}^k D_{R_+}(M)_{n-k} = 0$ for all $k \geq i$. Then, apply (1) with $t = n - l$ for all $l > i$ in order to see that $\mathcal{R}^{l-1} D_{R_+}(M/xM)_{n-(l-1)} = 0$ for all $l > i$. By induction we thus have $\mathcal{R}^{l-1} D_{R_+}(M/xM)_{m-(l-1)}$ for all $l > i$ and all $m \geq n$. Another use of (1) with $t = m - l$ for all $m > n$ and all $l > i$ now gives our claim. $\qquad\square$

3.4. Lemma. *Assume that the base ring R_0 is local with maximal ideal $\mathfrak{m}_0$. Let M be a finitely generated graded R-module with $M \neq \Gamma_{R_+}(M)$. Then*

a) $D_{R_+}(M) \neq 0$;

b) $\dim_R(M/\mathfrak{m}_0 M) = \max\{i + 1 \mid \mathcal{R}^i D_{R_+}(M) \neq 0\}$.

Proof Statement a) is obvious from the natural exact sequence $0 \to \Gamma_{R_+}(M) \to M \longrightarrow D_{R_+}(M)$. We now set $d := \dim_R(M/\mathfrak{m}_0 M)$ and show first that $\mathcal{R}^i D_{R_+}(M) = 0$ for all $i \geq d$. By (3.2) A), we have $d > 0$. As in the proof of (3.3) we may again assume that $R_0/\mathfrak{m}_0$ is infinite and $\Gamma_{R_+}(M) = 0$. So, again we may choose an M-regular element $x \in R_1$ such that $\dim((M/xM)/\mathfrak{m}_0 (M/xM)) = d - 1$ and consider the sequences (1) of the previous proof. If $d = 1$, (3.2) A) tells us that M/xM is R_+-torsion and hence again that $\mathcal{R}^{i-1} D_{R_+}(M/xM) = 0$ for all $i \geq d$. If $d > 1$ this same equality holds by induction. As $\mathcal{R}^i D_{R_+}(M)_t \cong H^{i+1}_{R_+}(M)_t = 0$ for all $t \gg 0$ (s. [B–S, (12.4.5) (iii), (15.1.5) (ii)]) we may apply the sequences (1) of the previous proof to conclude that $\mathcal{R}^i D_{R_+}(M) = 0$ for all $i \geq d$.

It remains to show that $\mathcal{R}^{d-1} D_{R_+}(M) \neq 0$. If $d = 1$, this follows from statement a). So, let $d > 1$. The graded short exact sequence $0 \longrightarrow \mathfrak{m}_0 M \longrightarrow M \longrightarrow M/\mathfrak{m}_0 M \longrightarrow 0$ gives rise to an exact sequence

$$\mathcal{R}^{d-1} D_{R_+}(M) \longrightarrow \mathcal{R}^{d-1} D_{R_+}(M/\mathfrak{m}_0 M) \longrightarrow \mathcal{R}^d D_{R_+}(\mathfrak{m}_0 M).$$

As $\dim_R(\mathfrak{m}_0 M/\mathfrak{m}_0(\mathfrak{m}_0 M)) \leq \dim_R(M/\mathfrak{m}_0^2 M) = d$ our previously shown vanishing statement gives $\mathcal{R}^d D_{R_+}(\mathfrak{m}_0 M) = 0$. As $\mathcal{R}^{d-1} D_{R_+}(M/\mathfrak{m}_0 M) \cong H^d_{R_+}(M/\mathfrak{m}_0 M) \cong H^d_{(R/m_0 R)_+}(M/\mathfrak{m}_0 M) \neq 0$ (see [B–S, (12.4.5) (iii), (13.1.8), (17.1.10)]), we get our result. $\square$

Now, we are ready to prove the main result of this section.

3.5. **Proposition.** *Let $\mathcal{F} \neq 0$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then, the cohomological pattern $P(\mathcal{F})$ of $\mathcal{F}$ is a combinatorial pattern with $w(P(\mathcal{F})) = d(\mathcal{F})$.*

Proof There is a finitely generated and graded R-module M such that $\mathcal{F} = \tilde{M}$. We may assume that $\Gamma_{R_+}(M) = 0$. Let $\mathfrak{p}_0 \subset R_0$ be a prime ideal. Then $R_{\mathfrak{p}_0} = \oplus_{n \geq 0}(R_n)_{\mathfrak{p}_0}$ is a positively graded homogeneous noetherian ring with local base ring $(R_0)_{\mathfrak{p}_0}$, and $M_{\mathfrak{p}_0} = \oplus_{n \in \mathbb{Z}}(M_n)_{\mathfrak{p}_0}$ is a finitely generated and graded $R_{\mathfrak{p}_0}$-module with $\Gamma_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0}) = \Gamma_{R_+}(M)_{\mathfrak{p}_0} = 0$.

