Deformation Quantization and Reduction

Alberto S. Cattaneo

Abstract. This note is an overview of the Poisson sigma model (PSM) and its applications in deformation quantization. Reduction of coisotropic and pre-Poisson submanifolds, their appearance as branes of the PSM, quantization in terms of L_∞- and A_∞-algebras, and bimodule structures are recalled. As an application, an "almost" functorial quantization of Poisson maps is presented if no anomalies occur. This leads in principle to a novel approach for the quantization of Poisson–Lie groups.

1. Introduction

The Poisson sigma model (PSM) [**I, SS**] is a two-dimensional topological field theory with target a Poisson manifold (M, π). It is defined by the action functional

$$S = \int_\Sigma \eta \, \mathrm{d}X + \frac{1}{2}\pi(X)\eta\,\eta, \tag{1.1}$$

where the pair (X, η) is a bundle map $T\Sigma \to T^*M$, and Σ is a two-manifold.

The Hamiltonian study of the PSM on $\Sigma = [0, 1] \times \mathbb{R}$ leads [**CF01**] to the construction of the symplectic groupoid of M whenever T^*M is an integrable Lie algebroid. The functional-integral perturbative quantization yields [**CF00**] Kontsevich's deformation quantization [**K**] if Σ is the disk and one chooses vanishing boundary conditions for η.

General boundary conditions ("branes") compatible with symmetries and the perturbative expansion were studied in [**CF04**] and turned out to correspond to coisotropic submanifolds. In [**Ca, CFa05**] (see also references therein) general boundary conditions compatible just with symmetries were classified; they correspond to what we now call pre-Poisson submanifolds (a generalization of the familiar notion of pre-symplectic submanifolds in the symplectic case).

The perturbative functional-integral quantization with branes gives rise to deformation quantization of the reduced spaces [**CF04, CFa05**]. The many-brane

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case leads to the construction of bimodules and morphisms between them [**CF04**] and potentially to a method for quantizing Poisson maps in an almost functorial way (see Section 8). This problem is also discussed in [**B04**] where results (and obstructions) for the symplectic case are given. The linear case (i.e., the case when the Poisson manifold is the dual of a Lie algebra and the coisotropic submanifold is an affine subspace) leads to many interesting results in Lie theory [**CF04, CT**].

A more general approach leads to a cohomological description of coisotropic submanifolds by L_∞-algebras naturally associated to them [**OP**] and to a deformation quantization in terms of (possibly nonflat) A_∞-algebras. A natural way to reinterpret these results is by associating a graded manifold to the coisotropic submanifold and use a graded version of Kontsevich's L_∞-quasi-isomorphism [**CF07, LS**]. This may also be regarded as a duality for the PSM with target a graded manifold [**CF07, C06**].

In the absence of the so-called anomaly—i.e., when the A_∞-algebra can be made flat—one can go down to cohomology.

This paper is an overview of all these themes. It contains a more extended discussion, Section 7, of deformation quantization for graded manifolds and an introduction, Section 8, to methods for an almost functorial quantization of morphisms whenever the anomaly is not present.

Plan of the paper. In Section 2 we recall properties and reduction of coisotropic, presymplectic and pre-Poisson submanifolds collecting results in [**Ca, CFa05, CZ, CZbis**]. In Section 3 we give a simple derivation of the L_∞-structure associated [**OP**] to a coisotropic submanifold; we also recall the BFV (Batalin–Fradkin–Vilkovisky) formalism [**BF, BV**] and its relation [**S**] to the L_∞-structure. In Section 4 we recall the Hamiltonian description of the PSM and how coisotropic [**CF04**] and pre-Poisson [**Ca, CFa05**] submanifolds show up as boundary conditions (branes) for the PSM. Perturbative functional-integral quantization of the PSM [**CF00**] is recalled in Section 5 with results for coisotropic [**CF04**] and pre-Poisson [**CFa05**] branes. Section 6 deals with the reinterpretation of these results in terms of a duality for the PSM with target a graded manifold [**CF07, C06**]. In Section 7 we recall the Formality Theorem for graded manifolds [**K, CF07**] and its applications in deformation quantization [**CF07, LS**]; the potential anomaly[**CF04, CF07**] is discussed; subsection 7.3 contains a description of the induced properties in cohomology and is original; subsection 7.5 contains a new description of methods for quantizing the inclusion map of a Poisson submanifold; in particular, it contains a proof of a conjecture by [**CR**] that the deformation quantization of a Poisson submanifold determined by central constraints may be obtained by quotienting Kontsevich's deformation quantization by the ideal generated by the image of the constraints under the Duflo–Kirillov–Kontsevich map. Finally, Section 8 recalls the many-brane case [**CF04**] and, in the absence of anomalies, presents a novel method for the quantization of Poisson maps compatible with compositions; the application to the quantization of Poisson–Lie groups is briefly described.

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2. Reduction

We recall reduction in the context of symplectic and Poisson manifolds. We will essentially follow [**CZ**] (see also [**CZbis**]). Let us first recall the linear case.

A symplectic space is a (possibly infinite dimensional) linear space V equipped with a nondegenerate skew-symmetric bilinear form ω, called a symplectic form. By nondegenerate we mean here that $\omega(v,w)=0\ \forall v\in V \Rightarrow w=0$ (this is sometimes called a weak symplectic form). We denote by $\omega^\sharp$ the induced linear map $V\to V^*$: $v\mapsto \omega(v,\ )$. With this notation, ω is nondegenerate iff $\omega^\sharp$ is injective.

The restriction of ω to a subspace W is in general degenerate, the kernel of $(\omega|_W)^\sharp$ being $W^\perp\cap W$, where

$$W^\perp := \{v\in V : \omega(v,w)=0\ \forall w\in W\}$$

is the symplectic orthogonal to W. Thus, the reduced space $\underline{W} := W/(W\cap W^\perp)$ is automatically endowed with a symplectic form.

An extreme case is when $W\cap W^\perp = 0$; i.e., when $(\omega|_W)^\sharp$ is nondegenerate. In this case, W is called a symplectic subspace and $\underline{W}=W$. In finite dimensions, one has the additional properties that $W^\perp$ is also symplectic and that

$$W\oplus W^\perp = V. \tag{2.1}$$

That is, in finite dimensions a symplectic subspace may be equivalently described as a subspace which admits a symplectic complement. For this reason, it may as well be called a cosymplectic subspace.

The other nontrivial extreme case is when $W^\perp\subset W$. In this case W is called a coisotropic subspace and $\underline{W}=W/W^\perp$. The coisotropic case is in some sense the only case one has to consider, as we have the following simple

Lemma 2.1. *Let W be a subspace of a finite dimensional symplectic space V. Then it is always possible to find a symplectic subspace W' of V which contains W as a coisotropic subspace.*

The Lemma is proved by adding W what is missing to make it symplectic. Any complement to $W+W^\perp$ will do (see Lemma 3.1 in [**CZ**] for details).

A symplectic manifold is a smooth manifold endowed with a closed, nondegenerate 2-form. Here nondegenerate means that the bundle map $\omega^\sharp\colon TM\to T^*M$ induced by the 2-form ω on M is injective. The restriction of ω to a submanifold C is in general degenerate. The kernel of $(\omega|_C)^\sharp$ is the characteristic distribution $T^\perp C\cap TC$ which in general is not smooth. Here $T^\perp C$ is the bundle of symplectic orthogonal spaces to TC in the restriction T_CM of TM to C. The submanifold N is called presymplectic if its characteristic distribution is smooth (i.e., $T^\perp C\cap TC$ is a subbundle of TC); by the closedness of ω it is automatically integrable. The corresponding leaf space $\underline{C}$ is called the reduced space. If it is a manifold, it is endowed with a unique symplectic form whose pullback to C is equal to the restriction of ω. The extreme nontrivial examples of presymplectic submanifolds are the symplectic and coisotropic submanifolds. A submanifold C is called symplectic (coisotropic) if T_xC is symplectic (coisotropic) in T_xM for every $x\in C$. The reduction of a symplectic submanifold is the manifold itself. If C is a submanifold of a finite dimensional manifold M, by Lemma 2.1 we may find a symplectic subspace of T_xM containing T_xC as a coisotropic subspace for every $x\in C$. These subspaces may be chosen to be glued together smoothly if the submanifold is presymplectic. Namely, we have the following

Proposition 2.2 ([**CZ**]). *Let C be a presymplectic submanifold of a symplectic manifold M. Then it is always possible to find a symplectic submanifold which contains C as a coisotropic submanifold.*

Moreover, one can show that these symplectic extensions are neighborhood equivalent. Namely, given two such extensions C' and C'', there exists a tubular neighborhood U of C such that $U \cap C'$ and $U \cap C''$ are related by a symplectomorphisms (i.e., a diffeomorphism compatible with the symplectic form) of U which fixes C.

We now move to the Poisson case. We will only work in finite dimensions. A Poisson space is a linear space V endowed with a bivector π (i.e., an element of $\Lambda^2 V$). We denote by $\pi^\sharp$ the induced linear map $V^* \to V$, $\alpha \mapsto \pi(\alpha, \ )$. A finite dimensional symplectic space (V, ω) is automatically Poisson with $\pi^\sharp = (\omega^\sharp)^{-1}$. The Poisson generalization of the notion of symplectic orthogonal of a subspace W is the image under $\pi^\sharp$ of its annihilator $W^0 := \{\alpha \in V^* : \alpha(w) = 0\ \forall w \in W\}$. In the symplectic case, $\pi^\sharp(W^0) = W^\perp$. Condition (2.1) may be replaced by $W \oplus \pi^\sharp(W^0) = V$; a subspace W satisfying it is called cosymplectic; it may equivalently be characterized by the condition that the projection of π to $\Lambda^2(V/W)$ is symplectic. A coisotropic subspace is analogously defined as a subspace W with $\pi^\sharp(W^0) \subset W$.

