FROM LOCAL TO GLOBAL DEFORMATION QUANTIZATION OF POISSON MANIFOLDS

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Dedicated to James Stasheff on the occasion of his 65th birthday

ABSTRACT. We give an explicit construction of a deformation quantization of the algebra of functions on a Poisson manifolds, based on Kontsevich's local formula. The deformed algebra of functions is realized as the algebra of horizontal sections of a vector bundle with flat connection.

1. INTRODUCTION

Let M be a paracompact smooth d-dimensional manifold. The Lie bracket of vector fields extends to a bracket, the Schouten–Nijenhuis bracket, on the graded commutative algebra $\Gamma(M, \wedge^{\cdot} TM)$ of multivector fields so that:

$$\begin{aligned}[\,\alpha_1 \wedge \alpha_2, \alpha_3] &= \alpha_1 \wedge [\,\alpha_2, \alpha_3] + (-1)^{m_2(m_3-1)}[\,\alpha_1, \alpha_3] \wedge \alpha_2, \\ [\,\alpha_1, \alpha_2] &= -(-1)^{(m_1-1)(m_2-1)}[\,\alpha_2, \alpha_1],\end{aligned}$$

if $\alpha_i \in \Gamma(M, \wedge^{m_i} TM)$. This bracket defines a graded super Lie algebra structure on $\Gamma(M, \wedge^{\cdot} TM)$, with the shifted grading $\deg'(\alpha) = m - 1$, $\alpha \in \Gamma(M, \wedge^m TM)$.

A Poisson structure on M is a bivector field $\alpha \in \Gamma(M, \wedge^2 TM)$ obeying $[\alpha, \alpha] = 0$. This identity for α, which we can regard as a bilinear form on the cotangent bundle, implies that $\{f, g\} = \alpha(df, dg)$ is a Poisson bracket on the algebra $C^\infty(M)$ of smooth real-valued function. If such a bivector field is given, we say that M is a Poisson manifold.

Following [2], we introduce the notion of (deformation) quantization of the algebra of functions on a Poisson manifold.

Definition. A *quantization* of the algebra of smooth functions $C^\infty(M)$ on the Poisson manifold M is a topological algebra A over the ring of formal power series $\mathbb{R}[[\epsilon]]$ in a formal variable ϵ with product $\star$, together with an $\mathbb{R}$-algebra isomorphism $A/\epsilon A \to C^\infty(M)$, so that

(i) A is isomorphic to $C^\infty(M)[[\epsilon]]$ as a topological $\mathbb{R}[[\epsilon]]$-module.

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(ii) There is an $\mathbb{R}$-linear section $a \mapsto \tilde{a}$ of the projection $A \to C^\infty(M)$ so that $\tilde{f} \star \tilde{g} = \widetilde{fg} + \sum_{j=1}^\infty \epsilon^j \widetilde{P_j(f,g)}$ for some bidifferential operators $P_j : C^\infty(M)^2 \to C^\infty(M)$ with $P_j(f,1) = P_j(1,g) = 0$ and $P_1(f,g) - P_1(g,f) = 2\alpha(df, dg)$.

If we fix a section as in (ii), we obtain a *star product* on $C^\infty(M)$, i.e. a formal series $P_\epsilon = \epsilon P_1 + \epsilon^2 P_2 + \cdots$ whose coefficients P_j are bidifferential operators $C^\infty(M)^2 \to C^\infty(M)$ so that $f \star_M g := fg + P_\epsilon(f,g)$ extends to an associative $\mathbb{R}[[\epsilon]]$-bilinear product on $C^\infty(M)[[\epsilon]]$ with unit $1 \in C^\infty(M)$ and such that $f \star_M g - g \star_M f = 2\epsilon\alpha(df, dg) \mod \epsilon^2$.

Remark. One can replace (i) by the equivalent condition that A is a Hausdorff, complete, ϵ-torsion free $\mathbb{R}[[\epsilon]]$-module, see [4], [8] and Appendix A.

M. Kontsevich gave in [9] a quantization in the case of $M = \mathbb{R}^d$, in the form of an explicit formula for a star product, as a special case of his formality theorem for the Hochschild complex of multidifferential operators. This theorem is extended in [9] to general manifolds by abstract arguments, yielding in principle a star product for general Poisson manifolds.

In this paper we give a more direct construction of a quantization, based on the realization of the deformed algebra of functions as the algebra of horizontal sections of a bundle of algebras. It is similar in spirit to Fedosov's deformation quantization of symplectic manifolds [5]. It has the advantage of giving in principle an explicit construction of a star product on any Poisson manifold.

We turn to the description of our results.

We construct two vector bundles with flat connection on the Poisson manifold M. The second bundle should be thought of as a quantum version of the first.

The first bundle E_0 is a bundle of Poisson algebras. It is the vector bundle of infinite jets of functions with its canonical flat connection D_0. The fiber over $x \in M$ is the commutative algebra of infinite jets of functions at x. The Poisson structure on M induces a Poisson algebra structure on each fiber, and the canonical map $C^\infty(M) \to E_0$ is a Poisson algebra isomorphism onto the Poisson algebra $H^0(E_0, D_0)$ of D_0-horizontal sections of E_0.

The second bundle E is a bundle of associative algebras over $\mathbb{R}[[\epsilon]]$ and is obtained by quantization of the fibers of E_0. Its construction depends on the choice $x \mapsto \varphi_x$ of an equivalence class of formal coordinate systems $\varphi_x : (\mathbb{R}^d, 0) \to (M, x)$, defined up to the action of $GL(d, \mathbb{R})$, at each point x of M and depending smoothly on x. As a bundle of $\mathbb{R}[[\epsilon]]$-modules, $E \simeq E_0[[\epsilon]]$ is isomorphic to the bundle of formal power series in ϵ whose coefficients are infinite jets of functions. The associative product on the fiber of E over $x \in M$ is defined by applying Kontsevich's star product formula for $\mathbb{R}^d$ with respect to the coordinate system φ_x. Thus the sections of E form an algebra. We say that a connection on a bundle of algebras is compatible if the covariant derivatives are derivations of the algebra of sections. If a connection is compatible then horizontal sections form an algebra. Our first main result is:

Theorem 1.1. *There exists a flat compatible connection $\bar{D} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots$ on E, so that the algebra of horizontal sections $H^0(E, \bar{D})$ is a quantization of $C^\infty(M)$.*

The construction of the connection is done in two steps. First one constructs a deformation D of the connection D_0 in terms of integrals over configuration spaces of the upper half-plane. This connection is compatible with the product as a consequence of Kontsevich's formality theorem on $\mathbb{R}^d$. Moreover the same theorem gives a formula for its curvature, which is the commutator $[F^M, \cdot]_\star$ with some E-valued two-form F^M, and also implies the Bianchi identity $DF^M = 0$. In the second step, we use these facts to show, following Fedosov's method [5], that there is an E-valued one-form γ so that $\bar{D} = D + [\gamma, \cdot]_\star$ is flat. This means that γ is a solution of the equation

$$F^M + \epsilon\omega + D\gamma + \gamma \star \gamma = 0. \tag{1}$$

Here ω is any E-valued two-form such that $D\omega = 0$ and $[\omega, \cdot]_\star = 0$.

To prove that the algebra of horizontal sections is a quantization of $C^\infty(M)$ one constructs a *quantization map*

$$\rho : C^\infty(M) \simeq H^0(E_0, D_0) \to H^0(E, \bar{D}),$$

extending to an isomorphism of topological $\mathbb{R}[[\epsilon]]$-modules $C^\infty(M)[[\epsilon]] \to H^0(E, \bar{D})$. We give two constructions of such a map. In the first construction, ρ is induced by a chain map $(\Omega^\cdot(E_0), D_0) \to (\Omega^\cdot(E), \bar{D})$ between the complexes of differential forms with values in E_0 and E, respectively. In the second construction, ρ is only defined at the level of cohomology, but behaves well with respect to the center.

Theorem 1.2. *Let $Z_0 = \{f \in \mathbb{C}^\infty(M) \,|\, \{f, \cdot\} = 0\}$ be the algebra of Casimir functions and $Z = \{f \in H^0(E, \bar{D}) \,|\, [f, \cdot]_\star = 0\}$ be the center of the algebra $H^0(E, \bar{D})$. Then there exists a quantization map ρ that restricts to an algebra isomorphism $Z_0[[\epsilon]] \to Z$.*

The local version of this theorem is a special case of the theorem on compatibility of the cup product on the tangent cohomology [9]. This global version is based on two further special cases of the formality theorem for $\mathbb{R}^d$.

By using the second quantization map ρ, we may represent the central two-form ω as $\rho(\omega_0)$, where ω_0 is a D_0-closed E_0-valued two-form which is Poisson central in the sense that $\{\omega_0, \cdot\} = 0$. A further advantage of this quantization map is that it allows us to define a map from Hamiltonian vector fields to inner derivations of the global star product.

Our construction depends on the choice of a class of local coordinate systems $\varphi^{\mathrm{aff}} = ([\varphi_x])_{x\in M}$, a Poisson central D_0-closed two-form ω_0 and a solution γ of (1). It turns out that different choices (at least within a homotopy class) lead to isomorphic algebra bundles with flat connection (and in particular to isomorphic algebras of horizontal sections) if the central two-forms are in the same cohomology class in the subcomplex of $(\Omega^{\cdot}(E_0), D_0)$ formed by Poisson central differential forms. Thus, up to isomorphism, our construction depends only on the cohomology class of the Poisson central two-form. This will be the subject of a separate publication.

