UNIFIED SO(3) QUANTUM INVARIANTS FOR RATIONAL HOMOLOGY 3–SPHERES

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Abstract. Let M be a rational homology 3–sphere with $|H_1(M,\mathbb{Z})| = b$. For any odd divisor c of b, we construct a unified invariant $I_{M,c}$ lying in a cyclotomic completion of a certain polynomial ring, which dominates Witten–Reshetikhin–Turaev $SO(3)$ invariants of M at all roots of unity whose order r satisfies $(r,b) = c$. For $c = 1$, we recover the unified invariant constructed by Le and Beliakova–Le. If $b = 1$, our invariant coincides with Habiro's invariant of integral homology 3–spheres. New structural properties of the set of quantum invariants at roots of unity not coprime to the torsion are the main applications of our construction.

Introduction

The $SO(3)$ Witten–Reshetikhin–Turaev invariant $\tau_M(\xi) \in \mathbb{Q}(\xi)$ is defined for any closed oriented 3–manifold M and any root of unity ξ of odd order [16], [8]. If, in addition, the order of ξ is prime, then by the results of Murakami [15] and Masbaum–Roberts [13], $\tau_M(\xi)$ is an algebraic integer. This integrality result was the starting point for the construction of integral TQFTs, representations of the mapping class group over $\mathbb{Z}[\xi]$ [5], and categorification of quantum 3–manifold invariants [9]. The proofs in [15] and [13] depend heavily on the arithmetic of $\mathbb{Z}[\xi]$ for a prime root of unity ξ and do not extend to other roots of unity.

Recently, for any integral homology 3–sphere, Habiro [6] constructed a unified invariant whose evaluation at any root of unity coincides with the value of the Witten–Reshetikhin–Turaev invariant at that root. Habiro's unified invariant is an element of the following ring

$$\widehat{\mathbb{Z}[q]} := \lim_{\overleftarrow{n}} \frac{\mathbb{Z}[q]}{((1-q)(1-q^2)...(1-q^n))}\,.$$

Every element $f(q) \in \widehat{\mathbb{Z}[q]}$ can be written as an infinite sum

$$f(q) = \sum_{k\geq 0} f_k(q)\,(1-q)(1-q^2)...(1-q^k),$$

with $f_k(q) \in \mathbb{Z}[q]$. In particular, for a root of unity ξ, the evaluation $\mathrm{ev}_\xi(f(q)) \in \mathbb{Z}[\xi]$. Thus, for any integral homology sphere M, $\tau_M(\xi)$ is an algebraic integer at *any root of unity* ξ. The fact that the unified invariant belongs to $\widehat{\mathbb{Z}[q]}$ is stronger, than just integrality of $\tau_M(\xi)$. We will refer to it as "strong" integrality.

In [2] Laplace transform method for constructing unified invariants of rational homology 3–spheres was developed. An application of this method is the following result in [3].

Theorem (Beliakova–Le)**.** *For every closed 3–manifold M and any root of unity ξ of odd order, $\tau_M(\xi) \in \mathbb{Z}[\xi]$.*

Strong integrality of quantum invariants for rational homology 3–spheres was studied in [10] and [3]. In [10], for a rational homology 3–sphere M with $|H_1(M,\mathbb{Z})| = b$, Le constructed an invariant I_M which dominates $SO(3)$ quantum invariants of M at roots of unity of order coprime to b. Habiro's universal ring was modified by inverting b and cyclotomic polynomials of order not coprime to b. In [3], it was proved that I_M has even stronger integrality, i.e. it belongs to a smaller ring. Moreover, a rational surgery formula for this invariant was given.

In this paper, we extend the theory to the case, where the orders of the root of unity and of the torsion are not coprime. Our method uses Andrews's identities generalizing those of Rogers–Ramanujan.

Results. Let $\mathcal{M}_b$ be the set of rational homology 3–spheres with $|H_1(M,\mathbb{Z})| = b$. Let us fix an odd divisor c of b and let $b = \prod_{i=1}^n p_i^{k_i}$ be the prime decomposition of b, i.e. the p_i's are distinct. We have $H_1(M;\mathbb{Z}) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} \mathbb{Z}_{p_i^{k_{ij}}}$ with $\sum_{j=1}^{m_i} k_{ij} = k_i$.

We renormalize $\tau(M)$ as

$$\tau'_M(\xi) = \frac{\tau_M(\xi)}{\prod_{i=1}^{n}\prod_{j=1}^{m_i} \tau_{L(p_i^{k_{ij}},1)}(\xi)} .$$

We say that a closed 3–manifold is of diagonal type if it can be obtained by an integral surgery along an algebraically split link.

Lemma 1. *There are connected sums of lens spaces M_{odd} and M_{ev} with links inside, such that M_{odd} is uniquely determined by M and $M' := M\#M_{odd}\#M_{ev}$ is of diagonal type. Moreover, $\tau'_{M_{odd}}(\xi)$ is invertible in $\mathbb{Z}[\xi]$.*

Suppose $c = \prod_i p_i^{k_{i,c}}$ is the prime decomposition of c. Define $c' := \prod_i p_i^{k_{i,c}}$ and $b'' := \prod_i p_i^{k_i - k_{i,c}}$, where the products are taken over i with $2k_{i,c} < k_i$ only. Put $t := q^{\frac{c'}{b''}}$ and $b' = b/c$. We denote by $\Phi_i(x)$ the i–the cyclotomic polynomial in x and define a subring of $\mathbb{Q}(q^{1/b''})$ by

$$\mathcal{R}_{b,c} := \mathbb{Z}[q^{\pm 1}, t^{\pm 1}][\Phi_n^{-1}(t) \text{ if } (n,b') \neq 1,\ \Phi_j^{-1}(q) \text{ if } c \nmid j\ ]$$

and let

$$\widehat{\mathcal{R}}_{b,c} := \varprojlim_k \frac{\mathcal{R}_{b,c}}{((q;q)_k)}$$

be its cyclotomic completion, where $(a;b)_k = \prod_{i=0}^{k-1}(1 - ab^i)$.

Let $S = \{r \in \mathbb{N} \,|\, (r,b) = c\}$. For $f \in \widehat{\mathcal{R}}_{b,c}$ and a root of unity ξ with $\operatorname{ord}(\xi) \in S$, we define $\operatorname{ev}_\xi(f)$ by sending q to ξ and $t = q^{c'/b''}$ to $(\xi^{c'})^d$, where $b''d = 1$ modulo $\operatorname{ord}(\xi)/c'$. Note that $\operatorname{ev}_\xi(f) \in \mathbb{Z}[1/b][\xi]$ in general.

We single out a subring $\widehat{\Gamma}_{b,c}$ of $\widehat{\mathcal{R}}_{b,c}$, such that $\operatorname{ev}_\xi(\Gamma_{b,c}) = \mathbb{Z}[c^{-1}][\xi]$, if b is odd, and $\operatorname{ev}_\xi(\Gamma_{b,c}) = \mathbb{Z}[(2c)^{-1}][\xi]$ if b is even. We put $b_{ij} = p_i^{k_{ij}}$, $c_{ij} = p_i^{k_{ij,c}} = (c, b_{ij})$ and $t_{ij} = q^{p_i^{2k_{ij,c}-k_{ij}}}$. Notice, that t_{ij} are powers of t for all i and j.

Any $f \in \widehat{\Gamma}_{b,c}$ admits the following presentation

$$f = \sum_{k=0}^{\infty} f_k(t,q) x_k \quad \text{with} \quad f_k(t,q) \in \Gamma_{b,c,k},$$

where $x_k := (q^{k+1};q)_{k+1}/(1-q)$ and, for $k \in \mathbb{N}$, $\Gamma_{b,c,k}$ is the subring of $\mathcal{R}_{b,c}$ generated over $\mathbb{Z}[c^{-1}][t^{\pm 1}, q^{\pm 1}]$ if b is odd, and over $\mathbb{Z}[(2c)^{-1}][t^{\pm 1}, q^{\pm 1}]$, if b is even, by

$$\prod_{i,j \text{ with } 2k_{ij,c} < k_{ij}} \frac{(t_{ij};t_{ij})_{k+1+\lfloor \frac{k+1}{c_{ij}} \rfloor}}{(q^{c_{ij}};q^{c_{ij}})_{k+1+\lfloor \frac{k+1}{c_{ij}} \rfloor}}$$

Notice, that the ideals $((q;q)_k)$ and (x_k) are cofinal in $\mathcal{R}_{b,c}$.

The following theorem states the main result of this paper.

Theorem 2. *For $M \in \mathcal{M}_b$, there exists a unique unified invariant $I_{M,c}(t) \in \widehat{\Gamma}_{b,c}$, such that for any root of unity ξ with $\operatorname{ord}(\xi) \in S$,*

$$\tau'_M(\xi) = (\tau'_{M_{odd}}(\xi))^{-1}\ \operatorname{ev}_\xi\,(I_{M,c})$$

Let $Q = \{q_1, q_2, \ldots, q_l\}$ be the set of all prime numbers which divide c but not $\frac{b}{c}$. We deduce the following.

Corollary 3. *For $M \in \mathcal{M}_b$ and any prime p not dividing b, the set of quantum invariants $\{\tau_M(\xi) \mid \operatorname{ord}(\xi) \in S\}$ is as powerful as the set $\{\tau_M(\xi) \mid \operatorname{ord}(\xi) = c\prod_{i=1}^l q_i^{l_i} p^e,\ l_i, e \in \mathbb{N}\}$.*

An interesting open problem is to verify whether the Le–Murakami–Ohtsuki invariant [11] determines the $SO(3)$ quantum invariants at the chosen family of roots for $c > 1$. In Proposition 12 (Part d) we show that the subrings of $\widehat{\mathcal{R}}_{b,c}$ admit injective Taylor expansions.

Plan of the paper. In Section 1 we recall known results and definitions and compute quantum invariants of lens spaces. In Section 2 we state the results on cyclotomic completions of polynomial rings, needed in the proofs of Theorem 2 and Corollary 3. The rest of the paper is devoted to the proof of Theorem 2.

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1. Quantum invariants

Let us first fix the notation.

$$\{n\} = q^{n/2} - q^{-n/2}, \quad \{n\}! = \prod_{i=1}^{n} \{i\}, \quad [n] = \frac{\{n\}}{\{1\}}, \quad \left[\begin{array}{c} n \\ k \end{array} \right] = \frac{\{n\}!}{\{k\}!\{n-k\}!}.$$

We denote the set $\{1, 2, 3, \dots\}$ with $\mathbb{N}$ and the set $\{0, 1, 2, 3, \dots\}$ with $\mathbb{N}_0$.

