Two-sided estimates of the modeling error for elliptic homogenization problems

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Abstract

In this paper, we derive new two-sided estimates of modeling errors for linear elliptic boundary value problems with periodic coefficients solved by homogenization method. Our approach is based on the concept of functional a posteriori error estimation. The estimates are obtained for the energy norm and use solely the global flux of the non-oscillatory solution of the *homogenized model* and solution of a boundary value problem on the cell of periodicity. Numerical tests illustrate the efficiency of the estimates.

2000 *Mathematics Subject Classification:* 35J15, 35B27, 65N15

Key words and phrases: periodic structures, homogenization, elliptic boundary value problem, a posteriori error estimate, modeling error

1 Introduction

Boundary value problems with periodic structures arise in various applications. Such structures are well known in industry (e.g., in composite materials). Homogenization theory is the major tool used to quantitatively analyze media with periodic structures. Within the framework of the theory (see, e.g., [10], [15]), the behavior of a heterogeneous media is described with the help of a certain homogenized problem, which is typically a boundary value problem with smooth coefficients, and the solution of a specially constructed problem with periodic boundary conditions. It has been proved that the functions reconstructed by this procedure converge to the exact solution as the cell size ε tends to zero. Moreover, known a priori error estimates qualified the convergence rate in terms of ε. The goal of this paper is to derive two-sided estimates of the modeling error generated by homogenization, i.e., to estimate the difference between the exact solution of the original problem and its approximation obtained by the corresponding homogenized model.

Our method is based on the theory of functional a posteriori estimates (see [20] - [26]), in which estimates of the difference between the exact solution of boundary value problems and arbitrary functions from the corresponding energy space has been derived by purely functional methods. As a result, the estimates contain no mesh dependent constants and are applicable for any function from the corresponding energy space. In [22] - [27] these properties have been used for the analysis of modeling errors. In [25] and [28], it was suggested a combined adaptive numerical strategy, which is based on simplification (defeaturing) of problems having complicated and irregular coefficients. This strategy takes into account both modeling and approximation errors. It was demonstrated that it is efficient for problems having rapidly changing (oscillating but non-periodical) diffusion coefficients.

In the present paper, we consider a different case related to fine periodical structures, i.e., we are concerned with the homogenized model of an elliptic boundary value problem with periodical coefficients.

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and $\Omega = \bigcup_{\mathbf{i}} \Pi^{\varepsilon}_{\mathbf{i}}$, where

$$\Pi^{\varepsilon}_{\mathbf{i}} = \mathbf{x_i} + \varepsilon\,\widehat{\Pi} = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \frac{\mathbf{x} - \mathbf{x_i}}{\varepsilon} \in \widehat{\Pi} \right\},$$

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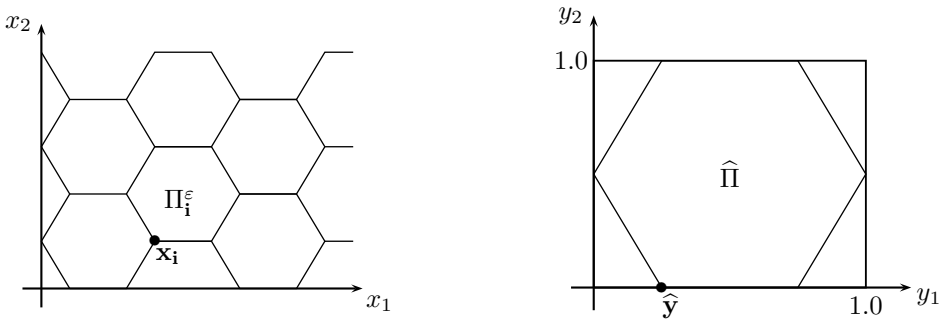
denotes the basic "cell" (repeating element of the periodic structure, see Fig. 1), $\mathbf{x_i}$ is the reference point of $\Pi_{\mathbf{i}}^{\varepsilon}$. By $\mathbf{x}$ we denote the global (Cartesian) coordinate system in $\mathbb{R}^d$ and by $\mathbf{i} = (i_1, i_2, ...i_d)$ the counting multi-indices for the cells. The notations $\bigcup_{\mathbf{i}}$ and $\sum_{\mathbf{i}}$ are shorthands for the union and summation over all cells. It is assumed that the overall amount of $\Pi_{\mathbf{i}}^{\varepsilon}$ in Ω is bounded from above by the quantity

$$c_0\,\varepsilon^{-d}, \quad \text{where} \quad c_0 = \mathcal{O}(1). \tag{1.1}$$

In the basic cell (see Fig. 1), we use local Cartesian coordinates $\mathbf{y} \in \mathbb{R}^d$. For any $\Pi_{\mathbf{i}}^{\varepsilon}$, local and global coordinates are joined by the relation

$$\mathbf{y} = \frac{\mathbf{x} - \mathbf{x_i}}{\varepsilon} \in \widehat{\Pi} \qquad \forall \mathbf{x} \in \Pi_{\mathbf{i}}^{\varepsilon}\ \forall \mathbf{i}.$$

Figure 1: Periodic structure (left) and its basic cell (right)



On $\widehat{\Pi}$, we define a matrix function $\widehat{A} \in L^{\infty}(\widehat{\Pi}, \mathbb{R}^{d\times d}_{\rm sym})$, where $\mathbb{R}^{d\times d}_{\rm sym}$ denotes the set of symmetric $d \times d-$ matrices. We assume that

$$c_1|\xi|^2 \le \widehat{A}(\mathbf{y})\xi\cdot\xi \le c_2|\xi|^2 \qquad \forall\,\xi \in \mathbb{R}^d \quad \forall \mathbf{y} \in \widehat{\Pi}, \tag{1.2}$$

where $0 < c_1 \le c_2 < \infty$. The global matrix

$$A_\varepsilon(\mathbf{x}) := \widehat{A}\left(\frac{\mathbf{x} - \mathbf{x_i}}{\varepsilon}\right) \qquad \forall \mathbf{x} \in \Pi_{\mathbf{i}}^{\varepsilon}\ \forall \mathbf{i}, \tag{1.3}$$

defines the periodic structure on Ω, where ε is a small parameter (geometrical size of a cell). In view of (1.2), A_ε (and its inverse A_ε^{-1}) satisfy similar two-sided estimates for any ε.

For

$$f \in L^2(\Omega), \tag{1.4}$$

we consider the second-order elliptic equation

$$-\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega$$

with homogeneous Dirichlet boundary conditions. The corresponding generalized solution $u_\varepsilon \in H_0^1(\Omega)$ is defined by the variational formulation

$$\int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla w = \int_\Omega f w \qquad \forall w \in H_0^1(\Omega). \tag{1.5}$$

For any $\varepsilon > 0$, the solution u_ε exists and is unique. It is known (see, e.g., [7], [10], [15]) that there exists the so-called homogenized matrix $A_0 \in \mathbb{R}^{d\times d}_{sym}$ (cf. (2.9)), which is positive definite (cf. [7], pp.17-18), such that

$$u_\varepsilon \to u_0 \quad \text{in } L^2(\Omega) \qquad \text{and} \qquad u_\varepsilon \rightharpoonup u_0 \quad \text{in } H_0^1(\Omega) \quad \text{for } \varepsilon \to 0,$$

where $u_0 \in H_0^1(\Omega)$ satisfies the relation

$$\int_\Omega A_0 \nabla u_0 \cdot \nabla w = \int_\Omega f w \qquad \forall w \in H_0^1(\Omega). \tag{1.6}$$

The homogenized problem (1.6) is well studied in the context of asymptotic analysis (see, e.g., [7], [15]). In particular, it was shown that it is possible to find the approximation

$$u_\varepsilon^1(\mathbf{x}) \;=\; u_{01}\left(\mathbf{x},\, \frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\right) \qquad \forall \mathbf{x} \in \Pi_{\mathbf{i}}^\varepsilon \;\forall \mathbf{i}, \tag{1.7}$$

where

$$u_{01}(\mathbf{x},\, \mathbf{y}) \;=\; u_0(\mathbf{x}) \;+\; \varepsilon\, u_1(\mathbf{x},\, \mathbf{y}) \qquad \forall \mathbf{x} \in \Omega,\; \forall \mathbf{y} \in \widehat{\Pi} \tag{1.8}$$

and $u_1(\mathbf{x},\cdot)$ is a $\widehat{\Pi}-$periodic function such that (cf. [15])

$$\|u_\varepsilon - u_\varepsilon^1\|_{H^1(\Omega)} \;\le\; c\,\sqrt{\varepsilon}.$$

Derivation of error indicators for homogenized problems is a topic of vivid research. Here, we first of all mention residual type error indicators that develop the ideas suggested in [2, 3] for finite element approximations. Since our approach is based on a different technique, we will sketch here only briefly some relevant literature on residual based estimation and refer for a detailed review, e.g., to [14]. A posteriori error estimates for the heterogeneous multiscale discretization (HMM) of elliptic problems in a periodic setting can be found in [18] and [13]. In [1], an a posteriori estimate of residual type for general, possibly non-periodic, diffusion tensors with micro-scales is presented while a residual-type a posteriori error estimate for more general diffusion tensors has been developed in [14]. Also, we mention the papers [4, 5, 8, 9, 17, 29, 30], which are closely related to the topic.