Also, the action of an extension of the Lie algebra of Poisson vector fields on the deformed algebra and a discussion of special cases, such as the case of a divergence-free Poisson bivector field [6] and the symplectic case will be presented elsewhere.

Our construction is also inspired by the quantum field theoretical description [3] of deformation quantization. In that approach, the quantization is defined by a path integral of a topological sigma model which should be well-defined for any Poisson manifold. The star product is obtained by a perturbation expansion in Planck's constant which requires to consider Taylor expansions at points of M. This suggests that a global version of the star product should be constructed in terms of a deformation of the bundle of infinite jets of functions. The deformation of the transition functions can be expressed in terms of Ward identities for the currents associated to infinitesimal diffeomorphisms [10]. As shown in [3], Ward identities correspond to identities of Kontsevich's formality theorem.

The organization of this paper is as follows. In Section 2 we recall the main notions of formal geometry, which we use to patch together objects defined locally. Section 3 is a short description of Kontsevich's formality theorem on $\mathbb{R}^d$. We formulate four special cases of this theorem, which are the ingredients of our construction. We then describe the quantization using the theory of compatible connections on bundles of algebras in Section 4, by adapting a construction of Fedosov [5] to our situation. In particular, we give a proof of Theorem 1.1. We study the relation between Casimir sections of E_0 and central sections of E, and give a proof of Theorem 1.2 in Section 5. The notion of topological $\mathbb{R}[[\epsilon]]$-module, appearing in the definition of quantization, is reviewed in Appendix A. In Appendix B, we prove some (well-known) cohomology vanishing results, by giving a canonical homotopy, similar to Fedosov's in the symplectic case. In particular we give a representation of cocycles as coboundaries, giving in principle an algorithm to compute star products of functions.

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2. FORMAL GEOMETRY

Formal geometry [7], [1] provides a convenient language to describe the global behavior of objects defined locally in terms of coordinates. The idea is to consider the "space of all local coordinate systems" on M with its transitive action of the Lie algebra of formal vector fields. More precisely, let M^{coor} be the manifold of jets of coordinates systems on M. A point in M^{coor} is an infinite jet at zero of local diffeomorphisms $[\varphi] : U \subset \mathbb{R}^d \to M$ defined on some open neighborhood U of $0 \in \mathbb{R}^d$. Two such maps define the same infinite jet iff their Taylor expansions at zero (for any choice of local coordinates on M) coincide. We have a projection $\pi : M^{\mathrm{coor}} \to M$ sending $[\varphi]$ to $\varphi(0)$. The group G_0 of formal coordinate transformations of $\mathbb{R}^d$ preserving the origin acts freely and transitively on the fibers. The tangent space to M^{coor} at a point $[\varphi]$ may be identified with the Lie algebra

$$\mathcal{W} = \left\{ \sum_{j=1}^{d} v_j \frac{\partial}{\partial y^j} \,\middle|\, v_j \in \mathbb{R}[[y^1, \dots, y^d]] \right\},$$

of vector fields on the formal neighborhood of the origin in $\mathbb{R}^d$: if $\xi \in T_{[\varphi]}M^{\mathrm{coor}}$ and $[\varphi_t]$ is a path in M^{coor} with tangent vector ξ at $t = 0$, then

$$\hat{\xi}(y) = \text{Taylor expansion at } 0 \text{ of } \left. -(d\varphi)(y)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t(y) \right|_{t=0}$$

is a vector field in $\mathcal{W}$ which only depends on the infinite jet of φ_t. We will often omit the bracket in $[\varphi]$ for simplicity when no confusion arises. The map $\omega_{\mathrm{MC}}(\varphi) \colon \xi \mapsto \hat{\xi}$ is in fact an isomorphism from the tangent space at φ of M^{coor} to $\mathcal{W}$ and defines the $\mathcal{W}$-valued *Maurer–Cartan form* $\omega_{\mathrm{MC}} \in \Omega^1(M^{\mathrm{coor}}, \mathcal{W})$ on M^{coor}. Its inverse defines a Lie algebra homomorphism $\mathcal{W} \mapsto \{\text{vector fields on } M^{\mathrm{coor}}\}$, which means that $\mathcal{W}$ acts on M^{coor}, and is equivalent to the fact that ω_{MC} obeys the *Maurer–Cartan equation*

$$d\omega_{\mathrm{MC}} + \frac{1}{2}[\omega_{\mathrm{MC}}, \omega_{\mathrm{MC}}] = 0, \tag{2}$$

where the bracket is the Lie bracket in $\mathcal{W}$ and the wedge product of differential forms. Moreover, ω_{MC} is $\mathcal{W}$-equivariant:

$$\mathcal{L}_{\hat{\xi}} \omega_{\mathrm{MC}} = \mathrm{ad}_{\xi} \omega_{\mathrm{MC}}, \qquad \xi \in \mathcal{W} \tag{3}$$

The action of $\mathcal{W}$, restricted to the subalgebra $\mathcal{W}_0$ of vector fields vanishing at the origin, can be integrated to an action of G_0. In particular, the subgroup $\mathrm{GL}(d, \mathbb{R})$ of linear diffeomorphisms in G_0 acts on M^{coor} and we set $M^{\mathrm{aff}} = M^{\mathrm{coor}}/\mathrm{GL}(d, \mathbb{R})$. We will need the fact that the fibers of the bundle $M^{\mathrm{aff}} \to$

M are contractible so that there exist sections $\varphi^{\text{aff}} : M \to M^{\text{aff}}$. Over M^{coor} we have the trivial vector bundle $M^{\text{coor}} \times \mathbb{R}[[y^1, \dots, y^d]]$. It carries a canonical flat connection, $d+\omega_{\text{MC}}$, which has the property that its horizontal sections are precisely the Taylor expansions of smooth functions on M: if $f \in C^\infty(M)$, then $\varphi \mapsto$ (Taylor expansion at zero of $f \circ \varphi$) is a horizontal section and all horizontal sections are obtained in this way.

Since the Maurer–Cartan form is $\mathrm{GL}(d, \mathbb{R})$-equivariant, the canonical connection induces a connection on the vector bundle $\tilde{E}_0 = M^{\text{coor}} \times_{\mathrm{GL}(d,\mathbb{R})} \mathbb{R}[[y^1, \dots, y^d]]$ over M^{aff}, as will be seen in detail in Lemma 4.1 below. Let $\varphi^{\text{aff}} : M \to M^{\text{aff}}$ be a section of the fiber bundle $M^{\text{aff}} \to M$. Then $E_0 = \varphi^{\text{aff}}\tilde{E}_0$ is a vector bundle over M, with fiber $\mathbb{R}[[y^1, \dots, y^d]]$: a point in the fiber of E_0 over x is a $\mathrm{GL}(d, \mathbb{R})$-orbit of pairs (φ, f) where φ is a representative of the class $\varphi^{\text{aff}}(x)$ and $f \in \mathbb{R}[[y^1, \dots, y^d]]$. The action of $g \in \mathrm{GL}(d, \mathbb{R})$ id $(\varphi, f) \mapsto (\varphi \circ g, f \circ g)$. The pull-back of the canonical connection is a flat connection D_0 on E_0.

This vector bundle has also a description independent of the choice of section which we turn to describe. Let $J(M)$ be the vector bundle of infinite jets of functions on M: the fiber over $x \in M$ consists of equivalence classes of smooth functions defined on open neighborhoods of x, where two functions are equivalent iff they have the same Taylor series at x (with respect to any coordinate system). It is easy to see that the map $J(M) \to E_0$ sending the jet p at x to $(\varphi$, Taylor expansion at 0 of $(p \circ \varphi))$, $\varphi \in \varphi^{\text{aff}}(x)$ is an isomorphism. The pull-back of the connection induces a canonical connection on $J(M)$ which is independent of the choice of φ^{aff}.

3. THE KONTSEVICH STAR PRODUCT AND FORMALITY THEOREM ON $\mathbb{R}^d$

Let $\alpha = \sum \alpha^{ij}(y)\frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}$ be a Poisson structure on $\mathbb{R}^d$. The Kontsevich star product of two functions f, g on $\mathbb{R}^d$ is given by a series $f \star g = fg + \sum_{j=1}^\infty \frac{\epsilon^j}{j!} U_j(\alpha, \dots, \alpha) f \otimes g$. The operator $U_j(\alpha_1, \dots, \alpha_j)$ is a multilinear symmetric function of j arguments $\alpha_k \in \Gamma(\mathbb{R}^d, \wedge^2 T\mathbb{R}^d)$, taking values in the space of bidifferential operator $C^\infty(\mathbb{R}^d) \otimes C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$. In fact $U_j(\alpha_1, \dots, \alpha_j)$ is defined more generally as a multilinear graded symmetric function of j multivector fields $\alpha_k \in \Gamma(\mathbb{R}^d, \wedge^{m_k} T\mathbb{R}^d)$, with values in the multidifferential operators $C^\infty(\mathbb{R}^d)^{\otimes r} \to C^\infty(\mathbb{R}^d)$, where $r = \sum_k m_k - 2j + 2$. The maps U_j are $\mathrm{GL}(d, \mathbb{R})$-equivariant and obey a sequence of quadratic relations (amounting to the fact that they are Taylor coefficients of an L_∞ morphism) of which the associativity of the star product is a special case.