A Poisson manifold is a manifold M endowed with a bivector field π (i.e., a section of $\Lambda^2 TM$) such that the bracket $\{\, f\, ,\, g\,\} := \pi(\mathrm{d}f, \mathrm{d}g)$, $f, g \in C^\infty(M)$ satisfies the Jacobi identity. Equivalently, $[\,\pi\, ,\, \pi\,] = 0$ with the Schouten–Nijenhuis bracket. This also amounts to saying that $(C^\infty(M), ., \{\ \ ,\ \ \})$ is a Poisson algebra. We denote by $\pi^\sharp \colon T^*M \to TM$ the corresponding bundle map. A finite dimensional symplectic manifold (M, ω) is Poisson with $\pi^\sharp = {\omega^\sharp}^{-1}$. A submanifold C is called cosymplectic (coisotropic) if T_xC is so in T_xM $\forall x \in C$. We will denote by N^*C—the conormal bundle—the vector bundle over C with fiber at x given by $N_x^*C := (T_xC)^0 \subset T_x^*M$. Observe that $\pi^\sharp(N^*C)$ is a (singular) distribution on C if C is coisotropic. Invariant functions form naturally a Poisson algebra. If the leaf space $\underline{C}$ is a manifold, it is naturally a Poisson manifold. A cosymplectic manifold C is also automatically a Poisson manifold (not a Poisson submanifold though). One way to see this is to think of a Poisson manifold as a manifold foliated by symplectic leaves. A cosymplectic submanifold intersects symplectic leaves cleanly, and each intersection is a symplectic submanifold of the symplectic leaf; thus, a cosymplectic submanifold is foliated in symplectic leaves, which makes it into a Poisson manifold. The induced Poisson bracket may also be obtained by Dirac's procedure.

We now wish to describe a Poisson generalization of the notion of presymplectic submanifold leading to a generalization of Proposition 2.2. Namely, we call a submanifold C pre-Poisson if $\pi^\sharp(N^*C) + TC$ has constant rank along C. (In [**CFa05**] and references therein, this was called a "submanifold with strong regularity constraints".) In the symplectic case, this is equivalent to C being presymplectic.

An equivalent definition of a pre-Poisson submanifold C amounts to asking that the bundle map $\phi \colon N^*C \to NC$ obtained by composing the injection $N^*C \to T_C^*M$ with $\pi^\sharp$ and finally with the projection $T_CM \to NC$ should have constant rank. As special cases we recognize coisotropic submanifolds ($\phi = 0$) and cosymplectic submanifolds (ϕ surjective). We have the following

Proposition 2.3 ([**CFa05, CZ**]). *Let C be a pre-Poisson submanifold of a Poisson manifold M. Then it is always possible to find a cosymplectic submanifold*

M' which contains C as a coisotropic submanifold. Moreover, this extension M' is unique up to neighborhood equivalence.

In the following we will also need a description in terms of Lie algebroids. Recall that a Lie algebroid is a vector bundle E over a smooth manifold M endowed with a bundle map $\rho\colon E \to TM$ (the anchor map) and a Lie algebra structure on $\Gamma(E)$ with the property $[\,a\,,\,fb\,] = f\,[\,a\,,\,b\,] + \rho(a)f\,b$ for every $a, b \in \Gamma(E)$, $f \in C^\infty(M)$. The tangent bundle itself is a Lie algebroid. The cotangent bundle of Poisson manifold is also a Lie algebroid with $\rho = \pi^\sharp$ and $[\,\mathrm{d}f\,,\,\mathrm{d}g\,] := \mathrm{d}\,\{\,f\,,\,g\,\}$. If C is a pre-Poisson submanifold, then $AC := N^*C \cap {\pi^\sharp}^{-1}TC$ is a Lie subalgebroid [**CFa05**]. Its anchor is defined just by restriction; as for the bracket, one has to extend sections of AC to sections of T^*M in a neighborhood of C, use the bracket on T^*M and finally restrict; it is not difficult to check that the result is independent of the extension. Observe that for C coisotropic, one simply has $AC = N^*C$.

3. Cohomological descriptions

The complex $\Gamma(\Lambda TM)$ of multivector fields on a smooth manifold M is naturally endowed with a Lie bracket of degree -1, the Schouten–Nijenhuis bracket. Let $\mathcal{V}(M) := \Gamma(\Lambda TM)[1]$ be the corresponding graded Lie algebra (GLA). A Poisson bivector field is now the same as an element π of $\mathcal{V}(M)$ of degree one which satisfies $[\,\pi\,,\,\pi\,] = 0$. In general, a self-commuting element of degree 1 in a GLA is called a Maurer–Cartan (MC) element. Observe that $[\,\pi\,,\ \ ]$ is a differential; actually, it is the Lie algebroid differential corresponding to the Lie algebroid structure on T^*M.

Let C be a submanifold and NC its normal bundle, canonically defined as the quotient by TC of the restriction T_CM of TM to C. It is also the dual bundle of the conormal bundle N^*C introduced above. Let A be the graded commutative algebra $\Gamma(\Lambda NC)$. Denote by $P\colon \Gamma(\Lambda TM) \to A$ the composition of the restriction to C with the projection $T_CM \to NC$. It is not difficult to check that $\operatorname{Ker} P$ is a Lie subalgebra. Moreover, C is coisotropic iff $\pi \in \operatorname{Ker} P$. In this case, there is a well-defined differential δ on A acting on $X \in A$ by

$$\delta X := P\left[\,\pi\,,\,\tilde{X}\,\right], \quad \tilde{X} \in P^{-1}(X).$$

This is the differential corresponding to the Lie algebroid structure on N^*C. Observe that H^0_δ is the algebra of invariant functions on C.

The projection P admits a section if M is a vector bundle over C. In particular, this is true if M is the normal bundle of C. The section i for $P\colon \Gamma(\Lambda T(NC)) \to A$ is given as follows: If f is a function on C, one defines if as the pullback of f by the projection $NC \to C$. If X is a section of NC, one defines iX as the unique vertical vector field $\tilde{X}$ on NC which is constant along the fibers and such that $P\tilde{X} = X$. Finally, i is extended to sections of ΛNC so that it defines an algebra morphism. It turns out that $i(\Gamma(NC))$ is an abelian subalgebra.

Let us now choose an embedding of NC into our Poisson manifold (M, π). By restriction π defines a Poisson structure on NC, so a MC element in $\mathcal{V}(NC)$. Following Voronov [**V**], we define the derived brackets

$$\begin{aligned} &\lambda_k\colon A^{\otimes k} \to A, \\ &\lambda_k(a_1, \ldots, a_k) := P\left([\,[\,\ldots[\,[\,\pi\,,\,ia_1\,]\,,\,ia_2\,]\,,\,\ldots\,]\,,\,ia_k\,]\right), \end{aligned} \tag{3.1}$$

and $\lambda_0 := P(\pi) \in A$. Observe that $\lambda_0 = 0$ iff C is coisotropic; in this case, $\delta = \lambda_1$. In general Voronov proved the following

Theorem 3.1. *Let $\mathfrak{g}$ be a GLA, $\mathfrak{h}$ an abelian subalgebra, i the inclusion map. Let P be a projection to $\mathfrak{h}$ such that $P \circ i = id$ and that* $\operatorname{Ker} P$ *is a Lie subalgebra. Then the derived brackets of every MC element of $\mathfrak{g}$ define an L_∞-structure on $\mathfrak{h}$.*

This means that the operations λ_k satisfy certain quadratic relations which in particular for $\lambda_0 = 0$ imply that λ_1 is a differential for λ_2 and that λ_2 satisfies the Jacobi identity up to λ_1-homotopy. So the λ_1-cohomology inherits the structure of a GLA.

In our case A is also a graded commutative algebra and the multibrackets are multiderivations. In this case we say that we have a P_∞-algebra (P for Poisson). In particular, when C is coisotropic, $H_{\lambda_1} = H_\delta$ is a graded Poisson algebra. The Poisson structure in degree zero—i.e., on invariants functions—is the same as the one described in the previous Section.

Observe that the P_∞-structure depends on a choice of embedding $NC \hookrightarrow M$. It is possible to show that different choices lead to L_∞-isomorphic algebras. We will return on this in a forthcoming paper [**CS**].

The P_∞-structure appeared first in [**OP**] as a tool to describe deformations of a coisotropic submanifold. In [**CF07**] it was rediscovered as the semiclassical limit of the quantization of coisotropic submanifolds. We will return on quantization in Section 7.

3.1. The BFV method. The Batalin–Fradkin–Vilkovisky (BFV) method is an older method to describe coisotropic submanifold in terms of a differential graded Poisson algebra (DGPA) [**BF, BV**]. Its advantage is that DGPAs are more manageable than P_∞-algebras. The disadvantage is that it works only under certain assumptions. We will essentially follow Stasheff's presentation [**S97**].

Suppose first that the coisotropic submanifold C is given by global constraints, or, equivalently, that the normal bundle of C is trivial. Let $\{y^\mu\}_{\mu=1,\dots,n}$, $n = \operatorname{codim} C$, be a basis of constraints (equivalently, a basis of sections of NC or a basis of transverse coordinates). Let $\mathcal{B}$ be the free graded commutative algebra generated by a set $\{b^\mu\}$ of odd variables of degree 1 (the "ghost momenta"). On $C^\infty(NC) \otimes \mathcal{B}$ one defines the Koszul differential $\delta_0 := y^\mu \frac{\partial}{\partial b^\mu}$. The δ_0 cohomology is then concentrated in degree 0 and is $C^\infty(C)$. Let us also introduce a set $\{c_\mu\}$ of odd variables of degree -1 (the "ghosts") and the free graded commutative algebra $\mathcal{C}$ they generate. On $C^\infty(NC) \otimes \mathcal{B} \otimes \mathcal{C}$ one has a unique Poisson structure with $\{\, b^\mu \,,\, c_\nu \,\} = \{\, c_\nu \,,\, b^\mu \,\} = \delta^\mu_\nu$. Then δ_0 is a Hamiltonian vector field with Hamiltonian function $\Omega_0 = y^\mu c_\mu$. On this enlarged algebra the δ_0-cohomology is $C^\infty(NC) \otimes \mathcal{C} \simeq \Gamma(\Lambda NC)$.

