Topological quantum field theory and invariants of graphs for quantum groups

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Abstract: On basis of generalized 6j-symbols we give a formulation of topological quantum field theories for 3-manifolds including observables in the form of coloured graphs. It is shown that the 6j-symbols associated with deformations of the classical groups at simple even roots of unity provide examples of this construction. Calculational methods are developed which, in particular, yield the dimensions of the state spaces as well as a proof of the relation, previously announced for the case of $SU_q(2)$ by V.Turaev, between these models and corresponding ones based on the ribbon graph construction of Reshetikhin and Turaev.

1 Introduction

In ref. [TV] a novel combinatorial approach to 3-dimensional topological quantum field theory was proposed. Its basis is the observation that the 6j-symbols of $SU_q(2)$ obey the symmetries of a tetrahedron and satisfy identities which may also be interpreted geometrically in terms of glued tetrahedra and which lead to the possibility of associating state sums (partition functions) with 3-dimensional triangulated manifolds which are independent of the triangulation, i.e. they are topological invariants.

This approach was generalized in [DJN, D] to a large class of algebras (replacing $SU_q(2)$) with associated generalized 6j-symbols thus leading to a class of (unitary) 3-dimensional topological quantum field theories satisfying all the standard properties (see [Wi], [At]).

In the case of $SU_q(2)$ a second generalization was introduced in [KS] by including observables in the form of coloured graphs on the boundary or the interiour

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of the manifolds. This leads to effective calculational methods which were used to calculate e.g. the dimensions of the state spaces in this model.

In this paper we combine the approaches of [DJN, D] and [KS] by providing a simple geometric interpretation of the state sums introduced in [KS]. This leads to a simplification of the discussion in [KS] in a more general setting, including the case of 6j-symbols associated to quantum deformations of an arbitrary classical group.

As an application we calculate the dimensions of the state spaces in the general case in terms of the fusion matrices and we give a proof that the partition function $Z(M)$ associated to a closed 3-manifold M is related to the invariant $\tau(M)$ introduced by Reshetikhin and Turaev [RT2] in terms of ribbon graphs by

$$Z(M) = |\tau(M)|^2 \tag{1.1}$$

for quantum deformations of the classical groups at the even simple roots of unity.

The paper is organized as follows. In section 2 we formulate a general system of axioms for 6j-symbols appropriate for our construction. In section 3 we construct the state sums $Z(M, G_{\underline{x}})$ where M is a 3-manifold and G is a graph on the boundary ∂M whose lines are coloured by labels indicated by $\underline{x}$, and we discuss the geometric meaning of $Z(M, G_{\underline{x}})$. Section 4 is devoted to an analysis of the properties of $Z(M, G_{\underline{x}})$, as e.g. its behaviour under cutting of handles or removal of tubes in M. This analysis yields the desired calculational tools which are applied in section 5 to evaluate the dimensions of the state spaces and to establish eq. (1.1). In section 6 we establish the properties of 6j-symbols stated in section 2 in the case of a quantum group by ribbon graph techniques. In addition, we prove that the state sum for a planar graph coincides with the corresponding ribbon graph invariant of [RT1].

2 Abstract 6j-symbols

In this section we list the defining properties of the abstract 6j-symbols to be used in the subsequent construction.

Let I be a finite set with involution $* : I \to I$ $(i \mapsto i^*)$ and a distinguished element $0 = 0^*$. The elements in I will be called colours. To each triple of colours $(i, j, k) \in I^3$ there is associated a finite dimensional complex vector space V_{ij}^k of dimension N_{ij}^k, and we assume there exist canonical isomorphisms

$$V_{ij}^k \simeq V_{jk^*}^{i^*}\,, \quad V_{j^*i^*}^{k^*} \simeq (V_{ij}^k)^* \tag{2.1}$$

where $(V_{ij}^k)^*$ denotes the dual vector space of V_{ij}^k, and

$$V_{ij}^k \simeq V_{ji}^k\,. \tag{2.2}$$

In the following these isomorphisms will be used without further notice to identify these spaces. Moreover, we assume that

$$N^0_{ij^*} = \delta_{ij} \ . \tag{2.3}$$

In [DJN] a rather general framework for the construction of Hilbert spaces V^k_{ij} fulfilling these properties was given, where I labels a set of irreducible representations of an algebra satisfying certain properties. The isomorphism (2.2) (see [DJN, section 5]) is, however, not needed for the construction of a topological quantum field theory for manifolds without graphs, but is crucial when graphs with vertices of order 4 are present, as will be seen below.

The first isomorphism in (2.1) allows us to associate a vector space with each coloured oriented 2-simplex (triangle) as follows. Let σ^2 be an oriented 2-simplex with boundary links $\sigma^1_1, \sigma^1_2, \sigma^1_3$ decorated by arrows as indicated in Fig.1,

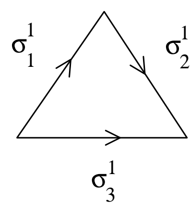


Fig.1

where the orientation of σ^2 is supposed to be clockwise. A colouring of the links in σ^2 is a mapping $j : \sigma^1_i \mapsto j(\sigma^1_i) \in I$, $i = 1,2,3$. Given a colouring j of σ^2 we associate with (σ^2, j) the vector space

$$V(\sigma^2, j) = V^{j(\sigma^1_3)}_{j(\sigma^1_1)j(\sigma^1_2)} \ . \tag{2.4}$$

If an arrow on a link σ^1_i is reversed we replace in this definition $j(\sigma^1_i)$ by $j^*(\sigma^1_i)$. The first isomorphism in (2.1) shows that this definition is invariant under rotations of the triangle in R^2 and the second shows that the vector space associated with the same triangle with reversed orientation is the dual of the original one.

Next, we assume the existence of a mapping $i \mapsto \omega_i$ from I to $\mathbf{C}/\{0\}$ such that $\omega_0 = 1, \omega^2_k = \omega^2_{k^*}$ and

$$\sum_k \omega^2_k N^k_{ij} = \omega^2_i \omega^2_j \ , \quad \sum_i {\omega_i}^4 = \omega^2 \neq 0 \ . \tag{2.5}$$

In [DJN] ω^2_i was denoted by F^{-1}_i and in the case of a quantum group, to be discussed later, it equals the q-dimension of the representation i up to a sign. The assumption $\omega^2_k = \omega^2_{k^*}$ then expresses the reality of the q-dimension while the first relation in (2.5) reflects the multiplicativity of the q-dimension w.r.t. tensor products.

Finally, to each ordered 6-tuple $(i,j,k,l,m,n) \in I^6$, is associated an abstract 6j-symbol,

$$\left| \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right| \in V_{ij}^k \otimes V_{kl}^m \otimes V_{i^*m}^n \otimes V_{j^*n}^l. \tag{2.6}$$

By the association of vector spaces with coloured 2-simplexes discussed above we see that the tensor product in (2.6) may be associated to the boundary of the coloured tetrahedron depicted in Fig.2.

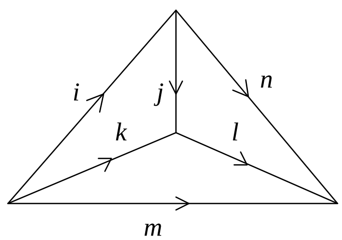


Fig.2

In order that the 6j-symbol in (2.6) define a unique vector associated to the tetrahedron it must be invariant under the tetrahedral symmetry group, i.e.

$$\left| \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right| = \left| \begin{matrix} k^* & i & j^* \\ n & l & m \end{matrix} \right| = \left| \begin{matrix} l & m^* & k^* \\ i & j^* & n^* \end{matrix} \right|. \tag{2.7}$$

In addition, we shall assume that

$$\left| \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right| = \left| \begin{matrix} j & i & k \\ m^* & l^* & n^* \end{matrix} \right|, \tag{2.8}$$

which is seen to imply that reversal of orientation of the tetrahedron is equivalent to applying the involution $*$ to all labels.

Besides the symmetry relations (2.7-8) the 6j-symbols are assumed to satisfy the following four relations, where products of 6j-symbols mean (unordered) tensor products together with contraction w.r.t. mutually dual pairs of vector spaces associated with certain pairs of factors:

1) Orthogonality:

$$\sum_k \omega_k^2 \left| \begin{matrix} i & j & k \\ C & B & A \end{matrix} \right| \left| \begin{matrix} i & j & k \\ C & B & A' \end{matrix} \right|^* = \omega_A^{-2} \delta_{AA'} 1_{V_{iA}^B \otimes V_{jc}^A}, \tag{2.9}$$

where

$$\left| \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right|^* = \left| \begin{matrix} i^* & j^* & k^* \\ l^* & m^* & n^* \end{matrix} \right|,$$

2) Biedenharn-Elliot relations:

$$\sum_n \omega_n^2 \left| \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right| \left| \begin{matrix} i & n & m \\ D & A & C \end{matrix} \right| \left| \begin{matrix} j & l & n \\ D & C & B \end{matrix} \right| = \left| \begin{matrix} i & j & k \\ B & A & C \end{matrix} \right| \left| \begin{matrix} k & l & m \\ D & A & B \end{matrix} \right|, \tag{2.10}$$

3) Racah identities: There exists a mapping $k \mapsto q_k$ from I to $\mathbf{C}/\{0\}$, such that $q_0 = 1$, $q_k = q_{k^*}$ and

$$\frac{q_k}{q_i q_j} \begin{vmatrix} i & j & k \\ A & B & C \end{vmatrix} = \sum_D \omega_D^2 \frac{q_A q_B}{q_C q_D} \begin{vmatrix} i & A & D \\ j & B & C \end{vmatrix} \begin{vmatrix} j & i & k \\ A & B & D \end{vmatrix}. \tag{2.11}$$

Moreover, we assume that the replacement of all q_i in (2.11) by q_i^{-1} leads to another identity.

4) Considering $1_{V_{iA}^B}$ as a vector in $V_{iA}^B \otimes (V_{iA}^B)^*$ and identifying V_{i0}^i, $i \in I$, with $\mathbf{C}$ according to (2.3) we have

$$\begin{vmatrix} i' & A' & B \\ A^* & i & 0 \end{vmatrix} = \frac{\delta_{AA'}\delta_{ii'}}{\omega_i \omega_A} 1_{V_{iA}^B} \ . \tag{2.12}$$

The 6j-symbols defined in [DJN] with the notation

$$F_{pq} \begin{bmatrix} i & j \\ k & l \end{bmatrix} = \omega_p^2 \begin{vmatrix} i & j & q \\ l & k & p \end{vmatrix} \tag{2.13}$$

for the quantum groups obtained as deformations of the universal enveloping algebra $U_q\mathfrak{g}$ of the classical semisimple Lie algebra $\mathfrak{g}$, were shown to satisfy the assumptions (2.7), (2.9) and (2.10) if q is an even simple root of unity and I is a certain set of irreducible representations of $U_q\mathfrak{g}$. These were shown to be sufficient for the construction of a topological quantum field theory for manifolds without graphs. The additional relations (2.8), (2.12) and Racah identities will turn out to be of importance for the inclusion of graphs with vertices of oder 4. These relations were not discussed in [DJN], but they are relatively easy to establish in the case of quantum groups by the methods developed there, with

$$q_i^2 = i(c)$$

for each representation $i \in I$, where c is a certain central element in the quantum group (acting as multiplication by a scalar q_i^2, since i is irreducible). In fact, it is straight forward to check (2.12) and it is possible to see that (2.8) follows from the Racah identities.

As to the latter one defines the braiding operator

$$B_{pq} \begin{bmatrix} i & j \\ k & l \end{bmatrix} : V_{jp}^k \otimes V_{jl}^p \to V_{jq}^k \otimes V_{il}^q$$

by setting its matrix element to be given as

$$< B_{pq} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \alpha \otimes \beta, \delta \otimes \gamma >= \alpha \circ \beta \circ \check{R}_{12}^{ij} \circ \gamma^* \circ \delta^* \tag{2.14}$$

for $\alpha \in V_{jp}^k, \beta \in V_{jl}^p, \delta \in V_{jq}^k, \gamma \in V_{il}^q$, where the RHS is an intertwiner between k and k and hence a number, and, if V_i denotes the representation space of i, $\check{R}_{12}^{ji} : V_j \otimes V_i \to V_i \otimes V_j$ is given by (see [DJN],[D])

$$\check{R}_{12}^{ji} = \sigma \circ (j \otimes i)(R)\,, \tag{2.15}$$

where $\sigma(x \otimes y) = y \otimes x$ for $x \in V_j, y \in V_i$ and $R \in U_q\mathsf{g} \otimes U_q\mathsf{g}$ is the standard R-matrix.

Defining the isomorphism $V_{ij}^k \simeq V_{ji}^k$ by

$$\beta \mapsto \hat{\beta} = \frac{q_k}{q_i q_j} \beta \circ \check{R}_{12}^{ij} \tag{2.16}$$

we have $\beta = \hat{\hat{\beta}}$ and one finds that

$$\omega_p^{-2} B_{pq} \left[\begin{array}{cc} i & j \\ k & l \end{array} \right] = \frac{q_p q_q}{q_k q_l} \left| \begin{array}{ccc} i & l & q \\ j & k & p \end{array} \right| \tag{2.17}$$

which is the vector that will be associated to a 4-vertex (or rectangle) (see Fig.6 below).