Moreover, in view of the natural isomorphisms mentioned in (3.1) C) and in view of the base change isomorphisms of [B-S, (12.4.5) (iii), (15.2.2) (v) and (vi)], we get isomorphisms of $(R_0)_{\mathfrak{p}_0}$-modules

$$(2) \qquad H^i(X, \mathcal{F}(n))_{\mathfrak{p}_0} \cong \mathcal{R}^i D_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n, \quad (\forall i, n \in \mathbb{Z}).$$

Moreover, if $M_{\mathfrak{p}_0} \neq 0$ we have $M_{\mathfrak{p}_0} \neq \Gamma_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})$ and so the first equivalence of (3.2) A) gives $\dim_{R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}/\mathfrak{p}_0 M_{\mathfrak{p}_0}) > 0$. So, if $x_0 \in X_0 = Spec(R_0)$

corresponds to $\mathfrak{p}_0$, the last observation made in (3.1) A) allows to say

$$(3) \qquad M_{\mathfrak{p}_0} \neq 0 \Longrightarrow \dim_{X(x_0)}(\mathcal{F}(x_0)) = \dim_{R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}/\mathfrak{p}_0 M_{\mathfrak{p}_0}) - 1 \,.$$

As $\mathcal{F} \neq 0$, there is some essential graded prime $\mathfrak{p} \subset R$ such that the homogeneous localization $M_{(\mathfrak{p})}$ of M at $\mathfrak{p}$ does not vanish and hence $M_{\mathfrak{p} \cap R_0} \neq 0$. In view of (3) we thus get $d(\mathcal{F}) \geq 0$. We set $w = d(\mathcal{F})$.

Choose $\mathfrak{p}_0$ such that $\dim_{R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}/\mathfrak{p}_0 M_{\mathfrak{p}_0}) = w + 1$. If we apply (3.4) to the graded ring $R_{\mathfrak{p}_0}$ and the graded R_{p_0}-module $M_{\mathfrak{p}_0}$ we find two integers m, n with $\mathcal{R}^0 D_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_m \neq 0$ and $\mathcal{R}^w D_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n \neq 0$. In view of (2) we thus have $(0, m), (w, n) \in P(\mathcal{F})$. So, $P(\mathcal{F})$ satisfies axiom (π_1).

Next, let $(i, n) \in P(\mathcal{F})$. In view of (2) we then find some prime $\mathfrak{p}_0 \subset R_0$ with $\mathcal{R}^i D_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n \neq 0$. By (3.4) b) and (3) we now may conclude $i \leq \dim_{R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}/\mathfrak{p}_0 M_{\mathfrak{p}_0}) - 1 \leq w$. So, axiom (π_2) holds for the set $P(\mathcal{F})$.

Again, let $(i, n) \in P(\mathcal{F})$ and choose $\mathfrak{p}_0 \in Spec(R_0)$ such that $\mathcal{R}^i D_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n \neq 0$. If we apply (3.3) a) to the graded ring $R_{\mathfrak{p}_0}$ and the graded $R_{\mathfrak{p}_0}$-module $M_{\mathfrak{p}_0}$ with $n+i+1$ instead of n, we must have some $j \leq i$ with $\mathcal{R}^j D_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_{n+i+1-j} \neq 0$. By (2) it follows $(j, n + i - j + 1) \in P(\mathcal{F})$. So $P(\mathcal{F})$ satisfies axiom (π_3).

The validity of axiom (π_4) is shown similarly on use of (3.3) b) instead of (3.3) a).

Finally, axiom (π_5) is the wellknown fact that $H^i(X, \mathcal{F}(n)) = 0$ for all $i > 0$ and all $n \gg 0$ (cf. [B–S, (20.4.6)]).

$\square$

4. Tame Cohomoligical Patterns

We keep all the previous notation and hypotheses. We first introduce some notion.

4.1. **Definition and Remark.** A) A graded R-module $T = \oplus_{n \in \mathbb{Z}} T_n$ is said to be *asymptotically gap free* if

$$\#\{n \in \mathbb{Z}_{\leq 0} \mid T_n \neq 0, \quad T_{n+1} = 0\} < \infty \,.$$

B) Clearly all noetherian and all artinian graded R-modules are asymptotically gap free. $\bullet$

4.2. **Lemma.** *Let the base ring R_0 be semilocal and of dimension ≤ 1. Let $i \in \mathbb{N}_0$. Then, the module $H^i_{R_+}(M)$ is asymptotically gap free.*

Proof Assume first that $\dim(R_0) = 0$, so that $\dim(R/R_+) = 0$. Then $H^i_{R_+}(M)$ is artinian (cf. [B–S, (7.1.4)]) and our claim follows from (4.1) B).

