THE BARNES G FUNCTION AND ITS RELATIONS WITH SUMS AND PRODUCTS OF GENERALIZED GAMMA CONVOLUTION VARIABLES

A. NIKEGHBALI AND MARC YOR

ABSTRACT. We give a probabilistic interpretation for the Barnes G-function which appears in random matrix theory and in analytic number theory in the important moments conjecture due to Keating-Snaith for the Riemann zeta function, via the analogy with the characteristic polynomial of random unitary matrices. We show that the Mellin transform of the characteristic polynomial of random unitary matrices and the Barnes G-function are intimately related with products and sums of gamma, beta and log-gamma variables. In particular, we show that the law of the modulus of the characteristic polynomial of random unitary matrices can be expressed with the help of products of gamma or beta variables, and that the reciprocal of the Barnes G-function has a Lévy-Khintchin type representation. These results lead us to introduce the so called generalized gamma convolution variables.

1. INTRODUCTION, MOTIVATION AND MAIN RESULTS

The Barnes G-function, which was first introduced by Barnes in [3] (see also [1]), may be defined via its infinite product representation:

$$G(1+z) = (2\pi)^{z/2} \exp\left[-\frac{1}{2}\left[(1+\gamma)z^2 + z\right]\right] \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^n \exp\left[-z + \frac{z^2}{2n}\right] \tag{1.1}$$

where γ is the Euler constant.

From (1.1), one can easily deduce the following (useful) development of the logarithm of $G(1+z)$ for $|z| < 1$:

$$\log G(1+z) = \frac{z}{2}(\log(2\pi) - 1) - \frac{1}{2}(1+\gamma)z^2 + \sum_{n=3}^{\infty}(-1)^{n-1}\zeta(n-1)\frac{z^n}{n} \tag{1.2}$$

where ζ denotes the Riemann zeta function.

This Barnes G-function has recently occurred in the work of Keating and Snaith [14] in their celebrated moments conjecture for the Riemann zeta

function. More precisely, they consider the set of unitary matrices of size N, endowed with the Haar probability measure, and they prove the following results:

Proposition 1.1 (Keating-Snaith [14])**.** *If Z denotes the characteristic polynomial of a generic random unitary matrix, considered at any point of the unit circle (for example 1), then the following hold:*

(1) *For λ any complex number satisfying $\mathfrak{Re}(\lambda) > -1$:*

$$\mathbb{E}_N\left[|Z|^{2\lambda}\right] = \prod_{j=1}^{N} \frac{\Gamma(j)\,\Gamma(j+2\lambda)}{(\Gamma(j+\lambda))^2} \tag{1.3}$$

(2) *For $\mathfrak{Re}(\lambda) > -1$:*

$$\lim_{N\to\infty} \frac{1}{N^{\lambda^2}} \mathbb{E}_N\left[|Z|^{2\lambda}\right] = \frac{(G(1+\lambda))^2}{G(1+2\lambda)} \tag{1.4}$$

Then, using a random matrix analogy (now called the "Keating-Snaith philosophy"), they make the following conjecture for the moments of the Riemann zeta function (see [14],[16]):

$$\lim_{T\to\infty} \frac{1}{(\log T)^{\lambda^2}} \frac{1}{T} \int_0^T \mathrm{d}t \left|\zeta\left(\frac{1}{2}+\imath t\right)\right|^{2\lambda} = M(\lambda)\,A(\lambda)$$

where $M(\lambda)$ is the "random matrix factor"

$$M(\lambda) = \frac{(G(1+\lambda))^2}{G(1+2\lambda)}$$

and $A(\lambda)$ is the arithmetic factor

$$A(\lambda) = \prod_{p\in\mathcal{P}} \left[\left(1-\frac{1}{p}\right)^{\lambda^2} \left(\sum_{m=0}^{\infty} \left(\frac{\Gamma(\lambda+m)}{m!\Gamma(\lambda)}\right)^2 p^{-m}\right)\right]$$

where, as usual, $\mathcal{P}$ is the set of prime numbers.

Due to the importance of this conjecture, as discussed in several papers in [16], it seems interesting to obtain probabilistic interpretations of the non arithmetic part of the conjecture. More precisely, the aim of this paper is twofold:

- to give a probabilistic interpretation of the "random matrix factor" $M(\lambda)$, and more generally of the Barnes G-function;
- to understand better the nature of the limit theorem (1.4) and its relations with (generalized) gamma variables.

To this end, we first give a probabilistic translation in Theorem 1.2 of the infinite product (1.1) in terms of a limiting distribution involving gamma variables (we note that, concerning the Gamma function, similar translations

have been presented in [10] and [9]). Let us recall that a gamma variable γ_a with parameter $a > 0$ is distributed as:

$$\mathbb{P}\left(\gamma_a \in \mathrm{d}t\right) = \frac{t^{a-1}\exp\left(-t\right)\mathrm{d}t}{\Gamma\left(a\right)}, \quad t > 0 \tag{1.5}$$

and has Laplace transform

$$\mathbb{E}\left[\exp\left(-\lambda\gamma_a\right)\right] = \frac{1}{\left(1+\lambda\right)^a}, \quad \mathfrak{Re}(\lambda) > -1 \tag{1.6}$$

and Mellin transform:

$$\mathbb{E}\left[\left(\gamma_a\right)^s\right] = \frac{\Gamma\left(a+s\right)}{\Gamma\left(a\right)}, \quad \mathfrak{Re}(s) > -a \tag{1.7}$$

Theorem 1.2. *If $(\gamma_n)_{n \geq 1}$ are independent gamma random variables with respective parameters n, then for z such that $\mathfrak{Re}(z) > -1$:*

$$\lim_{N \to \infty} \frac{1}{N^{\frac{z^2}{2}}} \mathbb{E}\left[\exp\left(-z\left(\sum_{n=1}^{N}\left(\frac{\gamma_n}{n}\right) - N\right)\right)\right] = \left(A^z \exp\left(\frac{z^2}{2}\right) G\left(1+z\right)\right)^{-1} \tag{1.8}$$

where

$$A = \sqrt{\frac{\mathrm{e}}{2\pi}}. \tag{1.9}$$

The next theorem gives an identity in law for the characteristic polynomial which shall lead to a probabilistic interpretation of the "random matrix factor":