1.1. **The colored Jones polynomial.** Suppose L is a framed, oriented link in S^3 with m ordered components. For every positive integer n there is a unique irreducible sl_2–module V_n of dimension n. For positive integers $n_1, \dots, n_m$ one can define the quantum invariant $J_L(n_1, \dots, n_m) := J_L(V_{n_1}, \dots, V_{n_m})$ known as the colored Jones polynomial of L (see e.g. [16]). Let us recall here a few well–known formulas. For the unknot U with 0 framing one has

$$J_U(n) = [n] = \{n\}/\{1\}. \tag{1}$$

If L' is obtained from L by increasing the framing of the i–th component by 1, then

$$J_{L'}(n_1, \dots, n_m) = q^{(n_i^2-1)/4} J_L(n_1, \dots, n_m). \tag{2}$$

In general, $J_L(n_1, \dots, n_m) \in \mathbb{Z}[q^{\pm 1/4}]$. However, there is a number $a \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ such that $J_L(n_1, \dots, n_m) \in q^a \mathbb{Z}[q^{\pm 1}]$.

1.2. **Evaluation map and Gauss sum.** Throughout this paper let ξ be a primitive root of unity of *odd* order r. We first define, for each ξ, the evaluation map ev_ξ, which replaces q by ξ. Suppose $f \in \mathbb{Q}[q^{\pm 1/d}]$, where d is coprime with r, the order of ξ. There exists an integer d_*, unique modulo r, such that $(\xi^{d_*})^d = \xi$. Then we define

$$\mathrm{ev}_\xi f := f|_{q^{1/d} = \xi^{d_*}}.$$

Suppose $f(q; n_1, \dots, n_m)$ is a function of variables q and integers $n_1, \dots, n_m$. Let

$$\sum_{n_i}^{\xi} f := \sum_{n_i} \mathrm{ev}_\xi(f),$$

where in the sum all the n_i run through the set of *odd* numbers between 0 and $2r$. A variation $\gamma_d(\xi)$ of the Gauss sum is defined by

$$\gamma_d(\xi) := \sum_{n}^{\xi} q^{d\frac{n^2-1}{4}}.$$

It is known that, for odd r, $|\gamma_d(\xi)| = \sqrt{r}$, and hence is never 0.

We define

$$F_L(\xi) := \sum_{n_i}^{\xi} J_L(n_1, \dots, n_m) \prod_{i=1}^{m} [n_i].$$

The following result is well–known (compare [10]).

Lemma 4. *For the unknot $U^{\pm}$ with framing ± 1, one has $F_{U^\pm}(\xi) \neq 0$. Moreover,*

$$F_{U^\pm}(\xi) = \mp 2 \gamma_{\pm 1}(\xi)\, \mathrm{ev}_\xi \left(\frac{q^{\mp 1/2}}{\{1\}} \right). \tag{3}$$

1.3. **Definition of the $SO(3)$ invariant of 3–manifolds.** All 3–manifolds in this paper are supposed to be closed and oriented. Every link in a 3–manifold is framed, oriented, and has components ordered.

Suppose M is an oriented 3–manifold obtained from S^3 by surgery along a framed, oriented link L. (Note that M does not depend on the orientation of L). Let σ_+ (respectively, σ_-) be the number of positive (resp. negative) eigenvalues of the linking matrix of L. Suppose ξ is a root of unity of odd order r. Then the quantum $SO(3)$ invariant is defined by

$$\tau_M(\xi) = \tau_M^{SO(3)}(\xi) := \frac{F_L(\xi)}{(F_{U^+}(\xi))^{\sigma_+}\,(F_{U^-}(\xi))^{\sigma_-}}.$$

For connected sum, one has $\tau_{M\#N}(\xi) = \tau_M(\xi)\tau_N(\xi)$.

1.4. **Laplace transform.** In [2], Laplace transform method for computing $\tau_M(\xi)$ was developed. In [3], it was generalize to the case where r is not coprime with the torsion. Let us recall these results.

Suppose r and b are positive integers, and r is odd. Let $c := (r, b)$ and $\mathcal{L}_{b;n} : \mathbb{Z}[q^{\pm n}, q^{\pm 1}] \to \mathbb{Z}[q^{\pm 1/b}]$ be the $\mathbb{Z}[q^{\pm 1}]$–linear operator, called Laplace transform, defined by

$$\mathcal{L}_{b;n}(q^{\pm na}) := \begin{cases} 0 & \text{if } c \nmid a; \\ q^{-a^2/b} & \text{if } a = ca', \end{cases} \tag{4}$$

The following lemma was proven in [3].

Lemma 5 (Beliakova–Le)**.** *Suppose $f \in \mathbb{Z}[q^{\pm n}, q^{\pm 1}]$. Then*

$$\sum_n^{\xi} q^{b\frac{n^2-1}{4}} f = \gamma_b(\xi)\,\mathrm{ev}_\xi(\mathcal{L}_{b;n}(f)).$$

1.5. **Lens spaces.** Applying Laplace transform method [2], it is easy to see that

$$\tau_{L_{b,1}}(\xi) = \frac{\gamma_b(\xi)}{\gamma_{\mathrm{sn}(b)}(\xi)}\,\mathrm{ev}_\xi\left(\frac{(1-q^{-\mathrm{sn}(b)/b})^{\chi(c)}}{1-q^{-\mathrm{sn}(b)}}\right), \tag{5}$$

where $c = (b, r)$, $\chi(c) = 0$ if $c > 1$ and $\chi(c) = 1$ if $c = 1$. Note that (compare [12])

$$\frac{\gamma_b(\xi)}{\gamma_{\mathrm{sn}(b)}(\xi)} = \sqrt{c}\left(\frac{\frac{b}{c}}{\frac{r}{c}}\right)\left(\frac{\mathrm{sn}(b)}{r}\right)\frac{\epsilon(r')}{\epsilon(r)}\,\mathrm{ev}_\xi\left(q^{\frac{\mathrm{sn}(b)-b}{4}}\right) \tag{6}$$

where $r' := \frac{r}{c}$ and $\left(\frac{x}{y}\right)$ denotes the Jacobi symbol. Further $\epsilon(x) = 1$ if $x \equiv 1 \pmod 4$ and $\epsilon(x) = I$ if $x \equiv 3 \pmod 4$.

Let us define $(L(p, s), K_d)$ as described in Figure 1.

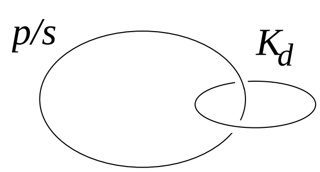

Figure 1. The lens space $(L(p, s), K_d)$ is obtained by p/s surgery on the first component of the Hopf link, the second component is the knot K colored by d.

Assume p/s is given by the continued fraction

$$\frac{p}{s} = m_n - \cfrac{1}{m_{n-1} - \cfrac{1}{m_{n-2} - \dots \cfrac{1}{m_2 - \cfrac{1}{m_1}}}}$$

with all $m_i \geq 2$. Choose s^* and p^*, such that $pp^* + ss^* = 1$ with $0 < s^* < p$.

Lemma 6. *For $c = (p, r)$ and d such that $c \mid (d - s^*)$, we have*

$$\frac{\tau_{(L(p,s),K_d)}(\xi)}{\tau_{L(p,1)}(\xi)} = \left(\frac{s}{c}\right) \mathrm{ev}_\xi \left(q^{\frac{3(n-1)-\sum_i m_i + p}{4} - \frac{s(d-s^*)^2}{4p} - \frac{p^*(s^*-2)}{4}} \right)$$

Proof. The formula follows from Lemmas 4.12 and 4.21 in [12][1] after replacing e_r by $\xi^{1/4}$. □

Proof of Lemma 1. The manifold M determines a linking pairing ϕ_M on $H_1(M) = \bigoplus_{i=1}^n \bigoplus_{j=1}^{m_i} \mathbb{Z}_{p_i^{k_{ij}}}$, which can be expressed as a direct sum of linking pairings $\phi(p_i^{k_{ij}}, s_{ij})$ on $\mathbb{Z}/p_i^{k_{ij}}$ and E_0^k, E_1^k on $\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k$ for $p_i = 2$. These linking pairings generate the abelian semigroup of linking pairings under block sum (see [14] and [17] for more details). Let us split $\phi_M = \phi_M^{\text{odd}} \oplus \phi_M^{\text{ev}}$, where $\phi_M^{\text{odd}} = \bigoplus_{i,j,p_i \neq 2} \phi(p_i^{k_{ij}}, s_{ij})$.

On ϕ_M^{odd}, there is only one relation between generators: $\phi(p^k, s) \oplus \phi(p^k, s) = \phi(p^k, 1) \oplus \phi(p^k, 1)$. We use this relation to minimize the number of $s_{ij} \neq 1$. The corresponding presentation of ϕ_M^{odd} as a direct sum of generators is called minimal.

We define M_{odd} as follows. Let $\phi_M^{\text{odd}} = \bigoplus_{i,j,p_i \neq 2} \phi(p_i^{k_{ij}}, s_{ij})$ be the minimal presentation. Then

$$M_{\text{odd}} = \#_{i,j,p_i \neq 2} \left(L(p_i^{k_{ij}}, s_{ij}), K_{d_{ij}}\right)$$

where $d_{ij} = 1$ if $s_{ij} = 1$ or $c_{ij} = 1$ and $d_{ij} = s_{ij}^* + c_{ij}$ otherwise. In this form, M_{odd} is uniquely determined by M.

Similarly, M_{ev} is defined by taking $L(2^k, s)$ for any generator $\phi_{2^k,s}$ of ϕ_M^{ev} and $L(2^k, -1)$ and $L(2^k, 3)$ for E_0^k and E_1^k, respectively. By Proposition 3.5 of [10], $M' = M \# M_{\text{ev}} \# M_{\text{odd}}$ is of diagonal type.