Our goal is to deduce estimates of a different type, which provide guaranteed computable bounds of the modeling error and does not contain unknown constants. These estimates (error majorants and minorants) reflect the decomposition (1.8). The majorant is based on the homogenized problem and its solution and, in addition, depends on free functions defined on the cell of periodicity. They should be chosen such that the majorant becomes as small as possible and can either be computed as the solution of a certain boundary value problem with periodic boundary conditions on the basic cell or by minimizing the error majorant. In general, the estimates have the form

$$\mathcal{M}_\ominus\left(u_\varepsilon^1;\, \Theta\right) \;\le\; \|\nabla\left(u_\varepsilon - u_\varepsilon^1\right)\|_{A_\varepsilon} \;\le\; \mathcal{M}_\oplus\left(u_\varepsilon^1;\, \boldsymbol{\eta},\, \boldsymbol{\lambda},\, s\right), \tag{1.9}$$

where

$$\|\boldsymbol{q}\|_{A_\varepsilon} := \left(\int_\Omega A_\varepsilon\, \boldsymbol{q}\cdot\boldsymbol{q}\right)^{1/2}. \tag{1.10}$$

The majorant $\mathcal{M}_\oplus$ and the minorant $\mathcal{M}_\ominus$ depend on the solution of (1.6), the small parameter ε, and some other functions, defined on $\widehat{\Pi}$. Technically the derivation is based on a posteriori error estimates of functional type (see, e.g., [21]-[27]).

The structure of this paper is as follows. In Section 2, we briefly overview the results in the homogenization theory of second order elliptic operators which are significant for subsequent analysis. In Section 3, we prove the main result, which yields computable upper and lower bounds of the modeling error. Numerical tests are exposed in Section 4. They confirm the efficiency of the above derived estimates.

2 Homogenization of second order elliptic operators

On each cell $\Pi_{\mathbf{i}}^\varepsilon$, the operator $-\mathrm{div}\left(A_\varepsilon\,\nabla u_\varepsilon^1\right)(\mathbf{x})$ can be represented in a different form:

$$-\mathrm{div}\left(A_\varepsilon\,\nabla u_\varepsilon^1\right)(\mathbf{x}) =: \left(\tilde{\mathcal{A}}_\varepsilon u_{01}\right)\left(\mathbf{x},\, \frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\right) \qquad \forall \mathbf{x} \in \Pi_{\mathbf{i}}^\varepsilon,$$

where the convention on summation of repeating indices is adopted and u_{01} is defined by (1.8). We have

$$\tilde{\mathcal{A}}_\varepsilon = -\left(\frac{\partial}{\partial x_i} \;+\; \varepsilon^{-1}\,\frac{\partial}{\partial y_i}\right)\,\widehat{A}_{ij}\,\left(\frac{\partial}{\partial x_j} \;+\; \varepsilon^{-1}\,\frac{\partial}{\partial y_j}\right) = \varepsilon^{-2}\mathcal{A}_1 + \varepsilon^{-1}\mathcal{A}_2 + \mathcal{A}_3, \tag{2.1}$$

and

$$\mathcal{A}_1 = -\frac{\partial}{\partial y_i}\left(\widehat{A}_{ij}(\mathbf{y})\,\frac{\partial}{\partial y_j}\right), \tag{2.2}$$

$$\mathcal{A}_2 = -\frac{\partial}{\partial y_i}\left(\widehat{A}_{ij}(\mathbf{y})\,\frac{\partial}{\partial x_j}\right) - \widehat{A}_{ij}(\mathbf{y})\,\frac{\partial^2}{\partial x_i \partial y_j},$$

$$\mathcal{A}_3 = -\widehat{A}_{ij}(\mathbf{y})\,\frac{\partial^2}{\partial x_i\,\partial x_j}.$$

Within the framework of the homogenization theory, the construction of an efficient approximation of the desired function u_ε is based on the form (1.7) - (1.8). Then

$$\tilde{\mathcal{A}}_\varepsilon u_\varepsilon^1\Big|_{\Pi_{\mathbf{i}}^\varepsilon} = \varepsilon^{-1}\big(\mathcal{A}_1\,u_1 + \mathcal{A}_2\,u_0\big) + \big(\mathcal{A}_3\,u_0 + \mathcal{A}_2\,u_1\big) + \varepsilon\,\mathcal{A}_3\,u_1. \tag{2.3}$$

The natural requirement that $\tilde{\mathcal{A}}_\varepsilon u_\varepsilon^1$ must be uniformly bounded as ε tends to zero, leads to the condition $\mathcal{A}_1\,u_1 + \mathcal{A}_2\,u_0 = 0$. Whence,

$$-\frac{\partial}{\partial y_i}\left(\widehat{A}_{ij}(\mathbf{y})\,\frac{\partial}{\partial y_j}u_1(\mathbf{x},\,\mathbf{y})\right) = \frac{\partial}{\partial y_i}\left(\widehat{A}_{ij}(\mathbf{y})\,\frac{\partial}{\partial x_j}u_0(\mathbf{x})\right).$$

This equation defines a problem on the cell of periodicity, where the right-hand side depends on $\mathbf{x}$ as a parameter.

For each $k = 1, ..., d$, let $N_k(\mathbf{y})$ be the unique solution of the auxiliary problem

$$\begin{aligned}
&\frac{\partial}{\partial y_i}\left(\widehat{A}_{ij}(\mathbf{y})\,\frac{\partial N_k}{\partial y_j}\right) = \frac{\partial}{\partial y_i}\,\widehat{A}_{ik}(\mathbf{y}) \quad \text{in} \quad \widehat{\Pi},\\
&N_k \ \text{ satisfies periodic boundary conditions ,}\\
&\int_{\widehat{\Pi}} N_k = 0.
\end{aligned} \tag{2.4}$$

Then $u_1(\mathbf{x},\,\mathbf{y})$ can be written (cf., e.g., [6], [7], [15]) as

$$u_1(\mathbf{x},\,\mathbf{y}) = -N_k(\mathbf{y})\,\frac{\partial u_0\,(\mathbf{x})}{\partial x_k}.$$

Therefore, u_ε^1 as defined in (1.7) has the form

$$u_\varepsilon^1\,(\mathbf{x}) \;=\; u_0(\mathbf{x}) \;-\varepsilon\,N_k(\mathbf{y})\,\frac{\partial u_0\,(\mathbf{x})}{\partial x_k}, \quad \forall \mathbf{y} \in \widehat{\Pi} \quad \forall \mathbf{x} \in \Pi_{\mathbf{i}}^\varepsilon \ \forall \mathbf{i} \tag{2.5}$$

and

$$\mathcal{A}_3\,u_0 + \mathcal{A}_2\,u_1 = -\widehat{A}_{ij}\frac{\partial^2 u_0}{\partial x_i \partial x_j} + \left(\frac{\partial}{\partial y_i}\Big(\widehat{A}_{ij}\,\frac{\partial}{\partial x_j}\Big) + \widehat{A}_{ij}\,\frac{\partial^2}{\partial x_i \partial y_j}\right)\left(N_k\,\frac{\partial u_0}{\partial x_k}\right)$$

$$= -\widehat{A}_{ij}\frac{\partial^2 u_0}{\partial x_i \partial x_j} + \frac{\partial}{\partial y_i}\left(\widehat{A}_{ij}\,N_k\right)\frac{\partial^2 u_0}{\partial x_j \partial x_k} + \widehat{A}_{ij}\frac{\partial N_k}{\partial y_j}\,\frac{\partial^2 u_0}{\partial x_i \partial x_k}$$

$$= \left(-\widehat{A}_{ij} + \widehat{A}_{ik}\frac{\partial N_j}{\partial y_k}\right)\,\frac{\partial^2\,u_0}{\partial x_i \partial x_j} + \frac{\partial}{\partial y_i}\left(\widehat{A}_{ij}\,N_k\right)\frac{\partial^2 u_0}{\partial x_j \partial x_k}. \tag{2.6}$$

Let $\omega \subset \Omega$ be a measurable subset. For a function $\zeta \in L^1(\omega)$ the integral mean is given by

$$\langle \zeta \rangle_\omega := \frac{1}{|\omega|} \int\limits_\omega \zeta. \tag{2.7}$$

If we write $\int_\omega \langle \zeta \rangle_\omega$ we consider this average as a constant function on ω (for vector-valued functions, we apply this definition componentwise). We denote the error caused by the average (2.7) by

$$\delta_\omega \zeta := \| \zeta - \langle \zeta \rangle_\omega \|_\omega,$$

where $\|\cdot\|_\omega$ denotes the standard L^2-norm on ω. For vector-valued functions $\boldsymbol{\zeta} = (\zeta_k)_{k=1}^d \in L^1\left(\omega, \mathbb{R}^d\right)$ and $\boldsymbol{\phi} = (\phi_k)_{k=1}^d \in L^1\left(\Omega, \mathbb{R}^d\right)$ we define the local and piecewise averages by

$$\delta_\omega \boldsymbol{\zeta} := \Big(\| \zeta_k - \langle \zeta_k \rangle_\omega \|_\omega \Big)_{k=1}^d, \qquad \delta_\Omega^{\mathrm{pw}} \boldsymbol{\phi} := \varepsilon^{d/2} \Big(\sum_{\mathbf{i}} \| \phi_k - \langle \phi_k \rangle_{\Pi_\mathbf{i}^\varepsilon} \|_{\Pi_\mathbf{i}^\varepsilon} \Big)_{k=1}^d$$

and

$$(\delta_\omega \boldsymbol{\zeta})^2 := \Big(\| \zeta_k - \langle \zeta_k \rangle_\omega \|_\omega^2 \Big)_{k=1}^d, \qquad (\delta_\Omega^{\mathrm{pw}} \boldsymbol{\phi})^2 := \varepsilon^d \left(\left(\sum_{\mathbf{i}} \| \phi_k - \langle \phi_k \rangle_{\Pi_\mathbf{i}^\varepsilon} \|_{\Pi_\mathbf{i}^\varepsilon} \right)^2 \right)_{k=1}^d.$$