So, let $\dim(R_0) = 1$ and let $\mathfrak{m}_{01}, \ldots, \mathfrak{m}_{0r}$ be the maximal ideals of R_0. Let $\mathfrak{p}_{01}, \ldots, \mathfrak{p}_{0s}$ be the non-maximal primes of R_0. Assume first, that the set

$$\mathbb{M} := \{n \in \mathbb{Z}_{\leq 0} \mid supp_{R_0}(H^i_{R_+}(M)_n) \nsubseteq \{\mathfrak{m}_{01}, \ldots, \mathfrak{m}_{0r}\}\}$$

is infinite. Then, there is some index $j \in \{0, \ldots, s\}$ such that $(H^i_{R_+}(M)_n)_{\mathfrak{p}_{0j}} \neq 0$ for infinitely many values $n \leq 0$. Now in view of the graded flat base change property of local cohomology we have natural isomorphisms of $(R_0)_{\mathfrak{p}_{0j}}$-modules (s. [B–S, (15.2.2) (iv)])

$$(4) \qquad\qquad (H^i_{R_+}(M)_n)_{\mathfrak{p}_{0j}} \cong H^i_{(R_{\mathfrak{p}_{0j}})_+}(M_{\mathfrak{p}_{0j}})_n, \quad (\forall n \in \mathbb{Z}) \,.$$

These isomorphisms show that $H^i_{(R_{\mathfrak{p}_{0j}})_+}(M_{\mathfrak{p}_{0j}})_n \neq 0$ for infinitely many values $n \leq 0$. As the base $(R_0)_{\mathfrak{p}_{0j}}$ ring of the positively graded homogeneous noetherian ring $R_{\mathfrak{p}_{0j}}$ is artinian, the previous argument shows that $H^i_{(R_{\mathfrak{p}_{0j}})_+}(M_{\mathfrak{p}_{0j}})_n \neq 0$ for all $n \ll 0$. Another use of (4) shows that $(H^i_{R_+}(M)_n)_{\mathfrak{p}_{0j}} \neq 0$ for all $n \ll 0$ and hence that $H^i_{R_+}(M)_n \neq 0$ for all $n \ll 0$. So $H^i_{R_+}(M)$ is asymptotically gap free.

Assume now, that $\mathbb{M}$ is finite. Choose $x \in \bigcap_{j=1}^r \mathfrak{m}_{0j} \setminus \bigcup_{l=1}^s \mathfrak{p}_{0l}$. Then, the standard graded exact sequence.

$$H^{i-1}_{R_+}(M)_x \longrightarrow H^i_{(R_+,x)}(M) \longrightarrow H^i_{R_+}(M) \xrightarrow{\eta_x} H^i_{R_+}(M)_x$$

in which η_x denotes the natural homomorphism (cf. [B–S, (13.1.12)]) gives rise to a graded epimorphism $H^i_{(R_+,x)}(M) \longrightarrow \Gamma_{(x)}(H^i R_+(M)) \longrightarrow 0$. As x avoids all minimal primes of R_0,

$$\dim(R/(R_+,x)) \cong \dim(R_0/xR_0) = 0 \,,$$

so that $H^i_{(R_+,x)}(M)$ is artinian, (s. [B–S, (7.1.4)]). Therefore $\Gamma_{(x)}(H^i_{R_+}(M))$ is artinian, too. So $\Gamma_{(x)}(H^i_{R_+}(M))$ is asymptotically gap free (s. (4.1) B)). As $\mathbb{M}$ is finite and as x is contained in all maximal ideals $\mathfrak{m}_{01}, \ldots, \mathfrak{m}_{0r}$ of R_0, $H^i_{R_+}(M)_n$ is an xR_0-torsion R_0-module for all $n \ll 0$. Therefore,

$$H^i_{R_+}(M)_n = \Gamma_{xR_0}(H^i_{R_+}(M)_n) = \Gamma_{(x)}(H^i_{R_+}(M))_n \text{ for all } n \ll 0.$$

This shows that $H^i_{R_+}(M)$ is asymptotically gap free. $\square$

4.3. Theorem. $\mathcal{F}$ *Assume that the base ring R_0 is semilocal and of dimension ≤ 1. Then, the cohomological pattern $P(\mathcal{F})$ of an arbitrary coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is tame.*

Proof In view of (3.5) it suffices to show that $P(\mathcal{F})$ satisfies the axiom (π_6). We write $\mathcal{F} = \tilde{M}$, where M is an appropriately choosen finitely generated graded R-module. In view of the natural isomorphism $H^i(X, \mathcal{F}(n)) \cong \mathcal{R}^i D_{R_+}(M)_n$ of (3.1) C) it thus suffices to show, that the graded R_+-modules $\mathcal{R}^i D_{R_+}(M)$ are asymptotically gap free for all $i \in \mathbb{N}_0$. As M is finitely generated, this follows

by (4.2) from the graded exact sequence $M \longrightarrow D_{R_+}(M) \longrightarrow H^1_{R_+}(M) \longrightarrow 0$ ([B-S, (12.4.2)]) and the natural graded isomorphisms $\mathcal{R}^i D_{R_+}(M) \cong H^{i+1}_{R_+}(M)$ for all $i \in \mathbb{N}$ (s. [B-S, (12.4.5) (iii)]). $\qquad\square$

Next we show that – under certain restrictions on X – each tame cohomological pattern can be realized as the cohomological pattern of a coherent sheaf of $\mathcal{O}_X$-modules. First, we give some preliminaries.

4.4. Remark. A) If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, then clearly $P(\mathcal{F}(t)) = P(\mathcal{F})(t)$ for all $t \in \mathbb{Z}$.

Moreover, if $\mathcal{G}$ is a second coherent sheaf of $\mathcal{O}_X$-modules, then $P(\mathcal{F} \oplus \mathcal{G}) = P(\mathcal{F}) \cup P(\mathcal{G})$.

