

AN AVOIDANCE PRINCIPLE WITH AN APPLICATION TO THE ASYMPTOTIC BEHAVIOUR OF GRADED LOCAL COHOMOLOGY

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ABSTRACT. We present an Avoidance Principle for certain graded rings. As an application we fill a gap in the proof of a result by Brodmann, Rohrer and Sazeedeh about the antipolynomiality of the Hilbert-Samuel multiplicity of the graded components of the local cohomology modules of a finitely generated module over a Noetherian homogeneous ring with two-dimensional local base ring.

Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring (R_0, \mathfrak{m}_0) and irrelevant ideal $R_+ := \bigoplus_{n \in \mathbb{N}} R_n$, let $\mathfrak{q}_0 \subseteq R_0$ be an \mathfrak{m}_0 -primary ideal and let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded R -module. For $i \in \mathbb{N}_0$ we denote by $H_{R_+}^i(M)$ the i -th local cohomology module of M with respect to R_+ , and for $n \in \mathbb{Z}$ we denote by $H_{R_+}^i(M)_n$ its n -th graded component. By $e_{\mathfrak{q}_0}(T)$ we denote the Hilbert-Samuel multiplicity of a finitely generated R_0 -module T with respect to \mathfrak{q}_0 .

If A is a ring, $\mathfrak{a} \subseteq A$ is an ideal and T is an A -module, by $\text{Gr}(\mathfrak{a}) := \bigoplus_{n \in \mathbb{N}_0} \mathfrak{a}^n / \mathfrak{a}^{n+1}$ and $\text{Gr}(\mathfrak{a}, T) := \bigoplus_{n \in \mathbb{N}_0} \mathfrak{a}^n T / \mathfrak{a}^{n+1} T$ we denote the graded ring associated with \mathfrak{a} and the graded module associated with T with respect to \mathfrak{a} respectively.

If (A, \mathfrak{m}) is a local ring, by an unramified A -algebra we mean an A -algebra B which is a local ring with maximal ideal $\mathfrak{m}B$.

The following Theorem is stated in [B-R-S, Theorem 5.7]:

Theorem

Let $i \in \mathbb{N}_0$, let $\dim(R_0) = 2$ and let $\dim_{R_0}(H_{R_+}^i(M)_n) \geq 1$ for all $n \ll 0$. Then, there exists a polynomial $Q \in \mathbb{Q}[X]$ such that $\deg(Q) < i$ and $e_{\mathfrak{q}_0}(H_{R_+}^i(M)_n) = Q(n)$ for all $n \ll 0$.

But Alas!, there is a gap in the proof of this Theorem in [B-R-S]. In this note we will fix this. We will not reproduce the entire proof and refer (as we do for unexplained notations) to the article cited above.

The argument lacks in the case where $\dim_{R_0}(H_{R_+}^i(M)_n) = 1$ for all $n \ll 0$ and under the additional assumptions that $\Gamma_{\mathfrak{m}_0 R}(M) = 0$ and that R_0 is complete. For $n \in \mathbb{Z}$ let $T_n := H_{R_+}^i(M)_n / \Gamma_{\mathfrak{m}_0}(H_{R_+}^i(M)_n)$. In order to continue the proof there must exist – possibly after a replacement of R_0 by an appropriate ring – an element $x \in \mathfrak{m}_0 \cap \text{NZD}_{R_0}(M)$ which is a parameter for R_0 and a non-zero-divisor on T_n for

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$n \ll 0$ such that for all $n \ll 0$ the following equations hold:

$$e_{\mathfrak{q}_0}(H_{R_+}^i(M)_n) = e_{\mathfrak{q}_0}(T_n) = e_{\mathfrak{q}_0}(T_n/xT_n) =$$

$$\text{length}_{R_0}(T_n/xT_n) = \text{length}_{R_0}(H_{R_+}^i(M)_n / (xH_{R_+}^i(M)_n + \Gamma_{\mathfrak{m}_0}(H_{R_+}^i(M)_n))).$$

In [B-R-S] it was not shown that the second equation $e_{\mathfrak{q}_0}(T_n) = e_{\mathfrak{q}_0}(T_n/xT_n)$ holds for the element x given there. We begin with a remark which reminds of some facts on superficial elements (cf. [N, §§22f.]) in a form suitable to our needs.