Let $S_{\ell, n-\ell}$ be the subset of the group S_n of permutations of n letters consisting of permutations such that $\sigma(1) < \cdots < \sigma(\ell)$ and $\sigma(\ell + 1) < \cdots <$

$\sigma(n)$. For $\sigma \in S_{\ell,n-\ell}$ let

$$\varepsilon(\sigma) = (-1)^{\sum_{r=1}^{\ell} m_{\sigma(r)}(\sum_{s=1}^{\sigma(r)-1} m_s - \sum_{s=1}^{r-1} m_{\sigma(s)})}.$$

The formality theorem for $\mathbb{R}^d$ is (with the signs computed in [3]):

Theorem 3.1 (Kontsevich [9]). *Let $\alpha_j \in \Gamma(\mathbb{R}^d, \wedge^{m_j} T\mathbb{R}^d)$, $j = 1, \dots, n$ be multivector fields. Let $\varepsilon_{ij} = (-1)^{(m_1+\dots+m_{i-1})m_i + (m_1+\dots+m_{i-1}+m_{i+1}+\dots+m_{j-1})m_j}$.*

Then, for any functions $f_0, \dots, f_m$,

$$\begin{aligned}\sum_{\ell=0}^{n}\sum_{k=-1}^{m}\sum_{i=0}^{m-k}(-1)^{k(i+1)+m}\sum_{\sigma\in S_{\ell,n-\ell}}\varepsilon(\sigma)U_\ell(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(\ell)})(f_0\otimes\cdots\otimes f_{i-1}\\ \otimes U_{n-\ell}(\alpha_{\sigma(\ell+1)},\dots,\alpha_{\sigma(n)})(f_i\otimes\cdots\otimes f_{i+k})\otimes f_{i+k+1}\otimes\cdots\otimes f_m)\\ = \sum_{i<j}\varepsilon_{ij}U_{n-1}([\,\alpha_i,\alpha_j],\alpha_1,\dots,\widehat{\alpha}_i,\dots,\widehat{\alpha}_j,\dots,\alpha_n)(f_0\otimes\cdots\otimes f_m).\end{aligned}$$

Here $[\ ,\ ]$ denotes the Schouten–Nijenhuis bracket and a caret denotes omission.

Of this theorem we will need some special cases, namely the cases involving vector fields and a Poisson bivector field.

Let $\alpha \in \Gamma(\mathbb{R}^d, \wedge^2 T\mathbb{R}^d)$ be a Poisson bivector field and ξ, η be vector fields. Let us introduce the formal series

$$\begin{aligned} P(\alpha) &= \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} U_j(\alpha, \dots, \alpha) \\ A(\xi, \alpha) &= \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} U_{j+1}(\xi, \alpha, \dots, \alpha) \\ F(\xi, \eta, \alpha) &= \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} U_{j+2}(\xi, \eta, \alpha, \dots, \alpha). \end{aligned}$$

The coefficients of the series P, A, F are, respectively, bidifferential operators, differential operators and functions. They obey the relations of the formality theorem. To spell out these relations it is useful to introduce the Lie algebra cohomology differential.

Definition. A *local polynomial map* from $\Gamma(\mathbb{R}^d, \wedge^2 T\mathbb{R}^d)$ to the space of multidifferential operators on $\mathbb{R}^d$, is a map $\alpha \mapsto U(\alpha) \in \oplus_{r=0}^{\infty} C^\infty(\mathbb{R}^d) \otimes \mathbb{R}[\partial/\partial y^1, \dots, \partial/\partial y^d]^{\otimes r}$, so that the coefficients of $U(\alpha)$ at $y \in \mathbb{R}^d$ are polynomials in the partial derivatives of the coordinates $\alpha^{ij}(y)$ of α at y. We denote by $\mathfrak{U}$ the space of these local polynomial maps.

The Lie algebra W of vector fields on $\mathbb{R}^d$ acts on $\mathfrak{U}$ and we can form the Lie algebra cohomology complex $C^\cdot(W, \mathfrak{U}) = \mathrm{Hom}_{\mathbb{R}}(\wedge^\cdot W, \mathfrak{U})$. An element of $C^k(W,\mathfrak{U})$ sends $\xi_1\wedge\cdots\wedge\xi_k$, for any vector fields ξ_j, to a multidifferential operator $S(\xi_1,\dots,\xi_k,\alpha)$ depending polynomially on α. Then $P\in C^0(W,\mathfrak{U})[[\epsilon]]$, $A\in C^1(W,\mathfrak{U})[[\epsilon]]$ and $F\in C^2(W,\mathfrak{U})[[\epsilon]]$. The differential (extended to formal power series by $\mathbb{R}[[\epsilon]]$-linearity) will be denoted by δ. If Φ^t_ξ denotes the flow of the vector field ξ, we have

$$\begin{aligned}\delta S(\xi_1,\dots,\xi_{p+1},\alpha) &= -\sum_{i=1}^{p+1}(-1)^{i-1}\frac{d}{dt}\Big|_{t=0} S(\xi_1,\dots,\hat{\xi}_i,\dots,\xi_{p+1},(\Phi^t_{\xi_i})_*\alpha)\\ &\quad+\sum_{i<j}(-1)^{i+j}S([\xi_i,\xi_j],\xi_1,\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_{p+1},\alpha).\end{aligned}$$

Corollary 3.2.

(i) $P(\alpha)\circ(A(\xi,\alpha)\otimes\mathrm{Id}+\mathrm{Id}\otimes A(\xi,\alpha))-A(\xi,\alpha)\circ P(\alpha)=\delta P(\xi,\alpha)$.
(ii) $P(\alpha)\circ(F(\xi,\eta,\alpha)\otimes\mathrm{Id}-\mathrm{Id}\otimes F(\xi,\eta,\alpha))-A(\xi,\alpha)\circ A(\eta,\alpha)+A(\eta,\alpha)\circ A(\xi,\alpha)=\delta A(\xi,\eta,\alpha)$.
(iii) $-A(\xi,\alpha)\circ F(\eta,\zeta,\alpha)-A(\eta,\alpha)\circ F(\zeta,\xi,\alpha)-A(\zeta,\alpha)\circ F(\xi,\eta,\alpha)=\delta F(\xi,\eta,\zeta,\alpha)$.

These relations can be deduced from Theorem 3.1, by noticing that some terms vanish owing to the Jacobi identity $[\alpha,\alpha]=0$ and that $[\xi,\alpha]$ is the Lie derivative of α in the direction of the vector field ξ.

Remark. The relations, together with the associativity relations $P\circ(P\otimes\mathrm{Id}-\mathrm{Id}\otimes P)=0$ may be written compactly in the Maurer–Cartan form $\delta S+\frac{1}{2}[S,S]=0$, where $S=P+A+F$ and the bracket is composed of the Gerstenhaber bracket on Hochschild cochains, see [9], and the cup product in the Lie algebra cohomology complex.

Remark. Relation (i) gives the behavior of the Kontsevich star product under coordinate transformations: if we do an infinitesimal coordinate transformation, the star product changes to an equivalent product.

We will also need the form of the lowest order terms of P, A, F and their action on $1\in\mathbb{R}[[y^1,\dots,y^d]]$. The following results are essentially contained in [9]. They amount to an explicit calculation of certain integrals over configuration spaces of points in the upper half-plane.

Proposition 3.3.

(i) $P(\alpha)(f\otimes g)=fg+\epsilon\alpha(df,dg)+O(\epsilon^2)$.
(ii) $A(\xi,\alpha)=\xi+O(\epsilon)$, *where we view* ξ *as a first order differential operator.*
(iii) $A(\xi,\alpha)=\xi$, *if* ξ *is a linear vector field.*
(iv) $F(\xi,\eta,\alpha)=O(\epsilon)$
(v) $P(\alpha)(1\otimes f)=P(\alpha)(f\otimes 1)=f$
(vi) $A(\xi,\alpha)1=0$.

Remark. As the coefficients of the multidifferential operators U_j are polynomial functions of the derivatives of the coordinates of the multivector fields, all results in this section continue to hold in the formal context, namely if we replace $C^\infty(\mathbb{R}^d)$ by $\mathbb{R}[[y^1,\dots,y^d]]$ and take the coordinates of the tensors α,ξ,η,ζ also in $\mathbb{R}[[y^1,\dots,y^d]]$.

4. Deformation quantization of Poisson manifolds

4.1. **A deformation of the canonical connection.** Let $\tilde{E}$ be the bundle of $\mathbb{R}[[\epsilon]]$-modules

$$M^{\mathrm{coor}}\times_{\mathrm{GL}(d,\mathbb{R})}\mathbb{R}[[y^1,\dots,y^d]][[\epsilon]]\to M^{\mathrm{aff}},$$

and let φ^{aff} be a section of the projection $p: M^{\mathrm{aff}}\to M$. Such a section is defined by a family $(\varphi_x)_{x\in M}$ of infinite jets at zero of maps $\varphi_x:\mathbb{R}^d\to M$ such that $\varphi_x(0)=x$, defined modulo $\mathrm{GL}(d,\mathbb{R})$ transformations.