By choosing an embedding $NC \hookrightarrow M$, we get a Poisson structure on $C^\infty(NC)$ which we can extend to $C^\infty(NC) \otimes \mathcal{B} \otimes \mathcal{C}$. We now consider the sum of the two Poisson structure. Let $F_0 := \frac{1}{2}\{\, \Omega_0 \,,\, \Omega_0 \,\} = \frac{1}{2}\{\, y^\mu \,,\, y^\nu \,\}\, c_\mu c_\nu$. Observe that $\{F_0, \delta_0, \{\ ,\ \}\}$ defines a nonflat L_∞-structure[1] on $C^\infty(NC) \otimes \mathcal{B} \otimes \mathcal{C}$. Moreover, F_0

[1]Originally L_∞-algebras were always assumed to miss the 0th term. When L_∞-algebras with the 0th term turned out to be interesting, various terminologies were introduced. An L_∞-algebra without the 0th term is called nowadays flat (resp. strict), while one where the 0th term is there is called curved (resp. weak). If no assumption on the 0th term is made (i.e., it may or may not vanish), we simply speak of an L_∞-algebra.

is δ_0-exact iff C is coisotropic. In this case, one can use cohomological perturbation theory to kill F_0 as follows.

Let $h := b^\mu \frac{\partial}{\partial y^\mu}$. Then $[\,\delta_0\,,\,h\,] = E := b^\mu \frac{\partial}{\partial b^\mu} + y^\mu \frac{\partial}{\partial y^\mu}$. One then has a homotopy s for δ_0 (i.e. $s\delta_0 + \delta_0 s = id - pr$, with pr the projection to a subspace isomorphic to the δ_0-cohomology) defined by $s\alpha = h\alpha/|\alpha|$ if $E\alpha = |\alpha|\alpha$ and $s\alpha = 0$ if $E\alpha = 0$. Let us then define $\Omega := \sum_{i=0}^{\infty} \Omega_i$ by induction as follows: Let $R_k := \sum_{i=0}^{k} \Omega_i$. Then define $\Omega_{k+1} = -\frac{1}{2}\, s\,(\{\,R_k\,,\,R_k\,\})$. If C is coisotropic, the following hold:

1. $\{\,\Omega\,,\,\Omega\,\} = 0$, so $\{D, \{\ \ ,\ \ \}\}$ defines a DGPA structure on $C^\infty(NC)\otimes\mathcal{B}\otimes\mathcal{C}$, where $D := \{\,\Omega\,,\ \ \} = \delta_0 + \cdots$.
2. The D-cohomology H_D is isomorphic to the Lie algebroid cohomology H_δ of N^*C.
3. This way H_δ gets the structure of a GPA.

It is natural to ask whether the GPA structure on H_δ is the same as the one induced before. The answer is affirmative. Actually, there is a stronger result:

THEOREM 3.2 (Schätz [**S**]). *$(C^\infty(NC)\otimes\mathcal{B}\otimes\mathcal{C}, D, \{\ \ ,\ \ \})$ is L_∞-quasi-isomorphic to $(A, \{\lambda_k\})$.*

Observe that the L_∞-structure $(A, \{\lambda_k\})$ can be defined for every submanifold, while the BFV formalism as presented above requires that the normal bundle should be trivial, or at least flat. A more general version of the BFV formalism allows for linearly dependent global constraints and uses the Koszul–Tate resolution. There is however a generalization by Bordemann and Herbig [**B00**] of the construction presented above which needs no assumption on the normal bundle and just requires the choice of a connection. The above Theorem holds also in the general case [**S**].

4. The reduced space of the Poisson sigma model

Given a finite dimensional manifold M, we denote by PM the Banach manifold of C^1-paths in M; viz., $PM := C^1(I, M)$, $I = [0,1]$. We denote by T^*PM the vector bundle over PM with fiber at $X \in PM$ the space of C^0 sections of $T^*I \otimes X^*T^*M$. Using integration over I and the canonical pairing between TM and T^*M, one can endow T^*PM with a symplectic structure. Let now π be Poisson bivector field on M. We define

$$\mathcal{C}(M,\pi) := \{(X,\zeta) \in T^*PM : \mathrm{d}X + \pi^\sharp(X)\zeta = 0\}.$$

These infinite-dimensional manifolds naturally appear when considering the PSM (1.1) on $\Sigma = I \times \mathbb{R}$. Namely, the map $X\colon \Sigma \to M$ may be regarded as a path in PM. On the other hand, introducing the coordinate t on $\mathbb{R}$, we may write $\eta = \zeta + \lambda \mathrm{d}t$, and reinterpret the pair (X,ζ) as a path in T^*PM. The bilinear term in X and ζ in the action functional (1.1) yields the canonical symplectic structure on T^*PM, while λ appears linearly as a Lagrange multiplier leading to the constraints that define $\mathcal{C}(M,\pi)$.

THEOREM 4.1 ([**CF01**]). *$\mathcal{C}(M,\pi)$ is a coisotropic submanifold of T^*PM. Its reduced space $\underline{\mathcal{C}}(M,\pi)$ is the source simply connected symplectic groupoid of M, whenever M is integrable (i.e., T^*M is an integrable Lie algebroid).*

We now want to discuss boundary conditions. Given two submanifolds C_0 and C_1 of M, let

$$\mathcal{C}(M,\pi; C_0, C_1) := \{(X,\zeta) \in \mathcal{C}(M,\pi) : X(0) \in C_0,\ X(1) \in C_1\}.$$

Observe that $\mathcal{C}(M,\pi;C_0,C_1)$ has natural maps to C_0 and C_1 (evaluation of X at 0 and 1). We call points in the image in $C_0\times C_1$ connectable.

Its symplectic orthogonal bundle may be explicitly computed. For simplicity we choose local coordinates on M. (The correct, but more cumbersome, description involves choosing a torsion-free connection for TM.) One gets [**Ca**]

$$T^\perp_{(X,\zeta)}\mathcal{C}(M,\pi;C_0,C_1)=\{(-\pi^\sharp(X)\beta,\mathrm{d}\beta_i+\partial_i\pi^{jk}(X)\zeta_j\beta_k):\\ \beta\in\Gamma(X^*T^*M),\ \beta(0)\in N^*_{X(i)}C_i,\ i=0,1\}. \tag{4.1}$$

Intersection with $T\mathcal{C}(M,\pi;C_0,C_1)$ just forces the boundary conditions for β to be such that the X-variations are tangent to the submanifolds: viz., $\pi^\sharp(X(i))\beta(i)\in T_{X(i)}C_i$, $i=0,1$. Thus, $\mathcal{C}(M,\pi;C_0,C_1)$ is presymplectic iff the pre-Poisson condition is satisfied at all connectable points [**CFa05, Ca**], and it is coisotropic iff the coisotropicity condition is satisfied at all connectable points [**CF04**]. The sections of $\delta X\oplus\delta\zeta$ of the characteristic distribution are parametrized by a section β of X^*T^*M with $\beta(i)\in A_{X(i)}C_i$, $i=0,1$:[2]

$$\delta X=-\pi^\sharp(X)\beta, \tag{4.2}$$

$$\delta\zeta_i=\mathrm{d}\beta_i+\partial_i\pi^{jk}(X)\zeta_j\beta_k, \tag{4.3}$$

where for simplicity we have written the second equation using local coordinates on M. (A more invariant, but more cumbersome, possibility is to write it upon choosing a torsion-free connection for TM and observing that the distribution does not depend on this choice.)

To remove the condition on connectable points, one can e.g. take the second submanifold to be the whole manifold M. So we have

Theorem 4.2.

(1) *$\mathcal{C}(M,\pi;C,M)$ is coisotropic in T^*PM iff C is coisotropic in M* [**CF04**].
(2) *$\mathcal{C}(M,\pi;C,M)$ is presymplectic in T^*PM iff C is pre-Poisson in M* [**CFa05, Ca**].

Point 2) says in particular that pre-Poisson submanifolds are the most general boundary conditions for the Poisson sigma model compatible with symmetries. A submanifold chosen as a boundary condition is usually called a brane. Point 1) is important as it describes boundary conditions compatible with symmetries *and* perturbation theory around $\pi=0$. Namely, consider the family of Poisson structures $\pi_\epsilon:=\epsilon\pi$, $\epsilon\geq 0$. Let C be a pre-Poisson submanifold for π. Then it is pre-Poisson for π_ϵ, $\forall\epsilon\geq 0$. In particular, it is coisotropic for $\epsilon=0$. One may indeed check that the codimension of the characteristic distribution of $\mathcal{C}(M,\pi_\epsilon;C,M)$ is the same for all $\epsilon>0$ but it may change for $\epsilon=0$. It stays the same iff C is coisotropic for π (and hence coisotropic for π_ϵ for all ϵ).

5. Expectation values in perturbation theory

We now consider the perturbative functional-integral quantization of the Poisson sigma model. Namely, we consider integrals of the form

$$\langle\,\mathcal{O}\,\rangle:=\int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\mathcal{O}, \tag{5.1}$$

[2] Actually, this is the image of the anchor of an infinite-dimensional Lie algebroid over $\mathcal{C}(M,\pi;C,M)$. Explicitly this is described in [**BC**] and in [**BCZ**].

where $\mathcal{O}$ is a function of the fields and the integration is over all fields (X,η), $X \in \mathrm{Map}(\Sigma, M)$, $\eta \in \Gamma(T\Sigma \otimes X^*T^*M)$. If Σ has a boundary, the discussion in Section 4 shows that we have to choose a pre-Poisson submanifold C of M (a brane) and impose the conditions $X(\partial\Sigma) \subset C$, $\iota_\partial^*\eta \in \Gamma(T^\Sigma \otimes X^*AC)$, where ι_∂ is the inclusion map $\partial\Sigma \hookrightarrow \Sigma$.