Using the definition (2.14) it is quite easy to verify eq. (2.11) by using complete reducibility of tensor products of representations in I and the properties of R (eq. (5.4) in [DJN]). Replacing in (2.16) $q_i \to q_i^{-1}$ and $R \to R^{-1}$ in a similar way we obtain the second Racah identity.

In section 6 we describe an alternative way of establishing the properties of the 6j-symbols.

3 State sums for manifold with coloured graphs

Let be M an oriented compact 3-manifold with triangulation X whose 1-simplexes (links) are assumed to be decorated by arrows in some arbitrary way, and let $\underline{j} : X \ni \sigma^1 \mapsto j(\sigma^1) \in I$ be a colouring of 1-simplexes in X. Furthermore, let G be an oriented graph on the boundary ∂M, i.e. a finite 1-dimensional simplicial complex without boundary, which is compatible with X, i.e. the 1-skeleton of G is contained in the 1-skeleton of ∂X. We assume that G has only 2-, 3- and 4-vertices. The 4-vertices can be of two types:

which we shall call inverse to each other. Orientations of links on opposite sides of 2- and 4-vertices are assumed to coincide. A colouring of G is a mapping $\underline{x}$ from its lines (maximal connected sets of links joined by vertices of order 2) such that the colours on opposite sides of 4-vertices are identical. The graph G coloured by $\underline{x}$ will be denoted by $G_{\underline{x}}$.

We associate now with $(M, X, G_{\underline{x}})$ a new triangulated pseudo-manifold $(M_{G_{\underline{x}}}, X_{G_{\underline{x}}})$ as follows: Let $\mathcal{C} = \{c_1, \ldots, c_n\}$ denote the set of connected components of $\partial M/G$, which we shall identify with new additional vertices. For each triangle $\sigma^2 \in c_i$ we glue onto ∂M a new tetrahedron, which has base σ^2 and an opposite vertex c_i. Furthermore, we glue these tetrahedra together along triangles, which they share, i.e. for two triangles $\sigma_1^2 \in c_i$ and $\sigma_2^2 \in c_i$ with $\sigma_1^2 \cap \sigma_2^2 = [AB]$ the corresponding tetrahedra are glued along the triangle (ABc_i) (see Fig.3).

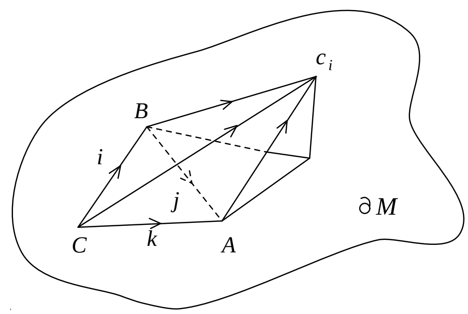


Fig.3

Next, for each k-coloured link $\sigma^1 \in \partial X$, contained in an x-coloured line of $G_{\underline{x}}$, and which is contained in the boundaries of the components c_i, c_j in $\mathcal{C}$ (Fig. 4), we glue on a tetrahedron along the two triangles containing σ^1 and c_i, respectively c_j, and continually glue along common triangles of the added tetrahedra.

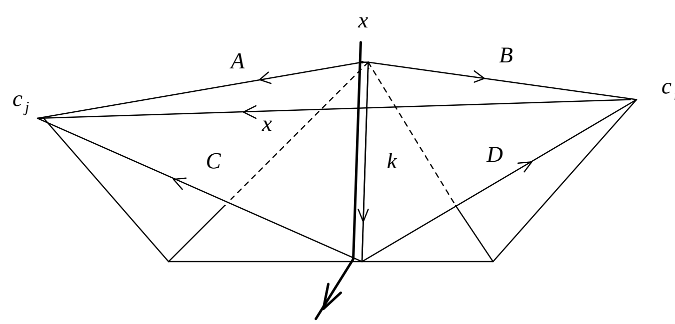


Fig.4

To the so obtained simplicial complex we glue for each vertex v of order 3 a new tetrahedron which contains the vertices (v, c_i, c_j, c_k), where c_i, c_j, c_k are the components in $\mathcal{C}$ sharing v (Fig.5).

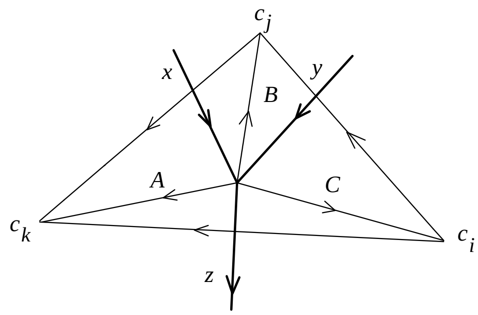


Fig.5

This finishes the construction of the coloured triangulated pseudo-manifold $(M_{G_{\underline{x}}}, X_{G_{\underline{x}}})$. The corresponding uncoloured triangulated pseudo-manifold will be denoted (M_G, X_G). Its principal feature is that it contains (M, X) in its interiour except for the original 4-vertices in G, which are now 4-vertices in ∂X_G. It is convenient to view the four triangles in ∂X_G sharing such a 4-vertex v as making up a rectangle with v as center. Then ∂X_G consists of a set of triangles and rectangles, namely one triangle for each 3-vertex in G and one rectangle for each 4-vertex in G, and hence ∂X_G may be viewed as a dual graph to G in ∂M.

We next associate with $(M, G_{\underline{x}})$ the state sum

$$Z(M, G_{\underline{x}}) = Z(M_{G_{\underline{x}}}), \tag{3.1}$$

where $Z(M_{G_{\underline{x}}})$ is defined in analogy with [TV, DJN] as follows: Given a colouring $\underline{j}$ of X, a factor ω^{-2} is attached to each interiour vertex $\sigma^0 \in int X_{G_{\underline{x}}}$, a factor ω_i^2 to each i-coloured interiour link $\sigma^1 \in int X_{G_{\underline{x}}}$ and a 6j-symbol to each tetrahedron $\sigma^3 \in X_{G_{\underline{x}}}$ as described above. Finally, to each coloured rectangle in $\partial X_{G_{\underline{x}}}$, as depicted in Fig.6,

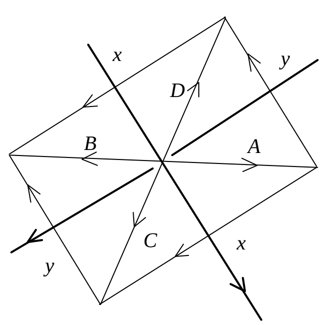


Fig.6

whose orientation is assumed to be clockwise, we attach the braiding operator

$$\frac{q_A q_B}{q_C q_D} \begin{vmatrix} y & A & D \\ x & B & C \end{vmatrix} \in V^D_{Ay} \otimes V^B_{Dx} \otimes V^C_{By^*} \otimes V^A_{Cx^*} \,. \tag{3.2}$$

For the inverse 4-vertex the braiding operator can be obtained from (3.2) by replacement of all q_i by q_i^{-1}. We then form the product of all factors so associated to $X_{G_{\underline{x}}}$, contract w.r.t. all interiour triangles as well as the triangles inside the rectangles in $\partial X_{G_{\underline{x}}}$ and take the direct sum over all colourings $\underline{j}$ of X, thus obtaining the desired vector

$$Z(M_{G_{\underline{x}}}) \in V(\partial M, G_{\underline{x}}) \,, \tag{3.3}$$

where $V(\partial M, G_{\underline{x}})$ is the tensor product of vector spaces associated to the coloured triangles in $\partial M_{G_{\underline{x}}}$, which are dual to 3-vertices in $G_{\underline{x}}$. Note that $Z(M_{G_{\underline{x}}})$ is independent of the initial distribution of arrows on the links in X, since $i \mapsto i^*$ is a bijective mapping on I.

This definition of $Z(M_{G_{\underline{x}}})$ is identical to the one in [DJN] except for the issues concerning 4-vertices in G and that no factors ω^{-1} are associated to vertices in $\partial X_{G_{\underline{x}}}$. By the same arguments as in [DJN] it then follows that $Z(M_{G_{\underline{x}}})$ *is independent of the interiour of the triangulation* $X_{G_{\underline{x}}}$ as a consequence of the identities (2.9-10). In particular, it is independent of the triangulation X of M and is invariant under homotopy changes of $G_{\underline{x}}$, since the latter clearly only affect the interiour of $X_{G_{\underline{x}}}$.

In general M_G is not a manifold. For example, in case G is empty one point c_i will be added to X for each connected component of ∂M and the points corresponding to components not homeomorphic to S^2 will be non-manifolds points of M_G. On the other hand, if G is sufficiently "large" then M_G will be homeomorphic to M. A particular graph of this type on a connected surface Σ of genus $g \geq 1$ may be obtain by first choosing a homology basis $a_1, ..., a_g, b_1, ..., b_g$ whose representatives have one point P in common and then deforming it such that P is separated into $4g-2$ vertices of order 3. We shall adopt the notation of [KS] and call this graph G^Σ. The dual graph to G^Σ on Σ is easily seen to yield a proper triangulation[2] of Σ and M_{G^Σ} is obtained from M, $\partial M = \Sigma$, by gluing on a cylinder $\Sigma \times [0,1]$ as is easy to verify.

In general, we have for a graph G *without vertices of order 4* such that M_G is homeomorphic to M that the partition function $Z'(M)$ of [DJN] is given by

$$Z'(M) = \omega^{-N_G} \otimes_{\underline{x}} Z(M, G_{\underline{x}}) \,, \tag{3.4}$$

where N_G is the number of connected components of $\partial M / G$ and the triangulation of ∂M is given by the dual graph to G. In particular,

$$Z'(M) \in V(\partial M, G) \,, \tag{3.5}$$

where

$$V(\partial M, G) \equiv \oplus_{\underline{x}} V(\partial M, G_{\underline{x}}) \tag{3.6}$$

is the vector space associated to the surface ∂M triangulated by the dual graph to G.

[2] We use here a more general notion of triangulation than the standard one, in that we allow identifications of subsimplexes of co-dimension > 1 in a simplex of maximal dimension.

Given a closed, compact, oriented surface Σ, a graph G in Σ and a compact oriented 3-manifold M with $\partial M = \Sigma$, we call G a *canonical graph* in Σ if M_G is homeomorphic to M, and we note that this property only depends on Σ and G but not on M. In particular, a graph G on Σ with no vertices of order 4 is canonical if and only if its dual graph in Σ yields a triangulation of Σ.

The correspondence (3.4) allows us to rephase results of [DJN] in terms of manifolds with graphs. Of particular interest is the gluing property (axiom (3) of [At]) whose triangulated version holds by construction for $Z'(M)$ (see [DJN]) and may be reformulated as follows.

Theorem 3.1 *Let M be a (connected or disconnected) oriented compact 3-manifold with $\partial M = \Sigma_1 \cup \Sigma_2 \cup \Sigma \cup \Sigma^*$ (disjoint union), where $\Sigma_1, \Sigma_2, \Sigma, \Sigma^*$ are closed oriented surfaces such that there exists an orientation reversing diffeomorphism $F : \Sigma \to \Sigma^*$. Moreover, let $G^1_{\underline{x_1}} \subseteq \Sigma_1$ and $G^2_{\underline{x_2}} \subseteq \Sigma_2$ be arbitrary coloured graphs and let $G \subseteq \Sigma$ be a canonical graph without vertices of order 4 and G^F its image in Σ^* by F. Then, if M_Σ denotes the 3-manifold obtained by gluing M along Σ and Σ^* by F, we have*

$$Z(M_\Sigma, G^1_{\underline{x_1}} \cup G^2_{\underline{x_2}}) = \omega^{-2N_G} \sum_{\underline{x}} \prod_{l \subseteq G} \omega^2_{x_l} (Z(M, G^1_{\underline{x_1}} \cup G^2_{\underline{x_2}} \cup G_{\underline{x}} \cup G^F_{\underline{x}}))_{G_{\underline{x}}}\,, \quad (3.7)$$

where the product on the RHS is over lines l in G and $(\cdot)_{G_{\underline{x}}}$ indicates a contraction w.r.t. all pairs of vector spaces associated to coloured 3-vertices in $G_{\underline{x}}$ and $G^F_{\underline{x}}$.

This theorem generalizes Thm. 7.1 of [KS] and is identical to eq. (4.6) in [DJN] in case G^1 and G^2 are canonical graphs, but its validity for arbitrary G^1 and G^2 follows, as in [DJN], immediately from the construction of $Z(M_G)$ and its independence of the interiour of the triangulation X_G of M_G.

It should be noted that in eq. (3.4) we have not exhibited the dependence of $Z'(M) \equiv Z'(M, G)$ on G for the following reason. Let G be as in eq. (3.4), i.e. G is a canonical graph in $\partial M \equiv \Sigma$ without 4-vertices, and consider the cylindrical 3-manifold $\Sigma \times [0,1]$ oriented such that we may identify Σ^* with $\Sigma \times \{0\}$ and Σ with $\Sigma \times \{1\}$. Then

$$P_{(\Sigma,G)} \equiv Z'(\Sigma \times [0,1], G \times \{0\}, G \times \{1\}) \in V(\Sigma, G)^* \otimes V(\Sigma, G) \simeq Hom(V(\Sigma, G), V(\Sigma, G))\,.$$

Since $M \cup_\Sigma (\Sigma \times [0,1])$ is homeomorphic to M it follows from eq. (3.7) that

$$Z'(M, G) = P_{(\Sigma,G)} Z'(M, G) \in P_{(\Sigma,G)} V(\Sigma, G) \equiv V'(\Sigma, G)\,.$$

Here we note that factors ω^2_x in eq. (3.7) associated to the coloured lines in $G_{\underline{x}}$ are included in the definition of the bilinear pairing of $V(\Sigma^*, G)$ and $V(\Sigma, G)$ (see eq. (3.30) in [DJN]).