B) If $R_0 = K$ is a field and if $R = K[\mathbf{x}_0, \ldots, \mathbf{x}_w]$ is a polynomical ring, we get $X = \mathbb{P}^w_K$. Then, for any coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules we have the following equivalence (cf. [B–S, (20.4.22)])

$$(i, n) \notin P(\mathcal{F}), \ \forall i < w, \ \forall n \ll 0 \Longleftrightarrow \mathcal{F} \text{ is locally free}.$$

Thus, in particular we can say:

$$P(\mathcal{F}) \in \mathcal{P}_{\min}, \ w(P(\mathcal{F})) = w \Longrightarrow \mathcal{F} \quad \text{is locally free}.$$

Finally, we have the equivalence

$$\left. \begin{array}{l} \{0\} \times \mathbb{Z} \not\subseteq P(\mathcal{F}) \text{ and} \\ (\{1, \ldots, w - 1\} \times \mathbb{Z}) \cap P(\mathcal{F}) = \emptyset \end{array} \right\} \Longleftrightarrow \exists a_1, \ldots, a_r \in \mathbb{Z} : \mathcal{F} = \oplus^r_{i=1} \mathcal{O}_{\mathbb{P}^w_k}(a_i).$$

(Indeed, the left hand side condition is equivalent to the fact that $D_{R_+}(M)$ is a graded free R-module of finite rank.) $\qquad\bullet$

4.5. Proposition. *Let K be a field and let $w \in \mathbb{N}$. Let $P \in \mathcal{P}_{\min}$ with $w(P) = w$. Then, there is an indecomposable locally free sheaf $\mathcal{E}$ of $\mathcal{O}_{\mathbb{P}^w_K}$-modules with rank $(\mathcal{E}) \leq w!$ and such that $P(\mathcal{E}) = P$.*

Proof (cf. [B-Ma-Mi, Constr. 8, Rem. 5.4]). By (2.7) there are uniquely determined integers $r_1 < r_2 < \cdots < r_w$ such that $P = P\langle r_1, r_2, \ldots, r_w \rangle$. Now, let $\mathbf{x}_i, \mathbf{y}_i$ $(i = 1, \ldots, w)$ be indeterminates and consider the Segre product ring (all tensor products are to be taken over K),

$$R := \sigma^w_{i=1} K[\mathbf{x}_i, \mathbf{y}_i] = \oplus_{n \geq 0} \otimes^w_{i=1} K[\mathbf{x}_i, \mathbf{y}_i]_n,$$

which may be identified naturally with the positively graded homogeneous noetherian domain

$$K\Big[\mathop{\pi}_{\substack{j \in \mathbb{M} \\ j \in \mathbb{W} \setminus \mathbb{M}}} \mathbf{x}_i \mathbf{y}_j \mid \mathbb{M} \subseteq \mathbb{W}\Big] \subseteq K\big[\mathbf{x}_1, \ldots, \mathbf{x}_w, \mathbf{y}_1, \ldots, \mathbf{y}_w\big],$$

where $\mathbb{W} := \{1, \ldots, w\}$.

Next, consider the finitely generated and graded R-module

$$M := \overset{w}{\underset{i=1}{\sigma}} (K[\mathbf{x}_i, \mathbf{y}_i](r_i)) = \underset{n \geq 0}{\oplus} \overset{w}{\underset{i=1}{\otimes}} K[\mathbf{x}_i, \mathbf{y}_i]_{n+r_i} \, ,$$

e.g. the Segre product of the modules $K[\mathbf{x}_i, \mathbf{y}_i](r_i)$. Then by the Künneth-relations (cf. [St–V, (2.10)], [F]) we get

$$\mathcal{R}^j D_{R_+}(M)_n \cong \underset{j_1+\cdots+j_w=j}{\oplus} \overset{w}{\underset{i=1}{\otimes}} \mathcal{R}^{j_i} D_{K[\mathbf{x}_i, \mathbf{y}_i]_+}(K[\mathbf{x}_i, \mathbf{y}_i](r_i))_n \, .$$

As, for each $i \in \mathbb{W}$,

$$\mathcal{R}^l D_{K[\mathbf{x}_i, \mathbf{y}_i]_+}(K[\mathbf{x}_i, \mathbf{y}_i](r_i)) = 0 \text{ for all } l > 1 \, ,$$
$$\mathcal{R}^0 D_{K[\mathbf{x}_i, \mathbf{y}_i]_+}(K[\mathbf{x}_i, \mathbf{y}_i](r_i))_n \neq 0 \Longleftrightarrow n \geq -r_i \, ,$$
$$\mathcal{R}^1 D_{K[\mathbf{x}_i, \mathbf{y}_i]_+}(K[\mathbf{x}_i, \mathbf{y}_i](r_i))_n \neq 0 \Longleftrightarrow n < -r_i - 1 \, ,$$

and as $r_1 < r_2 < \cdots < r_w$, it follows readily

(5) $$\mathcal{R}^j D_{R_+}(M)_n \neq 0 \Longleftrightarrow (j, n) \in P\langle r_1, \ldots, r_w \rangle = P$$

