CASTELNUOVO REGULARITY AND DEGREES OF GENERATORS OF GRADED SUBMODULES

MARKUS BRODMANN

Institute of Pure Mathematics
University of Zürich
Winterthurerstrasse 190
8057 Zürich, Switzerland

brodmann@math.unizh.ch

Abstract. We extend the regularity criterion of Bayer-Stillman for a graded ideal $\mathfrak{a}$ of a polynomial ring $K[\underline{\mathbf{x}}] := K[\underline{\mathbf{x}}_0, \cdots, \mathbf{x}_r]$ over an infinite field K, to the situation of a graded submodule M of a finitely generated graded module U over a noetherian homogeneous ring $R = \bigoplus_{n \geq 0} R_n$, whose base ring R_0 has infinite residue fields. If R_0 is artinian, we give a polynomial $P_U^{\sim} \in \mathbb{Q}[\mathbf{x}]$, which depends only on the Hilbert polynomial of U such that $\operatorname{reg}(M) \leq P_U^{\sim}(\max\{d(M), \operatorname{reg}(U)+1\})$, where $d(M)$ is the generating degree of M. This extends the regularity bound of Bayer-Mumford for a graded ideal $\mathfrak{a} \subseteq K[\underline{\mathbf{x}}]$ over a field K to the pair $M \subseteq U$.

1. Introduction

Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous noetherian ring and let $M \neq 0$ be a finitely generated graded R-module. For $i \in \mathbb{N}_0$ and $n \in \mathbb{N}$ let $H^i_{R_+}(M)_n$ denote the n-th graded component of the i-th local cohomology module $H^i_{R_+}(M)$ of M with respect to the irrelevant ideal $R_+ = \bigoplus_{n>0} R_n$ of R. The *(Castelnuovo-Mumford) regularity* $\operatorname{reg}(M)$ of M is defined by

$$\operatorname{reg}(M) := \inf\{m \in \mathbb{Z} \mid H^i_{R_+}(M)_{m-i+1} = 0 \quad \forall i \in \mathbb{N}_0\}. \tag{1.1}$$

1991 *Mathematics Subject Classification.* primary 13D45; secondary 13D40.

Key words and phrases. Castelnuovo-Mumford regularity, filter-regular sequences, Hilbert polynomials, generating degrees.

Upper bounds on $\mathrm{reg}(M)$ in terms of other invariants of M are of fundamental significance in algebraic geometry, commutative algebra and computational algebraic geometry (cf [3]).

So, in the theory of Hilbert and Piccard schemes one is lead to bound the regularity of a graded submodule M of a graded free module F over a polynomial ring in terms of the Hilbert polynomial of M, the generating degree and the rank of F, (cf [13], [14], [15], [22]).

On the other hand if the base ring R_0 is artinian, $\mathrm{reg}(M)$ and various other cohomological invariants of M may be bounded in terms of the *diagonal values* $\mathrm{length}_{R_0}\left(H^i_{R_+}(M)_{-i}\right)$ $(i = 0, 1, \cdots)$ of cohomology (cf [5], [6], [7]). In close relation to these bounds of diagonal type, the mere vanishing and non-vanishing of the graded components $H^i_{R_+}(M)_n$ is completely governed by a few simple combinatorial conditions, if R_0 is semilocal and of dimension ≤ 1 (cf [4]).

If $R = K[\mathbf{x}_0, \cdots, \mathbf{x}_r] =: K[\underline{\mathbf{x}}]$ is a polynomial ring over a field, $\mathrm{reg}(M)$ gives an upper bound on the generating degrees of the syzygies of M and hence is of crucial significance for the classical *problem of "the finitely many steps"* (cf [16], [17]). In more recent terms: $\mathrm{reg}(M)$ governs the computational complexity of calculating the syzygies of the finitely generated graded $K[\underline{\mathbf{x}}]$-module M (cf [9]).

Let us recall that the problem of "the finitely many steps" consists in constructing in a predictable number of steps, a minimal graded free resolution of M from a minimal graded free presentation $F_1 \to F_0 \to M \to 0$. This problem can be solved as the regularity $\mathrm{reg}(M)$ of a graded submodule M of the free module $K[\underline{\mathbf{x}}]^{\oplus s}$ can be bounded in terms of r, s and the generating degree $d(M)$ of M. This was essentially shown by Hermann [17] on use of ideas of Henzelt-Noether [16]. (Note that the bounds calculated by Hermann are not correct; for correctly calculated bounds see [19], for example.) In the spirit of this, Bayer and Mumford have shown that for a graded ideal $\mathfrak{a} \subseteq K[\underline{\mathbf{x}}]$ one has the bound (cf [1])

$$\mathrm{reg}(\mathfrak{a}) \leq (2d(\mathfrak{a}))^{r!}\,. \tag{1.2}$$

In [5] we have extended this bound by showing that for a graded submodule $M \subseteq K[\underline{\mathbf{x}}]^{\oplus s}$ it holds

$$\mathrm{reg}(M) \leq s^{e_r}\,(2d(M))^{r!}\,, \tag{1.3}$$

where the numbers e_r are defined recursively by $e_0 = 0$ and $e_r := e_{r-1}\cdot r+1$, if $r > 0$. It also should be noted that the bounds given in (1.2) and (1.3) still appear to be rather far away from being sharp: namely, if $\mathrm{Char}(K) = 0$ one has $\mathrm{reg}(\mathfrak{a}) \leq (2d(\mathfrak{a}))^{2^{r-1}}$ (cf [11], [12]), and by the examples of Mayr and Meyer (cf [21]) this latter bound is about to be of best possible type.

One basic aim of this paper is to extend the regularity bounds of (1.2) and (1.3) to a much more general situation. We namely consider an arbitrary finitely generated graded module U over a noetherian homogeneous ring $R = \bigoplus_{n\geq 0} R_n$ with artinian base ring R_0. Then we show (cf Theorem 5.7)

(1.4) *There is a polynomial* $P_U^{\sim} \in \mathbb{Q}[\mathbf{x}]$ (*of degree* $\dim(U)!$) *which depends only on the Hilbert polynomial* P_U *of* U, *such that for each graded submodule* $M \subseteq U$ *we have* $\operatorname{reg}(M) \leq P_U^{\sim}(\max\{d(M), \operatorname{reg}(U)+1\})$.

If in addition $\dim(U) = \dim(R)$ and $d(M) + \operatorname{reg}(M) \leq \operatorname{reg}(U) + 1$, we may replace $P_U^{\sim}$ by a polynomial $P_U^{*} \in \mathbb{Q}[\mathbf{x}]$ which is such that we get the bounds of (1.3) if we choose $R = K[\underline{\mathbf{x}}]$ and $U = K[\underline{\mathbf{x}}]^{\oplus s}$.

In [1], the bound of (1.2) is deduced on use of the *regularity criterion of Bayer-Stillman* (cf [2]). In fact it turns out, that the bound (1.2), and its extension (1.3), may be deduced without using this criterion (cf [5]). But nevertheless, our proof of the bound (1.3) (resp. its extension (1.4)) is closely related to the regularity criterion of Bayer-Stillman, as both rely on the technique of *(saturated) filter-regular sequences of linear forms.* In section 3 we give a criterion - in terms of such sequences - for detecting whether a graded submodule M of a finitely generated graded module U over a homogeneous noetherian ring $R = \bigoplus_{n\geq 0} R_n$ is m-regular, (cf Theorem 3.8). If the base ring R_0 has infinite residue fields, our criterion extends the corresponding criterion of Bayer-Stillman for a graded ideal $\mathfrak{a} \subseteq K[\underline{\mathbf{x}}]$ to the case of a graded submodule $M \subseteq U$ (cf Theorem 4.7).

2. Some Preliminaries

In this section we recall a few generalities on graded rings and graded modules. We use $\mathbb{N}_0$ (resp. $\mathbb{N}$) to denote the set of non-negative (resp. positive) integers.

2.1. **Definition and Remark.** A) By a *homogeneous ring* we mean a (commutative unitary) $\mathbb{N}_0$-graded ring $R = \bigoplus_{n\geq 0} R_n$ which is generated over its base ring R_0 by linear forms, thus with $R = R_0[R_1]$. Keep in mind that the $\mathbb{N}_0$-graded ring $R = \bigoplus_{n\geq 0} R_n$ is homogeneous and noetherian, if and only if R_0 is noetherian and there are finitely many linear forms $f_0, \cdots, f_r \in R_1$ such that $R = R_0[f_0, \cdots, f_r]$.