It follows, moreover, from eq. (3.7) (see section 4 of [DJN] for details) that $P_{(\Sigma,G)}$ is a projection and that $V'(\Sigma, G_1)$ and $V'(\Sigma, G_2)$ for two arbitrary canonical graphs in Σ may be consistently identified by the mappings

$$P_{(\Sigma,G_1),(\Sigma,G_2)} \equiv Z'(\Sigma \times [0,1], G_1 \times \{0\}, G_2 \times \{1\}) \in V(\Sigma, G_1)^* \otimes V(\Sigma, G_2) \simeq$$

$$Hom(V(\Sigma, G_1), V(\Sigma, G_2))$$

by which the vectors $Z(M, G_1)$ and $Z(M, G_2)$ are also identified.

In force of these identifications we thus obtain the vector space

$$V_\Sigma \simeq V'(\Sigma, G) \subseteq V(\Sigma, G)$$

containing the vector

$$Z'(M) \equiv Z'(M, G)$$

for all 3-manifolds M with $\partial M = \Sigma$.

By eq. (3.7) and the fact that $P_{(\Sigma,G)}$ is a projection we have

$$dim V_\Sigma = tr P_{(\Sigma,G)} = Z'(\Sigma \times S^1) = Z(\Sigma \times S^1)\,. \tag{3.8}$$

For later use we note the following simple lemma, valid for any compact oriented 3-manifolds M, M_1, M_2, and where we use the notation

$$Z(M) \equiv Z(M, \emptyset)\,,$$

with $\emptyset$ denoting the empty graph.

Lemma 3.2 *i) If $D^3 \subseteq intM$ is diffeomorphic to the 3-ball, then, for any graph $G \subseteq \partial M$,*

$$Z(M, G) = \omega^{-2} Z(M/D^3, G)\,. \tag{3.9}$$

ii)

$$Z(S^3) = \omega^{-2}, \quad Z(D^3) = 1\,. \tag{3.10}$$

iii) If M^* denotes M with opposite orientation, then

$$Z(M) = Z(M^*)\,. \tag{3.11}$$

iv) If M is the connected sum of M_1 and M_2, i.e. $M = (M_1/D_1^3) \cup_{S^2} (M_2/D_2^3)$, where $D_1^3 \subseteq intM_1$ and $D_2^3 \subseteq intM_2$ are diffeomorphic to the 3-ball and whose boundaries S^2 are identified, then

$$Z(M, G_1 \cup G_2) = \omega^2 Z(M_1, G_1) \otimes Z(M_2, G_2) \tag{3.12}$$

for arbitrary graphs $G_1 \subseteq \partial M_1$ and $G_2 \subseteq \partial M_2$.

Proof:

i) Follows since $M_G = (M/D^3)_G$, but no factor ω^{-2} is associated to the vertex c corresponding to the component $\partial D^3 \subseteq \partial(M/D^3)$ in the definition of $Z(M/D^3, G)$.

ii) By triangulating S^3 by two tetrahedra the first relation follows from (2.9) and the definition of ω^2. The second relation then follows from *i)* applied to $M = S^3$.

iii) Follows from (2.8) since all colours are summed over and $i \mapsto i^*$ is a bijective mapping from I to I.

iv) We first note that M is diffeomorphic to

$$(M_1/D_1^3) \underset{S^2\times\{0\}}{\cup} (S^2 \times [0,1]) \underset{S^2\times\{1\}}{\cup} (M_2/D_2^3)\,. \tag{3.13}$$

Next we note that the graph $\overline{G}$ in S^2 consisting of two 3-vertices connected by three lines is canonical since its dual yields a triangulation of S^2 by two triangles. We can obtain a triangulation of $S^2 \times [0,1]$ such that $S^2 \times \{0\}$ and $S^2 \times \{1\}$ are both triangulated by two triangles by taking two prisms, each triangulated in a standard fashion by three tetrahedra, and gluing then together along their sides. Using this triangulation a straightforward calculation using (2.9) and (2.5) yields

$$Z(S^2 \times [0,1], \overline{G}_{\underline{x}} \times \{0\}, \overline{G}_{\underline{y}} \times \{1\}) = \omega^2 1_{V^{x_3}_{x_1x_2}} \otimes 1_{V^{y_3}_{y_1y_2}}\,, \tag{3.14}$$

where $\underline{x} = (x_1, x_2, x_3)$ and $\underline{y} = (y_1, y_2, y_3)$ are the colours of the lines (suitably oriented) in $\overline{G} \times \{0\}$ and $\overline{G} \times \{1\}$, respectively. Applying eq. (3.7) to (3.13) the result follows from (3.14). □

We close this section with some remarks on a few concepts that will be of importance in the following (see also [KS]).

An *interiour graph* G in a 3-manifold M consists of 1) a core $c(\mathcal{G})$ which is a finite unoriented graph embedded in $intM$, 2) a tubular neighbourhood $\mathcal{T}_\mathcal{G}$ of $c(\mathcal{G})$ in $intM$ and 3) an oriented graph G in $\partial\mathcal{T}_\mathcal{G}$ as introduced at the beginning of this section. If G is coloured by $\underline{x}$ we say that $\mathcal{G}$ is coloured by $\underline{x}$ and denote the coloured interiour graph by $\mathcal{G}_{\underline{x}}$. A priori there need not be any connection between $c(\mathcal{G})$ and G, although this will be the case in the applications below. As a particular example we mention the case in which each component k_i of $c(\mathcal{G})$ is homeomorphic to the circle S^1 and the corresponding component $\mathcal{T}_i$ of $\mathcal{T}_\mathcal{G}$ contains exactly one component l_i of G and l_i is homotopic to k_i in $\mathcal{T}_i$. In this case $\mathcal{G}$ is a framed link in M, and will usually be denoted by $\mathcal{L}$.

For a compact oriented 3-manifold M with an interiour coloured graph $\mathcal{G}_{\underline{x}}$ as above we define

$$Z(M, \mathcal{G}_{\underline{x}}) \equiv Z(M/\mathcal{T}_\mathcal{G}, G_{\underline{x}})\,. \tag{3.15}$$

Of course, this definition can be generalized to the case where in addition a graph is present in ∂M.

Of particular relevance for the following is the concept of a meridian and of left- and righthanded lines. Assume that the 3-manifold M contains an empty tube $T = S^1 \times [0,1] \subseteq \partial M$, i.e. M can be obtained from a 3-manifold $\tilde{M}$ by removing a cylinder C diffeomorphic to $D^2 \times [0,1]$, where a D^2 is the two-dimensional unit disc. A *meridian* on T is an oriented circle $m = S^1 \times \{p\}, p \in]0,1[$.

A 3-manifold M containing a set of empty tubes $T_m, T'_{m'}, ...$ with meridians $m, m', ...$ will be denoted by $M(T_m, T'_{m'}, ...)$. Given a graph G on ∂M the lines in G may over- or undercross the meridians $m, m', ...$, such that these together with G constitute a graph $G \cup m \cup m' \cup ...$. For each colouring $\underline{x}$ of G we may thus define [KS]

$$Z(M(T_m, T'_{m'}, ...), G_{\underline{x}}) = \sum_{a,a',...} \frac{\omega_a^2}{\omega^2} \frac{\omega_{a'}^2}{\omega^2} ... Z(M, G_{\underline{x}} \cup m_a \cup m'_{a'} \cup ...) , \qquad (3.16)$$

where the sum is over colourings of the meridians $m, m', ...$.

If a line l in G intersects a meridian m exactly once we say that l is left-, respectively right-, handed w.r.t. m if l over-, respectively under-, crosses m. For the particular case of a framed link $\mathcal{L}$ as defined above the corresponding *left-, respectively right-, handed link* $\mathcal{L}_L$*, resp.* $\mathcal{L}_R$, is obtained by introducing a meridian m_i on each of the tubes $\mathcal{T}_i$ such that each component l_i is left-, respectively right-, handed w.r.t. m_i. The state sum associated to this left-, respectively right-, handed framed link is then given by (3.16) with the $\mathcal{T}_i$ replacing $T_m, T'_{m'}, ...$ and will be denoted by $Z(M, \mathcal{L}_L)$, resp. $Z(M, \mathcal{L}_R)$.

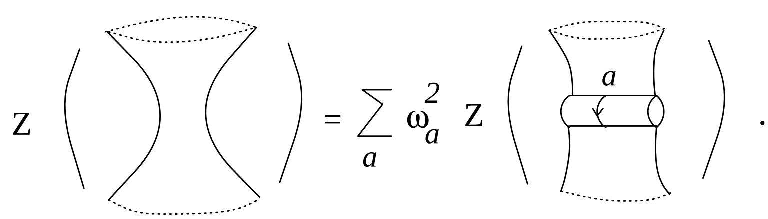
The main reason for introducing the definition (3.16) is illustrated by the following lemma, which asserts that filling an empty tube is equivalent (up to the factor ω^2) to introducing a meridian on the tube.

Lemma 3.3 *Let* $\tilde{M}$ *be a compact oriented 3-manifold that contains a cylinder* $C = D^2 \times [0,1]$*, with* $D^2 \times \{0,1\} \subseteq \partial\tilde{M}$*, and let* M *denote the manifold obtained by removing* C *from* $\tilde{M}$*. Then*

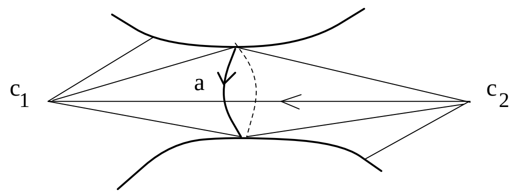
$$Z(\tilde{M}, G) = \omega^2 Z(M(T_m), G) \qquad (3.17)$$

for any graph $G \subseteq \partial\tilde{M}$ *such that* $G \cap (D^2 \times \{0,1\}) = \emptyset$ *and where* T_m *denotes the tube* $\partial D^2 \times [0,1] \subseteq \partial M$ *with meridian* m*.*

Pictorially (3.17) may be written as



Proof: This follows simply by realizing that the cylinder C gets reinserted automatically when $M_{G\cup m}$ is constructed. More explicitly, for a given triangulation X of M, $M_{G_{\underline{x}}\cup m_a}$ equals $\tilde{M}_{G_{\underline{x}}}$ with a triangulation in which the (interiour) link connecting the vertices c_1 and c_2 associated with the components in $\partial M/(G\cup m)$ on either side of m has fixed colour a, as indicated on the following figure.



Thus inserting the factor ω_a^2 and summing over a one obtains $Z(\tilde{M}_{G_{\underline{x}}})$ according to the definition of the latter. □

On the other hand the following result shows that a handle in M can be cut without changing the state sum.

Lemma 3.4 *If the 3-manifold M contains a cylinder $C = D^2 \times [0,1]$ with $\partial D^2 \times [0,1] \subseteq \partial M$, and $\tilde{M}$ denotes the manifold obtained be removing C from M, then*

$$Z(M,G) = Z(\tilde{M},G) \tag{3.18}$$

for any graph $G \subseteq \partial M$ such that $G \cap (\partial D^2 \times [0,1]) = \emptyset$.

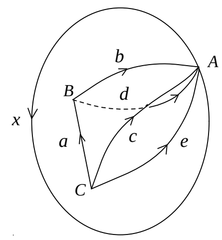
Proof: By choosing a triangulation X of M such that the handle C can be cut along a single triangle t (with $\partial t \subseteq \partial D^2 \times]0,1[$), the result follows from eq. (3.14) by an argument essentially identical to the one used in the proof of Lemma 3.2 □

Whereas Lemma 3.4 deals with cutting of handles disjoint from G the following lemma can be applied to handles traversed by a single line in G.

Lemma 3.5 *Let M be a compact oriented 3-manifold and $G_{\underline{x}}$ a coloured graph in ∂M. If an x-coloured line L in $G_{\underline{x}}$ does not separate two different components in $\partial M/G$ and there exists in $\partial M/G$ a contractible loop in M intersecting L (transversally) only one, then*

$$Z(M,G_{\underline{x}}) = 0, \quad \textit{when } x \neq 0\ . \tag{3.19}$$

Proof: Let X be a triangulation of M and denote by A the vertex in X_G corresponding to the component of $\partial M/G$ containing L in its boundary. Then there is an x-coloured link l in X_G (dual to L) whose end-points both equal A. The tetrahedron in $X_{G_{\underline{x}}}$ containing l and a a-coloured link in L as opposite links then looks as follows:



By the assumption the two triangles (ABC) may be connected by a thickened disc. By retriangulation of this thickened disk one may reach a (singular) triangulation, in which the two triangles (ABC) are identical. The link (BC) is then only contained in one tetrahedron. Summing over its colour a one obtains a contribution

$$\sum_a \omega_a^2 \begin{vmatrix} a & b & c \\ x & c & b \end{vmatrix} ,$$

contracted w.r.t. the dual pair of spaces $V_{ab}^c, V_{a^*c}^b$. But this expression equals 0 for $x \neq 0$, since it may be rewritten by eqs. (2.12) and (2.9) as:

$$\sum_a \omega_a^2 \begin{vmatrix} a & b & c \\ x & c & b \end{vmatrix} \begin{vmatrix} a & b & c \\ 0 & c & b \end{vmatrix} = \delta_{x0} 1_{V_{c0}^c \otimes V_{b0}^b} . \quad \square$$

Remark 3.6 In Lemma 4.2 we shall see that a line with colour 0 can be deleted from G without changing $Z(M,G)$. Together with Lemmas 3.4-5 this implies that if a single x-coloured line traverses a handle, then the state sum is either zero (if $x \neq 0$) or (if $x = 0$) the line can be deleted and the handle cut.