As $\dim(R) = w + 1$, we find elements $\mathbf{z}_0, \ldots, \mathbf{z}_w \in R_1$ such that R is a finite integral extension of its subring $S := K[\mathbf{z}_0, \ldots, \mathbf{z}_w]$ and such that $\mathbf{z}_0, \ldots, \mathbf{z}_w$ are algebraically independent over K. As $R_+ = \sqrt{S_+}$, by graded base ring independence (s. [B–S, (13.1.6)]) we thus get isomorphisms of graded S-modules $\mathcal{R}^j D_{S_+}(M) \cong \mathcal{R}^j D_{R_+}(M)$ for all $j \in \mathbb{N}_0$. If we apply what is said in (3.1) C) to S and the finitely generated graded S-module M, and keep in mind that $Proj(S) = \mathbb{P}^w_K$, the above statement (5) gives $P(\mathcal{F}) = P$, where $\mathcal{F} := \tilde{M}$ is the coherent sheaf of $\mathcal{O}_{\mathbb{P}^w_k}$-modules induced by M. Now, by the second observation made in (4.2) B), $\mathcal{F}$ is locally free, hence a vector bundle.

It is easy to see that the Hilbert polynomial of the R-module M has degree w and leading coefficient 1. So, the same holds for M as a graded S-module. Therefore, $\mathcal{F}$ is of rank $w!$. Finally, by the second observation made in 4.4 A), $\mathcal{F}$ must have an indecomposable factor $\mathcal{E}$ with $P(\mathcal{E}) = P$. $\square$

4.6. Corollary. *Let K be a field and let $d \in \mathbb{N}$. Let P be a tame combinatorial pattern with $w(P) \leq d$. Then, there is a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}^d_K}$-modules with $P(\mathcal{F}) = P$.*

Proof By (2.5) there are finitely many minimal patterns $P_1, \ldots, P_r \in \mathcal{P}_{\min}$ such that $P = P_1 \cup \cdots \cup P_r$. Let $w_i := w(P_i)$. Then, clearly $w_i \leq d$. So, for each $i \in \{1, \ldots, r\}$ there is a subspace $\mathbb{P}^{w_i}_K \subseteq \mathbb{P}^d_K$. Let $\alpha_i : \mathbb{P}^{w_i}_K \longrightarrow \mathbb{P}^d_K$ be the inclusion morphism. If $w_i = 0$, let $\mathcal{F}_i = \mathcal{O}_{\mathbb{P}^{w_i}_K}$. If $w_i > 0$ use (4.5) to find a locally free sheaf $\mathcal{F}_i$ of $\mathcal{O}_{\mathbb{P}^{w_i}_K}$-modules such that $P(\mathcal{F}_i) = P_i$. As $H^j(\mathbb{P}^{w_i}_K, \mathcal{F}_i(n)) \cong H^j(\mathbb{P}^d_K, \alpha_{i*}\mathcal{F}(n))$ for all $j, n \in \mathbb{Z}$ and all $i \in \{1, \ldots, r\}$, and as $P(\mathbb{P}^0_K, \mathcal{O}_{\mathbb{P}^0_K}) = \{0\} \times \mathbb{Z}$ we get $P(\mathcal{F}) = P$, if we set $\mathcal{F} := \oplus_{i=1}^r \alpha_{i*}\mathcal{F}_i$. $\square$

4.7. Corollary. *Assume that one of the special fibers of the natural morphism $X \longrightarrow X_0$ is a projective space of dimension d and let $P \in \mathcal{P}'$ be a tame combinatorial pattern with $w(P) \leq d$. Then, there is a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ such that $P(\mathcal{F}) = P$.*

Proof By our hypothesis there is a closed point $x_0 \in X_0$ and a closed immersion $\alpha : \mathbb{P}^d_{\kappa(x_0)} \longrightarrow X$. By (4.5) there is a coherent sheaf of $\mathcal{O}_{\mathbb{P}^d_{\kappa(x_0)}}$-modules $\mathcal{G}$ with $P(\mathcal{G}) = P$. It suffices to choose $\mathcal{F} := \alpha_* \mathcal{G}$. $\qquad\qquad\square$

4.8. Theorem. *Assume that R_0 is a noetherian semilocal ring of dimension ≤ 1 and let $d \in \mathbb{N}_0$. Then, the cohomological patterns of coherent sheaves of $\mathcal{O}_{\mathbb{P}^d_{R_0}}$-modules are precisely the tame combinatorial patterns of width $\leq d$.*

Proof Easy by (4.3) and (4.7). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.9. Remark and Problems. A) Let K be a field and let $w \in \mathbb{N}$. Let $P \in \mathcal{P}_{\min}$ with $w(P) = w$. By (4.5) there is an indecomposable vector bundle of rank $\leq w!$ whose cohomological pattern equals P. Obviously, the rank estimate in this statement is very rough. So, we are lead to ask the following question:

What is the least value of rank $(\mathcal{E})$ *if $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_{\mathbb{P}^w_K}$-modules with $P(\mathcal{E}) = P$?*