It remains to show that $\tau'_{M_{\text{odd}}}$ is invertible in $\mathbb{Z}[\xi]$. According to the definition, $\tau'_{M_{\text{odd}}}$ is a product of terms computed by Lemma 6. It is not difficult to see that each such term is a power of ξ. Indeed, 4 is invertible modulo r (since r is odd), and $p_i^{k_{ij}}/c_{ij}$ is invertible modulo r/c_{ij}. Observe that $c_{ij}^2 \mid (d_{ij} - s_{ij}^*)^2$. □

1.6. **Habiro's cyclotomic expansion of the colored Jones polynomial.** In [6], Habiro defined new bases P'_k and $\tilde{P}'_k$, $k = 0, 1, 2, \ldots$, for the Grothendieck ring of finite–dimensional sl_2–modules, where

$$P'_k := \frac{1}{\{k\}!} \prod_{i=1}^{k} (V_2 - q^{(2i-1)/2} - q^{-(2i-1)/2}) \qquad \text{and} \qquad \tilde{P}'_k = q^{\frac{1}{4}k(k-1)} P'_k.$$

Further, he defined for $k \geq 0$

$$\mathcal{P}_k = \mathrm{Span}_{\mathbb{Z}[q^{\pm 1}]} \{\tilde{P}'_n \mid n \geq k\}.$$

For any link L, using the linearity of J_L, one has

$$J_L(n_1, \ldots, n_m) = \sum_{0 \leq k_i \leq n_i - 1} J_L(P'_{k_1}, \ldots, P'_{k_m}) \prod_{i=1}^{m} \begin{bmatrix} n_i + k_i \\ 2k_i + 1 \end{bmatrix} \{k_i\}! \tag{7}$$

Since there is a denominator in the definition of P'_k, one might expect that $J_L(P'_{k_1}, \ldots, P'_{k_m})$ also has non–trivial denominator. A difficult and important integrality result of Habiro is Theorem 8.2 in [6]. This result can be generalized to

Theorem 7. *Let L be an m–component algebraically split 0–framed link in S^3 and L' be an l–component 0–framed one. Assume that the ith component L'_i of L' is colored by V_{j_i} and j_i is either odd or $lk(L_j, L'_i) = 0 \pmod 2$ for all j. Then, for $x_i \in \mathcal{P}_{k_i}$, $k_i \geq 0$, we have*

$$J_{L \cup L'}(x_1, \ldots, x_m, V_{j_1}, \ldots, V_{j_l}) \in \frac{(q^{k+1}; q)_{k+1}}{1 - q} \, \mathbb{Z}[q^{\pm 1}],$$

where $k = \max\{k_1, \ldots, k_m\}$.

[1]There are misprints in Lemma 4.21: $q^* \pm n$ should be replace by $q^* \mp n$.

The proof is given in Appendix 2. Thus, $J_{L\cup L'}(\tilde{P}'_{k_1},\dots,\tilde{P}'_{k_m},V_{j_1},\dots,V_{j_l})$ is not only integral, but also divisible by $(q;q)_k$.

Suppose L is an algebraically split link with 0–framing on each component. Then we have

(8)
$$\mathrm{ev}_\xi(J_{L\cup L'}(n_1,\dots,n_m,j_1,\dots,j_l)) = \mathrm{ev}_\xi\left(\sum_{k_1,\dots,k_m=0}^{(r-3)/2} J_L(P'_{k_1},\dots,P'_{k_m},V_{j_1},\dots,V_{j_l})\prod_{i=1}^m \left[\begin{array}{c} n_i+k_i \\ 2k_i+1\end{array}\right]\{k_i\}!\right).$$

2. Cyclotomic completions of polynomial rings

Let R be a commutative ring with unit. We assume that R is an integral domain of characteristic zero. Let $R[q]$ be the polynomial ring over R. For each $n\in\mathbb{N}$, let

$$\Phi_n(q) := \prod_{(i,n)=1}(q-e_n^i)$$

denotes the n–th cyclotomic polynomial, where e_n is a primitive n–th root of unity. If $S\subset\mathbb{N}$, we set $\Phi_S=\{\Phi_n(q)\mid n\in S\}$. Let Φ^*_S denote the multiplicative set in $\mathbb{Z}[q]$ generated by Φ_S and directed with respect to the divisibility relation. The principal ideals $(f(q))\subset R[q]$ for $f(q)\in\Phi^*_S$ define a linear topology of the ring $R[q]$. In [7], Habiro defined the (S–)cyclotomic completion ring $R[q]^S$ as follows:

$$R[q]^S := \varprojlim_{f(q)\in\Phi^*_S} \frac{R[q]}{(f(q))}. \tag{9}$$

For example, since the sequence $(q)_n$, $n\in\mathbb{N}$, is cofinal to $\Phi^*_{\mathbb{N}}$, we have

$$\widehat{\mathbb{Z}[q]}\simeq\mathbb{Z}[q]^{\mathbb{N}}.$$

Note that if S is finite, then $R[q]^S$ is identified with the ($\prod\Phi_S$–)adic completion of $R[q]$. In particular,

$$R[q]^{\{1\}}\simeq R[[q-1]],\quad R[q]^{\{2\}}\simeq R[[q+1]].$$

Suppose $S'\subset S$, then $\Phi^*_{S'}\subset\Phi^*_S$, hence there is a natural map

$$\rho^R_{S,S'}: R[q]^S\to R[q]^{S'}.$$

Recall important results concerning $R[q]^S$ from [7]. Two positive integers n,n' are called *adjacent* if and only if $n'/n=p^e$ with $e\in\mathbb{Z}$, for a prime p, such that the ring R is p–adically separated. A set of positive integers is *connected* if for any two distinct elements n,n' there is a sequence $n=n_1,n_2\dots,n_{k-1},n_k=n'$ in the set, such that any two consecutive numbers of this sequence are adjacent. Theorem 4.2 of [7] says that if S is connected, then for any subset $S'\subset S$, the natural map $\rho^R_{S,S'}: R[q]^S\hookrightarrow R[q]^{S'}$ is an embedding.

If ζ is a root of unity of order in S, then for every $f(q)\in R[q]^S$ the evaluation $\mathrm{ev}_\zeta(f(q))\in R[\xi]$ can be defined by sending $q\to\zeta$. For a set Ξ of roots of unity whose orders form a subset $\mathcal{T}\subset S$, one defines the evaluation

$$\mathrm{ev}_\Xi: R[q]^S\to\prod_{\zeta\in\Xi}R[\zeta].$$

Theorem 6.1 of [7] shows that if $R\subset\mathbb{Q}$, S is connected, and there exists $n\in S$ that is adjacent to infinitely many elements in $\mathcal{T}$, then ev_Ξ is injective.

In this section we apply Habiro's methods to study the ring $\mathcal{R}_{b,c}$ defined in the introduction. Recall that for any positive integer b, we fix an odd divisor c of b. We put $b':=b/c$. Let $b=\prod_i p_i^{k_i}$ and $c=\prod_i p_i^{k_{i,c}}$ be the prime decompositions of b and c. Then we define $c':=\prod_i p_i^{k_{i,c}}$ and $b'':=\prod_i p_i^{k_i-k_{i,c}}$, where the products are taken over i with $2k_{i,c}<k_i$ only. Let $t=q^{\frac{c'}{b''}}$. We fix $S=\{cl\mid l\in\mathbb{N},\,(l,b')=1\}$ for the rest of the paper.

Then

$$\mathcal{R}_{b,c}:=\mathbb{Z}[q^{\pm1},t^{\pm1}][\Phi_n^{-1}(t)\text{ if }(n,b')\neq1,\ \Phi_j^{-1}(q)\text{ if }c\nmid j\,]$$

and its cyclotomic completion

$$\widehat{\mathcal{R}}_{b,c}:=\varprojlim_n \frac{\mathcal{R}_{b,c}}{((q;q)_n)}.$$

Notice, that $\widehat{\mathcal{R}}_{b,c}$ has a well–defined evaluation at roots of unity of order in $S = \{cl \mid l \in \mathbb{N},\, (l, b') = 1\}$, which sends $q \mapsto \xi$ where $\mathrm{ord}(\xi) = cl \in S$, and $t \mapsto (\xi^{c'})^d$, where $b''d = 1 \pmod{cl/c'}$.

Lemma 8.

$$\widehat{\mathcal{R}}_{b,c} = \lim_{\overleftarrow{f(q)\in\Phi^*_S(q)}} \frac{\mathcal{R}_{b,c}}{(f(q))}$$

Proof. We have to show that the systems of ideals $(f(q)) \in \Phi^*_{\mathbb{N}}$ and $(f(q)) \in \Phi^*_S$ in $\mathcal{R}_{b,c}$ are cofinal. We refer to them as systems of ideals with respect to $\mathbb{N}$ and S, respectively.

Since, all $\Phi_j(q)$, $j \in \mathbb{N}$, $c \nmid j$ are invertible in $\mathcal{R}_{b,c}$, the completion with respect to $\mathbb{N}$ coincides with the completion w.r.t. to $\{cl \mid l \in \mathbb{N}\}$ or with the completion w.r.t. ideals generated by $(q^c; q^c)_i$ for $i \in \mathbb{N}$. The last system of ideals is cofinal to the one with respect to S, since up to units, for $c'' = c/c'$, we have $\prod_{j|l} \Phi_{cj}(q) = (1 - q^{cl}) = \prod_{i|c''l,(i,b')=1} \Phi_i(t)$ and hence, for any $k \mid b'$, the ideals generated by $(1 - q^{clk})$ and $(1 - q^{cl})$ coincide in $\mathcal{R}_{b,c}$. □

Denote $\Lambda_b := \mathbb{Z}[1/b][t^{\pm 1}, q^{\pm 1}]$ and put

$$\Lambda_b^S := \lim_{\overleftarrow{f(q)\in\Phi^*_S(q)}} \frac{\Lambda_b}{(f(q))}\,.$$

Our main theorem is

Theorem 9. *The ring $\widehat{\mathcal{R}}_{b,c}$ is isomorphic to Λ_b^S.*

We split the proof of this theorem into lemmas. Let I_{cl} be the ideal of $\mathcal{R}_{b,c}$ generated by $(1-q^{cl})$.

