Poincaré inequalities for maps with target manifold of negative curvature

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Abstract

We prove that for any given homotopic C^1-maps $u, v : G \to M$ in a nontrivial homotopy class from a metric graph into a closed manifold of negative sectional curvature, the distance between u and v can be bounded by $3\,(\mathrm{length}(u) + \mathrm{length}(v)) + C(\kappa, \varrho/20)$ where $\varrho > 0$ is a lower bound of the injectivity radius and $-\kappa < 0$ an upper bound for the sectional curvature of M. The constant $C(\kappa, \varepsilon)$ is given by

$$C(\kappa, \varepsilon) = 8sh_\kappa^{-1}(1) + 8sh_\kappa^{-1}\,(1/sh_\kappa(\varepsilon))$$

with $sh_\kappa(t) = \sinh\,(\sqrt{\kappa}\ t)$. Various applications are given.

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0 Introduction

Let G be a finite graph and $M = X/\Gamma$ a *complete* Riemannian manifold with universal cover X and Γ as group of deck transformations. Assume that M has negative sectional curvature bounded from above by $-\kappa < 0$ and injectivity radius bounded from below by $\varrho > 0$. A map $u : G \to M$ is called C^1 if the restriction of u to every edge is a C^1-map. In an obvious way one defines the length $L(u)$ of a C^1-map $u : G \to M$ by summing up the lengths of the restriction of u to any of the edges of G. Denote by $N(u, v)$ the distance between two homotopic C^1-maps $u, v : G \to M$,

$$N(u, v) = \inf_H \{\sup_{z \in G} \ell_H(z)\}$$

where the infimum is taken over all C^1-homotopies $H : G \times [0, 1] \to M$ between u and v and $\ell_H(x)$ is the length of the curve $s \mapsto H(x, s)$.

Theorem 0.1 *Let $\kappa > 0$ and $\varrho > 0$ be given. Then for any Riemannian manifold M with sectional curvature bounded from above by $-\kappa < 0$ and injectivity radius bounded from below by $\varrho > 0$, for any finite graph G and for any homotopic C^1-maps $u, v : G \to M$, which are not in the trivial homotopy class*

$$N(u, v) \leq 3\,(L(u) + L(v)) + C(\kappa, \varrho/20) \tag{0.1}$$

where $C(\kappa, \varepsilon) := 8sh_\kappa^{-1}(1) + 8sh_\kappa^{-1}\left(1/sh_\kappa^{-1}(\varepsilon)\right)$ and $sh_\kappa(t) = \sinh(\kappa t)$.

Remark: For C^1-maps $u, v : G \to M$ in the trivial homotopy class inequality (0.1) is not true. Assuming that M is closed, one obtains in this case an estimate of the form $N(u, v) \leq \frac{1}{2}\,(L(u) + L(v)) + \operatorname{diam}(M)$ where $\operatorname{diam}(M)$ denotes the diameter of M.

As an application of Theorem 0.1 we obtain a Poincaré inequality for homotopic C^1-maps $u, v : M' \to M$ where M' is a closed Riemannian manifold. To state it we need to introduce some further notation. For any $1 \leq p < \infty$ and arbitrary homotopic C^1-maps $u, v : M' \to M$ introduce the distance function

$$\begin{aligned} N_p(u, v) := \inf \big\{ N_p(H) \;|H : M' \times [0, 1] \to M \\ C^1 - \text{homotopy between } u \text{ and } v \big\} \end{aligned}$$

where

$$N_p(H) := \left(\int_M \ell_H(x)^p d\mathrm{vol}(x) \right)^{1/p},$$

and $\ell_H(x) = \int_0^1 \| \frac{d}{ds} H(x,s) \| ds$ as above. Finally we introduce the energy $E(u)$ of a C^1-map $u : M' \to M$,

$$E(u) := \int_M \|d_x u\|^2 d\mathrm{vol}(x)$$

where $\|d_x u\|$ denotes the Hilbert-Schmidt norm of the differential $d_x u : T_x M' \to T_{u(x)} M$.

Theorem 0.2 *Let M and M' be closed Riemannian manifolds and assume that M has negative sectional curvature. Then there exists $C_2 > 0$ depending only on the geometry of M and M' so that for any homotopic C^1-maps $u, v : M' \to M$*

$$N_2(u,v) \le C_2 \left(E(u)^{1/2} + E(v)^{1/2} + 1 \right). \tag{0.2}$$

Related work: In [KKS1], by different methods, inequality (0.2) is proved for target manifolds M with nonpositive sectional curvature with a constant C_2 which depends on the geometry of M and M' and, in addition, on the homotopy class of the maps u, v considered.

Theorem 0.2 can be applied to improve Theorem 0.2 in [KKS1] on perturbations of the harmonic map equation for maps $u : M' \to M$,

$$\tau(u) + F(x,u) + L(x,u) u_* G(x,u) = 0. \tag{0.3}$$

For the compactness result of Theorem 0.2 in [KKS1] to hold, the bound C_* on the size of the perturbation $L(x,u)u_*G(x,u)$,

$$\max_{\substack{x \in M' \\ y \in M}} \|L(x,y)\| \, \|G(x,y)\| \le C_*$$

can now be chosen independently of the homotopy class of maps considered if M has negative sectional curvature. Here $\tau(u)$ denotes the tension field, $F(x,y)$ is an x-dependent vector field on M, $L(x,y)$ an x-dependent linear operator on the tangent space $T_y M$ and $G(x,y)$ an y-dependent vector field on M'.

It is planned to extend Theorem 0.1 and Theorem 0.2 to other classes of target manifolds and to investigate how the corresponding constants depend on the geometric data of M in these cases.

The paper is organized as follows: Theorem 0.1 is proved in section 2 and the applications mentioned above, including Theorem 0.2, are treated in section 3. In section 1 we show estimates on the displacement functions needed in the proof of Theorem 0.1. To make the paper selfcontained we have included two appendices.

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1 Estimates of the displacement function

Assume that (M, g) is a complete Riemannian manifold of negative sectional curvature with

$$K \leq -\kappa < 0 \tag{1.1}$$

for some constant $\kappa > 0$ and of injectivity radius $\operatorname{inj}(M)$ bounded from below,

$$\operatorname{inj}(M) \geq \varrho > 0. \tag{1.2}$$

Then $M \cong X/\Gamma$ where X is the universal covering of M and Γ is the group of deck transformations of M. The main result of this section is Proposition 1.5 which states an estimate for displacement functions used in the proof of Theorem 0.1 - see section 3.