Theorem 1.3. *Let Λ denote the generic matrix of $U\left(N\right)$, the set of unitary matrices, fitted with the Haar probability measure, and $Z_N\left(\Lambda\right) = \det\left(I - \Lambda\right)$. Then the following hold:*

(1) *For $\mathfrak{Re}(t) > -1$, we have:*

$$\mathbb{E}\left[\left|Z_N\left(\Lambda\right)\right|^t\right] = \prod_{j=1}^{N}\frac{\Gamma\left(j\right)\Gamma\left(j+t\right)}{\left(\Gamma\left(j+\frac{t}{2}\right)\right)^2} \tag{1.10}$$

(2) *Equivalently, in probabilistic terms:*

$$\prod_{j=1}^{N}\gamma_j \stackrel{law}{=} \left|Z_N\left(\Lambda\right)\right|\prod_{j=1}^{N}\sqrt{\gamma_j\gamma_j'} \tag{1.11}$$

where all variables in sight are assumed to be independent, and γ_j, γ_j' are gamma random variables with parameter j.

The Barnes G-function now comes into the picture via the following limit results:

Theorem 1.4. *Let $(\gamma_n)_{n \geq 1}$ be independent gamma random variables with respective parameters n; then the following hold:*

(1) *for any λ, with $\mathfrak{Re}(\lambda) > -1$, we have:*

$$\lim_{N\to\infty} \frac{1}{N^{\lambda^2/2}} \mathbb{E}\left[\left(\prod_{j=1}^{N} \gamma_j\right)^\lambda\right] \exp\left(-\lambda \sum_{j=1}^{N} \psi(j)\right) = \left(A^\lambda G(1+\lambda)\right)^{-1}$$

(1.12)

where $A = \sqrt{\dfrac{\mathrm{e}}{2\pi}}$ and $\psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$.

(2) *consequently, from (1.12), together with (1.11), we recover the limit theorem (1.4) of Keating and Snaith: for $\mathfrak{Re}(\lambda) > -1$:*

$$\lim_{N\to\infty} \frac{1}{N^{\lambda^2}} \mathbb{E}_N\left[|Z|^{2\lambda}\right] = \frac{(G(1+\lambda))^2}{G(1+2\lambda)}$$

We then naturally extend Theorem 1.2 to the more general case of sums of the form:

$$S_N = \sum_{n=1}^{N} (Y_n - \mathbb{E}[Y_n])$$

(1.13)

where

$$Y_n = \frac{1}{n}\left(y_1^{(n)} + \ldots + y_n^{(n)}\right),$$

and where $\left(y_i^{(n)}\right)_{1\le i\le n<\infty}$ are independent, with the same distribution as a given random variable Y, where Y is a generalized gamma convolution variable (in short GGC), that is an infinitely divisible $\mathbb{R}_+$-valued random variable whose Lévy measure is of the form

$$\nu(\mathrm{d}x) = \left(\frac{\mathrm{d}x}{x}\right)\left(\int \mu(\mathrm{d}\xi) \exp(-x\xi)\right),$$

(1.14)

where $\mu(\mathrm{d}\xi)$ is a Radon measure on $\mathbb{R}_+$, called the Thorin measure associated to Y. We shall further assume that

$$\int \mu(\mathrm{d}\xi) \frac{1}{\xi^2} < \infty,$$

which, as we shall see is equivalent to the existence of a second moment for Y.

The GGC variables have been studied by Thorin [19] and Bondesson [6], see, e.g., [12] for a recent survey of this topic.

Theorem 1.5. *Let Y be a GGC variable, and let (S_N) as in (1.13). We note*

$$\sigma^2 = \mathbb{E}\left[Y^2\right] - (\mathbb{E}[Y])^2.$$

Then the following limit theorem for (S_N) holds: if $\lambda > 0$,

$$\frac{1}{N^{\frac{\lambda^2\sigma^2}{2}}} \mathbb{E}\left[\exp(-\lambda S_N)\right] \xrightarrow[N\to\infty]{} \mathcal{H}(\lambda),$$

(1.15)

where the function $\mathcal{H}(\lambda)$ is given by:

$$\mathcal{H}(\lambda) = \exp\left\{\frac{\lambda^2\sigma^2}{2}(1+\gamma) + \int_0^\infty \frac{dy}{y}\Sigma_\mu(y)\left(\exp(-\lambda y) - 1 + \lambda y - \frac{\lambda^2 y^2}{2}\right)\right\},$$

$$(1.16)$$

with

$$\Sigma_\mu(y) = \int \mu(d\xi) \frac{1}{\left(2\sinh\left(\frac{\xi y}{2}\right)\right)^2}.$$

$$(1.17)$$

Limit results such as (1.15) are not standard in Probability theory, and we intend to develop a systematic study in a forthcoming paper [11]. The rest of the paper is organized as follows:

- in Section 2, we prove Theorems 1.2, 1.3 and 1.4. We also give an interpretation of Theorem 1.2 in terms of Bessel processes as well as an important Lévy-Khintchine type representation for $1/G(1+z)$.
- in Section 3, we compare Theorems 1.2 and 1.4, which help us find adequate extensions for each of them (in particular we shall introduce there generalized gamma convolution variables and prove Theorem 1.5);

2. Proofs of Theorems 1.2, 1.3 and 1.4 and additional probabilistic aspects of the Barnes G-function

2.1. **Proof of Theorem 1.2 and interpretation in terms of Bessel processes.**

2.1.1. *Proof of Theorem 1.2.* To prove Theorem 1.2, we simply use the fact that

$$\mathbb{E}\left[\exp\left(-z\frac{\gamma_n}{n}\right)\right] = \frac{1}{\left(1+\frac{z}{n}\right)^n}.$$