Also observe that the Poisson sigma model has symmetries (see (4.2)). This makes the integrand $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}$ invariant under the Lagrangian extension of the symmetries which has the form:

$$\delta X = -\pi^\sharp(X)\beta, \tag{5.2}$$

$$\delta\eta_i = \mathrm{d}\beta_i + \partial_i\pi^{jk}(X)\eta_j\beta_k, \tag{5.3}$$

with $\beta \in \Gamma(X^*T^*M)$, $\iota_\partial^*\beta \in \Gamma(X^*AC)$. As a consequence the integral (5.1) cannot possibly converge (it would not even if we were in finite dimensions). The trick is to select representatives of the fields modulo symmetries in a consistent way. This is done using the BV formalism [**BV**] which in particular guarantees that the result is independent of the choice involved ("gauge fixing"). Its application to the PSM is described in [**CF00**]. An important issue is that an expectation value as in (5.1) is also gauge-fixing independent if $\mathcal{O}$ is invariant under symmetries. We will consider only boundary observables. Namely, let f be a function on C. To it we associate the observable $\mathcal{O}_{f,u}(X,\eta) := f(X(u))$, where u is some point of $\partial\Sigma$. The observable is invariant iff f is invariant constant along the characteristic distribution of C. It will be interesting for us to take two such functions f and g and consider $\mathcal{O} = \mathcal{O}_{f,u}\mathcal{O}_{g,v}$.

We compute integrals as in (5.1) in perturbation theory, i.e., by expanding S around its critical points and then computing the integral formally in terms of the momenta of the Gaussian distribution associated to the Hessian of S (observe that after gauge fixing this is no longer degenerate). We will pick only the trivial critical point; viz., the class of solutions $\{X \text{ constant}, \eta = 0\}$. We fix the constant by imposing X to be equal to some $x \in C$ at some boundary point which we denote by ∞. Finally, we select Σ to be a disk[3] and compute

$$f \star g(x;u,v) := \int_{X(\infty)=x} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\mathcal{O}_{f,u}\mathcal{O}_{g,v}, \tag{5.4}$$

where the points u, v, ∞ are cyclically ordered. It turns out that such an integral may be explicitly computed in perturbation theory. The results are as follows.

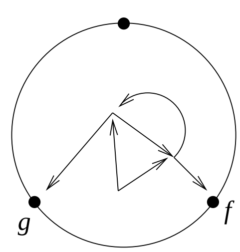

FIGURE 1. An example of a Kontsevich graph

[3]Higher genera would be interesting to consider, but this has not been done so far, the main difficulty being the appearance of nontrivial cohomological classes in the space of solutions.

The case $C = M$ has been studied in [**CF00**]. The result turns out to be independent of u and v and to define a star product [**BFFLS**] on M: viz., an associative product $\star$ on $C^\infty(M)[[\epsilon]]$ (in the above computation $\epsilon = \mathrm{i}\hbar/2$) deforming the pointwise product in the direction of the Poisson bracket:

$$f \star g = fg + \epsilon\{\, f\,,\, g\,\} + \sum_{n=2}^{\infty} B_n(f,g), \tag{5.5}$$

where the B_ns are bidifferential operator with the property $B_n(1,\bullet) = B_n(\bullet,1) = 0$. Moreover, this star product is exactly the one defined by Kontsevich in [**K**].

The case C pre-Poisson has been considered in [**CFa05**]. It turns out that the PSM on M with boundary conditions on C is equivalent to the PSM on any M' as in Prop. 2.3 with the same boundary conditions. As a result, it is enough to consider coisotropic submanifolds.

The case C coisotropic has been considered in [**CF04**]. A few interesting, but unpleasant phenomena appear. First, it might happen that the result of (5.4) depends on u and v. Second, the result regarded as a function on $C^\infty(C)$ may not be invariant. Third, associativity may not be guaranteed. Under suitable conditions, one may ensure that everything works. We return on this in subsection 7.4, where we show that the vanishing of the first and second Lie algebroid cohomologies is a sufficient condition.

6. Super PSM and duality

To deal with symmetries in the PSM consistently, one has to resort to the BV formalism. In the case at hand the recipe (following from the AKSZ formalism [**AKSZ**] adapted to the PSM with boundary [**CF01b**]) consists just of formally replacing the fields (X,η) in the action by "superfields" $(\mathsf{X},\boldsymbol{\xi})$. To simplify the discussion, let us choose a coordinate chart on M. Then we may regard X as an element of $\Omega^0(\Sigma)\otimes\mathbb{R}^m$, $m = \dim M$, and η as an element of $\Omega^1(\Sigma)\otimes(\mathbb{R}^m)^*$. The action in local coordinate reads

$$S = \int_\Sigma \eta_i \mathrm{d}X^i + \frac{1}{2}\pi^{ij}(X)\eta_i\eta_j,$$

where we use Einstein's convention on repeated indices.

The superfields are then $\mathsf{X} \in V := \Omega(\Sigma)\otimes\mathbb{R}^m$, $\boldsymbol{\eta} \in W := \Omega(\Sigma)\otimes(\mathbb{R}^m)^*$ and one just puts them in the action instead of the classical fields. The integral selects by definition the two-form component.[4] The superaction is a function on $V \oplus W$ regarded as a graded vector space. Observe that for consistency, we have to consider elements of V as even and those of W as odd. It is also useful to introduce a $\mathbb{Z}$-grading and assign V degree 0 and W degree 1, while integration over Σ is given degree -2.

If we only consider the "kinetic term" $S_0 = \int_\Sigma \boldsymbol{\eta}_i \mathrm{d}\mathsf{X}^i$, it is quite natural to observe that there is a duality obtained by exchanging $\boldsymbol{\eta}$ and X. The only problem is that we want to think of the zero form in X as a map. However, the zero form in $\boldsymbol{\eta}$ has to be considered as an odd element. The problem is solved if we allow the target M to be a graded manifold itself. The duality then exchanges M with a

[4] An invariant description requires introducing graded manifolds and regarding the space of superfields as the graded infinite-dimensional manifold $\mathrm{Map}(T[1]\Sigma, T^*[1]M)$.

dual manifold $\tilde{M}$. If we now take into account the term involving π, we see that it corresponds to a similar term for a multivector field on $\tilde{M}$.

If Σ has a boundary, the superfields have to be assigned boundary conditions. We choose a coisotropic submanifold C of M and require X on the boundary to take values in C, while $\boldsymbol{\eta}$ has to take values in the pullback of the conormal bundle of C. As we work in perturbation theory, we may actually consider just a formal neighborhood of C. Namely, we choose an embedding of the normal bundle NC of C into M and regard NC as the Poisson manifold. The dual manifold turns out to be the graded manifold $N^*[1]C$. To π there corresponds a multivector field $\tilde{\pi}$ of total degree 2. The dual theory can be mathematically understood in terms of a formality theorem for graded manifolds.

7. Formality Theorem

Kontsevich's star product is an application of his formality theorem stating that the DGLA of multidifferential operators on a smooth manifold is formal [**K**]. His explicit local formula for the L_∞-quasi-isomorphism may also be obtained in terms of expectation values of boundary and bulk observables in the PSM with zero Poisson structure [**CF00**]. This construction may be generalized to smooth graded manifolds [**CF07**] (see also [**C06**] for the PSM version). An application [**CF07, LS**] is the deformation quantization of coisotropic submanifolds [**CF04**].

Let $\mathcal{M}$ be a smooth graded manifold $A = \bigoplus_{k\in\mathbb{Z}} A^k$ its graded algebra of functions, $\mathcal{V}(\mathcal{M})$ its complex of graded multivector fields and $\mathcal{D}(\mathcal{M})$ its complex of graded multidifferential operators (regarded as a subcomplex of the Hochschild complex of A). We consider $\mathcal{V}(\mathcal{M})$ as DGLAs by taking the total degree (shifted by one); viz.:

$$\mathcal{V}^k(\mathcal{M}) := \{r\text{-multivector fields of homogeneous degree } s \text{ with } r+s=k+1\},$$
$$\mathcal{D}^k(\mathcal{M}) := \{r\text{-multidiff. operators of homogeneous degree } s \text{ with } r+s=k+1\}.$$

Theorem 7.1 (Formality Theorem). *There exists an L_∞-quasi-isomorphism $U\colon \mathcal{V}(\mathcal{M}) \rightsquigarrow \mathcal{D}(\mathcal{M})$, hence the DGLA $\mathcal{D}(\mathcal{M})$ is formal. Moreover, U may be chosen such that the degree-one component $U_1\colon \mathcal{V}(\mathcal{M}) \mapsto \mathcal{D}(\mathcal{M})$ is the Hochschild–Kostant–Rosenberg (HKR) map.*

See [**K**] for the proof in the ordinary case and [**CF07**] for the case of graded manifolds.

One may also extend U to formal power series in a parameter ϵ and get an L_∞-quasi-isomorphism $\mathcal{V}_\epsilon(\mathcal{M}) \rightsquigarrow \mathcal{D}_\epsilon(\mathcal{M})$, where

$$\mathcal{V}_\epsilon(\mathcal{M}) := \epsilon\mathcal{V}(\mathcal{M})[[\epsilon]], \quad \mathcal{D}_\epsilon(\mathcal{M}) := \epsilon\mathcal{D}(\mathcal{M})[[\epsilon]].$$

In particular, since MC elements are mapped to MC elements by an L_∞-morphism (with no convergence problems in the setting of formal power series), the Formality Theorem also shows that every MC element of $\mathcal{V}(\mathcal{M})$ gives rise to a MC element of $\mathcal{D}_\epsilon(\mathcal{M})$. In the case of ordinary manifolds, the former MC element is the same as a Poisson structure, while the latter defines a deformation quantization of the former. (A classification theorem also follows by the fact that an L_∞-quasi-isomorphism actually yields an isomorphism of the sets of MC elements modulo gauge transformations).