Let $E=(\varphi^{\mathrm{aff}})^*\tilde{E}$ be the pull-back bundle. As the Kontsevich product is $\mathrm{GL}(d,\mathbb{R})$-equivariant, it descends to a product, also denoted by $\star$, on $\Gamma(E)$.

Let us describe this product. For simplicity, we suppose that an open covering of M, consisting, say, of contractible sets has been fixed and that representatives φ_x of the $\mathrm{GL}(d,\mathbb{R})$-equivalence classes have been fixed on each open set of the covering. In this way, we may pretend that the bundle $E\to M$ is trivial with fiber $\mathbb{R}[[y^1,\dots,y^d]][[\epsilon]]$. Since all formulae are $\mathrm{GL}(d,\mathbb{R})$-equivariant, all statements will have a global meaning. A section f of E is then locally a map $x\mapsto f_x$, where $f_x=f_x(y)\in\mathbb{R}[[y^1,\dots,y^d]][[\epsilon]]$. The product of two sections f, g of $\Gamma(E)$ is $(f\star g)_x=P(\alpha_x)(f_x\otimes g_x)$, where $\alpha_x=(\varphi_x^{-1})_*\alpha$ is the expression of α in the coordinate system φ_x. Thus

$$(f\star g)_x(y)=f_x(y)g_x(y)+\epsilon\sum_{i,j=1}^d\alpha_x^{ij}(y)\frac{\partial f_x(y)}{\partial y^i}\frac{\partial g_x(y)}{\partial y^j}+\cdots$$

We now introduce a connection $D:\Gamma(E)\to\Omega^1(M)\otimes_{C^\infty(M)}\Gamma(E)$ on $\Gamma(E)$. We first assume that M is contractible and that a section $\varphi: M\to M^{\mathrm{coor}}$ is fixed. We set

$$(Df)_x=d_xf+A_x^Mf,$$

where d_xf is the de Rham differential of f, viewed as a function of $x\in M$ with values in $\mathbb{R}[[y^1,\dots,y^d]][[\epsilon]]$, and, for $\xi\in T_xM$,

$$A_x^M(\xi)=A(\hat{\xi}_x,\alpha_x),\qquad \hat{\xi}_x=\varphi^*\omega_{MC}(\xi).$$

Lemma 4.1. *Let $\varphi,\varphi': M\mapsto M^{\mathrm{coor}}$ be sections of M^{coor} such that $\varphi'_x=\varphi_x\circ g(x)$ for some smooth map $g: M\to\mathrm{GL}(d,\mathbb{R})$, and let D, D' be the corresponding connections. Then $D'(f\circ g)=(Df)\circ g$.*

Proof: Let $f : M \to \mathbb{R}[[y^1, \dots, y^d]]$ be a section and set $f'_x = f_x \circ g(x)$. We have $D' = d_x + A(\varphi'^* \omega_{\mathrm{MC}}(x), ({\varphi'_x}^{-1})_* \alpha)$. Let us choose local coordinates x^i on U. Then the covariant derivative in the direction of $\partial/\partial x^i$ is

$$D'_i f'_x = \frac{\partial}{\partial x^i}(f_x \circ g(x)) + A(\varphi'^* \omega_{\mathrm{MC}} \left(\frac{\partial}{\partial x^i}\right), ({\varphi'_x}^{-1})_* \alpha).$$

By the chain rule, we have, for $x \in U$,

$$\frac{\partial}{\partial x^i}(f_x(g(x)y)) = \frac{\partial f_x}{\partial x^i}(g(x)y) + \theta_i(f_x \circ g(x))(y), \qquad \theta_i(y) = g(x)^{-1} \frac{\partial}{\partial x^i} g(x) y.$$

The vector-valued function $y \mapsto \theta_i(y)$ is viewed here as an element of $\mathcal{W}$. On the other hand,

$$\varphi'^* \omega_{\mathrm{MC}} \left(\frac{\partial}{\partial x^i}\right) = (g(x)^{-1})_* \varphi^* \omega_{\mathrm{MC}} \left(\frac{\partial}{\partial x^i}\right) - \theta_i,$$

as can be seen from the definition of the Maurer–Cartan form. Also $\alpha'_x = ({\varphi'_x}^{-1})_* \alpha = (g(x)^{-1})_* (\varphi_x^{-1})_* \alpha$. Using the $GL(d, \mathbb{R})$-equivariance of A, we then obtain

$$D'_i f'_x = (D_i f_x) \circ g(x) + \theta_i f'_x - A(\theta_i, \alpha'_x) f'_x.$$

The point is that since θ_i is a *linear* vector field, we have $A(\theta_i, \alpha'_x) = \theta_i$, by Prop. 3.3, (iii). □

Let now M be a general manifold. Suppose that a section of $M^{\mathrm{aff}} \to M$ is given. Its restriction to a contractible open set U is an equivalence class of sections $\varphi : U \to U^{\mathrm{coor}}$, $x \mapsto \varphi_x$. Two sections φ, φ' are equivalent if there exists a map $g : U \to GL(d, \mathbb{R})$ such that $\varphi'_x = \varphi_x \circ g(x)$. If we change φ to φ' then the same section f of $\varphi^{\mathrm{aff}} \tilde{E}$ is described by a map $x \mapsto f'_x = f_x \circ g(x)$. The above lemma shows that D is independent of the choice of representatives and therefore induces a globally defined connection, which we also denote by D, on $E = (\varphi^{\mathrm{aff}})^* \tilde{E}$.

Let us extend D to the $\Omega^{\cdot}(M)$-module $\Omega^{\cdot}(E) = \Omega^{\cdot}(M) \otimes_{C^\infty(M)} \Gamma(E)$ by the rule $D(ab) = (d_x a)b + (-1)^p a Db$, $a \in \Omega^p(M)$, $b \in \Omega^{\cdot}(E)$. The wedge product on $\Omega^{\cdot}(E)$ and the star product on the fibers induce a product, still denoted by $\star$, on $\Omega^{\cdot}(E)$.

Proposition 4.2. *Let $F^M \in \Omega^2(E)$ be the E-valued two-form $x \mapsto F^M_x$, with $F^M_x(\xi, \eta) = F(\hat{\xi}_x, \hat{\eta}_x, \alpha_x)$, $\xi, \eta \in T_x M$. Then, for any $f, g \in \Gamma(E)$,*

(i) $D(f \star g) = Df \star g + f \star Dg$
(ii) $D^2 f = F^M \star f - f \star F^M$
(iii) $DF^M = 0$

These identities are obtained by translating the the identities of Corollary 3.2, using the following fact:

Lemma 4.3. *Let* $\varphi : M \mapsto M^{\mathrm{coor}}$ *be a section of* M^{coor} *and denote by* $\mathcal{D}$ *the vector space of formal multidifferential operators on* $\mathbb{R}^d$*. The map* $(\mathrm{Hom}(\wedge^{\cdot}\mathcal{W}, \mathfrak{U}), \delta) \to (\Omega^{\cdot}(M, \mathcal{D}), d_{\text{de Rham}})$, $\sigma \mapsto \sigma^M$ *with*

$$\sigma_x^M(\xi_1, \dots, \xi_p) = \sigma(\varphi^*\omega_{\mathrm{MC}}(\xi_1), \dots, \varphi^*\omega_{\mathrm{MC}}(\xi_p), (\varphi_x^{-1})_*\alpha),$$

is a homomorphism of complexes.

Proof: Suppose that σ is a homogeneous polynomial of degree k in α. Then there exists a $C^\infty(M)$-multilinear graded symmetric, multidifferential operator-valued function S of p vector fields and k bivector fields such that

$$\sigma(\eta_1, \dots, \eta_p, \alpha) = S(\eta_1, \dots, \eta_p, \alpha, \dots, \alpha).$$

Let us work locally and introduce coordinates $x^1, \dots, x^d$. Let $\psi_j = \varphi^*\omega_{\mathrm{MC}}(\partial/\partial x^j)$. The Maurer–Cartan equation (2) is then

$$\frac{\partial}{\partial x^\mu}\psi_\nu - \frac{\partial}{\partial x^\nu}\psi_\mu + [\psi_\mu, \psi_\nu] = 0.$$

With the abbreviation $\alpha_x = (\varphi^{-1})_*\alpha$, we then have

$$\begin{aligned}
&d_{\text{de Rham}}\sigma_x^M\left(\frac{\partial}{\partial x^{\mu_1}}, \dots, \frac{\partial}{\partial x^{\mu_{p+1}}}\right) \\
&= \sum_{j=1}^{p+1}(-1)^{j-1}\frac{\partial}{\partial x^{\mu_j}}\sigma_x^M\left(\frac{\partial}{\partial x^{\mu_1}}, \dots, \widehat{\frac{\partial}{\partial x^{\mu_j}}}, \dots, \frac{\partial}{\partial x^{\mu_{p+1}}}\right) \\
&= \sum_{i\neq j=1}^{p+1}(-1)^{j-1}S(\psi_{\mu_1}, \dots, \frac{\partial}{\partial x^{\mu_j}}\psi_{\mu_i}, \dots, \widehat{\psi_{\mu_j}}, \dots, \psi_{\mu_{p+1}}, \alpha_x, \dots, \alpha_x) \\
&\quad + \sum_{j=1}^{p+1}(-1)^{j-1}\sum_{l=1}^{k}S(\psi_{\mu_1}, \dots, \widehat{\psi_{\mu_j}}, \dots, \psi_{\mu_{p+1}}, \alpha_x, \dots, \frac{\partial}{\partial x^{\mu_j}}\alpha_x, \dots, \alpha_x).
\end{aligned}$$

The claim follows by using the Maurer–Cartan equation and the relation

$$\frac{\partial}{\partial x^\mu}\alpha_x + [\psi_\mu, \alpha_x] = 0.$$

which is an expression of the fact that α_x is the Taylor expansion of a globally defined tensor. □

By the property (i), the space of horizontal sections $\mathrm{Ker}\, D$ is an algebra. However D has curvature, so we need to modify it in such a way as to kill the curvature, still preserving (i). This can be done by a method similar to the one adopted by Fedosov [5], which we turn to describe in a slightly more general setting. We will come back to our case in Subsection 4.3.