The mean value of the right-hand side of (2.6) with respect to $\mathbf{y}$ is given by

$$\langle \mathcal{A}_3\, u_0 + \mathcal{A}_2\, u_1 \rangle_{\widehat{\Pi}} = \left\langle -\widehat{A}_{ij} + \widehat{A}_{ik} \frac{\partial N_j}{\partial y_k} \right\rangle_{\widehat{\Pi}} \frac{\partial^2 u_0}{\partial x_i \partial x_j} =: -\,(A_0)_{i\,j}\, \frac{\partial^2 u_0}{\partial x_i \partial x_j}. \tag{2.8}$$

Hence, for the homogenized matrix, we obtain the representation

$$A_0 = \left\langle \widehat{A} \left(I - \nabla \boldsymbol{N} \right) \right\rangle_{\widehat{\Pi}} \qquad \text{with} \quad \mathbf{N} := (N_k)_{k=1}^d. \tag{2.9}$$

In general, u_ε^1 defined by (2.5) does not satisfy the boundary conditions. We introduce the boundary corrected approximation w_ε^1 of u_ε by

$$w_\varepsilon^1(\mathbf{x})\Big|_{\Pi_\mathbf{i}^\varepsilon} := u_0(\mathbf{x}) - \varepsilon \psi^\varepsilon(\mathbf{x})\, N_k\left(\frac{\mathbf{x} - \mathbf{x_i}}{\varepsilon} \right) \frac{\partial\, u_0(\mathbf{x})}{\partial x_k} \quad \forall \mathbf{x} \in \Pi_\mathbf{i}^\varepsilon\ \forall \mathbf{i}, \tag{2.10}$$

where the cutoff function $\psi^\varepsilon(\mathbf{x})$ satisfies the following conditions:

$$\begin{array}{ll} \psi^\varepsilon \in W_0^{1,\infty}(\Omega), & \psi^\varepsilon \equiv 1 \,\text{in}\, \Omega_\varepsilon^{in} := \{\, x \in \Omega \,|\, \operatorname{dist}(x,\, \partial\Omega) > \varepsilon\}, \\ 0 \le \psi^\varepsilon \le 1, & \varepsilon\, |\nabla \psi^\varepsilon| \le c \,\text{in}\, \Omega \text{ for some } c \text{ independent of } \varepsilon. \end{array} \tag{2.11}$$

Since Ω is a Lipschitz domain a possible choice of ψ^ε which satisfies these assumptions is given by

$$\psi^\varepsilon(\mathbf{x}) := \min\{1,\, \frac{1}{\varepsilon} \operatorname{dist}(\mathbf{x},\, \partial\Omega)\}.$$

Below we summarize three steps necessary for computing the augmented approximation w_ε^1 of u_ε below. Note that our error majorant will depend on this precomputed function.

1) The solutions N_k of the cell problems

$$\begin{array}{l} \operatorname{div}\Big(\widehat{A}\, \nabla\, N_k \Big) = \Big(\operatorname{div}\, \widehat{A} \Big)_k \quad \text{in} \quad \widehat{\Pi}, \\ N_k \ \text{ is periodic in } \widehat{\Pi}, \\ \displaystyle\int_{\widehat{\Pi}} N_k = 0 \end{array} \tag{2.12}$$

must be computed. With the help of them we define the homogenized matrix in the general case (cf. (2.9)):

$$A_0 = \left\langle \widehat{A} \left(I - \nabla \boldsymbol{N} \right) \right\rangle_{\widehat{\Pi}}.$$

2) Solve the homogenized problem: Find $u_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega A_0 \nabla u_0 \cdot \nabla w \;=\; \int_\Omega f\, w \qquad \forall w \in H_0^1(\Omega). \tag{2.13}$$

3) With the help of u_0 and N_k, we obtain the approximation w_ε^1 of u_ε. It is defined by the relation

$$w_\varepsilon^1(\mathbf{x}) \;:=\; u_0(\mathbf{x}) \;-\; \varepsilon \psi^\varepsilon(\mathbf{x})\, N_k\left(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\right) \frac{\partial\, u_0(\mathbf{x})}{\partial x_k} \quad \forall \mathbf{x} \in \Pi_{\mathbf{i}}^\varepsilon,\ \forall \mathbf{i} \tag{2.14}$$

where $\psi^\varepsilon := \min\{1,\, \frac{1}{\varepsilon}\, \mathrm{dist}(\mathbf{x},\, \partial\Omega)\}$ is a cutoff function.

To prove optimal a priori convergence rates for the difference $u_\varepsilon - w_\varepsilon^1$ with respect to the energy norm, some regularity of the homogenized problems is typically required. The following assumptions are taken from [15], p.28: The domain Ω, the diffusion matrix $\widehat{A}$, and the right-hand side f are assumed to be sufficiently smooth such that

$$u_0 \;\in\; W^{2,\,\infty}(\overline{\Omega}), \tag{2.15}$$

and

$$\frac{\partial\, N_k}{\partial y_j} \in L^\infty(\widehat{\Pi}). \tag{2.16}$$

Then, it is proved (see, e.g., [7, Rem. 5.13], [15, p. 28], [11]) that the following error estimates hold:

$$\|u_\varepsilon - w_\varepsilon^1\|_{H^1(\Omega)} \;\le\; \widetilde{c}\sqrt{\varepsilon} \tag{2.17}$$

and

$$\|A_\varepsilon\, \nabla\, u_\varepsilon - \mathbf{v_0} - \varepsilon\, \mathbf{v_1}\| \;\le\; \widehat{c}\sqrt{\varepsilon}, \tag{2.18}$$

where[1]

$$\begin{aligned}
\mathbf{v_0} &:= (I - \mathrm{curl}_{\mathbf{y}} \widetilde{N})\, \boldsymbol{\mu},\\
\boldsymbol{\mu} &:= \langle A_0^{-1}\, (I - \mathrm{curl}_{\mathbf{y}} \widetilde{N})\rangle_{\widehat{\Pi}}^{-1}\, \nabla u_0,\\
\mathbf{v_1} &:= -\mathrm{curl}_{\mathbf{x}}(\widetilde{N}\, \boldsymbol{\mu}),
\end{aligned}$$

and the $d \times d$ matrix $\widetilde{N}$ with columns $\widetilde{N}_k$ is the solution of the auxiliary problem

$$\begin{aligned}
&\mathrm{curl}\, A_0^{-1} \left(\mathrm{curl}\, \widetilde{N_k}(\mathbf{y})\right) = \mathrm{curl}\left(A_0^{-1}\right)_k \quad \text{in} \quad \widehat{\Pi},\\
&\mathrm{div}\, \widetilde{N}_k = 0,\\
&\widetilde{N}_k \ \text{ is periodic in } \widehat{\Pi},\\
&\int_{\widehat{\Pi}} \widetilde{N}_k = 0.
\end{aligned} \tag{2.19}$$

Relation (2.17) provides an a priori estimate of the modeling error evaluated in terms of the parameter ε. In the next section, we deduce two-sided guaranteed a posteriori error estimates of

$$\|\nabla(u_\varepsilon - w_\varepsilon^1)\|_{A_\varepsilon}$$

which employ the computed functions N_k, $\widetilde{N}_k$, as well as the homogenized solution u_0.

[1] The columns of the matrix $\mathrm{curl}_{\mathbf{y}} \tilde{N}$ are given by $\mathrm{curl}_{\mathbf{y}} \tilde{N}_k$, $k = 1, 2, \ldots, d$.

3 Two-sided error estimate of the modeling error

3.1 Upper bound of the error

In this section, we first prove a subsidiary result which states an upper bound of the L^2-product of a globally defined function and a periodic function defined on the cell. For a vector $\boldsymbol{\mu} = (\mu_i)_{i=1}^d \in (\mathbb{R}_{>0})^d$ and $s \in \mathbb{R}$ we denote by $\boldsymbol{\phi}^s$ the componentwise application of the power s, i.e., $\boldsymbol{\mu}^s = (\mu_i^s)_{i=1}^d$.

Lemma 3.1 *For all $\mathbf{g} \in L^2(\Omega)^d$, $\boldsymbol{\eta} \in L^2(\widehat{\Pi})^d$, and all $\boldsymbol{\lambda} = (\lambda_d)_{k=1}^d \in (\mathbb{R}_{>0})^d$ it holds*

$$\sum_{\mathbf{i}} \int_{\Pi_{\mathbf{i}}^\varepsilon} \mathbf{g}(\mathbf{x}) \cdot \boldsymbol{\eta}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\,\mathbf{dx} \le |\Omega|\,\langle \mathbf{g}\rangle_\Omega \cdot \langle \boldsymbol{\eta}\rangle_{\widehat{\Pi}} \;+\; \frac{\boldsymbol{\lambda}}{2}\cdot(\delta_\Omega^{\mathrm{pw}}\mathbf{g})^2 \;+\; \frac{\boldsymbol{\lambda}^{-1}}{2}\cdot(\delta_{\widehat{\Pi}}\boldsymbol{\eta})^2. \tag{3.1}$$

Proof. For any $\mathbf{g} \in L^2(\Omega)^d$, we have

$$\begin{aligned}\mathcal{I} &:= \sum_{\mathbf{i}} \int_{\Pi_{\mathbf{i}}^\varepsilon} \mathbf{g}(\mathbf{x}) \cdot \boldsymbol{\eta}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\,\mathbf{dx} = \sum_{k=1}^d \sum_{\mathbf{i}} \int_{\Pi_{\mathbf{i}}^\varepsilon} g_k(\mathbf{x})\,\eta_k\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\,\mathbf{dx}\\ &= \sum_{k=1}^d \sum_{\mathbf{i}} \int_{\Pi_{\mathbf{i}}^\varepsilon} \big(g_k(\mathbf{x}) - \langle g_k\rangle_{\Pi_{\mathbf{i}}^\varepsilon}\big)\,\eta_k\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\,\mathbf{dx} \;+\; \sum_{k=1}^d \sum_{\mathbf{i}} \int_{\Pi_{\mathbf{i}}^\varepsilon} \langle g_k\rangle_{\Pi_{\mathbf{i}}^\varepsilon}\,\eta_k\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\,\mathbf{dx}.\end{aligned}$$