The case of graded manifold is important for coisotropic submanifolds. As we recalled in Section 3, the algebra $A := \Gamma(NC)$ naturally appears in the cohomological description of a coisotropic submanifold C of a Poisson manifold (M, π). Now one can reinterpret A as the algebra of functions on the graded manifold $N^*[1]C$ (where [1] denotes a shift by 1 in the fiber coordinates). Moreover, the restriction of π to NC (obtained upon choosing an embedding of NC into M) may be regarded as a MC element on NC but also on $N^*[1]C$.

One way to see this is via Roytenberg's Legendre mapping theorem [**R**] which states the existence of a canonical antisymplectomorphism between $T^*[n]E$ and $T^*[n](E^*[n])$ for every integer n and for every graded vector bundle E. In particular, for $n = 1$ and observing that multivector fields on a graded manifold may be reinterpreted as functions on its tangent bundle shifted by 1, one gets an anti-isomorphism between the GLAs $\mathcal{V}(E)$ and $\mathcal{V}(E[1])$; in particular, this yields an isomorphism of the sets of MC elements. Specializing to $E = NC$ (as a graded vector bundle concentrated in degree zero) yields the sought for result.

A more direct way is just to observe that each derived bracket λ_k in (3.1) is a multiderivation on the algebra A and so a k-vector field on $N^*[1]C$. Their linear combination is the desired MC element. We now consider a more general situation.

7.1. Maurer–Cartan elements for graded manifolds. In general, given a graded manifold $\mathcal{M}$ and a MC element π in $\mathcal{V}(\mathcal{M})$, by Theorem 3.1 (with $\mathfrak{g} = \mathcal{V}(\mathcal{M})$, $\mathfrak{h} = C^\infty(\mathcal{M})$, P and i the natural projection and inclusion), we may construct a P_∞-algebra on $A := C^\infty(\mathcal{M})$ with multibrackets as in (3.1). Taking care of degrees, one sees that λ_k is a multiderivation of degree $2 - i$, i.e.:

$$
\begin{aligned}
&\lambda_0 = P_0 \in A^2,\\
&\lambda_1 \colon A^j \to A^{j+1},\\
&\lambda_2 \colon A^{j_1} \otimes A^{j_2} \to A^{j_1+j_2},\\
&\dots\\
&\lambda_i \colon A^{j_1} \otimes \cdots \otimes A^{j_i} \to A^{j_1+\dots j_i+2-i},\\
&\dots
\end{aligned}
$$

If $P_0 = 0$, then $\lambda_0 = 0$ and we have a flat L_∞-algebra. In this case, λ_1 is a differential and its cohomology

$$\mathsf{A}^\bullet := H^\bullet_{\lambda_1}(A)$$

is a graded Poisson algebra. In the following we will denote by $\pmb{\{}\ ,\ \pmb{\}}$ the induced Poisson bracket on A. In particular, A^0 is a Poisson algebra (but in general it is not the algebra of functions on a smooth manifold) whose Poisson bracket we will simply denote by $\{\ ,\ \}$.

Definition 7.2. A graded manifold $\mathcal{M}$ with a (flat) P_∞-structure on $A = C^\infty(\mathcal{M})$ is called a (flat) P_∞-manifold.

We now turn to multidifferential operators. An element m in $\mathcal{D}^1_\epsilon(\mathcal{M})$ consists of a sequence m_i, where m_i is an i-multidifferential operators of degree $2-i$, i.e.,

$$
\begin{aligned}
&m_0 \in \epsilon A^2[[\epsilon]],\\
&m_1\colon A^j[[\epsilon]] \to \epsilon A^{j+1}[[\epsilon]],\\
&m_2\colon A^{j_1}[[\epsilon]] \otimes A^{j_2}[[\epsilon]] \to \epsilon A^{j_1+j_2}[[\epsilon]],\\
&\dots\\
&m_i\colon A^{j_1}[[\epsilon]] \otimes \cdots \otimes A^{j_i}[[\epsilon]] \to \epsilon A^{j_1+\dots j_i+2-i}[[\epsilon]],\\
&\dots
\end{aligned}
$$

Let us denote by χ the (extension to $A[[\epsilon]]$) of the multiplication on A and define $\mu = \chi + m$; viz.:

$$
\mu_i = \begin{cases} m_i & i \neq 2,\\ \chi + m_2 & i = 2. \end{cases} \tag{7.1}
$$

The element m is MC, i.e.,

$$
\delta m + \frac{1}{2}[\, m\,,\, m\,] = 0, \qquad (\delta = [\,\chi\,,\ \ ]),
$$

iff the operations μ_i define an A_∞-structure on $A[[\epsilon]]$. If in addition μ_0 vanishes, one says that the A_∞-algebra is flat. In this case, $H^\bullet_{\mu_1}$ is an associative algebra.

Let $\tilde\mu$ denote the skew-symmetrization of μ. Then $(A[[\epsilon]], \tilde\mu)$ is an L_∞-algebra. Since the multiplication χ is graded commutative, the $\tilde\mu$ will take values in $\epsilon A[[\epsilon]]$. So, dividing by ϵ and working modulo ϵ, they define operations $\underline{\tilde\mu_i}$ on A which make it into an L_∞-algebra. It may be shown, see [**CF07**], that the $\underline{\tilde\mu_i}$s are actually multiderivations, so we have a P_∞-algebra structure on A.

7.2. Deformation quantization of P_∞-manifolds. Let A be the algebra of functions on a P_∞-manifold $\mathcal{M}$. We denote by χ the graded commutative product and by λ the multibrackets.

Definition 7.3. A deformation quantization of $\mathcal{M}$ consists of an A_∞-structure on $(A[[\epsilon]], \mu)$ with $\mu = \chi + m$, such that:

(1) m is of order ϵ;
(2) m consists of multidifferential operators;
(3) m vanishes when one of its arguments is the unit in A; (i.e., $m_i(a_1, \dots, a_i) = 0$ if $a_j = 1$ for some $j \in \{1, \dots, i\}$, $i > 0$);
(4) the induced P_∞-structure $\tilde\mu$ is equal to λ.

If $\mathcal{M}$ is flat, by deformation quantization in the flat sense we mean that in addition the condition $\mu_0 = 0$ is fulfilled.

The formality theorem for graded manifolds then implies the following

Theorem 7.4. *Every P_∞-manifold $\mathcal{M}$ admits a deformation quantization.*

Proof. Let $P \in \mathcal{V}_\epsilon(\mathcal{M})$ be the MC element with derived brackets λ. Define m by applying the L_∞-quasi-isomorphism U to ϵP:

$$
m = \sum_{n=0}^{\infty} \frac{1}{n!} U_n(P, \dots, P).
$$

This is a MC element in $\epsilon\mathcal{D}_\epsilon(\mathcal{M})$ with the desired properties. $\square$

Observe however that the problem of quantizing flat P_∞-manifolds in the flat sense is on the other hand not solved. In fact, we are not able to conclude that $P_0 = 0$ implies $m_0 = 0$ but only that $m_0 = O(\epsilon^2)$. This is the deformation quantization version of a (potential) anomaly (see [**CF07**]).

There are some cases when m_0 actually happens to vanish (see [**CF04**]). There are also cases [**CF07**] where m_0 can be killed by shifting the A_∞-structure without changing its classical limit. Namely, for $\gamma \in \epsilon A^1[[\epsilon]]$ define

$$\hat{m}_0 = m_0 + m_1(\gamma) + \frac{1}{2} m_2(\gamma,\gamma) + \cdots,$$
$$\hat{m}_1(a) = m_1(a) + m_2(a,\gamma) \pm m_2(\gamma, a) + \cdots.$$

Then, $\forall\gamma$, $\mu+\hat{m}$ is again an A_∞-structure on $A[[\epsilon]]$ that induces the same P_∞-structure on A. One may then look for a γ such that $\hat{m}_0 = 0$. It is not difficult to prove by induction, see [**CF07**], that a sufficient condition for this to happen is that $\mathsf{A}^2 = \{0\}$. So we have the following

THEOREM 7.5. *Every flat P_∞-manifold $\mathcal{M}$ with $\mathsf{A}^2 = \{0\}$ admits a deformation quantization in the flat sense.*

7.3. Quantization of cohomology. Assume now that the flat P_∞-manifold $\mathcal{M}$ has been quantized in the flat A_∞ sense. We denote by χ the graded commutative multiplication on the algebra of functions A, by λ_i the multibrackets defining the flat P_∞-structure on A and by μ_i the multibrackets defining the flat A_∞-structure on $A[[\epsilon]]$. We will also denote by d the differential λ_1 and by A the d-cohomology of A.

The μ_1-cohomology is an associative algebra. However, in view of quantizing $\mathsf{A}^\bullet$, or at least A^0, this is not the algebra we are in general interested in since the projection modulo ϵ is not a chain map. We have instead to proceed as follows. Observe that by appropriately rescaling the multibrackets, we get a new A_∞-structure; viz., for given integers s_i, define $\tau_i = \epsilon^{s_i}\mu_i$, where we assume μ_i to be divisible by ϵ^{-s_j} whenever $s_j < 0$. If moreover $s_i + s_j$ is a function of $i+j$, the multibrackets τ_i also define an A_∞-algebra.

In particular, we may take $s_i = i - 2$ since $\mu_1 = O(\epsilon)$. Namely, we define the new A_∞-algebra structure

$$\tau_i := \epsilon^{i-2}\mu_i.$$

Observe that

$$\tau_1 = \mathrm{d} + O(\epsilon), \qquad \tau_2 = \chi + O(\epsilon), \qquad \tau_i = O(\epsilon^{i-1}),\ i > 2.$$

By construction, the graded skew-symmetrization $\tilde{\tau}_2$ of τ_2 is divisible by ϵ; so there is a unique operation ψ with $\tilde{\tau}_2 = \epsilon\psi$. The bracket λ_2 on A is then ψ modulo ϵ.

We will also denote by $\delta = \sum_{n=0}^\infty \epsilon \mathrm{d}_n$, $\mathrm{d}_0 = \mathrm{d}$, the differential τ_1 and by B the δ-cohomology of $A[[\epsilon]]$. Observe that the differential δ is a deformation of the differential d.