First we need to introduce some more notation and establish three lemmas. For any $\gamma \in \Gamma$ denote by $d_\gamma : X \to \mathbb{R}$ the displacement function, $d_\gamma(x) := d(x, \gamma x)$, where d denotes the distance function on X and by $\mathrm{MIN}(\gamma)$ the closed subset

$$\mathrm{MIN}(\gamma) := \{x \in X \mid d_\gamma(x) = \inf_X d_\gamma\}.$$

Assumptions (1.1) - (1.2) imply that for any nontrivial $\gamma \in \Gamma$, $\mathrm{MIN}(\gamma)$ consists of one geodesic curve - see Appendix B where we collect results needed about such manifolds. As d_γ is convex $\mathrm{MIN}(\gamma)$ is a convex set. This allows to define the metric projection $\pi_\gamma : X \to \mathrm{MIN}(\gamma)$ with $\pi_\gamma x$ the unique point in $\mathrm{MIN}(\gamma)$ satisfying

$$s_\gamma(x) := d(x, \pi_\gamma x) = \min\{d(x, y) \mid y \in \mathrm{MIN}(\gamma)\}.$$

Given a complete geodesic $A \subseteq X$, considered as a closed subset of X, and a unit speed geodesic $c : \mathbb{R} \to X$, consider the distance function $r(t) := d\,(c(t), A)$. As $t \mapsto r(t)$ is a convex function, the set $r^{-1}([0, \varepsilon])$, consisting of all $t \in \mathbb{R}$ with $c(t)$ in or on the tube of given radius $\varepsilon > 0$ around A, is either empty or an interval $[a, b] \cap \mathbb{R}$ where $a := \inf r^{-1}([0, \varepsilon]) \in \mathbb{R} \cup \{-\infty\}, b := \sup r^{-1}([0, \varepsilon]) \in \mathbb{R} \cup \{+\infty\}$. The following result says that $r(t)$ grows at least linearly outside $[a, b] \cap \mathbb{R}$. Recall that

$$sh_\kappa(t) := \sinh\left(\sqrt{\kappa}\ t\right).$$

Lemma 1.1 *Assume* (1.1) - (1.2) *and let* $C_1(\kappa, \varepsilon) := sh_\kappa^{-1}\left(1/sh_\kappa(\varepsilon)\right)$. *For* $\varepsilon > 0$ *with* $[a, b] \cap \mathbb{R} \neq \emptyset$ *the following statements hold:*

(i) *If* $a > -\infty$, *then* $r(a - t) \geq t - C_1(\kappa, \varepsilon)\ \forall t \geq 0$.

(ii) *If* $b < +\infty$, *then* $r(b + t) \geq t - C_1(\kappa, \varepsilon)\ \forall t \geq 0$.

Remark: $C_1(\kappa, \varepsilon)$ is strictly decreasing in both κ and ε.

Proof: (i) and (ii) are proved in the same fashion so we consider (ii) only. Denote by x and $y = y(t)$ the orthogonal projections of $c(b)$ and $c(b+t)$ onto the geodesic A and consider the geodesic quadrilateral $x, c(b), c(b+t), y(t)$. As $r(t)$ is convex, $r\mid_{[b,b+t]}$ is monotone increasing and hence the angles at the points $x, c(b)$ and $y(t)$ are $\geq \pi/2$. By Lemma A.1,

$$sh_\kappa\left(d(x, y)\right) \leq 1/sh_\kappa(\varepsilon). \tag{1.3}$$

On the other hand, as the angle at $c(b)$ is $\geq \pi/2, t = d\left(c(b), c(b+t)\right)$ satisfies $t \leq d\left(x, c(b+t)\right)$ and by the triangle inequality,

$$d\left(x, c(b+t)\right) \leq r(b+t) + d(x, y).$$

Combining these inequalities with (1.3), one obtains

$$t \leq r(b+t) + sh_\kappa^{-1}\left(1/sh_\kappa(\varepsilon)\right)$$

as claimed. ■

Recall that for $\gamma \in \Gamma, s_\gamma(x)$ denotes the distance of x to $MIN(\gamma)$, $s_\gamma(x) = d(x, \pi_\gamma x)$.

Lemma 1.2 *Assume* (1.1) - (1.2) *holds. Then for any* $\gamma \in \Gamma \backslash id, x \in X$

$$d_\gamma(x) \geq \inf_X d_\gamma + 2s_\gamma(x) - C_2(\kappa, \varrho)$$

where $C_2(\kappa, \varepsilon) := 4sh_\kappa^{-1}(1) + 2sh_\kappa^{-1}\left(1/sh_\kappa(\varepsilon)\right)$.

Proof: Let $c : [0, d_\gamma(x)] \to X$ be an arclength parametrization of the unique geodesic $[x, \gamma x]$ from x to γx and $t \in [0, d_\gamma(x)]$ the parameter so that for $z := c(t)$

$$d\left(z, \mathrm{MIN}(\gamma)\right) = \inf_{0 \leq s \leq d_\gamma(x)} d\left(c(s), \mathrm{MIN}(\gamma)\right).$$

Let us treat first the case where $z \neq x$ and $z \neq \gamma x$. Denote by x', y', z' the projections of $x, \gamma x$, and z respectively onto $\mathrm{MIN}(\gamma)$. Then the geodesic from z to z' intersects $\mathrm{MIN}(\gamma)$ and $[x, \gamma x]$ orthogonally. Further note that, by the definition of ϱ,

$$\max\left(d(z', y'), d(z', x')\right) \geq \frac{1}{2} \inf_X d_\gamma \geq \varrho.$$

As $sh_\kappa(t)$ is increasing in t, Lemma A.1 leads to the following upper bound for $d(z, z')$,

$$sh_\kappa\left(d(z, z')\right) \leq 1/sh_\kappa(\varrho). \tag{1.4}$$

Let x'' be the projection of x' onto the geodesic $[x, z']$ connecting x and z' and y'' the projection of y' onto the geodesic $[\gamma x, z']$. As the geodesic $[x', x'']$ intersects $[x, z']$ orthogonally and $[x, x']$ intersects $\mathrm{MIN}(\gamma)$ orthogonally one can apply Lemma A.2 either to the geodesic triangle (x, x', x'') or (x', x'', z') to conclude that

$$sh_\kappa\left(d(x', x'')\right) \leq 1. \tag{1.5}$$

Arguing in the same way one gets

$$sh_\kappa\left(d(y', y'')\right) \leq 1. \tag{1.6}$$

Inequalities (1.4) - (1.6) are now used to obtain the claimed statement: First note that

$$d_\gamma(x) = d(x, \gamma x) = d(x, z) + d(z, \gamma x).$$

By the triangle inequality

$$d(x, z) \geq d(x, z') - d(z, z'); \;\; d(z, \gamma x) \geq d(\gamma x, z') - d(z, z').$$

As $d(x,z') = d(x,x'') + d(x'',z')$ and $d(\gamma x,z') = d(\gamma x,y'') + d(y'',z')$ it then follows again by the triangle inequality

$$d(x,z') \geq d(x,x') + d(x',z') - 2d(x',x'')$$

and

$$d(\gamma x,z') \geq d(\gamma x,y') + d(y',z') - 2d(y',y'').$$

Combining these inequalities with (1.4) - (1.6) and using that $d(x',z') + d(z',y') = d(x',y') = \inf_X d_\gamma$ as well as $d(\gamma x,y') = d(x,x') = s_\gamma(x)$ it then follows that

$$d_\gamma(x) \geq \inf_X d_\gamma + 2s_\gamma(x) - C_2(\kappa,\varrho)$$

where

$$C_2(\kappa,\varepsilon) := 4sh_\kappa^{-1}(1) + 2sh_\kappa^{-1}\left(1/sh_\kappa(\varepsilon)\right).$$