B) If $R = \bigoplus_{n\geq 0} R_n$ is a $\mathbb{N}_0$-graded ring, we shall denote by R_+ the *irrelevant ideal* of R, thus $R_+ := \bigoplus_{n>0} R_n$. Recall that R is homogeneous if and only if R_+ is generated by linear forms, thus if and only if $R_+ = R_1 \cdot R$.

C) If $R = \bigoplus_{n\geq 0} R_n$ is a $\mathbb{N}_0$-graded ring, we use $\operatorname{Proj}(R)$ to denote the *projective spectrum* of R, e.g. the set of all graded primes $\mathfrak{p} \subseteq R$ with $R_+ \not\subseteq \mathfrak{p}$. •

2.2. **Definition.** A) Let $R = \underset{n\geq 0}{\oplus} R_n$ be a $\mathbb{N}_0$-graded ring and let $T = \underset{n\in\mathbb{N}}{\oplus} T_n$ be a graded R-module. We define the *beginning* and the *end* of T respectively by

$$\mathrm{beg}(T) := \inf\{n \in \mathbb{Z} \mid T_n \neq 0\}, \quad \mathrm{end}(T) := \sup\{n \in \mathbb{Z} \mid T_n \neq 0\},$$

where "inf" and "sup" are formed in $\mathbb{Z} \cup \{\pm\infty\}$ with the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

B) Let R and T be as in part A) and let $m \in \mathbb{Z}$. We define the m-th *left-truncation* and the m-th *right-truncation* of T respectively as the following R_0-submodules of T:

$$T_{\geq m} := \underset{n\geq m}{\oplus} T_n \ ; \quad T_{\leq m} := \underset{n\leq m}{\oplus} T_n.$$

As R is $\mathbb{N}_0$-graded, $T_{\geq m}$ is a (graded) R-submodule of T.

C) Let R and T be as above. We denote the *generating degree* of T by $d(T)$, so that

$$d(T) := \inf\{m \in \mathbb{Z} \mid T = T_{\leq t} \cdot R\},$$

where "inf" is formed under the same convention as in part A). •

2.3. **Definition and Remark.** (cf [8]). A) Let $R = \underset{n\geq 0}{\oplus} R_n$ be a homogeneous noetherian ring and let $M = \underset{n\in\mathbb{Z}}{\oplus} M_n$ be a graded R-module. Then, for each $i \in \mathbb{N}_0$, the i-th *local cohomology module* $H^i_{R_+}(M)$ of M with respect to the irrelevant ideal R_+ of R carries a natural grading. For all $n \in \mathbb{Z}$ we use $H^i_{R_+}(M)_n$ to denote the n-th *graded component* of $H^i_{R_+}(M)$.

B) Let $R = \underset{n\geq 0}{\oplus} R_n$ and $M = \underset{n\in\mathbb{Z}}{\oplus} M_n$ be as in part A) but assume in addition that the R-module M is finitely generated. Then, for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ the R_0-module $H^i_{R_+}(M)_n$ is finitely generated and vanishes for all $n \gg 0$. Moreover $H^i_{R_+}(M)$ vanishes for all $i > \dim(M)$. So, for each $k \in \mathbb{N}_0$ we may define the (*Castelnuovo-Mumford*) *regularity of M at and above level k by*

$$\mathrm{reg}^k(M) := \sup\{\mathrm{end}\big(H^i_{R_+}(M)\big) + i \mid i \geq k\},$$

and obtain $\mathrm{reg}^k(M) \in \mathbb{Z} \cup \{-\infty\}$.

C) Let R and M be as in part B). The (*Castelnuovo-Mumford*) *regularity of M* is defined as (cf (1.1))

$$\mathrm{reg}(M) := \mathrm{reg}^0(M),$$

where $\mathrm{reg}^0(M)$ is defined according to part B). It is important to keep in mind, that the generating degree and the regularity of M are related by the inequality (cf [8, 15.3.1])

$$d(M) \leq \mathrm{reg}(M).$$

D) Let R and M be as in part B) and let $k \in \mathbb{N}_0, m \in \mathbb{N}$. Then, the following equivalence is known to hold (cf [8, 15.2.5])

$$\operatorname{reg}^k(M) \leq m \Longleftrightarrow H^i_{R_+}(M)_{m-i+1} = 0 \quad \forall i \geq k.$$

If $\operatorname{reg}^k(M) \leq m$ we say that M is *m-regular at and above level k*. If $\operatorname{reg}(M) \leq m$, e.g. if M is m-regular at and above level 0, we say that M is *m-regular.* •

2.4. **Remark.** (*Faithfully flat base change*) A) Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous noetherian ring and let R'_0 be a noetherian faithfully flat R_0-algebra. Then, the faithfully flat R-algebra $R'_0 \underset{R_0}{\otimes} R = \bigoplus_{n\geq 0}(R'_0 \underset{R_0}{\otimes} R_n)$ is a homogeneous noetherian ring in a natural way and $(R'_0 \underset{R_0}{\otimes} R)_+ = R_+(R'_0 \underset{R_0}{\otimes} R)$.

B) Keep the notations and hypotheses of part A), let $T = \bigoplus_{n\in\mathbb{Z}} T_n$ be a graded R-module and $S = \bigoplus_{n\in\mathbb{Z}} S_n \subseteq T$ a graded submodule. Then $R'_0 \underset{R_0}{\otimes} T = \bigoplus_{n\in\mathbb{Z}} R'_0 \underset{R_0}{\otimes} T_n$ is a graded $(R'_0 \underset{R_0}{\otimes} R)$-module in a natural way and $R'_0 \underset{R_0}{\otimes} S = \bigoplus_{n\in\mathbb{Z}} R'_0 \underset{R_0}{\otimes} S_n \subseteq R'_0 \underset{R_0}{\otimes} T$ becomes a graded submodule. Clearly if T is finitely generated, then the $R'_0 \underset{R_0}{\otimes} R$-module $R'_0 \underset{R_0}{\otimes} T$ is finitely generated, too. Moreover $d(R'_0 \underset{R_0}{\otimes} T) = d(T)$.

C) Let R and R'_0 be as in part A) and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. Then, the graded flat base-change property of local cohomology yields a natural isomorphism of graded $R'_0 \underset{R_0}{\otimes} R$-modules

$$H^i_{(R'_0 \underset{R_0}{\otimes} R)_+}(R'_0 \underset{R_0}{\otimes} M) \cong R'_0 \underset{R_0}{\otimes} H^i_{R_+}(M),$$

(cf [8, 15.2.3]). As a consequence we have

$$\operatorname{reg}^k(R'_0 \underset{R_0}{\otimes} M) = \operatorname{reg}^k(M) \quad \forall k \in \mathbb{N}_0.$$

D) (*Replacement argument*) Let R and R'_0 be as above. Let M be a finitely generated graded R-module and $N \subseteq M$ a graded submodule. Then, the previous observations allow to replace M and N by $R'_0 \underset{R_0}{\otimes} M$ resp. $R'_0 \underset{R_0}{\otimes} N$ whenever we wish to prove a statement on regularities and generating degrees of M and N. •

For further unexplained notation and terminology from commutative algebra we refer to [10], [20].

3. Filter-Regular Sequences and Regularity

Let $R = \underset{n\geq 0}{\oplus} R_n$ be a homogeneous noetherian ring, let U be a finitely generated graded R-module and let $M \subseteq U$ be a graded submodule. Let $m \in \mathbb{Z}$ and let $f_1, \cdots, f_r \in R_1$ be a sequence of linear forms. We prove a criterion for the condition that M is m-regular and $f_1, \cdots, f_r$ form a saturated filter-regular sequence with respect to U/M. We briefly recall the notion of filter-regular sequence.

3.1. **Reminder and Remark.** (cf [8, Chapt. 18]). A) Let $R \underset{n\geq 0}{\oplus} R_n$ be a homogeneous noetherian ring and let $T = \underset{n\in\mathbb{Z}}{\oplus} T_n$ be a finitely generated and graded R-module. A homogeneous element $f \in R$ is said to be (R_+-) *filter-regular with respect to* T if it is a non-zero divisor with respect to $T/H^0_{R_+}(T)$. It is equivalent to say that f avoids all $\mathfrak{p} \in \mathrm{Ass}_R(T) \cap \mathrm{Proj}(R)$. Clearly, f is filter-regular with respect to T if and only if the annihilator $0 \underset{T}{:} f$ of f in T is contained in $H^0_{R_+}(T)$, thus if and only if $\mathrm{end}(0 \underset{T}{:} f) < \infty$.

