TRACE DECATEGORIFICATION OF CATEGORIFIED QUANTUM $\mathfrak{sl}_2$

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Abstract. The trace or the 0th Hochschild–Mitchell homology of a linear category $\mathcal{C}$ may be regarded as a kind of decategorification of $\mathcal{C}$. We compute the traces of the two versions $\dot{\mathcal{U}}$ and $\mathcal{U}^*$ of categorified quantum $\mathfrak{sl}_2$ introduced by the third author. The trace of $\mathcal{U}$ is isomorphic to the split Grothendieck group $K_0(\dot{\mathcal{U}})$, and the higher Hochschild–Mitchell homology of $\dot{\mathcal{U}}$ is zero. The trace of $\mathcal{U}^*$ is isomorphic to the idempotented integral form of the current algebra $\mathbf{U}(\mathfrak{sl}_2[t])$.

1. Introduction

Categorification is a process of transforming a set-like structure into a category-like structure, which is inverse to decategorification, usually defined as a functor from category-like structures to set-like structures. For example, the functor from the category of small categories to the category of sets, mapping a category $\mathcal{C}$ to the set $\mathrm{Ob}(\mathcal{C})/\cong$ of isomorphism classes of objects in $\mathcal{C}$, is considered to be a decategorification. In this example, the categorification problem is to find, for a given set S, a nice category $\mathcal{C}$ and a bijection between $\mathrm{Ob}(\mathcal{C})/\cong$ and S. The notion of niceness depends on the circumstances, and categorification is not a well-defined notion in general. Usually, a decategorification functor maps categories with structures, such as monoidal categories and 2-categories, to sets with structures, such as monoids and categories. The corresponding categorification problem for a monoid M is to find a nice monoidal category $\mathcal{C}$ with the monoid $\mathrm{Ob}(\mathcal{C})/\cong$ isomorphic to M.

Another example of decategorification functor is the Grothendieck group G_0, which is a functor from small abelian categories to abelian groups. This functor is enhanced to a functor from small abelian monoidal categories to associative, unital rings. The categorification problem for a ring R is to find a nice abelian monoidal category $\mathcal{C}$ with a ring isomorphism $G_0(\mathcal{C}) \cong R$. For an additive category $\mathcal{C}$, which is not necessarily abelian, the usual Grothendieck group is not defined, but there is the split Grothendieck group, denoted $K_0(\mathcal{C})$, which is generated by the isomorphism classes of objects in $\mathcal{C}$, with relations $[x \oplus y] = [x] + [y]$ for $x, y \in \mathrm{Ob}(\mathcal{C})$ (see Section 3 for details).

One can apply the split Grothendieck group K_0 to an additive 2-category $\mathbf{C}$ to obtain a linear category $K_0(\mathbf{C})$. (By a *linear category* we mean a category enriched over the category $\mathbf{Ab}$ of abelian groups.) Here, the objects in $K_0(\mathbf{C})$ are the objects in $\mathbf{C}$, and for objects x and y the hom-module $K_0(\mathbf{C})(x, y)$ is defined as the split Grothendieck group $K_0(\mathbf{C}(x, y))$ of the hom-category $\mathbf{C}(x, y)$.

In [8], the third author defined an additive 2-category $\dot{\mathcal{U}}$, which categorifies the Beilinson–Lusztig–MacPherson idempotented integral form ${}_{\mathcal{A}}\dot{\mathbf{U}}$ of the quantum group $\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_2)$ [1]. That $\dot{\mathcal{U}}$ categorifies ${}_{\mathcal{A}}\dot{\mathbf{U}}$ means here that the split Grothendieck group $K_0(\dot{\mathcal{U}})$ is isomorphic to ${}_{\mathcal{A}}\dot{\mathbf{U}}$. For an elementary introduction to the 2-category $\dot{\mathcal{U}}$ see [9].

In this paper, we consider another decategorification functor on linear categories, called the *trace*. The trace of a linear category $\mathcal{C}$ is defined to be the coend of the Hom functor $\mathrm{Hom}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Ab}$,

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i.e., the abelian group

$$\mathrm{Tr}(\mathcal{C}) = \int^{x\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(x,x) = \left(\bigoplus_{x\in\mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(x,x)\right) / \,\mathrm{Span}\{fg-gf\}.$$

where the subgroup $\mathrm{Span}\{fg-gf\}$ is spanned by $fg-gf$ for all pairs $f\colon x\to y$, $g\colon y\to x$ with $x,y\in\mathrm{Ob}(\mathcal{C})$. The trace $\mathrm{Tr}(\mathcal{C})$ is also known as the 0th Hochschild–Mitchell homology $\mathrm{HH}_0(\mathcal{C})$ of $\mathcal{C}$.

Unlike the split Grothendieck group, the trace can be defined for any linear category, not necessarily additive. The trace of a linear category $\mathcal{C}$ is naturally isomorphic to the trace of the additive closure of $\mathcal{C}$. Moreover, the trace $\mathrm{Tr}(\mathrm{Kar}(\mathcal{C}))$ of the Karoubi envelope $\mathrm{Kar}(\mathcal{C})$ is isomorphic to $\mathrm{Tr}(\mathcal{C})$, see Section 3.

Similarly to K_0, the trace Tr can be applied to linear 2-categories. The trace $\mathrm{Tr}(\mathbf{C})$ of a linear 2-category is a linear category such that $\mathrm{Ob}(\mathrm{Tr}(\mathbf{C}))=\mathrm{Ob}(\mathbf{C})$, and

$$\mathrm{Tr}(\mathbf{C})(x,y) = \mathrm{Tr}(\mathbf{C}(x,y))$$

for $x,y\in\mathrm{Ob}(\mathcal{C})$.

For an additive category $\mathcal{C}$, the trace $\mathrm{Tr}(\mathcal{C})$ and the split Grothendieck group $K_0(\mathcal{C})$ are related by the linear map

$$h_{\mathcal{C}}\colon\ K_0(\mathcal{C})\to\mathrm{Tr}(\mathcal{C})$$

which maps the equivalence class $[X]$ of $X\in\mathcal{C}$ to the equivalence class $[1_X]$ of the identity morphism $1_X\colon\ X\to X$.

For an additive 2-category $\mathbf{C}$, we have a linear functor

$$h_{\mathbf{C}}\colon\ K_0(\mathbf{C})\to\mathrm{Tr}(\mathbf{C}),$$

which maps each object to itself and maps morphisms by

$$h_{\mathbf{C}(x,y)}\colon\ K_0(\mathbf{C}(x,y))\to\mathrm{Tr}(\mathbf{C}(x,y)).$$

In this paper we compute the traces of the additive 2-categories $\dot{\mathcal{U}}$ and $\mathcal{U}^*$, where $\mathcal{U}^*$ is another version of the categorified quantum $\mathfrak{sl}_2$, introduced in [8]. Here we recall that the objects of $\dot{\mathcal{U}}$ (and, in fact, of $\mathcal{U}^*$) are the integers, which are regarded as the elements of the weight lattice of $\mathfrak{sl}_2$. For more details about $\dot{\mathcal{U}}$, see Section 5.

The first main result of this paper is the following.

Theorem 1.1. The linear functor

$$h_{\dot{\mathcal{U}}}\colon\ K_0(\dot{\mathcal{U}}) \longrightarrow \mathrm{Tr}(\dot{\mathcal{U}}) = \mathrm{HH}_0(\dot{\mathcal{U}})$$

is an isomorphism. Hence $\mathrm{Tr}(\dot{\mathcal{U}})$ is isomorphic to ${}_{\mathcal{A}}\dot{\mathbf{U}}$. Moreover, for $i>0$, and $n,m\in\mathbb{Z}$, the Hochschild–Mitchell homology group $\mathrm{HH}_i(\dot{\mathcal{U}}(n,m))$ of $\dot{\mathcal{U}}(n,m)$ is zero.

Now we consider the trace of the additive 2-category $\mathcal{U}^*$.

The Lie algebra $\mathfrak{sl}_2[t]=\mathfrak{sl}_2\otimes\mathbb{Q}[t]$ is the non-negative part of the loop Lie algebra $\mathfrak{sl}_2[t,t^{-1}]$. The universal enveloping algebra $\mathbf{U}(\mathfrak{sl}_2[t])$ of $\mathfrak{sl}_2[t]$ is called the *current algebra* of $\mathfrak{sl}_2$. Thus, $\mathbf{U}(\mathfrak{sl}_2[t])$ is the $\mathbb{Q}$-algebra generated by E_i, F_i and H_i for $i\geq 0$, where $X_i = X\otimes t^i$, subject to the following relations:

$$[H_i,H_j]=[E_i,E_j]=[F_i,F_j]=0,$$
$$[H_i,E_j]=2E_{i+j},\quad [H_i,F_j]=-2F_{i+j},\quad [E_i,F_j]=H_{i+j}.$$

An integral basis of the loop algebra $\mathbf{U}(\mathfrak{s}l_2[t,t^{-1}])$ was constructed in [6]. The following is our second main result.

Theorem 1.2. As a linear category, $\mathrm{Tr}(\mathcal{U}^*)$ is isomorphic to the idempotented integral form ${}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t])$ of the current algebra $\mathbf{U}(\mathfrak{sl}_2[t])$.

For the definition of ${}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t])$, see Section 8.

We conclude by summarizing some advantages of Tr compared with K_0:

- The trace is defined for linear categories, not only for additive categories.
- The trace is sometimes richer than K_0.
- The trace of the category is isomorphic to the trace of its Karoubi envelope. This property does not hold for the split Grothendieck group in general.

The paper is organized as follows. After recalling general facts about traces for (linear) categories and bicategories in the first two sections, we introduce (strongly) upper-triangular linear categories and compute all Hochschild–Mitchell homology groups for them. Then we recall the definitions of the additive 2-categories $\mathcal{U}$, $\dot{\mathcal{U}}$ and $\mathcal{U}^*$ and show that $\dot{\mathcal{U}}$ is strongly upper-triangular, yielding the proof of Theorem 1.1. In Section 7, we give an explicit presentation of the categories $\mathcal{U}^*(m,n)$ by generators and relations. Finally, we provide the proof of Theorem 1.2.

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2. Traces of categories and bicategories

In this section we define the traces of categories and bicategories.

2.1. **Traces of categories.** Let $\mathcal{C}$ be a small category. We denote by $\mathcal{C}(x,y)$ the set of morphisms from x to y in $\mathcal{C}$. For an object $x \in \mathrm{Ob}(\mathcal{C})$, set $\mathrm{End}_{\mathcal{C}}(x) = \mathcal{C}(x,x)$, the set of endomorphism of x, which has a monoid structure. Set $\mathrm{End}(\mathcal{C}) = \coprod_{x\in \mathrm{Ob}(\mathcal{C})} \mathrm{End}_{\mathcal{C}}(x)$.

The *trace* of $\mathcal{C}$ is defined to be the coend of the Hom functor

$$\mathrm{Tr}(\mathcal{C}) = \int^{x\in\mathcal{C}} \mathcal{C}(x,x) \in \mathbf{Set}.$$

By abuse of notation, we will also write $\mathrm{Tr}\,\mathcal{C}$ in later sections. Unravelling the definition of the coend (see e.g. [11, p. 226]) $\mathrm{Tr}(\mathcal{C})$ may be defined also by

$$\mathrm{Tr}(\mathcal{C}) = \mathrm{End}(\mathcal{C})/\sim,$$

where the equivalence relation $\sim$ is generated by $fg \sim gf$ for all pairs $f\colon x \to y$, $g\colon y \to x$ with $x, y \in \mathrm{Ob}(\mathcal{C})$. For $f \in \mathrm{End}(\mathcal{C})$, let $[f] \in \mathrm{Tr}(\mathcal{C})$ denote the corresponding equivalence class.

The trace gives rise to a functor

$$\mathrm{Tr}\colon\ \mathbf{Cat} \to \mathbf{Set},$$

from the category of small categories to the category of sets, which maps a functor $F\colon\ \mathcal{C} \to \mathcal{D}$ to the map $\mathrm{Tr}(F)\colon\ \mathrm{Tr}(\mathcal{C}) \to \mathrm{Tr}(\mathcal{D})$ defined by

$$\mathrm{Tr}(F)([f]) = [F(f)] \in \mathrm{Tr}(\mathcal{D})$$

for $f\colon\ x \to x$ in $\mathcal{C}$.

Here we summarize some useful facts:

- For $f \in \mathrm{End}_{\mathcal{C}}(x)$ and an isomorphism $a\colon\ y \xrightarrow{\simeq} x$, we have $[f] = [a^{-1}fa]$ in $\mathrm{Tr}(\mathcal{C})$.

- If $\sigma\colon F \overset{\simeq}{\Longrightarrow} G$ is a natural isomorphism between two functors $F, G\colon \mathcal{C} \to \mathcal{D}$, then $\mathrm{Tr}(F) = \mathrm{Tr}(G)$. Thus, the functor $\mathrm{Tr}\colon \mathbf{Cat} \to \mathbf{Set}$ can be refined to a 2-functor from the 2-category of categories, functors and natural isomorphisms to the 2-category of sets, functions and identities of functions.
- Equivalence of categories $\mathcal{C} \simeq \mathcal{D}$ induces an isomorphism $\mathrm{Tr}(\mathcal{C}) \cong \mathrm{Tr}(\mathcal{D})$.

2.2. **Products.** Recall that the categories **Cat** and **Set** have finite products given by direct products for categories and sets, respectively.

Lemma 2.1. The functor $\mathrm{Tr}\colon \mathbf{Cat} \to \mathbf{Set}$ preserves finite products.

Proof. For $\mathcal{C}, \mathcal{D} \in \mathbf{Cat}$, we have a bijection

$$\varphi\colon \mathrm{Tr}(\mathcal{C}) \times \mathrm{Tr}(\mathcal{D}) \to \mathrm{Tr}(\mathcal{C} \times \mathcal{D}) \tag{2.1}$$

defined by $\varphi([f],[g]) = [(f,g)]$ for $f \in \mathrm{End}(\mathcal{C})$, $g \in \mathrm{End}(\mathcal{D})$. Note also that the trace $\mathrm{Tr}(\mathcal{T})$ of the terminal object $\mathcal{T}$ in **Cat**, which is a category with one object t and one morphism 1_t, is a one-element set $\{[1_t]\}$, which is terminal in **Set**. □

By Lemma 2.1, the functor Tr defines a (strong) monoidal functor between cartesian monoidal categories

$$\mathrm{Tr}\colon (\mathbf{Cat}, \times) \to (\mathbf{Set}, \times).$$

2.3. **Traces of bicategories.** Recall that a (small) *bicategory* $\mathbf{C}$ consists of

- a set $\mathrm{Ob}(\mathbf{C})$ of *objects* in $\mathbf{C}$,
- a small category $\mathbf{C}(x,y)$ for $x, y \in \mathrm{Ob}(\mathbf{C})$,
- a functor $\circ\colon \mathbf{C}(y,z) \times \mathbf{C}(x,y) \to \mathbf{C}(x,z)$ for $x,y,z \in \mathrm{Ob}(\mathbf{C})$,
- an object $I_x \in \mathbf{C}(x,x)$ for $x \in \mathrm{Ob}(\mathbf{C})$,
- natural isomorphisms

$$\alpha_{h,g,f}\colon \ (h \circ g) \circ f \xrightarrow{\simeq} h \circ (g \circ f),$$

$$\lambda_f\colon \ I_y \circ f \xrightarrow{\simeq} f\ ,$$

$$\rho_f\colon \ f \circ I_x \xrightarrow{\simeq} f\ ,$$

for objects $f \in \mathrm{Ob}(\mathbf{C}(x,y))$, $g \in \mathrm{Ob}(\mathbf{C}(y,z))$, $h \in \mathrm{Ob}(\mathbf{C}(z,w))$,

which satisfy certain relations (pentagon etc.). For more details see [3, Chapter 7] or [2]. A 2-category is a bicategory such that the $\alpha_{h,g,f}$, λ_f and ρ_f are identities.

For a bicategory $\mathbf{C}$, we define a category $\mathrm{Tr}(\mathbf{C})$ with $\mathrm{Ob}(\mathrm{Tr}(\mathbf{C})) = \mathrm{Ob}(\mathbf{C})$ as follows. For $x, y \in \mathrm{Ob}(\mathbf{C})$, set $\mathrm{Tr}(\mathbf{C})(x,y) = \mathrm{Tr}(\mathbf{C}(x,y))$. For $x,y,z \in \mathrm{Ob}(\mathbf{C})$, define the composition map

$$\circ\colon \ \mathrm{Tr}(\mathbf{C})(y,z) \times \mathrm{Tr}(\mathbf{C})(x,y) \to \mathrm{Tr}(\mathbf{C})(x,z)$$

as the composite

$$\mathrm{Tr}(\mathbf{C}(y,z)) \times \mathrm{Tr}(\mathbf{C}(x,y)) \xrightarrow[\simeq]{\varphi} \mathrm{Tr}(\mathbf{C}(y,z) \times \mathbf{C}(x,y)) \xrightarrow{\mathrm{Tr}(\circ)} \mathrm{Tr}(\mathbf{C}(x,z))\ , \tag{2.2}$$

where φ is the isomorphism as defined in equation (2.1). For $\sigma \in \mathrm{End}_{\mathbf{C}(x,y)}$ and $\tau \in \mathrm{End}_{\mathbf{C}(y,z)}$, we have $[\tau] \circ [\sigma] = [\tau \circ \sigma]$. The identity morphism for $x \in \mathrm{Ob}(\mathrm{Tr}(\mathbf{C})) = \mathrm{Ob}(\mathbf{C})$ is given by $[1_{I_x}]$. The associativity and unitality of composition in $\mathrm{Tr}(\mathbf{C})$ follows from the natural isomorphisms α, λ and ρ. For example, for 2-morphisms $\sigma\colon f \Rightarrow f$, $\tau\colon g \Rightarrow g$ and $\rho\colon h \Rightarrow h$ in $\mathbf{C}$, we have

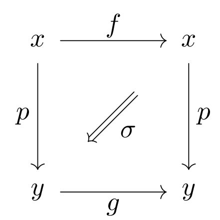
$$([\rho] \circ [\tau]) \circ [\sigma] = [(\rho \circ \tau) \circ \sigma] = [\alpha_{h,g,f}((\rho \circ \tau) \circ \sigma)\alpha^{-1}_{h,g,f}] = [\rho \circ (\tau \circ \sigma)] = [\rho] \circ ([\tau] \circ [\sigma]).$$

Remark 2.2. Recall that a monoidal category may be regarded as a bicategory with one object. Hence the above definition of the trace of bicategory specializes to the trace of monoidal category, which takes values in categories with one object, i.e., monoids. An internal notion of trace for 1-morphisms inside of a bicategory is given in [5].