Hence, for $z \geq 0$, the quantity

$$\mathbb{E}\left[\exp\left(-z\left(\sum_{n=1}^N \left(\frac{\gamma_n}{n}\right) - N\right)\right)\right]$$

equals

$$\frac{1}{\prod_{n=1}^N \left(1+\frac{z}{n}\right)^n \exp(-z)} \equiv \frac{1}{D_N}.$$

We then write

$$D_N \exp\left(\frac{z^2}{2}\log N\right) = D_N \exp\left(\frac{z^2}{2}\sum_{n=1}^N \frac{1}{n}\right)\exp\left(\frac{z^2}{2}\left(\log N - \sum_{n=1}^N \frac{1}{n}\right)\right)$$

which from (1.1) converges, as $N \to \infty$, towards

$$G(1+z)(2\pi)^{-z/2}\exp\left(\frac{z+z^2}{2}\right),$$

which proves formula (1.8), for $z \geq 0$, and thus for $\mathfrak{Re}(z) > -1$ by analytic continuation.

2.1.2. *An interpretation of Theorem 1.2 in terms of Bessel processes.* Let $(R_{2n}(t))_{t \geq 0}$ denote a BES($2n$) process, starting from 0, with dimension $2n$; we need to consider the sequence $(R_{2n})_{n=1,2,\ldots}$ of such independent processes. It is well known (see [17] for example) that:

$$\text{for fixed } t > 0, \quad R_{2n}^2(t) \;\overset{\text{law}}{=}\; (2t)\,\gamma_n.$$

Moreover, we have:

$$\text{for fixed } t > 0, \quad R_{2n}^2(t) = 2\int_0^t \sqrt{R_{2n}^2(s)}\mathrm{d}\beta_s^{(n)} + 2nt$$

the stochastic differential equation of R_{2n}^2, driven by a Brownian Motion $\beta^{(n)}$. Thus

$$\sum_{n=1}^N \frac{R_{2n}^2(t)}{2n} \;\overset{\text{law}}{=}\; t\left(\sum_{n=1}^N \frac{\gamma_n}{n}\right).$$

We now write Theorem 1.2, for $\mathfrak{Re}(z) > -1/t$ as:

$$\lim_{N\to\infty} \frac{1}{N^{\frac{t^2 z^2}{2}}}\mathbb{E}\left[\exp\left(-z\sum_{n=1}^N \frac{R_{2n}^2(t)-2nt}{2n}\right)\right] = \left(A^{tz}\exp\left(\frac{t^2 z^2}{2}\right)G(1+tz)\right)^{-1} \tag{2.1}$$

where

$$A = \sqrt{\frac{\mathrm{e}}{2\pi}}.$$

We now wish to write the LHS of (2.1) in terms of functional of a sum of squared Orstein-Uhlenbeck processes; indeed, if we write:

$$\sum_{n=1}^N \frac{R_{2n}^2(t)-2nt}{2n} = 2\sum_{n=1}^N \int_0^t \frac{1}{2n}\sqrt{R_{2n}^2(s)}\mathrm{d}\beta_s^{(n)},$$

the RHS appears as a martingale in t, with increasing process

$$\sum_{n=1}^N \int_0^t \frac{1}{n^2}R_{2n}^2(s)\,\mathrm{d}s,$$

and we obtain that (2.1) may be written as:

$$\lim_{N\to\infty} \frac{1}{N^{\frac{t^2 z^2}{2}}}\mathbb{E}^{(z)}\left[\exp\left(-\frac{z}{2}\sum_{n=1}^N \int_0^t \frac{1}{n^2}R_{2n}^2(s)\,\mathrm{d}s\right)\right] = \left(A^{tz}\exp\left(\frac{t^2 z^2}{2}\right)G(1+tz)\right)^{-1}$$

where, under $\mathbb{P}^{(z)}$ the process $(R_{2n}(t))_{t\geq 0}$ satisfies (by Girsanov's theorem)

$$
\begin{aligned}
R_{2n}^2(t) &= 2\int_0^t \sqrt{R_{2n}^2(s)}\left(\mathrm{d}\widetilde{\beta}_s^{(n)} - \frac{z}{n}\sqrt{R_{2n}^2(s)}\mathrm{d}s\right) + 2nt \\
&= 2\int_0^t \sqrt{R_{2n}^2(s)}\mathrm{d}\widetilde{\beta}_s^{(n)} - \frac{2z}{n}\int_0^t R_{2n}^2(s)\,\mathrm{d}s + 2nt
\end{aligned}
$$

That is, under $\mathbb{P}^{(z)}$, $\left(R_{2n}^2(t)\right)_{t\geq 0}$ now appears as the square of a one dimensional Ornstein-Uhlenbeck process, with parameter $\left(-\dfrac{z}{n}\right)$.

2.2. **Proof of Theorem 1.3.** Formula (1.10) is due to Keating-Snaith [14]. Formula (1.11) follows from (1.10) once one recalls formula (1.7).

2.3. **The characteristic polynomial and beta variables.** One can use the beta-gamma algebra (see, e.g., Chaumont-Yor [8], p.93-98, and the references therein) and (1.11) to represent (in law) the characteristic polynomial as products of beta variables. More precisely,

Theorem 2.1. *With the notations of Theorem 1.3, we have:*

$$
|Z_N| \overset{law}{=} 2^N \left(\prod_{j=1}^N \sqrt{\beta_{\frac{j}{2},\frac{j}{2}}}\right)\left(\prod_{j=2}^N \sqrt{\beta_{\frac{j+1}{2},\frac{j-1}{2}}}\right) \tag{2.2}
$$

where all the variables in sight are assumed to be independent beta variables.