Since

$$\begin{aligned}\sum_{\mathbf{i}} \int_{\Pi_{\mathbf{i}}^\varepsilon} \langle g_k\rangle_{\Pi_{\mathbf{i}}^\varepsilon}\,\eta_k\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\,\mathbf{dx} &= \varepsilon^d \int_{\widehat{\Pi}} \eta_k \sum_{\mathbf{i}} \frac{1}{|\Pi_{\mathbf{i}}^\varepsilon|}\int_{\Pi_{\mathbf{i}}^\varepsilon} g_k \;=\; \varepsilon^d \int_{\widehat{\Pi}} \eta_k \sum_{\mathbf{i}} \frac{1}{\varepsilon^d\,|\widehat{\Pi}|}\int_{\Pi_{\mathbf{i}}^\varepsilon} g_k\\ &= \int_{\widehat{\Pi}} \eta_k\,\frac{1}{|\widehat{\Pi}|}\int_\Omega g_k \;=\; |\Omega|\,\langle g_k\rangle_\Omega\,\langle \eta_k\rangle_{\widehat{\Pi}}\end{aligned}$$

and

$$\begin{aligned}\sum_{\mathbf{i}} \int_{\Pi_{\mathbf{i}}^\varepsilon} \big(g_k(\mathbf{x}) - \langle g_k\rangle_{\Pi_{\mathbf{i}}^\varepsilon}\big)\,\eta_k\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\,\mathbf{dx} &= \sum_{\mathbf{i}} \int_{\Pi_{\mathbf{i}}^\varepsilon} \big(g_k(\mathbf{x}) - \langle g_k\rangle_{\Pi_{\mathbf{i}}^\varepsilon}\big)\,\Big(\eta_k\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big) - c_k\Big)\,\mathbf{dx}\\ &\le \Big(\sum_{\mathbf{i}} \|g_k - \langle g_k\rangle_{\Pi_{\mathbf{i}}^\varepsilon}\|_{\Pi_{\mathbf{i}}^\varepsilon}\Big)\,\Big(\int_{\Pi_{\mathbf{i}}^\varepsilon}\Big(\eta_k\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big) - c_k\Big)^2\,\mathbf{dx}\Big)^{1/2}\\ &= \Big(\sum_{\mathbf{i}} \|g_k - \langle g_k\rangle_{\Pi_{\mathbf{i}}^\varepsilon}\|_{\Pi_{\mathbf{i}}^\varepsilon}\Big)\;\varepsilon^{d/2}\,\|\eta_k - c_k\|_{\widehat{\Pi}},\end{aligned}$$

we find that

$$\begin{aligned}\mathcal{I} &\le \sum_k \big(|\Omega|\,\langle g_k\rangle_\Omega\,\langle \eta_k\rangle_{\widehat{\Pi}} \;+\; \big(\delta_\Omega^{\mathrm{pw}} g\big)_k\,\|\eta_k - c_k\|_{\widehat{\Pi}}\big)\\ &\le \sum_k \Big(|\Omega|\,\langle g_k\rangle_\Omega\,\langle \eta_k\rangle_{\widehat{\Pi}} \;+\frac{\lambda_k}{2}\,(\delta_\Omega^{\mathrm{pw}} g)_k^2 \;+\frac{1}{2\,\lambda_k}\,\|\eta_k - c_k\|_{\widehat{\Pi}}^2\Big)\\ &= |\Omega|\,\langle \mathbf{g}\rangle_\Omega \cdot \langle \boldsymbol{\eta}\rangle_{\widehat{\Pi}} \;+\; \frac{1}{2}\,\boldsymbol{\lambda}\cdot(\delta_\Omega^{\mathrm{pw}}\mathbf{g})^2 \;+\; \frac{1}{2}\int_{\widehat{\Pi}} \sum_k \frac{1}{\lambda_k}\,\Big(\eta_k - c_k\Big)^2,\end{aligned}$$

for any $\boldsymbol{\lambda} \in \mathbb{R}^d_{>0}$ and $(c_k)_{k=1}^d \in \mathbb{R}^d$ are arbitrary vectors . In particular, we set $c_k = \langle \eta_k\rangle_{\widehat{\Pi}}$, and this implies (3.1). □

In order to present the main estimate in a transparent form, we define the functions

$$\mathbf{g}_{\boldsymbol{\tau_0}}(\mathbf{x}) \;:=\; A_\varepsilon\,\nabla w_\varepsilon^1 \;-\; \boldsymbol{\tau_0} \tag{3.2}$$

and

$$\begin{aligned}\mathcal{F}\Big(w_\varepsilon^1;\,\boldsymbol{\tau_0},\,\boldsymbol{\eta},\boldsymbol{\lambda},\,s\Big) := \|\mathbf{g}_{\boldsymbol{\tau_0}}\|^2_{A_\varepsilon^{-1}} + 2\,\varepsilon^s\,|\Omega|\,\langle \mathbf{g}_{\boldsymbol{\tau_0}}\rangle_\Omega\cdot\,\langle\boldsymbol{\eta}\rangle_{\widehat{\Pi}} +\\ + \varepsilon^s\Big(\boldsymbol{\lambda}^{-1}\cdot(\delta_{\widehat{\Pi}}\boldsymbol{\eta})^2 \;+\; \boldsymbol{\lambda}\cdot\big(\delta_\Omega^{\mathrm{pw}}(\mathbf{g}_{\boldsymbol{\tau_0}})\big)^2\Big) + c_0\varepsilon^{2s}\;\|\boldsymbol{\eta}\|^2_{\widehat{A}^{-1},\widehat{\Pi}},\end{aligned} \tag{3.3}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^d_{>0},\ s \in \mathbb{R}$,

$$\boldsymbol{\tau_0} \in H(\Omega,\, \mathrm{div}) := \left\{ \boldsymbol{\vartheta} \in \left(L^2(\Omega)\right)^d,\, \mathrm{div}\, \boldsymbol{\vartheta} \in L^2(\Omega) \right\} \tag{3.4}$$

and

$$\boldsymbol{\eta} \in H_0(\widehat{\Pi},\, \mathrm{div}) := \left\{ \boldsymbol{\vartheta} \in H(\widehat{\Pi},\, \mathrm{div}),\ \langle \mathrm{div}\, \boldsymbol{\vartheta} \rangle_{\widehat{\Pi}} = 0 \right\}.$$

Now, we formulate our main result.

Theorem 3.1 *Let A_ε be defined by (1.3), let the conditions (1.1), (1.2) be satisfied, and let the reference cell $\widehat{\Pi}$ be convex. We assume that the right-hand side in (1.5) satisfies (1.4) and u_ε is the exact solution. The solution u_0 of the homogenized problem is required to satisfy (2.15). Also, we assume that the approximation w^1_ε is defined by (2.14) with ψ^ε as in (2.11). Then,*

$$\begin{aligned} \|\nabla(u_\varepsilon - w^1_\varepsilon)\|_{A_\varepsilon} \ \le\ \mathcal{M}_\oplus\Big(w^1_\varepsilon,\, \boldsymbol{\tau_0},\, \boldsymbol{\eta},\, \boldsymbol{\lambda},\, s\Big) := \mathcal{F}^{1/2}\Big(w^1_\varepsilon;\ \boldsymbol{\tau_0},\, \boldsymbol{\eta}, \boldsymbol{\lambda},\, s\Big) + \ \widetilde{C}_{F_\Omega}\, \|\mathrm{div}\boldsymbol{\tau}_0 + f\| \\ +\ \varepsilon^s\, \widetilde{C}\, \|\mathrm{div}\, \boldsymbol{\eta}\|_{\widehat{\Pi}}, \end{aligned} \tag{3.5}$$

where $\mathcal{F}$, $\widetilde{C}_{F_\Omega}$ and $\widetilde{C}$ are defined by (3.3) and (3.10), respectively, $\boldsymbol{\eta}$ and $\boldsymbol{\tau}_0$ are arbitrary functions in $H_0(\widehat{\Pi},\, \mathrm{div})$, and $H(\Omega,\, \mathrm{div})$, respectively, and $\boldsymbol{\lambda} \in \mathbb{R}^d_{>0}$ and $s \in \mathbb{R}$ are free parameters.