2.4. **Horizontal trace of a bicategory.** This subsection is a digression on "horizontal traces" of bicategories, which is not necessary for the rest of the paper. The motivation for considering "horizontal traces" is given at the end of this subsection.

For a bicategory $\mathbf{C}$, there is another structure which may be called the "trace" of $\mathbf{C}$. In graphical explanations, a 1-morphism in a bicategory is sometimes drawn as a horizontal arrow, and a 2-morphism between two 1-morphisms are "vertical face" bounded by two horizontal arrows. The trace $\mathrm{Tr}(\mathbf{C})$ of $\mathbf{C}$ is obtained from $\mathbf{C}$ by taking traces in the "vertical direction", i.e., in 2-morphisms. Thus one might call $\mathrm{Tr}(\mathbf{C})$ the *vertical trace* of $\mathbf{C}$.

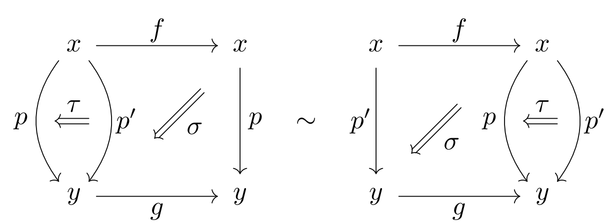
Here we introduce another notion of "trace" of a bicategory $\mathbf{C}$, which we call the "horizontal trace" $\mathrm{Tr}^{\mathrm{hor}}(\mathbf{C})$, which is obtained from $\mathbf{C}$ by taking trace in the "horizontal direction". More precisely, $\mathrm{Tr}^{\mathrm{hor}}(\mathbf{C})$ is the category defined as follows. Here, for simplicity, our bicategory $\mathbf{C}$ is a 2-category. The objects in $\mathrm{Tr}^{\mathrm{hor}}(\mathbf{C})$ are 1-endomorphisms $f\colon x \to x$, $x \in \mathrm{Ob}(\mathbf{C})$. Morphisms from $f\colon x \to x$ to $g\colon y \to y$ are equivalence classes $[p,\sigma]$ of pairs (p,σ) of a morphism $p\colon x \to y$ in $\mathbf{C}$ and a 2-morphism $\sigma\colon p \circ f \Rightarrow g \circ p\colon x \to y$, depicted by



where the equivalence relation on such pairs is generated by

$$(p,(g\circ\tau)\sigma) \sim (p', \sigma(\tau\circ f))$$

or



for $p, p'\colon x \to y$, $\sigma\colon p\circ f \Rightarrow g\circ p'\colon x \to y$, $\tau\colon p' \Rightarrow p\colon x \to y$. The composite of two morphisms $[p,\sigma]\colon (f\colon x\to x) \to (g\colon y \to y)$ and $[q,\tau]\colon (g\colon y \to y) \to (h\colon z \to z)$ is defined by

$$[q,\tau][p,\sigma] = [qp, (\tau\circ p)(q\circ\sigma)].$$

The identity morphism is given by

$$1_f = [1_x, 1_f]$$

for $f\colon x \to x$. It is not difficult to check that $\mathrm{Tr}^{\mathrm{hor}}(\mathbf{C})$ is a well-defined category.

The notion of the horizontal trace of a bicategory may be regarded as a categorification of the notion of the trace of a category.

There is a canonical functor $\mathrm{i}\colon \mathrm{Tr}(\mathbf{C}) \to \mathrm{Tr}^{\mathrm{hor}}(\mathbf{C})$ defined as follows. For an object $x \in \mathrm{Ob}(\mathbf{C})$ in $\mathrm{Tr}(\mathbf{C})$, set

$$\mathrm{i}(x) = (1_x\colon\ x \to x).$$

For a morphism $[\sigma\colon\ f \Rightarrow f\colon\ x \to y]\colon\ x \to y$ in $\mathrm{Tr}(\mathbf{C})$, set

$$\mathrm{i}([\sigma]) = [f, \sigma\colon\ f \circ 1_x \Rightarrow 1_y \circ f].$$

It is easy to see that the functor i is full and faithful. Thus, $\mathrm{Tr}^{\mathrm{hor}}(\mathbf{C})$ has more information about $\mathbf{C}$ than $\mathrm{Tr}(\mathbf{C})$.

The horizontal trace may be loosely regarded as a generalization to bicategories of the *tube algebra* of monoidal category (see for example [14, 4]). If our bicategory $\mathbf{C}$ admits biadjoints, then the hom-sets $\mathrm{Tr}^{\mathrm{hor}}(\mathbf{C})(f\colon\ x \to x, g\colon\ y \to y)$ can be naturally regarded as a "skein module of $\mathbf{C}$-diagrams" on the annulus $A = S^1 \times [0,1]$ with the 1-morphism x put on a point $(\mathrm{pt}, 0) \in \partial A$ and y on $(\mathrm{pt}, 1) \in \partial A$. It is natural to consider skein modules of $\mathbf{C}$-diagrams on more general surfaces with specified 1-morphisms on boundary points. The horizontal trace $\mathrm{Tr}^{\mathrm{hor}}(\mathbf{C})$ plays an analogous role for studying such general skein modules, as the tube algebra does to the skein modules associated to a monoidal category with duals. It would be interesting to consider the horizontal trace and the skein modules for $\mathbf{C} = \mathcal{U}^*$.

3. Traces of linear (bi)categories

3.1. **Linear and additive categories.** A *linear* category (also called **Ab**-category or *preadditive* category) is a category enriched in the category **Ab** of abelian groups. This means it is a category whose hom-sets are equipped with structures of abelian groups and the composition maps are bilinear (compare [11, p. 276]).

A *linear functor* between two linear categories $\mathcal{C}$ and $\mathcal{D}$ is a functor F from $\mathcal{C}$ to $\mathcal{D}$ such that for $x, y \in \mathrm{Ob}(\mathcal{C})$, the map $F\colon\ \mathcal{C}(x,y) \longrightarrow \mathcal{D}(F(x), F(y))$ is an abelian group homomorphism.

An *additive* category is a linear category equipped with a zero object and direct sums. For a linear category $\mathcal{C}$, there is a universal additive category generated by $\mathcal{C}$, called the *additive closure* $\mathcal{C}^{\oplus}$, in which the objects are formal finite direct sums of objects in $\mathcal{C}$ and the morphisms are matrices of morphisms in $\mathcal{C}$. There is a canonical fully faithful functor $i\colon\ \mathcal{C} \to \mathcal{C}^{\oplus}$. Every linear functor $F\colon\ \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to an additive category $\mathcal{D}$ factors through i uniquely up to natural isomorphism.

3.2. **Traces of linear categories.** Similarly to the case of categories which are not linear, for a linear category $\mathcal{C}$, the *trace* $\mathrm{Tr}(\mathcal{C})$ of $\mathcal{C}$ is defined to be the coend of the Hom functor

$$\mathrm{Tr}(\mathcal{C}) = \int^{x \in \mathcal{C}} \mathcal{C}(x,x) = \bigoplus_{x \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(x,x) / \operatorname{Span}\{fg - gf\},$$

where f and g runs through all pairs of morphisms $f\colon\ x \longrightarrow y$, $g\colon\ y \longrightarrow x$ with $x, y \in \mathrm{Ob}(\mathcal{C})$.

The trace Tr gives a functor from the category of small linear categories to the category of abelian groups.

Lemma 3.1. If $\mathcal{C}$ is an additive category, then for $f\colon\ x \to x$ and $g\colon\ y \to y$, we have

$$[f \oplus g] = [f] + [g].$$

Proof. Since $f \oplus g = (f \oplus 0) + (0 \oplus g)\colon\ x \oplus y \to x \oplus y$, we have

$$[f \oplus g] = [f \oplus 0] + [0 \oplus g].$$

Now we have $[f \oplus 0] = [ifp] = [pif] = [f]$ where $p\colon\ x \oplus y \to x$ and $i\colon\ x \to x \oplus y$ are the projection and the inclusion. Similarly, we have $[0 \oplus g] = [g]$. Hence the result. □

3.3. **Traces of linear bicategories.** A *linear bicategory* is a bicategory $\mathbf{C}$ such that

(1) for $x, y \in \mathrm{Ob}(\mathbf{C})$, the category $\mathbf{C}(x, y)$ is equipped with a structure of a linear category,
(2) for $x, y, z \in \mathrm{Ob}(\mathbf{C})$, the functor $\circ\colon \mathbf{C}(y, z) \times \mathbf{C}(x, y) \to \mathbf{C}(x, z)$ is "bilinear" in the sense that the functors $- \circ f\colon \mathbf{C}(y, z) \to \mathbf{C}(x, z)$ for $f \in \mathrm{Ob}(\mathbf{C}(x, y))$ and $g \circ -\colon \mathbf{C}(x, y) \to \mathbf{C}(x, z)$ for $g \in \mathrm{Ob}(\mathbf{C}(y, z))$ are linear functors.

The trace $\mathrm{Tr}(\mathbf{C})$ of a linear bicategory $\mathbf{C}$ is defined similarly to the trace of bicategory, and is equipped with a linear category structure.

3.4. **Traces of additive closures.** Here we consider the trace of the additive closure $\mathcal{C}^{\oplus}$ of a linear category $\mathcal{C}$. The homomorphism

$$\mathrm{Tr}(i)\colon\ \mathrm{Tr}(\mathcal{C}) \to \mathrm{Tr}(\mathcal{C}^{\oplus})$$

induced by the canonical functor $i\colon \mathcal{C} \to \mathcal{C}^{\oplus}$ is an isomorphism. In fact, the inverse $\mathrm{tr}\colon\ \mathrm{Tr}(\mathcal{C}^{\oplus}) \to \mathrm{Tr}(\mathcal{C})$ is defined by

$$\mathrm{tr}([(f_{k,l})_{k,l}]) = \sum_k [f_{k,k}]$$

for an endomorphism in $\mathcal{C}^{\oplus}$

$$(f_{k,l})_{k,l \in \{1,\dots,n\}}\colon\ x_1 \oplus \cdots \oplus x_n \to x_1 \oplus \cdots \oplus x_n$$

with $f_{k,l}\colon\ x_l \to x_k$ in $\mathcal{C}$.

3.5. **Traces and the Karoubi envelope.** Let $\mathcal{C}$ be a linear category. An idempotent $e\colon\ x \to x$ in $\mathcal{C}$ is said to *split* if there is an object y and morphisms $g\colon\ x \to y$, $h\colon\ y \to x$ such that $hg = e$ and $gh = 1_y$.

The *Karoubi envelope* $\mathrm{Kar}(\mathcal{C})$ (also called *idempotent completion*) of $\mathcal{C}$ is the category whose objects are pairs (x, e) of objects $x \in \mathrm{Ob}(\mathcal{C})$ and an idempotent endomorphism $e : x \to x$, $e^2 = e$, in $\mathcal{C}$ and whose morphisms

$$f : (x, e) \to (y, e')$$

are morphisms $f\colon\ x \to y$ in $\mathcal{C}$ such that $f = e' f e$. Composition is induced by the composition in $\mathcal{C}$ and the identity morphism is $e : (x, e) \to (x, e)$. $\mathrm{Kar}(\mathcal{C})$ is equipped with a linear category structure. It is well known that the Karoubi envelope of an additive category is additive.

There is a natural fully faithful linear functor

$$\iota : \mathcal{C} \to \mathrm{Kar}(\mathcal{C})$$

such that $\iota(x) = (x, 1_x)$ for $x \in \mathrm{Ob}(\mathcal{C})$ and $\iota(f\colon\ x \to y) = f$. The Karoubi envelope $\mathrm{Kar}(\mathcal{C})$ is universal in the sense that if $F\colon\ \mathcal{C} \to \mathcal{D}$ is a linear functor from $\mathcal{C}$ to a linear category $\mathcal{D}$ with split idempotents, then F extends to a functor from $\mathrm{Kar}(\mathcal{C})$ to $\mathcal{D}$ uniquely up to natural isomorphism [3, Proposition 6.5.9].

Proposition 3.2. The functor $\iota\colon\ \mathcal{C} \longrightarrow \mathrm{Kar}(\mathcal{C})$ induces an isomorphism

$$\mathrm{Tr}(\iota)\colon\ \mathrm{Tr}(\mathcal{C}) \xrightarrow{\cong} \mathrm{Tr}(\mathrm{Kar}(\mathcal{C})).$$

Proof. We can construct a map $u\colon\ \mathrm{Tr}(\mathrm{Kar}(\mathcal{C})) \longrightarrow \mathrm{Tr}(\mathcal{C})$ such that, for $f\colon\ (x, e) \longrightarrow (x, e)$ in $\mathrm{Kar}(\mathcal{C})$, $[f] \in \mathrm{Tr}(\mathrm{Kar}(\mathcal{C}))$ is mapped to $[f] \in \mathrm{Tr}(\mathcal{C})$. Then one can check that u is an inverse to $\mathrm{Tr}(\iota)$. □

Given a linear bicategory $\mathbf{C}$ we define the Karoubi envelope $\mathrm{Kar}(\mathbf{C})$ as the linear category with $\mathrm{Ob}(\mathrm{Kar}(\mathbf{C})) = \mathrm{Ob}(\mathbf{C})$, and for $x, y \in \mathrm{Ob}(\mathbf{C})$ the hom-categories are given by $\mathrm{Kar}(\mathbf{C})(x, y) := \mathrm{Kar}(\mathbf{C}(x, y))$. The composition functor $\mathrm{Kar}(\mathbf{C})(y, z) \times \mathrm{Kar}(\mathbf{C})(x, y) \to \mathrm{Kar}(\mathbf{C})(x, z)$ is induced by the universal property of the Karoubi envelope from the composition functor in $\mathbf{C}$. The fully-faithful additive functors $\mathbf{C}(x,y) \to \mathrm{Kar}(\mathbf{C}(x,y))$ combine to form an additive 2-functor $\mathbf{C} \to \mathrm{Kar}(\mathbf{C})$ that is universal with respect to splitting idempotents in the hom-categories $\mathbf{C}(x,y)$.

3.6. **Split Grothendieck groups and traces.** For an additive category $\mathcal{C}$, the *split Grothendieck group* $K_0(\mathcal{C})$ of $\mathcal{C}$ is the abelian group generated by the isomorphism classes of objects of $\mathcal{C}$ with relations $[x \oplus y]_{\cong} = [x]_{\cong} + [y]_{\cong}$ for $x, y \in \mathrm{Ob}(\mathcal{C})$. Here $[x]_{\cong}$ denotes the isomorphism class of x. The split Grothendieck group K_0 is functorial.

Define a homomorphism

$$h_{\mathcal{C}} \colon K_0(\mathcal{C}) \longrightarrow \mathrm{Tr}(\mathcal{C})$$

by

$$h_{\mathcal{C}}([x]_{\cong}) = [1_x]$$

for $x \in \mathrm{Ob}(\mathcal{C})$. Indeed, one can easily check that

$$h_{\mathcal{C}}([x \oplus y]_{\cong}) = [1_x] + [1_y]\,.$$

The maps $h_{\mathcal{C}}$ form a natural transformation

$$h \colon K_0 \Rightarrow \mathrm{Tr} \colon \mathbf{AdCat} \to \mathbf{Ab},$$

where **AdCat** denote the category of small additive category.

Given a linear bicategory $\mathbf{C}$, we define the split Grothendieck category $K_0(\mathbf{C})$ of $\mathbf{C}$ as the linear category with $\mathrm{Ob}(K_0(\mathbf{C})) = \mathrm{Ob}(\mathbf{C})$ and $K_0(\mathbf{C})(x,y) := K_0(\mathbf{C}(x,y))$ for any two objects $x, y \in \mathrm{Ob}(\mathbf{C})$. For $[f]_{\cong} \in \mathrm{Ob}(K_0(\mathbf{C})(x,y))$ and $[g]_{\cong} \in \mathrm{Ob}(K_0(\mathbf{C})(y,z))$ the composition in $K_0(\mathbf{C})$ is defined by $[g]_{\cong} \circ [f]_{\cong} := [g \circ f]_{\cong}$.

The homomorphisms $h_{\mathbf{C}(x,y)}$ taken over all objects $x, y \in \mathrm{Ob}(\mathbf{C})$ give rise to a linear functor

$$h_{\mathbf{C}} \colon \mathcal{K}_0(\mathbf{C}) \to \mathrm{Tr}(\mathbf{C}) \tag{3.1}$$

which is the identity map on objects and sends $K_0(\mathbf{C})(x,y) \to \mathrm{Tr}(\mathbf{C})(x,y)$ via the homomorphism $h_{\mathbf{C}(x,y)}$. It is easy to see that this assignment preserves composition since

$$h_{\mathbf{C}}([g]_{\cong} \circ [f]_{\cong}) = h_{\mathbf{C}}([g \circ f]_{\cong}) = [1_{g \circ f}] = [1_g \circ 1_f] = [1_g] \circ [1_f] = h_{\mathbf{C}}([g]_{\cong}) \circ h_{\mathbf{C}}([f]_{\cong}). \tag{3.2}$$

3.7. **Tools for computing the trace.** In this subsection we collect a few results which will be needed later.

Proposition 3.3. Let $\mathcal{C}$ be a linear category and $B \subset \mathrm{Ob}(\mathcal{C})$ such that every object x of $\mathcal{C}$ is isomorphic to direct sum of elements of B. Let $\mathcal{C}|_B$ be the full subcategory of $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}|_B) = B$. Then $\mathrm{Tr}(\mathcal{C})$ is isomorphic to $\mathrm{Tr}(\mathcal{C}|_B)$.