Lemma 10. *For any l coprime to b', the inclusion $\mathcal{R}_{b,c} \hookrightarrow \mathcal{R}_{b,c}[1/b]$ induces an isomorphism*

$$\frac{\mathcal{R}_{b,c}}{I_{cl}} \hookrightarrow \frac{\mathcal{R}_{b,c}[1/b]}{I_{cl}}\,.$$

Proof. Injectivity is obvious. For surjectivity we have to show that any prime factor p of b is invertible modulo I_{cl} in $\mathcal{R}_{b,c}$. Recall that $(\Phi_m(q)) + (\Phi_n(q)) = (1)$ in $\mathbb{Z}[q]$ if and only if m/n is not a power of a prime. If $m/n = p^e$ with prime p, then $(p) \in (\Phi_n(q), \Phi_m(q))$ and there is a j such that $\Phi_n^j(q) \in (p, \Phi_m(q))$ by Lemma 4.1 in [7]. Moreover, $j = 1$ if $e < 0$.

Assume $p \mid c$, then $c = p\tilde{c}$ and there exists j, such that $\Phi^j_{\tilde{c}i}(q) \in (p, \Phi_{ci}(q))$. It follows that p is invertible modulo $\Phi_{ci}(q)$ in $\mathcal{R}_{b,c}$ for any i, i.e. modulo I_{cl}.

If $p \mid b'$, we have $\Phi_{pi}(t) \in (p, \Phi_i(t))$. Since $\Phi_{pi}(t)$ is invertible in $\mathcal{R}_{b,c}$, we have that p is invertible modulo $\Phi_i(t)$ in $\mathcal{R}_{b,c}$ for any i with $(i, b') = 1$. □

Lemma 11. *The localization*

$$\frac{\Lambda_b}{I_{cl}} \to \frac{\mathcal{R}_{b,c}[1/b]}{I_{cl}}$$

is an isomorphism.

Proof. By Proposition 2.1 in [4], any localization at a non–zero divisor is injective. Since the cyclotomic polynomials we invert in $\mathcal{R}_{b,c}$ are coprime to I_{cl}, we do not have zero divisors.

For surjectivity we have to show that $\Phi_j(q)$ with $j \in \mathbb{N}$, $c \nmid j$, and $\Phi_k(t)$, for $k \in \mathbb{N}$, $(k, b') \neq 1$, are invertible in $\mathbb{Z}[1/b][t^{\pm 1}, q^{\pm 1}]$ modulo I_{cl}. Since, for j with $c \nmid j$, $\frac{j}{cl}$ is either not a power of a prime, or a power of a divisor of c, which is invertible in $\mathbb{Z}[1/b]$, we deduce that $\Phi_j(q)$ is invertible modulo I_{cl} in $\mathbb{Z}[1/b][t^{\pm 1}, q^{\pm 1}]$.

Similarly, for k with $(k, b') \neq 1$, $\frac{k}{i}$ with $(i, b') = 1$ is either not a power of a prime, or a power of a divisor of b', which is invertible in $\mathbb{Z}[1/b]$. □

The main theorem follows.

Let $Q = \{q_1, q_2, \dots, q_l\}$ be the set of all prime numbers which divide c but not $\frac{b}{c}$. Assume p is a prime not dividing b, let $T_{\mathbf{k}}(p) = \{c \prod_i q_i^{k_i} p^e \mid k_i, e \in \mathbb{N}, q_i \in Q\}$ and $S_{\mathbf{k}} = \{c \prod_i q_i^{k_i} l \mid k_i, l \in \mathbb{N}, (l, b) = 1), q_i \in Q\}$. We have $S = \cup_{\mathbf{k}} S_{\mathbf{k}}$.

Proposition 12. *(a) For $S_{\mathbf{k}} \neq S_{\mathbf{k}'}$ and $a \in S_{\mathbf{k}}$, $a' \in S_{\mathbf{k}'}$, we have $(\Phi_a(q), \Phi_{a'}(q)) = (1)$ in Λ_b.*

(b) We have

$$\Lambda_b^S = \prod_{\mathbf{k}} \Lambda_b^{S_{\mathbf{k}}}.$$

c) Suppose $f, g \in \Lambda_b^{S_{\mathbf{k}}}$ such that $\mathrm{ev}_\xi(f) = \mathrm{ev}_\xi(g)$ for any root of unity ξ with $ord(\xi) \in T_{\mathbf{k}}(p)$, then $f = g$.

d) For $\lambda_{\mathbf{k}} = c \prod_i q_i^{k_i}$, the map

$$\Lambda_b^{S_{\mathbf{k}}} \to \Lambda_b[[1 - q^{\lambda_{\mathbf{k}}}]]$$

is injective.

Note that Corollary 3 follows from Proposition 12.

Proof. (a) We put $a = c \prod_i q_i^{k_i} l \in S_{\mathbf{k}}$ and $a' = c \prod_i q_i^{k'_i} l' \in S_{\mathbf{k}'}$. There exists an i, such that $k_i \neq k'_i$. Without lost of generality, we can assume $i = 1$. If $l \neq l'$, then $\frac{a}{a'}$ is never a prime power. In the case of $l = l'$, we have $\frac{a}{a'} = q_1^{k_1 - k'_1} \prod_{i>1} q_i^{k_i - k'_i}$. This can only be a prime power if $k_i = k'_i$ for $i > 1$. But q_1 is invertible in $\mathbb{Z}[\frac{1}{b}]$ and therefore the claim holds.

(b) Since $S = \cup_{\mathbf{k}} S_{\mathbf{k}}$, we can write $f \in \Phi_S^*$ as $f = \prod f_{\mathbf{k}}$ with $f_{\mathbf{k}} \in \Phi_{S_{\mathbf{k}}}^*$. From (a) we know, that the $f_{\mathbf{k}}$'s are pairwise coprime. Applying the Chinese remainder theorem we get

$$\Lambda_b/(f) = \prod_{\mathbf{k}} \Lambda_b/(f_{\mathbf{k}}).$$

Taking the inverse limit we get the claim.

(c) This is an adaptation of the proof of Theorem 6.1 in [7] to our case.

Let $c \prod_i q_i^{k_i} = \lambda_{\mathbf{k}}$, then $S_{\mathbf{k}} = \{\lambda_{\mathbf{k}}\, l \mid (l, b) = 1, l \in \mathbb{N}\}$. Suppose for contradiction that there is a nonzero element $a \in \Lambda_b^{S_{\mathbf{k}}}$ with $\mathrm{ev}_\Xi(a) = 0$, where Ξ is a set of roots of unity with orders in $T_{\mathbf{k}}(p)$. Since $S_{\mathbf{k}}$ is connected, the map $\rho : \Lambda_b^{S_{\mathbf{k}}} \to \Lambda_b^{\{\lambda_{\mathbf{k}}\}}$ is injective ([7, Proposition 4.2]). Therefore, $\rho(a) = \sum_{j=l}^\infty a_j \Phi_{\lambda_{\mathbf{k}}}^j(q)$, with $a_j \in \Lambda_b$ and for some l, $0 \neq a_l \notin (\Phi_{\lambda_{\mathbf{k}}}(q))$. Since the evaluation of a at any root of unity of order in $T_{\mathbf{k}}(p)$ vanish, we have that $\Phi_{\lambda_{\mathbf{k}} p^e}(q) | a$ for infinitely many e. Moreover, $\Phi_{\lambda_{\mathbf{k}} p^e}(q) \mid \rho(a)$ for any e. Note that $\Phi_{\lambda_{\mathbf{k}} p^e} \in (p, \Phi_{\lambda_{\mathbf{k}}})$. Therefore, $\bar{a}_l = a_l \pmod{\Phi_{\lambda_{\mathbf{k}}}(q)}$ is divisible infinitely many times by p in $\Lambda_b/(\Phi_{\lambda_{\mathbf{k}}}(q))$. Any element of the last ring can be written as $\sum_{i=0}^{d-1} \sum_{j=0}^{g-1} f_{i,j} q^i t^j$ where $f_{i,j} \in \mathbb{Z}[1/b]$, $d = \deg \Phi_{\lambda_{\mathbf{k}}}(q)$ and $g = \deg \Phi_{\lambda_{\mathbf{k}}/c'}(t)$.

From the divisibility of $\bar{a}_l$ infinitely many times by p, we deduce that $\bar{a}_l = 0$ and $a_l \in \Phi_{\lambda_{\mathbf{k}}}(q)$ which is a contradiction.

$d)$ By [7, Proposition 4.2], the map $\rho : \Lambda_b^{S_{\mathbf{k}}} \to \Lambda_b^{\{\lambda_{\mathbf{k}}\}}$ is injective. Any element f of the last ring can be written in the form $f = \sum_j a_j \Phi_{\lambda_{\mathbf{k}}}^j(q)$ with $a_j \in \Lambda_b/(\Phi_{\lambda_{\mathbf{k}}}(q))$. Since for any divisor i of $\lambda_{\mathbf{k}}$, $\Phi_i(q)$ is invertible in $\Lambda_b/(\Phi_{\lambda_{\mathbf{k}}}(q))$, we can rewrite $f = \sum_j \tilde{a}_j (1 - q^{\lambda_{\mathbf{k}}})^j \in \Lambda_b[[1 - q^{\lambda_{\mathbf{k}}}]]$ by using $q^n - 1 = \prod_{d|n} \Phi_{n/d}(q)$.

□

3. Unified invariant

3.1. **Technical results.** As before let $b' = b/c$. We choose t such that $t^{b'} = q^c$ and define the subring $\tilde{\Gamma}_{b,c,k}$ of $\mathbb{Q}(q^{\frac{1}{b'}})$ as generated over $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ by $A_k(t,q)$, if b is odd, and by $\left((-t;t)_{k+1}\right)^{-1} A_k(t,q)$, if b is even, where

$$A_k(t,q) \;=\; \frac{(t;t)_{k+1+\lfloor \frac{k+1}{c} \rfloor}}{(q^c;q^c)_{k+1+\lfloor \frac{k+1}{c} \rfloor}} \;\; \frac{1}{\widetilde{(q;q)}_{c(k+1)}}\,.$$

Here we use the notation

$$\widetilde{(q^a;q^b)}_k = \prod_{i=0, c\nmid a+bi}^{k-1} \left(1 - q^{a+bi}\right).$$

The following proposition is the main technical result of the paper.