Remark 1

Let A be a Noetherian ring, let $\mathfrak{a} \subseteq A$ be an ideal, let T be a finitely generated A -module and let $x \in \mathfrak{a} \cap \text{NZD}_A(T)$ be such that $x + \mathfrak{a}^2 \in \text{Gr}(\mathfrak{a})_1 = \mathfrak{a}/\mathfrak{a}^2$ is a quasi-non-zerodivisor on $\text{Gr}(\mathfrak{a}, T)$. Then, there is some $m_0 \in \mathbb{N}_0$ such that $\frac{1}{x}\mathfrak{a}^m T \cap \mathfrak{a}^{m-2}T = (\mathfrak{a}^m T :_{\mathfrak{a}^{m-2}T} x) = \mathfrak{a}^{m-1}T$ for all $m \geq m_0$. By induction it follows $\frac{1}{x}\mathfrak{a}^m T \cap \mathfrak{a}^{m_0}T \subseteq \mathfrak{a}^{m-1}T$ for all $m \geq m_0$. By Artin-Rees there is some $m_1 \geq m_0$ such that $\mathfrak{a}^{m_1} \frac{1}{x}T \cap T \subseteq \mathfrak{a}^{m_0}T$. So $\frac{1}{x}\mathfrak{a}^m T \cap T \subseteq \mathfrak{a}^{m-1}T$ and hence $(\mathfrak{a}^m T :_T x) = \mathfrak{a}^{m-1}T$ for all $m \geq m_1$. Therefore, the exact sequences

$$0 \rightarrow T/(\mathfrak{a}^m T :_T x) \rightarrow T/\mathfrak{a}^m T \rightarrow (T/xT)/\mathfrak{a}^m(T/xT) \rightarrow 0$$

yield isomorphisms of A -modules $(T/xT)/\mathfrak{a}^m(T/xT) \cong (T/\mathfrak{a}^m T)/(T/\mathfrak{a}^{m-1}T)$ for all $m \gg 0$.

If (A, \mathfrak{m}) is local, \mathfrak{a} is \mathfrak{m} -primary and $\dim_A(T) > 0$, it follows for the Hilbert-Samuel polynomials $P_{\mathfrak{a}, T/xT}(\mathbf{x}) = P_{\mathfrak{a}, T}(\mathbf{x} - 1) - P_{\mathfrak{a}, T}(\mathbf{x} - 2)$ and hence $e_{\mathfrak{a}}(T/xT) = e_{\mathfrak{a}}(T)$. •

By Remark 1, for the equations $e_{\mathfrak{q}_0}(T_n) = e_{\mathfrak{q}_0}(T_n/xT_n)$ as well as the other conditions to hold it is sufficient for x to meet the following conditions:

- (1) x is a parameter for R_0 ;
- (2) $\forall n \ll 0 : x \in \text{NZD}_{R_0}(T_n)$;
- (3) $\forall k \in \mathbb{Z} : x \in \text{NZD}_{R_0}(M_k)$;
- (4) $x \in \mathfrak{q}_0$;
- (5) $\forall n \ll 0 : x + \mathfrak{q}_0^2 \in \text{Gr}(\mathfrak{q}_0)_1$ is a quasi-non-zerodivisor on $\text{Gr}(\mathfrak{q}_0, T_n)$.

We shall indeed look for an element x which satisfies the following stronger conditions:

- (1') $x \notin \bigcup \text{Min}(R_0)$;
- (2') $\forall n \in \mathbb{Z} : x \notin \bigcup \text{Ass}_{R_0}(T_n)$;
- (3') $\forall k \in \mathbb{Z} : x \notin \bigcup \text{Ass}_{R_0}(M_k)$;
- (4') $x \in \mathfrak{q}_0 \setminus \mathfrak{m}_0 \mathfrak{q}_0$;
- (5') $\forall n \in \mathbb{Z} : x + \mathfrak{q}_0^2 \notin \bigcup \text{Ass}_{\text{Gr}(\mathfrak{q}_0)}(\text{Gr}(\mathfrak{q}_0, T_n)) \setminus \text{Var}(\text{Gr}(\mathfrak{q}_0)_+)$.