Proof. For any $v, w \in H^1_0(\Omega)$ and $\boldsymbol{\tau} \in H(\Omega, \mathrm{div})$, we have

$$\begin{aligned} \int_\Omega A_\varepsilon \nabla(u_\varepsilon - v) \cdot \nabla w &= \int_\Omega (-A_\varepsilon \nabla v \cdot \nabla w + f\, w) \\ &= \int_\Omega (\boldsymbol{\tau} - A_\varepsilon \nabla v) \cdot \nabla w \ + \int_\Omega (\mathrm{div}\boldsymbol{\tau} + f)\, w. \end{aligned} \tag{3.6}$$

We set $w = u_\varepsilon - v$ and estimate the first term in (3.6) as follows:

$$\int_\Omega (\boldsymbol{\tau} - A_\varepsilon \nabla v) \cdot \nabla(u_\varepsilon - v) \le \|\nabla(u_\varepsilon - v)\|_{A_\varepsilon}\, \|A_\varepsilon \nabla v - \boldsymbol{\tau}\|_{A_\varepsilon^{-1}}. \tag{3.7}$$

Henceforth, we select $\boldsymbol{\tau}$, in a special form, namely, on any $\Pi^\varepsilon_{\mathbf{i}}$ we set

$$\boldsymbol{\tau}(\mathbf{x}) = \boldsymbol{\tau}_0(\mathbf{x}) - \varepsilon^s \boldsymbol{\eta}\left(\frac{\mathbf{x} - \mathbf{x_i}}{\varepsilon}\right), \tag{3.8}$$

where

$$\boldsymbol{\eta} \in H_0(\widehat{\Pi},\, \mathrm{div}).$$

Since

$$\mathrm{div}\, \boldsymbol{\tau}\, (\mathbf{x}) = \mathrm{div}\, \boldsymbol{\tau}_0\, (\mathbf{x}) - \varepsilon^s\, \mathrm{div}\, \boldsymbol{\eta}\left(\frac{\mathbf{x} - \mathbf{x_i}}{\varepsilon}\right) \quad \forall \mathbf{x} \in \Pi^\varepsilon_{\mathbf{i}}\ \forall \mathbf{i}$$

and

$$\left\langle \mathrm{div}\, \boldsymbol{\eta}\left(\frac{\cdot - \mathbf{x_i}}{\varepsilon}\right)\right\rangle_{\Pi^\varepsilon_{\mathbf{i}}} = \varepsilon^{d-1}\, \langle \mathrm{div}\, \boldsymbol{\eta}\rangle_{\widehat{\Pi}} = 0,$$

we obtain

$$\begin{aligned} \int_\Omega (\mathrm{div}\, \boldsymbol{\tau} + f)\, (u_\varepsilon - v) &= \int_\Omega (\mathrm{div}\, \boldsymbol{\tau}_0 + f)\, (u_\varepsilon - v) - \sum_{\mathbf{i}} \int_{\Pi^\varepsilon_{\mathbf{i}}} \varepsilon^s \mathrm{div}\, \boldsymbol{\eta}\left(\frac{\cdot - \mathbf{x_i}}{\varepsilon}\right)\ (u_\varepsilon - v) \\ &\le\ C_{F_\Omega}\, \|\mathrm{div}\boldsymbol{\tau}_0 + f\|\, \|\nabla(u_\varepsilon - v)\| + \varepsilon^s \sum_{\mathbf{i}} \varepsilon^{d/2-1} \|\mathrm{div}\, \boldsymbol{\eta}\|_{\widehat{\Pi}}\ C_{\Pi^\varepsilon_{\mathbf{i}}}\, \|\nabla(u_\varepsilon - v)\|_{\Pi^\varepsilon_{\mathbf{i}}}, \end{aligned}$$

where C_{F_Ω} is a constant in the Friedrich's inequality for Ω and $C_{\Pi^\varepsilon_{\mathbf{i}}}$ is a constant in the Poincare's inequality for $\Pi^\varepsilon_{\mathbf{i}}$. It is known (cf. [19]) that for convex $\Pi^\varepsilon_{\mathbf{i}}$

$$C_{\Pi^\varepsilon_{\mathbf{i}}} \le \frac{\mathrm{diam}\, \Pi^\varepsilon_{\mathbf{i}}}{\pi} \quad \text{for any} \quad d \ge 1.$$

In our case

$$\operatorname{diam}\Pi_{\mathbf{i}}^{\varepsilon} = \varrho\varepsilon,$$

where ρ is a certain number of the order 1 depending on geometric properties of $\widehat{\Pi}$ (e.g., if it is the unit cube, then we can set $\rho = \sqrt{d}$).

We use (1.1) and arrive at the estimate

$$\begin{aligned}\int_{\Omega} (\operatorname{div}\boldsymbol{\tau} + f)\,(u_\varepsilon - v) \,&\le C_{F_\Omega}\,\|\operatorname{div}\boldsymbol{\tau}_0 + f\|\,\|\nabla(u_\varepsilon - v)\| + \,\varepsilon^s\,\varepsilon^{\frac{d}{2}-1}\,\|\operatorname{div}\boldsymbol{\eta}\|_{\widehat{\Pi}}\,\sqrt{c_0}\,\varepsilon^{-\frac{d}{2}}\,\varepsilon\,\frac{\varrho}{\pi}\,\|\nabla(u_\varepsilon - v)\| \\ &= C_{F_\Omega}\,\|\operatorname{div}\boldsymbol{\tau}_0 + f\|\,\|\nabla(u_\varepsilon - v)\| \;+\; \varepsilon^s\,\frac{\varrho}{\pi}\,\sqrt{c_0}\,\|\operatorname{div}\boldsymbol{\eta}\|_{\widehat{\Pi}}\,\|\nabla(u_\varepsilon - v)\|.\end{aligned}$$

In view of (1.2), we obtain

$$\int_{\Omega} (\operatorname{div}\boldsymbol{\tau} + f)\,(u_\varepsilon - v) \;\le\; \widetilde{C}_{F_\Omega}\,\|\operatorname{div}\boldsymbol{\tau}_0 + f\|\,\|\nabla(u_\varepsilon - v)\|_{A_\varepsilon} + \varepsilon^s\,\widetilde{C}\,\|\operatorname{div}\boldsymbol{\eta}\|_{\widehat{\Pi}}\,\|\nabla(u_\varepsilon - v)\|_{A_\varepsilon}, \tag{3.9}$$

where

$$\widetilde{C}_{F_\Omega} \;:=\; \frac{C_{F_\Omega}}{\sqrt{c_1}}, \qquad \text{and} \qquad \widetilde{C} \;:=\; \frac{\varrho}{\pi}\sqrt{\frac{c_0}{c_1}}. \tag{3.10}$$

Now (3.6), (3.7), and (3.9) imply the estimate

$$\|\nabla(u_\varepsilon - v)\|_{A_\varepsilon} \;\le\; \|A_\varepsilon\nabla v - \boldsymbol{\tau}_0 - \varepsilon^s\boldsymbol{\eta}\|_{A_\varepsilon^{-1}} \;+\; \widetilde{C}_{F_\Omega}\,\|\operatorname{div}\boldsymbol{\tau}_0 + f\| \;+\; \varepsilon^s\,\widetilde{C}\,\|\operatorname{div}\boldsymbol{\eta}\|_{\widehat{\Pi}}. \tag{3.11}$$

Consider the first term in the right-hand side of the estimate (3.11). We have

$$\begin{aligned}\|A_\varepsilon\nabla v - \boldsymbol{\tau}_0 - \varepsilon^s\boldsymbol{\eta}\|^2_{A_\varepsilon^{-1}} = \sum_{\mathbf{i}} \int_{\Pi_{\mathbf{i}}^{\varepsilon}} \widehat{A}^{-1}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\,\Big(\widehat{A}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\nabla v(\mathbf{x}) - \boldsymbol{\tau}_0(\mathbf{x}) + \varepsilon^s\,\boldsymbol{\eta}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\Big) \\ \cdot\Big(\widehat{A}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\nabla v(\mathbf{x}) - \boldsymbol{\tau}_0(\mathbf{x}) + \varepsilon^s\,\boldsymbol{\eta}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\Big)\;\mathbf{dx}.\end{aligned}$$

We set $v = w_\varepsilon^1$ (cf. (2.10)) and obtain with the help of (3.2)

$$\begin{aligned}\|A_\varepsilon\nabla w_\varepsilon^1 - \boldsymbol{\tau}_0 - \varepsilon^s\boldsymbol{\eta}\|^2_{A_\varepsilon^{-1}} = \sum_{\mathbf{i}} \int_{\Pi_{\mathbf{i}}^{\varepsilon}} \Big(\varepsilon^{2s}\,\widehat{A}^{-1}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\,\boldsymbol{\eta}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\cdot\boldsymbol{\eta}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big) \\ +\; 2\,\widehat{A}^{-1}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\,\varepsilon^s\,\mathbf{g}_{\boldsymbol{\tau_0}}(\mathbf{x})\cdot\boldsymbol{\eta}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big) \\ +\;\widehat{A}^{-1}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\,\mathbf{g}_{\boldsymbol{\tau_0}}(\mathbf{x})\cdot\mathbf{g}_{\boldsymbol{\tau_0}}(\mathbf{x})\Big)\;\mathbf{dx}.\end{aligned}$$

Now we apply Lemma 3.1 for the second term of the right-hand side and arrive at (3.5). □

We note that the estimate (3.11) also holds in a more general setting and can be applied to any reconstruction v (including numerical one) of u_ε with the requirement that $v \in H^1(\Omega)$.