Proposition 13. *For $b \in \mathbb{Z}_{\neq 0}$, $c \geq 1$, ξ a r–th root of unity and $k \leq (r-3)/2$, there are $F_k(q,t,b,c) \in q^{\frac{k^2+3k+2}{4}} \tilde{\Gamma}_{b,c,k}$, such that*

$$\sum_n^{\xi} q^{-b\left(\frac{n^2-1}{4}\right)} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}!\{n\} = 2\gamma_{-b}(\xi) \mathrm{ev}_\xi \left((-1)^{(k+1)(1-\mathrm{sn}(b))/2} F_k(q^{\mathrm{sn}(b)}, t^{\mathrm{sn}(b)}, |b|, c) \right).$$

We prove Proposition 13 in Appendix 1 and define now $F_k(q,t,b,c)$ for odd $b>0$. For the even case we refer to Appendix 1. Put $a := \frac{c+1}{2}$, $d := \lfloor \frac{b'-1}{2} \rfloor$, $N := k+1$ and $m := \lfloor \frac{N}{c} \rfloor$. Let ω be a primitive b'–th root of unity and $p := (c+1)\frac{b'}{2} + 1$. We define

$$\begin{aligned}
F_k(q,t,b,c) \quad := \quad & q^{\frac{1}{4}(k^2+3k+2)} \frac{(t;t)_{N+m}}{(q^c;q^c)_{N+m}} \frac{1}{\widetilde{(q;q)}_{cN}} \sum_{n_p \geq n_{p-1} \geq \cdots \geq n_1 = 0} \frac{(t^{-N})_{n_p} (t^{m+1})_{N-m+n_a}}{(t^{-N})_{n_{a+1}} (t^{m+1})_{n_{a+1}}} \\
& \cdot (-1)^{n_p} t^{x'} \cdot \frac{\prod_{j=1}^{a-1} (t^{2m})_{\nu_j} \cdot (t^{-N+m})_{\nu_a} \cdot \prod_{i=1}^{d} (t^{2m+1})_{\nu_{a+i}} (t^{-1})_{\nu_{\gamma_1+i}} \prod_{j=1}^{c-1} (t^{2m})_{\nu_{a+jd+i}}}{\prod_{i=1}^{p-1} (t)_{\nu_i}} \\
& \cdot (t^{-m})_{n_a} \cdot \prod_{j=1}^{a-1} (t^{\frac{-N+u_j}{c}})_{n_j} (t^{\frac{-N+c-u_j}{c}})_{n_j} (t^{\frac{N+c-u_j}{c} + n_{j+1}})_{N-n_{j+1}} (t^{\frac{N+u_j}{c} + n_{j+1}})_{N-n_{j+1}} \\
& \cdot \prod_{i=1}^{d} (\omega^i t^{-m})_{n_{a+i}} (\omega^{-i} t^{-m})_{n_{a+i}} (1 + \omega^{\pm i} t^{n_{\gamma_1+i}})^{(-\nu_{\gamma_1+i}+1) \left\lceil \frac{n_{\gamma_1+i}}{1+n_{\gamma_1+i}} \right\rceil} (1+\omega^{\pm i})^{\left\lfloor \frac{1}{1+n_{\gamma_1+i+1}} \right\rfloor} \\
& \cdot \prod_{i=1}^{d} (\omega^{\pm i} t^{m+1+n_{a+i+1}})_{N-n_{a+i+1}} \cdot \prod_{i=1}^{d} \prod_{j=1}^{c-1} (\omega^i t^{\frac{N+c-u_j}{c} + n_{a+jd+i+1}})_{N-n_{a+jd+i+1}} \\
& \cdot \prod_{i=1}^{d} \prod_{j=1}^{c-1} (\omega^{-i} t^{\frac{N+u_j}{c} + n_{a+jd+i+1}})_{N-n_{a+jd+i+1}} (\omega^i t^{\frac{-N+u_j}{c}})_{n_{a+jd+i}} (\omega^{-i} t^{\frac{-N+c-u_j}{c}})_{n_{a+jd+i}}
\end{aligned}$$

The notation is explained in Appendix 1.

3.2. **Definition of the unified invariant.** Let $M \in \mathcal{M}_b$. In this subsection we define $I_{M,c} \in \hat{\Gamma}_{b,c}$, such that

$$\tau'_M(\xi) \;=\; (\tau'_{M_{\mathrm{odd}}}(\xi))^{-1} \, \mathrm{ev}_\xi \, (I_{M,c})$$

where M_{odd} is a connected sum of lens spaces with links inside defined in Proof of Lemma 1, $\tau'_{M_{\mathrm{odd}}}(\xi)$ is a power of ξ. The invariant $I_{M,c}$ is multiplicative with respect to the connected sum of spaces. Let us consider 3 cases.

Case 1: $M = L(2^k, a)$. Then we define

$$I_{M,1} := q^{3s(1,2^k) - 3s(a,2^k)} ,$$

where $s(b,a)$ is the Dedekind sum. By Lemma 6 or equation (3) in [10], $\mathrm{ev}_\xi(I_{M,1}) = \tau'_M$. Note that $3s(1,2^k) - 3s(a,2^k) \in \mathbb{Z}$ and I_M is invertible in $\widehat{\Gamma}_{b,c}$.

Case 2: $M \in \mathcal{M}_b$ is obtained by surgery along an algebraically split link L with m–components and an l–component link L' colored by $V_{j_1}, \cdots, V_{j_l}$ sits inside M. Assume that the linking pairing on M decomposes into a direct sum of generators as follows:

$$\phi_M = \oplus_{i=1}^m \; \phi(|b_i|, s_i)$$

where b_i is the framing on L_i. Let us denote the link L with all framings switched to zero as L_0. Further suppose that c is a fixed odd divisor of $b = \prod_i b_i$, $c_i = (c, b_i)$ and $t_i = q^{c_i^2/b_i}$.

We can now give a closed formula for the $I_{(M,L'),c}$.

$$\begin{aligned}
(10) \qquad I_{(M,L'),c} \quad = \quad & \sum_{k_i=0}^{\infty} J_{L_0 \cup L'}(P'_{k_1}, \ldots, P'_{k_m}, V_{j_1}, \ldots, V_{j_l}) (-1)^{\sum_{i=1}^{m} \frac{k_i(1+\mathrm{sn}(b_i))}{2}} \\
& \cdot \prod_{i=1}^{m} q^{\mathrm{sn}(b_i)/2} \frac{(1 - q^{-\mathrm{sn}(b_i)})}{(1 - t_i^{-\mathrm{sn}(b_i)})^{\chi(c_i)}} F_{k_i}(q^{-\mathrm{sn}(b_i)}, t_i^{-\mathrm{sn}(b_i)}, |b_i|, c_i)
\end{aligned}$$

where $F_{k_i}(q,t_i,b_i,c_i)$ is defined as above and $\chi(c)=1$ if $c=1$ and zero otherwise.

Theorem 14. *For (M,L') as above, the unified invariant $I_{(M,L'),c}$ given by (10) satisfies*

$$\tau'_{(M,L')}(\xi) = \mathrm{ev}_\xi\left(I_{(M,L'),c}\right)$$

Proof. Applying (2) and (8) to the definition of $F_L(\xi)$ we conclude

$$F_{L_0\cup L'}(\xi) = \sum_{k_1,\ldots,k_m=0}^{(r-3)/2} \mathrm{ev}_\xi(J_{L_0\cup L'}(P'_{k_1},\ldots,P'_{k_m},V_{j_1},\ldots,V_{j_l}))\sum_{n_i}^{\xi}\left(\prod_{i=1}^m q^{-b_i\left(\frac{n_i^2-1}{4}\right)}\left[\begin{array}{c} n_i+k_i\\ 2k_i+1\end{array}\right]\{k_i\}![n_i]\right).$$

Further we use Proposition 13 to obtain

$$F_{L_0\cup L'}(\xi) = (-2)^m\prod_{i=1}^m \gamma_{b_i}(\xi)\,\mathrm{ev}_\xi\left(\sum_{k_i}^{\infty} J_{L_0\cup L'}(P'_{k_1},\ldots,P'_{k_m},V_{j_1},\ldots,V_{j_l})\prod_{i=1}^m \frac{F_{k_i}(q^{-\operatorname{sn}(b_i)},t_i^{-\operatorname{sn}(b_i)},|b_i|,c_i)}{(-1)^{\frac{(k_i+1)(1+\operatorname{sn}(b_i))}{2}}\{1\}}\right).$$

Therefore we get for the WRT invariant the formula

$$\begin{aligned}\tau_{(M,L')}(\xi) \quad &= \quad \prod_{i=1}^m \frac{\gamma_{b_i}(\xi)}{\gamma_{\operatorname{sn}(b_i)}(\xi)}\cdot\\ &\mathrm{ev}_\xi\left(\sum_{k_i}^{\infty} J_{L_0\cup L'}(P'_{k_1},\ldots,V_{j_l})\prod_{i=1}^m q^{\operatorname{sn}(b_i)/2}(-1)^{k_i(1+\operatorname{sn}(b_i))/2}F_{k_i}(q^{-\operatorname{sn}(b_i)},t_i^{-\operatorname{sn}(b_i)},|b_i|,c_i)\right)\end{aligned}$$

Using the definition of τ', (6) and (5), we get the claim. □

Case 3: For any $M\in\mathcal{M}_d$, we define

$$I_{M,c} = I_{(M',L'),c}(I_{M_{\mathrm{ev}},1})^{-1}$$

where $(M',L') = M\#M_{\mathrm{odd}}\#M_{\mathrm{ev}}$ is of diagonal type and M_{ev} is a connected sum of lens spaces considered in Case 1.

3.3. **Proof of Theorem 2.** It remains to show that $I_{M,c}\in\widehat{\Gamma}_{b,c}$ or $I_{(M',L'),c}\in\widehat{\Gamma}_{b,c}$. By Theorem 7, we have $J_{L_0\cup L}(P'_{k_1},\ldots,P'_{k_m},V_{j_1},\ldots,V_{j_l})\in\frac{(q^{k+1};q)_{k+1}}{1-q}\prod_{i=1}^m q^{\frac14 k_i(k_i-1)}\,\mathbb{Z}[q^{\pm1}]$ for $k=\max_i\{k_i\}$. Moreover, $k_i(k_i-1)\equiv k_i^2+3k_i \pmod 4$. From Proposition 13, we see that $I_{(M',L'),c}$ can be written in the form

$$\sum_k f_k(t,q)x_k \quad\text{with}\quad f_k(t,q)\in\prod_i\tilde{\Gamma}_{k,b_i,c_i}\,.$$

Since the coefficient $f_k(t,q)$ can be reduced modulo (x_k), we have $I_{(M',L'),c}\in\widehat{\Gamma}_{b,c}$. □

Appendix 1

Proof of Proposition 13. We have to calculate the Laplace transform

$$\mathcal{L}_{-b;n}\left(\left[\begin{array}{c} n+k\\ 2k+1\end{array}\right]\{k\}!\{n\}\right),$$

which is equal to $2\frac{(-1)^{k+1}}{\{k+1\}!}S_k(q,b)$, where

$$S_k(q,b) := 1+\sum_{n=1}^{\infty}\frac{q^{(k+1)cn}(q^{-k-1};q)_{cn}}{(q^{k+2};q)_{cn}}(1+q^{cn})q^{\frac{c^2n^2}{b}}. \tag{11}$$

See [3], Lemma 2.4 for details. Because of the term $(q^{-k-1})_{cn}$, we can assume $cn\le k+1$ (the summands are otherwise zero) and the sum is therefore finite.