The cases where $z = x$ or $z = \gamma x$ are treated in a similar way - in fact they are easier. ■

Given $\gamma_1,\gamma_2 \in \Gamma\backslash id$ with $\mathrm{MIN}(\gamma_1) \neq \mathrm{MIN}(\gamma_2)$ and a unit speed geodesic $c : \mathbb{R} \to X$, consider the distance function

$$r_i(t) := d\left(c(t), \mathrm{MIN}(\gamma_i)\right) \quad (1 \leq i \leq 2)$$

and denote by I_ε the set of all $t \in \mathbb{R}$ with $c(t)$ in the ε-tube around $\mathrm{MIN}(\gamma_1)$ *and* $\mathrm{MIN}(\gamma_2)$,

$$I_\varepsilon := r_1^{-1}([0,\varepsilon]) \cap r_2^{-1}([0,\varepsilon]).$$

As r_1 and r_2 are both convex and the intersection of convex sets is again convex, I_ε is convex. The following result gives an estimate of the length of I_ε.

Lemma 1.3 *Assume that* (1.1) - (1.2) *hold and* $\gamma_1,\gamma_2 \in \Gamma\backslash id$ *satisfy* $MIN(\gamma_1) \neq MIN(\gamma_2)$. *Then for any* $0 < \varepsilon < inj(M)/10$,

$$\mathrm{length}(I_\varepsilon) \leq \inf_X d_{\gamma_1} + \inf_X d_{\gamma_2}.$$

Proof: Assume that the contrary holds. As I_ε is connected we may assume without loss of generality that c is parametrized in such a fashion that $[0,a] \subseteq$

I_ε where $a := a_1 + a_2$ and $a_i := \inf_X d_{\gamma_i}$. Denote by $\pi_i \equiv \pi_{\gamma_i} : X \to \mathrm{MIN}(\gamma_i)$ the metric projection onto $\mathrm{MIN}(\gamma_i)$ and let

$$x_i := \pi_i c(0), \ y_i := \pi_i \left(c(a)\right).$$

Let $c_i = c_{\gamma_i} : \mathbb{R} \to X$ be unit speed parametrizations of $\mathrm{MIN}(\gamma_i)$ such that $c_i(0) = x_i$ and $c_i\left(d(x_i, y_i)\right) = y_i$ and choose $\overline{\gamma}_i \in \{\gamma_i, \gamma_i^{-1}\}$ so that

$$\overline{\gamma}_i \left(c_i(t)\right) = c_i(t + a_i) \quad \forall t \in \mathbb{R}.$$

By the triangle inequality

$$d\left(x_1, x_2\right) \le d\left(x_1, c(0)\right) + d\left(c(0), x_2\right) \le 2\varepsilon \tag{1.7}$$

and similarly $d(y_1, y_2) \le 2\varepsilon$. Further for $i = 1, 2$,

$$d\left(c(0), c(a)\right) - 2\varepsilon \le d(x_i, y_i) \le d\left(c(0), c(a)\right) + 2\varepsilon.$$

As $d\left(c(0), c(a)\right) = a$, this means that $|d(x_j, y_j) - a| < 2\varepsilon$. Together with the fact that $d(x_j, y_j) = d\left(x_j, c_j(a)\right) \pm d\left(c_j(a), y_j\right)$ one then concludes that $d\left(c_j(a), y_j\right) \le 2\varepsilon$. Hence

$$d\left(c_1(a), c_2(a)\right) \le d\left(c_1(a), y_1\right) + d(y_1, y_2) + d\left(y_2, c_2(a)\right) \le 6\varepsilon. \tag{1.8}$$

As $d\left(c_1(t), c_2(t)\right)$ is convex in t (cf [BGS, Theorem 1.3]) it then follows from (1.7) - (1.8) that

$$d\left(c_1(t), c_2(t)\right) \le 6\varepsilon \quad \forall t \in [0, a].$$

We claim that for $x := c(0)$,

$$d(\overline{\gamma}_2\overline{\gamma}_1 x, \overline{\gamma}_1\overline{\gamma}_2 x) \le 20\varepsilon. \tag{1.9}$$

Hence the projection of the geodesic $[x, (\overline{\gamma}_2\overline{\gamma}_1)^{-1}\overline{\gamma}_1\overline{\gamma}_2 x] \subseteq X$ leads to a closed geodesic loop in M of length $20\varepsilon < 2\mathrm{inj}(M)$. This implies that $\overline{\gamma}_1\overline{\gamma}_2 = \overline{\gamma}_2\overline{\gamma}_1$ and, by Lemma B.2, $\mathrm{MIN}(\overline{\gamma}_1) = \mathrm{MIN}(\overline{\gamma}_2)$. As $\overline{\gamma}_i \in \{\gamma_i, \gamma_i^{-1}\}, \mathrm{MIN}(\overline{\gamma}_i) = \mathrm{MIN}(\gamma_i)$ and hence $\mathrm{MIN}(\gamma_1) = \mathrm{MIN}(\gamma_2)$, contradicting our assumption.

It remains to prove (1.9). As $\overline{\gamma}_1 c_1(0) = c_1(a_1)$,

$$d\left(\overline{\gamma}_1 x, c_1(a_1)\right) = d\left(x, c_1(0)\right) \le \varepsilon$$

and therefore

$$d\left(\overline{\gamma}_1 x, c_2(a_1)\right) \le d\left(\overline{\gamma}_1 x, c_1(a_1)\right) + d\left(c_1(a_1), c_2(a_1)\right) \le 7\varepsilon.$$

As $\overline{\gamma}_2 c_2(a_1) = c_2(a_1+a_2) = c_2(a)$ this leads to

$$d\left(\overline{\gamma}_2\overline{\gamma}_1 x, c_2(a)\right) \leq 7\varepsilon.$$

Similarly one gets

$$d\left(\overline{\gamma}_1\overline{\gamma}_2 x, c_1(a)\right) \leq 7\varepsilon$$

and thus, by the triangle inequality

$$\begin{aligned}
&d\left(\overline{\gamma}_2\overline{\gamma}_1 x, \overline{\gamma}_1\overline{\gamma}_2 x\right)\\
&\leq d\left(\overline{\gamma}_2\overline{\gamma}_1 x, c_1(a)\right) + d\left(c_1(a), c_2(a)\right) + d\left(c_2(a), \overline{\gamma}_1\overline{\gamma}_2 x\right)\\
&\leq 20\varepsilon. \quad \blacksquare
\end{aligned}$$

The following estimate of the displacement function is the main ingredient into our proof of Proposition 1.5 stated below.