4 Calculational methods

In this section we establish a number of invariance properties or transformation rules for the state sums $Z(M,G)$ under local changes of the graph G. These are generalizations of the results derived in [KS], but we give different and, we hope, more transparent proofs.

In the following let M be a compact oriented 3-manifold and $G_{\underline{x}}$ a coloured graph in ∂M.

Lemma 4.1 *If $G_{\underline{x}} = G^1_{\underline{x_1}} \cup G^2_{\underline{x_2}}$, where G^1 is contained in a disc $D^2 \subseteq \partial M$ such that $G^2 \cap D^2 = \emptyset$, i.e. G contains an isolated planar subgraph G^1, then*

$$Z(M, G_{\underline{x}}) = Z(G^1_{\underline{x_1}}) \otimes Z(M, G^2_{\underline{x_2}}), \tag{4.1}$$

where

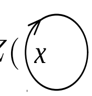
$$Z(G_{\underline{x}}) \equiv Z(D^3, G_{\underline{x}}) \tag{4.2}$$

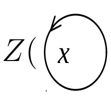
and D^3 denotes a 3-ball.

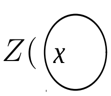
Proof:

By choosing a suitable triangulation X of M we can represent $(M, G_{\underline{x}})$ as two manifolds $(M', G^1_{\underline{x_1}})$ and $(D^3, G^2_{\underline{x_2}})$, which are glued together along the triangle $t \subseteq \partial M' \cap \partial D^3$ with $\partial t \subseteq \partial M$, and such that M' is homeomorphic to M. Then M_G equals the connected sum of $D^3_{G^1}$ and M'_{G^2}, obtained by first removing from $D^3_{G^1}$, resp. from M'_{G^2}, the tetrahedron with base t and opposite vertex c, resp. c', corresponding to the connected component of $\partial D^3/G^1$, resp. $\partial M'/G^2$, containing t, and subsequently gluing the two resulting manifolds together along the two copies of S^2 (the boundaries of the removed tetrahedra), such that c and c' are identified. Eq. (4.1) then follows from eq. (3.14) as in the proof of Lemma 3.2 *iv)*, taking into account that no factors ω^{-2} are associated to $c \in D^3_{G^1}$ and $c' \in M'_{G^2}$ nor to the vertex resulting from their identification in M_G. $\square$

Examples: 1) For a coloured circle S^1_x we have

$Z(S^1_x) = Z($ 

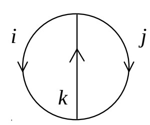
$) = Z($ 

$) = Z($ 

$) = \omega_x^2\,.$ (4.3)

Observe that, for a given triangulation X of D^3, $(D^3_{S^1}, X_{S^1})$ equals (S^3, X_{S^1}) where X_{S^1} contains two distinguished vertices c and c' connected by a link with colour x. Triangulating S^3 by two tetrahedra eq. (4.3) follows from eq. (2.9).

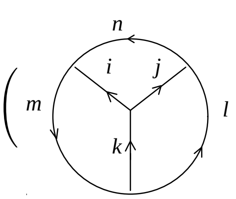
2)

$Z\Big($ 

$\Big) = 1_{V^k_{ij}}\,.$ (4.4)

Note that D^3_G, for the graph in (4.4), equals a 3-ball with boundary triangulated by two triangles and thus can be obtained by gluing two tetrahedra along three common triangles. Eq. (4.4) is then again a consequence of eq. (2.9).

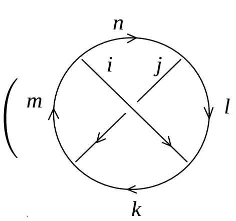
3)

$Z\Big($ 

$\Big) = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}.$ (4.5)

For the graph in (4.5) D^3_G equals the 3-ball whose boundary is triangulated by the dual graph to G and hence can be obtained as a single tetrahedron. Eq. (4.5) then follows from the definition of $Z(M, G)$.

4)

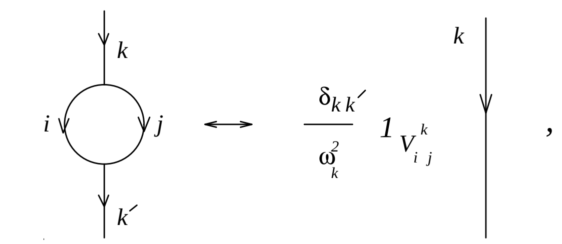
$Z\Big($ 

$\Big) = \frac{q_l q_m}{q_k q_n} \begin{vmatrix} j & l & n \\ i & m & k \end{vmatrix}.$ (4.6)

We leave the verification of this identity to the reader.

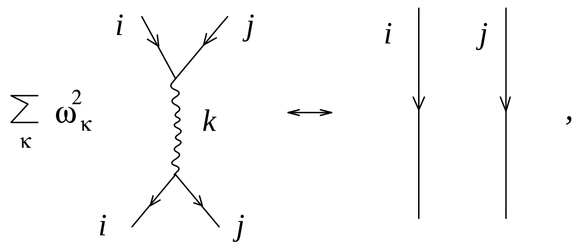
Lemma 4.2 *The state sum $Z(M,G)$ is invariant under the following local changes of $G_{\underline{x}}$:*

i)



(4.7)

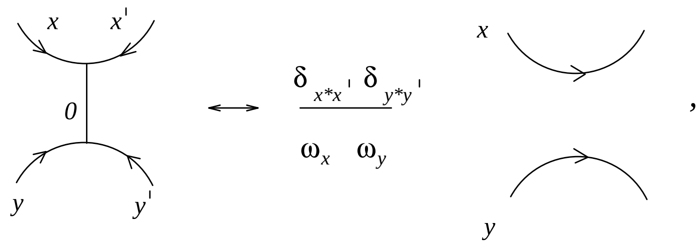
ii)



(4.8)

where the curly line indicates the contraction with respect to the dual pair of spaces associated to the 3-vertices it connects.

iii)

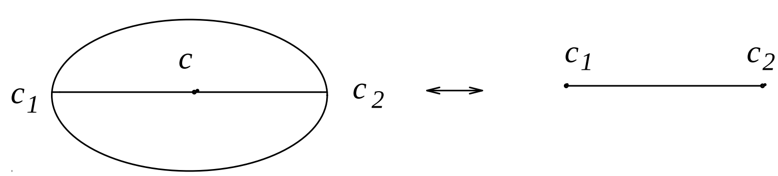


(4.9)

where the vector spaces $V^0_{xx^*}$ and $V^0_{yy^*}$ have been identified with $\mathbf{C}$.

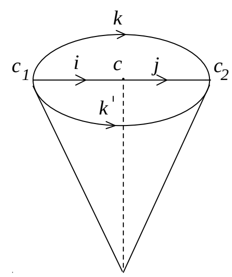
Proof:

i) The substitution (4.7) corresponds to the following local transformation of the triangulation on the boundary of M_G:

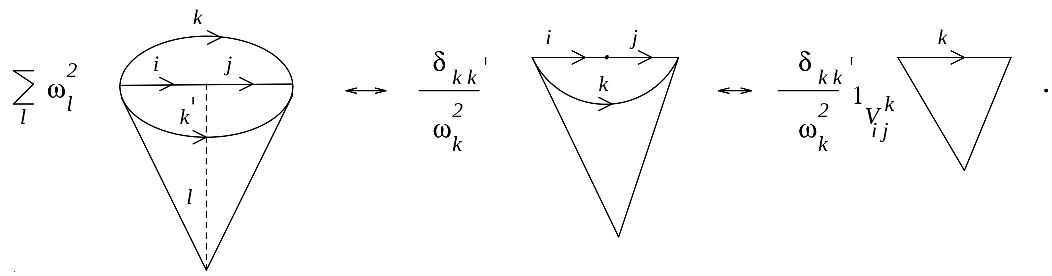


where c_1, c and c_2 are the vertices associated to the left, middle and the right components of $\partial M/G$, respectively on the lefthand side of (4.7), and similarly for the righthand side.

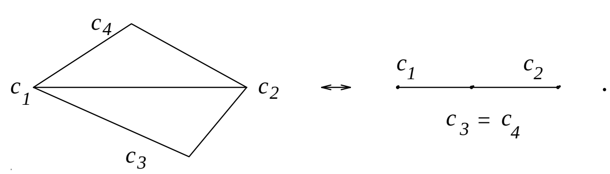
Choosing a triangulation of $M_{G_{\underline{x}}}$ which in the vicinity of c_1, c, c_2 looks as



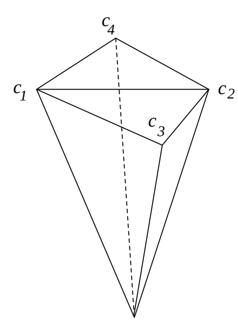
the result follows from the orthogonality relations, which can be interpreted in terms of a collapsing of two tetrahedra glued along two common triangles [DJN]:



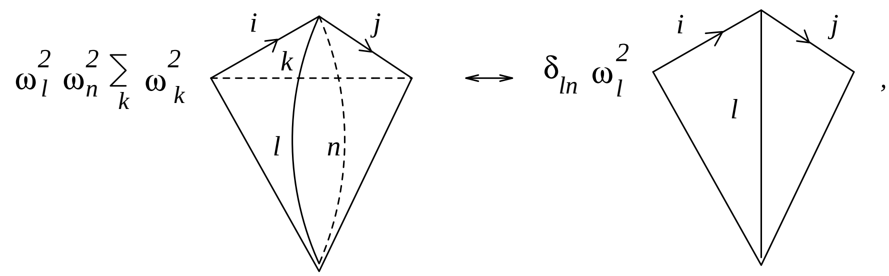
ii) The substitution (4.8) corresponds to the following local transformation of the triangulation on the boundary $\partial M_{G_{\underline{x}}}$:



Choosing a triangulation of M_G which in the vicinity of c_1, c_2, c_3, c_4 looks as

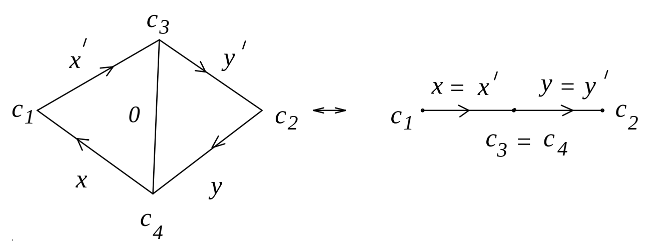


the contraction on the lefthand side of (4.8) corresponds to gluing the two triangles $(c_1c_2c_4)$ and $(c_1c_2c_3)$ together. Having done so we may again apply eq. (2.9) as above and obtain:



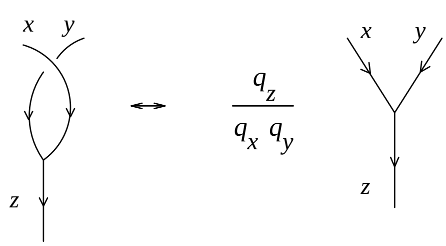
which yields the desired result.

iii) The substitution (4.9) corresponds to the following local transformation of the triangulation of the boundary of M_G:



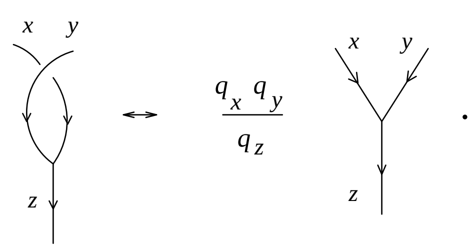
By choosing a suitable triangulation of M_G one finds that (4.9) is a simple consequence of eq. (2.12). □

Next we note that the Racah identity (2.11) implies invariance of $Z(M, G)$ under the following local substitution in G:



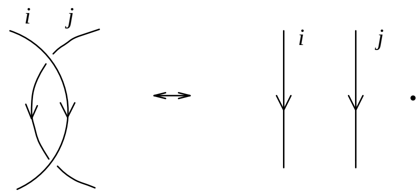
(4.10)

which is easily seen in terms of a suitable local choice of triangulation of M_G. Similarly, the second Racah identity yields invariance under the substitution



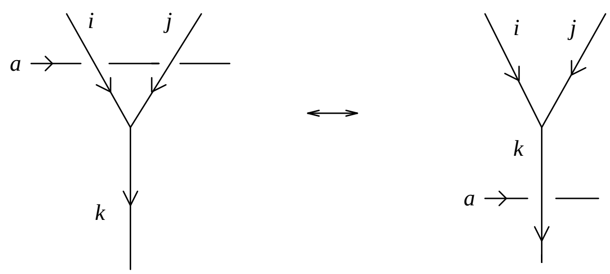
(4.11)

By combining these two identities with (4.8) one obtains invariance of $Z(M, G)$ under the following local change of G:



(4.12)

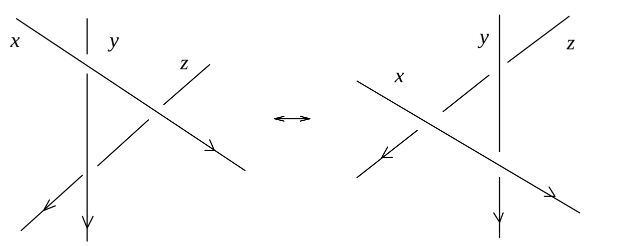
The Biedenharn-Elliot relations give rise to invariance under the substitution



(4.13)

which follows in a similar way as above by a convenient choice of triangulation of M_G.