Proof. To deduce the result stated in the theorem from (1.11), we use the following factorization:

$$
\gamma_j \overset{law}{=} \sqrt{\gamma_j\gamma_j'}\xi_j, \tag{2.3}
$$

where on the RHS, we have:

$$
\text{for} \quad j=1 \qquad \xi_1 \overset{law}{=} 2\sqrt{\beta_{\frac{1}{2},\frac{1}{2}}}
$$

$$
\text{for} \quad j>1 \qquad \xi_j \overset{law}{=} 2\sqrt{\left(\beta_{\frac{j}{2},\frac{j}{2}}\right)\left(\beta_{\frac{j+1}{2},\frac{j-1}{2}}\right)}.
$$

Indeed, starting from (2.3), then multiplying both sides by $\sqrt{\gamma_j\gamma_j'}$ and using the beta-gamma algebra, we obtain that (2.3) is equivalent to:

$$
\gamma_j \overset{law}{=} 2\sqrt{\gamma_{\frac{j}{2}}\gamma_{\frac{j+1}{2}}'}
$$

which easily follows from the duplication formula for the gamma function (again, see, e.g., Chaumont-Yor [8], p.93-98). $\qquad\square$

Remark 2.2. In the above Theorem, the factor 2^N can be explained by the fact that the characteristic polynomial of a unitary matrix is, in modulus, smaller than 2^N. Hence the products of beta variables appearing on the RHS of the formula (2.2) measures how the modulus of the characteristic polynomial deviates from the largest value it can take.

2.4. **A Lévy-Khintchine type representation of** $1/G(1+z)$. In this subsection, we give a Lévy-Khintchine representation type formula for $1/G(1+z)$ which will be used next to prove Theorem 1.4.

Proposition 2.3. *For any $z \in \mathbb{C}$, such that $\mathfrak{Re}(z) > -1$, one has:*

$$\frac{1}{G(1+z)} = \exp\left\{-\frac{1}{2}(\log(2\pi)-1)z + (1+\gamma)\frac{z^2}{2} + \int_0^\infty \frac{\mathrm{d}u\left(\exp(-zu) - 1 + zu - \frac{u^2 z^2}{2}\right)}{u\left(2\sinh(u/2)\right)^2}\right\}$$

$$(2.4)$$

Before proving Proposition 2.3, a few remarks are in order.

Remark 2.4. Note that formula (2.4) cannot be considered exactly as a Lévy-Khintchine representation. Indeed, the integral featured in (2.4) consists in integrating the function $\left(\exp(-zu) - 1 + zu - \frac{u^2 z^2}{2}\right)$ against the measure $\dfrac{\mathrm{d}u}{u\left(2\sinh(u/2)\right)^2}$, which is not a Lévy measure. Indeed, Lévy measures integrate $(u^2 \wedge 1)$, which is not the case here because of the equivalence: $u\left(2\sinh(u/2)\right)^2 \sim u^3$, when $u \to 0$. Also, due to this singularity, one cannot integrate $\left(\exp(-zu) - 1 + zu\mathbf{1}_{(u\leq 1)}\right)$ with respect to this measure, and one is forced to "bring in" under the integral sign the companion term $\frac{u^2 z^2}{2}$.

Proof. From the consideration of the series

$$\mathcal{L}(z) := \sum_{n=3}^\infty (-1)^{n-1}\zeta(n-1)\frac{z^n}{n} \tag{2.5}$$

featured in (1.2), it seems natural to introduce a random variable Q taking values in $\mathbb{R}_+$, with Mellin transform:

$$\mathbb{E}[Q^s] = \left(\frac{\mathbb{E}\left[\mathbf{e}^{s+2}\right]}{2}\right)\left(\frac{\zeta(s+2)}{\zeta(2)}\right) \tag{2.6}$$

where $\mathbf{e}$ denotes a standard exponential variable. A little more analytically, formula (2.6) may be presented as

$$\mathbb{E}[Q^s] = \frac{3}{\pi^2}\Gamma(s+3)\zeta(s+2). \tag{2.7}$$

We first show the existence of the random variable Q by computing its density (for an example of occurrence of the random variable Q in the theory of stochastic processes and some relation with the theory of the Riemann Zeta function, see e.g. [5] and [4]).

From the definition of Q via its Mellin transform, we get for $f : \mathbb{R}_+ \to \mathbb{R}_+$:

$$\mathbb{E}[f(Q)] = \frac{3}{\pi^2}\sum_{n=1}^\infty \left(\frac{1}{n^2}\int_0^\infty \mathrm{d}t\, t^2 \exp(-t) f\left(\frac{t}{n}\right)\right)$$

Hence by some elementary change of variables:

$$\mathbb{P}\left(Q \in \mathrm{d}u\right) = \frac{3}{\pi^2} u^2 \left(\sum_{n=1}^{\infty} n \exp\left(-nu\right)\right) \mathrm{d}u.$$

Now, since $\sum_{n=1}^{\infty} \exp\left(-nu\right) = \dfrac{1}{\exp\left(u\right) - 1}$, we get:

$\sum_{n=1}^{\infty} n \exp\left(-nu\right) = \dfrac{1}{\left(2 \sinh\left(\frac{u}{2}\right)\right)^2}$. Hence,

$$\mathbb{P}\left(Q \in \mathrm{d}u\right) = \frac{3}{\pi^2} \left(\frac{u/2}{\sinh\left(\frac{u}{2}\right)}\right)^2 \mathrm{d}u \tag{2.8}$$

Now, we consider the following series development for $|z| < 1$:

$$\mathbb{E}\left[\frac{1}{Q^3}\left(\sum_{n=1}^{\infty} \frac{(-zQ)^n}{n!}\right)\right] = \frac{3}{\pi^2} \sum_{n=3}^{\infty} (-1)^n \zeta\left(n-1\right) \frac{z^n}{n},$$

hence, in comparison with formula (2.5) we obtain:

$$
\begin{aligned}
\mathcal{L}\left(z\right) &= -\frac{\pi^2}{3} \mathbb{E}\left[\frac{1}{Q^3}\left(\sum_{n=1}^{\infty} \frac{(-zQ)^n}{n!}\right)\right] \\
&= -\frac{\pi^2}{3} \mathbb{E}\left[\frac{1}{Q^3}\left(\exp\left(-zQ\right) - 1 + zQ - \frac{Q^2 z^2}{2}\right)\right].
\end{aligned}
$$