B) Let R and T be as in part A). A sequence of homogeneous elements $f_1, \cdots, f_r \in R$ is called a *filter-regular sequence with respect to* T if f_i is filter-regular with respect to $T/\sum_{j=1}^{i-1} f_j T$ for all $i \in \{1, \cdots, r\}$. If in addition $f_1, \cdots, f_r \in R_1$, we speak of a *filter-regular sequence of linear forms.* If $W \subseteq H^0_{R_+}(T)$ is a graded submodule, a sequence $f_1, \cdots, f_r$ of homogeneous elements in R is filter-regular with respect to T if and only if it is with respect to T/W. •

3.2. **Lemma.** *Let $R = \underset{n\geq 0}{\oplus} R_n$ be a homogeneous noetherian ring, let $T = \underset{n\in\mathbb{Z}}{\oplus} T_n$ be a finitely generated graded R-module, let $f_1, \cdots, f_r \in R_1$ be a filter-regular sequence with respect to T and let $i \in \{0, \cdots, r\}$. Then*

a) $\mathrm{reg}\big(T/\sum_{j=1}^{i} f_j T\big) \leq \mathrm{reg}(T)$ *;*

b) $\mathrm{end}\,\big(H^i_{R_+}(T)\big) + i \leq \mathrm{end}\Big(H^0_{R_+}\big(T/\sum_{j=1}^{i} f_j T\big)\Big)$.

Proof: "a)": Follows from [8, (18.3.11)].

"b)": The case $i = 0$ is obvious. So, let $i > 0$. As $f_2, \cdots, f_r$ is a filter-regular sequence with respect to T/f_1T, by induction

$$\mathrm{end}\big(H^{i-1}_{R_+}(T/f_1T)\big) + i - 1 \leq \mathrm{end}\big(H^0_{R_+}(T/\sum_{j=1}^{i} f_j T)\big) =: e.$$

Let $\overline{T} := T/H^0_{R_+}(T)$. Then, the graded epimorphism $H^{i-1}_{R_+}(T/f_1T) \twoheadrightarrow H^{i-1}_{R_+}(\overline{T}/f_1\overline{T})$ shows that $\operatorname{end}\big(H^{i-1}_{R_+}(\overline{T}/f_1\overline{T})\big) + i - 1 \leq e$. But now, the exact sequences

$$H^{i-1}_{R_+}(\overline{T}/f_1\overline{T})_{n+1} \longrightarrow H^i_{R_+}(\overline{T})_n \xrightarrow{f_1} H^i_{R_+}(\overline{T})_{n+1}$$

and the vanishing of $H^i_{R_+}(\overline{T})_n$ for all $n \gg 0$ show that

$$\operatorname{end}\big(H^i_{R_+}(\overline{T})\big) \leq \operatorname{end}\big(H^{i-1}_{R_+}(\overline{T}/f_1\overline{T}))\big) - 1 \leq e - i.$$

In view of the graded isomorphism $H^i_{R_+}(T) \cong H^i_{R_+}(\overline{T})$ we get our claim. ■

In order to prove and to formulate the announced regularity criterion we introduce the notion of saturated filter-regular sequence.

3.3. **Definition and Remark.** A) Let $R = \bigoplus_{n\geq 0} R_n$ and $T = \bigoplus_{n\in\mathbb{Z}} T_n$ be as in 3.1. A filter-regular sequence $f_1, \cdots, f_r$ with respect to T is *saturated* if $f_1, \cdots, f_r \in R_+$ and if $T/\sum_{j=1}^{r} f_jT$ is an R_+-torsion module. It is equivalent to say that $\sum_{j=1}^{r} f_jR \subseteq R_+ \subseteq \sqrt{0 \underset{R}{:} T/\sum_{j=1}^{r} f_jT}$ or else that $\sqrt{(0 \underset{R}{:} T) + R_+} = \sqrt{(0 \underset{R}{:} T) + \sum_{j=1}^{r} f_jR}$.

B) As a consequence of this we can say (cf [8, 2.1.9]):
If $f_1, \cdots, f_r \in R$ is a saturated filter-regular sequence with respect to T, there are natural isomorphisms $H^i_{R_+}(T) \cong H^i_{(f_1,\cdots,f_r)}(T)$ for all $i \in \mathbb{N}_0$. So, in this situation we have $H^i_{R_+}(T) = 0$ for all $i > r$. •

3.4. **Proposition.** *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous noetherian ring, let $T = \bigoplus_{n\in\mathbb{Z}} T_n$ be a finitely generated graded R-module, let $f_1, \cdots, f_r \in R_1$ and let $m \in \mathbb{Z}$. Then, the following statements are equivalent:*

(i) $\operatorname{reg}(T) < m$ *and $f_1, \cdots, f_r$ is a saturated filter-regular sequence with respect to T;*

(ii) $\operatorname{end}(0 \underset{T/\sum_{j=1}^{i-1} f_jT}{:} f_i) < m$ *for all $i \in \{1, \cdots, r\}$ and* $\operatorname{end}(T/\sum_{j=1}^{r} f_jT) < m$.

Proof: "(i) $\Longrightarrow$ (ii)": Assume that condition (i) holds. Then, 3.2 a) shows that $\operatorname{end}\big(H^0_{R_+}(T/\sum_{j=1}^{k} f_jT)\big) \leq \operatorname{reg}(T/\sum_{j=1}^{k} f_jT) \leq \operatorname{reg}(T) < m$ for all $k \in \{1, \cdots, r\}$. As f_i is filter-regular with respect to $T/\sum_{j=1}^{i-1} f_jT$, we obtain

$$\operatorname{end}(0 \underset{T/\sum_{j=1}^{i-1} f_jT}{:} f_i) \leq \operatorname{end}\big(H^0_{R_+}(T/\sum_{j=1}^{i-1} f_jT)\big) < m, \quad \forall i \in \{1, \cdots, r\}.$$

As the sequence $f_1, \cdots, f_r$ is saturated, we have $T/\sum_{j=1}^{r} f_jT = H^0_{R_+}(T/\sum_{j=1}^{r} f_jT)$ and hence obtain $\operatorname{end}(T/\sum_{j=1}^{r} f_jT) < m$.

"(ii) $\Longrightarrow$ (i)": Assume that condition (ii) holds. As $\operatorname{end}(0 \underset{T/\sum_{j=1}^{i-1} f_jT}{:} f_i) < \infty$ for $i = 1, \cdots, r$, it follows that the sequence $f_1, \cdots, f_r$ is filter-regular with respect to T. As $\operatorname{end}(T/\sum_{j=1}^{r} f_jT) < \infty$ this sequence is saturated. In particular we have $H^i_{R_+}(T) = 0$ for all $i > r$ (cf 3.3 B)). If we apply 3.2 b) with $i = 1, \cdots, r$ we obtain $\operatorname{reg}(T) < m$. •

3.5. **Corollary.** *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous noetherian ring, let $m \in \mathbb{Z}$, let U be a finitely generated graded R-module such that $\operatorname{reg}(U) < m$. Let $M \subseteq U$ be a graded submodule and let $f_1, \cdots, f_r \in R_1$. Then, the following statements are equivalent:*

(i) $\operatorname{reg}(M) \leq m$ *and $f_1, \cdots, f_r$ is a saturated filter-regular sequence with respect to U/M.*

(ii) $\big((M + \sum_{j=1}^{i-1} f_jU) \underset{U}{:} f_i\big)_{\geq m} = (M + \sum_{j=1}^{i-1} f_jU)_{\geq m}$ *for all $i \in \{1, \cdots, r\}$ and $(M + \sum_{j=1}^{r} f_jU)_{\geq m} = U_{\geq m}$.*

Proof: Let $T := U/M$. Then, the graded exact sequence $0 \to M \to U \to T \to 0$ shows that $\operatorname{reg}(M) \leq \max\{\operatorname{reg}(U), \operatorname{reg}(T)+1\}$ and $\operatorname{reg}(T) \leq \max\{\operatorname{reg}(U), \operatorname{reg}(M)-1\}$ (cf [8, 15.2.15]). So, statement (i) of 3.4 is equivalent to statement (i) of 3.5. It is immediate that statement (ii) of 3.4 is equivalent to statement (ii) of 3.5. ■

The announced regularity criterion turns the criterion 3.5 into a "persistency result": the comparison of graded components in all degrees $\geq m$ which appears in statement 3.5 (ii) may be replaced by a comparison in degree m. To prove this, we use the following lemma:

3.6. **Lemma.** *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous noetherian ring. Let U be a finitely generated graded R-module, let $m \in \mathbb{Z}$ and let $M, N \subseteq U$ be two graded submodules such that $d(M), d(N) \leq m$ and* $\operatorname{reg}(M + N) < m$. *Then, $d(M \cap N) \leq m$.*