Proposition 3.1 *The free parameters in $\mathcal{M}_\oplus$ can be selected such that it possess the same asymptotic properties as the true error (cf. (2.18)).*

Proof: For a function $\mathbf{v}_1 : \Omega \times \widehat{\Pi} \to \mathbb{R}$, we introduce the notation

$$\mathbf{v}_1^{\star}(\mathbf{x}) := \mathbf{v}_1\left(\mathbf{x}, \frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\right) \qquad \forall\mathbf{x}\in\Pi_{\mathbf{i}}^{\varepsilon} \quad \forall\mathbf{i}.$$

We choose

$$\boldsymbol{\tau_0} := \mathbf{v_0} + \varepsilon\,\mathbf{v_1}^{\star}, \tag{3.12}$$

where $\mathbf{v_0}$, $\boldsymbol{\mu}$, and $\mathbf{v_1}$ are defined by (2). By using these definitions we obtain

$$\operatorname{div}\boldsymbol{\tau}_0 = \operatorname{div}\mathbf{v_0} + \varepsilon\,\operatorname{div}\mathbf{v}^*{}_{\mathbf{1}} = \operatorname{div}\mathbf{v}_0 + \varepsilon\left((\operatorname{div}_{\mathbf{x}}\mathbf{v}_1)^{\star} + \varepsilon^{-1}(\operatorname{div}_{\mathbf{y}}\mathbf{v}_1)^{\star}\right).$$

Since $\mathbf{v_1} := -\mathrm{curl}_{\mathbf{x}}(\widetilde{N}\,\boldsymbol{\mu})$, (cf. 2) the first term in the brackets vanishes and for the second one we use the fact that

$$(\mathrm{div}_{\mathbf{y}}\,\mathbf{v_1})^{\star} = f + \mathrm{div}_{\mathbf{x}}\,\mathbf{v_0}$$

(see, e.g., [7], p.65), and obtain

$$\mathrm{div}\,\boldsymbol{\tau}_0 = \mathrm{div}\,\mathbf{v_0} - f - \mathrm{div}\,\mathbf{v_0} = -f. \tag{3.13}$$

Therefore,

$$\|\nabla(u_\varepsilon - w_\varepsilon^1)\|_{A_\varepsilon} \;\le\; \mathcal{M}_{\oplus}\Big(w_\varepsilon^1, \boldsymbol{\tau}_0,\, \boldsymbol{\eta},\, \boldsymbol{\lambda},\, s\Big) = \mathcal{F}^{1/2}\Big(w_\varepsilon^1;\; \boldsymbol{\tau}_0,\, \boldsymbol{\eta}, \boldsymbol{\lambda},\, s\Big) \;+\; \varepsilon^s\,\widetilde{C}\,\|\mathrm{div}\,\boldsymbol{\eta}\|_{\widehat{\Pi}}, \tag{3.14}$$

where $\mathcal{F}$ is defined by (3.3) with

$$\mathbf{g}_{\boldsymbol{\tau}_0}(\mathbf{x}) \;=\; A_\varepsilon\,\nabla w_\varepsilon^1 - (\mathbf{v_0} + \varepsilon\,\mathbf{v_1}).$$

Then, with the help of (2.17), (2.18), and the triangle inequality, we find that

$$\begin{aligned}\|\mathbf{g}_{\boldsymbol{\tau}_0}(\mathbf{x})\|_{A_\varepsilon^{-1}} &= \|A_\varepsilon\,\nabla(w_\varepsilon^1 - u_\varepsilon + u_\varepsilon) - (\mathbf{v_0} - \varepsilon\,\mathbf{v_1})\|_{A_\varepsilon^{-1}} \\ &\le \|A_\varepsilon\,\nabla(w_\varepsilon^1 - u_\varepsilon)\|_{A_\varepsilon^{-1}} + \|A_\varepsilon\,\nabla u_\varepsilon - (\mathbf{v_0} + \varepsilon\,\mathbf{v_1})\|_{A_\varepsilon^{-1}} \;\le \widetilde{c}\,\sqrt{\varepsilon}.\end{aligned}$$

We set

$$\boldsymbol{\eta} = \mathbf{0}, \quad \lambda \to \mathbf{0}, \qquad \text{and} \qquad s = 1. \tag{3.15}$$

Then,

$$\mathcal{M}_{\oplus} \le c\,\varepsilon^{1/2}. \tag{3.16}$$

□

It is worth noting that in some special cases this asymptotic result can be proved in a simpler way. For example, if

$$A_0 = \langle \widehat{A}^{-1}\rangle_{\widehat{\Pi}}^{-1}$$

(which is always the situation in the one-dimensional case or if $\mathrm{curl}\,\widehat{A}^{-1} = \mathbf{0}$) the simple choice

$$\boldsymbol{\tau}_0 = A_0\,\nabla u_0,$$

implies $\mathrm{div}\,\boldsymbol{\tau}_0 = -f$ and the error $u_\varepsilon - w_\varepsilon^1$ can be estimated by

$$\|\nabla(u_\varepsilon - w_\varepsilon^1)\|_{A_\varepsilon} \;\le\; \mathcal{M}_{\oplus}\Big(w_\varepsilon^1,\, \boldsymbol{\tau}_0,\, \boldsymbol{\eta},\, \boldsymbol{\lambda},\, s\Big) := \mathcal{F}^{1/2}\Big(w_\varepsilon^1;\; \boldsymbol{\tau}_0,\, \boldsymbol{\eta}, \boldsymbol{\lambda},\, s\Big) \;+\; \varepsilon^s\,\widetilde{C}\,\|\mathrm{div}\,\boldsymbol{\eta}\|_{\widehat{\Pi}}, \tag{3.17}$$

where $\mathcal{F}$ is defined by (3.3) with

$$\mathbf{g}_{\boldsymbol{\tau}_0}(\mathbf{x}) \;=\; \varepsilon\,(1-\psi^\varepsilon)\nabla \boldsymbol{N}\Big(\frac{\mathbf{x}-\mathbf{x_i}}{\varepsilon}\Big)\,\nabla u_0(\mathbf{x}) + \;\varepsilon\,\nabla\psi^\varepsilon(\mathbf{x})\,\boldsymbol{N}(\mathbf{y})\cdot\nabla u_0(\mathbf{x}) \;+\; \varepsilon\,\psi^\varepsilon(\mathbf{x})\,\nabla\Big(\nabla u_0(\mathbf{x})\Big)\,\boldsymbol{N}(\mathbf{y}),$$

for all $\mathbf{y} \in \widehat{\Pi}$, $\mathbf{x} \in \Pi_{\mathbf{i}}^{\varepsilon}$, and all cell indices $\mathbf{i}$. Choosing again the parameter set (3.15), we obtain (3.16).

Remark 3.1 *The right-hand side of the majorant (3.5) is the sum of three non-negative terms, which include a global function $\boldsymbol{\tau}_0$ and a function $\boldsymbol{\eta}$ defined on the cell of periodicity. This reflects the specifics of the considered class of problems. Hence, the computation of the majorant is based on the flux of the homogenized solution and a proper selection (cf. Section 4) of the function $\boldsymbol{\eta}$ defined on the cell of periodicity. The scalar parameters λ_i and the power s can be selected in order to minimize the overall value of the majorant. We emphasize that the majorant does not require an approximation of the flux associated with the original periodic problem.*

Remark 3.2 *For the practical use of the majorant $M_{\oplus}$ in (3.5), the free functions and parameters have to be chosen in a proper way which balances the cost for their computation with the gain of having a sharper majorant. One very general strategy in this direction would be to replace*

a) the minimization with respect to $\tau \in H(\Omega, \mathrm{div})$ *by a minimization over a finite dimensional subspace* $S_h(\Omega) \subset H(\Omega, \mathrm{div})$ *(e.g. a finite element space) and*

b) also the minimization with respect to $\eta \in H_0(\widehat{\Pi}, \mathrm{div})$ *by a minimization over a finite dimensional subspace* $S_h(\widehat{\Pi}) \subset H_0(\widehat{\Pi}, \mathrm{div})$.

Then the minimization over the functions $\boldsymbol{\tau}$, $\boldsymbol{\eta}$ *and parameters* $\boldsymbol{\lambda}$, s *can be calculated via an iteration with starting guess*

$$\boldsymbol{\tau}_0^{(0)} = A_0 \nabla u_0 \quad \textit{and} \quad \boldsymbol{\eta}^{(0)} = 0. \tag{3.18}$$

Note that in our numerical experiments we have chosen $\boldsymbol{\tau}$ *and* $\boldsymbol{\eta}$ *according to (3.18) and used the simplified error estimator*

$$\|\nabla(u_\varepsilon - w_\varepsilon^1)\|_{A_\varepsilon} \;\le\; \overline{\mathcal{M}}_\oplus\Big(u_0,\, \varepsilon\Big) := \left| \sum_{\mathbf{i}} \int_{\Pi_{\mathbf{i}}^\varepsilon} \widehat{A}\left(\frac{\mathbf{x} - \mathbf{x_i}}{\varepsilon}\right) \mathbf{g}_{\boldsymbol{\tau}_0}(\mathbf{x}) \cdot \mathbf{g}_{\boldsymbol{\tau}_0}(\mathbf{x})\, \mathbf{dx} \right|^{1/2}, \tag{3.19}$$

where $\mathbf{g}_{\boldsymbol{\tau}_0}(\mathbf{x})$ *is defined by (3.2). These choice always led to satisfactory sharp estimates. If a periodic structure is coarse and consists of relatively few cells (e.g., 25-100) and/or the coefficients of the matrix* $\widehat{A}$ *have jumps, oscillations, etc. then the term* $\varepsilon^s \boldsymbol{\eta}$ *may augment the homogenized flux substantially. If the periodic structure is fine, then the correction term is less significant and its influence can be diminished by increasing values of* s.

Lemma 3.2 *In the one-dimensional case, (3.19) holds as equality provided that*

$$\int_\Omega \left(A_\varepsilon^{-1} \int_0^x f \right) = \int_\Omega \left(A_0^{-1} \int_0^x f \right). \tag{3.20}$$

The proof follows by some tedious but straightforward calculations and will be skipped here.