B) We fix two integers u, v with $u + w \geq v$. Then $P := (\{w\} \times \mathbb{Z}_{\leq u}) \cup (\{0\} \times \mathbb{Z}_{\geq v})$ is a non-minimal tame cohomological pattern of width w and we have

$$P = P(\mathbb{P}^w_K, \mathcal{O}_{\mathbb{P}^w_K}(v) \oplus \mathcal{O}_{\mathbb{P}^w_K}(u + w + 1)).$$

Moreover, by the last observation made in (4.4) B), P cannot be realized by an indecomposable coherent sheaf. So, there are tame combinatorial patterns, which cannot be realized by an indecomposable coherent sheaf of $\mathcal{O}_{\mathbb{P}^w_K}$-modules. This leads to the question:

Is there a purely combinatorial characterization of those patterns, which are realized by locally free and indecomposable sheaves of $\mathcal{O}_{\mathbb{P}^w_K}$-modules? $\qquad\bullet$

5. The Tameness Problem

We keep the previous notations and hypothesis. In (4.3) we have seen that the cohomological pattern $P(\mathcal{F})$ of an arbitrary coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is tame, provided that the base ring R_0 is semilocal and of dimension at most one. It appears to us, that the restriction on the dimension of R_0 is unnecessary to guarantee the tameness of $P(\mathcal{F})$. So, we like to pose the following problem.

5.1. Problem. *Assume that the base ring R_0 is semilocal. Is $P(\mathcal{F})$ tame for any coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$?*

5.2. Remark. A) In fact, we do not know of any example at all, in which $P(\mathcal{F})$ is not tame. But in view of the example presented in [Si] it might be

too daring to ask the question in (5.1) in case of an arbitrary noetherian base ring R_0.

B) Clearly, in view of the isomorphisms (3.1) C) the above *tameness problem* is equivalent to ask whether the local cohomology modules $H^i_{R_+}(M)$ are asymptotically gap free for all $i \in \mathbb{N}_0$ and all finitely generated graded R-modules M.

C) For each finitely generated graded R-module M and each $i \in \mathbb{N}_0$ we may consider the following statements

(i) $Ass_{R_0}(H^i_{R_+}(M)_n)$ is *asymptotically stable* for $n \longrightarrow -\infty$ (e.g. there is an $n_0 \in \mathbb{Z}$ such that $Ass_{R_+}(H^i_{R_+}(M)_n) = Ass_{R_0}(H^i_{R_+}(M)_{n_0})$ for all $n \leq n_0$);

(ii) $Ass_{R_0}(H^i_{R_+}(M)_n)$ is *asymptotically increasing* for $n \longrightarrow -\infty$ (e.g there is an $n_0 \in \mathbb{Z}$ such that $Ass_{R_0}(H^i_{R_+}(M)_n) \subseteq Ass_{R_0}(H^i_{R_+}(M)_{n+1})$ for all $n \leq n_0$);

(iii) $H^i_{R_+}(M)$ *is asymptotically gap free.*

Then obviously, we have the following implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). •

5.3. **Notation and Remark.** A) Let M be a finitely generated and graded R-module. Then, we may consider the *finiteness dimension of M relative to R_+:*

$$f_{R_+}(M) := \inf\{i \in \mathbb{N} \mid H^i_{R_+}(M) \text{ is not finitely generated}\}.$$

B) As the R_0-modules $H^i_{R_+}(M)_n$ are finitely generated for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ and vanish for all $n \gg 0$ (s. [B–S, (15.1.5)]) we can write

$$f_{R_+}(M) = \inf\{i \in \mathbb{N}_0 \mid \#\{n \leq 0 \mid H^i_{R_+}(M)_n \neq 0\} = \infty\} =$$
$$= \inf\{i \in \mathbb{N}_0 \mid \#\{n \in \mathbb{Z} \mid H^i_{R_+}(M)_n \neq 0\} = \infty\}.$$

•

5.4. **Lemma.** *Assume that the base ring R_0 is local. Let M be a finitely generated and graded R-module and let $i \in \mathbb{N}_0$ be such that the R-module $H^j_{R_+}(M)$ is finitely generated for all $j < i$. Then $Ass_{R_0}(H^i_{R_+}(M)_n)$ is asymptotically stable for $n \longrightarrow -\infty$.*

Proof Let $\mathfrak{m}_0$ be the maximal ideal of R_0. Let $R'_0 := R_0[\mathbf{x}]_{\mathfrak{m}_0 R[\mathbf{x}]}$, where $\mathbf{x}$ is an indeterminate. Then, in the notation of (3.2) B), the faithful R_0-flatness of R'_0 gives rise to isomorphisms of R'_0-modules

$$H^k_{R'_+}(M')_n \cong H^k_{R_+}(M)_n \otimes_{R_0} R'_0 \text{ for all } k \in \mathbb{N}_0 \text{ and all } n \in \mathbb{Z}$$

(s. [B–S, (15.2.2) (iv)]). These show that the R_0-module $H^j_{R'_+}(M')$ is finitely generated for all $j < i$ (cf. (5.3) B)) and that $Ass_{R_0}(H^i_{R_+}(M)_n) = \{\mathfrak{p}'_0 \cap R_0 \mid \mathfrak{p}'_0 \in Ass_{R'_0}(H^i_{R'_+}(M')_n)\}$ for all $n \in \mathbb{Z}$ (cf. [M, (23.2) (ii)]). This allows to

replace R and M respectively by R' and M' and hence to assume that $R_0/\mathfrak{m}_0$ is infinite.