Observe that by construction τ_2, $\tilde{\tau}_2$ and ψ are chain maps and we will denote by $[\tau_2]$, $[\tilde{\tau}_2]$ and $[\psi]$ the induced operations in cohomology. The operation $[\tau_2]$ is an associative product which we will also denote by $\bigstar$ (and by $\star$ its restriction to B^0).

Whenever needed we denote δ-cohomology classes by $[\ ]_\delta$ and d-cohomology classes by $[\ ]_\mathrm{d}$. With these notations we have

$$[a]_\delta \bigstar [b]_\delta = [\tau_2]([a]_\delta, [b]_\delta) = [\tau_2(a,b)]_\delta,$$

where $[\tau_2]$ is the map induced by τ_2 in cohomology, while a and b are representatives of the classes $[a]_\delta$ and $[b]_\delta$. By construction $[\tilde\tau_2]$ defines the graded $\star$-commutator $[\ ,\ ]$ on B:

$$[\,[a]_\delta\,,\,[b]_\delta\,] = [\tilde\tau_2]([a]_\delta, [b]_\delta) = [\tilde\tau_2(a,b)]_\delta.$$

We will denote simply by $[\ ,\ ]$ its restriction to B^0. Observe that multiplication by ϵ commutes with δ. So B is also an $\mathbb{R}[[\epsilon]]$-module with $\epsilon[a]_\delta = [\epsilon a]_\delta$. Moreover, we have

$$[\,[a]_\delta\,,\,[b]_\delta\,] = \epsilon[\psi]([a]_\delta, [b]_\delta).$$

If B (B^0) is ϵ-torsion free—i.e., multiplication by ϵ is injective—, then $[\psi]$ is the unique operation with the above property. Consider now the $\mathbb{R}$-linear projection

$$\begin{array}{cccc} \varpi\colon & A[[\epsilon]] & \to & A \\ & \sum_{n=0}^\infty \epsilon^n a_n & \mapsto & a_0. \end{array}$$

Observe that $\varpi\colon (A[[\epsilon]], \delta) \to (A, \mathrm{d})$ is a chain map. We will denote by $[\varpi]\colon \mathsf{B} \to \mathsf{A}$ the induced map in cohomology. Since ϖ is an algebra homomorphism, so is $[\varpi]$. The relation with the Poisson structure on A is clarified by the now obvious formula

$$[\varpi]([\psi]([a]_\delta, [b]_\delta)) = \{\,[\varpi]([a]_\delta)\,,\,[\varpi]([b]_\delta)\,\}, \qquad \forall [a]_\delta, [b]_\delta \in \mathsf{B}$$

which immediately implies the

Proposition 7.6. *The algebra homomorphism $[\varpi]$ has the following additional properties:*

1. *The image C (C^0) of $[\varpi]$ is a Poisson subalgebra of A (A^0).*
2. *If B is ϵ-torsion free, then*
$$[\varpi]\left(\frac{[\,[a]_\delta\,,\,[b]_\delta\,]}{\epsilon}\right) = \{\,[\varpi]([a]_\delta)\,,\,[\varpi]([b]_\delta)\,\}, \qquad \forall [a]_\delta, [b]_\delta \in \mathsf{B}.$$
3. *If B^0 is ϵ-torsion free, then the above formula holds for all $[a]_\delta, [b]_\delta \in \mathsf{B}^0$.*

Since obviously $\operatorname{Ker}\varpi = \epsilon A[[\epsilon]]$, we have the short exact sequence

$$0 \to A[[\epsilon]] \xrightarrow{\epsilon} A[[\epsilon]] \xrightarrow{\varpi} A \to 0$$

which induces the long exact sequence

$$\cdots \to \mathsf{A}^{i-1} \xrightarrow{\partial} \mathsf{B}^i \xrightarrow{\epsilon} \mathsf{B}^i \xrightarrow{[\varpi]} \mathsf{A}^i \xrightarrow{\partial} \mathsf{B}^{i+1} \to \cdots$$

in cohomology. Immediately we then get the

Proposition 7.7. *The following statements are equivalent:*

1. *$[\varpi]$ is surjective;*
2. *B is ϵ-torsion free;*
3. *∂ is trivial.*

To continue our study of the problem, we now need Lemma A.1 of [**CFT**], which we state in a slightly modified version:

Lemma 7.8. *Let $\Bbbk$ be a field and M a $\Bbbk[[\epsilon]]$-module endowed with the $\Bbbk[[\epsilon]]$-adic topology. Then $\mathsf{M} \simeq_\Bbbk \mathsf{M}_0[[\epsilon]]$ for some $\Bbbk$-vector space M_0 iff M is Hausdorff, complete and ϵ-torsion free. Moreover, $\mathsf{M}_0 \simeq_\Bbbk \mathsf{M}/\epsilon\mathsf{M}$.*

We finally have the

THEOREM 7.9. *If* B *is Hausdorff and complete in the* ϵ*-adic topology, then* B *is a deformation quantization of* A *iff any (and so all) of the statements in Proposition 7.7 holds.*

PROOF. If B is a deformation quantization of A, then in particular $\mathsf{B} \simeq_{\mathbb{R}} \mathsf{A}[[\epsilon]]$. So by Lemma 7.8 B is ϵ-torsion free. On the other hand, the statements in Proposition 7.7 imply that $\mathsf{A} \simeq_{\mathbb{R}} \mathsf{B}/\operatorname{Ker}[\varpi] = \mathsf{B}/\epsilon\mathsf{B}$; so $\mathsf{B} \simeq_{\mathbb{R}} \mathsf{A}[[\epsilon]]$ by Lemma 7.8. Statement (2) in Proposition 7.6 completes the proof. □

We are not able to show that $\operatorname{Im}\delta$ is closed in the ϵ-adic topology, so we must put the extra condition in the Theorem. We wish however to make the following[5]

CONJECTURE 7.10. *The* $\mathbb{R}[[\epsilon]]$*-module* B *is Hausdorff and complete in the* ϵ*-adic topology.*

In general we do not expect B to be a deformation quantization of A. However, it is often enough to have B^0 as a deformation quantization of A^0. We end this Section by exploring some sufficient conditions for this to happen.

LEMMA 7.11. *If* B^i *is Hausdorff and complete, then it is isomorphic to* $\mathsf{C}^i[[\epsilon]]$ *as an* $\mathbb{R}$*-vector space iff it is* ϵ*-torsion free.*

PROOF. The "only if" implication is obvious. As for the "if" part, by Lemma 7.8 we conclude that B^i is isomorphic to $(\mathsf{B}^i/\epsilon\mathsf{B}^i)[[\epsilon]]$. On the other hand the long exact sequence gives $0 \to \mathsf{B}^i \xrightarrow{\epsilon} \mathsf{B}^i \xrightarrow{[\varpi]} \mathsf{C}^i \to 0$ which completes the proof. □

By statement (3) in Proposition 7.6, we then get

COROLLARY 7.12. *If* B^0 *is Hausdorff, complete and* ϵ*-torsion free, then it is a deformation quantization of* C^0.

The long exact sequence yields the

LEMMA 7.13. *If* $\mathsf{A}^{i-1} = \{0\}$*, then* B^i *is* ϵ*-torsion free.*

Thus, $A^{-1} = \{0\}$ is a sufficient condition for B^0 to be a deformation quantization of C^0 provided it is Hausdorff and complete. Finally, we have the

LEMMA 7.14. *If* $\mathsf{A}^{i+1} = \{0\}$*, then* $[\varpi]^i \colon \mathsf{B}^i \to \mathsf{A}^i$ *is surjective.*

PROOF. This is a standard proof in cohomological perturbation theory. Let $[a_0]_{\mathrm{d}} \in \mathsf{A}^i$. Choose one of its representatives $a_0 \in A^i$. We look for a δ-closed $a = \sum_{n=0}^{\infty} \epsilon^n a_n \in A^i[[\epsilon]]$. The equations we have to solve have the form

$$\mathrm{d}a_n = -\sum_{\substack{r+s=n+1\\ r>0}} \mathrm{d}_r a_s, \tag{7.2}$$

and we may solve them by induction. Namely:

(1) For $n = 0$, the equation is just $\mathrm{d}a_0 = 0$ which is satisfied by assumption.
(2) Assume now that all equations for a_k, $k < n$, have been solved. This implies that that the r.h.s. of (7.2) is d-closed. Since we assume that $\mathsf{A}^{i+1} = \{0\}$, we may then find a_n satisfying the equation.

□

[5] Observe that the Conjecture is general false if we drop the conditions that A is the algebra of functions on a graded manifold and that the components d_n of δ are differential operators.

REMARK 7.15. Observe that, if we knew that B^{i+1} were Hausdorff (e.g., if Conjecture 7.10 were true), then we could derive Lemma 7.14 directly from the long exact sequence (this is just a variant of Nakayama's Lemma). In fact, $A^{i-1} = \{0\}$ implies that multiplication by ϵ on B^{i+1} is surjective. So every element $a \in \mathsf{B}^{i+1}$ may be written as $a = \epsilon a_1$. Continuing this process, we get a sequence a_n with $a = \epsilon^n a_n$. The sequence $\epsilon^n a_n$ converges to zero, so $a = 0$ since B^{i+1} is Hausdorff. Thus, $\mathsf{B}^{i+1} = \{0\}$ and by the long exact sequence again we get the result.

Putting together Theorem 7.5, Corollary 7.12, Lemmata 7.13 and 7.14, we get the following

THEOREM 7.16. *If $\mathsf{A}^2 = \mathsf{A}^1 = \mathsf{A}^{-1} = \{0\}$, then B^0 is a deformation quantization of A^0 provided it is Hausdorff and complete.*

If Conjecture 7.10 holds, we get the sufficient conditions in a much nicer way which makes reference only to the d-cohomology:

"THEOREM". *A sufficient condition for a deformation quantization of A^0 to exist is $\mathsf{A}^2 = \mathsf{A}^1 = \mathsf{A}^{-1} = \{0\}$.*

Observe that the sufficient condition is by no means necessary. For example, in the case $\lambda_i = 0\ \forall i$, we have $\mathsf{A}^\bullet = A^\bullet$. On the other hand, $\delta = 0$ and $\mathsf{B}^\bullet = A^\bullet[[\epsilon]]$ is a deformation quantization of $\mathsf{A}^\bullet$.