Remark 3.3 *In certain cases, we may know only numerical approximations to the solutions* N_k, $\widetilde{N}_k$ *and* u_0 *of the auxiliary cell problems (cf. (2.12), (2.19)) and of the homogenized equation (cf. (2.13)). The corresponding approximation errors can be estimated by error majorants of similar types (see [20] - [25] and references therein). Then, the overall error majorant will include both, approximation and modeling errors. A combined modeling-discretization strategy is suggested in [25] (where the modeling error is generated by defeaturing of a complicated structure) and should be used in this case. This topic deserves a separate investigation and lies beyond the framework of this paper which is focused on the principal structure of the guaranteed error bound for homogenized problems.*

3.2 Lower bound of the error

Lower bounds of the modeling error allows us to estimate numerically the sharpness of the error majorant and to evaluate the *efficiency* of error estimation. A lower bound of the energy error norm can be derived by means of the well known relation (see, e.g., [23], pp. 85-86):

$$\|\nabla(u_\varepsilon - v)\|_{A_\varepsilon}^2 \;=\; \sup_{w \in H_0^1(\Omega)} \mathcal{M}_\ominus^2(v;\, w) := \sup_{w \in H_0^1(\Omega)} \int_\Omega \Big(2\,(f\, w \,-\, A_\varepsilon\, \nabla v \cdot \nabla w) \;-\; A_\varepsilon \nabla w \cdot \nabla w \Big). \tag{3.21}$$

Clearly, for any $w \in H_0^1(\Omega)$ it holds $\|\nabla(u_\varepsilon - v)\|_{A_\varepsilon} \;\ge\; \mathcal{M}_\ominus(v;\, w)$. We set $v = w_\varepsilon^1$ (cf. 2.10) and make the ansatz $w = \rho_{\max} z$ for some function $z \in H_0^1(\Omega)$, where the normalization $\rho_{\max}$ is defined by the relation

$$\rho_{\max} \;=\; \frac{\int_\Omega \left(f\, \omega \,-\, A_\varepsilon\, \nabla w_\varepsilon^1 \cdot \nabla z\right)}{\int_\Omega A_\varepsilon \nabla z \cdot \nabla z}$$

so that $\mathcal{M}_\ominus^2(w_\varepsilon^1;\, \rho z)$ is maximal as a quadratic function with respect to ρ. Inserting this into the definition of $\mathcal{M}_\ominus^2(v;\, w)$, we get (with a slight abuse of notation)

$$\mathcal{M}_\ominus(w_\varepsilon^1;\, z) \;:=\; \frac{\left|\int_\Omega \left(f\, z \,-\, A_\varepsilon\, \nabla w_\varepsilon^1 \cdot \nabla z\right)\right|}{\|\nabla z\|_{A_\varepsilon}} = \frac{\left|\int_\Omega \left(A_0\, \nabla u_0 \,-\, A_\varepsilon\, \nabla w_\varepsilon^1\right) \cdot \nabla z\right|}{\|\nabla z\|_{A_\varepsilon}} \tag{3.22}$$

and this is a lower bound of the modeling error $\|\nabla(u_\varepsilon - w_\varepsilon^1)\|_{A_\varepsilon}$.

We note that for the periodic functions f the choice of a cell periodic test function z is preferable. More precisely, we can make the ansatz

$$z := w_\varepsilon^1 - u_0. \tag{3.23}$$

In this case, we obtain

$$\mathcal{M}_{\ominus,\,\mathrm{per}}(w_\varepsilon^1;\ \mathbf{0}) = \frac{\left|\int_\Omega \mathbf{q}\cdot\nabla\varphi_\varepsilon^0\right|}{\|\nabla\,\varphi_\varepsilon^0\|_{A_\varepsilon}}, \tag{3.24}$$

where

$$\mathbf{q} := (A_0 - A_\varepsilon)\nabla\, u_0 \ -\ \varepsilon\, A_\varepsilon\, \nabla\varphi_\varepsilon^0.$$

An alternative choice of the test function z is $z = u_0$. Then, $\mathcal{M}_\ominus$ in (3.22) becomes

$$\mathcal{M}_{\ominus,\,\mathrm{aper}}(w_\varepsilon^1;\ \mathbf{0}) = \frac{\left|\int_\Omega \mathbf{q}\cdot\nabla u_0\right|}{\|\nabla\, u_0\|_{A_\varepsilon}}\,. \tag{3.25}$$

Remark 3.4 *By using the upper and lower bounds (denoted by $M_\oplus$ and $M_\ominus$, respectively) of the modeling error, we define the number*

$$\kappa := \frac{M_\oplus}{M_\ominus}\,, \tag{3.26}$$

which is a computable upper bound of the efficiency index

$$i_{\mathrm{eff}}^{\mathrm{up}} := \frac{M_\oplus}{\|\nabla(u_\varepsilon - w_\varepsilon^1)\|_{A_\varepsilon}}$$

and gives insights of the quality of the error estimation. We also denote by $i_{\mathrm{eff}}^{\mathrm{low}}$ the efficiency index of the lower bound defined by

$$i_{\mathrm{eff}}^{\mathrm{low}} := \frac{M_\ominus}{\|\nabla(u_\varepsilon - w_\varepsilon^1)\|_{A_\varepsilon}}$$

4 Numerical experiments

In this section, we apply the theory introduced in Section 3 to several one- and two-dimensional test problems. For this purpose, we select problems used in publications related to analysis of homogenized and interface problems, e.g., see [8, 14, 16, 17, 30]. Our main goal is to validate the sharpness of the upper bound $\overline{\mathcal{M}}_\oplus\Big(u_0,\,\varepsilon\Big)$ (cf.(3.19)) setting $\boldsymbol{\tau}_0 = A_0\,\nabla u_0$, and lower bounds $\mathcal{M}_{\ominus,\,\mathrm{per}}(w_\varepsilon^1;\ \mathbf{0})$ (cf. (3.24)) and $\mathcal{M}_{\ominus,\,\mathrm{aper}}(w_\varepsilon^1;\ \mathbf{0})$ (cf. (3.25)). In the first series of tests, we set $d = 1$ and $\Omega = (0,\,1)$. Then, $u_\varepsilon \in H_0^1(\Omega)$ is defined by the relation

$$\int_0^1 A_\varepsilon\, u_\varepsilon'\, v' = \int_0^1 f v \quad \forall v \in H_0^1\,(\Omega)\,. \tag{4.1}$$

Test 1.1 $\widehat{A}\,(y) := \begin{cases} 1, \text{ if } 0 < y \le 1/2, \\ 2, \text{ if } 1/2 < y < 1, \end{cases}$ and A_ε is defined as in (1.3). The right-hand side is given by $f := \sin\,(2\,\pi\frac{x}{\varepsilon})$.

Test 1.2 $A_\varepsilon(x) = 2 + \cos\,(2\,\pi\frac{x}{\varepsilon})$, $f := \mathrm{e}^{10\,x}$.

In Tests 1.1 and 1.2, the explicit forms of A_0, u_0', $\frac{\mathrm{d}N}{\mathrm{d}y}$ and N are known $\Big($they can be found from (2.9) and (2.4)$\Big)$:

Test 1.1 $A_0(x) = \frac{4}{3}$, $u_0' = \frac{3\varepsilon}{8\pi}\cos(2\pi x \varepsilon^{-1})$, $\frac{\mathrm{d}N}{\mathrm{d}y}(y) = \begin{cases} -\frac{1}{3}, & \text{if } 0 < y \le 1/2, \\ \frac{1}{3}, & \text{if } 1/2 < y < 1, \end{cases}$

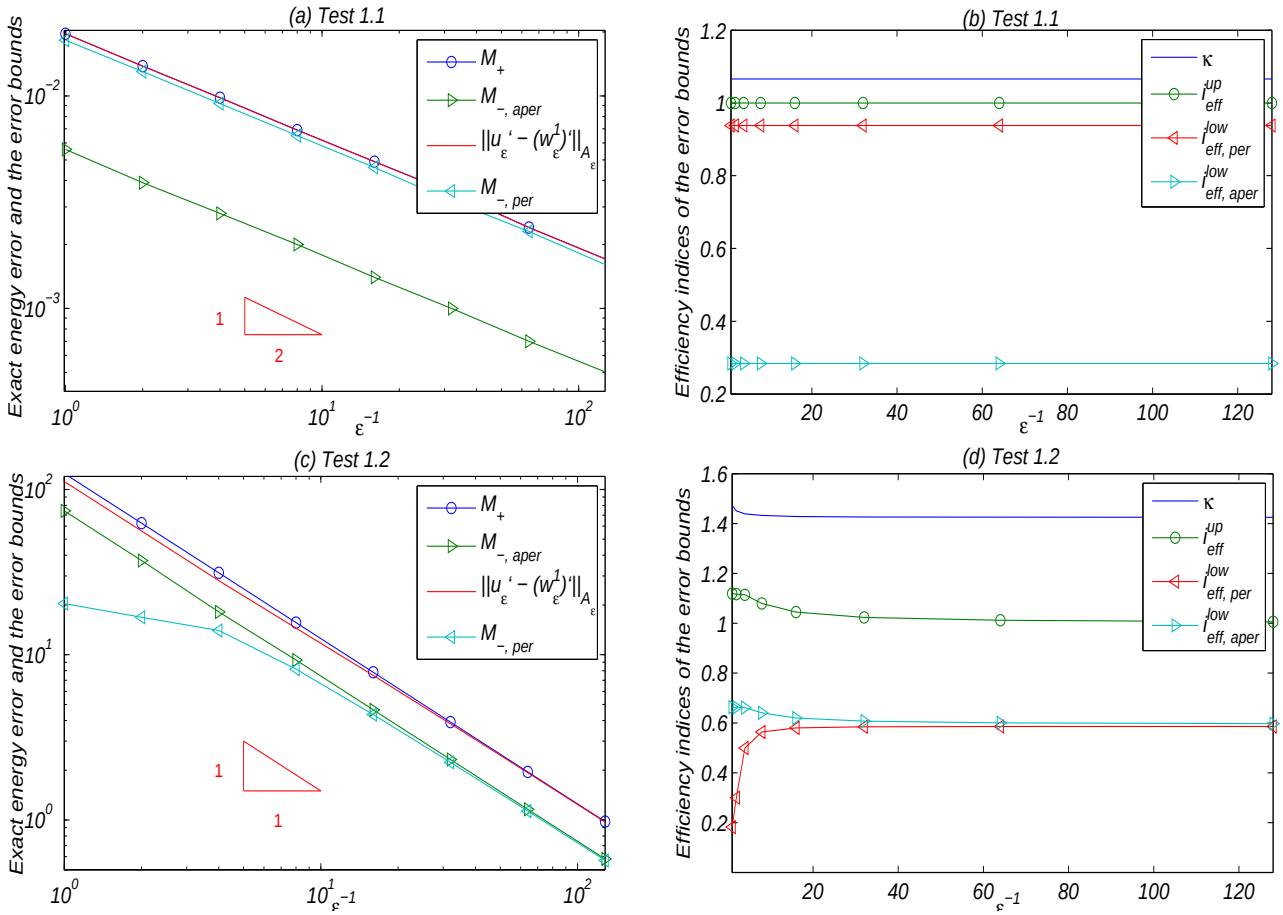
and $N(y) = \begin{cases} -\frac{y}{3} + \frac{1}{12}, & \text{if } 0 < y \le 1/2, \\ \frac{y}{3} - \frac{1}{4}, & \text{if } 1/2 < y < 1. \end{cases}$