4.2. **Connections on bundles of algebras.** If $E \to M$ is a bundle of associative algebras over the ring $R = \mathbb{R}[[\epsilon]]$ or $R = \mathbb{R}$, then the space of sections $\Gamma(E)$ with fiberwise multiplication is also an associative algebra over R and a module over $C^\infty(M)$. The product of sections is denoted by $\star$, and we also consider the commutator $[\,a, b]_\star = a \star b - b \star a$ of sections. Let $D : \Gamma(E) \to \Omega^1(M) \otimes_{C^\infty(M)} \Gamma(E)$ be a connection on E, i.e., a linear map obeying $D(fa) = df \otimes a + fDa$, $f \in C^\infty(M)$, $a \in \Gamma(E)$. Extend D to the $\Omega^\cdot(M)$-module $\Omega^\cdot(E) = \Omega^\cdot(M) \otimes_{C^\infty(M)} \Gamma(E)$ in such a way that $D(\beta a) = (d\beta)a + (-1)^p \beta Da$ if $\beta \in \Omega^p(M)$, $a \in \Omega^\cdot(E)$. The space $\Omega^\cdot(E)$ with product $(\beta \otimes a) \star (\gamma \otimes b) = (\beta \wedge \gamma) \otimes (a \star b)$ is a graded algebra. We say that D is a compatible connection if $D(a \star b) = Da \star b + a \star Db$ for all $a, b \in \Gamma(E)$. A connection D is compatible iff its extension on $\Omega^\cdot(E)$ is a (super) derivation of degree 1, i.e.,

$$D(a \star b) = Da \star b + (-1)^{\deg(a)} a \star Db, \qquad a, b \in \Omega^\cdot(E).$$

If this holds, then the curvature D^2 is a $C^\infty(M)$-linear derivation of the algebra $\Omega^\cdot(E)$.

Definition. A *Fedosov connection* D with *Weyl curvature* $F \in \Omega^2(E)$ is a compatible connection on a bundle of associative algebras such that $D^2 a = [\,F, a]_\star$ and $DF = 0$.

Note that the Weyl curvature of a Fedosov connection is not uniquely determined by the connection: Weyl curvatures corresponding to the same connection differ by a two-form with values in the center.

Proposition 4.4. *If D is a Fedosov connection on E and $\gamma \in \Omega^1(E)$ then $D + [\,\gamma, \cdot]_\star$ is a Fedosov connection with curvature*

$$F + D\gamma + \gamma \star \gamma$$

Proof: Let $\bar{D} = D + [\gamma, \cdot]_\star$. If $a \in \Gamma(E)$,

$$\begin{aligned}
\bar{D}^2 a &= [\,F, a]_\star + D[\,\gamma, a]_\star + [\,\gamma, D(a)]_\star + [\,\gamma, [\,\gamma, a]_\star]_\star \\
&= [\,F, a]_\star + [\,D\gamma, a]_\star + [\,\gamma, [\,\gamma, a]_\star]_\star \\
&= [\,F + D\gamma + \frac{1}{2}[\,\gamma, \gamma]_\star, a]_\star.
\end{aligned}$$

In the last step we use the Jacobi identity. Now,

$$\begin{aligned}
\bar{D}(F + D\gamma + \frac{1}{2}[\,\gamma, \gamma]_\star) &= D^2\gamma + \frac{1}{2}[\,D\gamma, \gamma]_\star - \frac{1}{2}[\,\gamma, D\gamma]_\star + [\,\gamma, F + D\gamma]_\star \\
&= [\,F, \gamma]_\star + [\,\gamma, F]_\star = 0.
\end{aligned}$$

The term $[\,\gamma, [\,\gamma, \gamma]_\star]_\star$ vanishes by the Jacobi identity. $\square$.

Definition. A Fedosov connection is *flat* if $D^2 = 0$.

If D is a flat Fedosov connection, we may define cohomology groups $H^j(E,D) = \mathrm{Ker}(D : \Omega^j(E) \to \Omega^{j+1}(E))/\mathrm{Im}(D : \Omega^{j-1}(E) \to \Omega^j(E))$.

If E_0 is a vector bundle over M, let $E_0[[\epsilon]]$ be the associated bundle of $\mathbb{R}[[\epsilon]]$-modules. Sections of $E_0[[\epsilon]]$ are formal power series in ϵ whose coefficients are sections of E_0. Let us suppose that $E = E_0[[\epsilon]]$ as a bundle of $\mathbb{R}[[\epsilon]]$-modules, and that D is a Fedosov connection on E. Then we have expansions

$$D = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots, \qquad F = F_0 + \epsilon F_1 + \epsilon^2 F_2 + \cdots$$

where D_0 is a Fedosov connection on the bundle of $\mathbb{R}$-algebras E_0 with Weyl curvature F_0.

Lemma 4.5. *Suppose that $F_0 = 0$ and that $H^2(E_0, D_0) = 0$. Then there exists a $\gamma \in \epsilon\Omega^1(E)$ such that $D + [\,\gamma, \cdot\,]_\star$ has zero Weyl curvature.*

Proof: By Prop. 4.4, we need to solve the equation $F + D\gamma + \gamma \star \gamma = 0$ for $\gamma \in \epsilon\Omega^1(E)$. If $\gamma = 0$ this equation holds modulo ϵ. Assume by induction that $\gamma^{(k)} = \epsilon\gamma_1 + \cdots + \epsilon^k\gamma_k$ obeys

$$\bar{F}^{(k)} := F + D\gamma^{(k)} + \gamma^{(k)} \star \gamma^{(k)} = 0 \mod \epsilon^{k+1}$$

Then, for any choice of $\gamma_{k+1} \in \Omega^1(E)$, $\bar{F}^{(k+1)} = \bar{F}^{(k)} + \epsilon^{k+1} D_0\gamma_{k+1} \mod \epsilon^{k+2}$. By Prop. 4.4, $D\bar{F}^{(k)} + [\,\gamma^{(k)}, \bar{F}^{(k)}]_\star = 0$. Since $\bar{F}^{(k)} = 0 \mod \epsilon^{k+1}$, we then have $D_0\bar{F}^{(k)} = 0 \mod \epsilon^{k+2}$. Since the second cohomology is trivial, we can choose γ_{k+1} so that $D_0\gamma_{k+1} = -\epsilon^{-k-1}\bar{F}^{(k)}|_{\epsilon=0}$, and we get $\bar{F}^{(k+1)} = 0 \mod \epsilon^{k+2}$. The induction step is proved, and $\gamma = \sum_{j=1}^{\infty} \epsilon^j\gamma_j$ has the required properties. □

If D_0 is a flat connection on E_0 then the differential forms with values in the vector bundle $\mathrm{End}(E_0)$ of fiber endomorphisms form a differential graded algebra $\Omega^{\cdot}(\mathrm{End}(E_0))$ acting on $\Omega^{\cdot}(E_0)$. The differential is the super commutator $D_0(\Phi) = D_0 \circ \Phi - (-1)^p \Phi \circ D_0$, $\Phi \in \Omega^p(\mathrm{End}(E_0))$.

If $D = D_0 + \epsilon D_1 + \cdots$ is a connection on $E = E_0[[\epsilon]]$ then clearly $D_j \in \Omega^1(\mathrm{End}(E_0))$ for $j \geq 1$.

Lemma 4.6. *Suppose that $D = D_0 + \epsilon D_1 + \cdots$ is a flat Fedosov connection on $E = E_0[[\epsilon]]$ and that $H^1(\mathrm{End}(E_0), D_0) = 0$. Then there exists a formal series $\rho = \mathrm{Id} + \epsilon\rho_1 + \epsilon^2\rho_2 + \cdots$, with coefficients $\rho_i \in \Omega^0(\mathrm{End}(E_0))$ which induces an isomorphism of topological $\mathbb{R}[[\epsilon]]$-modules $H^0(E_0, D_0)[[\epsilon]] \to H^0(E, D)$. If B is an algebra (not necessarily with unit) subbundle of $\mathrm{End}(E_0)$ so that (i) $\Omega^{\cdot}(B)$ is a subcomplex of $\Omega^{\cdot}(\mathrm{End}(E_0))$, (ii) $D_j \in \Omega^1(B)$, $j \geq 1$, (iii) $H^1(B, D_0) = 0$, then the ρ_j may chosen in $\Omega^0(B)$.*

Proof: The proof is very similar to the proof of the previous lemma. We construct recursively a solution $\rho = \mathrm{Id} + \epsilon\rho_1 + \cdots \in \Omega^0(B)[[\epsilon]]$ of the equation

$$D \circ \rho - \rho \circ D_0 = 0. \tag{4}$$

Since the series ρ starts with the identity map, it is then automatically invertible as a power series with coefficients in $\Omega^0(B)$ and the claim follows.