Now, formula (2.4), for $|z| < 1$, follows from (2.8) and the fact that:

$$
\begin{aligned}
\frac{1}{G\left(1+z\right)} &= \exp\left(\log G\left(1+z\right)\right) \\
&= \exp\left\{-\frac{z}{2}\left(\log\left(2\pi\right) - 1\right) + \frac{1}{2}\left(1+\gamma\right)z^2 - \mathcal{L}\left(z\right)\right\}.
\end{aligned}
$$

The formula extends by analytic continuation to the case $\mathfrak{Re}(z) > -1$. $\quad\square$

2.5. **Proof of Theorem 1.4.** To prove Theorem 1.4, we shall use the following lemma:

Lemma 2.5. *For any $a > 0$, the random variable $\log\left(\gamma_a\right)$ is infinitely divisible and its Lévy-Khintchine representation is given, for $\mathfrak{Re}(\lambda) > -a$, by:*

$$
\begin{aligned}
\mathbb{E}\left[\gamma_a^\lambda\right] &= \frac{\Gamma\left(a+\lambda\right)}{\Gamma\left(a\right)} \\
&= \exp\left\{\lambda\psi\left(a\right) + \int_0^{\infty} \frac{\exp\left(-au\right)\left(\exp\left(-\lambda u\right) - 1 + \lambda u\right)}{u\left(1 - \exp\left(-u\right)\right)} \mathrm{d}u\right\}
\end{aligned}
$$

where

$$\psi\left(a\right) \equiv \frac{\Gamma'\left(a\right)}{\Gamma\left(a\right)}.$$

Proof. This is classical: it follows from (1.7) and some integral representation of the ψ-function, see, e.g. Lebedev [15] and Carmona-Petit-Yor [7] where this lemma is also used.   □

We are now in a position to prove Theorem 1.4. We start by proving the first part, i.e. formula (1.12). Let us write

$$I_N(\lambda) := \frac{1}{N^{\lambda^2/2}} \mathbb{E}\left[\left(\prod_{j=1}^{N} \gamma_j\right)^{\lambda}\right].$$

Thus with the help of Lemma 2.5, we obtain:

$$N^{\lambda^2/2} I_N(\lambda) = \exp\left\{\lambda \sum_{j=1}^{N} \psi(j) + \int_0^{\infty} \frac{\sum_{j=1}^{N} \exp(-ju)}{u(1-\exp(-u))}(\exp(-\lambda u) - 1 + \lambda u)\,du\right\}$$
(2.9)

Note that:

$$\frac{1}{u(1-\exp(-u))}\sum_{j=1}^{N}\exp(-ju) = \frac{\exp(-u)(1-\exp(-Nu))}{u(1-\exp(-u))^2}$$

and we may now write:

$$I_N(\lambda) = \exp\left\{\lambda \sum_{j=1}^{N} \psi(j) + J_N(\lambda)\right\}$$
(2.10)

where

$$J_N(\lambda) = \left\{\int_0^{\infty} \frac{du\,\exp(-u)}{u(1-\exp(-u))^2}(1-\exp(-Nu))(\exp(-\lambda u)-1+\lambda u)\right\} - \frac{\lambda^2}{2}\log N$$
(2.11)

Next, we shall show that:

$$J_N(\lambda) \xrightarrow[N\to\infty]{} J_{\infty}(\lambda)$$

together with some integral expression for $J_{\infty}(\lambda)$, from which it will be easily deduced how $J_{\infty}(\lambda)$ and $G(1+\lambda)$ are related thanks to Proposition 2.3.

We now write (2.11) in the form:

$$J_N(\lambda) = \int_0^{\infty} \frac{du\,\exp(-u)}{u(1-\exp(-u))^2}(1-\exp(-Nu))\left(\exp(-\lambda u)-1+\lambda u-\frac{\lambda^2 u^2}{2}\right)$$
$$+ \frac{\lambda^2}{2}\left(\int_0^{\infty} \frac{du\,u\,\exp(-u)}{(1-\exp(-u))^2}(1-\exp(-Nu)) - \log N\right)$$
(2.12)

Now letting $N \to \infty$, we obtain:

$$J_N(\lambda) \to J_{\infty}(\lambda),$$
(2.13)

with

$$J_\infty(\lambda) = \int_0^\infty \frac{du \exp(-u)}{u(1-\exp(-u))^2} \left(\exp(-\lambda u) - 1 + \lambda u - \frac{\lambda^2 u^2}{2}\right) + \frac{\lambda^2}{2}C$$

$$\equiv \widetilde{J}_\infty(\lambda) + \frac{\lambda^2}{2}C$$

where

$$C = \lim_{N\to\infty} \left(\int_0^\infty \frac{du\, u \exp(-u)}{(1-\exp(-u))^2}(1-\exp(-Nu)) - \log N\right),$$

a limit which we shall show to exist and identify to be $1 + \gamma (= C)$ with the following lemma:

Lemma 2.6. *We have:*

$$\int_0^\infty \frac{u \exp(-u)}{(1-\exp(-u))^2}(1-\exp(-Nu))\, du = \log N + (1+\gamma) + o(1).$$

Consequently, we have:

$$C = 1 + \gamma.$$

Proof.

$$\int_0^\infty \frac{u \exp(-u)}{(1-\exp(-u))^2}(1-\exp(-Nu))\, du = \int_0^\infty \frac{u \exp(-u)}{(1-\exp(-u))} \left(\sum_{k=0}^{N-1} \exp(-ku)\right) du$$

$$= \sum_{k=1}^N \int_0^\infty \frac{u}{(1-\exp(-u))} \exp(-ku)\, du = \sum_{k=1}^N \int_0^\infty du\, \exp(-ku)\, u \sum_{r=0}^\infty \exp(-ru)$$

$$= \sum_{k=1}^N \sum_{r=0}^\infty \int_0^\infty du\, u \exp(-(r+k)u) = \sum_{k=1}^N \sum_{r=0}^\infty \frac{1}{(r+k)^2}$$