First notice that we can restrict our calculations to the case when $b>0$, since

$$S_k(q,b) = S_k(q^{-1},-b).$$

Therefore, let us assume b to be positive for the rest of the proof.

Let us choose t such that $t^{b'} = q^c$ and a b'–th primitive root of unity ω where $b' := \frac{b}{c}$ and put $N = k+1$. Using the equalities

$$\begin{aligned}(q^x;q)_{cl} &= \prod_{j=0}^{c-1}(q^{x+j};q^c)_l \\ (q^{xc};q^c)_l &= \prod_{i=0}^{b'-1}(\omega^i t^x;t)_l\end{aligned}$$

we can see that

$$S_k(q,b) = 1 + \sum_{n=1}^{\infty}\prod_{i=0}^{b'-1}\prod_{j=0}^{c-1}\frac{(\omega^i t^{\frac{-N+j}{c}})_n}{(\omega^i t^{\frac{N+1+j}{c}})_n}(1+t^{b'n})t^{n^2+b'Nn}$$

Here we use the notation $(t)_l := (t;t)_l$.

The sum $S_k(q,b)$ can be identified with the LHS of the Andrew's identity (3.43) of [1]

$$\sum_{n\geq 0}(-1)^n\alpha_n t^{-\binom{n}{2}+pn+Nn}\frac{(t^{-N})_n}{(t^{N+1})_n}\prod_{i=1}^{p}\frac{(b_i)_n(c_i)_n}{b_i^n c_i^n(\frac{t}{b_i})_n(\frac{t}{c_i})_n} =$$

$$\frac{(t)_N(\frac{q}{b_pc_p})_N}{(\frac{t}{b_p})_N(\frac{t}{c_p})_N}\sum_{n_p\geq n_{p-1}\geq\cdots\geq n_1\geq 0}\beta_{n_1}\frac{t^{n_p}(t^{-N})_{n_p}(b_p)_{n_p}(c_p)_{n_p}}{(t^{-N}b_pc_p)_{n_p}}\prod_{i=1}^{p-1}\frac{t^{n_i}}{b_i^{n_i}c_i^{n_i}}\frac{(b_i)_{n_i}(c_i)_{n_i}}{(\frac{t}{b_i})_{n_{i+1}}(\frac{t}{c_i})_{n_{i+1}}}\frac{(\frac{t}{b_ic_i})_{n_{i+1}-n_i}}{(t)_{n_{i+1}-n_i}}$$

with the parameters chosen as follows.

We use the special Bailey pair

$$\begin{aligned}\alpha_0 &= 1, & \alpha_n &= (-1)^n t^{\frac{n(n-1)}{2}}(1+t^n)\\ \beta_0 &= 1, & \beta_n &= 0 \text{ for } n\geq 1.\end{aligned}$$

and define further $a := \frac{c+1}{2}$, $d := \lfloor\frac{b'-1}{2}\rfloor$ and $m := \lfloor\frac{N}{c}\rfloor$. Notice, that for $j\in\{1,\ldots,a-1\}$, there exist unique $u_j\in\{0,\ldots,c-1\}$, such that $u_j = j+N \pmod c$.

There are two cases: b odd and b even.

Case b odd. We define

$$\begin{aligned}
b_j &= t^{\frac{-N+u_j}{c}}, & c_j &= t^{\frac{-N+c-u_j}{c}}, && j=1,\ldots,a-1\\
b_a &= t^{-m}, & c_a &= t^{N+1} &&\\
b_{a+i} &= \omega^i t^{-m}, & c_{a+i} &= \omega^{-i}t^{-m}, && i=1,\ldots,d\\
b_{a+jd+i} &= \omega^i t^{\frac{-N+u_j}{c}}, & c_{a+jd+i} &= \omega^{-i}t^{\frac{-N+c-u_j}{c}}, && i=1,\ldots,d \text{ and } j=1,\ldots,c-1\\
b_{\gamma_1+i} &= -\omega^i t, & c_{\gamma_1+i} &= -\omega^{-i}t && i=1,\ldots,d\\
b_p &\to \infty, & c_p &\to \infty. &&
\end{aligned}$$

where $\gamma_1 := a+cd$. As a result, we have $p = (c+1)\frac{b}{2}+1$.

Case b even. We choose a square root ν of ω and define

$$\begin{aligned}
b_j &= t^{\frac{-N+u_j}{c}}, & c_j &= t^{\frac{-N+c-u_j}{c}}, && j=1,\ldots,a-1\\
b_a &= t^{-m}, & c_a &= t^{N+1} &&\\
b_{a+i} &= \omega^i t^{-m}, & c_{a+i} &= \omega^{-i}t^{-m}, && i=1,\ldots,d\\
b_{a+jd+i} &= \omega^i t^{\frac{-N+u_j}{c}}, & c_{a+jd+i} &= \omega^{-i}t^{\frac{-N+c-u_j}{c}}, && i=1,\ldots,d \text{ and } j=1,\ldots,c-1\\
b_{\gamma_1+j} &= -t^{\frac{-N+u_j}{c}} & c_{\gamma_1+j} &= -t^{\frac{-N+c-u_j}{c}} && j=1,\ldots,a-1\\
b_{\gamma_2+i} &= -\nu^{2i-1}t, & c_{\gamma_2+i} &= -\nu^{-(2i-1)}t && i=1,\ldots,d+1\\
b_{\gamma_2+d+2} &= -t^{-m}, & c_{\gamma_2+d+2} &= -t^0 = -1 &&\\
b_p &\to \infty, & c_p &\to \infty, &&
\end{aligned}$$

where $\gamma_1 := a+cd$ and $\gamma_2 := 2a+cd-1$. As a result, we have $p = (c+1)\frac{b}{2}+2$.

We now look at the case b odd and since the calculations for the even case are very similar, we omit these details and state the result after the calculations for the odd case.

The RHS of the identity gives

$$
\begin{aligned}
&(t)_N \sum_{n_p\geq n_{p-1}\geq\cdots\geq n_1=0} \frac{t^x\cdot(t^{-N})_{n_p}(b_p)_{n_p}(c_p)_{n_p}}{\prod_{i=1}^{p-1}(t)_{n_{i+1}-n_i}(t^{-N}b_pc_p)_{n_p}}\\
&\cdot\prod_{j=1}^{a-1}\frac{(t^{\frac{-N+u_j}{c}})_{n_j}(t^{\frac{-N+c-u_j}{c}})_{n_j}(t^{2m})_{n_{j+1}-n_j}}{(t^{\frac{N+c-u_j}{c}})_{n_{j+1}}(t^{\frac{N+u_j}{c}})_{n_{j+1}}}\cdot\frac{(t^{-m})_{n_a}(t^{N+1})_{n_a}(t^{-N+m})_{n_{a+1}-n_a}}{(t^{m+1})_{n_{a+1}}(t^{-N})_{n_{a+1}}}\\
&\cdot\prod_{i=1}^{d}\frac{(\omega^it^{-m})_{n_{a+i}}(\omega^{-i}t^{-m})_{n_{a+i}}(t^{2m+1})_{n_{a+i+1}-n_{a+i}}}{(\omega^it^{m+1})_{n_{a+i+1}}(\omega^{-i}t^{m+1})_{n_{a+i+1}}}\frac{(-\omega^it)_{n_{\gamma_1+i}}(-\omega^{-i}t)_{n_{\gamma_1+i}}(t^{-1})_{n_{\gamma_1+i+1}-n_{\gamma_1+i}}}{(-\omega^i)_{n_{\gamma_1+i+1}}(-\omega^{-i})_{n_{\gamma_1+i+1}}}\\
&\cdot\prod_{i=1}^{d}\prod_{j=1}^{c-1}\frac{(\omega^it^{\frac{-N+u_j}{c}})_{n_{a+jd+i}}(\omega^{-i}t^{\frac{-N+c-u_j}{c}})_{n_{a+jd+i}}(t^{2m})_{n_{a+jd+i+1}-n_{a+jd+i}}}{(\omega^it^{\frac{N+c-u_j}{c}})_{n_{a+jd+i+1}}(\omega^{-i}t^{\frac{N+u_j}{c}})_{n_{a+jd+i+1}}}
\end{aligned}
$$

where

$$x=n_p+\sum_{j=1}^{a-1}2m\,n_j+(m-N)\,n_a+\sum_{i=1}^{d}(2m+1)\,n_{a+i}+\sum_{i=1}^{d}\sum_{j=1}^{c-1}2m\,n_{a+jd+i}-\sum_{i=1}^{d}n_{\gamma_1+i}.$$

For $c=1$, we use the convention, that empty products are set to be 1.