(For $n \in \mathbb{Z}$, keep in mind that, as $\text{Gr}(\mathfrak{q}_0)$ is Noetherian and homogeneous and $\text{Gr}(\mathfrak{q}_0, T_n)$ is finitely generated, $x + \mathfrak{q}_0^2$ is a quasi-non-zerodivisor on $\text{Gr}(\mathfrak{q}_0, T_n)$ if and only if it avoids all essential primes associated to $\text{Gr}(\mathfrak{q}_0, T_n)$.)

As R_0 is complete, the Countable Prime Avoidance Principle (cf. [Sh-V, Corollary 2.2]) allows to find an $x \in \mathfrak{m}_0$ which satisfies conditions (1'), (2') and (3'). This was already used in the proof of [B-R-S, Theorem 5.7]. In order to show the existence of an element x which satisfies moreover conditions (4') and (5') we will reduce the problem to a case which allows to conclude with the Avoidance Principle presented below (Proposition 3). Its proof needs the following well-known fact.

Remark 2

Let K be a field, let V be a vector space over K of finite dimension and let $(V_i)_{i \in I}$ be a family of proper subspaces $V_i \subsetneq V$ such that I has cardinality strictly less than K . Then $\bigcup_{i \in I} V_i \subsetneq V$. \bullet

Proposition 3

Let (A, \mathfrak{m}) be a local ring with infinite residue field K and let $\mathfrak{a} \subseteq A$ be a finitely generated ideal with $\mathfrak{a} \neq 0$. Let S be a set of ideals of A not containing \mathfrak{a} and let T be a set of ideals of $\text{Gr}(\mathfrak{a})$ not containing $\text{Gr}(\mathfrak{a})_+$. Assume that S and T have cardinality strictly less than K . Then, there is an element $x \in \mathfrak{a} \setminus (\mathfrak{m}\mathfrak{a} \cup \bigcup S)$ such that $x + \mathfrak{a}^2 \in \text{Gr}(\mathfrak{a})_1 \setminus \bigcup T$.

Proof. Consider the family $F := ((\mathfrak{b} \cap \mathfrak{a} + \mathfrak{m}\mathfrak{a})/\mathfrak{m}\mathfrak{a})_{\mathfrak{b} \in S}$ of K -subspaces of $\mathfrak{a}/\mathfrak{m}\mathfrak{a}$. Nakayama's Lemma implies $\mathfrak{b} \cap \mathfrak{a} + \mathfrak{m}\mathfrak{a} \subsetneq \mathfrak{a}$ for each $\mathfrak{b} \in S$. Thus, F consists of proper K -subspaces of $\mathfrak{a}/\mathfrak{m}\mathfrak{a}$.

For each $\mathfrak{c} \in T$ there is an ideal $\mathfrak{d}_{\mathfrak{c}} \subseteq A$ such that $\mathfrak{a}^2 \subseteq \mathfrak{d}_{\mathfrak{c}} \subseteq \mathfrak{a}$ and $\mathfrak{c} \cap \text{Gr}(\mathfrak{a})_1 = \mathfrak{c} \cap (\mathfrak{a}/\mathfrak{a}^2) = \mathfrak{d}_{\mathfrak{c}}/\mathfrak{a}^2$. As $\text{Gr}(\mathfrak{a})_+ \not\subseteq \mathfrak{c}$, we have $\mathfrak{d}_{\mathfrak{c}} \subsetneq \mathfrak{a}$. So, Nakayama's Lemma implies $\mathfrak{d}_{\mathfrak{c}} + \mathfrak{m}\mathfrak{a} \subsetneq \mathfrak{a}$, and $G := ((\mathfrak{d}_{\mathfrak{c}} + \mathfrak{m}\mathfrak{a})/\mathfrak{m}\mathfrak{a})_{\mathfrak{c} \in T}$ is a family of proper K -subspaces of $\mathfrak{a}/\mathfrak{m}\mathfrak{a}$. As the family $F \amalg G$ has cardinality strictly less than K , by Remark 2 we find some $x \in \mathfrak{a} \setminus \mathfrak{m}\mathfrak{a}$ such that $x + \mathfrak{m}\mathfrak{a} \in \mathfrak{a}/\mathfrak{m}\mathfrak{a}$ avoids all members of $F \amalg G$. This element x has the requested properties. \square

The reduction will be based on a lifting result for complete local rings. We begin with the following result about the existence of certain flat algebras over a discrete valuation ring (DVR). For its proof we refer to [ÉGA, Proposition 0.6.8.2, Corollaire 0.6.8.3].