Now, we write:

$$\sum_{k=1}^N \sum_{r=0}^\infty \frac{1}{(r+k)^2} = \zeta(2) + \sum_{k=1}^{N-1} \left(\zeta(2) - \sum_{s=1}^k \frac{1}{s^2}\right)$$

$$= N\zeta(2) - \sum_{k=1}^{N-1} \sum_{s=1}^k \frac{1}{s^2}$$

$$= N\zeta(2) - \sum_{s=1}^N \frac{N-s}{s^2}$$

$$= N \sum_{s=N+1}^\infty \frac{1}{s^2} + \frac{1}{N} + \sum_{s=1}^N \frac{1}{s} - \frac{1}{N}$$

The result now follows easily from the facts:

$$\lim_{N \to \infty} N \sum_{s=N+1}^{\infty} \frac{1}{s^2} = 1$$

$$\sum_{s=1}^{N} \frac{1}{s} = \log N + \gamma + o(1)$$

$\square$

We have thus proved so far that:

$$\lim_{N \to \infty} \frac{1}{N^{\lambda^2/2}} \mathbb{E}\left[\left(\prod_{j=1}^{N} \gamma_j\right)^{\lambda}\right] \exp\left(-\lambda \sum_{j=1}^{N} \psi(j)\right) = \exp\left(J_{\infty}(\lambda)\right), \quad (2.14)$$

where

$$J_{\infty}(\lambda) = \int_0^{\infty} \frac{\mathrm{d}u \exp(-u)}{u \left(1 - \exp(-u)\right)^2} \left(\exp(-\lambda u) - 1 + \lambda u - \frac{\lambda^2 u^2}{2}\right) + \frac{\lambda^2}{2}(1 + \gamma).$$

We can still rewrite $J_{\infty}(\lambda)$ as:

$$J_{\infty}(\lambda) = \int_0^{\infty} \frac{\mathrm{d}u}{u \left(2 \sinh(u/2)\right)^2} \left(\exp(-\lambda u) - 1 + \lambda u - \frac{\lambda^2 u^2}{2}\right) + \frac{\lambda^2}{2}(1 + \gamma).$$

$$(2.15)$$

Now comparing (2.4) and (2.15), we obtain:

$$\exp\left(J_{\infty}(\lambda)\right) = \left(A^{\lambda} G(1 + \lambda)\right)^{-1} \tag{2.16}$$

Plugging (2.16) in (2.14) yields the first part of Theorem 1.4:

$$\lim_{N \to \infty} \frac{1}{N^{\lambda^2/2}} \mathbb{E}\left[\left(\prod_{j=1}^{N} \gamma_j\right)^{\lambda}\right] \exp\left(-\lambda \sum_{j=1}^{N} \psi(j)\right) = \left(\sqrt{\frac{2\pi}{\mathrm{e}}}\right)^{\lambda} \frac{1}{G(1 + \lambda)}$$

To prove the second part of Theorem 1.4, we use formula (1.11) together with formula (1.12). Formula (1.11) yields:

$$\frac{1}{N^{2\lambda^2}} \mathbb{E}\left[\left(\prod_{j=1}^{N} \gamma_j\right)^{2\lambda}\right] = \left(\frac{1}{N^{\lambda^2}} \mathbb{E}_N\left[|Z_N|^{2\lambda}\right]\right) \left(\left(\frac{1}{N^{\lambda^2/2}} \mathbb{E}\left[\left(\prod_{j=1}^{N} \gamma_j\right)^{\lambda}\right]\right)^2\right)$$

$$(2.17)$$

Multiplying both sides by $\exp\left(-2\lambda \sum_{j=1}^{N} \psi(j)\right)$ and using (1.12) we obtain:

$$\lim_{N \to \infty} \frac{1}{N^{\lambda^2}} \mathbb{E}_N\left[|Z|^{2\lambda}\right] = \frac{(G(1 + \lambda))^2}{G(1 + 2\lambda)},$$

which completes the proof of Theorem 1.4.

3. Generalized Gamma Convolutions

3.1. Definition and examples.
We recall the definition of a GGC variable.

Definition 3.1. A random variable Y is called a GGC variable if it is infinitely divisible with Lévy measure ν of the form:

$$\nu\left(\mathrm{d}x\right) = \frac{\mathrm{d}x}{x} \int \mu\left(\mathrm{d}\xi\right) \exp\left(-\xi x\right), \tag{3.1}$$

where $\mu\left(\mathrm{d}\xi\right)$ is a Radon measure on $\mathbb{R}_+$, called the Thorin measure of Y.

Remark 3.2. Y is a selfdecomposable random variable because its Lévy measure can be written as $\nu\left(\mathrm{d}x\right) = \dfrac{\mathrm{d}x}{x} h\left(x\right)$ with h a decreasing function (see, e.g. [18], p.95).

Remark 3.3. We shall require Y to have finite first and second moments; these moments can be easily computed with the help of the Thorin measure $\mu\left(\mathrm{d}\xi\right)$:

$$\mathbb{E}\left[Y\right] \;=\; \mu_{-1} = \int \mu\left(\mathrm{d}\xi\right) \frac{1}{\xi}$$

$$\sigma^2 = \mathbb{E}\left[Y^2\right] - \left(\mathbb{E}\left[Y\right]\right)^2 \;=\; \mu_{-2} = \int \mu\left(\mathrm{d}\xi\right) \frac{1}{\xi^2}$$

Now we give some examples of GGC variables. Of course, γ_a falls into this category with $\mu\left(\mathrm{d}\xi\right) = a\delta_1\left(\mathrm{d}\xi\right)$ where $\delta_1\left(\mathrm{d}\xi\right)$ is the Dirac measure at 1.

More generally, the next proposition gives a large set of such variables:

Proposition 3.4. *Let f be a nonnegative Borel function such that*

$$\int_0^\infty \mathrm{d}u \log\left(1 + f\left(u\right)\right) < \infty,$$

and let (γ_u) denote the standard gamma process. Then the variable Y defined as

$$Y = \int_0^\infty \mathrm{d}\gamma_u f\left(u\right) \tag{3.2}$$

is a GGC variable.