Notice, that

$$\frac{(b_p)_{n_p}(c_p)_{n_p}}{(t^{-N}b_pc_p)_{n_p}}=(-1)^{n_p}t^{\frac{n_p(n_p-1)}{2}}t^{Nn_p},\qquad \lim_{c\to\infty}\left(\frac{t}{c}\right)_n=1.$$

Now, from the second last term in the sum we can follow that $n_{\gamma_1+i+1}-n_{\gamma_1+i}\leq 1$. Therefore, we have

$$\prod_{i=1}^{d}\frac{(-\omega^{\pm i}t)_{n_{\gamma_1+i}}}{(-\omega^{\pm i})_{n_{\gamma_1+i+1}}}=\prod_{i=1}^{d}(1+\omega^{\pm i}t^{n_{\gamma_1+i}})^{(n_{\gamma_1+i}-n_{\gamma_1+i+1}+1)\left\lceil\frac{n_{\gamma_1+i}}{1+n_{\gamma_1+i}}\right\rceil}(1+\omega^{\pm i})^{\left\lfloor\frac{1}{1+n_{\gamma_1+i+1}}\right\rfloor}.$$

Notice, that $n_p\leq N$ and therefore $n_i\leq N$ for all i. We multiply the numerator and denominator of the sum by

$$
\begin{aligned}
&\prod_{i=1}^{d}(\omega^{\pm i}t^{m+1+n_{a+i+1}})_{N-n_{a+i+1}}\prod_{j=1}^{a-1}(t^{\frac{N+c-u_j}{c}+n_{j+1}})_{N-n_{j+1}}(t^{\frac{N+u_j}{c}+n_{j+1}})_{N-n_{j+1}}\\
&\cdot\prod_{i=1}^{d}\prod_{j=1}^{c-1}(\omega^it^{\frac{N+c-u_j}{c}+n_{a+jd+i+1}})_{N-n_{a+jd+i+1}}(\omega^{-i}t^{\frac{N+u_j}{c}+n_{a+jd+i+1}})_{N-n_{a+jd+i+1}}
\end{aligned}
$$

such that we achieve in the denominator the term $\prod_{i=0}^{b-1}\prod_{j=1}^{c-1}(\omega^it^{\frac{N+u_j}{c}})_N\cdot\prod_{i=1}^{b-1}(\omega^it^{m+1})_N$ which is equal to

$$\prod_{j=1}^{c-1}(t^{b\frac{N+u_j}{c}};t^b)_N\cdot\frac{(t^{b(m+1)};t^b)_N}{(t^{m+1};t)_N}=(\widetilde{q^{N+1};q})_{cN}\frac{(\widehat{q^{N+1};q})_{cN}}{(t^{m+1};t)_N}.$$

Here we use the notation $(\widehat{q^a;q^b})=\frac{(q^a;q^b)}{(\widetilde{q^a;q^b})}$.

To get the wanted Laplace transform, we still have to multiply by $\frac{(-1)^{k+1}}{\{k+1\}!}$. Notice that $\{k+1\}!=(-1)^{k+1}q^{-\frac{(k+1)(k+2)}{4}}(q;q)_N$. Using the notation $\nu_j:=n_{j+1}-n_j$, we get

$$q^y \frac{(t^{m+1};t)_N}{(\widehat{q^{N+1};q})_{cN}} \frac{1}{(\widetilde{q^{N+1};q})_{cN}} \sum_{n_p \geq n_{p-1} \geq \cdots \geq n_1 = 0} \frac{(-1)^{n_p} t^{x'} \cdot (t)_N (t^{-N})_{n_p} (t^{N+1})_{n_a}}{(t^{-N})_{n_{a+1}} (t^{m+1})_{n_{a+1}}}$$
$$\frac{\prod_{j=1}^{a-1} (t^{2m})_{\nu_j} \cdot (t^{-N+m})_{\nu_a} \cdot \prod_{i=1}^{d} (t^{2m+1})_{\nu_{a+i}} (t^{-1})_{\nu_{\gamma_1+i}} \prod_{j=1}^{c-1} (t^{2m})_{\nu_{a+jd+i}}}{\prod_{i=1}^{p-1} (t)_{\nu_i}}$$
$$\cdot (t^{-m})_{n_a} \cdot \prod_{j=1}^{a-1} (t^{\frac{-N+u_j}{c}})_{n_j} (t^{\frac{-N+c-u_j}{c}})_{n_j} (t^{\frac{N+c-u_j}{c}+n_{j+1}})_{N-n_{j+1}} (t^{\frac{N+u_j}{c}+n_{j+1}})_{N-n_{j+1}}$$
$$\cdot \prod_{i=1}^{d} (\omega^i t^{-m})_{n_{a+i}} (\omega^{-i} t^{-m})_{n_{a+i}} (1+\omega^{\pm i} t^{n_{\gamma_1+i}})^{(-\nu_{\gamma_1+i}+1)\left\lceil \frac{n_{\gamma_1+i}}{1+n_{\gamma_1+i}} \right\rceil} (1+\omega^{\pm i})^{\left\lfloor \frac{1}{1+n_{\gamma_1+i+1}} \right\rfloor}$$
$$\cdot \prod_{i=1}^{d} (\omega^{\pm i} t^{m+1+n_{a+i+1}})_{N-n_{a+i+1}} \cdot \prod_{i=1}^{d} \prod_{j=1}^{c-1} (\omega^i t^{\frac{N+c-u_j}{c}+n_{a+jd+i+1}})_{N-n_{a+jd+i+1}}$$
$$\cdot \prod_{i=1}^{d} \prod_{j=1}^{c-1} (\omega^{-i} t^{\frac{N+u_j}{c}+n_{a+jd+i+1}})_{N-n_{a+jd+i+1}} (\omega^i t^{\frac{-N+u_j}{c}})_{n_{a+jd+i}} (\omega^{-i} t^{\frac{-N+c-u_j}{c}})_{n_{a+jd+i}}$$

where $x' = x + \frac{n_p(n_p-1)}{2} + Nn_p$ and $y = -\frac{(k+1)(k+2)}{4}$.

Notice, that

$$(t)_N (t^{N+1})_{n_a} = (t)_{N+n_a} = (t)_m (t^{m+1})_{N-m+n_a}.$$

From the term $(t^{-N+m})_{n_{a+1}-n_a}$ follows that $n_{a+1} - n_a \leq N - m$, and therefore $(t^{m+1})_{n_{a+1}}$ divides $(t^{m+1})_{N-m+n_a}$. It is now easy to see, that the denominator of the summands divides its numerator and we get as a result

$$\mathcal{L}\left(\begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}!\{n\} \right) = 2q^y \cdot \frac{(t;t)_{N+m}}{(q^c;q^c)_{N+m}} \frac{1}{(\widetilde{q;q})_{cN}} \sum_{n_p \geq n_{p-1} \geq \cdots \geq n_1 = 0} \frac{(t^{-N})_{n_p} (t^{m+1})_{N-m+n_a}}{(t^{-N})_{n_{a+1}} (t^{m+1})_{n_{a+1}}}$$
$$\cdot (-1)^{n_p} t^{x'} \cdot \frac{\prod_{j=1}^{a-1} (t^{2m})_{\nu_j} \cdot (t^{-N+m})_{\nu_a} \cdot \prod_{i=1}^{d} (t^{2m+1})_{\nu_{a+i}} (t^{-1})_{\nu_{\gamma_1+i}} \prod_{j=1}^{c-1} (t^{2m})_{\nu_{a+jd+i}}}{\prod_{i=1}^{p-1} (t)_{\nu_i}}$$
$$\cdot (t^{-m})_{n_a} \cdot \prod_{j=1}^{a-1} (t^{\frac{-N+u_j}{c}})_{n_j} (t^{\frac{-N+c-u_j}{c}})_{n_j} (t^{\frac{N+c-u_j}{c}+n_{j+1}})_{N-n_{j+1}} (t^{\frac{N+u_j}{c}+n_{j+1}})_{N-n_{j+1}}$$
$$\cdot \prod_{i=1}^{d} (\omega^i t^{-m})_{n_{a+i}} (\omega^{-i} t^{-m})_{n_{a+i}} (1+\omega^{\pm i} t^{n_{\gamma_1+i}})^{(-\nu_{\gamma_1+i}+1)\left\lceil \frac{n_{\gamma_1+i}}{1+n_{\gamma_1+i}} \right\rceil} (1+\omega^{\pm i})^{\left\lfloor \frac{1}{1+n_{\gamma_1+i+1}} \right\rfloor}$$
$$\cdot \prod_{i=1}^{d} (\omega^{\pm i} t^{m+1+n_{a+i+1}})_{N-n_{a+i+1}} \cdot \prod_{i=1}^{d} \prod_{j=1}^{c-1} (\omega^i t^{\frac{N+c-u_j}{c}+n_{a+jd+i+1}})_{N-n_{a+jd+i+1}}$$
$$\cdot \prod_{i=1}^{d} \prod_{j=1}^{c-1} (\omega^{-i} t^{\frac{N+u_j}{c}+n_{a+jd+i+1}})_{N-n_{a+jd+i+1}} (\omega^i t^{\frac{-N+u_j}{c}})_{n_{a+jd+i}} (\omega^{-i} t^{\frac{-N+c-u_j}{c}})_{n_{a+jd+i}}$$