Combining (4.13) with (4.8) and (4.10) we get in addition invariance under the substitution

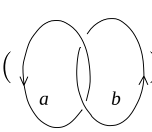


(4.14)

Of course, invariance of $Z(M, G)$ also holds under the substitution analogous to (4.12) (resp. (4.13)), where the j-line (resp. a-line) overcrosses the i-line (resp. i- and j-lines) on the lefthand side.

In order to formulate the following lemma, which will be of importance in the next section, we define the matrix S by (see [KS])

$$S_{ab} = \omega^{-1} Z(\quad$$



$$) = \omega^{-1} \sum_c \frac{\omega_c^2 q_c^2}{q_a^2 q_b^2} N_{ab}^c \tag{4.15}$$

and set

$$\omega_b^2(a) = \frac{\omega}{\omega_a^2} S_{ab} \ . \tag{4.16}$$

The last expression in eq. (4.15) follows by first applying (4.8) to the double line in the middle of the graph in (4.15), then applying (4.10) and finally (4.4). We note that

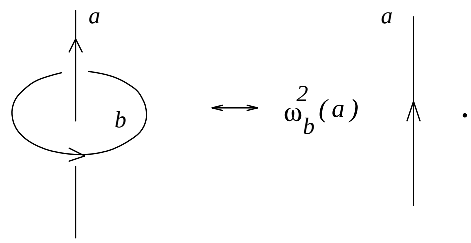
$$S_{ab} = S_{ba} = S_{a^*b^*} \ , \tag{4.17}$$

since $q_{a^*} = q_a, \omega_{c^*} = \omega_c$ and $N^c_{ab} = N^c_{ba} = N^{c^*}_{a^*b^*}$, and that

$$\omega_b^2(0) = \omega S_{0b} = \omega_b^2 \tag{4.18}$$

since $q_0 = \omega_0 = 1$ and $N^c_{0b} = \delta_{cb}$.

Lemma 4.3 *i) The state sum $Z(M, G_{\underline{x}})$ is invariant under the following local substitution in $G_{\underline{x}}$*



(4.19)

ii) We have

$$\sum_b N^b_{cd} \omega_b^2(a) = \omega_c^2(a)\omega_d^2(a), \tag{4.20}$$

i.e. $\omega_c^2(a), a \in I$, is an eigenvalue of the matrix N_c defined by

$$(N_c)^b_d = N^b_{cd}, \quad b, d \in I \ , \tag{4.21}$$

for each $c \in I$, with eigenvector $(\omega_d^2(a))_{d\in I}$ (provided the latter is non-zero).

iii) If $q_a^2 \neq 1$ for all $a \in I/\{0\}$ and

$$\triangle = \sum_c q_c^2 \omega_c^4 \neq 0, \tag{4.22}$$

then the following formulae hold:

$$\sum_b \omega_b^2 \omega_b^2(a) = \omega^2 \delta_{a0}, \tag{4.23}$$

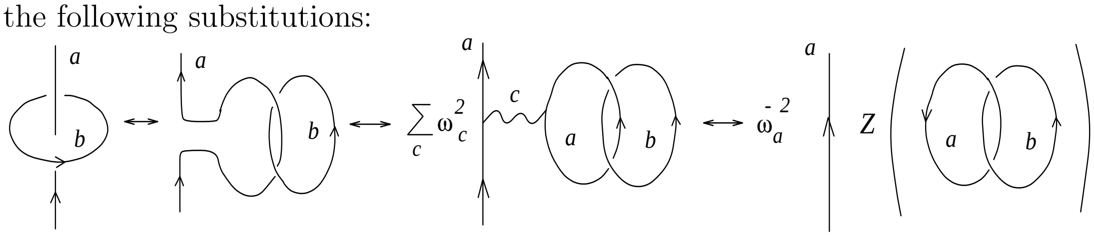
$$\sum_b S_{ab} S_{bc^*} = \delta_{ac} \ , \tag{4.24}$$

$$N^c_{ab} = \sum_d \frac{S_{ad} S_{bd} S_{c^*d}}{S_{0d}} \ . \tag{4.25}$$

Proof:

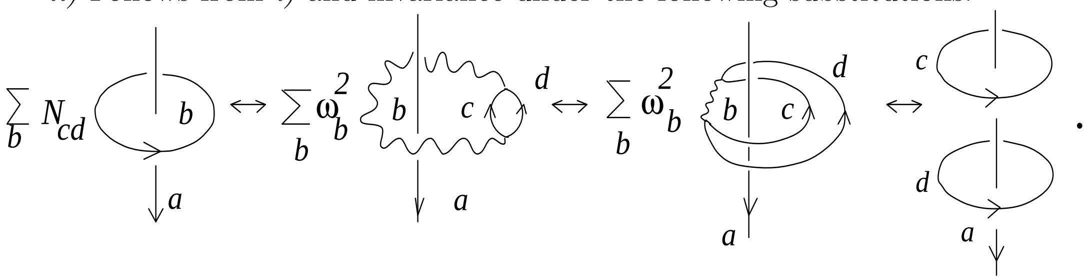
The statements follow by rather obvious modifications of the arguments in [KS], Appendix A. For completeness we give some details:

i) Using (4.8), Lemma 3.5, (4.9) and Lemma 4.1 we obtain invariance under the following substitutions:



as desired.

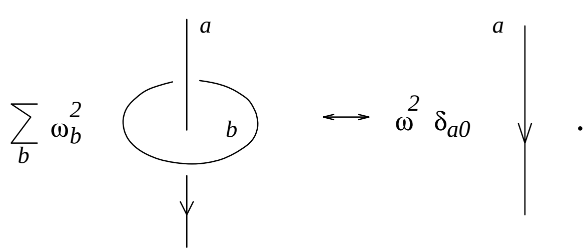
ii) Follows from *i)* and invariance under the following substitutions:



iii) It is easy to see (Lemma A.2 in [KS]) that the assumption $q_a^2 \neq 1$ and $\Delta \neq 0$ imply the existence of a $b \in I$, such that $\omega_b^2 \neq \omega_b^2(a)$. Then (4.23) follows for $a \neq 0$ from

$$(\omega_b^2 - \omega_b^2(a)) \sum_c \omega_c^2 \omega_{c^*}^2(a) = \sum_{cd} (N_{bc}^d \omega_d^2 \omega_{c^*}^2(a) - N_{bc^*}^d \omega_d^2(a) \omega_c^2) = 0\,,$$

where we have used *ii)* and the fact that $N_{bc^*}^d = N_{bd^*}^c$ by (2.1-2) as well as $\omega_{c^*}^2 = \omega_c^2$. For $a = 0$ eq. (4.23) follows from the definition of ω^2. Eq. (4.24) coincides with eq. (4.23) for $a = 0$ or $c = 0$ in view of eqs. (4.16-18). Multiplying by ω_b^2 and summing over $b \in I$ in (4.19) we get from eq. (4.23) invariance under the substitution



(4.26)

Using this eq. (4.24) follows easily for general $a, b \in I$ as in [KS], Lemma A.3.

Finally, eq. (4.25) follows immediately from eqs. (4.20) and (4.16) using (4.24). □

We note that, in view of eq. (4.24), eq. (4.25) states that the matrix S *diagonalizes all the matrices* $N_a; a \in I$

We end this section by discussing some useful properties of the state sums for manifolds with interiour graphs, which can be derived easily from the rules developed in this and the previous sections.

1) Suppose that two meridians m and m' are introduced on an empty tube T in M disjoint from G. According to Lemma 3.3 this is equivalent up to a factor ω^4 to filling the tube by two cylinders in the vicinity of m and m'. Between the cylinders there will then be an empty 3-ball. Filling this 3-ball is equivalent to multiplying the state sum by ω^{-2} by Lemma 3.2 *i)*. Thus, it follows that the presence of two (or more) meridians on T is equivalent to the presence of just

one, i.e.

$$Z(M(T_m, T_{m'}), G) = Z(M(T_m), G). \tag{4.27}$$

In fact, the purpose of the factors ω^{-2} in (3.16) is to ensure this projection property of the meridians.

2) Consider in M a branching of an empty tube T_1 into two empty tubes T_2 and T_3, which are all disjoint from G (see Fig.7).

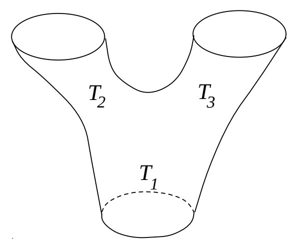
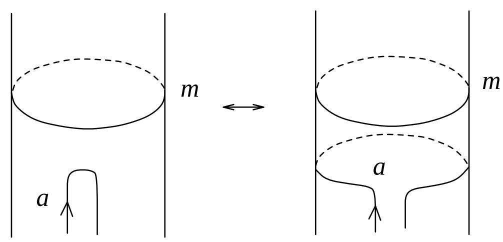


Fig.7

An argument similar to the previous one implies equivalence of any two configurations of meridians on the tubes T_1, T_2, T_3, if only at least two of the tubes contain a meridian.

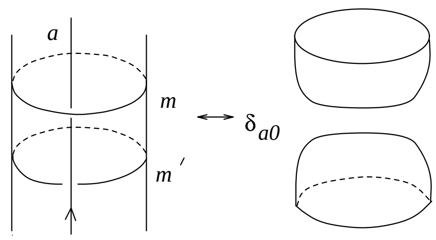
3) Similarly one sees that lines in G along a tube containing a meridian may be deformed non-trivially as follows:



4) The statements in 1-3) may generalized to the case where lines in G (in addition to the a-line in 3)) are traversing the tubes, if only all these lines overcross (or all undercross) the meridians (and the a-line in 3)). This follows that by repeated use of (4.8) and (4.12- 4.14).

5) Combining the generalized versions of 2) and 3) with Lemma 3.3 it is easy to show that if T_m and $T'_{m'}$ are two empty tubes in M such that T, resp. T', is traversed by a lefthanded , resp. righthanded, line then T_m and $T'_{m'}$ have trivial braiding, i.e. they may be moved through each other.

6) Using (4.26) and 3) as well as Lemma 3.4 it follows that if a tube in M is traversed by an a-coloured line in G which overcrosses one meridian and undercrosses another then $Z(M, G)$ vanishes unless $a = 0$, in which case the line and the meridian can be deleted and the tube filled:



(4.28)

We leave the details of these arguments to the reader. Alternatively, [KS] can be consulted.

Finally, we note the following lemma, which we shall need in the next section.

Lemma 4.4 *Let $\mathcal{L}$ and $\mathcal{L}'$ be arbitrary disjoint framed links in S^3. Then the following holds.*

i)

$$Z(S^3, \mathcal{L}_L \cup \mathcal{L}'_R) = \omega^2 Z(S^3, \mathcal{L}_L) Z(S^3, \mathcal{L}'_R)\,. \tag{4.29}$$

ii) For any colouring $\underline{a}$ of $\mathcal{L}$ we have

$$Z(S^3, (\mathcal{L}_{\underline{a}})_L) = \omega^{-2} Z(G_L(\mathcal{L})_{\underline{a}}) \quad , \tag{4.30}$$

where $G_L(\mathcal{L})$ is the planar graph (with no 3-vertices) naturally obtained from the framed link $\mathcal{L}$ by projecting onto a plane (and specified more precisely below).

iii) For any colouring $\underline{a}$ of $\mathcal{L}$ we have

$$Z(S^3, (\mathcal{L}_{\underline{a}})_R) = \omega^{-2} Z(G_R(\mathcal{L})_{\underline{a}}) \quad , \tag{4.31}$$

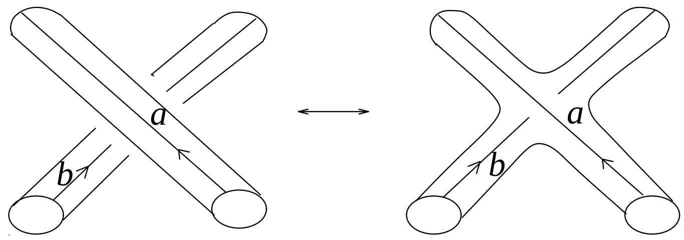
where $G_R(\mathcal{L})$ is obtained from $G_L(\mathcal{L})$ by replacing all 4-vertices by their inverses.

Proof:

i) follows immediately from 5) above and lemma 3.2 *iv).*

ii) Consider $\mathcal{L}$ as embedded into $\mathbf{R}^3 \subseteq S^3$ and choose a plane π such that $\mathcal{L}$ lies on one side of π. We may then deform $\mathcal{L}$ such that the circles on the tori constituting the boundary of the tubular neighbourhood $\mathcal{T}_{\mathcal{L}}$ can be obtained as translates of the corresponding cores in the direction orthogonal to and away from π. (In terms of ribbons given by the framing of $\mathcal{L}$ this is equivalent to deforming the ribbons such that no twists are present (see [RT1]).) The coloured graph $G_L(\mathcal{L})_{\underline{a}}$ is then obtained by projecting the circles on $\partial\mathcal{T}_{\mathcal{L}}$ onto π taking into account over- and undercrossings as well as orientation and colouring $\underline{a}$ of $\mathcal{L}$.