Proposition 1.4 *Assume* (1.1) - (1.2) *holds. Then for any* $\gamma_1, \gamma_2 \in \Gamma\backslash id$ *with* $\gamma_1\gamma_2 \neq \gamma_2\gamma_1$ *and any* $x, y \in X$,

$$\max_{1\leq j\leq 2}\left(d_{\gamma_j}(x) + d_{\gamma_j}(y)\right) \geq d(x,y) - C_4(\kappa, \varrho/20)$$

where

$$C_4(\kappa,\varepsilon) := 4C_1(\kappa,\varepsilon) + 2C_2(\kappa,\varepsilon).$$

Proof: Let $\varepsilon := \varrho/20$ and for any given $x, y \in X$, denote by $c : [0, d(x,y)] \to X$ the unit speed parametrization of the geodesic $[x,y]$. First consider the case where there exists $\gamma \in \{\gamma_1, \gamma_2\}$ with $d\left(c(t), \mathrm{MIN}(\gamma)\right) > \varepsilon$ for any $0 \leq t \leq d(x,y)$. Denote by $t_0 \in [0, d(x,y)]$ the parameter so that $\varepsilon_1 := d\left(c(t_0), \mathrm{MIN}(\gamma)\right)$ is the minimal value of $r(t) := d\left(c(t), \mathrm{MIN}(\gamma)\right)$. By Lemma 1.1, applied with $(a := t_0, b := t_0)$ if $0 < t_0 < d(x,y)$, with $(a < 0, b := 0)$ if $0 = t_0$, and with $(a := t_0, b > t_0)$ if $t_0 = d(x,y)$ one gets

$$r(0) \geq t_0 - C_1(\kappa, \varepsilon_1)$$

and

$$r\left(d(x,y)\right) \geq \left(d(x,y) - t_0\right) - C_1(\kappa,\varepsilon_1)$$

As $\varepsilon \leq \varepsilon_1$, one has $C_1(\kappa,\varepsilon) > C_1(\kappa,\varepsilon_1)$ and thus, adding the two inequalities above,

$$r(0) + r\left(d(x,y)\right) \geq d(x,y) - 2C_1(\kappa,\varepsilon).$$

As $s_\gamma(x) = r(0)$ and $s_\gamma(y) = r\left(d(x,y)\right)$, it then follows from Lemma 1.2,

$$\begin{aligned} d_\gamma(x) + d_\gamma(y) &\geq 2\inf_X d_\gamma + 2r(0) + 2r\left(d(x,y)\right) - 2C_2(\kappa,\varrho) \\ &\geq 2\inf_X d_\gamma + 2d(x,y) - C_4(\kappa,\varepsilon) \end{aligned}$$

and the claimed estimate is proved in this case. In the case where no such γ exists it follows that for the convex functions $(i = 1,2)$

$$r_i(t) := d\left(c(t), \mathrm{MIN}(\gamma_i)\right) \quad 0 \leq t \leq d(x,y),$$

$J_i := r_i^{-1}\left([0,\varepsilon]\right) \neq \emptyset$ is an interval, $J_i = [a_i, b_i]$ with $0 \leq a_i \leq b_i \leq d(x,y)$. By Lemma 1.1 one obtains in the case $0 < a_i$

$$r_i(0) \geq a_i - C_1(\kappa,\varepsilon) \tag{1.10}$$

and, similarly, if $b_i < d(x,y)$

$$r_i\left(d(x,y)\right) \geq \left(d(x,y) - b_i\right) - C_1(\kappa,\varepsilon). \tag{1.11}$$

As $-C_1(\kappa,\varepsilon) \leq 0$, (1.10) and (1.11) trivially hold in the case $a_i = 0$ and $b_i = d(x,y)$ respectively. Hence

$$r_i(0) + r_i\left(d(x,y)\right) \geq d(x,y) - \mathrm{length}(J_i) - 2C_1(\kappa,\varepsilon).$$

As $s_{\gamma_i}(x) = r_i(0)$ and $s_{\gamma_i}(y) = r_i\left(d(x,y)\right)$ it then follows from Lemma 1.2,

$$\begin{aligned} d_{\gamma_i}(x) + d_{\gamma_i}(y) &\geq 2\inf_X d_{\gamma_i} + 2r_i(0) + 2r_i\left(d(x,y)\right) - 2C_2(\kappa,\varrho) \\ &\geq 2\inf_X d_{\gamma_i} + 2d(x,y) - 2\ \mathrm{length}(J_i) - C_4(\kappa,\varepsilon) \end{aligned} \tag{1.12}$$

where for the last inequality we used that as $\varepsilon < \varrho$

$$\begin{aligned} C_4(\kappa,\varepsilon) :=& 4C_1(\kappa,\varepsilon) + 2C_2(\kappa,\varepsilon) \\ &\leq 4C_1(\kappa,\varepsilon) + 2C_2(\kappa,\varrho). \end{aligned}$$

Now add the inequalities (1.12) for $i = 1$ and 2. As $\gamma_1\gamma_2 \neq \gamma_2\gamma_1$ one has by Lemma B.2 $MIN(\gamma_1) \neq MIN(\gamma_2)$ and hence Lemma 1.3 leads to

$$\inf_X d_{\gamma_1} + \inf_X d_{\gamma_2} \geq \mathrm{length}(J_1 \cap J_2)$$

and

$$2\max_{1\le i\le 2}\left(d_{\gamma_i}(x)+d_{\gamma_i}(y)\right)\ge\sum_{i=1}^{2}\left(d_{\gamma_i}(x)+d_{\gamma_i}(y)\right)$$

to obtain

$$\begin{aligned}&2\max_{1\le i\le 2}\left(d_{\gamma_i}(x)+d_{\gamma_i}(y)\right)\ge\\&\ge 4d(x,y)+2\left(\mathrm{length}(J_1\cap J_2)-\mathrm{length}J_1-\mathrm{length}J_2\right)-2C_4(\kappa,\varepsilon)\\&\ge 2\left(d(x,y)-C_4(\kappa,\varepsilon)\right)\end{aligned}$$

leading to the claimed inequality. ■

Given elements $\gamma_1,\dots,\gamma_n$ in $\Gamma\backslash id$, recall that $Z(\gamma_1,\dots,\gamma_n)$ denotes the centralizer of $\{\gamma_1,\dots,\gamma_n\}$.