Proof: Write R as a graded homomorphic image of a polynomial ring $R_0[\underline{\mathbf{x}}] = R_0[\mathbf{x}_0, \cdots, \mathbf{x}_r]$ and observe that neither the generating degree nor the regularity of a finitely generated graded R-module V change their values, if we consider V as an $R_0[\underline{\mathbf{x}}]$-module. Therefore we may and do assume that $R = R_0[\underline{\mathbf{x}}]$ is a polynomial ring. Now, we may proceed as in the proof of [5, 2.4], where our result is shown in the special case in which R is a polynomial ring over a field. Namely, as $d(M), d(N) \leq m$

there are graded epimorphisms $\pi : F \to M \to 0$, $\varrho : G \to N \to 0$ in which F and G are graded free R-modules of finite rank with $d(F), d(G) \leq m$. As $\mathrm{reg}(R) = 0$ we thus obtain $\mathrm{reg}(F \oplus G) \leq m$ and the graded short exact sequence

$$0 \to \mathrm{Ker}(\pi + \varrho) \to F \oplus G \xrightarrow{\pi+\varrho} M + N \to 0$$

yields that $\mathrm{reg}\big(\mathrm{Ker}(\pi + \varrho)\big) \leq m$, thus $d\big(\mathrm{Ker}(\pi + \varrho)\big) \leq m$ (cf 2.3 C)). Now, the commutative diagram

$$\begin{array}{ccc} M \oplus N & \xrightarrow{\sigma := id_M + id_N} & M + N \\ \uparrow{\scriptstyle \pi \oplus \varrho} & & \uparrow{\scriptstyle \pi + \varrho} \\ F \oplus G & = & F \oplus G \end{array}$$

shows that $(\pi \oplus \varrho)\big(\mathrm{Ker}(\pi + \varrho)\big) = \mathrm{Ker}(\sigma)$ and thus $d\big(\mathrm{Ker}(\sigma)\big) \leq m$. In view of the graded isomorphism $M \cap N \cong \mathrm{Ker}(\sigma)$ we get our claim. ■

3.7. **Lemma.** *Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous noetherian ring and let $m \in \mathbb{Z}$. Let U be a finitely generated graded R-module, let $M \subseteq U$ be a graded submodule and let $f \in R_1$ be filter-regular with respect to U. Assume that $d(M), \mathrm{reg}(U), \mathrm{reg}(M + fU) \leq m$. Then, $d(M \underset{U}{:} f) \leq m$.*

Proof: As $d(fU) \leq d(U) + 1 \leq \mathrm{reg}(U) + 1 \leq m + 1$, Lemma 3.6 implies $d(M \cap fU) \leq m + 1$. As $M \cap fU = f(M \underset{U}{:} f)$ we have a graded short exact sequence

$$0 \to (0 \underset{U}{:} f) \to (M \underset{U}{:} f) \to (M \cap fU)(1) \to 0.$$

As f is filter-regular with respect to U, we have $(0 \underset{U}{:} f) \subseteq H^0_{R_+}(U)$ and hence $d(0 \underset{U}{:} f) \leq \mathrm{end}(0 \underset{U}{:} f) \leq \mathrm{end}\big(H^0_{R_+}(U)\big) \leq \mathrm{reg}(U) \leq m$. Now, the above exact sequence yields $d(M \underset{U}{:} f) \leq m$. ■

Now, we are ready to formulate and to prove the main result of this section.

3.8. **Theorem.** *Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous noetherian ring and let $m \in \mathbb{Z}$. Let U be a finitely generated graded R-module, let $M \subseteq U$ be a graded submodule, let $f_1, \cdots, f_r \in R_1$ be filter-regular elements with respect to U and assume that* $\mathrm{reg}(U) < m$ *and $d(M) \leq m$. Then, the following statements are equivalent:*

(i) $\mathrm{reg}(M) \leq m$ *and $f_1, \cdots, f_r$ is a saturated filter-regular sequence with respect to U/M;*

(ii) $\big((M + \sum_{j=1}^{i-1} f_jU) \underset{U}{:} f_i\big)_m = (M + \sum_{j=1}^{i-1} f_jU)_m$ *for all* $i \in \{1, \cdots, r\}$ *and* $(M + \sum_{j=1}^{r} f_jU)_m = U_m$.

Proof: "(i) $\Longrightarrow$ (ii)": Clear by 3.5.

"(ii) $\Longrightarrow$ (i)": We proceed by induction on r. First, let $r = 1$. By statement (ii) we have $(M + f_1U)_m = U_m$. As $d(U) \leq \mathrm{reg}(U) \leq m$ it follows $(M + f_1U)_{\geq m} = U_{\geq m}$, hence $\mathrm{end}\big(U/(M + f_1U)\big) < m$. In view of the graded short exact sequence $0 \to (M + f_1U) \to U \to U/(M + f_1U) \to 0$ it follows $\mathrm{reg}(M + f_1U) \leq m$. By Lemma 3.7 we get $d(M \underset{U}{:} f_1) \leq m$. By statement (ii), we have $(M \underset{U}{:} f_1)_m = M_m$; it follows $(M \underset{U}{:} f_1)_{\geq m} = M_{\geq m}$. By the implication "(ii) $\Longrightarrow$ (i)" of Corollary 3.5 we get $\mathrm{reg}(M) \leq m$ and that f_1 constitutes a saturated filter-regular sequence with respect to U/M.

Now, let $r > 1$ and assume that statement (ii) holds. As $d(f_1U) \leq d(U) + 1 \leq \mathrm{reg}(U) + 1 \leq m$, we have $d(M + f_1U) \leq m$. We apply induction to the graded submodule $M + f_1U \subseteq U$ and the sequence $f_2, \cdots, f_r \in R_1$. In doing so, we thus see that $\mathrm{reg}(M + f_1U) \leq m$ and that $f_2, \cdots, f_r$ is a saturated filter-regular sequence with respect to $U/(M + f_1U)$. So, by 3.5 we have $\big((M + \sum_{j=1}^{i-1} f_jU) \underset{U}{:} f_i)_{\geq m} = (M + \sum_{j=1}^{i-1} f_jU)_{\geq m}$ for all $i \in \{2, \cdots, r\}$ and $(M + \sum_{j=1}^{r} f_jU)_{\geq m} = U_{\geq m}$. By 3.7 we also have $d(M \underset{U}{:} f_1) \leq m$. As $(M \underset{U}{:} f_1)_m = M_m$ and $d(M) \leq m$, it follows $(M \underset{U}{:} f_1)_{\geq m} = M_{\geq m}$. Now, another use of 3.5 gives statement (i). ■

4. Extending the Regularity Criterion of Bayer-Stillman

Let $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \cdots, \mathbf{x}_t]$ be a polynomial ring over an infinite field K and let $\mathfrak{a} \subseteq K[\underline{\mathbf{x}}]$ be a graded ideal. Let $m \in \mathbb{N}$. In [2, 1.10] Bayer and Stillman proved that $\mathfrak{a}$ is m-regular if and only if there is a sequence of linear forms $f_1, \cdots, f_r \in K[\underline{\mathbf{x}}]_1$ such that statement (ii) of Theorem 3.8 holds with $M = \mathfrak{a}$ and $U = K[\underline{\mathbf{x}}]$. The aim of this section is to extend this regularity criterion of Bayer-Stillman to a situation closely as general as in 3.8. To do so, we obviously need that there are saturated filter-regular sequences of linear forms with respect to arbitrary finitely generated modules over the considered homogeneous noetherian ring $R = \bigoplus_{n \geq 0} R_n$. To ensure the existence of such sequences, we shall subject the base ring R_0 to an appropriate condition.

4.1**. Definition and Remark.** A) A Ring R_0 is said to have *infinite residue fields* if the field $R_0/\mathfrak{m}_0$ is infinite for each $\mathfrak{m}_0 \in \mathrm{Max}(R_0)$ or - equivalently - if $R_0/\mathfrak{p}_0$ is an infinite domain for each $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)$.