In order to prove (4.30) we then consider the crossings of tubes in $\mathcal{L}$ (see the LHS of the figure below) corresponding to 4-vertices in $G_L(\mathcal{L})$. By Lemma 3.3 and 2) above we may connect the two tubes at each crossing by a tube at the cost of a factor ω^2. Using the lefthandedness and the general version of 3) above (or (4.8)) it follows that the two tubes may be joined as indicated in the following figure.



One thus obtains an interiour graph in S^3 whose core is a copy of $G_L(\mathcal{L})$ and whose corresponding coloured graph on the boundary of its tubular neighborhood is a copy of $G_L(\mathcal{L})_{\underline{a}}$ and all of whose lines are lefthanded.

Now it is not difficult to see that all meridians can be eliminated, partly by use of 2) above and partly by using lemmas 3.5 and 4.2 at the cost of factors ω^{-2}. Finally, handles (corresponding to the faces of the graph $G_L(\mathcal{L})$) may be cut by lemma 3.4 thus obtaining D^3. Keeping track of the ω-factors one arrives at (4.30).

iii) Follows from *ii)* by observing that in the case of righthanded lines the 4-vertex on the RHS of the figure above must be replaced by its inverse. □

5 Dimension of the state spaces and factorization of state sums

This section is devoted to calculating the dimensions of the state space V_Σ associated to a closed, compact and oriented surface Σ, and to showing that the state sum $Z(M)$ for a closed, compact, oriented 3-manifold M factorizes into a lefthanded and a righthanded contribution, which in the quantum group case equal $\tau(M)$ and $\tau(M^*)$, respectively, where $\tau(M)$ is the invariant introduced in [RT2].

Let us first note that by (3.8), (3.14) and (2.5) we have, for Σ connected of genus 0,

$$dimV_{S^2} = Z(S^2 \times S^1) = \omega^{-4} \sum_{a,b,c} N^c_{ab} \omega_a^2 \omega_b^2 \omega_c^2 = 1\,. \qquad (5.1)$$

Let the matrix $|\vec{N}|^2$ be defined by

$$|\vec{N}|^2 = \sum_a N_a^t N_a = \sum_a N_{a^*} N_a \qquad (5.2)$$

where the upper index t denotes transposition and the last equality follows from (2.1-2).

Theorem 5.1 *Assume that $q_a^2 \neq 1$ for $a \neq 0$ and that $\Delta \neq 0$. Let Σ be a connected, closed, compact, oriented surface of genus $g \geq 1$. Then*

$$dimV_\Sigma = (tr(|\vec{N}|^{2(g-1)}))^2\,. \qquad (5.3)$$

Proof:

Given the computational rules developed in sections 3 and 4 the proof of (5.3) is identical to the proof of the Thm. 7.4 (without punctures) in [KS] with only minor modifications. It suffices therefore here to indicate the main line of argument: According to eq. (3.8) we have to calculate $Z(\Sigma \times S^1)$. Using Thm. 3.1 we may cut $\Sigma \times S^1$ along $\Sigma \times \{a\}$, for some $a \in S^1$, thus obtaining a manifold homeomorphic to $\Sigma \times [0,1]$ with a graph $(G \times \{0\}) \cup (G \times \{1\})$ on its boundary, where G is a canonical graph on Σ without 4-vertices, e.g. $G = G^{\Sigma}$ (see section 3). By (4.28) we can then introduce an empty tube connecting $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ thus obtaining a handlebody with $2g$ handles. Using (4.8) the graph on its boundary can be reduced to one without 3-vertices and using in addition Lemmas 3.4-5 and (4.9) the handles can be cut yielding a 3-ball with a graph on its boundary. The corresponding state sum can then be evaluated and shown to equal the RHS of (5.3) by Lemma 4.3 *iii)*. □

We note here that, as a consequence of eqs. (4.24-25), $|\vec{N}|^2$ on the RHS of (5.3) can be replaced by

$$\vec{N}^2 \equiv \sum_a (N^a)^2 \,, \tag{5.4}$$

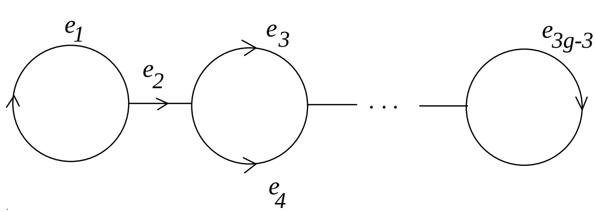
where the symmetric matrix N^a is defined by

$$(N^a)_{bc} = N^a_{bc}, \quad b, c \in I \,, \tag{5.5}$$

for each $a \in I$. Thus

$$dim V_{\Sigma} = (tr(\vec{N}^{2(g-1)}))^2 \,. \tag{5.6}$$

Let now G_g be the graph depicted below



with g circles and coloured by $\underline{e} = (e_1, ..., e_{3g-3})$ for $g > 1$ and $\underline{e} = (e_1)$ for $g = 1$. Notice that the dimension of the vector space $V(\Sigma, G_g)$ associated to the graph G_g embedded in a closed oriented surface Σ is given by

$$dim V(\Sigma, G_g) = (tr(\vec{N}^{2(g-1)}))^2 \,. \tag{5.7}$$

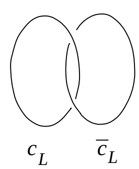
as is easily seen by use of Lemma 4.3.

Let $M_{\Sigma^g} \subseteq R^3$ be a handlebody with g handles and $\partial M_{\Sigma^g} = \Sigma^g$, containing in its interiour two disjoint (non-oriented) copies c_g^L and c_g^R of G_g as deformation retracts of Σ^g. Thus, in particular, the circles in c_g^L and c_g^R are non-contractible in M_{Σ^g}. We let $\mathcal{G}_g^L$ and $\mathcal{G}_g^R$ be two disjoint interiour graphs in M_{Σ^g} whose cores are

c_g^L and c_g^R, respectively, and whose associated graphs G_g^L and G_g^R are also copies of G_g on the boundaries of tubular neighborhoods $\mathcal{T}^L$ and $\mathcal{T}^R$, respectively, which are equipped with $3g-3$ meridians for $g>1$ and one for $g=1$ each such that all lines in G_g^L, resp. G_g^R, are lefthanded, resp. righthanded. Denote the vector spaces associated to $G_g^L \subseteq \partial\mathcal{T}^L$ and $G_g^R \subseteq \partial\mathcal{T}^R$ by V_g^L and V_g^R, respectively.

By introducing a canonical graph G without 4-vertices on Σ_g we may consider the partition function associated to M_{Σ_g} with interiour graphs $\mathcal{G}_g^L$ and $\mathcal{G}_g^R$ and boundary graph G as a linear mapping from $(V_g^L \otimes V_g^R)^*$ into V_Σ. It is in fact not hard to show that this mapping is an isomorphism. We shall not need this general result in the following, but restrict our attention to the case $g=1$.

In this case G_g is a circle and Σ_g is a torus T embedded in $R^3 \subseteq S^3$ and M_T is a solid torus, whose complement $\bar{M}_T$ in S^3 is also a solid torus with $\partial\bar{M}_T = T^*$. We thus consider copies c^L, c^R, resp. $\bar{c}^L$, $\bar{c}^R$, of the circle S^1 in M_T and $\bar{M}_T$, respectively, and interiour graphs $\mathcal{G}^L$ and $\mathcal{G}^R$, resp. $\bar{\mathcal{G}}^L$ and $\bar{\mathcal{G}}^R$, whose associated tubular neighborhoods $\mathcal{T}^L$ and $\mathcal{T}^R$, resp. $\bar{\mathcal{T}}^L$ and $\bar{\mathcal{T}}^R$, are solid tori, and whose associated graphs G^L, G^R, resp. $\bar{G}^L$, $\bar{G}^R$, are oriented circles, which we assume have zero linking number with c^L, c^R, $\bar{c}^L$, $\bar{c}^R$, respectively. Moreover, G^L and $\bar{G}^L$ are lefthanded and G^R, $\bar{G}^R$ are righthanded and we note that c^L and $\bar{c}^L$, resp. c^R and $\bar{c}^R$, are linked as indicated in the following picture:



Since the graphs G^L, $\bar{G}^L$, G^R, $\bar{G}^R$ have no 3-vertices and only one line the vector spaces associated to them may be identifyed with $\mathbf{C}^{|I|}$, where $|I|$ is the cardinality of I, (cf. eq.(5.7)).

The partition function Z'_T (as defined in eq.(3.4)) associated to M_T with interiour graphs $\mathcal{G}^L$ and $\mathcal{G}^R$ and a canonical graph G without 4-vertices on T thus yields a linear mapping

$$K : \mathbf{C}^{|I|} \otimes \mathbf{C}^{|I|} \to V_T$$

and similarly the partition function $\bar{Z}'_T$ associated to $\bar{M}_T$ with the interiour graphs $\bar{\mathcal{G}}^L$ and $\bar{\mathcal{G}}^R$ and boundary graph G on T^* yields a linear mapping

$$L : V_T \to \mathbf{C}^{|I|} \otimes \mathbf{C}^{|I|} .$$

Gluing M_T and $\bar{M}_T$ with interiour graphs along T yields, of course, S^3 with interiour graphs $\mathcal{G}^L$, $\mathcal{G}^R$, $\bar{\mathcal{G}}^L$, $\bar{\mathcal{G}}^R$. Thus the mapping $LK : \mathbf{C}^{|I|}\otimes\mathbf{C}^{|I|} \to \mathbf{C}^{|I|}\otimes\mathbf{C}^{|I|}$ is given by the martix

$$(LK)_{(e,e')(f,f')} = Z(S^3, \mathcal{G}_e^L \cup \mathcal{G}_f^R \cup \bar{\mathcal{G}}_{e'}^L \cup \bar{\mathcal{G}}_{f'}^R) = \omega^2 Z(S^3, \mathcal{G}_e^L \cup \bar{\mathcal{G}}_{e'}^L) Z(S^3, \mathcal{G}_f^R \cup \bar{\mathcal{G}}_{f'}^R) , \tag{5.8}$$

where we have used 5) at the end of section 4 and Lemma 4.4 *i)*. According to Lemma 4.4 *ii)*- *iii)* we have

$$Z(S^3, \mathcal{G}_e^L \cup \bar{\mathcal{G}}_{e'}^L) = \omega^{-1} S_{e,e'} \,, \tag{5.9}$$

$$Z(S^3, \mathcal{G}_f^R \cup \bar{\mathcal{G}}_{f'}^R) = \omega^{-1} S^{-1}_{f,f'} \,, \tag{5.10}$$

where $S_{e,e'}$ is defined by eq. (4.15). Thus eq. (5.8) reads

$$LK = \omega^2(\omega^{-1} S \otimes \omega^{-1} S^{-1}) \,, \tag{5.11}$$

and we conclude now from the fact that $dim V_T = dim(\mathbf{C}^{|I|} \otimes \mathbf{C}^{|I|})$, that L and K are isomorphisms and that

$$K(S^{-1} \otimes S)L = 1_{V_T} \,, \tag{5.12}$$

provided the assumptions in Thm. 5.1 are fulfilled.

It was first observed by M. Karowski and R. Schrader [KS1] that the remarkable property of the vector spaces V_Σ being factorized into lefthanded and righthanded ones should lead to a rather simple proof of the relation between two different constructions of the 3-manifolds invariants. This is the content of the following theorem where the context is assumed to be that of quantum deformations of a semisimple Lie algebra at an even root of unity, in which case the assumptions of Thm. 5.1 are known to be fulfilled and in addition $|q_a| = 1$ for all $a \in I$.

Theorem 5.2 *Let M be a closed, compact, oriented 3-manifold. Then*

$$Z(M) = \tau(M)\, \tau(M^*) \,, \tag{5.13}$$

where $\tau(M)$ is an invariant defined by eq. (5.17) below, which in fact equals the invariant introduced in [RT2], up to normalization.

Proof: As is well known M can be obtained (up to diffeomorphism) by surgery along a framed link in S^3: Let $\mathcal{L}_1, ..., \mathcal{L}_n$ be the components of $\mathcal{L}$, i.e. $\mathcal{L}_i$ is a framed circle, and let $\mathcal{T}_i$ be a tubular neighborhood of $\mathcal{L}_i$ such that $\cap_i \mathcal{T}_i = \emptyset$. We then obtain M as

$$(S^3/(\cup_i \mathcal{T}_i)) \cup_{F_1} \mathcal{T}_1 \cup_{F_2} ... \cup_{F_i} \mathcal{T}_i ... \cup_{F_n} \mathcal{T}_n \,,$$

i.e. by first removing all $\mathcal{T}_i$ from S^3 and then gluing them back with identification maps $F_i : \partial\mathcal{T}_i \to \partial\mathcal{T}_i \subseteq \partial(S^3/\cup_i \mathcal{T}_i)$, which are determined by the framing of $\mathcal{L}_i$. In other words if the framing of $\mathcal{L}_i$ is given by the circle L_i on $\partial\mathcal{T}_i$ with linking number N w.r.t. the core of $\mathcal{T}_i$, then F_i is composed of an inversion (which interchanges the cycles in a canonical homotopy basis for $\partial\mathcal{T}_i$) and an N-fold Dehn-twist. Equivalently, $L_i \subseteq \partial\mathcal{T}_i$ is identified with a meridian on $\partial\mathcal{T}_i \subseteq \partial(S^3/\cup_i \mathcal{T}_i)$.