Proof. Let $S := \{\bigoplus_i b_i \mid b_i \in B\}$, so $\mathcal{C}|_S$ is equivalent to $\mathcal{C}|_B{}^{\oplus}$. Since every object of $\mathcal{C}$ is equivalent to an element of S, we have $\mathcal{C} \simeq \mathcal{C}|_S \simeq \mathcal{C}|_B{}^{\oplus}$. So, $\mathrm{Tr}(\mathcal{C}) \simeq \mathrm{Tr}(\mathcal{C}_B{}^{\oplus}) \simeq \mathrm{Tr}(\mathcal{C}|_B)$. □

Proposition 3.4. Let $\mathcal{C}$ be a small linear category. Let $H := \bigoplus_{x \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(x,x)$, and let $K \subset H$ be a subgroup. Assume that there is a homomorphism $\pi \colon H \to K$ with the following properties:

(1) π is a projection,
(2) $[\pi(f)] = [f] \in \mathrm{Tr}(\mathcal{C})$ for every $f \in H$, and
(3) $\pi(gh) = \pi(hg)$ for every $g \in \mathcal{C}(x,y)$ and $h \in \mathcal{C}(y,x)$ $(x, y \in \mathrm{Ob}(\mathcal{C}))$;

then $\mathrm{Tr}(\mathcal{C})$ is isomorphic to K.

Proof. The inclusion $i \colon K \hookrightarrow H$ induces a homomorphism $i^* \colon K \to \mathrm{Tr}(\mathcal{C})$, $k \mapsto [k]$. The map π induces $\bar{\pi} \colon \mathrm{Tr}(\mathcal{C}) \to K$. It is easy to check that i^* and $\bar{\pi}$ are inverse to each other. Hence $\mathrm{Tr}(\mathcal{C}) \cong K$. □

4. The Hochschild–Mitchell homology of upper-triangular categories

In this section, we introduce (strongly) upper-triangular linear categories and compute their Hochschild–Mitchell homology groups.

4.1. Hochschild–Mitchell homology of linear categories.

Recall that for a ring R, the Hochschild homology $\mathrm{HH}_*(R)$ of R can be defined as the homology of the Hochschild complex (see [13]):

$$C_* = C_*(R): \qquad \ldots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow 0,$$

where $C_n(R) = R^{\otimes n+1}$ and

$$d_n(a_0 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

for $a_0, \ldots, a_n \in R$.

The Hochschild–Mitchell homology of a linear category can be similarly defined as follows. Let $\mathcal{C}$ be a small linear category. Define the *Hochschild–Mitchell complex* of $\mathcal{C}$

$$C_* = C_*(\mathcal{C}): \qquad \ldots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow 0,$$

where

$$C_n = C_n(\mathcal{C}) := \bigoplus_{x_0, \ldots, x_n \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(x_n, x_0) \otimes \mathcal{C}(x_{n-1}, x_n) \otimes \cdots \otimes \mathcal{C}(x_0, x_1),$$

$$d_n(f_n \otimes f_{n-1} \otimes \cdots \otimes f_0) := \left(\sum_{i=0}^{n-1} (-1)^i f_n \otimes \cdots \otimes f_{n-i} f_{n-i-1} \otimes \cdots \otimes f_0 \right) + (-1)^n f_0 f_n \otimes f_{n-1} \otimes \cdots \otimes f_1.$$

The Hochschild–Mitchell homology $\mathrm{HH}_*(\mathcal{C})$ of $\mathcal{C}$ is defined to be the homology of the chain complex $C_*(\mathcal{C})$. It is clear that $\mathrm{HH}_0(\mathcal{C}) = C_0/d_1(C_1)$ is isomorphic to $\mathrm{Tr}(\mathcal{C})$. Here we list some well-known properties, see [13].

Lemma 4.1.

(1) An equivalence $\mathcal{C} \simeq \mathcal{D}$ of linear categories induces an isomorphism of the Hochschild–Mitchell homology $\mathrm{HH}_*(\mathcal{C}) \cong \mathrm{HH}_*(\mathcal{D})$. (More generally, a Morita equivalence of linear categories induces an isomorphism on the Hochschild–Mitchell homology. We do not need this fact.)

(2) The inclusion functor $\mathcal{C} \to \mathcal{C}^{\oplus}$ of a linear category $\mathcal{C}$ into its additive closure $\mathcal{C}^{\oplus}$ induces an isomorphism $\mathrm{HH}_*(\mathcal{C}) \cong \mathrm{HH}_*(\mathcal{C}^{\oplus})$.

4.2. Upper-triangularity of linear categories.

Let us define a property of linear categories which allows us to compute all their Hochschild–Mitchell homology groups.

Definition 4.2. A small linear category $\mathcal{C}$ is said to be *upper-triangular* if there is a partial order $\leq$ on the set $\mathrm{Ob}(\mathcal{C})$ such that, for all $x, y \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}(x, y) \neq 0$ implies $x \leq y$.

Note that $\mathcal{C}$ is upper-triangular if and only if there is no sequence

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} x_n \xrightarrow{f_n} x_0 \quad (n \geq 1)$$

of nonzero morphisms in $\mathcal{C}$ unless $x_0 = x_1 = \cdots = x_n$.

Lemma 4.3. For an upper-triangular linear category $\mathcal{C}$, we have

$$\mathrm{HH}_*(\mathcal{C}) \cong \bigoplus_{x \in \mathrm{Ob}(\mathcal{C})} \mathrm{HH}_*(\mathcal{C}(x, x)). \tag{4.1}$$

Proof. From the definition of upper-triangularity, we have, for $x_0, \dots, x_n \in \mathrm{Ob}(\mathcal{C})$, $n \geq 0$,

$$\mathcal{C}(x_n, x_0) \otimes \mathcal{C}(x_{n-1}, x_n) \otimes \cdots \otimes \mathcal{C}(x_0, x_1) = 0$$

unless $x_0 = x_1 = \cdots = x_n$. Hence

$$C_n(\mathcal{C}) = \bigoplus_{x \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(x,x)^{\otimes n+1},$$

and the Hochschild–Mitchell complex of $\mathcal{C}$ decomposes as

$$C_*(\mathcal{C}) \cong \bigoplus_{x \in \mathrm{Ob}(\mathcal{C})} C_*(\mathcal{C}(x,x)).$$

Hence we have (4.1). □

Remark 4.4. In the situation of Lemma 4.1, similar isomorphisms hold for the cyclic homology [10]

$$\mathrm{HC}_*(\mathcal{C}) \cong \bigoplus_{x \in \mathrm{Ob}(\mathcal{C})} \mathrm{HC}_*(\mathcal{C}(x,x)).$$

Definition 4.5. A *strongly upper-triangular* linear category is an upper-triangular linear category $\mathcal{C}$ such that for all $x \in \mathrm{Ob}(\mathcal{C})$, we have $\mathcal{C}(x,x) \cong \mathbb{Z}$.

Proposition 4.6. For a strongly upper-triangular linear category $\mathcal{C}$, we have

$$\mathrm{HH}_i(\mathcal{C}) \cong \begin{cases} \mathbb{Z}\,\mathrm{Ob}(\mathcal{C}) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Here $\mathbb{Z}\,\mathrm{Ob}(\mathcal{C})$ denotes the free $\mathbb{Z}$-module spanned by $\mathrm{Ob}(\mathcal{C})$.

Proof. The result follows from Lemma 4.3 and

$$\mathrm{HH}_i(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

□

4.3. **Additive categories with strongly upper-triangular bases.** Let $\mathcal{C}$ be an additive category, and let $B \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Denote by $\mathcal{C}|_B$ the full subcategory of $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}|_B) = B$. The set B is called a *strongly upper-triangular basis* of $\mathcal{C}$ if the following two conditions hold.

(1) The inclusion functor $\mathcal{C}|_B \to \mathcal{C}$ induces equivalence of additive categories $(\mathcal{C}|_B)^{\oplus} \simeq \mathcal{C}$.
(2) $\mathcal{C}|_B$ is strongly upper-triangular.

Proposition 4.7. For an additive category $\mathcal{C}$ with a strongly upper-triangular basis B, we have

$$\mathrm{HH}_i(\mathcal{C}) \cong \begin{cases} \mathbb{Z}B & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Proof. Because of Lemma 4.1 and the definition of the strongly upper-triangular basis, the Hochschild–Mitchell homology of the category $\mathcal{C}$ is the same as that of the category $\mathcal{C}|_B$. Proposition 4.6 concludes the proof. □

5. The 2-categories $\mathcal{U}$, $\mathcal{U}^*$ and their Karoubi envelopes

In this section we define 2-categories $\mathcal{U}$ and $\mathcal{U}^*$ as well as their Karoubi envelopes $\dot{\mathcal{U}}$ and $\dot{\mathcal{U}}^*$. We recall some definitions and results of the "thick calculus" developed in [7].

5.1. **The algebras $\mathbf{U}$, ${}_{\mathcal{A}}\mathbf{U}$ and ${}_{\mathcal{A}}\dot{\mathbf{U}}$.** The quantum group $\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_2)$ is the unital, associative algebra over the rational function field $\mathbb{Q}(q)$ with generators E, F, K and K^{-1} subject to the relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

For $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$, the integral form ${}_{\mathcal{A}}\mathbf{U} = {}_{\mathcal{A}}\mathbf{U}(\mathfrak{sl}_2)$ is the $\mathcal{A}$-subalgebra of $\mathbf{U}$, generated by K, K^{-1} and the divided powers $E^{(k)} = E^k/[k]!$ and $F^{(k)} = F^k/[k]!$ for $k \geq 1$, where $[k]! = \prod_{i=1}^{k} \frac{q^i - q^{-i}}{q - q^{-1}}$.

The idempotented version ${}_{\mathcal{A}}\dot{\mathbf{U}} = {}_{\mathcal{A}}\dot{\mathbf{U}}(\mathfrak{sl}_2)$ of ${}_{\mathcal{A}}\mathbf{U}$, introduced in [1], can be obtained from ${}_{\mathcal{A}}\mathbf{U}$ by replacing the unit with a collection of mutually orthogonal idempotents 1_n for $n \in \mathbb{Z}$, such that

$$K1_n = 1_nK = q^n1_n, \quad E^{(k)}1_n = 1_{n+2k}E^{(k)}, \quad F^{(k)}1_n = 1_{n-2k}F^{(k)}.$$

One may regard ${}_{\mathcal{A}}\dot{\mathbf{U}}$ both as a non-unital algebra and as a linear category. In the latter case, we regard ${}_{\mathcal{A}}\dot{\mathbf{U}}$ as a linear category with objects in $\mathbb{Z}$ such that the abelian group of morphisms from $m \in \mathbb{Z}$ to $n \in \mathbb{Z}$ is the $\mathcal{A}$-submodule of the non-unital ring ${}_{\mathcal{A}}\dot{\mathbf{U}}$ consisting of the elements 1_nx1_m for $x \in {}_{\mathcal{A}}\mathbf{U}$.

The defining relations for the algebra ${}_{\mathcal{A}}\dot{\mathbf{U}}$ are

$$E^{(a)}E^{(b)}1_n = \begin{bmatrix} a+b \\ b \end{bmatrix} E^{(a+b)}1_n, \tag{5.1}$$

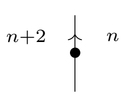
$$F^{(a)}F^{(b)}1_n = \begin{bmatrix} a+b \\ b \end{bmatrix} F^{(a+b)}1_n, \tag{5.2}$$

$$E^{(a)}F^{(b)}1_n = \sum_{j=0}^{\min(a,b)} \begin{bmatrix} a-b+n \\ j \end{bmatrix} F^{(b-j)}E^{(a-j)}1_n, \quad \text{for } n \geq b-a, \tag{5.3}$$

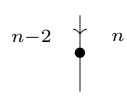
$$F^{(b)}E^{(a)}1_n = \sum_{j=0}^{\min(a,b)} \begin{bmatrix} b-a-n \\ j \end{bmatrix} E^{(a-j)}F^{(b-j)}1_n, \quad \text{for } n \leq b-a\,. \tag{5.4}$$

5.2. **The 2-category $\mathcal{U}$.** The 2-category $\mathcal{U} = \mathcal{U}(\mathfrak{sl}_2)$ is the additive 2-category consisting of

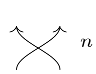
- objects n for $n \in \mathbb{Z}$,
- for a signed sequence $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$, with $\varepsilon_1, \dots, \varepsilon_m \in \{+, -\}$, define
$$\mathcal{E}_{\underline{\varepsilon}} := \mathcal{E}_{\varepsilon_1}\mathcal{E}_{\varepsilon_2}\dots\mathcal{E}_{\varepsilon_m}$$
where $\mathcal{E}_+ := \mathcal{E}$ and $\mathcal{E}_- := \mathcal{F}$. A 1-morphisms from n to n' is a formal finite direct sum of strings
$$\mathcal{E}_{\underline{\varepsilon}}\mathbf{1}_n\langle t\rangle = \mathbf{1}_{n'}\mathcal{E}_{\underline{\varepsilon}}\mathbf{1}_n\langle t\rangle$$
for any $t \in \mathbb{Z}$ and signed sequence $\underline{\varepsilon}$ such that $n' = n + 2\sum_{j=1}^{m}\varepsilon_j 1$.
- 2-morphisms are $\mathbb{Z}$-modules spanned by (vertical and horizontal) composites of identity 2-morphisms and the following tangle-like diagrams



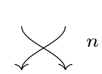
$: \mathcal{E}\mathbf{1}_n\langle t\rangle \to \mathcal{E}\mathbf{1}_n\langle t+2\rangle$



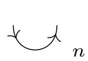
$: \mathcal{F}\mathbf{1}_n\langle t\rangle \to \mathcal{F}\mathbf{1}_n\langle t+2\rangle$



$: \mathcal{E}\mathcal{E}\mathbf{1}_n\langle t\rangle \to \mathcal{E}\mathcal{E}\mathbf{1}_n\langle t-2\rangle$



$: \mathcal{F}\mathcal{F}\mathbf{1}_n\langle t\rangle \to \mathcal{F}\mathcal{F}\mathbf{1}_n\langle t-2\rangle$



$: \mathbf{1}_n\langle t\rangle \to \mathcal{F}\mathcal{E}\mathbf{1}_n\langle t+1+n\rangle$

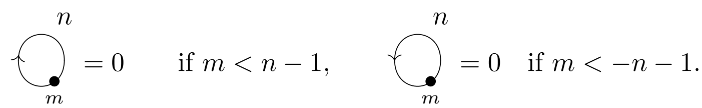
$: \mathbf{1}_n\langle t\rangle \to \mathcal{E}\mathcal{F}\mathbf{1}_n\langle t+1-n\rangle$

for every $n, t \in \mathbb{Z}$. The degree of a 2-morphism is the difference between degrees of the target and the source.

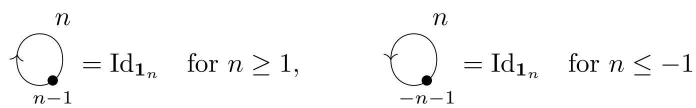
Diagrams are read from right to left and bottom to top. The rightmost region in our diagrams is usually colored by n. The identity 2-morphism of the 1-morphism $\mathcal{E}\mathbf{1}_n$ is represented by an upward oriented line (likewise, the identity 2-morphism of $\mathcal{F}\mathbf{1}_n$ is represented by a downward oriented line).

The 2-morphisms satisfy the following relations (see [8] for more details).

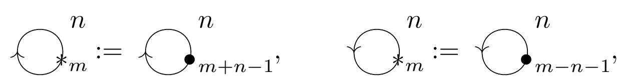
(1) The 1-morphisms $\mathcal{E}\mathbf{1}_n$ and $\mathcal{F}\mathbf{1}_n$ are biadjoint (up to a specified degree shift). Moreover, the 2-morphisms are cyclic with respect to this biadjoint structure.
(2) The $\mathcal{E}$'s carry an action of the nilHecke algebra. Using the adjoint structure this induces an action of the nilHecke algebra on the $\mathcal{F}$'s.
(3) Dotted bubbles of negative degree are zero, so that for all $m \geq 0$ one has

(5.5)


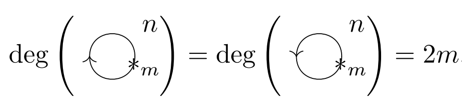
Dotted bubble of degree 0 are equal to the identity 2-morphism:



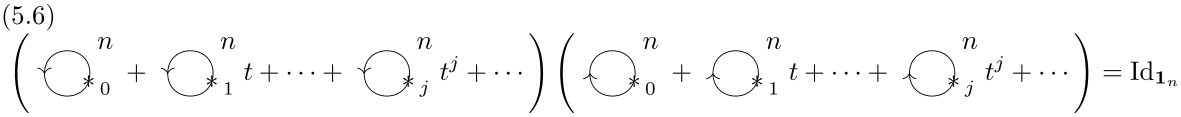
We use the following notation for the dotted bubbles:



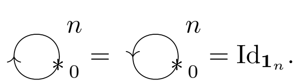
so that



We call a clockwise (resp. counterclockwise) bubble fake if $m+n-1 < 0$ and (resp. if $m-n-1 < 0$). The fake bubbles are defined recursively by the homogeneous terms of the equation

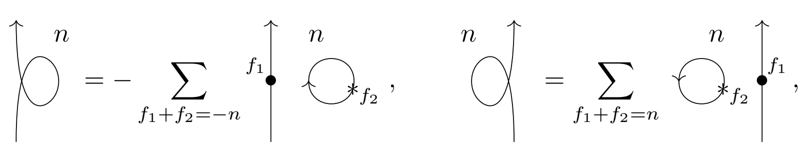
(5.6)


and the additional condition

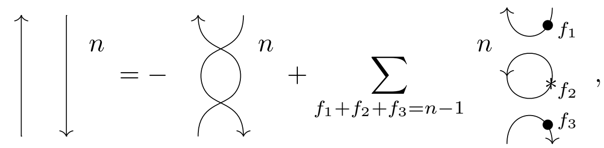


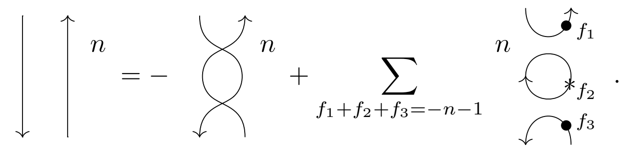
One can check that relation (5.6) holds also for the real bubbles. So we will not distinguish between them in what follows.