B) Clearly, if $f : R_0 \to R'_0$ is a homomorphism of rings and R_0 has infinite residue fields, then so has R'_0. In particular R_0 has infinite residue fields if it contains an infinite field. •

4.2. **Lemma.** *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous noetherian ring such that R_0 has infinite residue fields and let $\mathfrak{Q} \subseteq \mathrm{Proj}(R)$ be a finite set. Then $R_1 \not\subseteq \bigcup_{\mathfrak{q}\in\mathfrak{Q}} \mathfrak{q}$.*

Proof: We may assume that $\mathfrak{Q} \neq \emptyset$. For $\mathfrak{m}_0 \in \mathrm{Max}(R_0)$ set $\mathfrak{Q}(\mathfrak{m}_0) := \{\mathfrak{q} \in \mathfrak{Q} \mid \mathfrak{q} \cap R_0 \subseteq \mathfrak{m}_0\}$. Clearly, there is a finite set $\mathbb{M} \subseteq \mathrm{Max}(R_0)$ such that $\mathfrak{Q}(\mathfrak{m}_0) \neq \emptyset$ for each $\mathfrak{m}_0 \in \mathbb{M}$ and $\mathfrak{Q} = \bigcup_{\mathfrak{m}_0\in\mathbb{M}} \mathfrak{Q}(\mathfrak{m}_0)$. For each $\mathfrak{m}_0 \in \mathbb{M}$ and each $\mathfrak{q} \in \mathfrak{Q}(\mathfrak{m}_0)$ it follows by Nakayama that $\mathfrak{q} \cap R_1 + \mathfrak{m}_0 R_1 \subsetneqq R_1$. So, as $\mathfrak{Q}(\mathfrak{m}_0)$ is finite and $R_0/\mathfrak{m}_0$ is infinite, there is some $v_{\mathfrak{m}_0} \in R_1 \setminus \bigcup_{\mathfrak{q}\in\mathfrak{Q}(\mathfrak{m}_0)} (\mathfrak{q}_1 + \mathfrak{m}_0 R_1)$. For each $\mathfrak{m}_0 \in \mathbb{M}$ we find some element $a_{\mathfrak{m}_0} \in \big(\bigcap_{\mathfrak{n}_0\in\mathbb{M}\setminus\{\mathfrak{m}_0\}} \mathfrak{n}_0\big)\setminus\mathfrak{m}_0$. With $v := \sum_{\mathfrak{m}_0\in\mathbb{M}} a_{\mathfrak{m}_0} v_{\mathfrak{m}_0}$ it follows $v \in R_1 \setminus \bigcup_{\mathfrak{m}_0\in\mathbb{M}} \bigcup_{\mathfrak{q}\in\mathfrak{Q}(\mathfrak{m}_0)} (\mathfrak{q}_1 + \mathfrak{m}_0 R_1) = R_1 \setminus \bigcup_{\mathfrak{q}\in\mathfrak{Q}} \mathfrak{q}$. ■

4.3. **Lemma.** *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous noetherian ring such that R_0 has infinite residue fields and let $\mathcal{P} \subseteq \mathrm{Proj}(R)$ be a finite set. Let $r \in \mathbb{N}$ and let $T = \bigoplus_{n\in\mathbb{Z}} T_n$ be a finitely generated graded R-module. Then there is a sequence $(f_i)_{i\in\mathbb{N}} \subseteq R_1 \setminus \bigcup_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}$ such that $f_1, \cdots, f_r$ is a filter-regular sequence with respect to T for each $r \in \mathbb{N}$.*

Proof: If we apply 4.2 with $\mathfrak{Q} := \mathcal{P} \cap \mathrm{Ass}(T) \cap \mathrm{Proj}(R)$ we get an element $f_1 \in R_1 \setminus \bigcup_{\mathfrak{q}\in\mathcal{P}} \mathfrak{p}$ which is filter-regular with respect to T. On use of this observation, a sequence $(f_i)_{i\in\mathbb{N}}$ of the requested type is easily constructed by induction. ■

So, if the base ring R_0 has infinite residue fields, filter-regular sequence of arbitrary length and consisting of linear forms exist. Now, the existence of saturated filter-regular sequences follows easily.

4.4. **Lemma.** *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous noetherian ring and let T be a finitely generated graded R-module. Let $(f_i)_{i\in\mathbb{N}} \subseteq R_+$ be a sequence such that $f_1, \cdots, f_r$ is a filter-regular sequence with respect to T for each $r \in \mathbb{N}$. Then, there is some $r_0 \in \mathbb{N}$ such that the filter-regular sequence $f_1, \cdots, f_r$ is saturated for each $r \geq r_0$.*

Proof: If, for some $r \in \mathbb{N}$, the filter-regular sequence $f_1, \cdots, f_r$ is non-saturated, f_{r+1} avoids some member of $\mathrm{Ass}_R(T/\sum_{i=1}^{r} f_i T)$, so that $f_{r+1} \notin \sum_{i=1}^{r} f_i R$, hence $\sum_{i=1}^{r} f_i R \subsetneqq \sum_{i=1}^{r+1} f_i R$. As R is noetherian, we get our claim. ■

The possible values of the number r_0 in Lemma 4.4 can be bounded easily. In order to do so, let us recall some notion.

4.5. **Definition.** The *arithmetic rank* $\operatorname{ara}(\mathfrak{a})$ of an ideal $\mathfrak{a}$ of a noetherian ring R is defined as the minimum number of elements in R, which generate an ideal which is radically equal to $\mathfrak{a}$, thus

$$\operatorname{ara}(\mathfrak{a}) := \min\left\{r \in \mathbb{N}_0 \mid \exists a_1, \cdots, a_r \in R : \sqrt{\sum_{i=1}^{r} a_i R} = \sqrt{\mathfrak{a}}\ \right\}. \qquad \bullet$$

4.6. **Lemma.** *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous noetherian ring, let T be a finitely generated graded R-module and let $f_1, \cdots, f_r \in R_+$ be a filter-regular sequence with respect to T. Then:*

a) *If the filter-regular sequence $f_1, \cdots, f_r$ is saturated, $r \geq \operatorname{ara}\big((R/(0 \underset{R}{:} T))_+\big)$.*

b) *If $r \geq \dim(T)$, the filter-regular sequence $f_1, \cdots, f_r$ is saturated.*

c) *If R_0 is artinian, then the filter-regular sequence $f_1, \cdots, f_r$ is saturated if and only if $r \geq \dim(T)$.*

Proof: "a)": Clear by 3.3 A).

"b)": Assume that the sequence $f_1, \cdots, f_r$ is not saturated, so that $\sqrt{(0 \underset{R}{:} T) + R_+} \supsetneqq \sqrt{(0 \underset{R}{:} T) + \sum_{j=1}^{r} f_j R}$. Then, there is a prime $\mathfrak{p} \in \operatorname{Var}\big((0 \underset{R}{:} T) + \sum_{j=1}^{r} f_j R\big) \setminus \operatorname{Var}(R_+)$. Thus $f_1/1, \cdots f_r/1 \in \mathfrak{p}R_{\mathfrak{p}}$ is a regular sequence with respect to $T_{\mathfrak{p}}$ (cf [8, 18.3.8]), so that $r \leq \operatorname{depth}(T_{\mathfrak{p}}) \leq \dim(T_{\mathfrak{p}})$. As $\mathfrak{p} \subsetneqq \mathfrak{p}_0 + R_+ \in \operatorname{Spec}(R)$, we have $\dim(T_{\mathfrak{p}}) < \dim(T)$ and hence get $r < \dim(T)$.

"c)" As R_0 is artinian, we have $\dim\big(R/(0 \underset{R}{:} T)\big) = \operatorname{ara}\big((R/(0 \underset{R}{:} T))_+\big)$. Now, we conclude by statements a) and b). ∎

Next, we give the announced extension of the regularity criterion of Bayer-Stillman.

4.7. **Theorem.** *Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous noetherian ring such that R_0 has infinite residue fields. Let $m \in \mathbb{Z}$, let U be a finitely generated graded R-module and let $M \subseteq U$ be a graded submodule. Assume that $\operatorname{reg}(U) < m$ and $d(M) \leq m$. Then, the following statements are equivalent:*

(i) $\operatorname{reg}(M) \leq m$;

(ii) *there are elements $f_1, \cdots, f_r \in R_1$ which are filter-regular with respect to U and such that*

$$\big((M + \sum_{j=1}^{i-1} f_j U) \underset{U}{:} f_i\big)_m = (M + \sum_{j=1}^{i-1} f_j U)_m \quad \forall i \in \{1, \cdots, r\}$$

and

$$\big(M + \sum_{j=1}^{r} f_j U\big)_m = U_m.$$

Proof: "(ii) $\Longrightarrow$ (i)": Clear by Theorem 3.8.

"(i) $\Longrightarrow$ (ii)": If we apply 4.3 with $\mathcal{P} = \mathrm{Ass}_R(U) \cap \mathrm{Proj}(R)$ and keep in mind 4.4 we get a saturated filter-regular sequence $f_1, \cdots, f_r \in R_1$ with respect to U/M such that each f_i is filter-regular with respect to U. Now, we conclude by Theorem 3.8. ■

4.8**. Remark.** A) Keep the notations and all the hypotheses of 4.7. Let $f_1, \cdots, f_r \in R_1$ be filter-regular linear forms with respect to U. Then, in view of Theorem 3.8 the two conditions

$$\big((M + \sum_{j=1}^{i-1} f_j U) \underset{U}{:} f_i\big)_m = (M + \sum_{j=1}^{i-1} f_j U)_m \quad \forall i \in \{1, \cdots, r\}$$

and

$$(M + \sum_{j=1}^{r} f_j U)_m = U_m$$

hold if and only if $f_1, \cdots, f_r$ is a saturated filter-regular sequence with respect to U/M.