In the case b even, the calculation works similar and the result is

$$
\begin{aligned}
\mathcal{L}\left(\begin{bmatrix} n+k \\ 2k+1 \end{bmatrix}\{k\}!\{n\}\right) \; = \; & 2q^y \cdot \frac{(t;t)_{N+m}}{(q^c;q^c)_{N+m}} \frac{1}{\widetilde{(q;q)}_{cN}} \frac{1}{(-t)_N} \sum_{n_p \geq n_{p-1} \geq \cdots \geq n_1 = 0} (-1)^{n_p} t^{x''} \\
& \cdot \frac{(t^{-N})_{n_p} (t^{m+1})_{N-m+n_a}}{(t^{-N})_{n_a+1} (t^{m+1})_{n_a+1}} \cdot \frac{(t^{m+1})_{\nu_{\gamma_2+d+2}} \cdot \prod_{i=1}^{d+1} (t^{-1})_{\nu_{\gamma_2+i}}}{\prod_{i=\gamma_2}^{p-1} (t)_{\nu_i}} \\
& \cdot \frac{(t^{-N+\frac{N}{c}})_{\nu_a} \cdot \prod_{j=1}^{a-1} (t^{2m})_{\nu_j} (t^{2m})_{\nu_{\gamma_1+j}} \cdot \prod_{i=1}^{d} (t^{2m+1})_{\nu_{a+i}} \prod_{j=1}^{c-1} (t^{2m})_{\nu_{a+jd+i}}}{\prod_{i=1}^{\gamma_2-1} (t)_{\nu_i}} \\
& \cdot (t^{-m})_{n_a} \cdot (-t^{m+1+n_p})_{N-n_p} \cdot (-t^{-m})_{n_{\gamma_2+d+2}} \cdot (-t)_{n_{\gamma_2+d+2}-1} \cdot (-t^{n_p+1})_{N-n_p} \\
& \cdot \prod_{j=1}^{a-1} (t^{\frac{-N+u_j}{c}})_{n_j} (t^{\frac{-N+c-u_j}{c}})_{n_j} (t^{\frac{N+c-u_j}{c}+n_{j+1}})_{N-n_{j+1}} (t^{\frac{N+u_j}{c}+n_{j+1}})_{N-n_{j+1}} (-t^{\frac{-N+u_j}{c}})_{n_{\gamma_1+j}} \\
& \cdot \prod_{j=1}^{a-1} (-t^{\frac{-N+c-u_j}{c}})_{n_{\gamma_1+j}} (-t^{\frac{N+c-u_j}{c}+n_{\gamma_1+j+1}})_{N-n_{\gamma_1+j+1}} (-t^{\frac{N+u_j}{c}+n_{\gamma_1+j+1}})_{N-n_{\gamma_1+j+1}} \\
& \cdot \prod_{i=1}^{d} (\omega^i t^{-m})_{n_{a+i}} (\omega^{-i} t^{-m})_{n_{a+i}} (\omega^i t^{m+1+n_{a+i+1}})_{N-n_{a+i+1}} (\omega^{-i} t^{m+1+n_{a+i+1}})_{N-n_{a+i+1}} \\
& \cdot \prod_{i=1}^{d} \prod_{j=1}^{c-1} (\omega^i t^{\frac{-N+u_j}{c}})_{n_{a+jd+i}} (\omega^{-i} t^{\frac{-N+c-u_j}{c}})_{n_{a+jd+i}} (\omega^i t^{\frac{N+c-u_j}{c}+n_{a+jd+i+1}})_{N-n_{a+jd+i+1}} \\
& \cdot \prod_{i=1}^{d} \prod_{j=1}^{c-1} (\omega^{-i} t^{\frac{N+u_j}{c}+n_{a+jd+i+1}})_{N-n_{a+jd+i+1}} \cdot \prod_{i=1}^{d+1} (1+\nu^{\pm(2i-1)} t^{n_{\gamma_2+i}})^{(n_{\gamma_2+i}-n_{\gamma_2+i+1}+1)\left\lceil \frac{n_{\gamma_2+d+2}}{1+n_{\gamma_2+d+2}} \right\rceil} \\
& \cdot \prod_{i=1}^{d+1} \left((-\nu^{2i-1})_1 (-\nu^{-(2i-1)})_1\right)^{\left\lfloor \frac{1}{1+n_{a+cd+i+2}} \right\rfloor}
\end{aligned}
$$

where $x'' = x' + \sum_{j=1}^{a-1} 2m\, n_{\gamma_1+j} + n_{\gamma_2+d+2}(m+1) - \sum_{i=1}^{d} (n_{\gamma_2+i} + n_{\gamma_1+i})$.

Let us denote the sum in the above expressions by $T_k(q,t,b,c)$. Notice that the denominator of $T_k(q,t,b,c)$ divides its numerator. Therefore we proved that $T_k(q,t,b,c) \in \mathbb{Z}[t^{\pm 1}, t^{1/c}, \omega]$ if b odd and $T_k(q,t,b,c) \in \mathbb{Z}[t^{\pm 1}, t^{1/c}, \nu]$ if b even. Now, we look again at the odd case. Since

$$\frac{(-1)^{k+1}}{\{k+1\}!} S_k(q,b) = q^y \frac{(t^{m+1};t)_N}{\widehat{(q^{N+1};q)}_{cN}} \frac{1}{\widetilde{(q^{N+1};q)}_{cN}} T_k(q,t,b,c)$$

there are $f_0, g_0 \in \mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ such that $T_k(q,t,b,c) = \frac{f_0}{g_0}$. We need the following lemma to prove that $T_k(q,t,b,c)$ lies in $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$.

Lemma. *If $f, g \in \mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ and $f/g \in \mathbb{Z}[t^{\pm 1}, t^{1/c}, \omega]$ for ω a b'–th root of unity, we have $f/g \in \mathbb{Z}[t^{1/c}, t^{\pm 1}, q^{\pm 1}]$.*

Proof. Since f and g do not change under the ring automorphisms φ_{ij} which send ω^i to ω^j for $\mathrm{ord}(\omega^i) = \mathrm{ord}(\omega^j)$, f/g is unchanged as well. Therefore the ω^i's of the same order must have the same coefficients in f/g. Notice that the sum over the b'–th root of unity of the same order lies in $\mathbb{Z}$. We can conclude that $f/g \in \mathbb{Z}[t^{\pm 1}, t^{1/c}]$. □

Since f_0 and g_0 are independent of the choice of the c–th root of t (we only need $t^{1/c}$ when we replace (q^x, q) by $(t^{x/c};t)(\omega t^{x/c};t) \cdots (\omega^{b'-1} t^{x/c};t)$ and this equation is true for all c–th root of t) we can apply the argument of the Lemma to $t^{1/c}$ and get $T_k(q,t,b,c) \in \mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$.

The proof for the even case works similar.

Appendix 2

We first recall Habiro's setting. We denote by $U_h = U_h(sl_2)$ the h–adically complete $\mathbb{Q}[[h]]$–algebra, topologically generated by H, E and F, satisfying the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{v - v^{-1}},$$

where we set $K = v^H = e^{\frac{hH}{2}}$. Further, $\mathcal{U}_q$ ($\mathcal{U}_q^{\rm ev}$) denotes the subalgebra of U_h freely generated over $\mathbb{Z}[q^{\pm 1}]$ by $\tilde{F}^{(i)} K^j e^k$ ($\tilde{F}^{(i)} K^{2j} e^k$, respectively) for $i, k \geq 0, j \in \mathbb{Z}$, where

$$\tilde{F}^{(n)} = \frac{F^n K^n}{v^{\frac{n(n-1)}{2}}[n]!} \quad \text{and} \quad e = (v - v^{-1})E.$$

On $\mathcal{U}_q^{\rm ev}$, Habiro introduced the filtration $\mathcal{F}_n(\mathcal{U}_q^{\rm ev})$, which is spanned by $(\tilde{F}^{(k)} K^k) K^{2j} e^l$ over $\mathbb{Z}[v^{\pm 1}]$, and the completion

$$\widehat{\mathcal{U}}_q^{\rm ev} = \lim_{\overleftarrow{n}} \frac{\mathcal{U}_q^{\rm ev}}{\mathcal{F}_n(\mathcal{U}_q^{\rm ev})}.$$

Further, he denoted $\tilde{U}_q^{\rm ev}$ as the image of the map $\widehat{\mathcal{U}}_q^{\rm ev} \to U_h$, the "completion in U_h" of $\mathcal{U}_q^{\rm ev}$.

Lemma 15. *For odd k and $x \in \mathcal{U}_q$, we have*

$$(1 \otimes \mathrm{tr}_q^{V_k})(e^{\frac{h}{2} H \otimes H})(1 \otimes x) \in \mathcal{U}_q^{\rm ev}$$

Proof. For fixed k and x, we can find a basis $e_1, \ldots, e_k$ of V_k, such that

$$H = \begin{pmatrix} k-1 & & & 0 \\ & k-3 & & \\ & & \ddots & \\ 0 & & & 1-k \end{pmatrix}.$$

Therefore,

$$\begin{aligned} (1 \otimes \mathrm{tr}_q^{V_k})(e^{\frac{h}{2} H \otimes H})(1 \otimes x) &= \sum_m \frac{1}{m!} \left(\frac{h}{2}\right)^m H^m \otimes \mathrm{tr}^{V_k}(K^{-1} H^m x) \\ &= \sum_i \sum_m \frac{1}{m!} \left(\frac{h}{2}\right)^m H^m \otimes < e_i, K^{-1} H^m x e_i > \end{aligned}$$

is only nonzero if x contains summands of the form $\tilde{F}^{(l)} K^j e^l$ with $l \leq k$. In the last case, we have

$$\sum_m \frac{1}{m!} \left(\frac{h}{2}\right)^m H^m \otimes < e_i, K^{-1} H^m \tilde{F}^{(l)} K^j e^l e_i > = \sum_m \frac{1}{m!} \left(\frac{h}{2}\right)^m H^m \otimes < e_i, K^{-1} \tilde{F}^{(l)} K^j e^l H^m e_i >$$

$$= \sum_m \frac{1}{m!} \left(\frac{h}{2}\right)^m (2c_i)^m H^m \otimes < e_i, K^{-1} \tilde{F}^{(l)} K^j e^l e_i > = \sum_i K^{2c_i} \otimes < e_i, K^{-1} \tilde{F}^{(l)} K^j e^l e_i > \; \in \; \mathcal{U}_q^{\rm ev}$$

where $2c_i := k - 2i + 1$ is even because k is odd. □

Now, we prove Theorem 7.

Proof of Theorem 7. Let us open the first component of $L \cup L'$ and denote the resulting $(1,1)$–tangle by T. The universal invariant J_T of T is defined in Section 4 of [6]. We claim that

$$(1 \otimes \cdots \otimes 1 \otimes t_q^{V_{j_1}} \otimes \cdots \otimes t_q^{V_{j_l}}) J_T \in \tilde{\mathcal{U}}_q^{\rm ev}.$$

If $\mathrm{lk}(L_i, L'_j) = 0$ for all i, j, the result follows from Theorem 4.1 of Habiro [6]. Assume $\mathrm{lk}(L_i, L'_j) = \mathrm{lk}_{ij} \neq 0$. Then according to the definition, J_T is an infinite sum of terms

$$e^{h \sum_{i,j} \frac{\mathrm{lk}_{ij}}{2} H_{ij}} y_1 \otimes \cdots \otimes y_{m+l}$$

where $y_i \in \mathcal{U}_q$ for all i and H_{ij} is defined to be $1\otimes\cdots\otimes H\otimes 1\otimes\cdots\otimes 1\otimes H\otimes\cdots\otimes 1$ with everywhere a 1 expect an H at the i–th and the j–th position. By Lemma 15, if $\mathrm{lk}_{ij} = 0 \pmod 2$ or k_i is odd, we have

$$(1\otimes\cdots\otimes \mathrm{tr}^{V_{k_j}}\otimes\cdots\otimes 1)(e^{h\frac{\mathrm{lk}_{ij}}{2}H_{ij}})(1\otimes\cdots\otimes y_{m+j}\otimes\cdots\otimes 1)\in \mathcal{U}_q^{\mathrm{ev}}.$$

This proves the claim. The rest is analogous to the proof of Theorem 8.2 in [6].

□

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