(4) There are additional relations:

(5.7)


(5.8)

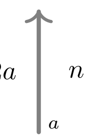




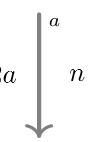
5.3. **The 2-category $\dot{\mathcal{U}}$.** The 2-category $\dot{\mathcal{U}}$ is defined as $\mathrm{Kar}(\mathcal{U})$, meaning that the categories $\mathcal{U}(m,n)$ are replaced by their Karoubi envelopes.

In $\mathcal{U}(m,n)$, for any $a,b \geq 0$, there are additional objects $\mathcal{E}^{(a)}\mathbf{1}_n$ and $\mathcal{F}^{(b)}\mathbf{1}_n$ defined by

(5.9) $$\mathcal{E}^{(a)}\mathbf{1}_n\langle t\rangle := \left(\mathcal{E}^a\mathbf{1}_n\left\langle t-\frac{a(a-1)}{2}\right\rangle, i_a\right) =: \quad n+2a$$

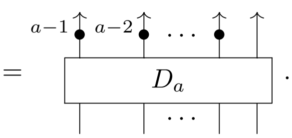


$$\mathcal{F}^{(a)}\mathbf{1}_n\langle t\rangle := \left(\mathcal{F}^a\mathbf{1}_n\left\langle t+\frac{a(a-1)}{2}\right\rangle, i'_a\right) =: \quad n-2a$$



where the idempotent i_a is defined as follows

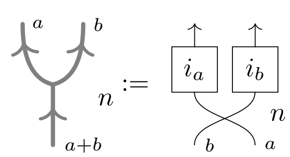
$$i_a := \delta_a D_a =$$



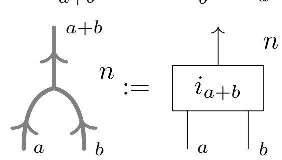
Here D_a is the longest braid on a-strands. The idempotents i'_a are obtained from i_a by a 180° rotation. We have $\mathcal{E}^a\mathbf{1}_n \cong \oplus_{[a]!}\mathcal{E}^{(a)}\mathbf{1}_n$ and $\mathcal{F}^{(b)}\mathbf{1}_n \cong \oplus_{[b]!}\mathcal{F}^{(b)}\mathbf{1}_n$. Here we use the standard notation

$$[a] := \frac{q^a-q^{-a}}{q-q^{-1}} = q^{a-1}+q^{a-3}+\cdots+q^{-a+1}, \quad [a]! = \prod_{i=1}^{a}[i], \quad \text{and} \quad \begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a]!}{[b]![a-b]!}.$$

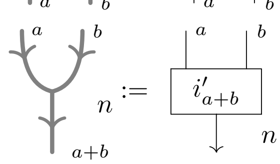
We define here some additional 2-morphisms in $\dot{\mathcal{U}}$, whose degrees can be read from the shift on the right-hand side.



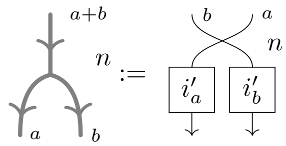
$: \mathcal{E}^{(a+b)}\mathbf{1}_n \to \mathcal{E}^{(a)}\mathcal{E}^{(b)}\mathbf{1}_n\langle -ab\rangle,$



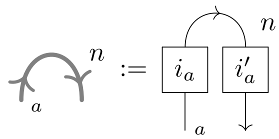
$: \mathcal{E}^{(a)}\mathcal{E}^{(b)}\mathbf{1}_n \to \mathcal{E}^{(a+b)}\mathbf{1}_n\langle -ab\rangle,$



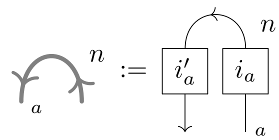
$: \mathcal{F}^{(a+b)}\mathbf{1}_n \to \mathcal{F}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n\langle -ab\rangle,$



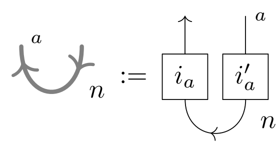
$: \mathcal{F}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n \to \mathcal{E}^{(a+b)}\mathbf{1}_n\langle -ab\rangle,$



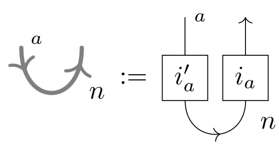
$: \mathcal{E}^{(a)}\mathcal{F}^{(a)}\mathbf{1}_n \to \mathbf{1}_n\langle a^2 - an\rangle,$



$: \mathcal{F}^{(a)}\mathcal{E}^{(a)}\mathbf{1}_n \to \mathbf{1}_n\langle a^2 + an\rangle,$



$: \mathbf{1}_n \to \mathcal{E}^{(a)}\mathcal{F}^{(a)}\mathbf{1}_n\langle a^2 - an\rangle,$



$: \mathbf{1}_n \to \mathcal{F}^{(a)}\mathcal{E}^{(a)}\mathbf{1}_n\langle a^2 + an\rangle.$

5.4. **The split Grothendieck group** $K_0(\dot{\mathcal{U}})$**.** Lusztig's canonical basis $\mathbb{B}$ of ${}_{\mathcal{A}}\dot{\mathbf{U}}$ is

(i) $E^{(a)}F^{(b)}1_n$ for $a, b \geq 0$, $n \in \mathbb{Z}$, $n \leq b - a$,
(ii) $F^{(b)}E^{(a)}1_n$ for a,$b \geq 0$, $n \in \mathbb{Z}$, $n \geq b - a$,

where $E^{(a)}F^{(b)}1_{b-a} = F^{(b)}E^{(a)}1_{b-a}$. Let ${}_m\mathbb{B}_n$ be set of elements in $\mathbb{B}$ belonging to $1_m({}_{\mathcal{A}}\dot{\mathbf{U}})1_n$. The defining relations for the algebra ${}_{\mathcal{A}}\dot{\mathbf{U}}$ all lift to explicit isomorphisms in $\dot{\mathcal{U}}$ (see [7, Theorem 5.1, Theorem 5.9]) after associating to each $x \in \mathbb{B}$ a 1-morphism in $\dot{\mathcal{U}}$ as follows:

$$x \mapsto \mathcal{E}(x) := \begin{cases} \mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n & \text{if } x = E^{(a)}F^{(b)}1_n, \\ \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n & \text{if } x = F^{(b)}E^{(a)}1_n. \end{cases}$$

Let $\mathcal{B} = \{\mathcal{E}(x) \mid x \in \mathbb{B}\}$ and ${}_m\mathcal{B}_n = \{\mathcal{E}(x) \mid x \in {}_m\mathbb{B}_n\}$. For later use, we note that there is a bijection

$$\mathcal{E}\colon \mathbb{B} \xrightarrow{\cong} \mathcal{B}. \tag{5.10}$$

Theorem 5.1. ([7]) Every 1-morphism f in $\dot{\mathcal{U}}$ is isomorphic to a unique sum of elements in $\mathcal{B}$. The map

$$\begin{array}{rccc} \gamma\colon & {}_{\mathcal{A}}\dot{\mathbf{U}} & \longrightarrow & K_0(\dot{\mathcal{U}}) \\ & x & \mapsto & [\mathcal{E}(x)]_{\cong}, \end{array} \tag{5.11}$$

is an isomorphism of linear categories.

5.5. **The 2-category $\mathcal{U}^*$.** The 2-category $\mathcal{U}^*$ is defined as follows. The objects and 1-morphisms of $\mathcal{U}^*$ are the same as those of $\mathcal{U}$. Given a pair of 1-morphisms $f, g\colon n \to m$, the abelian group $\mathcal{U}^*(n,m)(f,g)$ is defined by

$$\mathcal{U}^*(n,m)(f,g) := \bigoplus_{t\in\mathbb{Z}} \mathcal{U}(n,m)(f, g\langle t\rangle).$$

The category $\mathcal{U}^*(n,m)$ is additive and enriched over $\mathbb{Z}$-graded abelian groups. Alternatively, the linear category $\mathcal{U}^*(n,m)$ is obtained from $\mathcal{U}(n,m)$ by adding a family of natural isomorphisms $f \to f\langle 1\rangle$ for each object f of the category $\mathcal{U}(n,m)$.

In $\mathcal{U}^*(n,m)$ an object f and its translation $f\langle t\rangle$ are isomorphic via the 2-isomorphism

$$1_f \in \mathcal{U}(n,m)(f, f\langle 0\rangle) = \mathcal{U}(n,m)(f, (f\langle t\rangle)\langle -t\rangle) \subset \mathcal{U}^*(n,m)(f, f\langle t\rangle).$$

The inverse of the isomorphism $1_f\colon\ f \to f\langle t\rangle$ is given by

$$1_f\langle t\rangle \in \mathcal{U}(n,m)(f\langle t\rangle, f\langle t\rangle) = \mathcal{U}(n,m)(f\langle t\rangle, (f\langle 0\rangle)\langle t\rangle) \subset \mathcal{U}^*(n,m)(f\langle t\rangle, f).$$

These isomorphisms $f \cong f\langle t\rangle$ make the Grothendieck group $K_0(\mathcal{U}^*)$ into a $\mathbb{Z}$-module, rather than $\mathbb{Z}[q,q^{-1}]$-module since $[f]_{\cong} = [f\langle t\rangle]_{\cong}$ in $\mathcal{U}^*$.

The horizontal composition in $\mathcal{U}$ induces horizontal composition in $\mathcal{U}^*$. It follows that the $\mathcal{U}^*(n,m)$, $n,m\in\mathbb{Z}$, form an additive 2-category.

The Karoubi envelope $\mathrm{Kar}(\mathcal{U}^*)$ will be denoted by $\dot{\mathcal{U}}^*$, which is equivalent as an additive 2-category to the additive 2-category obtained from $\dot{\mathcal{U}}$ by applying the procedure where we obtained $\mathcal{U}^*$ from $\mathcal{U}$, i.e.,

$$\dot{\mathcal{U}}^*(n,m)(f,g) = \bigoplus_{t\in\mathbb{Z}} \dot{\mathcal{U}}(n,m)(f, g\langle t\rangle).$$

6. Strongly upper-triangular basis for $\dot{\mathcal{U}}$

Before we define a basis of the hom-modules in $\dot{\mathcal{U}}$, we need some basic facts about symmetric functions.

6.1. **Symmetric functions.** For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_a)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_a \geq 0$ let $|\lambda| := \sum_{i=1}^a \lambda_i$. We denote by $P(a,b)$ the set of all partitions λ with at most a parts (i.e. with $\lambda_{a+1} = 0$) such that $\lambda_1 \leq b$. Moreover, the set of all partitions with at most a parts (i.e. the set $P(a,\infty)$) we denote simply by $P(a)$. We will denote the collection of all partitions of arbitrary size by P. The dual (conjugate) partition of λ is the partition $\lambda^t = (\lambda^t_1, \lambda^t_2, \ldots)$ with $\lambda^t_j = \sharp\{i \mid \lambda_i \geq j\}$ which is given by reflecting the Young diagram of λ along the diagonal. For a partition $\lambda \in P(a,b)$ we define the *complementary partition* $\lambda^c = (b-\lambda_a, \ldots, b-\lambda_2, b-\lambda_1)$. Finally we define $\widehat{\lambda} := (\lambda^c)^t$.

Let us denote by S_k the symmetric group and $\mathrm{Sym}_k = \mathbb{Z}[x_1,\ldots,x_k]^{S_k}$ the ring of symmetric polynomials. For any k-tuple μ of natural numbers we define

$$a_\mu = \sum_{\sigma\in S_k} \mathrm{sign}(\sigma) x^{\sigma(\mu)} \in \mathrm{Sym}_k,$$

where $x^\mu = x_1^{\mu_1} x_2^{\mu_2} \ldots x_k^{\mu_k}$. Clearly, $a_\mu = 0$ if $\mu_i = \mu_j$ for $i \neq j$, and it changes the sign after permuting μ_i and μ_{i+1}. The Schur polynomials $s_\mu \in \mathrm{Sym}_k$ are then defined as

$$s_\mu := \frac{a_{\mu+\delta}}{a_\delta} = \frac{\det(x_i^{\mu_j+k-j})_{1\leq i,j\leq k}}{\det(x_i^{k-j})_{1\leq i,j\leq k}}. \tag{6.1}$$

Note that this definition make sense for any k-tuple $(\mu_1,\ldots,\mu_k) \in \mathbb{Z}^k$ with $\mu_i \geq i-k$. We will make use of this fact in what follows.

Let Sym be the ring of symmetric functions, defined as a subring of the inverse limit of the system $(\mathrm{Sym}_k)_{k\geq 0}$ (see e.g. [12]). The elementary symmetric functions

$$e_j := \sum_{1\leq i_1<i_2<\cdots<i_j} x_{i_1}\dots x_{i_j}\ ,$$

or the complete symmetric functions

$$h_j := \sum_{1\leq i_1\leq i_2\leq\cdots\leq i_j} x_{i_1}\dots x_{i_j}$$

generate Sym as a free commutative ring, i.e., $\mathrm{Sym} = \mathbb{Z}[e_1, e_2, \dots] = \mathbb{Z}[h_1, h_2, \dots]$. The power sum symmetric functions are defined by

$$p_t := \sum_i x_i^t\ .$$

We have the Newton identities:

$$je_j = \sum_{i=1}^{j}(-1)^{i-1}e_{j-i}p_i,$$

$$jh_j = \sum_{i=1}^{j} h_{j-i}p_i\ .$$

For $t, j \geq 0$, set

$$e_{t,j} := \sum_{1\leq i_1<i_2<\cdots<i_j} x_{i_1}^t\dots x_{i_j}^t\ .$$

Then we have

$$je_{t,j} = \sum_{i=1}^{j}(-1)^{i-1}e_{t,j-i}p_{it}\ .$$

We also have for $t, j \geq 0$

$$e_{t,j} = \frac{1}{j!}\sum_{\lambda\in P(j)} M_{j,\lambda}p_t^{\lambda_1}p_{2t}^{\lambda_2}p_{3t}^{\lambda_3}\cdots$$

where

$$M_{j,\lambda} = j!(-1)^{|\lambda|-l(\lambda)}/\prod_{i\geq 1} i^{m_i(\lambda)}m_i(\lambda). \tag{6.2}$$

Here $m_i(\lambda)$ denotes the number of parts in λ of size i.

By Pieri's formulas for a partition λ we have

$$s_\lambda e_j = \sum_{\mu\in\lambda\otimes 1^j} s_\mu, \tag{6.3}$$

where $\lambda\otimes 1^j$ is the set of partitions obtained by adding j boxes to λ at most one per row.

The ring Sym has a Hopf algebra structure with comultiplication

$$\Delta\colon\ \mathrm{Sym}\longrightarrow\mathrm{Sym}\otimes\mathrm{Sym}$$

given by

$$\Delta(s_\lambda) = \sum_{\mu,\nu\in P} N_{\mu\nu}^{\lambda} s_\mu\otimes s_\nu,$$

where $N^{\lambda}_{\mu\nu}$ are the Littlewood-Richardson coefficients, counit

$$\varepsilon\colon \mathrm{Sym} \longrightarrow \mathbb{Z}, \quad s_\lambda \mapsto \delta_{\lambda,0},$$

and antipode

$$S\colon \mathrm{Sym} \longrightarrow \mathrm{Sym}, \quad s_\lambda \mapsto (-1)^{|\lambda|} s_{\lambda^t}.$$

We will also use the standard notation for the skew Schur functions

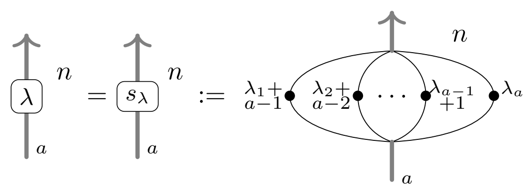
$$s_{\nu/\mu} = \sum_{\lambda} N^{\nu}_{\lambda\mu} s_\lambda \,.$$

We will need the following two bases for the ring Sym of symmetric functions:

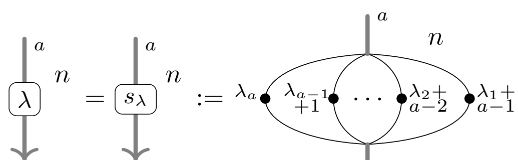
$$\{s_\lambda \mid \lambda \in P\}, \tag{6.4}$$

$$\{e_{t_1,j_1} \dots e_{t_s,j_s} \mid s \geq 0; t_1 > \dots > t_s \geq 1; j_1, \dots, j_s \geq 1\}. \tag{6.5}$$

6.2. **Basis for 2-morphisms in $\dot{\mathcal{U}}$.** For any partition $\lambda \in P(a)$ let us define

(6.6) $\mathcal{E}^{(a)}_{\lambda} \mathbf{1}_n = \mathcal{E}^{(a)}_{s_\lambda} \mathbf{1}_n =$  $: \mathcal{E}^{(a)}\mathbf{1}_n \to \mathcal{E}^{(a)}\mathbf{1}_n \langle 2|\lambda| \rangle$

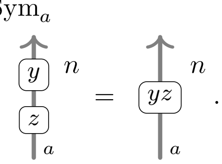
and analogously,

$\mathcal{F}^{(a)}_{\lambda} \mathbf{1}_n = \mathcal{F}^{(a)}_{s_\lambda} \mathbf{1}_n =$  $: \mathcal{F}^{(a)}\mathbf{1}_n \to \mathcal{F}^{(a)}\mathbf{1}_n \langle 2|\lambda| \rangle.$