Equation (4) is clearly satisfied modulo ϵ. Let us assume by induction that $\rho^{(k)} = \mathrm{Id} + \epsilon\rho_1 + \cdots + \epsilon^k\rho_k$ solves the equation modulo ϵ^{k+1}. The next term ρ_{k+1} must obey $\Phi^{(k)} + \epsilon^{k+1}D_0(\rho_{k+1}) \equiv 0 \mod \epsilon^{k+2}$, where $\Phi^{(k)} = D\circ\rho^{(k)} - \rho^{(k)}\circ D_0 \equiv 0 \mod \epsilon^{k+1}$. Since D and D_0 are flat, we have $D\circ\Phi^{(k)} + \Phi^{(k)}\circ D_0 = 0$. It follows that $D_0(\Phi^{(k)}) = D_0\circ\Phi^{(k)} + \Phi^{(k)}\circ D_0 \equiv 0 \mod \epsilon^{k+2}$. It then follows from the vanishing of $H^1(B, D_0)$ that such a ρ_{k+1} exists. $\square$

4.3. **Deformation quantization.** Let us return to our problem. Fix a section $\varphi^{\mathrm{aff}} : M \to M^{\mathrm{aff}}$ and let $E = (\varphi^{\mathrm{aff}})^*\tilde{E}$, as above. Let $D = D_0 + \epsilon D_1 + \cdots$ be the deformed canonical connection on E defined in 4.1.

Lemma 4.7. *For any $p > 0$, and any section of M^{aff}, $H^p(E_0, D_0) = 0$.*

This result is standard, but we give a proof below in Appendix B, which also gives an algorithm to represent canonically cycles as coboundaries.

By Prop. 4.2, D is a Fedosov connection with Weyl curvature F^M. By Prop. 3.3, (iv), its constant term vanishes. If we add to F^M a term $\epsilon\omega$ with $\omega \in \Omega^2(E)$ such that $D\omega = 0$ and $[\omega, \cdot]_\star = 0$, then we still get a Weyl curvature for D. We can thus apply Lemma 4.5 to find a solution $\gamma \in \epsilon\Omega^1(E)$ of (1). In particular, $\bar{D} = D + [\,\gamma, \cdot]_\star$ is flat. Then $H^0(E, \bar{D}) = \mathrm{Ker}\,\bar{D}$ is an algebra over $\mathbb{R}[[\epsilon]]$. Let B_k be the subbundle of $\mathrm{End}(E_0)$ consisting of differential operators of order $\le k$ vanishing on constants.

Lemma 4.8. *The differential forms with values in B_k form a subcomplex of $\Omega^\cdot(\mathrm{End}(E_0))$ and we have $H^p(B_k, D_0) = 0$ for $p > 0$.*

This lemma is proved in Appendix B. By using this lemma and the fact that the maps U_j are given by multidifferential operators, we deduce that $B = \cup_k B_k$ obeys the hypotheses of Lemma 4.6. Therefore, we have a homomorphism

$$\rho : H^0(E_0, D_0) \mapsto H^0(E, \bar{D}), \qquad \rho(f) = f + \epsilon\rho_1(f) + \epsilon^2\rho_2(f) + \cdots,$$

with $\rho_j \in \Omega^0(B)$, $j = 1, 2, \ldots$. Composing ρ with the canonical isomorphism $C^\infty(M) \to H^0(E_0, D_0)$ which sends a function to its Taylor expansions, we get a section $a \mapsto \tilde{a}$ of the projection $H^0(E, \bar{D}) \to C^\infty(M)$, $f \mapsto (x \mapsto f_x(0))$, with the property that the constant function 1 is sent to the constant section 1.

Proposition 4.9. *$H^0(E, \bar{D})$ is a quantization of the algebra of smooth functions on the Poisson manifold M.*

Proof: The section $a \mapsto \tilde{a}$ extends to an isomorphism $C^\infty(M)[[\epsilon]] \to H^0(E, \bar{D})$ by Lemma 4.6. So (i) in the definition of quantization is fulfilled.

To prove (ii), let f, $g \in C^\infty(M)$ and denote by $f_x(y), g_x(y)$ the Taylor expansions at $y = 0$ of $f \circ \varphi_x$, $g \circ \varphi_x$, respectively. Then, by construction, we have $\tilde{f} \star \tilde{g} = \tilde{h}$ with h of the form

$$h(x) = \sum_{j=0}^{\infty} \epsilon^j \sum_{J,K} a^j_{J,K}(x;y) \partial^J_y f_x(y) \partial^K_y g_x(y) \Big|_{y=0},$$

(J, K are multiindices). Since $D_0 f_x = 0 = D_0 g_x$, we may use these differential equations to replace partial derivatives with respect to y by partial derivatives with respect to x. Indeed, $D_0 f_x = 0$ is equivalent, in local coordinates, to

$$\frac{\partial f_x(y)}{\partial x^i} = \sum_{j,k} R^k_j(x,y) \frac{\partial \varphi^j_x(y)}{\partial x^i} \frac{\partial f_x(y)}{\partial y^k}.$$

The matrix R is the inverse of the Jacobian matrix $(\partial \varphi^i_x(y)/\partial y^j)$. Differentiating the identity $\varphi^j_x(0) = x^j$, we see that the matrix $(\partial \varphi^i_x(y)/\partial x^j)$ is invertible (as a matrix with coefficients in $\mathbb{R}[[y^1, \dots, y^d]]$). Thus h is expressed as a sum of bidifferential operators acting on $f_x(0) = f(x)$ and $g_x(0) = g(x)$.

Since ρ sends 1 to 1 and 1 is the identity for the Kontsevich product (Prop. 3.3 (v)), we deduce that $\tilde{1} \star \tilde{f} = \tilde{f} \star \tilde{1} = \tilde{f}$. Finally, by Prop. 3.3 (i), $\tilde{f} \star \tilde{g} = \tilde{h}$, with $h = fg + \epsilon\{\alpha(df, dg) + [\rho_1(f_x) g_x + \rho_1(g_x) f_x - \rho_1(f_x g_x)](y = 0)\} + O(\epsilon^2)$. Therefore the skew-symmetric part of P_1 is α. $\square$

This completes the proof of Theorem 1.1.

5. Casimir and central functions

In this section we discuss the relation between Casimir functions on the Poisson manifolds and the center of the deformed algebra. Let us first formulate a local version, due to Kontsevich, of Theorem 1.2. Suppose that α is a formal bivector field on $\mathbb{R}^d$ and f is a formal function on $\mathbb{R}^d$. Let

$$R(f, \alpha) = \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} U_{j+1}(f, \alpha, \dots, \alpha) \in \mathbb{R}[[y^1, \dots, y^d]][[\epsilon]].$$

Theorem 5.1 (Kontsevich [9])**.** *If α is a Poisson bivector field, then the map $f \mapsto R(f, \alpha)$ is a ring homomorphism from the ring $Z_0(\mathbb{R}^d)$ of Casimir functions to the center $Z(\mathbb{R}^d)$ of $(\mathbb{R}[[y^1, \dots, y^d]][[\epsilon]], \star)$.*

Since $U_1(f) = f$, R is a deformation of the identity map and therefore it extends by $\mathbb{R}[[\epsilon]]$-linearity to an isomorphism of $\mathbb{R}[[\epsilon]]$-algebras $Z_0(\mathbb{R}^d)[[\epsilon]] \to Z(\mathbb{R}^d)$.

To find a global version of this result, we need two more special cases of the formality theorem 3.1.

Corollary 5.2. *(Continuation of Cor. 3.2)*

(iv) $P(\alpha) \circ (R(f, \alpha) \otimes \mathrm{Id} - \mathrm{Id} \otimes R(f, \alpha)) = \epsilon A([\alpha, f], \alpha)$.

(v) $A(\xi, \alpha) R(f, \alpha) = \epsilon \sum_0^\infty \frac{\epsilon^j}{j!} U_{j+2}([\xi, \alpha], f, \alpha, \dots, \alpha) + R([\xi, f], \alpha) + \epsilon F([\alpha, f], \xi, \alpha)$.