Lemma 4

Let A be a DVR with residue field K and let L be an extension field of K . Then, there exists an unramified flat A -algebra which is a complete DVR with residue field isomorphic to L .

Proposition 5

Let A be a Noetherian complete local ring with residue field K and let L be an extension field of K . Then, there is a diagram of rings

$$A'' \xleftarrow{g} A' \xrightarrow{f} A$$

such that A' and A'' are complete regular local rings, f is surjective and g turns A'' into an unramified flat A' -algebra with residue field isomorphic to L .

Proof. As A is complete, Cohen's Structure Theorem (cf. [M, §29]) gives rise to a surjective morphism of rings $f : A' \rightarrow A$, where $A' = B[[X_1, \dots, X_t]]$ is a ring of formal power series with $t \in \mathbb{N}_0$ and B a field or a complete DVR. In particular, A' is a complete regular local ring, and if K' denotes the residue field of A' , we have $K' \cong K$. In order to construct a morphism of the requested type $g : A' \rightarrow A''$ we consider the two possible cases separately:

Case 1: B is a field. Then $B \cong K' \cong K$, so that L is an extension field of B . Now, set $A'' := L[[X_1, \dots, X_t]]$ and let $g : A' \rightarrow A''$ be the morphism of rings induced by

the inclusion $B \subseteq L$. Then, A'' is a complete regular local ring and an unramified A' -algebra with residue field isomorphic to L . Moreover, by the Local Flatness Criterion (cf. [M, Theorem 22.1]) g is flat.

Case 2: B is a DVR. Let $\pi \in B$ be a uniformizer of B . As L is an extension field of $B/\pi B$, by Lemma 4 there is an unramified flat B -algebra C which is a complete DVR with residue field isomorphic to L . Now, set $A'' := C[[X_1, \dots, X_t]]$ and let $g : A' \rightarrow A''$ be the morphism of rings induced by the structure morphism $B \rightarrow C$. Then, A'' is a complete regular local ring with residue field isomorphic to L and again in the view of the Local Flatness Criterion g turns A'' into an unramified flat A' -algebra. \square

For the reduction to be applicable we will need the following Lemma. (It does not need any assumptions beyond those in the first paragraph of this article.)

Lemma 6

Given a diagram of Noetherian local rings

$$R_0'' \xleftarrow{g_0} R_0' \xrightarrow{f_0} R_0$$

such that f_0 is surjective and g_0 turns R_0'' into an unramified flat R_0' -algebra, for all $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ we have

$$e_{\mathfrak{q}_0}(H_{R_+}^i(M)_n) = e_{f_0^{-1}(\mathfrak{q}_0)R_0''}(H_{R_+ \otimes_{R_0'} R_0''}^i(M \otimes_{R_0'} R_0'')_n).$$

Proof. As R is Noetherian and homogeneous, there are finitely many elements $l_1, \dots, l_r \in R_1$ such that $R = R_0[l_1, \dots, l_r]$. Let $R' := R_0'[X_1, \dots, X_r]$ be the polynomial ring over R_0' furnished with the standard \mathbb{N}_0 -grading. Then, there is a surjective morphism of graded rings $f : R' \rightarrow R$ with component of degree 0 equal to f_0 .

Let $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$. As f is surjective, by the graded base ring independence property of local cohomology (cf. [B-Sh, Theorem 13.1.6]) there is an isomorphism of R_0' -modules $H_{R_+}^i(M)_n \upharpoonright_{R_0'} \cong H_{R_+}^i(M \upharpoonright_{R'})_n$ and $f_0^{-1}(\mathfrak{q}_0)$ is \mathfrak{m}_0 -primary. Now it is easy to see that

$$e_{\mathfrak{q}_0}(H_{R_+}^i(M)_n) = e_{f^{-1}(\mathfrak{q}_0)}(H_{R_+}^i(M \upharpoonright_{R'})_n).$$

Therefore, it suffices to prove the claim under the assumption that $R_0' = R_0$ and $f_0 = \text{Id}_{R_0'}$.