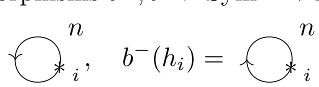
These notations for s_λ extends linearly to any elements of Sym_a. The correspondence

$$\mathrm{Sym}_a \ni y \mapsto \mathcal{E}^{(a)}_{y} \mathbf{1}_n$$

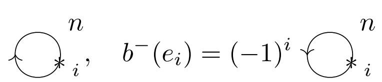
is multiplicative, i.e., we have for $y, z \in \mathrm{Sym}_a$



We will use the following algebra isomorphisms $b^+, b^-\colon \mathrm{Sym} \longrightarrow \mathrm{End}(\mathbf{1}_n)$ defined by

(6.7) $b^+(h_i) =$ 

It follows

(6.8) $b^+(e_i) = (-1)^i$ 

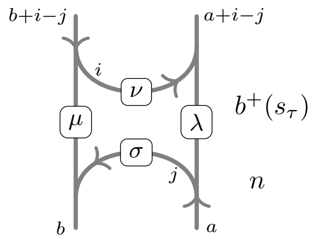
For $n \in \mathbb{Z}$, $a, b \geq 0$, $\delta \in \mathbb{Z}$, let us define the following sets:

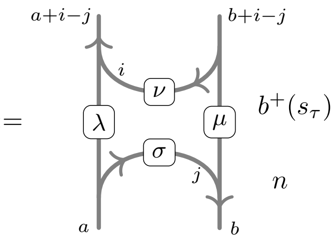
$$B^+(n,a,b,\delta) :=$$

$$\{f^{b,a,i,j}_{\lambda,\mu,\nu,\sigma,\tau} \mathbf{1}_n | 0 \leq i,j \leq \min(a,b), \delta = i-j, \lambda \in P(a-j), \mu \in P(b-j), \nu \in P(i), \sigma \in P(j), \tau \in P\},$$

$$B^{-}(n,a,b,\delta) :=$$
$$\{g^{a,b,i,j}_{\lambda,\mu,\nu,\sigma,\tau}\mathbf{1}_n | 0 \le i,j \le \min(a,b), \delta = i-j, \lambda \in P(a-j), \mu \in P(b-j), \nu \in P(i), \sigma \in P(j), \tau \in P\},$$

where

$f^{b,a,i,j}_{\lambda,\mu,\nu,\sigma,\tau}\mathbf{1}_n :=$ 

, $g^{a,b,i,j}_{\lambda,\mu,\nu,\sigma,\tau}\mathbf{1}_n :=$ 

Proposition 6.1 (Proposition 5.15 of [7]). Let $a,b \ge 0$, $\delta \in \mathbb{Z}$. The $\mathbb{Z}$-module

$$\dot{\mathcal{U}}(\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n, \mathcal{F}^{(b+\delta)}\mathcal{E}^{(a+\delta)}\mathbf{1}_n\langle t\rangle)$$

is free with basis given by the elements of $B^{+}(n,a,b,\delta)$ of degree t. Similarly, the $\mathbb{Z}$-module

$$\dot{\mathcal{U}}(\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{E}^{(a+\delta)}\mathcal{F}^{(b+\delta)}\mathbf{1}_n\langle t\rangle)$$

is free with basis given by the elements of $B^{-}(n,a,b,\delta)$ of degree t.

Corollary 6.2. The minimal degree of an element of $B^{+}(n,a,b,\delta)$ for $n \ge b-a$, and of $B^{-}(n,a,b,\delta)$ for $n \le b-a$ is at least δ^2. The only degree 0 element in $B^{+}(n,a,b,0)$ for $n \ge b-a$ (in $B^{-}(n,a,b,0)$ for $n \le b-a$) is the identity.

Proof. Let $n \le b-a$. Polynomials and bubbles can only increase the degree, so it is enough to compute the degree of $g^{a,b,i,j}_{0,0,0,0,0}\mathbf{1}_n$, which is

$$-j(b-j) - j(a-j) - i(b-j) - i(a-j) + i^2 + j^2 - i(n-2(b-j)) - j(n-2(b-j)) =$$

$$= -\delta(a+b-2j) + i^2 + j^2 - \delta(n-2(b-j)) = -\delta(n+a-b) + i^2 + j^2 \ge i^2 + j^2 \ge \delta^2\,.$$

For $\delta = 0$ it is 0 if and only if $i = 0$. The case when $n \ge b-a$ is similar. □

Corollary 6.3. Let $t,t' \in \mathbb{Z}$ and $x,y \in \mathbb{B}$. If $t'-t < 0$ or $t-t' = 0$ and $x \neq y$, then

$$\dot{\mathcal{U}}(\mathcal{E}(x)\langle t\rangle, \mathcal{E}(y)\langle t'\rangle) = 0.$$

The only elements in $\dot{\mathcal{U}}(\mathcal{E}(x)\langle t\rangle, \mathcal{E}(x)\langle t\rangle)$ are multiples of the identity.

6.3. **Strongly upper-triangular basis for $\dot{\mathcal{U}}(n,m)$.**

Proposition 6.4. For $m,n \in \mathbb{Z}$, the set ${}_n\mathcal{B}'_m := \{x\langle t\rangle \mid x \in {}_n\mathcal{B}_m, t \in \mathbb{Z}\}$ is a strongly upper-triangular basis for $\dot{\mathcal{U}}(n,m)$.

Proof. Let $B := {}_n\mathcal{B}'_m$. Corollary 6.3 ensures that $\dot{\mathcal{U}}(n,m)|_B$ is strongly upper-triangular. Proposition 6.1 ensures that the inclusion $\dot{\mathcal{U}}(n,m)|_B \to \dot{\mathcal{U}}(n,m)$ induces equivalence of additive categories $(\dot{\mathcal{U}}(n,m)|_B)^{\oplus} \simeq \dot{\mathcal{U}}(n,m)$. □

6.4. **Proof of Theorem 1.1.** Now we prove Theorem 1.1.

The homomorphism

$$h_{\dot{\mathcal{U}}(n,m)}\colon\ K_0(\dot{\mathcal{U}}(n,m)) \to \operatorname{Tr}(\dot{\mathcal{U}}(n,m))$$

is an isomorphism since it factors as the composition of isomorphisms

$$K_0(\dot{\mathcal{U}}(n,m)) \xrightarrow[\cong]{\gamma^{-1}} 1_{m\mathcal{A}}\dot{\mathbf{U}}1_n = \mathbb{Z}\,{}_n\mathbb{B}'_m \xrightarrow[\cong]{\mathbb{Z}\mathcal{E}'} \mathbb{Z}\,{}_n\mathcal{B}'_m \cong \operatorname{Tr}(\dot{\mathcal{U}}(n,m)).$$

Here γ is given in Theorem 5.1, $\mathcal{E}'\colon\ {}_n\mathbb{B}'_m \to {}_n\mathcal{B}'_m$ maps $x\langle t\rangle$ to $\mathcal{E}(x)\langle t\rangle$, and the last isomorphism follows from Propositions 4.7 and 6.4. Hence the linear functor $h_{\dot{\mathcal{U}}}\colon\ K_0(\dot{\mathcal{U}}) \to \operatorname{Tr}(\dot{\mathcal{U}})$ is an isomorphism.

Propositions 4.7 and 6.4 imply also that $\mathrm{HH}_i(\dot{\mathcal{U}}) = 0$ for $i > 0$.

□

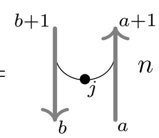
7. An alternative presentation of $\dot{\mathcal{U}}^*(n,m)$

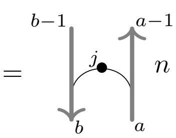
In this section we give an algebraic presentation of the Karoubi envelope $\dot{\mathcal{U}}^*(n,m)$ of $\mathcal{U}^*(n,m)$ obtained by reformulating results in [7].

If $n+m \geq 0$, then every object of $\dot{\mathcal{U}}^*(n,m)$ is isomorphic to a direct sum of finitely many copies of the objects $\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n$ with $2(a-b) = m-n$. In the following, we give a presentation of the full subcategory of $\dot{\mathcal{U}}^*(n,m)$ with objects $\{\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n \mid 2(a-b) = m-n\}$, which essentially gives a presentation of $\dot{\mathcal{U}}^*(n,m)$. Using symmetry, one can similarly define $\dot{\mathcal{U}}^*(n,m)$ when $n+m \leq 0$, by giving a presentation of the full subcategory of $\dot{\mathcal{U}}^*(n,m)$ with objects $\{\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n \mid 2(a-b) = m-n\}$.

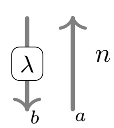
In the following we consider the full subcategory of $\dot{\mathcal{U}}^*(n,m)$ with objects $\{\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n \mid 2(a-b) = m-n\}$, where we do not assume $n+m \geq 0$ or $n+m \leq 0$.

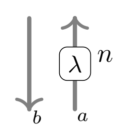
Let us define:

$t_j^{(b,a)} =$ 

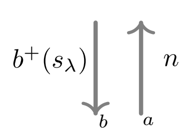
, $u_j^{(b,a)} =$ 

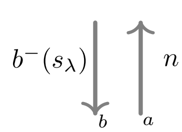
,

$d_\lambda^{(b,a)} =$ 

, $d'^{(b,a)}_\lambda =$ 

,

$b_\lambda^{(b,a)} = b^{+}{}_\lambda^{(b,a)} =$ 

, $b^{-}{}_\lambda^{(b,a)} =$ 

.

We extend these definitions by setting $u_j^{(b,a)} = 0$ for $a = 0$ or $b = 0$.

Theorem 7.1. Let n, m be integers with $n - m \in 2\mathbb{Z}$. The full subcategory of $\dot{\mathcal{U}}^*(n,m)$ with objects $\{\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n \mid 2(a-b) = m-n\}$ is generated as a linear category by the morphisms

$$t_j^{(b,a)}\colon\ \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n \to \mathcal{F}^{(b+1)}\mathcal{E}^{(a+1)}\mathbf{1}_n, \quad j \geq 0,$$
$$u_j^{(b,a)}\colon\ \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n \to \mathcal{F}^{(b-1)}\mathcal{E}^{(a-1)}\mathbf{1}_n, \quad j \geq 0,$$
$$d_\lambda^{(b,a)}\colon\ \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n \to \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n, \quad \lambda \in P(b),$$
$$d'^{(b,a)}_\lambda\colon\ \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n \to \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n, \quad \lambda \in P(a),$$
$$b_\lambda^{(b,a)}\colon\ \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n \to \mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n, \quad \lambda \in P$$

for $a, b \geq 0$, $2(a-b) = m-n$, subject to the relations

$$t_i t_j + t_j t_i = 0, \quad u_i u_j + u_j u_i = 0, \tag{7.1}$$

$$u_i t_j + t_j u_i = \widetilde{c}_{1+\frac{m+n}{2}+i+j}, \tag{7.2}$$

$$d'_\lambda t_i = \sum_{m\geq 0, \nu\in P} N^\lambda_{(m)\nu} t_{i+m} d'_\nu, \quad d_\lambda t_i = \sum_{m\geq 0, \nu\in P} N^\lambda_{(m)\nu} t_{i+m} d_\nu, \tag{7.3}$$

$$u_i d'_\lambda = \sum_{m\geq 0, \nu\in P} N^\lambda_{(m)\nu} d'_\nu u_{i+m}, \quad u_i d_\lambda = \sum_{m\geq 0, \nu\in P} N^\lambda_{(m)\nu} d_\nu u_{i+m}, \tag{7.4}$$

$$d_\mu d_\nu = \sum_\lambda N^\lambda_{\mu\nu} d_\lambda, \quad d'_\mu d'_\nu = \sum_\lambda N^\lambda_{\mu\nu} d'_\lambda, \quad b_\mu b_\nu = \sum_\lambda N^\lambda_{\mu\nu} b_\lambda, \tag{7.5}$$

$$[d'_\lambda, d_\mu] = [b_\lambda, t_i] = [b_\lambda, u_i] = [b_\lambda, d_\mu] = [b_\lambda, d'_\mu] = 0, \tag{7.6}$$

where we omit the superscripts always assuming that the last superscript in each relation is (b, a). (For example, the first relation in (7.1) is $t_i^{(b+1,a+1)} t_j^{(b,a)} + t_j^{(b+1,a+1)} t_i^{(b,a)} = 0$.) In (7.2), $\widetilde{c}_k$ for $k \in \mathbb{Z}$ is defined by

$$\widetilde{c}_k = \sum_{i,i',i''\geq 0, i+i'+i''=k} b_{(i)} (-1)^{i'} d_{(1^{i'})} d'_{(i'')}.$$

Note that $\widetilde{c}_k = 0$ for $k < 0$.

Proof. Here we give a sketch proof since the result is not used in the rest of the paper.

Let $V(n, m)$ be the linear category with $\mathrm{Ob}(V(n, m)) = \mathrm{Ob}(\dot{\mathcal{U}}^*(n, m))$ and with generators and relations as stated in the theorem. We define a linear functor

$$\mathcal{F}\colon V(n, m) \to \dot{\mathcal{U}}^*(n, m)|_{\{\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n \mid 2(a-b)=m-n\}}$$

which maps the objects and the generating morphisms in an obvious way. By checking the relations (7.1)–(7.6), one can verify that $\mathcal{F}$ is well defined. (For this verification, we need identities proved in [7].) Since $\mathcal{F}$ is identity on objects, it suffices to prove that $\mathcal{F}$ is full and faithful.

We first prove that $\mathcal{F}$ is full. For $i = 0, \dots, \min(a, b)$ and $\lambda \in P(i)$, we set

$$t_\lambda^{(b,a)} = t_{\lambda_1+i-1} t_{\lambda_2+i-2} \dots t_{\lambda_{i-1}+1} t_{\lambda_i}^{(b,a)},$$
$$u_\lambda^{(b,a)} = u_{\lambda_i} u_{\lambda_{i-1}+1} \dots u_{\lambda_2+i-2} u_{\lambda_1+i-1}^{(b,a)}.$$

where we omit the obvious superscripts. For $a, b \geq 0$, $\delta \in \mathbb{Z}$, let us define the following subsets of $V(n, m)$

$$B(n, a, b, \delta) :=$$
$$\{t_\nu d_\mu d'_\lambda u_\sigma b_\tau^{(b,a)} \mid 0 \leq i, j \leq \min(a, b), \delta = i-j, \lambda \in P(a-j), \mu \in P(b-j), \nu \in P(i), \sigma \in P(j), \tau \in P\}.$$

Using Lemma 7.2 below we see that the functor $\mathcal{F}$ sends $B(n, a, b, \delta)$ bijectively to the base $B^+(n, a, b, \delta)$ of $\dot{\mathcal{U}}^*(\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n, \mathcal{F}^{(b+\delta)}\mathcal{E}^{(a+\delta)}\mathbf{1}_n)$. Hence $\mathcal{F}$ is full.

To see that $\mathcal{F}$ is faithful, one has only to see that every morphism can be expressed as a linear combination of elements in $B(n, a, b, \delta)$. This is easily checked. □

The following lemma is a $\dot{\mathcal{U}}^*$-version of Proposition 6.1 that follows immediately.

Lemma 7.2. Let $a, b \geq 0$, $\delta \in \mathbb{Z}$. The $\mathbb{Z}$-graded module

$$\dot{\mathcal{U}}^*(\mathcal{F}^{(b)}\mathcal{E}^{(a)}\mathbf{1}_n, \mathcal{F}^{(b+\delta)}\mathcal{E}^{(a+\delta)}\mathbf{1}_n)$$

is free with basis given by the elements of $B^+(n, a, b, \delta)$. Similarly, the $\mathbb{Z}$-graded module

$$\dot{\mathcal{U}}^*(\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n, \mathcal{E}^{(a+\delta)}\mathcal{F}^{(b+\delta)}\mathbf{1}_n)$$

is free with basis given by the elements of $B^-(n,a,b,\delta)$.

8. The linear category ${}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t])$

8.1. **The $\mathbb{Q}$-algebra $\mathbf{U}(\mathfrak{sl}_2[t])$.** Recall that as a $\mathbb{Q}$-algebra, the current algebra $\mathbf{U}(\mathfrak{sl}_2[t])$ has the following presentation. The generators are E_i, F_i and H_i for $i \geq 0$, where $X_i = X \otimes t^i$. The relations are

$$[H_i, H_j] = [E_i, E_j] = [F_i, F_j] = 0,$$
$$[H_i, E_j] = 2E_{i+j}, \quad [H_i, F_j] = -2F_{i+j}, \quad [E_i, F_j] = H_{i+j}$$

for $i, j \geq 0$.

Let $\mathbf{U}^+(\mathfrak{sl}_2[t])$, $\mathbf{U}^-(\mathfrak{sl}_2[t])$ and $\mathbf{U}^0(\mathfrak{sl}_2[t])$ be subalgebras of $\mathbf{U}(\mathfrak{sl}_2[t])$ generated by $\{E_i \mid i \geq 0\}$, $\{F_i \mid i \geq 0\}$ and $\{H_i \mid i \geq 0\}$ respectively.

For every $i \geq 0$ we define

$$|E_i| = 2, \quad |F_i| = -2, \quad |H_i| = 0$$

and extend this definition of degree to $\mathbf{U}(\mathfrak{sl}_2[t])$ by setting $|xy| = |x| + |y|$.

The $\mathbb{Q}$-algebra $\mathbf{U}(\mathfrak{sl}_2[t])$ has a basis given by the elements

$$F_{i_1}^{a_1} \dots F_{i_r}^{a_r} H_{j_1}^{b_1} \dots H_{j_s}^{b_s} E_{k_1}^{c_1} \dots E_{k_t}^{c_t},$$

where

$$r \geq 0, \quad i_1 > \cdots > i_r \geq 0, \quad a_1, \dots, a_r \geq 1,$$
$$s \geq 0, \quad j_1 > \cdots > j_s \geq 0, \quad b_1, \dots, b_s \geq 1,$$
$$t \geq 0, \quad k_1 > \cdots > k_t \geq 0, \quad c_1, \dots, c_t \geq 1.$$

8.2. **The integral form ${}_{\mathbb{Z}}\mathbf{U}(\mathfrak{sl}_2[t])$ of $\mathbf{U}(\mathfrak{sl}_2[t])$.** For $a, i \geq 0$, define the divided powers of E_i and F_i by

$$E_i^{(a)} = \frac{1}{a!}E_i^a, \quad F_i^{(a)} = \frac{1}{a!}F_i^a.$$

Note that $E_i^{(0)} = F_i^{(0)} = 1$.