Inserting the identity operator in the form of (5.12) for each $\partial\mathcal{T}_i$ a simple calculation yields

$$Z(M) = \sum_{\underline{e},\underline{f},\underline{e}',\underline{f}'} Z(S^3, (\mathcal{L}_{\underline{e}})_L \cup (\mathcal{L}_{\underline{f}})_R) \prod_{i=1}^{n} S^{-1}_{e_i e'_i} \delta_{e'_i 0} S_{f_i f'_i} \delta_{f'_i 0} \,, \tag{5.14}$$

where the factors $\delta_{e'_i 0}\ \delta_{f'_i 0}$ arise because the inversions make the e'_i- and f'_i-lines traverse handles and hence their colours vanish by Lemma 4.2.

Using (4.18) and Lemma 4.4 *i)* in (5.14) we get

$$\begin{aligned} Z(M) &= (\omega^2)^{-n+1} \sum_{\underline{e}} \omega^2_{\underline{e}} Z(S^3, (\mathcal{L}_{\underline{e}})_L) \sum_{\underline{f}} \omega^2_{\underline{f}} Z(S^3, (\mathcal{L}_{\underline{f}})_R) \\ &= (\omega^2)^{-n+1} Z(S^3, \mathcal{L}_L) Z(S^3, \mathcal{L}_R) \,, \end{aligned} \tag{5.15}$$

where $\omega^2_{\underline{a}} = \prod_{i=1}^n \omega^2_{e_i}$. Eq. (5.15) may be rewritten as

$$Z(M) = \tau_L(M)\tau_R(M) \,, \tag{5.16}$$

if we define

$$\begin{aligned} \tau_L(M) &= \omega^{-n+1}(\Delta_L \omega^{-1})^{\sigma(\mathcal{L})} Z(S^3, \mathcal{L}_L) \\ &= \omega^{-n-1}(\Delta_L \omega^{-1})^{\sigma(\mathcal{L})} \sum_{\underline{e}} \omega^2_{\underline{e}} Z(G_L(\mathcal{L})_{\underline{e}}) \,, \end{aligned} \tag{5.17}$$

and correspondingly τ_R by replacing L by R in (5.17), where we have used Lemma 4.4 *ii)* and *iii)*. Here $\sigma(\mathcal{L})$ is the signature of a certain 4-manifold whose boundary is M, $\Delta_L = \Delta$ is defined by eq. (4.22) and Δ_R is defined by replacing q_i by $q_i^{-1} = \bar{q}_i$ in eq. (4.22). Since it is known (see [T, ch.2]) that

$$\Delta_L \Delta_R = \omega^2$$

it follows that (5.15) and (5.16) are equivalent.

In section 6 we show that

$$Z(G_L(\mathcal{L})_{\underline{e}}) = F(\Gamma(\mathcal{L}, \underline{e})) \,, \tag{5.18}$$

where F denotes the functor from the category of coloured ribbon graphs into the category of representations of the quantum group under consideration, which was introduced in [RT1]. Using the identity (5.18) in (5.17) one obtains the expression for the invariant $\tau(M)$ introduced in [RT2] (up to normalization) as given in [T, ch.2]. Thus $\tau_L(M) = \tau(M)$. Since, moreover, it is known that $\tau(M^*) = \tau(M)^*$ and it is easy to verify that $\tau_R(M) = \tau_L(M)^*$ as a consequence of the properties of 6j-symbols, we have proven (5.13). □

As a final topic in this section we shall briefly discuss how vector spaces can be associated to surfaces with punctures and the corresponding extension of Thm. 5.1.

Let Σ be a closed, compact, oriented surface with a set $\underline{p} = (p_1, ..., p_n)$ of n different, distinguished points, and consider a canonical graph G on Σ without 4-vertices, disjoint from $\{p_1, ..., p_n\}$. Adjoining to G lines $l_1, ..., l_n$ such that l_i connects p_i to a point in G, different from the vertices in G and otherwise not intersecting G, we obtain a graph $G(\underline{p})$ on Σ whose vertices are all of order 2 or 3 except for $p_1, ..., p_n$, which are of order 1. The dual graph to $G(\underline{p})$ on Σ, constructed in the standard way, then yields the proper triangulation of Σ except for the n discs $D_1, ..., D_n$ containing $p_1, ..., p_n$, respectively, whose boundaries consist of a single link each, namely the dual links to $l_1, ..., l_n$. Interpreting these discs as holes in Σ, what remains is a proper triangulation of $\Sigma / \cup_i D_i$, such that the boundary of each hole consists of a single link (see Fig.8)

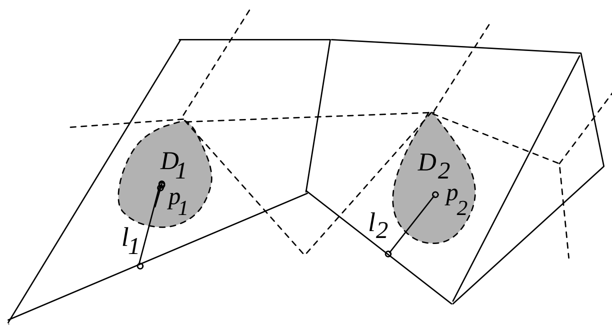


Fig.8

We call graphs of the form $G(\underline{p})$ canonical graphs for the punctured surface $\Sigma(p)$ and we shall assume that the discs $D_1, ..., D_n$ around $p_1, ..., p_n$, respectively, are fixed and only consider graphs G disjoint from $\cup_i D_i$. Moreover, we assume that l_i intersects ∂D_i at a (unique) fixed point q_i.

Fixing the colours of the boundary links in the triangulation, i.e. of the lines $l_1, ..., l_n$ to be $\underline{a} = (a_1, ..., a_n)$ we define, in analogy with $V(\Sigma, G)$, the vector space $V_{\underline{a}}(\Sigma(\underline{p}), G(\underline{p}))$ to be the direct sum over colourings of the remaining lines in $G(\underline{p})$ of the tensor product of spaces associated to the coloured 3-vertices in $G(\underline{p})$.

We next have to define a set of linear mappings between these spaces for different canonical graphs possessing properties corresponding to those of $P_{(\Sigma,G_1),(\Sigma,G_2)}$ discussed in section 3. For this purpose let G_1 and G_2 be two canonical graphs on Σ and consider the graphs $G_1(\underline{p}) \times \{0\}$ and $G_2(\underline{p}) \times \{1\}$ on $\partial(\Sigma \times [0,1])$. We construct the graph $G_1(\underline{p}) + G_2(\underline{p})$ on $\partial((\Sigma / \cup_i D_i) \times [0,1])$ by connecting the points $(q_i, 0)$ in $G_1(\underline{p}) \times \{0\}$ to $(q_i, 1)$ in $G_2(\underline{p}) \times \{1\}$ by the a-coloured line $q_i \times [0,1]$, suitably oriented. Finally, we introduce meridians m_i on the tubes $T^i = \partial D_i \times [0,1]$. It is then easy to see that $(G_1(\underline{p}) + G_2(\underline{p})) \cup \{m_i\}$ is a canonical graph on $\partial((\Sigma / \cup_i D_i) \times [0,1])$ and that the corresponding vector space with $\underline{a}$ fixed is

$$V_{\underline{a}}(\Sigma(\underline{p}), G_1(\underline{p}))^* \otimes V_{\underline{a}}(\Sigma(\underline{p}), G_2(\underline{p})) \,.$$

The partition function

$$Z'_{\underline{a}}(M(\{T^i_{m_i}\}), G_1(\underline{p}) + G_2(\underline{p}))$$

defined as in eq. (3.4) but with $\underline{a}$ fixed, may hence be considered as a mapping

$$P_{G_1,G_2}(\underline{a}) : V_{\underline{a}}(\Sigma(\underline{p}), G_1(\underline{p})) \to V_{\underline{a}}(\Sigma(\underline{p}), G_2(\underline{p}))\,.$$

Since this mapping depends on the chiralities of the $\underline{a}$-lines w.r.t. the meridians we change notation and denote by $\underline{a} = (a_1, ..., a_r)$, resp. $\underline{b} = (b_1, ..., b_s)$, the colours of the left-, resp. right-, handed lines and the correspondingly replace $\underline{a}$ everywhere by $\underline{a}, \underline{b}$.

It is a simple consequence of Thm. 3.1 and the projection property of meridians that for any three canonical graphs G_1, G_2, G_3 without 4-vertices on Σ we have

$$P_{G_2,G_3}(\underline{a}, \underline{b}) P_{G_1,G_2}(\underline{a}, \underline{b}) = P_{G_1,G_3}(\underline{a}, \underline{b})$$

and, in particular, $P_G(\underline{a}, \underline{b}) = P_{G,G}(\underline{a}, \underline{b})$ is a projection. Hence we let $V'_{\underline{a},\underline{b}}(\Sigma(\underline{p}), G(\underline{p}))$ be the support of this projection and conclude, as in section 3, that these spaces for different G may be consistently identified with a space $V_{\underline{a},\underline{b}}(\Sigma(\underline{p}))$ by the mappings $P_{G_1,G_2}(\underline{a}, \underline{b})$ and that

$$dim V_{\underline{a},\underline{b}}(\Sigma(\underline{p})) = Z(\Sigma \times S^1, \mathcal{L}^L_{\underline{a}} \cup \bar{\mathcal{L}}^R_{\underline{b}}) \tag{5.19}$$

where $\mathcal{L}$, resp. $\bar{\mathcal{L}}$, consists of r, resp. s, framed circles of the form $p \times S^1$ with trivial framing and $\mathcal{L}^L_{\underline{a}}$, resp. $\bar{\mathcal{L}}^R_{\underline{b}}$, denote the corresponding coloured left-, resp. right-, handed links.

On the basis of (5.17) a slight extension of the proof of Thm. 5.1 yields the following generalization (see Thm. 7.4 in [KS]).

Theorem 5.3 *Let $\Sigma(\underline{p})$ be a connected, closed, compact, oriented surface of genus $g \geq 1$ with n punctures $(p_1, ..., p_n) \equiv \underline{p}$, r of which are lefthanded with colours $\underline{a} = (a_1, ..., a_r)$ and s righthanded with colours $\underline{b} = (b_1, ..., b_s)$. Then*

$$dim V_{\underline{a},\underline{b}}(\Sigma(\underline{p})) = tr(N_{a_1} \dots N_{a_r} |\vec{N}|^{2(g-1)}) tr(N_{b_1} \dots N_{b_s} |\vec{N}|^{2(g-1)}) \tag{5.20}$$

We remark that in (5.18) we have assumed a certain common orientation of the $\underline{a}$ and $\underline{b}$ coloured lines in the definition of $V_{\underline{a},\underline{b}}(\Sigma(\underline{p}))$.

6 Ribbon graphs and 6j-symbols

The purpose of this section is partly to describe how the properties of the 6j-symbols of a quantum group can be demonstrated by the use of the notion of ribbon graphs and partly to verify eq. (5.18).

As was proven in [A], q-deformations of the universal enveloping algebras of complex simple finite-dimensional Lie algebras $U_q\mathsf{g}$ (quantum groups) with q-admissible modules $\{V_i\}$ constructed in [TW] for $q = \exp(i\pi/l)$ and l bigger then the Coxeter number of g give us natural examples of modular Hopf algebras. (For the Lie algebras of type $\mathfrak{g}_2$ l must also be odd and not divisible by 3.) For the reader's convenience we recall some of the basic notions that enter the definition of a modular Hopf algebra.

Definition [TW]: Let be (A, R, v) a ribbon Hopf algebra (see Def.3.3 in [RT1]) over $\mathbf{C}$, where A is a quasi triangular Hopf algebra, R a universal R-matrix and $v \in A$ a central element with special properties (see eq. (6.3) below). Assume the following data are given:

i) a finite set I with involution $i \mapsto i^* : I \to I$ and a preferred element $0 = 0^*$,

ii) a set of A-modules $\{V_i\}$ labeled by $i \in I$, where $V_0 = \mathbf{C}$ with the action of A determined by the counit $A \to \mathbf{C}$,

iii) a set of A-linear isomorphisms

$$\{w_i : (V_i)^* \to V_{i^*}, i \in I\}, \quad w_0 = \mathrm{id}_{\mathbf{C}}$$

The triple (A, R, v) together with these data is called a modular Hopf algebra if the following axioms (1-5) are satisfied:

1) The modules $\{V_i, i \in I\}$ are mutually non-isomorphic, irreducible (i.e. do not contain proper non-trivial A-submodules), have a finite $\mathbf{C}$-dimension and all have nonsero quantum dimension (see def. below).

2) For each $i \in I$ the homomorphism

$$w_i^* \circ (w_{i^*})^{-1} : V_i \to V_i^{**} = V_i$$

is the multiplication by $g = uv^{-1}$.

3) For any $k \geq 2$ and for any sequence $\Theta = (\lambda_1, \ldots, \lambda_k) \in I^k$ there exists an A-linear decomposition:

$$V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots V_{\lambda_k} = Z_\Theta \oplus \bigoplus_{\lambda \in I} (V_\lambda \otimes \Omega_\Theta^\lambda)$$

where $\{\Omega_\Theta^\lambda\}$ are vector spaces over $\mathbf{C}$ and Z_Θ is a certain A-module satisfying the next axiom (4).