7.4. Deformation quantization of coisotropic submanifolds. We now return to the case when C is a coisotropic submanifold of M (which is Poisson at least in a neighborhood of C). We take our graded manifold $\mathcal{M}$ to be $N^*[1]C$ with MC element the restriction of π to $\mathcal{V}(NC) = \mathcal{V}(N^*[1]C)$. Now $\mathsf{A}^{-1} = \{0\}$ since we do not have negative degrees, while A^0 is the Poisson algebra of functions on C that are invariant under the canonical distribution on C. If we denote by $\underline{C}$ the leaf space, we also write $C^\infty(\underline{C}) = \mathsf{A}^0$. Observe that in general $\underline{C}$ is not a manifold, so the above is just a definition. Using the notations of Section 7.3, we have $\mathsf{B}^0 = \operatorname{Ker}\delta$. So B^0 is Hausdorff and complete, and by Theorem 7.16 we have the

THEOREM 7.17 ([**CF07**]). *If the first and second Lie algebroid cohomology of N^*C vanish, the zeroth cohomology $C^\infty(\underline{C})$ has a deformation quantization.*

The second Lie algebroid cohomology is also the space where the usual BRST/BFV anomaly lives; see [**B04**] for this in the context of deformation quantization. The advantage of the present approach is that one has formulae for m_0, so one can in principle check whether it may be canceled.

Observe that the conditions of Theorem 7.17 are very strong and by no means necessary. At the moment we actually do not know a single example where the anomaly shows up, at least locally. There are examples where the Lie algebroid cohomology class of m_0 is nontrivial, see [**W**], but in these examples C is a point, so there is no problem in quantizing its reduced space (the problem arises however in the many-brane setting of Section 8).

As $C^\infty(\underline{C})$ might be rather poor (e.g., just constant functions), it might be interesting to quantize the whole Lie algebroid cohomology. Assuming that the A_∞-structure is flat, by Theorem 7.9 the deformed cohomology B yields a deformation quantization of A iff it is Hausdorff, complete and ϵ-torsion free.

7.5. Poisson submanifolds. A particular case of a coisotropic submanifold is a Poisson submanifold, i.e., a submanifold C of (M,π) such that the inclusion map is a Poisson map. Equivalently, a Poisson submanifold is a coisotropic submanifold with trivial characteristic distribution. In this case, $\underline{C}=C$, so the quantization of the reduction is not a problem. The interesting question is whether one can deform the pullback of the inclusion map to a morphism of associative algebras. To approach this problem, we associate to C a different graded manifold as in the coisotropic case.

If the submanifold C is determined by constraints, i.e., $C=\Phi^{-1}(0)$ with $\Phi\colon M\to V$ a given map to a vector space V, we set $\mathcal{M}:=M\times V[-1]$ and reinterpret Φ as an element of $C^\infty(M)\otimes V$ and so as a vector field X of degree 1 on $\mathcal{M}$. If we introduce coordinates $\{\mu^\alpha\}_{\alpha=1,\dots,k:=\dim V}$ of degree -1, we have $X=\Phi^\alpha(x)\frac{\partial}{\partial\mu^\alpha}$, where the Φ^αs are the components of Φ w.r.t. the same basis. Observe that $[\,X\,,\,X\,]=0$ and that $[\,\pi\,,\,X\,]$ vanishes on C. Moreover, the cohomology of $C^\infty(\mathcal{M})$ w.r.t the differential $\delta:=[\,X\,,\ \,]$ is $C^\infty(C)$. If C is not determined by constraints, we take $\mathcal{M}:=N[-1]C$ and X the vector field of degree 1 corresponding to the zero section. The crucial observation now is that $A:=C^\infty(\mathcal{M})=\bigoplus_{j\le 0}A^j$ is a nonpositively graded commutative algebra. As a consequence, $\mathsf{A}^i=0$, $i>0$, with the differential $\lambda_1=\delta$. Observe moreover that A^0 is just $C^\infty(M)$ and that $\mathsf{A}^0=A^0/I=C^\infty(C)$, with $I=\delta A^{-1}$ the vanishing ideal of C.

7.5.1. *Casimir functions.* A particularly simple case is when the components of Φ are Casimir functions: viz., $\{\,\Phi^\alpha\,,\,f\,\}=0\ \forall\alpha\ \forall f\in C^\infty(M)$. Equivalently $[\,\pi\,,\,\Phi^\alpha\,]=0\ \forall\alpha$, i.e., $[\,\pi\,,\,X\,]=0$. Thus $\hat\pi:=\pi+X$ is a MC element. The induced P_∞-structure has $\lambda_1=\delta$. Quantization has the following easy to check remarkable properties: *i)* $(A^0[[\epsilon]],\tau_2)$ is Kontsevich's deformation quantization A_M of $C^\infty(M)$; *ii)* $\tau_2(\mu^\alpha,f)=\tau_2(f,\mu^\alpha)=\mu^\alpha f$, $\forall\alpha$. As a consequence $\mathsf{I}:=\tau_1(A^{-1}[[\epsilon]])$ is the two-sided ideal generated by $\{\tau_1(\mu^\alpha)\}$. A further easy computation, using Stokes' theorem in the explicit expression of the coefficients by Kontsevich's graphs, shows that $\tau_1(\mu^\alpha)=D(\Phi^\alpha)$, where D is the Duflo–Kirillov–Kontsevich map

$$\begin{array}{rccc} D\colon & C^\infty(M) & \to & C^\infty(M)[[\epsilon]],\\ & f & \mapsto & \sum_{n=0}^\infty \frac{\epsilon^n}{n!}\,U_{n+1}(f,\pi,\dots,\pi). \end{array}$$

So we get a deformation quantization of C as $\mathsf{B}^0=A_M/\mathsf{I}$, and the projection $A_M\to\mathsf{B}^0$ is a quantization of the inclusion map. This proves a conjecture in [**CR**] (see also [**C06**, Sect. 5] for a previous sketch of this proof).

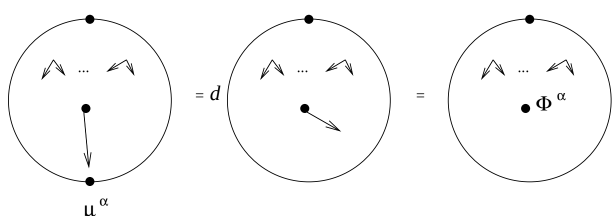

FIGURE 2. $\tau_1(\mu^\alpha)=D(\Phi^\alpha)$

7.5.2. *The general case.* In general $\pi+X$ is not a MC element. However, the fact that $[\,\pi\,,\,X\,]$ vanishes on C, makes $\pi+X$ a MC element up to δ-exact terms. As a consequence, one may use cohomological perturbation theory as in [**LS**] and find a MC element $\hat{\pi}$ of the form $\pi + X$+corrections in the ideal of multivector fields of total degree 2 generated by $\{\mu^\alpha\}$. Observe that the only vector field in $\hat{\pi}$ is X, so $\lambda_1 = \delta$. The vanishing of the cohomologies in positive degrees implies that we have an A_∞-structure on $A[[\epsilon]]$ and a surjective map $\mathsf{B}^0 \to A^0/I$. Moreover, the restriction of τ_2 in degree 0 makes $A^0[[\epsilon]]$ into an algebra with an algebra morphism to $\mathsf{B}^0 = A^0[[\epsilon]]/\mathsf{I}$, $\mathsf{I} := \tau_1(A^{-1}[[\epsilon]])$. The two problems of this constructions are the following: *i*) $(A^0[[\epsilon]], \tau_2)$ might not be associative; *ii*) B^0 might not be isomorphic to $A^0/I[[\epsilon]]$.

It may be shown that $(A^0[[\epsilon]], \tau_2)$ is associative iff $\hat{\pi}$ has the form $\pi + X + \pi'$ with π' a *bivector* field. If in addition one chooses the constraints to be linear (e.g., if one works with $\mathcal{M} = N[-1]C$), then one can also easily see that B^0 turns out to be a deformation quantization of $C^\infty(C)$. However, finding a π' as above is a highly nontrivial problem, and it is not clear under which conditions a solution may exist.

A very simple case is when C consists of a point x (a zero of the Poisson structure). A quantization of the inclusion map can then be reinterpreted as a character (i.e., an algebra morphism to the ground ring) of the deformed algebra (deforming the evaluation at x). Even in such a simple situation, the existence of a π' is guaranteed only for $\dim M = 2$ [**S2**]. In higher dimensions, it is an open problem.

8. Many branes

We now turn to the case when more than one coisotropic submanifold is chosen as a boundary condition. Namely, as in Section 5, we take Σ to be a disk. However, we now subdivide the boundary into closed, cyclically ordered intervals $I_1, \dots, I_n$ with exactly one intersection point between subsequent intervals. To the interval I_i we associate boundary conditions corresponding to a coisotropic submanifold C_i.

Assuming clean intersections, in [**CF04**] it is shown that the case $i = 2$ leads to the construction of a bimodule for the deformation quantizations of $\underline{C}_1$ and $\underline{C}_2$ if no anomaly appears.

In this Section we assume that no anomalies show up. If the submanifolds are pairwise transverse this amounts to asking that for each of them the anomaly discussed in subsection 7.4 vanishes. The vanishing of the second Lie algebroid cohomology for each submanifold is a sufficient condition by Theorem 7.17. Notice however that this is only a sufficient condition by no means necessary. A very simple example is when M is symplectic and C_1 is Lagrangian. In this case $\underline{C}_1$ is a point (assume C_1 to be connected), so there is no problem in quantizing it. The bimodule structure associated to $C_2 = M$ can always be found (upon choosing the star product appropriately). On the other hand the relevant Lie algebroid cohomology is the de Rham cohomology of C_1. Another example is the linear case $C_1 = \mathfrak{h}^0 \subset \mathfrak{g}^*$, where $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. It is shown in [**CF04**] that there is no anomaly in this case even if the cohomology may be very complicated. Observe however that an example of nonvanishing anomaly has recently been found in [**W**].