Now, we prove our claim by induction on i. The case $i = 0$ is clear as $H^0_{R_+}(M)_n = 0$ for all $n \ll 0$. So, let $i > 0$. In view of the natural graded isomorphisms $H^k_{R_+}(M) \cong H^k_{R_+}(M/\Gamma_{R_+}(M))$ for all $k \in \mathbb{N}$ (s. [B-S, (2.1.7) (iii)]) we may replace M by $M/\Gamma_{R_+}(M)$ and hence assume that $\mathrm{grade}_M(R_+) \neq 0$. So, there is some element $x \in R_1$ which is M-regular (cf. [B-S, (15.1.4)]). If we apply cohomology to the short exact sequence $0 \longrightarrow M(-1) \overset{x}{\longrightarrow} M \longrightarrow M/xM \longrightarrow 0$, we get graded exact sequences

$$(6) \qquad H^{k-1}_{R_+}(M) \longrightarrow H^{k-1}_{R_+}(M/xM) \longrightarrow H^k_{R_+}(M)(-1) \longrightarrow H^k_{R_+}(M) \,.$$

These sequences first show that $H^{j-1}_{R_+}(M/xM)$ is finitely generated for all $j < i$. So, by induction, there is some $n_1 \in \mathbb{Z}$ such that $Ass_{R_0}(H^{i-1}_{R_+}(M/xM)_n) = Ass_{R_0}(H^{i-1}_{R_+}(M/xM)_{n_1}) =: \mathcal{A}$ for all $n \leq n_1$. Moreover, there is some $n_2 < n_1$ such that $H^{i-1}_{R_+}(M)_{n+1} = 0$ for all $n \leq n_2$. So, for each $n \leq n_2$ we have an exact sequence of R_0-modules

$$0 \longrightarrow H^{i-1}_{R_+}(M/xM)_{n+1} \longrightarrow H^i_{R_+}(M)_n \longrightarrow H^i_{R_+}(M)_{n+1}$$

induced by (6). This shows that

$$\mathcal{A} \subseteq Ass_{R_0}(H^i_{R_+}(M)_n) \subseteq \mathcal{A} \cup Ass_{R_0}(H^i_{R_+}(M)_{n+1}) \text{ for all } n \leq n_2$$

and hence proves that

$$Ass_{R_0}(H^i_{R_+}(M)_n) \subseteq Ass_{R_0}(H^i_{R_+}(M)_{n+1}) \text{ for all } n < n_2 \,.$$

This proves our claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.5. **Remark.** A) Let M be a finitely generated and graded R-module and let $i \in \mathbb{N}_0$. As $H^i_{R_+}(M)$ is an R_+-torsion module, we have $Ass_R(H^i_{R_+}(M)) \subseteq Var(R_+)$ and this gives rise to a natural bijection

$$Ass_R(H^i_{R_+}(M)) \overset{\approx}{\longrightarrow} \bigcup_{n \in \mathbb{Z}} Ass_{R_0}(H^i_{R_+}(M)_n); \ (\mathfrak{p} \mapsto \mathfrak{p} \cap R_0) \,.$$

B) By the above observation $Ass_R(H^i_{R_+}(M))$ is finite if $Ass_{R_0}(H^i_{R_+}(M)_n)$ is asymptotically stable for $n \longrightarrow -\infty$. We do not know, whether the converse of this implication holds in general. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\bullet$

5.6. **Proposition.** *Let M be a finitely generated and graded R-module and let $f := f_{R_+}(M) \in \mathbb{N}$. Then $Ass_{R_0}(H^f_{R_+}(M)_n)$ is asymptotically stable for $n \longrightarrow -\infty$.*

Proof According to [B-L], the set $Ass_R(H^f_{R_+}(M))$ is finite. So, by (5.5) A) the set $\mathcal{S} = \bigcup_{n \in \mathbb{Z}} Ass_{R_0}(H^f_{R_+}(M)_n)$ is finite. Let $\mathfrak{p}_0 \in \mathcal{S}$.