Let ${}_{\mathbb{Z}}\mathbf{U}(\mathfrak{sl}_2[t])$ denote the $\mathbb{Z}$-subalgebra of $\mathbf{U}(\mathfrak{sl}_2[t])$ generated by $E_i^{(a)}$, $F_i^{(a)}$, $i \geq 0$, $a \geq 1$. Set

$${}_{\mathbb{Z}}\mathbf{U}^0(\mathfrak{sl}_2[t]) = {}_{\mathbb{Z}}\mathbf{U}(\mathfrak{sl}_2[t]) \cap \mathbf{U}^0(\mathfrak{sl}_2[t]),$$
$${}_{\mathbb{Z}}\mathbf{U}^+(\mathfrak{sl}_2[t]) = {}_{\mathbb{Z}}\mathbf{U}(\mathfrak{sl}_2[t]) \cap \mathbf{U}^+(\mathfrak{sl}_2[t]),$$
$${}_{\mathbb{Z}}\mathbf{U}^-(\mathfrak{sl}_2[t]) = {}_{\mathbb{Z}}\mathbf{U}(\mathfrak{sl}_2[t]) \cap \mathbf{U}^-(\mathfrak{sl}_2[t]),$$

which are $\mathbb{Z}$-subalgebras of ${}_{\mathbb{Z}}\mathbf{U}(\mathfrak{sl}_2[t])$.

For $j \geq 0$ set $H_{j,0} = 1$. For $j \geq 0$ and $b > 0$, set recursively

$$bH_{j,b} = \sum_{l=1}^{b} (-1)^{l-1} H_{j,b-l} H_{lj},$$

or explicitly

$$H_{j,b} = \frac{1}{b!} \sum_{\lambda \in P(b)} M_{b,\lambda} H_j^{\lambda_1} H_{2j}^{\lambda_2} H_{3j}^{\lambda_3} \cdots$$

with $M_{b,\lambda}$ as in (6.2).

Define a ring homomorphism $\phi\colon$ Sym $\to \mathbf{U}^0(\mathfrak{sl}_2[t])$ by

$$\phi(e_j) = H_{1,j}.$$

Then we have

$$H_j = \phi(p_j), \quad H_{i,b} = \phi(e_{i,b}). \tag{8.1}$$

Let ${}_{\mathbb{Z}}\mathbf{U}^P(\mathfrak{sl}_2[t]) \subset {}_{\mathbb{Z}}\mathbf{U}^0(\mathfrak{sl}_2[t])$ be the image of ϕ.

In [6], Garland defined an integral basis of the loop algebra.

Proposition 8.1 (Garland [6], Thm. 5.8)**.** The $\mathbb{Z}$-algebra ${}_{\mathbb{Z}}\mathbf{U}(\mathfrak{sl}_2[t])$ is a free abelian group with basis given by the elements

$$F_{i_1}^{(a_1)} \dots F_{i_r}^{(a_r)} H_{j_1,b_1} \dots H_{j_s,b_s} E_{k_1}^{(c_1)} \dots E_{k_t}^{(c_t)},$$

where

$$\begin{aligned} r \geq 0, \quad & i_1 > \dots > i_r \geq 0, \quad a_1, \dots, a_r \geq 1, \\ s \geq 0, \quad & j_1 > \dots > j_s \geq 0, \quad b_1, \dots, b_s \geq 1, \\ t \geq 0, \quad & k_1 > \dots > k_t \geq 0, \quad c_1, \dots, c_t \geq 1. \end{aligned}$$

From this proposition and the basis for symmetric functions given in equation (6.5) the next lemma follows easily.

Lemma 8.2. The $\mathbb{Z}$-subalgebra ${}_{\mathbb{Z}}\mathbf{U}^P(\mathfrak{sl}_2[t])$ is free with basis given by the elements

$$H_{j_1,b_1} \dots H_{j_s,b_s},$$

where

$$s \geq 0, \quad j_1 > \dots > j_s \geq 0, \quad b_1, \dots, b_s \geq 1.$$

Thus, the map $\phi\colon \ \mathrm{Sym} \to {}_{\mathbb{Z}}\mathbf{U}^P(\mathfrak{sl}_2[t])$ is an isomorphism.

By using the Schur functions as a base, we obtain the following corollary.

Corollary 8.3. The $\mathbb{Z}$-algebra ${}_{\mathbb{Z}}\mathbf{U}(\mathfrak{sl}_2[t])$ is free with basis given by the elements

$$F_{i_1}^{(a_1)} \dots F_{i_r}^{(a_r)} \phi(s_\tau) H_{0,b_0} E_{k_1}^{(c_1)} \dots E_{k_t}^{(c_t)},$$

where

$$\begin{gathered} r \geq 0, \quad i_1 > \dots > i_r \geq 0, \quad a_1, \dots, a_r \geq 1, \\ \tau \in P, \\ b_0 \geq 0, \\ t \geq 0, \quad k_1 > \dots > k_t \geq 0, \quad c_1, \dots, c_t \geq 1. \end{gathered}$$

8.3. **The idempotented version $\dot{\mathbf{U}}(\mathfrak{sl}_2[t])$.** The idempotented version $\dot{\mathbf{U}}(\mathfrak{sl}_2[t])$ of $\mathbf{U}(\mathfrak{sl}_2[t])$ is the $\mathbb{Q}$-linear category defined as follows. The objects of $\dot{\mathbf{U}}(\mathfrak{sl}_2[t])$ are integers. For $m,n \in \mathbb{Z}$, the $\mathbb{Q}$-module $\dot{\mathbf{U}}(\mathfrak{sl}_2[t])(m,n)$ is defined by

$$\dot{\mathbf{U}}(\mathfrak{sl}_2[t])(m,n) = \mathbf{U}(\mathfrak{sl}_2[t])/(\mathbf{U}(\mathfrak{sl}_2[t])(H_0-m) + (H_0-n)\mathbf{U}(\mathfrak{sl}_2[t])).$$

The element in $\dot{\mathbf{U}}(\mathfrak{sl}_2[t])(m,n)$ represented by $x \in \mathbf{U}(\mathfrak{sl}_2[t])$ is denoted by

$$1_n x 1_m = x 1_m = 1_n x,$$

which is zero unless $n - m = |x|$. Composition in $\dot{\mathbf{U}}(\mathfrak{sl}_2[t])$ is induced by multiplication in $\mathbf{U}(\mathfrak{sl}_2[t])$, i.e.,

$$(1_p x 1_n)(1_n y 1_m) = 1_p x y 1_m,$$

for $x, y \in \mathbf{U}(\mathfrak{sl}_2[t])$, $m,n,p \in \mathbb{Z}$, $p-n = |x|$, $n-m = |y|$. The identity morphism for $n \in \mathbb{Z}$ is denoted by 1_n.

8.4. **The idempotented integral form ${}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t])$.** The idempotented version ${}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t])$ of the integral form ${}_{\mathbb{Z}}\mathbf{U}(\mathfrak{sl}_2[t])$ is the linear subcategory of $\dot{\mathbf{U}}(\mathfrak{sl}_2[t])$, such that $\mathrm{Ob}({}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t])) = \mathbb{Z}$ and

$$ {}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t])(m,n) = 1_m({}_{\mathbb{Z}}\mathbf{U}(\mathfrak{sl}_2[t]))1_n \subset \dot{\mathbf{U}}(\mathfrak{sl}_2[t])(m,n). $$

It follows that ${}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t])(m,n)$ has a basis as a free $\mathbb{Z}$-module given by the elements

$$ 1_m F_{i_1}^{(a_1)} \dots F_{i_r}^{(a_r)} \phi(s_\tau) E_{k_1}^{(c_1)} \dots E_{k_t}^{(c_t)} 1_n, \tag{8.2} $$

$$ \begin{gathered} r \geq 0, \quad i_1 > \cdots > i_r \geq 0, \quad a_1, \dots, a_r \geq 1, \\ \tau \in P, \\ t \geq 0, \quad k_1 > \cdots > k_t \geq 0, \quad c_1, \dots, c_t \geq 1, \\ c_1 + \cdots + c_t - (a_1 + \cdots + a_r) = 2(n-m). \end{gathered} $$

The above base will be used for the case $m+n \geq 0$. For $m+n \leq 0$ we will use another base, obtained from previous one by acting with the automorphism $\Phi\colon \mathbf{U}(\mathfrak{sl}_2[t]) \to \mathbf{U}(\mathfrak{sl}_2[t])$ such that

$$ \Phi(E_i) = F_i, \quad \Phi(F_i) = E_i, \quad \Phi(H_i) = -H_i. $$

9. Trace of the 2-category $\mathcal{U}^*$

In this section we prove Theorem 1.2 by computing $\mathrm{Tr}(\dot{\mathcal{U}}^*) \cong \mathrm{Tr}(\mathcal{U}^*)$.

9.1. **Split Grothendieck group of $\dot{\mathcal{U}}^*$.** Recall that the addition of translations in the definition of $\mathcal{U}^*$ identifies every 1-morphism with all its shifts. Hence, on the level of the split Grothendieck group multiplication with q becomes a trivial operation. Thus the split Grothendieck group of $K_0(\dot{\mathcal{U}}^*)$ can be obtained from those of $\dot{\mathcal{U}}$ by setting $q=1$ and it coincides with the integral idempotented version of $\mathbf{U}$.

9.2. **Generators of the trace of $\dot{\mathcal{U}}^*$.** Let us introduce the following notation

$$ \mathsf{E}_\mu^{(a)}\mathbf{1}_n := [\mathcal{E}_\mu^{(a)}\mathbf{1}_n], \quad \mathsf{F}_\lambda^{(a)}\mathbf{1}_n := [\mathcal{F}_\lambda^{(a)}\mathbf{1}_n], \quad b^+(\tau) := b^+(s_\tau). $$

Recall that $\mathsf{E}_\mu^{(a)}$ is the trace of the 2-endomorphism of $\mathcal{E}^{(a)}$ given by the multiplication with the Schur function indexed by μ.

Proposition 9.1. (Triangular Decomposition) For $n+m \geq 0$, $\mathrm{Tr}\dot{\mathcal{U}}^*(n,m)$ is a free $\mathbb{Z}$-module with basis

$$ \mathsf{F}_\mu^{(b)} b^+(\tau) \mathsf{E}_\lambda^{(a)} \mathbf{1}_n \quad \text{for } n \in \mathbb{Z},\ a,b \geq 0,\ 2(a-b) = m-n,\ \lambda \in P(a),\ \mu \in P(b),\ \tau \in P, $$

and for $n+m \leq 0$ by

$$ \mathsf{E}_\lambda^{(a)} b^+(\tau) \mathsf{F}_\mu^{(b)} \mathbf{1}_n \quad \text{for } n \in \mathbb{Z},\ a,b \geq 0,\ 2(a-b) = m-n,\ \lambda \in P(a),\ \mu \in P(b),\ \tau \in P. $$

Note that if $n+m=0$ then both these bases can be used.

Proof. We will prove the proposition for $m+n \geq 0$. The other case is similar.

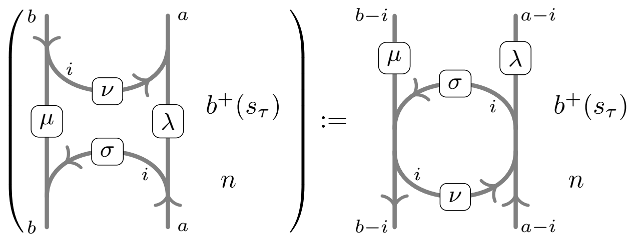
After bubble slides (Corollary 4.7 and Proposition 4.8 of [7]), we have to show that $\mathrm{Tr}\dot{\mathcal{U}}^*(n,m)$ has a basis given by

$$ \mathsf{F}_\mu^{(b)} \mathsf{E}_\lambda^{(a)} b^+(\tau) \mathbf{1}_n \quad \text{for} \quad n \in \mathbb{Z}, \quad a,b \geq 0, \quad 2(a-b) = m-n, \quad \lambda \in P(a), \quad \mu \in P(b). $$

Since every object of $\dot{\mathcal{U}}^*(n,m)$ is isomorphic to a direct sum of elements of ${}_m\mathcal{B}_n$ (Theorem 5.1), by Proposition 3.3 we have $\mathrm{Tr}\dot{\mathcal{U}}^*(n,m) \simeq \mathrm{Tr}(\dot{\mathcal{U}}^*(n,m)|_{{}_m\mathcal{B}_n})$. We will use Proposition 3.4 for the linear category $\mathcal{C} := \dot{\mathcal{U}}^*(n,m)|_{{}_m\mathcal{B}_n}$.

Let $H = \bigoplus_{x \in {}_m\mathcal{B}_n} \mathcal{C}(x,x)$. By Proposition 6.1, H is a free $\mathbb{Z}$-module generated by $\{f^{b,a,i,i}_{\lambda,\mu,\nu,\sigma,\tau}\mathbf{1}_n \mid 2(a-b) = n-m\}$. Let a subspace $K \subset H$ be generated by $\{f^{b,a,0,0}_{\lambda,\mu,0,0,\tau}\mathbf{1}_n \mid 2(a-b) = n-m\}$.

Define $p\colon H \to H$ as follows:

$p\left(f^{b,a,i,i}_{\lambda,\mu,\nu,\sigma,\tau}\mathbf{1}_n\right) = p$ 

It is obvious that $p(f) = f$ for $f \in K$, and that $[p(f)] = [f]$.

Using thick calculus relations to simplify the results of iteratively applying p, one can show that for every $f \in H$ there is $k \geq 0$ such that $p^k(f) \in K$. Let $\pi : H \to K$ be defined by $\pi(f) = p^k(f)$ where k is chosen as above. Since it is true for p, we have $\pi(f) = f$ for $f \in K$, and $[\pi(f)] = [f]$. So conditions (1) and (2) of Proposition 3.4 for the map π are satisfied. Condition (3) is proved in the following Lemma 9.2.

Thus, $\mathrm{Tr}\,\dot{\mathcal{U}}^* = \bigoplus_{n,m} \mathrm{Tr}\,\dot{\mathcal{U}}^*(n,m)$ is freely generated by $\{[f^{b,a,0,0}_{\lambda,\mu,0,0,\tau}\mathbf{1}_n] = \mathsf{F}^{(b)}_\lambda \mathsf{E}^{(a)}_\mu b^+(\tau)\mathbf{1}_n\}$ as desired. □

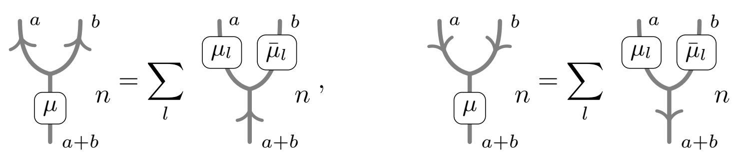
Lemma 9.2. For every $g \in (\dot{\mathcal{U}}^*(n,m))(s,t)$ and $h \in (\dot{\mathcal{U}}^*(n,m))(t,s)$, for $s,t \in {}_m\mathcal{B}_n$, the equation $\pi(gh) = \pi(hg)$ holds.

Proof. Let $s = \mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbf{1}_n$ and $t = \mathcal{E}^{(a+i-j)}\mathcal{F}^{(b+i-j)}\mathbf{1}_n$. By Proposition 6.1 it is enough to prove the statement for generators $g = f^{b,a,i,j}_{\lambda,\mu,\nu,\sigma,\tau}\mathbf{1}_n$ and $h = f^{b+i-j,a+i-j,i',j'}_{\lambda',\mu',\nu',\sigma',\tau'}\mathbf{1}_n$. Moreover, it is enough to prove it for:

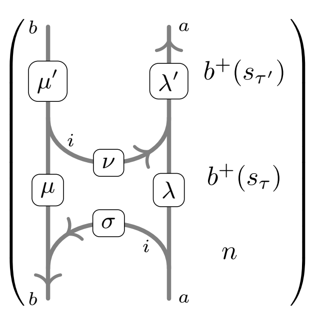
(1) $g = f^{b,a,i,i}_{\lambda,\mu,\nu,\sigma,\tau}\mathbf{1}_n$ and $h = f^{b,a,0,0}_{\lambda',\mu',0,0,\tau'}\mathbf{1}_n$,
(2) $g = f^{b,a,i,j}_{\lambda,\mu,\nu,\sigma,\tau}\mathbf{1}_n$ and $h = f^{b+i-j,a+i-j,i-j,0}_{0,0,\nu',0,0}\mathbf{1}_n$ and
(3) $g = f^{b,a,i,j}_{\lambda,\mu,\nu,\sigma,\tau}\mathbf{1}_n$ and $h = f^{b+i-j,a+i-j,0,j-i}_{0,0,0,\sigma',0}\mathbf{1}_n$,

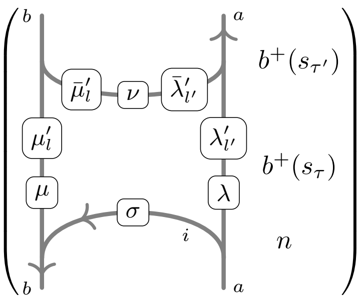
because all generators can be constructed from the above h's by vertical composition.