Proposition 1.5 *Assume that* (1.1) - (1.2) *hold. Let x, y be arbitrary points in X and $\gamma_1,\dots,\gamma_n\in\Gamma\backslash id$ with $n\ge 1$. Then there exist $\gamma\in\{\gamma_1,\dots,\gamma_n\}$ and $\alpha\in Z(\gamma_1,\dots,\gamma_n)$ such that*

$$d_\gamma(x)+d_\gamma(y)\ge d(x,\alpha y)-C_4(\kappa,\varrho/20).$$

Proof: Consider first the case where $Z(\gamma_1,\dots,\gamma_n)=\{id\}$. This implies in particular that $n\ge 2$ and that there are two elements $\gamma_i,\gamma_j\in\{\gamma_1,\dots,\gamma_n\}$ with $\gamma_i\gamma_j\ne\gamma_j\gamma_i$. Hence by Proposition 1.4, there exists $\gamma\in\{\gamma_i,\gamma_j\}$ so that

$$d_\gamma(x)+d_\gamma(y)\ge d(x,y)-C_4(\kappa,\varrho/20).$$

Thus in this case the conclusion holds with $\alpha=id$. In the case $Z(\gamma_1,\dots,\gamma_n)\ne\{id\}$, we have $\{\gamma_1,\dots,\gamma_n\}\subset Z(\gamma_1,\dots,\gamma_n)$ and this group is cyclic by Lemma B.2, i.e. there exists $\beta\in Z(\gamma_1,\dots,\gamma_n)$ with

$$Z(\gamma_1,\dots,\gamma_n)=\{\beta^n\mid n\in\mathbb{Z}\}.$$

Let $\pi_\beta: X\to\mathrm{MIN}(\beta)$ be the metric projection, set $x':=\pi_\beta(x), y':=\pi_\beta(y)$ and choose $\gamma\in\{\gamma_1,\dots,\gamma_n\}$ arbitrary. Recall that $s_\gamma(x)=d\left(x,\pi_\gamma(x)\right)$. Then $s_\gamma=s_\beta$ as $\mathrm{MIN}(\gamma)=\mathrm{MIN}(\beta)$ - see Lemma B.2 - and $\inf_X d_\beta\le\inf_X d_\gamma$ as γ is an element of $Z(\gamma_1,\dots,\gamma_n)$ and hence of the form $\gamma=\beta^i$ for some $i\in\mathbb{Z}$.

Further there exists $m \in \mathbb{Z}$ so that $d(\beta^m y', x') \leq \inf_X d_\beta$. Combining these inequalities one obtains

$$\begin{aligned} d(x, \beta^m y) &\leq d(x, x') + d(x', \beta^m y') + d(\beta^m y', \beta^m y) \\ &\leq s_\beta(x) + \inf_X d_\beta + s_\beta(y) \\ &\leq s_\gamma(x) + \inf_X d_\gamma + s_\gamma(y) \\ &\leq d_\gamma(x) + d_\gamma(y) + 2C_2(\kappa, \varrho) \end{aligned}$$

where for the last inequality we used Lemma 1.2. As $2C_2(\kappa, \varrho) \leq 2C_2(\kappa, \varrho/20) \leq C_4(\kappa, \varrho/20)$, the claimed statement holds in this case with $\alpha := \beta^m$. ■

2 Short homotopies between graphs

In this section we prove Theorem 0.1 as stated in the introduction. Let G be a finite graph. For simplicity of exposition only, we assume that G is a connected metric graph (i.e. every edge has some positive length) and has no terminals (i.e. that every edge is incident to at least two edges).

As above, let (M, g) denote a complete Riemannian manifold with

$$K \leq -\kappa < 0 \tag{2.1}$$

and

$$\mathrm{inj}(M) \geq \varrho > 0 \tag{2.2}$$

for some given constants $\varrho > 0, \kappa > 0$. A map $u : G \to M$ is called C^1 if the restriction of u to every edge is C^1. In an obvious way one defines the length $L(u)$ of a C^1-map $u : G \to M$ by summing the lengths of the restriction of u to any of the edges of G.

Theorem 2.1 *Assume that* (M, g) *satisfies* (2.1) - (2.2). *Then for any homotopic* C^1*-maps* $u, v : G \to M$ *which are not in the trivial homotopy class there exists a* C^1*-homotopy* $H : G \times [0,1] \to M$ *so that* $\sup_{z \in G} \ell_H(z) \leq 3\,(L(u) + L(v)) + C(\kappa, \varrho/20)$ *where* $C(\kappa, \varepsilon) := 8sh_\kappa^{-1}(1) + 8sh_\kappa^{-1}\,(1/sh_\kappa(\varepsilon))$ *with* $sh_\kappa(\varepsilon) := \sinh(\sqrt{\kappa}\,\varepsilon)$ *and* $\ell_H(t)$ *is the length of* $[0,1] \to M, s \mapsto H(t,s)$.

Remark 1 Note that the constant $C(\kappa, \varrho/20)$ is independent of G.

Remark 2 In the case where $u, v : G \to M$ are homotopic C^1-maps which are in the trivial homotopy class it is necessary to assume that M is compact. By lifting u and v to the universal cover X one verifies easily that for any closed Riemannian manifold (M, g) of *nonpositive* sectional curvature there exists a C^1-homotopy $H : G \times [0, 1] \to M$ so that

$$\sup_{z \in G} \ell_H(z) \leq \frac{1}{2} \left(L(u) + L(v) \right) + \operatorname{diam}(M)$$

where $\operatorname{diam}(M)$ denotes the diameter of M.

In the remainder of this section we prove Theorem 2.1. We begin arguing as in the proof of Theorem 5.1 in [KKS1]. Recall that the Euler characteristic $\chi(G)$ of G is defined by

$$\chi(G) := \sharp \text{ vertices} - \sharp \text{ edges} .$$

By a straightforward inductive argument one sees that $\chi(G) \leq 1$ as G is connected. Further, G is said to be a tree if it does not contain any loop. Again by a straight forward inductive argument one verifies that a connected graph G is a tree iff $\chi(G) = 1$. Let $T_1 \subseteq G$ be a maximal connected subgraph of G such that T_1 is in addition a tree. T_1 is obtained from G by removing m edges, denoted by $e_1, \ldots, e_m$. It then follows from the above characterization of trees that $m = 1 - \chi(G)$. Let $p_1, \ldots, p_m$ be the midpoints of $e_1, \ldots, e_m$ and consider the abstract metric tree T which is obtained from G by removing the points p_j and then completing the metric tree. A point p_i then gives rise to two points, p_i^+ and p_i^-, in T. Thus T is a metric tree whose terminals are the vertices $p_i^+, p_i^-, i = 1, \ldots, m$, and G is obtained from T by identifying p_i^+ with p_i^- for any $1 \leq i \leq m$. Let us denote by $\varphi : T \to G$ this identification map. We choose a base point t_0 in the interior of the tree T. For every terminal p_i^+, p_i^- of T there is a unique path $\sigma_i^+, \sigma_i^- : [0, 1] \to T$ parametrized proportionally to arclength from t_0 to p_i^+, p_i^-. By our assumption there exists a homotopy $H^G : G \times [0, 1] \to M$ with $H_0^G = v$ and $H_1^G = u$. Let $H^T : T \times [0, 1] \to M$ be the map