B) Keep the above notations and hypotheses. Assume that $\dim(U/M) \leq r$. Then, in view of 4.6 b) the two conditions mentioned in part A) hold if and only if the linear forms $f_1, \cdots, f_r$ form a filter-regular sequence with respect to U/M. Moreover, the above conditions never can hold if $r < \mathrm{ara}\big((R/(0 \underset{R}{:} T))_+\big)$ (cf 4.6 a)). In particular, for each $r \geq \dim(U/M)$ and for a "generic sequence" $f_1, \cdots, f_r \in R_1$ of linear forms, the above two conditions hold, whereas for $r < \mathrm{ara}\big((R/(0 \underset{R}{:} T))_+\big)$ they never hold simultaneously.

C) Let $K[\underline{\mathbf{x}}] = K[\underline{\mathbf{x}}_0, \cdots, \mathbf{x}_t]$ be a polynomial ring over an infinite field K, let $m, s \in \mathbb{N}$, let $U := K[\underline{\mathbf{x}}]^{\oplus s}$ and let $M \subseteq U$ a graded submodule with $d(M) \leq m$. As $\mathrm{reg}(U) = 0$ and as U is torsion-free, it follows from 4.7 that $\mathrm{reg}(M) \leq m$ if and only there are linear forms $f_1, \cdots, f_r \in K[\underline{\mathbf{x}}]_1 \backslash \{0\}$ such that the above two conditions hold. Moreover, if this is the case, these two conditions hold for a generic sequence $f_1, \cdots, f_r$ of linear forms whenever $r \geq \dim(U/M)$. This is precisely what is shown in [18, 1.10]. Choosing $s = 1$, we get the regularity criterion of Bayer-Stillman. •

5. Extending the Regularity bound of Bayer-Mumford

Let $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \cdots, \mathbf{x}_t]$ be a polynomial ring over a field K and let $\mathfrak{a} \subseteq K[\underline{\mathbf{x}}]$ be a graded ideal. In [1, 3.8] Bayer and Mumford have shown that $\mathrm{reg}(\mathfrak{a}) \leq \big(2d(\mathfrak{a})\big)^{n!}$. Our aim is to extend this bounding result to the case where $K[\underline{\mathbf{x}}]$ is replaced by an arbitrary finitely generated graded module U over a homogeneous noetherian ring $R = \bigoplus_{n \geq 0} R_n$ with artinian base ring R_0 and $\mathfrak{a}$ by a graded submodule M of U.

5.1. **Notation and Remark.** A) Let R_0 be an artinian ring and let V be a finitely generated R_0-module. We use $\ell(V) = \ell_{R_0}(V)$ to denote the length of V.

B) Let R_0 and V be as in part A). Let $\mathfrak{m}_1, \cdots, \mathfrak{m}_t$ be the different maximal ideals of R_0, let $\mathbf{x}$ be an indeterminate and set

$$R_0' := \big(R_0[\mathbf{x}] \setminus \bigcup_{i=1}^{t} \mathfrak{m}_i R_0[\mathbf{x}]\big)^{-1} R_0[\mathbf{x}].$$

Then, clearly R_0' is a faithfully flat artinian extension ring of R_0 with the different maximal ideals $\mathfrak{m}_i' = \mathfrak{m}_i R_0'$ $(i = 1, \cdots, t)$. Moreover we have $\ell_{R_0'}(R_0' \otimes_{R_0} V) = \ell_{R_0}(V)$. As $R_0'/\mathfrak{m}_i' \cong R_0/\mathfrak{m}_i(\mathbf{x})$ for all $i \in \{1, \cdots, t\}$, the ring R_0' has infinite residue fields. •

5.2. **Lemma.** *Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous noetherian ring such that R_0 is artinian, let U be a finitely generated graded R-module, let $M \subseteq U$ be a graded submodule and let $f \in R_1$ be filter-regular with respect to U and to U/M. Let $k \in \mathbb{Z}$ be such that $d(M), \mathrm{reg}(M + fU), \mathrm{reg}(U) + 1 \leq k$. Then*

a) $\mathrm{end}\big(H^i_{R_+}(M)\big) + i \leq k$ *for all* $i \neq 1$ *;*

b) $\mathrm{end}\big(H^1_{R_+}(M)\big) \leq \ell(U_k) + k - 1$.

Proof: Let $T := U/M$. The short exact sequence $0 \to (M + fU) \to U \to T/fT \to 0$ shows that $\mathrm{reg}(T/fT) \leq \max\{\mathrm{reg}(U), \mathrm{reg}(M + fU) - 1\} \leq k - 1$. As $f \in R_1$ is filter-regular with respect to T, it follows $\mathrm{reg}^1(T) \leq \mathrm{reg}(T/fT) \leq k - 1$ (cf [8, 18.3.11]) and the graded short exact sequence $0 \to M \to U \to T \to 0$ implies $\mathrm{reg}^2(M) \leq \max\{\mathrm{reg}^2(U), \mathrm{reg}^1(T) + 1\} \leq k$ (cf [8, 15.2.15]) and hence $\mathrm{end}\big(H^i_{R_+}(M)\big) + i \leq k$ for all $i \geq 2$. As $\mathrm{end}\big(H^0_{R_+}(M)\big) \leq \mathrm{end}\big(H^0_{R_+}(U)\big) \leq \mathrm{reg}(U) \leq k$, we have shown statement a).

It remains to prove statement b). In view of the graded short exact sequence $0 \to M \to U \to T \to 0$ and as $\mathrm{end}\big(H^1_{R_+}(U)\big) \leq \mathrm{reg}(U) - 1 \leq k - 1$, it suffices to show that $\mathrm{end}\big(H^0_{R_+}(T)\big) \leq \ell(U_k) + k - 1$. We have seen above that $\mathrm{reg}(T/fT) \leq k - 1$. So, if we apply cohomology to the graded short exact sequence $0 \to T/(0 :_T f) \xrightarrow{f} T(1) \to$

$(T/fT)(1) \to 0$ we get isomorphisms

$$H^0_{R_+}\big(T/(0 \underset{T}{:} f)\big)_n \cong H^0_{R_+}(T)_{n+1}, \quad \forall n \geq k-1.$$

If we apply cohomology to the graded short exact sequence $0 \to (0 \underset{T}{:} f) \to T \to T/(0 \underset{T}{:} f) \to 0$ and keep in mind that $(0 \underset{T}{:} f) \subseteq H^0_{R_+}(T)$ (cf 3.1 A)), we thus get exact sequences

$$0 \to (0 \underset{T}{:} f)_n \to H^0_{R_+}(T)_n \xrightarrow{\pi_n} H^0_{R_+}(T)_{n+1} \to 0, \quad \forall n \geq k-1.$$

By 3.7 we have $d(0 \underset{T}{:} f) = d(M \underset{U}{:} f) \leq k$ so that π_m becomes an isomorphism for all $m \geq n$, provided π_n is an isomorphism for some $n \geq k$. From this it follows that the length $\ell\big(H^0_{R_+}(T)_n\big)$ of the R_0-module $H^0_{R_+}(T)_n$ is strictly decreasing as a function of n in the range $n \geq k$ until its value becomes 0. This implies that $\operatorname{end}\big(H^0_{R_+}(T)\big) \leq \ell\big(H^0_{R_+}(T)_k\big) + k - 1$. As $H^0_{R_+}(T)_k$ is a subquotient of the R_0-module U_k we get $\operatorname{end}\big(H^0_{R_+}(T)\big) \leq \ell(U_k) + k - 1$. ∎

5.3. **Lemma.** *Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous noetherian ring such that R_0 is artinian and $\dim(R) = 1$. Let U be a finitely generated and graded R-module and let $M \subseteq U$ be a graded submodule. Let $k \in \mathbb{Z}$ be such that $d(M) + \operatorname{reg}(R)$ and $\operatorname{reg}(U) + 1 \leq k$. Then, $\operatorname{reg}(M) \leq k$.*