Then, by the graded flat base change property the $R_{\mathfrak{p}_0}$-module $H^j_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})$ is finitely generated for all $j < f$. Moreover, for each $n \in \mathbb{Z}$ we have

$$\mathfrak{p}_0 \in Ass_{R_0}(H^f_{R_+}(M)_n) \iff \mathfrak{p}_0(R_0)_{\mathfrak{p}_0} \in Ass_{(R_0)_{\mathfrak{p}_0}}(H^f_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})_n) \,.$$

So, if we apply (5.4) to each of the $R_{\mathfrak{p}_0}$-modules $M_{\mathfrak{p}_0}$, we get our claim. □

5.7. **Example and Remark.** A) Let $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ be indeterminates, let $R_0 = \mathbb{Z}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ and consider the positively graded homogeneous R_0-algebra $R = R_0[\mathbf{u}, \mathbf{v}, \mathbf{w}]/(\mathbf{ux} + \mathbf{vy} + \mathbf{wz})$ (furnished with the standard grading which assigns the degree 1 to the images of the three generators $\mathbf{u}, \mathbf{v}, \mathbf{w}$). Then, by [Si], we know that $\#\{\mathfrak{p} \cap \mathbb{Z} \mid \mathfrak{p} \in Ass_R(H^3_{R_+}(R))\} = \infty$. So, by (5.5) B) we see that $Ass_{R_0}(H^3_{R_+}(R)_n)$ is not asymptotically stable for $n \longrightarrow -\infty$.

B) Let R be as in part A) and let $\mathfrak{p} \in Spec(R) \setminus Var(\mathbf{x}R)$. Then $\mathbf{x}$ is invertible in $R_{\mathfrak{p}}$ so that $(R_+)_{\mathfrak{p}} = (\mathbf{u}, \mathbf{v}, \mathbf{w}) \, R_{\mathfrak{p}} = (\mathbf{v}, \mathbf{w})R_{\mathfrak{p}}$ and hence

$$H^3_{R_+}(R)_{\mathfrak{p}} \cong H^3_{(R_+)_{\mathfrak{p}}}(R_{\mathfrak{p}}) = H^3_{(\mathbf{v},\mathbf{w})R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0 \,.$$

Therefore, $\mathfrak{p} \notin Ass_R(H^3_{R_+}(R))$. If $\mathfrak{p} \notin Var(\mathbf{y}R)$ or $p \notin Var(\mathbf{z}R)$, the same conclusion holds. Therefore, we have $Ass_R(H^3_{R_+}(R)) \subseteq Var((\mathbf{x}, \mathbf{y}, \mathbf{z})R)$. If we write $\mathfrak{q}_0 := (\mathbf{x}, \mathbf{y}, \mathbf{z})R_0$ we thus obtain (cf. (5.5) A))

$$Ass_R(H^3_{R_+}(R)) \subseteq \{p\mathbb{Z} + \mathfrak{q}_0 + R_+ \mid p = 0 \text{ or } p \in \mathbb{N} \text{ prime } \} \,;$$

$$\bigcup_{n \in \mathbb{Z}} Ass_{R_0}(H^3_{R_+}(R)_n) \subseteq \{p\mathbb{Z} + \mathfrak{q}_0 \mid p = 0 \text{ or } p \in \mathbb{N} \text{ prime } \} \,.$$

C) Let $S = R_0[\mathbf{u}, \mathbf{v}, \mathbf{w}]$ be graded such that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ have degree 1. In view of the graded isomorphism

$$H^3_{S_+}(S) \cong R_0[\mathbf{u}^-, \mathbf{v}^-, \mathbf{w}^-] = \oplus_{n \le -3} \oplus_{\substack{\alpha,\beta,\gamma<0 \\ \alpha+\beta+\gamma=n}} \mathbf{u}^\alpha \mathbf{v}^\beta \mathbf{w}^\gamma R_0$$

(cf. [B–S, (12.4.1)]) and in view of the graded exact sequence

$$0 \longrightarrow H^2_{R_+}(R) \longrightarrow H^3_{S_+}(S)(-1) \xrightarrow{\mathbf{ux+vy+wz}} H^3_{S_+}(S) \longrightarrow H^3_{R_+}(R) \longrightarrow 0 \,,$$

it is easy to see that $H^2_{R_+}(R)_n = 0$ for all $n > -2$ and that $Ass_{R_0}(H^2_{R_+}(R)_n) = \{0\}$ for all $n \le -2$. The same sequences also show (after localization at $\mathbb{Z} \setminus \{0\}$ and on use of the observation of part B)) that $H^3_{R_+}(R)_n = 0$ for all $n > -3$ and that $\mathfrak{q}_0$ is the unique minimal member of $Ass_{R_0}(H^3_{R_+}(R)_n)$ for all $n \le -3$.

●

5.8. **Remark.** A) If our base ring R_0 is local, the problem whether $H^i_{R_+}(M)$ is asymptotically gap free for a finitely generated graded R-module M as well as the related problems of the asymptotic increasing (resp. stability) of the sets $Ass_{R_0}(H^i_{R_+}(M)_n)$ for $n \longrightarrow -\infty$ (cf. (5.2) C)) or the finiteness of the sets $Ass_R(H^i_{R_+}(M))$ (s. (5.5) B)) are all still open. Even in the special case

where $R = M$ is a Cohen-Macaulay ring we have not been able to answer any of these questions.

B) The state of the art concerning the problems mentioned in part A) is similar to the situation for the question whether the sets $Ass_R(H^i_{\mathfrak{a}}(R))$ are finite for an arbitrary ideal $\mathfrak{a}$ of a local CM-ring R (cf. [Hu]). The methods and results of [He] may be used to reduce the problems mentioned in A) to more specific cases. Notably, it would be interesting to solve the problems in the case where the irrelevant ideal R_+ has at most three generators. ●

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