$$H^T(t, s) = H^G\left(\varphi(t), s\right).$$

Since T is contractible, we can lift H^T to a map

$$\overline{H}^T : T \times [0, 1] \to X$$

where $\pi : X \to M$ is the universal covering of M. Since $H^T(p_i^+, s) = H^T(p_i^-, s)$ for any $i = 1, \ldots, m$ and $s \in [0,1]$, the points $\overline{H}^T(p_i^+, s)$ and $\overline{H}^T(p_i^-, s)$ are identified by deck transformations. Hence there are isometries $\gamma_1, \ldots, \gamma_m$ in the deck transformation group Γ so that for any $0 \le s \le 1$,

$$\gamma_i\left(\overline{H}^T(p_i^+, s)\right) = \overline{H}^T(p_i^-, s).$$

Introduce

$$L(\sigma_i^\pm, s) := \text{length}\left(\tau \mapsto \overline{H}^T(\sigma_i^\pm(\tau), s)\right)$$

and note that

$$L(\sigma_i^\pm, 0) \le L(v)$$

as well as

$$L(\sigma_i^\pm, 1) \le L(u).$$

Since we assume that u, v are not in the trivial homotopy class, $\{\gamma_1, \ldots, \gamma_m\} \cap \Gamma \backslash id \neq \emptyset$. W.l.o.g. assume that $\{\gamma_1, \ldots, \gamma_k\} = \{\gamma_j \mid 1 \le j \le m, \gamma_j \neq id\}$ where $k \ge 1$ and let

$$x := \overline{H}^T(t_0, 0) \in X;\ y := \overline{H}^T(t_0, 1) \in X. \tag{2.3}$$

By Proposition 1.5 there exists $\gamma \in \{\gamma_1, \ldots, \gamma_k\}$ and $\alpha \in Z(\gamma_1, \ldots, \gamma_k)$ so that

$$d_\gamma(x) + d_\gamma(y) \ge d(x, \alpha y) - C_4(\kappa, \varrho/20). \tag{2.4}$$

W.l.o.g. we may assume that $\gamma = \gamma_1$. Consider the pathes $\tau \mapsto \overline{H}^T\left(\sigma_1^-(\tau), 0\right)$ from $x = \overline{H}^T(t_0, 0)$ to $\overline{H}^T(p_1^-, 0)$ and $\tau \mapsto \gamma_1 \overline{H}^T\left(\sigma_1^+(\tau), 0\right)$ from $\gamma_1 x$ to $\gamma_1 \overline{H}^T(p_1^+, 0) = \overline{H}^T(p_1^-, 0)$. By the triangle inequality and the estimate above

$$\begin{aligned} d_{\gamma_1}(x) &\le \text{length}\left(\tau \mapsto \overline{H}^T(\sigma_1^-(\tau), 0)\right) \\ &\quad + \text{length}\left(\tau \mapsto \gamma_1 \overline{H}^T(\sigma_1^+(\tau), 0)\right) \\ &\le 2L(v) \end{aligned} \tag{2.5}$$

and similarly

$$d_{\gamma_1}(y) \le 2L(u). \tag{2.6}$$

Define the homotopy $\hat{H}^T : T \times [0,1] \to X$ given for any $t \in T$ by the geodesic $s \mapsto c_t(s)$ from $\overline{H}^T(t, 0)$ to $\alpha \overline{H}^T(t, 1)$, parametrized proportional to arclength with $\alpha \in Z(\gamma_1, \ldots, \gamma_k)$ given as above. Then $\hat{H}^T$ is a C^1-homotopy. We claim that for any $1 \leq i \leq m$ and any $0 \leq s \leq 1$

$$\gamma_i \hat{H}^T(p_i^+, s) = \hat{H}^T(p_i^-, s). \tag{2.7}$$

To see it, let $c_{p_i^\pm}$ be the geodesic from $\overline{H}^T(p_i^\pm, 0)$ to $\alpha\overline{H}^T(p_i^\pm, 1)$. Then $\gamma_i c_{p_i^+}$ is the geodesic from $\gamma_i \overline{H}^T(p_i^+, 0) = \overline{H}^T(p_i^-, 0)$ to

$$\gamma_i \alpha \overline{H}^T(p_i^+, 1) = \alpha \gamma_i \overline{H}^T(p_i^+, 1) = \alpha \overline{H}^T(p_i^-, 1)$$

as α is an element in the centralizer $Z(\gamma_1, \ldots, \gamma_m)$. Thus $\gamma_i c_{p_i^+}$ is the geodesic $c_{p_i^-}$ and hence (2.7) established. By (2.7), $\hat{H}^T$ induces a homotopy $H : G \times [0,1] \to M$. For $z_0 := \varphi(t_0) \in G$ (with $\varphi : T \to G$ the identification map) we have

$$\ell_H(z_0) = d\left(\overline{H}^T(t_0, 0),\ \alpha\overline{H}^T(t_0, 1)\right) = d(x, \alpha y).$$

Hence by (2.4) - (2.6)

$$\ell_H(z_0) \leq 2\left(L(v) + L(u)\right) + C_4(\kappa, \varrho/20)$$

where (cf Lemma 1.4, Lemma 1.1 and Lemma 1.2)

$$\begin{aligned} C_4(\kappa, \varepsilon) &= 4C_1(\kappa, \varepsilon) + 2C_2(\kappa, \varepsilon) \\ &= 4sh_\kappa^{-1}\left(1/sh_\kappa(\varepsilon)\right) + 8sh_\kappa^{-1}(1) + 4sh_\kappa^{-1}\left(1/sh_\kappa(\varepsilon)\right) \\ &= 8sh_\kappa^{-1}\left(1/sh_\kappa(\varepsilon)\right) + 8sh_\kappa^{-1}(1) \end{aligned}$$

which by definition equals $C(\kappa, \varepsilon)$. By the triangle inequality we then obtain for any $z \in G$,

$$\ell_H(z) \leq 3\left(L(u) + L(v)\right) + C(\kappa, \varrho/20)$$

as claimed. ■

3 Proof of Theorem 0.2

First we need to prove the following

Proposition 3.1 *Assume that M' has negative sectional curvature. Denote by $2r$ the convexity radius of M and let $x_0 \in M$ and $0 < \mu < 1$ be arbitrary. Then there exists a constant $C_3 > 0$ so that for any homotopic C^1-maps $u, v : M' \to M$ there is an open subset $A_{uv} \subseteq B_r(x_0)$ with*

$$\mathrm{vol}(A_{uv}) > \mu \mathrm{vol}\left(B_r(x_0)\right)$$

and the property that for any $z \in A_{uv}$ there exists a geodesic homotopy [1] *$H : M' \times [0,1] \to M$ from u to v satisfying*

$$\mathrm{length}\left(s \mapsto H(z,s)\right) \leq C_3 \left(E(u)^{1/2} + E(v)^{1/2} + 1\right).$$

The constant C_3 depends only on the geometry of M and M'.