Let \mathfrak{m}_0'' denote the maximal ideal of R_0'' . Note that by unramifiedness $\mathfrak{q}_0 R_0''$ is \mathfrak{m}_0'' -primary. Looking at chains of submodules, by flatness and unramifiedness it is easy to see that $\text{length}_{R_0''}(T \otimes_{R_0} R_0'') = \text{length}_{R_0}(T)$ for an R_0 -module T . Writing $R_+'' := R_+ \otimes_{R_0} R_0''$, by the graded flat base change property of local cohomology (cf. [B-Sh, Theorem 13.1.8]) we have

$$H_{R_+}^i(M)_n \otimes_{R_0} R_0'' \cong H_{R_+''}^i(M \otimes_{R_0} R_0'')_n.$$

Hence, again by flatness, we get an isomorphism $(H_{R_+}^i(M)_n / \mathfrak{q}_0^{m+1} H_{R_+}^i(M)_n) \otimes_{R_0} R_0'' \cong H_{R_+''}^i(M \otimes_{R_0} R_0'')_n / (f_0^{-1}(\mathfrak{q}_0)^{m+1} R_0'') H_{R_+''}^i(M \otimes_{R_0} R_0'')_n$ for every $m \in \mathbb{N}_0$. The above remark about the lengths and the definition of the Hilbert-Samuel multiplicity now yields the claim. \square

Now, we can use Proposition 5 to perform a replacement argument which brings us in a situation in which we have an element x satisfying the conditions (1') to (5'), as the following argument shows.

We choose an uncountable extension field L of $K := R_0/\mathfrak{m}_0$, for example the field of fractions of a polynomial ring in uncountably many indeterminates over R_0/\mathfrak{m}_0 . By Proposition 5 there is a diagram of rings

$$R_0'' \xleftarrow{g_0} R_0' \xrightarrow{f_0} R_0$$

such that (R_0', \mathfrak{m}_0') and $(R_0'', \mathfrak{m}_0'')$ are complete regular local rings with residue fields denoted by K' and K'' respectively, f_0 is surjective, g_0 turns R_0'' into an unramified flat R_0' -algebra and $K'' \cong L$. By Lemma 6, it follows

$$e_{\mathfrak{q}_0}(H_{R_+}^i(M)_n) = e_{f_0^{-1}(\mathfrak{q}_0)R_0''}(H_{R_+ \otimes_{R_0'} R_0''}^i(M \otimes_{R_0'} R_0'')_n).$$

Therefore, we may replace R and M by $R \otimes_{R_0'} R_0''$ and $M \otimes_{R_0'} R_0''$ respectively and hence assume that the residue field K of the base ring R_0 is uncountable.

Thus, we can use Proposition 3 as follows. For $n \in \mathbb{Z}$ we define

$$\mathcal{A} := \text{Min}(R_0) \cup \bigcup_{n \in \mathbb{Z}} \text{Ass}_{R_0}(T_n) \cup \bigcup_{k \in \mathbb{Z}} \text{Ass}_{R_0}(M_k) \subseteq \text{Spec}(R_0)$$

and

$$\mathcal{B} := \bigcup_{n \in \mathbb{Z}} \text{Ass}_{\text{Gr}(\mathfrak{q}_0)}(\text{Gr}(\mathfrak{q}_0, T_n)) \setminus \text{Var}(\text{Gr}(\mathfrak{q}_0)_+) \subseteq \text{Spec}(\text{Gr}(\mathfrak{q}_0)).$$

As $\text{Gr}(\mathfrak{q}_0)$ is Noetherian and $\text{Gr}(\mathfrak{q}_0, T_n)$ is finitely generated for $n \in \mathbb{Z}$, both \mathcal{A} and \mathcal{B} are countable sets of ideals of R_0 and $\text{Gr}(\mathfrak{q}_0)$ respectively. Furthermore, as $\Gamma_{\mathfrak{m}_0 R}(M) = 0$, no ideal in \mathcal{A} contains \mathfrak{q}_0 and no ideal in \mathcal{B} contains $\text{Gr}(\mathfrak{q}_0)_+$. By Proposition 3 there is an element $x \in \mathfrak{q}_0 \setminus (\mathfrak{m}_0 \mathfrak{q}_0 \cup \bigcup \mathcal{A})$ such that $x + \mathfrak{q}_0^2 \in \text{Gr}(\mathfrak{q}_0)_1 \setminus \bigcup \mathcal{B}$. Thus x meets conditions (1') to (5') from above, and Lo!, the gap is filled.

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