We also use the notation $A_{\underline{C}}$ for the PSM deformation quantization of the Poisson algebra $C^\infty(\underline{C})$, where C is a coisotropic submanifold of M. Whenever a distribution is trivial, we omit underlining the submanifold. Observe that A_M is Kontsevich's deformation quantization. With $B_{\underline{C_1 \cap C_2}}$ we then denote the $A_{\underline{C_1}}$-module-$A_{\underline{C_2}}$ corresponding to the picture on the left in fig. 3 (where the big black dot denotes the point at infinity which has to be sent to the point in the intersection where evaluation takes place).

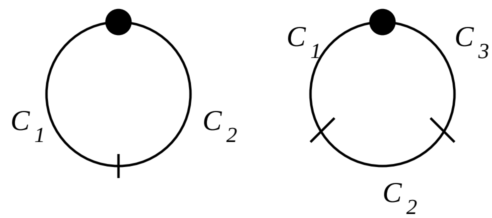

FIGURE 3. Two and three branes

The situation with three coisotropic branes C_1, C_2 and C_3—see the picture on the right in fig. 3—leads to a morphism of bimodules $B_{\underline{C_1 \cap C_2}} \otimes_{A_{\underline{C_2}}} B_{\underline{C_2 \cap C_3}} \to B_{\underline{C_1 \cap C_3}}$ in a neighborhood of a triple intersection. The construction of the corresponding Kontsevich graphs has been analyzed in [**CT**]. In subsection 8.2 we discuss a special case.

One may also consider many branes and, instead of just invariant functions, sections of the corresponding complexes of exterior algebras of normal bundles. One may hope that this would lead to A_∞-bimodules and morphisms thereof, but problems seem to arise with four and more branes.

In the rest of this Section we will concentrate on the special case when one of the branes is the whole manifold; we will consider only functions and no more than three branes.

8.1. Quantization of morphisms. Let ϕ be a Poisson map $M \to N$. Then its graph $\operatorname{Graph}\phi$ is a coisotropic submanifold of $\overline{M} \times N$, where $\overline{M}$ denotes M with opposite Poisson structure. We select the two branes $C_1 = \operatorname{Graph}\phi$ and $C_2 = M \times N$ in $\overline{M} \times N$. Consider the right $A_{\overline{M}\times N}$-module structure on $B_{\operatorname{Graph}\phi}$, forgetting about its left $A_{\underline{\operatorname{Graph}\phi}}$-module structure. Since $A_{\overline{M}} \otimes A_N$ is a subalgebra of $A_{\overline{M}\times N}$ and Kontsevich's star product has the property $A_{\overline{M}} = A_M^{\mathrm{opp}}$, we may regard $B_{\operatorname{Graph}\phi}$ as an A_M-bimodule-A_N. This bimodule has a distinguished element, the constant function 1, and the map $A_M \to B_{\operatorname{Graph}\phi}$, $f \mapsto f \cdot 1$ is an isomorphism of $\mathbb{R}[[\epsilon]]$-modules (since it is a deformation of the pullback of the diffeomorphism $p_1\colon \operatorname{Graph}\Phi \to M$ defined by projection). Thus, to every $f \in A_N$ we associate a unique element $\hat{\phi}(f)$ of A_M by the equation

$$1 \cdot f = \hat{\phi}(f) \cdot 1.$$

Observe that $\hat{\phi}$ is a deformation of the pullback ϕ^*. It is not difficult to see that it is a morphism of associative algebras. This way, with the assumption that there is no anomaly, we have found a quantization procedure for Poisson maps.

Observe that one can exchange the role of C_1 and C_2 using the symmetry corresponding to reflecting the disk through the line joining the two intersections of the intervals I_1 and I_2 ($\bullet$ and $|$ in fig. 3). In terms of Kontsevich graphs one gets the same formulae upon changing the sign of the Poisson structure; viz., we get the same result as before if we take $M \times \overline{N}$ as the ambient Poisson manifold.

8.2. Compositions. Now suppose we have two Poisson maps $\phi\colon M \to N$ and $\psi\colon N \to K$. With no anomalies, we may quantize ϕ, ψ and $\psi\circ\phi$. The next problem we wish to address concerns the relation between $\widehat{\psi \circ \phi}$ and $\hat{\phi} \circ \hat{\psi}$.

To deal with it we consider the three branes $C_1 = \mathrm{Graph}\,\phi \times K$, $C_2 = M \times N \times K$ and $C_3 = M \times \mathrm{Graph}\,\psi$ in $\overline{M} \times N \times \overline{K}$ as in fig.3. Let $p_1\colon M \times N \to M$ and $\tilde{p}_2\colon N \times K \to K$ be the canonical projections. To $f \in C^\infty(M)[[\epsilon]]$ and $g \in C^\infty(K)[[\epsilon]]$, we associate the element $D(f,g) \in C^\infty(\mathrm{Graph}\,\psi\circ\phi)[[\epsilon]]$ obtained by putting $p_1^* f \otimes 1$ and $1 \otimes \tilde{p}_2^* g$ at the two intersection points (here 1 is the constant function on $C^\infty(N)$). It is possible to check that D defines a morphism of A_M-bimodules-A_K $B_{\mathrm{Graph}\,\phi} \otimes_{A_N} \otimes B_{\mathrm{Graph}\,\psi} \to B_{\mathrm{Graph}\,\psi\circ\phi}$. So it is enough to compute $\sigma_{\phi,\psi} := D(1,1) = 1 + O(\epsilon^2) \in B_{\mathrm{Graph}\,\psi\circ\phi}$. Observe again that there is a unique element $\hat{\sigma}_{\phi,\psi} \in A_M$ such that $\hat{\sigma}_{\phi,\psi} \cdot 1 = \sigma_{\phi,\psi}$. Moreover, since $\hat{\sigma}_{\phi,\psi}$ is of the form $1 + O(\epsilon^2)$, it is invertible. Now for $h \in A_K$ we have the identities

$$\begin{aligned} \sigma_{\phi,\psi} \cdot h = D(1, 1 \cdot h) = D(1, \hat{\psi}(h) \cdot 1) = \\ = D(1 \cdot \hat{\psi}(h), 1) = D(\hat{\phi}(\hat{\psi}(h)) \cdot 1, 1) = \hat{\phi} \circ \hat{\psi} \cdot \sigma_{\phi,\psi}. \end{aligned}$$

Using $1 \cdot h = \widehat{\psi \circ \phi}(h) \cdot 1$ and the definition of $\hat{\sigma}_{\phi,\psi}$, we get

$$\begin{aligned} \sigma_{\phi,\psi} \cdot h = \hat{\sigma}_{\phi,\psi} \cdot 1 \cdot h = \hat{\sigma}_{\phi,\psi} \cdot (\widehat{\psi \circ \phi}(h) \cdot 1) = (\hat{\sigma}_{\phi,\psi} \star_M \widehat{\psi \circ \phi}(h)) \cdot 1, \\ \hat{\phi} \circ \hat{\psi}(h) \cdot \sigma_{\phi,\psi} = \hat{\phi} \circ \hat{\psi}(h) \cdot (\hat{\sigma}_{\phi,\psi} \cdot 1) = (\hat{\phi} \circ \hat{\psi}(h) \star_M \hat{\sigma}_{\phi,\psi}) \cdot 1, \end{aligned}$$

where $\star_M$ denotes the star product on M. Finally,

$$\widehat{\psi \circ \phi}(h) = \hat{\sigma}_{\phi,\psi}^{-1} \star_M (\hat{\phi} \circ \hat{\psi}(h)) \star_M \hat{\sigma}_{\phi,\psi},$$

where $\hat{\sigma}_{\phi,\psi}^{-1}$ is the inverse of $\hat{\sigma}_{\phi,\psi}$ w.r.t. $\star_M$.

Thus, upon conjugation (by an element which depends on the given Poisson maps), we see that the composition of quantizations is the quantization of compositions.

8.3. Quantum groups. We now want to apply the results of the last subsection to the case of a Poisson–Lie group G; i.e., a Poisson manifold and a Lie group such that the product $m\colon G \times G \to G$ and the inverse ${}^{-1}\colon \overline{G} \to G$ are Poisson maps. By Kontsevich we have an associative algebra A_G. On the other hand, if no anomaly is present, we get a morphism of algebras $\Delta := \hat{m}\colon A_G \to A_G \hat{\otimes} A_G := A_{G\times G}$. The associativity equation $m \circ (m \otimes id) = m \circ (id \otimes m)$ then yields

$$(\Delta \otimes id) \circ \Delta = \Phi^{-1} \star_{G^3} ((id \otimes \Delta) \circ \Delta) \star_{G^3} \Phi,$$

with $\Phi = \hat{\sigma}_{id\otimes m,m}^{-1} \star_{G^3} \hat{\sigma}_{m\otimes id,m} \in A_{G^3}$. In some lucky cases, Φ might turn out to be 1 (or at least central). In this case we would have quantized the Poisson–Lie group as a bialgebra. Otherwise, one may hope to get the right properties for the "associator" Φ in order to get a bialgebra out of it.

Finally, observe that we may also quantize the inverse and get a candidate for the antipode. If relations are preserved, we should get a Hopf algebra structure on A_G, i.e., the corresponding quantum group.

Just assuming that anomalies are absent is not enough to get a Hopf algebra as relations are preserved only up to conjugation by certain elements. If one cannot get rid of them, the resulting structure is probably that of a hopfish algebra as defined in [**BW**].

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Institut für Mathematik, Universität Zürich–Irchel, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

E-mail address: `alberto.cattaneo@math.unizh.ch`