Proof. It is easily shown, by approximating f by simple functions that

$$
\begin{aligned}
\mathbb{E}\left[\exp\left(-\lambda Y\right)\right] \;&=\; \exp\left(-\int_0^\infty \mathrm{d}u \int_0^\infty \frac{\mathrm{d}x}{x} \exp\left(-x\right)\left(1 - \exp\left(-\lambda f\left(u\right) x\right)\right)\right) \\
&=\; \exp\left(-\int_0^\infty \frac{\mathrm{d}y}{y}\left(\int_0^\infty \mathrm{d}u \exp\left(-\frac{y}{f\left(u\right)}\right)\right)\left(1 - \exp\left(-\lambda y\right)\right)\right)
\end{aligned}
$$

which yields the result. $\qquad\square$

For much more details on GGC variables, see [12].

3.2. **Proof of Theorem 1.5.** We now prove Theorem 1.5, which is a natural extension for Theorem 1.2. Recall that in (1.13), we have defined S_N as:

$$S_N = \sum_{n=1}^{N} (Y_n - \mathbb{E}[Y_n])$$

where

$$Y_n = \frac{1}{n}\left(y_1^{(n)} + \ldots + y_n^{(n)}\right),$$

and where $\left(y_i^{(n)}\right)_{1 \leq i \leq n < \infty}$ are independent, with the same distribution as a given GGC variable Y, which has a second moment. For any $\lambda \geq 0$, we have:

$$\mathbb{E}\left[\exp\left(-\lambda S_N\right)\right] = \prod_{n=1}^{N}\left(\widetilde{\varphi}\left(\frac{\lambda}{n}\right)\right)^n \tag{3.3}$$

where

$$\widetilde{\varphi}\left(\lambda\right) = \mathbb{E}\left[\exp\left(-\lambda\left(Y - \mathbb{E}[Y]\right)\right)\right]. \tag{3.4}$$

Now, using the form (3.1) of the Lévy-Khintchine representation for Y, we obtain:

$$
\begin{aligned}
\prod_{n=1}^{N}\left(\widetilde{\varphi}\left(\frac{\lambda}{n}\right)\right)^n &= \exp\left\{-\sum_{n=1}^{N} n \int \nu\left(\mathrm{d}x\right)\left(1 - \frac{\lambda}{n}x - \exp\left(-\frac{\lambda}{n}x\right)\right)\right\} \\
&= \exp\left(-\int \mu\left(\mathrm{d}\xi\right) I_N\left(\xi, \lambda\right)\right).
\end{aligned} \tag{3.5}
$$

where:

$$
\begin{aligned}
I_N\left(\xi, \lambda\right) &= \sum_{n=1}^{N} n \int_0^{\infty} \frac{\mathrm{d}x}{x}\exp\left(-\xi x\right)\left(1 - \frac{\lambda}{n}x - \exp\left(-\frac{\lambda}{n}x\right)\right) \\
&= \sum_{n=1}^{N} n \int_0^{\infty} \frac{\mathrm{d}y}{y}\exp\left(-n\xi y\right)\left(1 - \lambda y - \exp\left(-\lambda y\right)\right) \\
&= \int_0^{\infty} \frac{\mathrm{d}y}{y}\left(\sum_{n=1}^{N} n\exp\left(-n\xi y\right)\right)\left(1 - \lambda y - \exp\left(-\lambda y\right)\right)
\end{aligned}
$$

Some elementary calculations yield:

$$\sum_{n=1}^{N} n\exp\left(-na\right) = \frac{\exp\left(a\right)\left(1 - \exp\left(-aN\right)\right)}{\left(\exp\left(a\right) - 1\right)^2} - \frac{N\exp\left(-aN\right)}{\left(\exp\left(a\right) - 1\right)}.$$

Consequently, taking $a = \xi y$ in the formula for $I_N\left(\xi, \lambda\right)$, we can write it as:

$$I_N\left(\xi, \lambda\right) = J_N\left(\xi, \lambda\right) - R_N\left(\xi, \lambda\right) \tag{3.6}$$

where

$$J_N\left(\xi,\lambda\right) = \int_0^\infty \frac{\mathrm{d}y}{y}\left(1-\lambda y-\exp\left(-\lambda y\right)\right)\left[\frac{\exp\left(\xi y\right)\left(1-\exp\left(-\xi y N\right)\right)}{\left(\exp\left(\xi y\right)-1\right)^2}\right]$$

(3.7)

and

$$R_N\left(\xi,\lambda\right) = \int_0^\infty \frac{\mathrm{d}y}{y}\left(1-\lambda y-\exp\left(-\lambda y\right)\right)\left[\frac{N\exp\left(-\xi y N\right)}{\left(\exp\left(\xi y\right)-1\right)}\right] \qquad (3.8)$$

It is clear that

$$\lim_{N\to\infty} R_N\left(\xi,\lambda\right) = 0.$$

Now let us study $J_N\left(\xi,\lambda\right)$ when $N\to\infty$. Let us "bring in" the additional term $\dfrac{\lambda^2 y^2}{2}$; more precisely, we rewrite $J_N\left(\xi,\lambda\right)$ as:

$$J_N\left(\xi,\lambda\right) = \int_0^\infty \frac{\mathrm{d}y}{y}\left(1-\lambda y+\frac{\lambda^2 y^2}{2}-\exp\left(-\lambda y\right)\right)\left[\frac{\exp\left(\xi y\right)\left(1-\exp\left(-\xi y N\right)\right)}{\left(\exp\left(\xi y\right)-1\right)^2}\right]$$
$$-\frac{\lambda^2}{2}\int \mathrm{d}y\, y\frac{\exp\left(\xi y\right)\left(1-\exp\left(-\xi y N\right)\right)}{\left(\exp\left(\xi y\right)-1\right)^2}$$

(3.9)