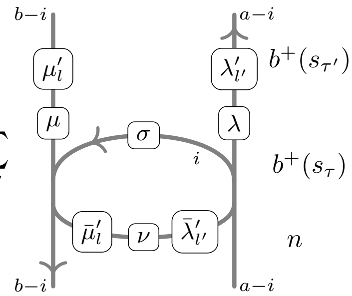
Schur functions can be slid through splitters using [7, equation (2.74)]. We will not need the precise form of these relations. In the calculations below it suffices to consider the generic formula of the form



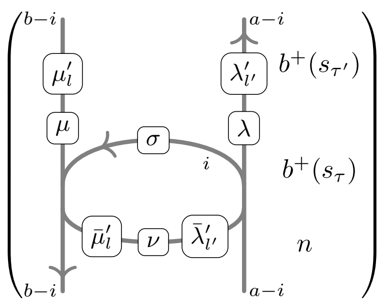
Without loss of generality let us assume $m \geq n$. Let us prove (1) by induction on b. If $b = 0$ it is obvious since $\pi(gh) = gh = hg = \pi(hg)$. If $b > 0$

$p(hg) = p$  $=$

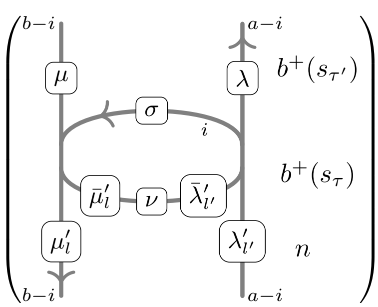
$= \sum_{l,l'} p$ 

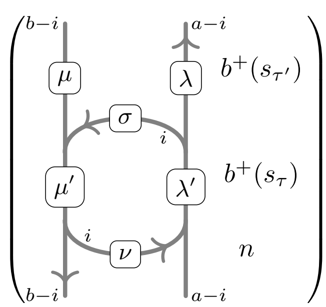
$= \sum_{l,l'}$ 

By acting with π on above relations and using assumption of the induction we get

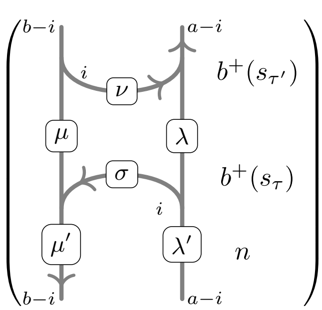
$\pi(hg) = \sum_{l,l'} \pi$ 

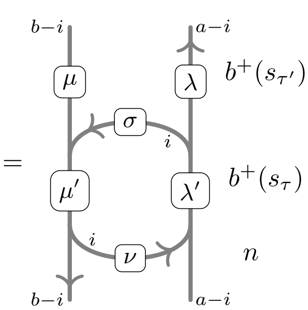
$=$

$= \sum_{l,l'} \pi$ 

$= \pi$ 

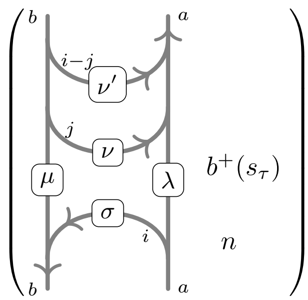
On the other side

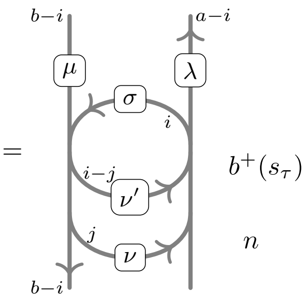
$p(gh) = p$ 

$=$ 

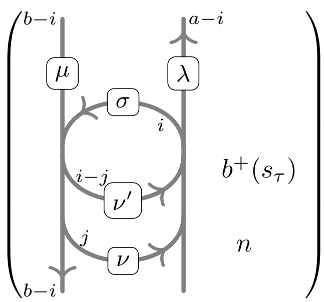
and by acting with π we get $\pi(hg) = \pi(gh)$.

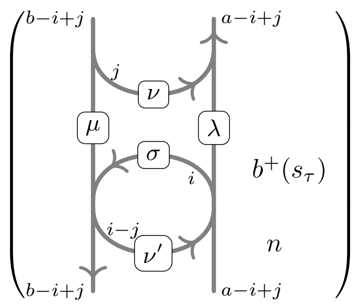
Let us prove (2) by induction on b. If $b = 0$ then $h = \mathbf{1}_n$, so it is obvious. Likewise, if $i - j = 0$ then $h = \mathbf{1}_n$. So let us assume $b > 0$ and $i - j > 0$.

$p(hg) = p$


$=$


By acting with π on above relations and using assumption of the induction we get

$\pi(hg) = \pi$


$= \pi$


$= \pi(gh).$

Case (3) is similar. □

The degree of $\mathsf{F}_\lambda^{(b)} b^+(\tau) \mathsf{E}_\mu^{(a)} \mathbf{1}_n$ is $2(|\lambda| + |\mu| + |\tau|) \geq 0$. So we have the following.

Corollary 9.3. The negative degree part of $\mathrm{Tr}\,\mathcal{U}^*$ is zero. For any positive integer i, the $2i$-degree part of $\mathrm{Tr}\,\dot{\mathcal{U}}^*(n,m)$ is freely generated by $\mathsf{F}_\lambda^{(a)} b^+(\tau) \mathsf{E}_\mu^{(b)} \mathbf{1}_n$ for $n + m \geq 0$ and by $\mathsf{E}_\mu^{(b)} b^+(\tau) \mathsf{F}_\lambda^{(a)} \mathbf{1}_n$ for $n + m \leq 0$. In both cases $2(a - b) = m - n$ and $|\lambda| + |\mu| + |\tau| = i$. In particular, the degree zero part of $\mathrm{Tr}\,\dot{\mathcal{U}}^*$ coincides with $K_0(\dot{\mathcal{U}}^*)$.

9.3. **Wedge product of symmetric polynomials.** As a preparation for the proof of Proposition 9.7 below, we need the following extension of the Schur polynomials to non-partition sequences.

For $a \geq 0$, set

$$\widetilde{P}(a) = \{(\widetilde{\mu}_1, \ldots, \widetilde{\mu}_a) \in \mathbb{Z}^a \mid \widetilde{\mu}_j \geq j - a\}.$$

Note that $P(a) \subset \widetilde{P}(a)$. As mentioned in Section 6.1, the definition (6.1) of the Schur polynomial extends to sequences in $\widetilde{P}(a)$ as follows. For $\widetilde{\mu} = (\widetilde{\mu}_1, \ldots, \widetilde{\mu}_a) \in \widetilde{P}(a)$, set

$$s_{\widetilde{\mu}} := \frac{a_{\widetilde{\mu}+\delta}}{a_\delta} = \frac{\det(x_i^{\widetilde{\mu}_j + a - j})_{1 \leq i,j \leq a}}{\det(x_i^{a-j})_{1 \leq i,j \leq a}}. \tag{9.1}$$

For $k = 1, \ldots, a-1$, we have

$$s_{(\widetilde{\mu}_1, \ldots, \widetilde{\mu}_a)} = -s_{(\widetilde{\mu}_1, \ldots, \widetilde{\mu}_{k-1}, \widetilde{\mu}_{k+1} - 1, \widetilde{\mu}_k + 1, \widetilde{\mu}_{k+2}, \ldots, \widetilde{\mu}_a)}. \tag{9.2}$$

Note that the sequence $(\widetilde{\mu}_1, \ldots, \widetilde{\mu}_{k-1}, \widetilde{\mu}_{k+1} - 1, \widetilde{\mu}_k + 1, \widetilde{\mu}_{k+2}, \ldots, \widetilde{\mu}_a)$ of the right hand side is obtained from $\widetilde{\mu}$ by permuting the kth and $k+1$st entries and adding ± 1 to them. Using (9.2), it is easily checked that for every $\widetilde{\mu} \in P(a)$ we have either $s_{\widetilde{\mu}} = 0$ or $s_{\widetilde{\mu}} = \pm s_\mu$ for some uniquely determined $\mu \in P(a)$. Consequently, $s_{\widetilde{\mu}} \in \mathrm{Sym}_a$.

For $\widetilde{\lambda} \in \mathbb{Z}^a$, $\widetilde{\mu} \in \mathbb{Z}^b$, set

$$\widetilde{\lambda} \cup \widetilde{\mu} = (\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_a, \widetilde{\mu}_1, \dots, \widetilde{\mu}_b) \in \mathbb{Z}^{a+b}.$$

Note that we have $\widetilde{\lambda} \cup \widetilde{\mu} \in P(a+b)$ if and only if $\widetilde{\lambda} \in P(a)$, $\widetilde{\mu} \in P(b)$ and $\widetilde{\lambda}_a \geq \widetilde{\mu}_1$.

Define the *wedge product* of symmetric polynomials

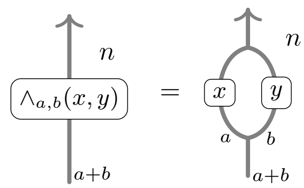
$$\wedge_{a,b} \colon \operatorname{Sym}_a \times \operatorname{Sym}_b \to \operatorname{Sym}_{a+b}$$

by

$$\wedge_{a,b}(s_\lambda, s_\mu) = s_{(\lambda - b)\cup\mu}$$

for $\lambda \in P(a)$, $\mu \in P(b)$, where $\lambda - b = (\lambda_1 - b, \dots, \lambda_a - b) \in \widetilde{P}(a)$. Since $(\lambda - b) \cup \mu \in \widetilde{P}(a+b)$, it follows that $s_{\lambda\cup\mu} \in \operatorname{Sym}_{a+b}$ is well defined. The maps $\wedge_{a,b}$, $a, b \geq 0$, form an algebra structure on the graded $\mathbb{Z}$-module $\bigoplus_{a\geq 0} \operatorname{Sym}_a$, which is isomorphic to the exterior algebra $\bigwedge \operatorname{Sym}_1 \cong \bigwedge \mathbb{Z}[h_1]$.

We have ([7, Proposition 2.9])



for $x \in \operatorname{Sym}_a, y \in \operatorname{Sym}_b$.

9.4. **Structure of the category** $(\operatorname{Tr}\dot{\mathcal{U}}^*)^+$**.** Let $\operatorname{Tr}(\dot{\mathcal{U}}^*)^+$ denote the linear subcategory of $\operatorname{Tr}(\dot{\mathcal{U}}^*)$ such that $\operatorname{Ob}(\operatorname{Tr}(\dot{\mathcal{U}}^*)^+) = \mathbb{Z}$ and the morphisms are generated by composites of $\mathsf{E}^{(a)}_\lambda \mathbf{1}_n$ for $n \in \mathbb{Z}$, $a \geq 0$, $\lambda \in P(a)$. Similarly, let $\operatorname{Tr}(\dot{\mathcal{U}}^*)^-$ denote the linear subcategory of $\operatorname{Tr}(\dot{\mathcal{U}}^*)$ such that $\operatorname{Ob}(\operatorname{Tr}(\dot{\mathcal{U}}^*)^-) = \mathbb{Z}$ and the morphisms are generated by $\mathsf{F}^{(a)}_\lambda \mathbf{1}_n$ for $n \in \mathbb{Z}$, $a \geq 0$, $\lambda \in P(a)$.

We define $\operatorname{Tr}(\dot{\mathcal{U}}^*)^0$ as the linear subcategory of $\operatorname{Tr}(\dot{\mathcal{U}}^*)$ with $\operatorname{Ob}(\operatorname{Tr}(\dot{\mathcal{U}}^*)^0) = \mathbb{Z}$ and the morphisms are generated by $b^+(\tau)\mathbf{1}_n$, $n \in \mathbb{Z}$, $\tau \in P$. It is easy to check that $\operatorname{Tr}(\dot{\mathcal{U}}^*)^0(n,m) = 0$ for $n \neq m$, and that $\operatorname{Tr}(\dot{\mathcal{U}}^*)^0(n,n)$ has basis given by $b^+(\tau)\mathbf{1}_n$ for $\tau \in P$.

For $x \in \operatorname{Sym}_a$, set

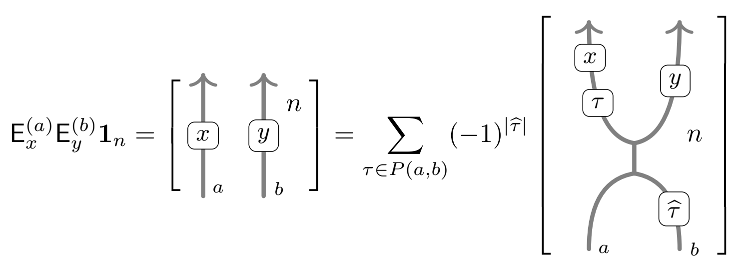
$$\mathsf{E}^{(a)}_x \mathbf{1}_n = \mathsf{E}^{(a)}(x)\mathbf{1}_n := [\mathcal{E}^{(a)}_x \mathbf{1}_n].$$

The following lemma gives the composition rule in $\operatorname{Tr}((\dot{\mathcal{U}}^*)^+)$.

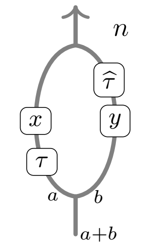
Lemma 9.4. For $x \in \operatorname{Sym}_a$, $y \in \operatorname{Sym}_b$, we have

$$\mathsf{E}^{(a)}_x \mathsf{E}^{(b)}_y \mathbf{1}_n = \sum_{\tau \in P(a,b)} (-1)^{|\widehat{\tau}|} \mathsf{E}^{(a+b)} \left(\wedge_{a,b}(x s_\tau \otimes s_{\widehat{\tau}} y) \right) \mathbf{1}_n.$$

Proof. We have



$$= \sum_{\tau \in P(a,b)} (-1)^{|\widehat{\tau}|} \left[\quad \right] = \sum_{\tau \in P(a,b)} (-1)^{|\widehat{\tau}|} \mathsf{E}^{(a+b)} \left(\wedge_{a,b}(x s_\tau \otimes s_{\widehat{\tau}} y) \right) \mathbf{1}_n.$$



□

Proposition 9.1 and Lemma 9.4 imply the following.

Proposition 9.5. Let $n, m \in \mathbb{Z}$, $a \geq 0$, and $m - n = 2a$. Then $\mathrm{Tr}(\dot{\mathcal{U}}^*)^+(n,m)$ is a free abelian group with basis

$$\{\mathsf{E}^{(a)}_\lambda \mathbf{1}_n \mid \lambda \in P(a)\}.$$

Our goal in this section is to obtain a presentation for $\mathrm{Tr}(\dot{\mathcal{U}}^*)^+$ and bases for $\mathrm{Tr}(\dot{\mathcal{U}}^*)^+(n,m)$ that facilitate a direct comparison with ${}_{\mathbb{Z}}\mathbf{U}^+(\mathfrak{sl}_2[t])$ and its basis given in Proposition 8.1. To do this we will need to relate the basis for $\mathrm{Tr}(\dot{\mathcal{U}}^*)^+(n,m)$ given in Proposition 9.5 to one given by composites of the form

$$\mathsf{E}^{(a_1)}_{l_1^{a_1}} \mathsf{E}^{(a_2)}_{l_2^{a_2}} \dots \mathsf{E}^{(a_p)}_{l_p^{a_p}} \mathbf{1}_n,$$

where $a_1 + a_2 + \dots + a_p = a$, $l_1 > l_2 > \dots > l_p \geq 0$, $a_i \geq 1$, $p \geq 0$.

In what follows we utilize the lexicographic order on partitions, where $\lambda > \mu$ implies $\lambda_1 > \mu_1$ or else $\lambda_i = \mu_i$ for $1 \leq i \leq k$ and $\lambda_{k+1} > \mu_{k+1}$ for some $k \geq 1$. The lemma below gives the leading term in this order for the change of basis from $\mathsf{E}^{(a)}_{j^a} \mathsf{E}^{(b)}_\lambda \mathbf{1}_n$ to $\mathsf{E}^{(a+b)}_\tau \mathbf{1}_n$.

Lemma 9.6. For $\lambda \in P(b)$, $j > \lambda_1$, we have

$$\mathsf{E}^{(a)}_{j^a} \mathsf{E}^{(b)}_\lambda \mathbf{1}_n - \mathsf{E}^{(a+b)}_{j^a \cup \lambda} \mathbf{1}_n \in \mathrm{Span}_{\mathbb{Z}} \{\mathsf{E}^{(a+b)}_\tau \mathbf{1}_n \mid \tau \in P(a+b), \tau < j^a \cup \lambda\}.$$

Proof. By Lemma 9.4, we have

$$\begin{aligned}
\mathsf{E}^{(a)}_{j^a} \mathsf{E}^{(b)}_\lambda \mathbf{1}_n &= \sum_{\tau \in P(a,b)} (-1)^{\widehat{\tau}} \mathsf{E}^{(a+b)} (\wedge_{a,b}(s_{j^a} s_\tau \otimes s_{\widehat{\tau}} s_\lambda)) \mathbf{1}_n \\
&= \sum_{\tau \in P(a,b)} (-1)^{\widehat{\tau}} \mathsf{E}^{(a+b)} (\wedge_{a,b}(s_{\tau + j} \otimes \sum_{\nu \in P(b)} N^\nu_{\widehat{\tau},\lambda} s_\nu)) \mathbf{1}_n \\
&= \mathsf{E}^{(a+b)} \left(\sum_{\tau \in P(a,b)} \sum_{\nu \in P(b)} (-1)^{\widehat{\tau}} N^\nu_{\widehat{\tau},\lambda} \wedge_{a,b} (s_{\tau+j} \otimes s_\nu) \right) \mathbf{1}_n \\
&= \mathsf{E}^{(a+b)} \left(\sum_{\tau \in P(a,b)} \sum_{\nu \in P(b)} (-1)^{\widehat{\tau}} N^\nu_{\widehat{\tau},\lambda} s_{(\tau_1 + j - b, \dots, \tau_a + j - b, \nu_1, \dots, \nu_b)} \right) \mathbf{1}_n,
\end{aligned}$$

where $\tau + j = (\tau_1 + j, \dots, \tau_a + j) \in P(a)$. Note that the term for $\tau = b^a$ in the above sum is exactly $s_{j^a \cup \lambda}$. One can check that the other terms are contained in $\mathrm{Span}_{\mathbb{Z}}\{s_\tau \mid \tau \in P(a+b), \tau < j^a \cup \lambda\}$. □

Proposition 9.7. The linear category $(\mathrm{Tr}\dot{\mathcal{U}}^*)^+$ has the following presentation.

- Objects are integers $n \in \mathbb{Z}$.
- Morphisms are generated by $\mathsf{E}^{(a)}_{l^a} \mathbf{1}_n \in (\mathrm{Tr}\dot{\mathcal{U}}^*)(n, n+2a)$ for $a, l \geq 0$, $n \in \mathbb{Z}$.