4) For any $k \geq 2, \Theta \in I^k$ and any A-linear homomorphism $f : Z_\Theta \to Z_\Theta$ the q-trace of f is equal to 0, where q-trace of the operator $f : V \to V$ is defined as the trace of an operator

$$x \mapsto uv^{-1} f(x) : V \to V$$

and the quantum dimension $\dim_q V$ equals $\mathrm{tr}_q(\mathrm{id}_V)$, where $u \in A$ is such that that ad_u equals the square of the antipode S of A (see also eq. (6.3) below).

5) Let S_{ij} the q-trace of the A-linear operator

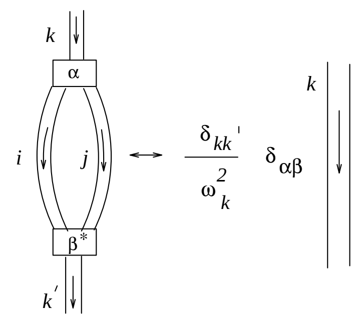
$$a \mapsto R_{21} R_{12} a : V_i \otimes V_j \to V_i \otimes V_j \,,$$

Then the matrix S_{ij} must be invertible.

We shall henceforth use the standard notations for ribbon graphs and let F denote the functor introduced in [RT1] mentioned previously.

In order to prove eq. (5.18) it is enough to verify that Z and F agree on the graphs depicted in (4.3-6), i.e. that the equations analogous to (4.3-6) with Z replaced by F hold. In fact, one uses repeatedly Lemma 3.2 and Lemma 4.2 *ii)-iii)* (most effectively in the form of the Wigner-Eckart type relations derived in [KS]) to decompose any planar graph G into pieces of the types in (4.3-6) and correspondingly obtains $Z(G)$ as a contraction of a linear combination of tensor products of partition functions of the pieces. Since the analogue of Lemma 4.2 also holds for the functor F (see below) the claim follows.

The validity of the analogues of eqs. (4.3-6) for F is, on the other hand, essentially obvious by inspection. In fact eq. (4.3) holds for F since ω_x^2 equals the quantum dimension of $V_x, x \in I$, up to a sign, and eq. (4.4) for F can be obtained by a suitable choice of dual bases $\{\alpha\}$ and $\{\alpha^*\}$ in the mutually dual intertwiner spaces V_{ij}^k (from $V_i \otimes V_j$ to V_k) and V_k^{ij} (from V_k to $V_i \otimes V_j$), so that

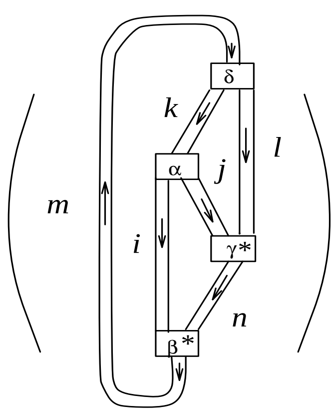


(6.1)

which is the ribbon graph version of (4.7).

We note that the dual basis $\{\alpha^*\}$ in V_k^{ij} to $\{\alpha\}$ in V_{ij}^k can be constructed w.r.t. a natural bilinear pairing or, alternatively by exploiting the natural inner product on V_{ij}^k (see [D]).

Eq. (4.5) can be written in terms of ribbon graphs as

F 

$$\doteq \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}^{\alpha\beta^*\gamma^*\delta} \qquad (6.2)$$

and the analogue of eq. (4.6) for F is given below in (6.6).

We next proceed to show that the 6j-symbols defined by (6.2) have the required properties (see section 2). First we note that due to the lemma 5.1 of

[RT2] ribbon graphs can be considered as two-side objects, where the down sides of the ribbons and annuli have the dual colour and opposite orientation compared to the up side. We shall use an additional operation on the ribbons and annuli – so-called half-twists, as proposed by [N]. The left half-twist is illustrated in Figs.9 and 10.

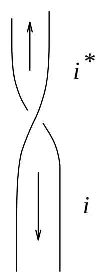


Fig.9

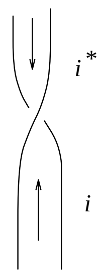


Fig.10

The transformation in Fig.9 is given by the action of the operator $\pi_{V_i}(\tau) : V_i \to (V_{i^*})^* = V_i$, and the one in Fig.10 by the action of $\pi_{V_i^*}(\tau)\colon V_i^* \to V_{i^*} = V_i^*$. We use here the standard notation:

$$\pi_{V_i^*}(a) = [\pi_{V_i}(S(a))]^t,\ a \in A,$$

where $S(a)$ denotes the antipode of a, t the transposition w.r.t. the canonical pairing $V_i^* \otimes V_i \to \mathbf{C}$ and τ an invertible element of the Hopf algebra, which satisfies the following identities:

$$\tau^2 = v^{-1},\, u = \mathrm{Ad}_\tau S(u),\, \varepsilon(\tau) = 1, \Delta(\tau) = (\tau \otimes \tau)R,\ S(\tau) = \tau^{-1}u^{-1}. \qquad (6.3)$$

τ corresponds to the element w^{-1} in the [RT1] notation. The definition of the right half-twists is obtained from the above by replacing of τ by τ^{-1}.

It turns out that in such a way defined half-twists have a nice geometrical property, namely they can be pulled off from left to right through all four annihilation and creation generators of the ribbon graph category:

This follows directly from the properties of τ for non self-dual colours. In the case, when $i = i^*$, there exists an intertwiner $T : V_i \to V_i^*$, and the above mentioned property can be reduced to the claim: $T = T^t$. This means that the two isomorphisms $V_i^* \otimes V_i \cong V_i \otimes V_i$ and $V_i \otimes V_i^* \cong V_i \otimes V_i$ give us the same invariant vector $\xi \in V_i \otimes V_i$, i.e.

$$Perm(\pi_{V_i}(g) \otimes 1)\xi = \xi\,. \qquad (6.4)$$

This was proven for quantum groups in [DJN]. In the terminology of [T] (6.4) distinguishes unimodular and modular Hopf algebras.

We may now construct isomorphisms $V_{ij}^k \simeq V_{k^*i}^{j^*} \simeq V_{jk^*}^{i^*}$ as follows (cf. also [DJN]). To each $\alpha \in V_{ij}^k$ we associate an $\tilde{\alpha} \in V_{k^*i}^{j^*}$ and an $\underset{\sim}{\alpha} \in V_{jk^*}^{i^*}$, such that

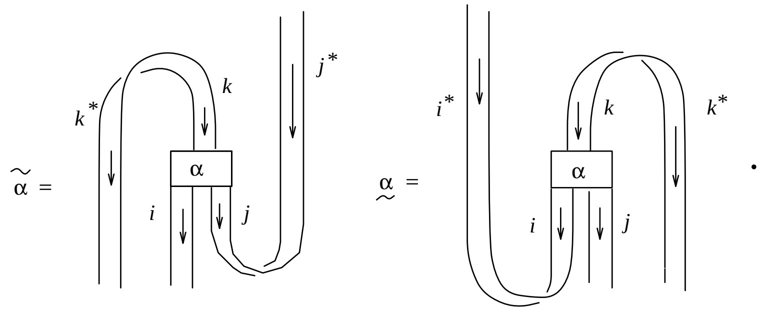


Fig.11

It is easy to see from the well known identities for ribbon graphs [RT1], that

$$\tilde{\tilde{\alpha}} = \underset{\sim}{\alpha}\ ,\ \tilde{\tilde{\tilde{\alpha}}} = \alpha\ ,\ \underset{\approx}{\alpha} = \tilde{\alpha}\,.$$

Due to the fact, that all admissible modules $\{V_i\}$ of a modular Hopf algebra have real q-dimension there exist isomorphisms $\alpha \to \alpha^G$ between V_{ij}^k and V_{ji}^k given by:

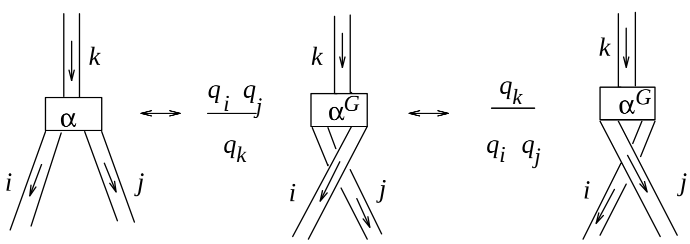


Fig.12

where $q_i = \sqrt{v_i^{-1}}$, $v_i = \pi_{V_i}(v)$, $v_i = v_{i^*}$. The modular property of ribbon Hopf algebras $R_{21}R_{12}(v^{-1} \otimes v^{-1})\Delta v = 1$ implies $\alpha^{GG} = \alpha$.

On the other hand the identity $\Delta(\tau) = (\tau \otimes \tau)R$ allows us to construct an isomorphism between V_{ij}^k and $V_{k^*}^{i^*j^*}$:

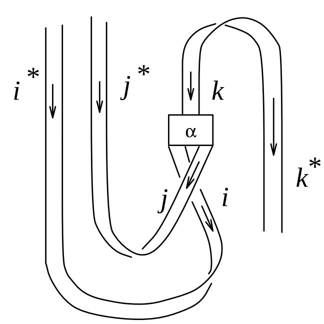
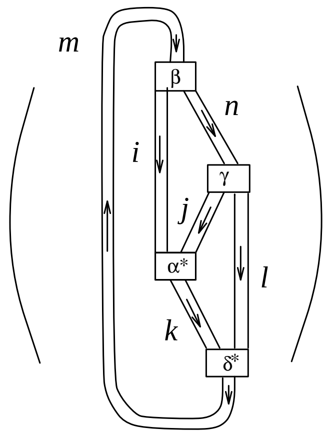


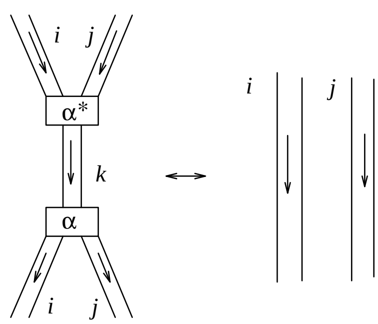
Fig.13

We denote by $\mathcal{V}_{ij}^k$ the **C**-module which is obtained by identifying $V_{ij}^k, V_{k^*i}^{j^*}, V_{jk^*}^{i^*}, V_{ji}^k$ and $V_{k^*}^{i^*j^*}$ by the isomorphisms constructed above. If we choose in (6.2) $\alpha \in \mathcal{V}_{ij}^k$, $\delta \in \mathcal{V}_{kl}^m$, $\gamma^* \in \mathcal{V}_n^{jl}$, $\beta^* \in \mathcal{V}_m^{in}$ the symmetry properties (2.7) follow directly from the isotopies of the corresponding ribbon graphs [RT1]. Additionally, we can show using the isomorphisms in Fig.12 and Fig.13, that

$$\begin{vmatrix} i^* & j^* & k^* \\ l^* & m^* & n^* \end{vmatrix}^{\alpha^*\beta\gamma\delta^*} = F \left(\quad \right) . \qquad (6.5)$$



Due to the properties 3) and 4) of the modular Hopf algebras for closed graphs we always have the equivalence



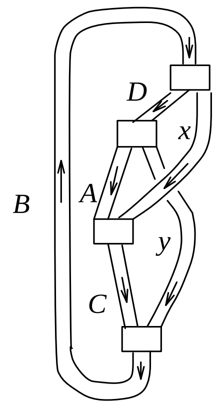
$$\sum_{k\,\alpha} \omega_k^2 \; (\ldots) \longleftrightarrow (\ldots) , \qquad (6.6)$$

which is the ribbon graph version of (4.8).

Taking (6.5), (6.1) and (6.6) into consideration one can quite easily prove the orthogonality, the Biedenharn-Elliot relations and the Racah identities for 6j-symbols defined by (6.2). Applying isomorphisms given by Fig.12 to each 3-vertex in (6.2) and using isotopies of ribbon graphs one obtains (2.8). The last condition (2.12) is a consequence of the normalization (6.1). In fact, to preserve (6.1) for $k = 0$ we must introduce the factor ω_i^{-1} after removing the 0-ribbon from $\alpha \in V_{ii^*}^0$.

Finally, one can analogously prove that

$$F\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right) = \frac{q_A q_B}{q_C q_D}\begin{vmatrix} y & A & D \\ x & B & C \end{vmatrix} \qquad (6.7)$$

which is the ribbon graph version of eq. (2.17).

7 Conclusion

We have in this paper developed calculational techniques applicable to a large class of 3-dimensional state sum models of topological quantum field theories, and extending those of [KS]. As a byproduct we have calculated the dimensions of the state spaces associated to surfaces and obtained a relatively simple proof of the relation between these models and those introduced in [RT1,2], and hence to continuum Chern-Simons theory with an arbitrary compact semisimple gauge group [Wi2]. Our arguments have been mainly based on simple geometrical considerations which we believe can be generalized rather straightforwardly to higher dimensions. Indeed, higher dimensional models of Turaev-Viro type have been suggested recently in the literature and it would be an obvious task to introduce observables into those models in the form of "higher dimensional graphs" by extending our construction in section 3. In order to develop such a construction into an effective calculational tool it would be crucial to have at disposal analogues of the Racah identities. We leave these issues for future investigation.

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