Proof: (of Proposition 3.1) Following the proof of Theorem 6.1 in [KKS1] word by word the claimed statement follows from Proposition 6.2 and Proposition 6.3 in [KKS1] together with Theorem 0.1. ■

Proof: (of Theorem 0.2) Following the proof of Proposition 3.2 in [KKS1] word for word the claimed statement follows from Proposition 3.1 in [KKS1] together with Theorem 0.1. ■

A Appendix: Hyperbolic trigonometry

In this appendix we collect elementary facts on hyperbolic geometry. Assume that (X, g) is a Hadamard manifold with bounded sectional curvature,

$$K(x) \leq -\kappa \quad \forall x \in X$$

where $\kappa > 0$ and denote by $d : X \times X \to X$ the distance function. Further let $\mathbb{H}^2_\kappa$ be the upper half plane with constant curvature $-\kappa$ and denote the corresponding distance function by d_κ

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[1] i.e. a homotopy so that for any $x \in M, s \mapsto H(x,s)$ is a geodesic parametrized proportional to arclength

Lemma A.1 *Let* $(x_j)_{0\le j\le 3}$ *be the four distinct corners of a geodesic quadrilateral in* X *so that for* $1 \le j \le 3$*, the angle* α_j *at* x_j *satisfies* $\alpha_j \ge \pi/2$*. Then* $a := d(x_1, x_2)$ *and* $b := d(x_2, x_3)$ *satisfy*

$$sh_\kappa(a) \cdot sh_\kappa(b) \le 1$$

where $sh_\kappa(t) := \sinh(\sqrt{\kappa}\ t)$*.*

Proof: Let $\overline{x}_2$ and $\overline{x}_0$ be points in the hyperbolic plane $\mathbb{H}^2_\kappa$ with $d_\kappa(\overline{x}_2, \overline{x}_0) = d(x_2, x_0)$ and choose $\overline{x}_1$ and $\overline{x}_3$ in $\mathbb{H}^2_\kappa$ so that the geodesic triangles $(\overline{x}_1, \overline{x}_2, \overline{x}_0)$ and $(\overline{x}_2, \overline{x}_3, \overline{x}_0)$ in $\mathbb{H}^2_\kappa$ have the same sidelengths as the triangles (x_1, x_2, x_0) and (x_2, x_3, x_0) respectively. The angles of these comparison triangles are not smaller than the corresponding ones of the original triangles. It then follows that the angles $\overline{\alpha}_j$ at x_j of the geodesic quadrilateral $(\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3)$ in $\mathbb{H}^2_\kappa$ satisfy $\overline{\alpha}_j \ge \pi/2$ for $1 \le j \le 3$. Elementary considerations in the hyperbolic plane show that the points $\overline{x}_0, \overline{x}_1$ and $\overline{x}_3$ can be moved to points $\tilde{x}_0, \tilde{x}_1, \tilde{x}_3 \in \mathbb{H}^2$ so that $a = d_\kappa(\tilde{x}_1, \overline{x}_2), b = d_\kappa(\overline{x}_2, \tilde{x}_3)$ and $\tilde{\alpha}_j = \pi/2$ for $1 \le j \le 3$ where $\tilde{\alpha}_j$ is the angle at $\tilde{x}_j$ of the geodesic quadrilateral $(\tilde{x}_0, \tilde{x}_1, \overline{x}_2, \tilde{x}_3)$. By hyperbolic trigonometry we conclude $sh_\kappa(a) \cdot sh_\kappa(b) \le 1$ (cf [Bu, 2.3.1 (i)]). ■

Lemma A.2 *Let* $(x_j)_{1\le j\le 3}$ *be the corners of a geodesic triangle in* X *with* $\alpha_2 \ge \pi/2$ *and* $\alpha_1 \ge \pi/4$ *where* α_j *denotes the angle at* x_j $(1 \le j \le 3)$*. Then* $a := d(x_1, x_2)$ *satisfies* $sh_\kappa(a) \le 1$*.*

Proof: Let $(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ be a geodesic triangle in $\mathbb{H}^2_\kappa$ with the same sidelengths as (x_1, x_2, x_3). It then follows that the angles of $(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ are not smaller than the corresponding angles of (x_1, x_2, x_3). By elementary considerations in $\mathbb{H}^2_\kappa$ one sees that the points $\overline{x}_1$ and $\overline{x}_3$ can be moved to points $\tilde{x}_1, \tilde{x}_3 \in \mathbb{H}^2$ so that the angles $\tilde{\alpha}_j$ at $\tilde{x}_j$ of the geodesic triangle $(\tilde{x}_1, \overline{x}_2, \tilde{x}_3)$ satisfy

$$\tilde{\alpha}_2 = \pi/2,\ \tilde{\alpha}_1 \ge \pi/4$$

and $d(\tilde{x}_1, \overline{x}_2) = a$. Using elementary hyperbolic trigonometry one concludes that

$$sh_\kappa(a) \le 1$$

(cf [Bu, 2.2.2 (iv)]). ■

B Appendix: Manifolds of negative sectional curvature

Assume that (M, g) is a complete Riemannian manifold of negative sectional curvature with

$$K \leq -\kappa < 0 \tag{B.1}$$

for some constant $\kappa > 0$. Then $M \cong X/\Gamma$ where X is the universal covering of M and Γ is the group of deck transformations of M. In particular any $\gamma \in \Gamma$ is an isometry of X. The universal covering is a Hadamard manifold, i.e. a complete and contractible Riemannian manifold of nonpositive - actually negative in the case at hand - sectional curvature (cf [BGS, §2]) and $\Gamma \cong \pi_1(M)$ acts on X freely (i.e. $\gamma \in \Gamma \backslash \{id\}$ has no fixed points) and discretely (i.e. for any $K \subseteq X$ compact, there are finitely many $\gamma \in \Gamma$ with $\gamma K \cap K \neq \emptyset$).

To a deck transformation $\gamma \in \Gamma$ we associate its displacement function $d_\gamma : X \to [0, \infty), d_\gamma(x) := d(x, \gamma x)$ where d is the distance function on X. The function d_γ is convex, i.e. $d_\gamma(x(t))$ is convex in t for any geodesic, parametrized proportional to arclength and, by the triangle inequality, 2-Lipschitz continuous

$$|d_\gamma(x) - d_\gamma(z)| \leq 2d(x, z) \quad (\forall x, z \in X).$$

Thus the set

$$\mathrm{MIN}(\gamma) := \{x \in X \mid d_\gamma(x) = \inf_{x \in X} d_\gamma(x)\}$$

is a closed, convex subset of X. The injectivity radius $\mathrm{inj}(M)$ of M is given by

$$\mathrm{inj}(M) = \frac{1}{2} \inf\{d_\gamma(x) \mid \gamma \in \Gamma \backslash id,\ x \in X\}.$$

We assume that for some given constant $\varrho > 0$,

$$\mathrm{inj}(M) \geq \varrho > 0. \tag{B.2}$$

By standard arguments, conditions (B.1) and (B.2) imply the following

Lemma B.1 *Assume* (B.1) - (B.2) *hold. Then any deck transformation* $\gamma \in \Gamma \backslash id$ *is hyperbolic, i.e.* $\inf\{d_\gamma(x) \mid x \in X\} > 0$ *and* $MIN(\gamma) \neq \emptyset$.