- The morphisms satisfy the following relations:

$$\mathsf{E}^{(a)}_{l^a}\mathsf{E}^{(b)}_{s^b}\mathbf{1}_n = \mathsf{E}^{(b)}_{s^b}\mathsf{E}^{(a)}_{l^a}\mathbf{1}_n \quad \text{for } n \in \mathbb{Z},\ a,b,l,s \geq 0, \tag{9.3}$$

$$\mathsf{E}^{(0)}_{l^0}\mathbf{1}_n = \mathbf{1}_n \quad \text{for } n \in \mathbb{Z},\ l \geq 0, \tag{9.4}$$

$$\mathsf{E}^{(a)}_{l^a}\mathsf{E}^{(b)}_{l^b}\mathbf{1}_n = \begin{pmatrix} a+b \\ a \end{pmatrix} \mathsf{E}^{(a+b)}_{l^{a+b}}\mathbf{1}_n \quad \text{for } n \in \mathbb{Z},\ a,b,l \geq 0. \tag{9.5}$$

Proof. First we prove that $\mathrm{Tr}(\dot{\mathcal{U}}^*)^+$ is spanned by composites of the morphisms $\mathsf{E}^{(a)}_{l^a}\mathbf{1}_n$, $n \in \mathbb{Z}$, $a, l \geq 0$. Suppose we are given $\mathsf{E}^{(a)}_{\lambda}\mathbf{1}_n$, $n \in Z$, $a \geq 0$, $\lambda \in P(a)$. We will show by induction on a and λ (in the lexicographic order) that it is a linear combination of composites of morphisms of the form $\mathsf{E}^{(i)}_{l^i}\mathbf{1}_n$ with $i, l \geq 0$ and $n \in \mathbb{Z}$. If $a = 0$, then there is nothing to prove. Suppose $a > 0$. If $\lambda = (\lambda_1)^a$, then we are done. Otherwise, let $k \geq 1$ be such that $\lambda_1 = \cdots = \lambda_k > \lambda_{k+1}$. Then, by Lemma 9.6, we have

$$\mathsf{E}^{(a)}_{\lambda}\mathbf{1}_n - \mathsf{E}^{(k)}_{\lambda_1^k}\mathsf{E}^{(a-k)}_{(\lambda_{k+1},\ldots,\lambda_a)}\mathbf{1}_n \in \mathrm{Span}_{\mathbb{Z}}\{\mathsf{E}^{(a)}_{\tau}\mathbf{1}_n \mid \tau \in P(a), \tau < \lambda\}.$$

By the induction hypothesis, it follows that $\mathsf{E}^{(a)}_{\lambda}\mathbf{1}_n$ is as desired.

Now we prove the relations given in the statement. The relation (9.4) is obvious. Relation (9.5) is proven using Lemma 9.4,

$$\begin{aligned}
\mathsf{E}^{(a)}_{l^a}\mathsf{E}^{(b)}_{l^b}\mathbf{1}_n &= \sum_{\tau \in P(a,b)} (-1)^{\widehat{\tau}} \mathsf{E}^{(a+b)}(\wedge_{a,b}(s_{l^a}s_\tau \otimes s_{\widehat{\tau}}s_{l^b}))\mathbf{1}_n \\
&= \sum_{\tau \in P(a,b)} (-1)^{\widehat{\tau}} \mathsf{E}^{(a+b)}(\wedge_{a,b}(s_{\tau+l} \otimes s_{\widehat{\tau}+l}))\mathbf{1}_n \\
&= \mathsf{E}^{(a+b)}\left(\sum_{\tau \in P(a,b)} (-1)^{\widehat{\tau}}(s_{(\tau+l)\cup(\widehat{\tau}+l)})\right)\mathbf{1}_n.
\end{aligned}$$

Since we have

$$s_{(\tau+l)\cup(\widehat{\tau}+l)} = (-1)^{\widehat{\tau}} s_{l^{a+b}}$$

for $\tau \in P(a,b)$ and we have $|P(a,b)| = \binom{a+b}{a}$ relation (9.5) follows. □

Corollary 9.8. As a $\mathbb{Z}$-module, $(\mathrm{Tr}\dot{\mathcal{U}}^*)^+(n, n+2a)$ has a basis given by

$$\mathsf{E}^{(a_1)}_{l_1^{a_1}}\mathsf{E}^{(a_2)}_{l_2^{a_2}}\ldots\mathsf{E}^{(a_p)}_{l_p^{a_p}}\mathbf{1}_n, \tag{9.6}$$

where $a_1 + a_2 + \cdots + a_p = a$, $l_1 > l_2 > \cdots > l_p \geq 0$, $a_i \geq 1$, $p \geq 0$.

Proof. Using the relation 9.5 it is clear that $(\mathrm{Tr}\dot{\mathcal{U}}^*)^+(n, n+2a)$ is spanned by the elements given in (9.6). To see that these elements are linearly independent, we use Lemma 9.6 repeatedly to deduce that

$$\mathsf{E}^{(a_1)}_{l_1^{a_1}}\mathsf{E}^{(a_2)}_{l_2^{a_2}}\ldots\mathsf{E}^{(a_p)}_{l_p^{a_p}}\mathbf{1}_n - \mathsf{E}^{(a)}_{(l_1^{a_1},l_2^{a_2},\ldots,l_p^{a_p})}\mathbf{1}_n \in \mathrm{Span}\{\mathsf{E}^{(a)}_{\tau}\mathbf{1}_n \mid \tau \in P(a), \tau < (l_1^{a_1}, l_2^{a_2}, \ldots, l_p^{a_p})\},$$

so that the change of basis between elements in (9.6) and those in Proposition 9.5 is upper triangular. □

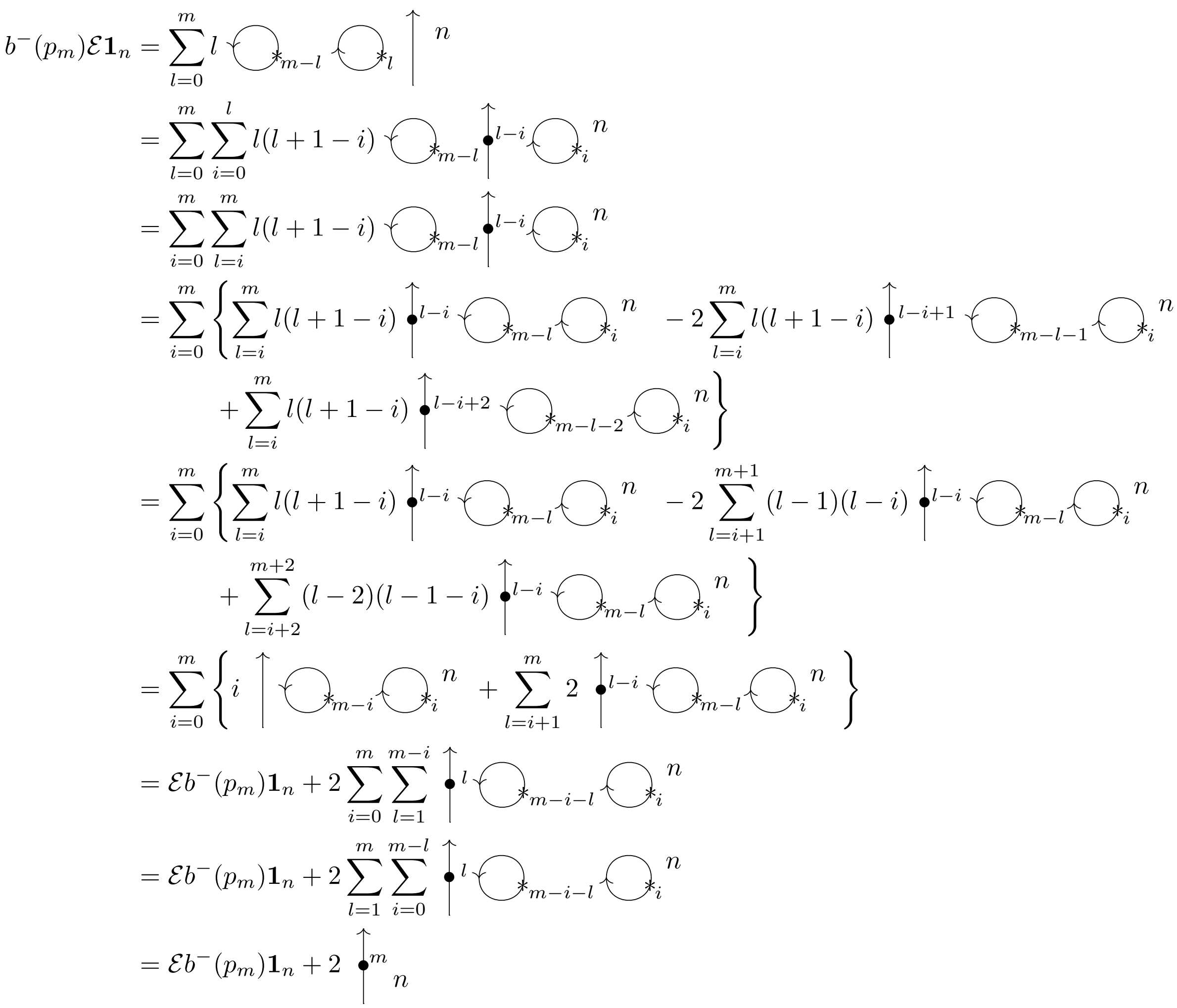
9.5. **Commutation relations in** $\mathrm{Tr}\,\mathcal{U}^*$**.** Before the main result we need the following two lemmas.

Lemma 9.9. The equation

$$b^-(p_m)\mathcal{E}\mathbf{1}_n - \mathcal{E}b^-(p_m)\mathbf{1}_n = 2\mathcal{E}_m\mathbf{1}_n$$

holds in the 2-category $\mathcal{U}$.

Proof. The proof is by direct computation using the relations in $\mathcal{U}$.



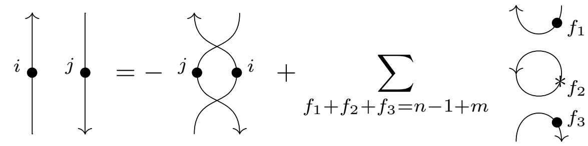
□

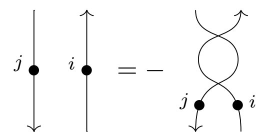
Lemma 9.10. The commutation relation

$$\mathsf{E}_i\mathsf{F}_j\mathbf{1}_n - \mathsf{F}_j\mathsf{E}_i\mathbf{1}_n = \begin{cases} n\mathbf{1}_n & \text{if } i+j=0 \\ b^-(p_{i+j})\mathbf{1}_n & \text{otherwise,} \end{cases}$$

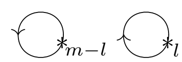
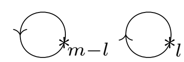
holds in $\mathrm{Tr}\,\mathcal{U}^*$.

Proof. Let $m := i + j$. For $n \geq 0$, the relations





imply

$$\mathsf{E}_i\mathsf{F}_j\mathbf{1}_n - \mathsf{F}_j\mathsf{E}_i\mathbf{1}_n = \sum_{l=0}^{m}(l+1)$$  $$= \sum_{l=0}^{m} l$$  $$=$$

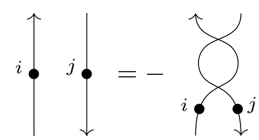
$$= \sum_{l=0}^{m}(-1)^{m-l} l b^-(h_l) b^-(e_{m-l})\mathbf{1}_n = b^-\left[\sum_{l=0}^{m}(-1)^{m-l} l h_l e_{m-l}\right]\mathbf{1}_n =$$

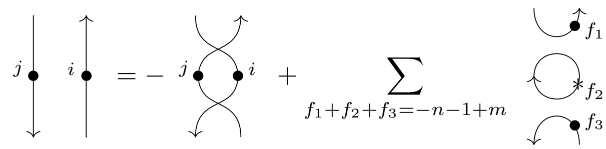
$$= b^-\left[s_m - s_{m-1,1} + s_{m-2,1^2} - \cdots + (-1)^{m-1}s_{1^m}\right]\mathbf{1}_n = b^-(p_m)\mathbf{1}_n,$$

where we use that $\sum_{l=0}^{m}(-1)^l h_{m-l}e_l = 0$ and the identity

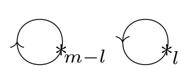
$$p_m = s_m - s_{m-1,1} + s_{m-2,1^2} - \cdots + (-1)^{m-1}s_{1^m} \in \mathrm{Sym}\,.$$

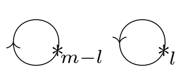
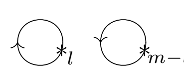
If $n \leq 0$ then the relations





imply

$$\mathsf{E}_i\mathsf{F}_j\mathbf{1}_n - \mathsf{F}_j\mathsf{E}_i\mathbf{1}_n = -\sum_{l=0}^{m}(l+1)$$  $$=$$

$$= \sum_{l=0}^{m}(m-l)$$  $$= \sum_{l=0}^{m} l$$  $$=$$

$$= \sum_{l=0}^{m}(-1)^{m-l} l b^-(h_l) b^-(e_{m-l})\mathbf{1}_n = b^-\left[\sum_{l=0}^{m}(-1)^{m-l} l h_l e_{m-l}\right]\mathbf{1}_n =$$

$$= b^-\left[s_m - s_{m-1,1} + s_{m-2,1^2} - \cdots + (-1)^{m-1}s_{1^m}\right]\mathbf{1}_n = b^-(p_m)\mathbf{1}_n.$$

□

9.6. **Main result.**

Proof of Theorem 1.2. We need to show that $\mathrm{Tr}\,\mathcal{U}^*$, as a linear category, is isomorphic to the idempotented integral form ${}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t])$. Because of Proposition 3.2 it is enough to prove that $\mathrm{Tr}\,\dot{\mathcal{U}}^*$ is isomorphic to ${}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t])$.

Let us define the functor $\mathcal{F} : \dot{\mathbf{U}}(\mathfrak{sl}_2[t]) \to \mathrm{Tr}\,\dot{\mathcal{U}}^* \otimes \mathbb{Q}$ to be identity on the objects and to send $E_i 1_n$ and $F_i 1_n$ to $\mathsf{E}_i \mathbf{1}_n$ and $\mathsf{F}_i \mathbf{1}_n$, respectively. Moreover, let

$$H_i 1_n \mapsto \begin{cases} n\mathbf{1}_n & i = 0, \\ b^-(p_i)\mathbf{1}_n & i > 0. \end{cases}$$

Let us check the relations on target, i.e. we need to show that the following relations hold true:

$$\begin{aligned} [b^-(p_i), b^-(p_j)] &= 0 \\ [\mathsf{E}_i\mathbf{1}_n, \mathsf{E}_j\mathbf{1}_n] &= [\mathsf{F}_i\mathbf{1}_n, \mathsf{F}_j\mathbf{1}_n] = 0 \\ [b^-(p_i), \mathsf{E}_j\mathbf{1}_n] &= 2\mathsf{E}_{i+j} \\ [b^-(p_i), \mathsf{F}_j\mathbf{1}_n] &= -2\mathsf{F}_{i+j} \\ [\mathsf{E}_i, \mathsf{F}_j] &= \begin{cases} n\mathbf{1}_n & \text{if} \quad i+j=0, \\ b^-(p_{i+j}\mathbf{1}_n) & \text{otherwise.} \end{cases} \end{aligned}$$

The first relation is obvious, the second follows from Proposition 9.7. The next one follows from Lemma 9.9. The last relation is shown in Lemma 9.10. So, the functor $\mathcal{F}$ is well defined.

It is clear that $\mathcal{F}$ sends

$$E_j^{(a)} \mapsto \mathsf{E}_{j^a}^{(a)}, \quad F_j^{(a)} \mapsto \mathsf{F}_{j^a}^{(a)}, \quad \phi(s_\tau) \mapsto b^+(\tau).$$

Proposition 9.1 and Corollary 9.8 give the basis of $\mathrm{Tr}\,\dot{\mathcal{U}}^*(n,m)$ for $n+m \geq 0$:

$$\mathsf{F}_{i_1^{a_1}}^{(a_1)} \dots \mathsf{F}_{i_r^{a_r}}^{(a_r)} b^+(\tau) \mathsf{E}_{k_1^{c_1}}^{(c_1)} \dots \mathsf{E}_{k_t^{c_t}}^{(c_t)} \mathbf{1}_n,$$

$$\begin{gathered} i_1 > \cdots > i_r \geq 0, \quad r \geq 0, \quad a_1, \dots, a_r \geq 1, \\ \tau \in P, \\ k_1 > \cdots > k_t \geq 0, \quad t \geq 0, \quad c_1, \dots, c_t \geq 1, \\ c_1 + \cdots + c_t - (a_1 + \cdots + a_r) = 2(n-m), \end{gathered}$$

and similar for $m+n \leq 0$.

We see that $\mathcal{F}$ sends basis elements (8.2) of ${}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t])(n,m)$ to the basis elements of $\mathrm{Tr}\,\dot{\mathcal{U}}^*(n,m)$, both for $n+m \geq 0$ and $n+m \leq 0$. So it is an isomorphism. Therefore, the restriction $\mathcal{F}|_{{}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t])}$: ${}_{\mathbb{Z}}\dot{\mathbf{U}}(\mathfrak{sl}_2[t]) \to \mathrm{Tr}\,\dot{\mathcal{U}}^*$ is also an isomorphism. □

By using Theorem 1.2, we can get rid of two cases in triangular decomposition 9.1 and have the following corollary.

Corollary 9.11 (Triangular Decomposition)**.** For every $n, m \in \mathbb{Z}$, $\mathrm{Tr}\,\dot{\mathcal{U}}^*(n,m)$ is free with basis

$$\mathsf{F}_\mu^{(b)} b^+(\tau) \mathsf{E}_\lambda^{(a)} \mathbf{1}_n \quad \text{for} \quad n \in \mathbb{Z}, \quad a, b \geq 0, \quad 2(a-b) = m-n, \quad \lambda \in P(a), \quad \mu \in P(b), \quad \tau \in P.$$

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