These universal identities may be translated to identities for objects on the Poisson manifold M. We fix as above a section φ^{aff} of M^{aff} and let D denote the deformation of the canonical connection D_0 on the algebra bundle E. We also choose locally representatives $\varphi : M \to M^{\mathrm{coor}}$ of φ^{aff}, and set $\alpha_x = (\varphi_x^{-1})_* \alpha$, $x \in M$. For $f \in \Omega^0(E_0)$, set

$$R^M(f) = R(f, \alpha_x) \in \Omega^0(E).$$

Let $\mathrm{Der}(E_0)$ be the Lie algebra bundle of derivations of the algebra bundle E_0. A section of $\mathrm{Der}(E_0)$ is represented locally via φ by a function on M with values in the Lie algebra $\mathcal{W}$ of formal vector fields on $\mathbb{R}^d$. For $\eta \in \Gamma(\mathrm{Der}(E_0))$, set

$$\begin{aligned} C^M(\eta) &= A(\eta, \alpha_x) \in \Omega^0(\mathrm{End}(E)) \\ G^M(\eta) &= F(\eta, \varphi^* \omega_{\mathrm{MC}}(\cdot), \alpha_x) \in \Omega^1(E). \end{aligned}$$

Proposition 5.3. *Let $f \in \Omega^0(E_0), g \in \Omega^{\cdot}(E)$.*

(i) $DR^M(f) = R^M(D_0 f) + \epsilon G^M([\alpha_x, f])$

(ii) $[R^M(f), g]_\star = \epsilon C^M([\alpha_x, f]) g$

The proof of this Proposition is similar to the proof of Prop. 4.2.

5.1. **A quantization map compatible with the center.** The idea is now to look for a quantization map of the form $\rho(f) = R^M(f) + \beta([\alpha, f])$, for some $\beta(\eta) \in \Omega^0(E)$, defined for Hamiltonian vector fields $[\alpha, f]$ on M. Such a ρ clearly restricts to a ring homomorphism from $Z_0(M) = \{f \in C^\infty(M) \,|\, [\alpha, f] = 0\}$ to the ring of sections of E taking values in the center. Let $\bar{D} = D + [\gamma, \cdot]_\star$ be a flat deformation of the canonical connection as above. We have to choose β so that ρ sends D_0-horizontal sections to $\bar{D}$-horizontal sections. Then, by Prop. 5.3, we have, for any $f \in \Omega^0(E_0)$,

$$\begin{aligned} \bar{D}(R^M(f)) &= R^M(D_0 f) + \epsilon G^M([\alpha_x, f]) + [\gamma, R^M(f)]_\star \\ &= R^M(D_0 f) + \epsilon G^M([\alpha_x, f]) - \epsilon C^M([\alpha_x, f]) \gamma. \end{aligned} \tag{5}$$

This formula suggests introducing, for any $\eta \in \Gamma(\mathrm{Der}(E_0))$, the one-form

$$H^M(\eta) = G^M(\eta) - C^M(\eta) \gamma \in \Omega^1(E).$$

Moreover $G^M(\eta) \in \epsilon \Omega^1(E)$, see Prop. 3.3, and $\gamma \in \epsilon \Omega^1(E)$, so $H^M(\eta) \in \epsilon \Omega^1(E)$

Lemma 5.4. *Let $\eta = [\alpha, f]$ be a Hamiltonian vector field on M. Let $\bar\eta \in \Gamma(\mathrm{Der}(E_0))$ be the Taylor expansion of η in the coordinates φ. Then $\bar D H^M(\bar\eta) = 0$.*

Proof: Apply $\bar D$ to (5). □

Remark. Lemma 5.4 holds more generally for Poisson vector fields, i.e., vector fields obeying $[\alpha, \eta] = 0$.

Since the first cohomology of D_0 vanishes, we may recursively find a solution $\beta(\eta) \in \epsilon\Omega^0(E)$ of the equation $\bar D\beta(\eta) = -H^M(\bar\eta)$. The solution is unique, if we impose the normalization condition

$$\beta(\eta)(y = 0) = 0. \tag{6}$$

By this uniqueness, β depends linearly on the Poisson vector field η. In particular, it defines a linear map $f \mapsto \beta([\alpha, f])$ from $C^\infty(M)$ to $\Omega^0(E)$.

We thus obtain the following result.

Proposition 5.5. *Let $\bar D = D + [\gamma, \cdot]_\star$ be a flat connection on E as in 4.3, and for a Poisson vector field η, let $\beta(\eta)$ be the solution of $\bar D\beta(\eta) = -H^M(\bar\eta)$ obeying the normalization condition* (6)*. Then the map $\rho : C^\infty(M) \simeq H^0(E_0, D_0) \to H^0(E, \bar D)$*

$$f \mapsto R^M(f) + \epsilon\beta([\alpha, f]) = f + O(\epsilon^2)$$

is a quantization map. Its restriction to the ring Z_0 of Casimir functions extends to an $\mathbb{R}[[\epsilon]]$-algebra isomorphism from $Z_0[[\epsilon]]$ to the center of $H^0(E, \bar D)$.

Proof: It remains to prove that ρ is a quantization map, i.e., that it defines (via the canonical identification of $C^\infty(M)$ with $H^0(E_0, D_0)$) a map $f \mapsto \tilde f$ obeying the condition (ii) in the definition of quantization given in the Introduction. We have $U_{j+1}(1, \alpha, \dots, \alpha) = \delta_{j,0}1$, as can immediately be seen from the definition. Thus ρ sends 1 to 1. Also $\rho(f) = f + O(\epsilon^2)$. So $P_1(f, g) = \alpha(df, dg)$.

We are left to prove that the product is given by bidifferential operators. The normalization condition (6) is imposed by using the Fedosov homotopy $b = k^{-1}d_0^*$, see (7), to solve recursively the equation $\bar D\beta(\eta) = -H(\bar\eta)$. It is then clear that $\beta([\alpha, f])$ is a power series whose coefficients are differential operators acting on the Taylor series of f. Since the same holds for R^M, the same reasoning as in the proof of Prop. 4.9 implies that all coefficients of the product are given by bidifferential operators. □

In particular, Theorem 1.2 holds.

5.2. **Quantization of Hamiltonian vector fields.** The quantization map ρ defined in Proposition 5.5 is compatible with the action of Hamiltonian vector fields in the following sense. For a given Poisson vector field ξ, we define

$$\tau(\xi) = \epsilon\rho^{-1} \circ (A(\xi_x, \alpha_x) + [\beta(\xi),\ ]_*) \circ \rho.$$

Then we have the following result.

Proposition 5.6. *τ maps Hamiltonian vector fields on M to inner derivation of the star product $\star_M$.*

Proof: Using Property (iv) of Corollary 5.2, we can prove for any $h, f \in C^\infty(M)$ that

$$\tau([\alpha,h])(f) = \epsilon\rho^{-1}(A([\alpha_x,h_x],\alpha_x)\rho(f) + [\beta([\alpha,h]),\rho(f)]_\star) = \\ = \rho^{-1}[R(h_x,\alpha_x) + \epsilon\beta([\alpha,h]),\rho(f)]_\star = [h,f]_{\star_M}.$$

From the associativity of $\star_M$, it follows then

$$\tau([\alpha,h])(f \star_M g) = [h,f]_{\star_M} \star_M g + f \star_M [h,g]_{\star_M}.$$

$\square$

5.3. **Central two-forms.** The space of sections $\Gamma(E_0)$ is a Poisson algebra. Denote by $Z_0(\Gamma(E_0))$ the subalgebra of Casimir sections. Define $Z_0(\Omega^\cdot(E_0)) = \Omega^\cdot(M) \otimes_{C^\infty(M)} Z_0(\Gamma(E_0))$. It is easy to see that $Z_0(\Omega^\cdot(E_0))$ is a subcomplex of $\Omega^\cdot(E_0)$ with differential D_0. Similarly, we define $Z(\Omega^\cdot(E)) = \Omega^\cdot(M) \otimes_{C^\infty(M)} Z(\Gamma(E))$, where $Z(\Gamma(E))$ is the algebra of central sections of E. This is again a subcomplex of $\Omega^\cdot(E)$ with differential $\bar{D}$. By (5), R^M establishes an isomorphism (of complexes of algebras) $Z_0(\Omega^\cdot(E_0))[[\epsilon]] \to Z(\Omega^\cdot(E))$.

In particular, to each $\bar{D}$-closed form $\omega \in Z(\Omega^2(E))$ considered in (1), there corresponds a unique D_0-closed $\omega_0 = (R^M)^{-1}(\omega)$ in $Z_0(\Omega^2(E_0))$.

APPENDIX A. TOPOLOGICAL $k[[\epsilon]]$-MODULES

Let $k[[\epsilon]]$ be the ring of formal power series $\sum_{j=0}^{\infty} a_j\epsilon^j$ with coefficients a_j is some field k. It is a topological ring with the translation invariant topology such that $\epsilon^j k[[\epsilon]]$, $j \geq 1$ form a basis of neighborhoods of 0. Thus a subset U of $k[[\epsilon]]$ is open if and only if for every $a \in U$ there exists a $j \geq 1$ so that $a + \epsilon^j k[[\epsilon]] \subset U$. With this topology, called the ϵ-adic topology, the ring operations are continuous. More generally, if M is a $k[[\epsilon]]$-module, we may define a translation invariant topology on M by declaring that the submodules $\epsilon^j M$ form a basis of neighborhoods of 0. This topology is Hausdorff if and only if $m \in \epsilon^j M$ for all j implies $m = 0$. In this case the ϵ-adic topology comes from a metric d on M: set $d(m,m') = \|m - m'\|$ where $\|m\| = 2^{-j}$ and j is the largest integer such that $m \in \epsilon^j M$. We say that M is *complete* if it is complete as a metric space. Moreover, M is called ϵ-*torsion free* if, for all $j \in \mathbb{Z}_{\geq 0}$, $\epsilon^j m = 0$ implies $m = 0$. If M is a $k[[\epsilon]]$-module, then $M/\epsilon M$ is a module over $k = k[[\epsilon]]/\epsilon k[[\epsilon]]$.