Proof: Assume $MIN(\gamma) = \emptyset$. Then there exists a point η in the ideal boundary of X such that γ leaves η and all horospheres centered at η invariant - see [BGS]. Choose $x \in X$ and $c : [0, \infty) \to X$ a ray from $x = c(0)$ to $\eta = \lim_{t \to \infty} c(t)$. Note that X is a CAT(-κ) space - see [BH], Theorem 1A.6. Thus by standard comparison arguments $d\left(c(t), \gamma \cdot c(t)\right) \to 0$ for $t \to \infty$ in contradiction to $\inf\{d_\gamma(x) \mid x \in X\} \geq 2\varrho > 0$. ■

As X admits no parallel geodesic in view of (B.1) (cf [BGS, Lemma 2.3]) one concludes that for any $\gamma \in \Gamma \backslash id$ (cf [BGS, Lemma 6.5])

$$\mathrm{MIN}(\gamma) = \{c_\gamma(t) \mid t \in \mathbb{R}\} \tag{B.3}$$

where $c_\gamma : \mathbb{R} \to X$ is a geodesic, parametrized by arclength. For any $\gamma_1, \ldots, \gamma_n \in \Gamma \backslash id$, denote by $Z(\gamma_1, \ldots, \gamma_n)$ the centralizer of $\{\gamma_1, \ldots, \gamma_n\}$, i.e.

$$Z(\gamma_1, \ldots, \gamma_n) = \{\alpha \in \Gamma \mid \alpha\gamma_i = \gamma_i\alpha \quad \forall 1 \leq i \leq n\}.$$

Lemma B.2 *Assume* (B.1) - (B.2) *hold.*

(i) For any $\gamma_1, \gamma_2 \in \Gamma \backslash id$,

$$\gamma_1\gamma_2 = \gamma_2\gamma_1 \text{ iff } MIN\,(\gamma_1) = MIN\,(\gamma_2).$$

(ii) For any $\gamma_1, \ldots, \gamma_n \in \Gamma \backslash id$ *with* $n \geq 1$ *either* $Z(\gamma_1, \ldots, \gamma_n) = \{id\}$ *or* $Z(\gamma_1, \ldots, \gamma_n) \cong \mathbb{Z}$. *In the latter case* $\{\gamma_1, \ldots, \gamma_n\} \subseteq Z(\gamma_1, \ldots, \gamma_n)$ *and* $MIN(\alpha) = MIN(\beta)\ \forall \alpha, \beta \in Z(\gamma_1, \ldots, \gamma_n) \backslash id$.

Proof: (i) Assume that γ_1 and γ_2 commute. Then $\mathrm{MIN}(\gamma_1)$ is left invariant by γ_2. As γ_2 is an isometry it then translates $\mathrm{MIN}(\gamma_1)$ and thus, by (B.3) and [BGS, Lemma 6.5], $\mathrm{MIN}(\gamma_1) = \mathrm{MIN}(\gamma_2)$. Conversely assume that $\mathrm{MIN}(\gamma_1) = \mathrm{MIN}(\gamma_2)$. Then γ_1 translates c_{γ_2} and it follows that

$$\gamma_1\gamma_2 \cdot x = \gamma_2\gamma_1 \cdot x \quad \forall x \in \mathrm{MIN}(\gamma_2)$$

hence $(\gamma_2\gamma_1)^{-1}\gamma_1\gamma_2$ has fixed points. As Γ acts freely on X, $(\gamma_2\gamma_1)^{-1}\gamma_1\gamma_2 = id$ or $\gamma_1\gamma_2 = \gamma_2\gamma_1$.

(ii) Assume that $Z(\gamma_1, \ldots, \gamma_n) \neq \{id\}$ and choose $\alpha \in Z(\gamma_1, \ldots, \gamma_n) \backslash id$ arbitrary. Then, for any $1 \leq i \leq n, \alpha\gamma_i = \gamma_i\alpha$, hence by statement (i),

$\mathrm{MIN}(\gamma_i) = \mathrm{Min}(\alpha)$ and, γ_i being an isometry, translates $\mathrm{MIN}(\alpha)$. In particular $\mathrm{MIN}(\gamma_i) = \mathrm{MIN}(\gamma_j)\ \forall i, j$, hence $\gamma_i \in Z(\gamma_1, \ldots, \gamma_n)$ for any $1 \leq i \leq n$ and as $\alpha \in Z(\gamma_1, \ldots, \gamma_n) \backslash id$ is arbitrary it follows that for any $\beta \in Z(\gamma_1, \ldots, \gamma_n) \backslash id$

$$\mathrm{MIN}(\beta) = \mathrm{MIN}(\gamma_i) = \mathrm{MIN}(\alpha).$$

Thus, again by (i), $\alpha\beta = \beta\alpha$ and therefore $Z(\gamma_1, \ldots, \gamma_n)$ is Abelian. Recall that $Z(\gamma_1, \ldots, \gamma_n)$ acts by translations on $\mathrm{MIN}(\alpha)$, hence it has no torsion elements, and $Z(\gamma_1, \ldots, \gamma_n)$ acts discretely on $\mathrm{Min}(\alpha)$ so that $Z(\gamma_1, \ldots, \gamma_n)$ cannot have more than one generator: Given any $\beta, \gamma \in Z(\gamma_1, \ldots, \gamma_n) \backslash id$ there exist $t_\beta, t_\gamma \in \mathbb{R}$ so that for any $t \in \mathbb{R}$

$$\beta \cdot c_\alpha(t) = c_\alpha(t + t_\beta);\ \gamma \cdot c_\alpha(t) = c_\alpha(t + t_\gamma).$$

As Γ acts freely, one has $t_\beta \neq 0, t_\gamma \neq 0$ and as Γ acts discretely, t_β and t_γ must be rationally dependent, $t_\beta / t_\gamma = m/n$ with $m, n \in \mathbb{Z} \backslash \{0\}$ relatively prime. Hence the linear congruence $mx = 1 \mathrm{mod} n$ has a solution $x \in \mathbb{Z}$, i.e. there exist $i, j \in \mathbb{Z}$ with $mi + nj = 1$. Let $s := t_\beta / m$ and note that $t_\beta = ms, t_\gamma = ns$, as well as

$$it_\beta + jt_\gamma = \text{ ims } + \text{ jns } = s.$$

Thus for $\alpha_0 := \beta^i \gamma^j$ one has $\alpha_0 c_\alpha(t) = c_\alpha(t + s)$ for any $t \in \mathbb{R}$ or $s = t_{\alpha_0}$. As Γ acts freely on X one concludes $\beta = \alpha_0^m, \gamma = \alpha_0^n$. ■

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