Proof: We may apply the replacement argument 2.4 D) with R'_0 defined according to 5.1 B) and thus may assume that R_0 has infinite residue fields. As $\operatorname{end}\big(H^0_{R_+}(M)\big) \leq \operatorname{end}\big(H^0_{R_+}(U)\big) < k$ and as $H^i_{R_+}(M) = 0$ for all $i > 1$ it remains to show that $\operatorname{end}\big(H^1_{R_+}(M)\big) \leq k-1$. Choosing $\mathcal{P} = \operatorname{Ass}_R(R) \cap \operatorname{Proj}(R)$ we conclude by 4.3 that there is a linear form $f \in R_1$ which is at the same time filter-regular with respect to U and to R. As f is filter-regular with respect to U, we have $\operatorname{end}(0 \underset{U}{:} f) \leq \operatorname{end}\big(H^0_{R_+}(U)\big) < k$. Therefore, the multiplication map $f : U_n \to U_{n+1}$ is injective for all $n \geq k$. As $\dim(R) = 1$ and as $f \in R_1$ avoids all minimal primes of R we have $R_+ \subseteq \sqrt{Rf}$ and R is a finitely generated graded module over its subring $R_0[f]$. In particular by the graded base ring independence of local cohomology, $\operatorname{reg}(R)$ does not change if we consider R as an $R_0[f]$-module. In doing so we obtain $d(R) \leq \operatorname{reg}(R) \leq k - d(M)$ so that $R_{n+1} = fR_n$ for all $n \geq k - d(M)$. Hence for each $n \geq k$ we obtain $M_{n+1} = R_{n-d(M)+1}M_{d(M)} = fR_{n-d(M)}M_{d(M)} = fM_n$. As $f : U_n \to U_{n+1}$ is injective for all $n \geq k$ it follows that $(M_{n+1} \underset{U_n}{:} f) = M_n$ for all such n. From this, we see that $\operatorname{end}(0 \underset{U/M}{:} f) < k$. As $f \in R_1$, it follows $\operatorname{end}\big(H^0_{R_+}(U/M)\big) < k$. If we apply cohomology to the graded exact sequence $0 \to M \to U \to U/M \to 0$ and keep in mind that $\operatorname{end}\big(H^1_{R_+}(U)\big) < \operatorname{reg}(U) < k$ it follows indeed that $\operatorname{end}\big(H^1_{R_+}(M)\big) < k$. ∎

In order to formulate our main result, we introduce some notation

5.4. **Definition and Remark.** A) Let $\mathbb{P}$ be the set of all polynomials $P \in \mathbb{Q}[\mathbf{x}]$ with the property that $P(n) \in \mathbb{N}_0$ for all integers $n \gg 0$. For $P \in \mathbb{P}$, let $\Delta P \in \mathbb{P}$ denote the difference polynomial $P(\mathbf{x}) - P(\mathbf{x}-1)$ of P.

B) For $P \in \mathbb{P}$ we recursively define a polynomial $P^* = P^*(\mathbf{x})$ by

$$P^*(\mathbf{x}) := \begin{cases} \mathbf{x}, & \text{if } \deg(P) \leq 0 \\ (\Delta P)^*(\mathbf{x}) + P\big((\Delta P)^*(\mathbf{x})\big), & \text{if } \deg(P) > 0. \end{cases}$$

It is easy to see, that $P^* \in \mathbb{P}$, whenever $P \in \mathbb{P}$.

C) Now, let $s \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Then clearly $s\binom{\mathbf{x}+r}{r} \in \mathbb{P}$ and $\Delta\left[s\binom{\mathbf{x}+r}{r}\right] = s\binom{\mathbf{x}+r-1}{r-1}$. We write $F_r(s,\mathbf{x}) := \left[s\binom{\mathbf{x}+r}{r}\right]^*$ so that

$$F_0(s,\mathbf{x}) = \mathbf{x} \text{ and } F_r(s,\mathbf{x}) = F_{r-1}(s,\mathbf{x}) + s\binom{F_{r-1}(s,\mathbf{x})+r}{r} \text{ for all } r > 0.$$

This means, that $F_r(s,\mathbf{x})$ is as in [5, 2.5 A)]. In particular, we have (cf [5, 2.5 B)]):

$$F_r(s,t) < s^{e_r}(2t)^{r!}, \quad (\forall s,t \in \mathbb{N}),$$

where the numbers e_r are defined inductively by

$$e_0 := 0 \text{ and } e_r := r \cdot e_{r-1} + 1 \text{ for } r > 0.$$

D) Also, for each $P \in \mathbb{P}$ we recursively define a polynomial $P^\sim \in \mathbb{P}$ by

$$P^\sim(\mathbf{x}) := \begin{cases} \mathbf{x}, & \text{if } P = 0 \\ (\Delta P)^\sim(\mathbf{x}) + P\big((\Delta P)^\sim(\mathbf{x})\big), & \text{if } P \neq 0. \end{cases}$$

It is easy to see that $\widetilde{P}(k) \geq P^*(k)$ for all $k \gg 0$. •

Finally let us recall a few facts on Hilbert polynomials.

5.5. **Reminder.** A) Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous noetherian ring such that R_0 is artinian and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R-module. We denote the Hilbert polynomial of M by P_M so that (cf [8, Chap. 17])

$$P_M(n) = \ell(M_n) \quad \forall n > \operatorname{reg}(M).$$

B) Also, if $f \in R_1$ is filter regular with respect to M, we have short exact sequences $0 \to M_{n-1} \xrightarrow{f} M_n \to (M/fM)_n \to 0$ for all $n \gg 0$ and these yield $P_{M/fM} = \Delta P_M$.

If R_0' is defined according to 5.1 B) and in the notation of 2.4 B) we have

$$P_{R_0' \underset{R_0}{\otimes} M} = P_M. \qquad \bullet$$

5.6. **Lemma.** *Let* $R \bigoplus_{n \geq 0} R_n$ *be a homogeneous noetherian ring such that* R_0 *is artinian. Let* U *be a finitely generated graded* R*-module and let* $k \in \mathbb{Z}$ *be such that* $\operatorname{reg}(U) < k$*. Then*

a) $k \leq (\Delta P_U)^*(k) \leq P_U^*(k)$ *;*

b) $k \leq (\Delta P_U)^{\sim}(k) \leq P_U^{\sim}(k)$*.*

Proof: In view of 2.4 D) and 5.5 B) we may assume that R_0 has infinite residue fields. We now proceed by induction on $\deg(P_U)$. If $P_U = 0$, we have $P_U^* = P_U^{\sim} = (\Delta P_U)^* = (\Delta P_U)^{\sim} = \mathbf{x}$, and our claims are obvious. If $\deg(P_U) = 0$ we have $P_U^* = (\Delta P_U)^* = (\Delta P_U)^{\sim} = \mathbf{x}$ and $P_U^{\sim} = \mathbf{x} + P_U(\mathbf{x})$. As P_U is a positive constant our claims follow. Let $\deg(P_U) > 0$. As R_0 has infinite residue fields there is a linear form $f \in R_1$ which is filter regular with respect to U. In particular we have $\Delta P_U = P_{U/fU}$ (cf 5.5 B)) and $\operatorname{reg}(U/fU) < k$ (cf 3.2 a)). So, by induction we have $k \leq (\Delta P_U)^*(k)$ and $k \leq (\Delta P_U)^{\sim}(k)$. In particular (cf 5.5 A)) $P_U\big((\Delta P_U)^*(k)\big) = \ell(U_{(\Delta P_U)^*(k)}) \geq 0$ and $P_U\big((\Delta P_U)^{\sim}(k)\big) = \ell(U_{(\Delta P_U)^{\sim}(k)}) \geq 0$. Now, both claims follow from the definitions of P_U^* and $P_U^{\sim}$. ■

Now, we prove the main result of this section.

5.7. **Theorem.** *Let* $R = \bigoplus_{n \geq 0} R_n$ *be a homogeneous noetherian ring such that* R_0 *is artinian . Let* U *be a finitely generated graded* R*-module and let* $M \subseteq U$ *be a graded submodule. Let* $k \in \mathbb{Z}$ *and assume that* $\operatorname{reg}(U) < k$*.*

a) If $d(M) \leq k$*, then* $\operatorname{reg}(M) \leq P_U^{\sim}(k)$*.*

b) If $\dim(R) = \dim(U)$ *and* $d(M) + \operatorname{reg}(R) \leq k$*, then* $\operatorname{reg}(M) \leq P_U^*(k)$*.*

Proof: In view of 2.4 D) and the last observation made in 5.5 B), we may assume that R_0 has infinite residue fields. We proceed by induction on $\dim(U)$. If $\dim(U) \leq 0$ we have $P_U = 0$ and $\operatorname{reg}(M) = \operatorname{end}\big(H^0_{R_+}(M)\big) \leq \operatorname{end}\big(H^0_{R_+}(U)\big) = \operatorname{reg}(U) < k = 0^*(k) = 0^{\sim}(k)$, which proves both claims in this case. Now, let $\dim(U) > 0$. From now on, we prove our two claims separately.