The category of topological $k[[\epsilon]]$-modules is the subcategory of the category of $k[[\epsilon]]$-modules whose objects are $k[[\epsilon]]$-modules and whose morphisms are continuous morphisms of $k[[\epsilon]]$-modules.

Lemma A.1. *A topological $k[[\epsilon]]$-module M is isomorphic to a module of the form $M_0[[\epsilon]]$ for some k-vector space M_0 if and only if M is Hausdorff, complete and ϵ-torsion free.*

Proof: Let M_0 be a k-vector space and let $M = M_0[[\epsilon]]$. Then M is clearly ϵ-torsion free. It is Hausdorff: if $a = \sum a_j \epsilon^j \neq b = \sum b_j \epsilon^j$ then $a \in U = \sum_{j=1}^N a_j \epsilon^j + \epsilon^{N+1} M$ and $b \in V = \sum_{j=1}^N b_j \epsilon^j + \epsilon^{N+1} M$ are open sets, which are disjoint if N is large enough. A sequence $x_1, x_2, \dots \in M$ is Cauchy iff for any given N, $x_n - x_m \in \epsilon^N M_0$ for all sufficiently large n, m. Then $x = x_1 + (x_2 - x_1) + (x_3 - x_2) + (x_4 - x_3) + \dots$ is a well-defined element of M, since the coefficient of ϵ^j, for any j, is determined by finitely many summands. Since, for any n, $x = x_n + (x_{n+1} - x_n) + \dots$, it follows that x_n converges to x. Thus M is complete.

Conversely, suppose that M is a Hausdorff, complete, ϵ-torsion free $k[[\epsilon]]$-module. Let $M_0 = M/\epsilon M$ and denote by $p : M \to M_0$ the canonical projection. Let us choose a k-linear section i.e. a k-linear map $s : M_0 \to M$ such that $p \circ s = \mathrm{id}$. Then s extends to a continuous $k[[\epsilon]]$-linear map

$$s : M_0[[\epsilon]] \to M, \qquad \sum_{j=0}^{\infty} a_j \epsilon^j \mapsto \sum_{j=0}^{\infty} s(a_j) \epsilon^j.$$

The series on the right converges since the partial sums form a Cauchy sequence and M is complete.

The kernel of s is trivial, since M is ϵ-torsion free: if $0 \neq a \in \mathrm{Ker}(s)$, then, for some j, $a = \epsilon^j (a_j + \epsilon a_{j+1} + \cdots)$ with $a_j \neq 0$ and $\epsilon^j (s(a_j) + \epsilon s(a_{j+1}) + \cdots) = 0$. Then $m = s(a_j) + \epsilon s(a_{j+1}) + \cdots = 0$ and thus $p(m) = a_j = 0$, a contradiction.

The image of s is M, since M is Hausdorff: let $m \in M$ and suppose inductively that there exist $a_0, \dots, a_j \in M_0$ so that $m = s(x_j) \mod \epsilon^{j+1} M$ where $x_j = \sum_{i=0}^j a_i \epsilon^i$. Thus $m - s(x_j) = \epsilon^{j+1} r$ for some $r \in M$. If we set $a_{j+1} = p(r)$, then $m = s(x_{j+1}) \mod \epsilon^{j+2} M$. It follows that $x = \sum_{j=0}^{\infty} a_j \epsilon$ obeys $s(x) - m \in \epsilon^j M$ for all j. Thus $s(x) = m$. $\square$

To appreciate the meaning of this lemma, it is instructive to have counterexamples if one of the hypotheses is removed. Here they are: The module of formal Laurent series $M = k((\epsilon))$ is ϵ-torsion free but not Hausdorff, since every Laurent series belongs to $\cap_{j \geq 0} \epsilon^j M$. If M_0 is an infinite-dimensional k-vector space, then $M = k[[\epsilon]] \otimes_k M_0$ is Hausdorff, ϵ-torsion free, but not complete: if $e_1, e_2, \dots \in M_0$ are linearly independent, the sums $\sum_1^n e_j \epsilon^j$ form a divergent Cauchy sequence. Finally, $k[[\epsilon]]/\epsilon^N k[[\epsilon]]$ is Hausdorff, complete, but not ϵ-torsion free.

Definition. A *topological algebra* over $k[[\epsilon]]$ is an algebra over $k[[\epsilon]]$ with continuous product $A \times A \to A$.

If $A = A_0[[\epsilon]]$ for some k-module A_0, then any k-bilinear map $A_0 \times A_0 \to A$ extends uniquely to a $k[[\epsilon]]$-bilinear map $A \times A \to A$, which is then continuous. Thus a topological algebra structure on the $k[[\epsilon]]$-module $A_0[[\epsilon]]$ with unit $1 \in A_0$ is the same as a series $P = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \cdots$ whose coefficients P_j are k-bilinear maps $A_0 \times A_0 \to A_0$ obeying the relations $\sum_{j=0}^m P_{m-j}(P_j(f,g),h) = \sum_{j=0}^m P_{m-j}(f, P_j(g,h))$, $P_m(1,f) = \delta_{m,0} f = P_m(1,f)$, for all $f,g,h \in A_0$, $m \in \{0,1,2,\dots\}$.

Appendix B. Vanishing of the cohomology

We compute the cohomology of $\Omega^\cdot(E_0)$ and $\Omega^\cdot(B_k)$, in particular proving Lemma 4.7 and Lemma 4.8. Let us start with E_0. For $k = 0, 1, \dots,$ let $\mathbb{R}[[y^1, \dots, y^d]]^k$ be the space of power series a vanishing at zero to order at least k, i.e., such that $a(ty^1, \dots, ty^d)$ is divisible by t^k. These subspaces are stable under $\mathrm{GL}(d, \mathbb{R})$ and form a filtration. Thus we have a filtration

$$E_0 = E_0^0 \supset E_0^1 \supset E_0^2 \supset \cdots .$$

From the local coordinate expression of the differential

$$D_0 = dx^i \left(\frac{\partial}{\partial x^i} - R_k^j(x,y) \frac{\partial \varphi_x^k(y)}{\partial x^i} \frac{\partial}{\partial y^j} \right), \qquad R(x,y)^{-1} = (\partial \varphi_x^i(y)/\partial y^j)_{i,j=1,\dots,d},$$

(sum over repeated indices) expanded in powers of y, we see that most terms do not decrease the degree in y except the constant part of the second expression, which decreases the degree by one. It follows that the spaces

$$F^k \Omega^p(E_0) = \Omega^p(E_0^{k-p}), \qquad k = p, p+1, \dots$$

form a decreasing filtration of subcomplexes of $\Omega^\cdot(E_0)$. The first term in the associated spectral sequence is the cohomology of $\oplus_k F^k \Omega^\cdot(E_0)/F^{k-1}\Omega^\cdot(E_0)$. The k-th summand may be identified locally, upon choosing a representative in the class φ^{aff}, with the space of differential forms with values in the homogeneous polynomials of degree k, with differential

$$d_0 = \sum_i dx^i R_i^j(x,0) \frac{\partial}{\partial y^j}.$$

As in [5], we introduce a homotopy (for $k > 0$): let

$$d_0^* = \sum_{i,j} y^i \frac{\partial \varphi_x^j(0)}{\partial y^i} \iota(\frac{\partial}{\partial x^j}), \tag{7}$$

where ι denotes interior multiplication. Then $d_0 d_0^* + d_0^* d_0 = k\,\mathrm{Id}$; so if $d_0 a = 0$, then $a = d_0 b$, with $b = k^{-1} d_0^* a$. Moreover $k^{-1} d_0^*$ is compatible with the action of $\mathrm{GL}(d, \mathbb{R})$ and is thus defined independently of the choice of representative of φ^{aff}. Thus the cohomology of d_0 is concentrated in degree 0 and the spectral sequence collapses. In degree 0, cocycles are sections that are constant as functions of y. Thus

$$H^p(E_0, D_0) = \begin{cases} C^\infty(M), & p = 0, \\ 0, & p > 0. \end{cases}$$

The calculation of the cohomology of $\Omega^\cdot(B_k)$ to prove Lemma 4.8 is similar. We first use the filtration $B_k \supset B_{k-1} \supset \cdots \supset B_0 = 0$, by the order of the differential operator, which leads us to computing $H^\cdot(B_j/B_{j-1}, D_0)$, $1 \leq j \leq k$. As B_j/B_{j-1} may be canonically identified with the jth symmetric power of the tangent bundle, the complex is $\Omega^\cdot(M, S^j T(\mathbb{R}^d))$, with differential $d_{\text{de Rham}} + L$, where the value of the one-form L on $\xi \in T_x M$ is the Lie derivative in the direction of $\varphi^* \omega_{\mathrm{MC}}(\xi)$. By using the filtration by the degree of the coefficients as above, we obtain $H^p(B_j/B_{j-1}, D_0) = 0$ for $p \geq 1$, $j \geq 1$. It follows that $H^p(B_k, D_0) = 0$ for all $k \geq 0$, $p \geq 1$.

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