Hence:

$$J_N\left(\xi,\lambda\right) = o(1)+\int_0^\infty \frac{\mathrm{d}y}{y}\frac{1}{\left(2\sinh\left(\xi y/2\right)\right)^2}\left(1-\lambda y+\frac{\lambda^2 y^2}{2}-\exp\left(-\lambda y\right)\right)-\frac{\lambda^2}{2}K_N\left(\xi\right),$$

(3.10)

where

$$
\begin{aligned}
K_N\left(\xi\right) &= \int \mathrm{d}y\, y\frac{\exp\left(\xi y\right)\left(1-\exp\left(-\xi y N\right)\right)}{\left(\exp\left(\xi y\right)-1\right)^2}\\
&= \frac{1}{\xi^2}\int_0^\infty \frac{u\exp\left(-u\right)}{\left(1-\exp\left(-u\right)\right)^2}\left(1-\exp\left(-N u\right)\right)du.
\end{aligned}
$$

Moreover, from Lemma 2.6, we have:

$$K_N\left(\xi\right) = \frac{1}{\xi^2}\left(\log N+1+\gamma+o(1)\right). \qquad (3.11)$$

Now using the above asymptotics, and integrating with respect to $\mu\left(\mathrm{d}\xi\right)$ (see (3.5)), we obtain:

$$\prod_{n=1}^N \left(\widetilde{\varphi}\left(\frac{\lambda}{n}\right)\right)^n =$$
$$\exp\left\{o(1)+\frac{\lambda^2}{2}\mu_{-2}\left(\log N+1+\gamma\right)+\int_0^\infty \frac{\mathrm{d}y}{y}\Sigma_\mu\left(y\right)\left(1-\lambda y+\frac{\lambda^2 y^2}{2}-\exp\left(-\lambda y\right)\right)\right\}$$

(3.12)

Hence we have finally proved that if $\lambda > 0$,

$$\frac{1}{N^{\frac{\lambda^2\sigma^2}{2}}}\mathbb{E}\left[\exp\left(-\lambda S_N\right)\right] \xrightarrow[N\to\infty]{} \mathcal{H}\left(\lambda\right), \tag{3.13}$$

where the function $\mathcal{H}\left(\lambda\right)$ is given by:

$$\mathcal{H}\left(\lambda\right) = \exp\left\{\frac{\lambda^2\sigma^2}{2}\left(1+\gamma\right) + \int_0^\infty \frac{\mathrm{d}y}{y}\Sigma_\mu\left(y\right)\left(\exp\left(-\lambda y\right) - 1 + \lambda y - \frac{\lambda^2 y^2}{2}\right)\right\}, \tag{3.14}$$

with

$$\Sigma_\mu\left(y\right) = \int \mu\left(\mathrm{d}\xi\right) \frac{1}{\left(2\sinh\left(\frac{\xi y}{2}\right)\right)^2}, \tag{3.15}$$

which is Theorem 1.5.

References

[1] V.S. Adamchik: *Contribution to the theory of the Barnes function*, preprint (2001).

[2] G.E. Andrews, R. Askey, R. Roy: *Special functions*, Cambridge University Press (1999).

[3] E.W. Barnes: *The theory of the G-function*, Quart. J. Pure Appl. Math. **31** (1899), 264-314.

[4] P. Biane, J. Pitman, M. Yor, *Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions* , Bull. Am. Math. Soc., New Ser., **38** No.4, 435-465 (2001).

[5] P. Biane, M. Yor, *Valeurs principales associées aux temps locaux Browniens* , Bull. Sci. Maths., 2e série, **111**, (1987), 23-101.

[6] L. Bondesson: *Generalized gamma convolutions and related classes of distributions and densities*, Springer (1992).

[7] P. Carmona, F. Petit, M. Yor: *On the distribution and asymptotic results for exponential functionals of Lévy processes* , in Exponential functionals and principal values associated related to Brownian Motion, Ed. M. Yor, Revista Matematica Iberoamericana (1997).

[8] L. Chaumont, M. Yor: *Exercises in Probability : a guided tour from measure theory to random processes, via conditioning*, Cambridge University Press (2003).

[9] A. Fuchs, G. Letta: *Un rsultat élémentaire de fiabilité. Application la formule de Weierstrass sur la fonction gamma* , Sém.Proba. XXV, Lecture Notes in Mathematics **1485**, 316-323 (1991).

[10] L. Gordon: *A stochastic Approach to the Gamma Function*, Amer. Math. Monthly 101 , p. 858-865 (1994).

[11] J. Jacod, A. Nikeghbali and M. Yor: in preparation.

[12] L. James, B. Roynette and M. Yor: *Generalized Gamma Convolutions, Dirichlet means, Thorin measures: some new results and explicit examples*, preprint available on the ArXiV.

[13] J.P. Keating: *L-functions and the characteristic polynomials of random matrices*, Recent Perspectives in Random Matrix Theory and Number Theory, eds. F. Mezzadri and N.C. Snaith, London Mathematical Society Lecture Note Series **322** (CUP), (2005) 251-278.

[14] J.P. Keating, N.C. Snaith: *Random matrix theory and $\zeta(1/2+it)$*, Commun. Math. Phys. **214**, (2000), 57-89.

[15] N. N. Lebedev: *Special functions and their applications*, Dover Publications (1972).

[16] F. Mezzadri, N.C. Snaith (editors): *Recent Perspectives in Random Matrix Theory and Number Theory*, London Mathematical Society Lecture Note Series **322** (CUP), (2005).

[17] D. Revuz, M. Yor: *Continuous martingales and Brownian motion*, Springer. Third edition (1999).

[18] K. Sato: *Lévy processes and infinitely divisible distributions*, Cambridge University Press **68**, (1999).

[19] O. Thorin: *On the infinite divisibility of the lognormal distributions*, Scand. Actuarial J., (1977), 121-148.

[20] A. Voros: *Spectral functions, special functions and the Selberg Zeta function*, Commun. Math. Phys. **110**, (1987), 439-465.

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

E-mail address: ashkan.nikeghbali@math.uzh.ch

Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, et CNRS UMR 7599, 175 rue du Chevaleret F-75013 Paris, France.