"a)": If we apply 4.3 with $\mathcal{P} := \operatorname{Ass}_R(U/M) \cap \operatorname{Proj}(R)$, we find a linear form $f \in R_1$ which is filter-regular with respect to U and U/M. As $\dim(U) > 0$, f avoids all minimal members of $\operatorname{Ass}_R(U)$ so that $\dim(U/fU) = \dim(U) - 1$. By 3.2 a) we have $\operatorname{reg}(U/fU) \leq \operatorname{reg}(U) < k$. Clearly $d\big((M + fU)/fU\big) \leq d(M) \leq k$. By 5.5 B) we also have $\Delta P_U = P_{U/fU}$. Now, by induction we have $\operatorname{reg}\big((M + fU)/fU\big) \leq (\Delta P)^{\sim}(k)$. As $(0 \underset{f}{:} U) \subseteq H^0_{R_+}(U)$ and in view of the graded isomorphism $fU \cong \big(U/(0 \underset{U}{:} f)\big)(-1)$

we get $\operatorname{reg}(fU) = \operatorname{reg}\big(U/(0 \underset{U}{:} f)\big) + 1 \leq \operatorname{reg}(U) + 1 \leq k$, hence $\operatorname{reg}(fU) \leq (\Delta P)^{\sim}(k)$, (cf 5.6 b)). The exact sequence $0 \to fU \to (M + fU) \to (M + fU)/fU \to 0$ yields $\operatorname{reg}(M + fU) \leq (\Delta P_U)^{\sim}(k) =: m$. If we keep in mind that $k \leq m$ we get $m \leq P_U^{\sim}(m)$ (cf 5.6 b)) and $\ell(U_m) = P_U(m)$ (cf 5.5 A)). So, if we apply 5.2 with m instead of k we get $\operatorname{end}\big(H^i_{R_+}(M)\big) + i \leq P_U^{\sim}(m)$ for all $i \neq 1$ and $\operatorname{end}\big(H^1_{R_+}(M)\big) + 1 \leq P_U(m) + m = (\Delta P_U)^{\sim}(k) + P_U\big((\Delta P_U)^{\sim}(k)\big) = P_U^{\sim}(k)$. Therefore $\operatorname{reg}(M) \leq P_U^{\sim}(k)$.

"b)": Assume first that $\dim(U) = 1$ and hence $\dim(R) = 1$. Then, 5.3 and 5.6 a) show that $\operatorname{reg}(M) \leq k \leq P_U^*(k)$. So, let $\dim(U) > 1$. Now apply 4.3 with $\mathcal{P} = \operatorname{Ass}_R(U/M) \cup \operatorname{Ass}_R(R) \cap \operatorname{Proj}(R)$ in order to obtain a linear form $f \in R_1$ which is at the same time filter-regular with respect to $U, U/M$ and R. As in the proof of statement a) we now get $\dim(R/fR) = \dim(U/fU) = \dim(U) - 1, \operatorname{reg}(U/fU) < k$ and $d\big((M + fU)/fU\big) + \operatorname{reg}(R/fR) \leq k$. Again, by 5.5 B) we have $\Delta P_U = P_{U/fU}$. Thus, by induction we obtain $\operatorname{reg}\big((M + fU)/fU\big) \leq (\Delta P)^*(k)$. Now, we may conclude literally in the same way as in the proof of statement a) if we replace $(\Delta P_U)^{\sim}$ by $(\Delta P_U)^*$ and $P_U^{\sim}$ by P_U^*. ∎

5.8. **Corollary.** *Let $R_0[\underline{\mathbf{x}}] = R_0[\mathbf{x}_0, \cdots, \mathbf{x}_r]$ be a polynomial ring over an artinian ring R_0. Let $w \in \mathbb{N}$ and let $M \subseteq R_0[\underline{\mathbf{x}}]^{\oplus w}$ be a graded submodule. Then*

$$\operatorname{reg}(M) \leq \big(\ell(R_0)w\big)^{e_r}\big(2d(M)\big)^{r!},$$

where e_r is defined according to 5.4 C).

Proof: If $d(M) = 0$, there is a graded isomorphism $M \cong M_0 \underset{R_0}{\otimes} R_0[\underline{\mathbf{x}}]$, so that $\operatorname{reg}(M) = 0$. Therefore we may assume that $d(M) > 0$. Let $R := R_0[\underline{\mathbf{x}}]$, $U := R_0[\underline{\mathbf{x}}]^{\oplus w}$. Then $\operatorname{reg}(U) = \operatorname{reg}(R) = 0$, $\dim(R) = \dim(U) = r$ and the fact that $P_U = \ell(R_0)w\binom{\mathbf{x} + r}{r}$ allow to conclude by 5.7 b) and 5.4 C). ∎

5.9. **Remark.** If in 5.8 we choose $R_0 = K$ to be a field, we get the bound given in [5, 2.7]. If we choose in addition $w = 1$, we get the bound of Bayer-Mumford [1, 3.8]. •

References

[1] D. BAYER and D. MUMFORD: *What can be computed in algebraic geometry?* in "Computational Algebraic Geometry and Commutative Algebra" Proc. Cortona 1991 (D. Eisenbud and L. Robbiano Eds.), Cambridge University Press (1993) 1 - 48.

[2] D. BAYER and M. STILLMAN: *A criterion for detecting m-regularity*, Invent. Math. 87 (1987) 1 - 11.

[3] M. BRODMANN: *Cohomological invariants of coherent sheaves over projective schemes - a survey* in "Local Cohomology and its Applications" M. Dekker Lecture Notes 226 (G. Lyubeznik Ed.), M. Dekker (2001) 91 - 120.

[4] M. BRODMANN and M. HELLUS: *Cohomological patterns of coherent sheaves over projective schemes*, J. of Pure and Applied Algebra 172 (2002) 165 - 182.

[5] M. BRODMANN and A. LASHGARI: *A diagonal bound for cohomological postulation numbers of projective schemes*, preprint.

[6] M. BRODMANN, C. MATTEOTTI and N.D. MINH: *Bounds for cohomological Hilbert-functions of projective schemes over artinian rings*, Vietnam J. of Math. 28, 4 (2000) 345 - 384.

[7] M. BRODMANN, C. MATTEOTTI and N.D. MINH: *Bounds for cohomological deficiency functions of projective schemes over artinian rings*, to appear in Vietnam J. of Math.

[8] M. BRODMANN and R.Y. SHARP: *Local cohomology - an algebraic introduction with geometric applications*. Cambridge Studies in Advanced Mathematics, 60, Cambridge University Press (1998).

[9] B. BUCHBERGER: *A note on the complexity of constructing Gröbner-Bases*, in "EUROCAL '83" (J.A. v. Hulzen Ed.), Springer LNCS 162 (1983).

[10] D. EISENBUD: *Commutative algebra with a view towards algebraic geometry*, Springer New York (1996).

[11] A. GALLIGO: *Théorème de division et stabilité en géométrie analytique locale*, Ann. Inst. Fournier 29 (1979) 107 - 184.

[12] M. GIUSTI: *Some effectivity problems in polynomial ideal theory* in "Eurosam 84", Springer Lecture Notes in Computer Sciences 174 (1984) 159 - 171.

[13] G. GOTZMANN: *Eine Bedingung für die Flachheit und das Hilbertpolynom eines gradierten Ringes*, Math. Z. 158 (1978), 61 - 70.

[14] G. GOTZMANN: *Durch Hilbertfunktionen definierte Unterschemata des Hilbert-Schemas*, Comment. Math. Helvetici 63 (1988) 114 - 149.

[15] A. GROTHENDIECK: *Séminaire de géométrie algébrique VI*, Springer Lecture Notes in Mathematics 225, Springer (1971).

[16] K. HENTZELT and E. NOETHER: *Zur Theorie der Polynomideale und Resultanten*, Math. Ann. 88 (1923), 53 - 79.

[17] G. HERMANN: *Über die Frage der endlich vielen Schritte in der Theorie der Polynomideale*, Math. Ann. 95 (1926), 736 - 788.

[18] A.F. LASHGARI: *The cohomology diagonal bounds the postulation numbers of a coherent sheaf over a projective scheme*, Dissertation, University of Zürich (2000).

[19] D. MASSER and G. WÜSTHOLZ: *Fields of large transcendence degree generated by values of elliptic functions*, Invent. Math. 72 (1983), 407 - 464.

[20] H. MATSUMURA: *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press (1986).

[21] E.W. MAYR and A.R. MEYER: *The complexity of the word problem for commutative semigroups and polynomial ideals*, Advances in Math. 46 (1982), 305 - 329.

[22] D. MUMFORD: *Lectures on curves on an algebraic surface*, Annals of Math. Studies 59, Princeton University Press (1966).