Test 1.2 $A_0(x) = \sqrt{3}$, $u_0' = -\frac{3^{-0.5}}{10}e^{10x} + \frac{3^{-0.5}}{100}e^{10}$, $\frac{\mathrm{d}N}{\mathrm{d}y}(y) := 1 - \sqrt{3}(2 + \cos(2\pi y))^{-1}$, and $N(y) = \int \left(1 - \sqrt{3}(2 + \cos(2\pi y))^{-1}\right) \mathrm{d}y$.

In Test 1.1, f is a periodic function. Therefore, it is natural to expect that the minorant $M_{\ominus,\mathrm{per}}$ (in which the periodicity is taken into account) will provide better results. In Test 1.2, the right-hand side is represented by a non-periodical function, and, therefore, we expect that $M_{\ominus,\mathrm{aper}}$ will be better (at least for problems with relatively small amount of cells). The corresponding numerical results are depicted in Figure 2 and confirm the proposed choice of the lower error bound.

We note that in Test 1.1 $\int_\Omega \left(A_\varepsilon^{-1}\int_0^x f\right) = \int_\Omega \left(A_0^{-1}\int_0^x f\right)$, and (as it follows from Lemma 3.2) the majorant (3.19) coincides with the error. This fact is confirmed numerically (see Fig. 2 a, b). Test 1.2 shows that the majorant and minorants are quite sharp if the number of cells is sufficiently large $\Big($regardless of the condition $\int_\Omega \left(A_\varepsilon^{-1}\int_0^x f\right) = \int_\Omega \left(A_0^{-1}\int_0^x f\right)\Big)$.

Figure 2: Error bounds (left) and efficiency indices (right) for Test 1.1 and Test 1.2.

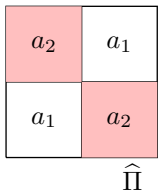


Now, we consider the case $d = 2$ and $\Omega = (0,1)^2$. Let $u_\varepsilon \in H_0^1(\Omega)$ be defined by the relation

$$\int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla v = \int_\Omega f v \qquad \forall v \in H_0^1(\Omega),$$

in which A_ε is derived from $\widehat{A} := aI$ as in (1.3) where the coefficient is defined as follows:

$$a := \begin{cases} a_1 > 0 & \text{in } \left(0, \frac{1}{2}\right)^2 \cup \left(\frac{1}{2}, 1\right)^2, \\ a_2 > 0 & \text{in } (0,1)^2 \setminus \left(\left(0, \frac{1}{2}\right)^2 \cup \left(\frac{1}{2}, 1\right)^2\right). \end{cases} \tag{4.2}$$



$\widehat{\Pi}$

f is chosen such that

$$u_0(\mathbf{x}) = x_1\, x_2\, (1 - x_1)(1 - x_2). \tag{4.3}$$

Then (see, e.g., [15], pp. 35 - 39.), $A_0 = \sqrt{a_1\, a_2}$.
We use results of [16], where the exact solution of the cell problem

$$\begin{aligned} &\frac{\partial}{\partial y_i}\left(\widehat{A}_{ij}(\mathbf{y})\,\frac{\partial N_k(\mathbf{y})}{\partial y_j}\right) = \frac{\partial}{\partial y_i}\,\widehat{A}_{ik}(\mathbf{y}) \quad \text{in } \widehat{\Pi} = (0,\,1)^2, \\ &N_k \text{ is periodic in } \widehat{\Pi}, \\ &\langle N_k \rangle_{\widehat{\Pi}} = 0, \end{aligned} \tag{4.4}$$

is found as

$$N_k(\mathbf{y}) = \nu\left(\frac{\mathbf{y}+1}{2}\right) + y_k, \quad k = 1,\,2, \tag{4.5}$$

where $\nu(\mathbf{y})$ is the unique solution of the problem

$$-\mathrm{div}\,(a\,\nabla\nu) \;=\; 0 \quad \text{in} \quad (-1,1)^2 \tag{4.6}$$

with homogenous Dirichlet boundary conditions and a is defined by (4.2). This solution is given in polar coordinates (r,θ) centered in the origin by the relation

$$\nu = r^\gamma\, \mu(\theta), \tag{4.7}$$

Here, $\mu(\theta)$ is a continuous, piecewise smooth function, and γ depends on $\frac{a_1}{a_2}$. We note that ν has a restricted regularity (namely, $\nu \in H^{1+\gamma-\epsilon}(\widehat{\Pi})$ for any $\epsilon > 0$).

In order to verify the efficiency of error majorant, we consider two tests in which the exact solution of (4.4) have different regularity properties.

Test 2.1 Let $a_1 = 5.0$, $a_2 = 1.0$. In this case, (4.6) yields $\gamma = 0.53544094560$ and $\theta = \frac{\pi}{2}$ (cf. system (3.2) in [16]) Then, $\nu \in H^{3/2}(\Omega)$.

Test 2.2 Now, we set $\gamma = 0.1$ and $\theta = \frac{\pi}{2}$. Then, (4.6) holds if $a_1 = 161.4476387975881$ and $a_2 = 1.0$ (cf. system (3.2) in [16]) In this case example, the exact solution belongs to $H^{1+\alpha}(\widehat{\Pi})$ with $0 < \alpha < 0.1$, i.e., has very low Sobolev regularity.

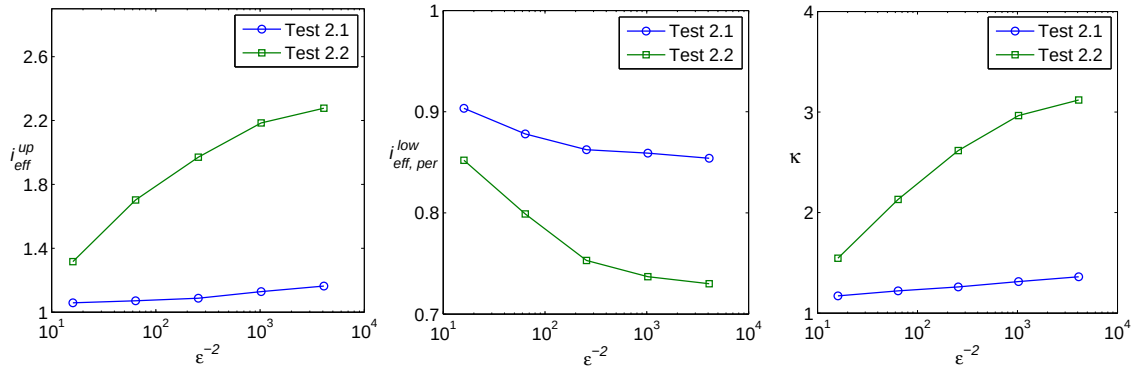
To quantify the efficiency of (3.19) and (3.24), we compare the error bounds with the exact error

$$e := \|\nabla(u_\varepsilon - w_\varepsilon^1)\|_{A_\varepsilon}. \tag{4.8}$$

Since the exact solutions u_ε are unknown, we compute (using linear finite elements) "reference" solutions u_{ref} on a mesh $\mathcal{T}_h$ with the mesh width $h \ll \varepsilon$. By using u_{ref} we define approximate computable efficiency indices via

$$i_{\mathrm{eff}}^{\mathrm{up}} = \frac{M_\oplus(w_\varepsilon^1;\,\mathbf{0},\,\mathbf{1},\,1)}{\|\nabla(u_{\mathrm{ref}} - w_\varepsilon^1)\|_{A_\varepsilon}} \quad \text{and} \quad i_{\mathrm{eff,\,per}}^{\mathrm{low}} = \frac{M_{\ominus,\,\mathrm{per}}(w_\varepsilon^1;\,\mathbf{0})}{\|\nabla(u_{\mathrm{ref}} - w_\varepsilon^1)\|_{A_\varepsilon}}. \tag{4.9}$$

Figure 3: *Efficiency of error majorant and minorant for Test 2.1 - 2.2.*



In Fig. 3 we show these quantities together with the quantity κ as in (3.26). We see that in both tests the estimates adequately reproduce values of the modeling error. As it can be expected a priorily, the estimates are better in the first case (related to a more regular ν). However, even if ν has minimal regularity the upper efficiency index does not exceed 2.3 and the lower one does not go below 0.7.

Acknowledgment: The authors are grateful to Swiss National Science Foundation for supporting this research under the grants 200021_119809 and 200020_134621. The first author wishes also to thank RFBR (grant 11-01-00531-a) for partial support. We are also indebted to Dr. Christian Wüst – the finite element computations have been performed on the basis of his program *JCFD*.

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