

3D SUPERGRAVITY IN THE BATALIN–VILKOVISKY FORMALISM

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ABSTRACT. Three-dimensional supergravity in the Batalin–Vilkovisky formalism is constructed by showing that the theory including the Rarita–Schwinger term is equivalent to an AKSZ theory.

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1. INTRODUCTION

In this paper we construct the BV action [BV81] for supergravity in three dimensions (see, e.g. [RN96; CCG16] and references therein). Our strategy is to start considering the minimal coupling of spin $\frac{3}{2}$

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fermions to BF theory (with Lie algebra $\mathfrak{so}(2, 1)$), where the BV action is easy to construct, since we realise this as an AKSZ theory [Ale+97]. Three-dimensional BF theory is on-shell equivalent to gravity in the Palatini–Cartan formalism [Wit88], and the off-shell equivalence in the BV formalism has been constructed in [CSS18]. Thus, the main contribution of this paper is the extension of the latter equivalence in the presence of spin $\frac{3}{2}$ fermions. From this point of view, supergravity is “discovered”, in the sense that the resulting BV operator turns out to contain the local supersymmetry transformations. More precisely, switching to zero all the ghosts but for the fermionic ghost ε , we get

$$\begin{aligned} Qe &= \bar{\psi} \rho \varepsilon, \\ Q\psi &= D\varepsilon, \end{aligned}$$

where e denotes the dreibein, ψ the spin $\frac{3}{2}$ field, ρ the spin representation, and D the covariant derivative (note that the spin connection is not affected by this transformation). Moreover, the ghost for diffeomorphisms accordingly transforms with a term $\frac{1}{2}\bar{\varepsilon} e^{-1}(\rho)\varepsilon$, which encodes the fact that two supersymmetry transformations yield a translation. The BV structure we obtain for supergravity in three dimensions is considerably simpler than in four, where the BV action is known to require non-linear terms in the antifields (see [Bau+90] for the half-shell formulation and [CF] for the Palatini–Cartan formulation). One reason for this may be that three-dimensional gravity is topological and remains so when adding spin $\frac{3}{2}$ fermions.

In this note we focus on the Minkowski signature and on the spin $\frac{3}{2}$ Majorana representation. However, these results are readily generalised to the Euclidean case (or to alternative signatures) and to other real representations of $\mathfrak{spin}(2, 1)$ or $\mathfrak{spin}(3)$, respectively.

1.1. Motivation. The Palatini–Cartan–Holst (PCH) formalism was conceived in order to bring Einstein’s general relativity closer to the language of gauge theories,⁽ⁱ⁾ and based on the work in [CS19a] it was proven in [CSS18] that in three dimensions Palatini–Cartan (PC) gravity is strongly equivalent to a BF theory, result that serves as a basis for the current paper. Here we will show that this strong equivalence persists even when we incorporate the Rarita–Schwinger term to the action,⁽ⁱⁱ⁾ which further allows us to define a Batalin–Vilkovisky (BV)

⁽ⁱ⁾ A general review of the Palatini formalism is found in [Rom93], one of the PCH formalism in [CS18], and Holst’s original work is [Hol96].

⁽ⁱⁱ⁾ For an introduction to supersymmetry, see [Con10], and see [Nas12] for an introduction to supergravity, while the reader is referred to [WB92] for a thorough exposition of both.

extension of 3D supergravity where both the invariance under space-time diffeomorphisms and the supersymmetry are explicitly encoded in their corresponding ghost fields.

Such result might serve as the starting point for the analysis of the boundary structure of this theory through the BV–BFV formalism [CMR14] developed by Cattaneo, Mnev and Reshetikhin, hence joining a broader effort to better understand how gravity coupled to different kinds of matter fields should be described, as well as how to quantize such theories, even in the presence of non-trivial boundaries. Recent papers in this line include [CS19b; Cat+24; CMS24; CCF22; Can+24], to name a few. A further development of the results of this paper would be its extension to all lower-dimensional strata, as was done for PC theory in [CS22], and the construction of its space of quantum states, which for Einstein–Hilbert theory was done in [CS24], leveraging the relation with PC theory.

Concerning the structure of the document, the second section consists of a short summary of the BV formalism [BV81] and of the AKSZ procedure [Ale+97]. The third section, in turn, presents BV and BF gravity, and leads to the fourth and last section, where we present supergravity and build its BV extension. Besides these sections, the reader can find an appendix where we will compile a series of results of graded geometry that we shall use in the rest of the article. Sources reviewing graded geometry are the review [CM20] or the books [Kef19; DeW92]. For an approach to three-dimensional supergravity in the context of graded geometry, see also [CCG16; CCG20]. Lastly, we let the reader know that this article is an improved adaptation of the content in the master thesis of the third author [Sma24].

1.2. Notation.

- \mathbb{N} denotes the set of natural numbers *including* 0.
- $\llbracket m, n \rrbracket = \{k \in \mathbb{Z} \mid m \leq k \leq n\}$ denotes a range of integers.
- Given a Lie group G , its associated Lie algebra is denoted by \mathfrak{g} .
- All forms of products are left implicit and deduced from the context, unless some ambiguity is present, e.g. $\lambda \cdot u \otimes v =: \lambda uv$.
- In the absence of parentheses, a derivation D acts only on the element directly adjacent to it: $aDbc := aD(b)c$.
- Einstein’s summation convention for pairs of upper and lower indices is generally assumed: $x^i y^i \neq x^i y_i := \sum_{i=j} x^i y_j$.
- Indices with a bar on them are not summed over: $x^{\bar{i}} y_{\bar{i}} \neq x^i y_i$.
- δ_j^i denotes Kronecker’s delta: $\delta_j^i = 1$ for $i = j$, $\delta_j^i = 0$ otherwise.

- In the context of gravity or special relativity, Greek letters designate spacetime indices while Latin indices designate Lorentz bundle indices.
- The equivalence sign “ \equiv ” is used to designate equality *on shell*.
- There are multiple fields associated to a field ψ :
 - ψ^* is its complex conjugate,
 - $\overline{\psi^\dagger}$ is its Hermitian conjugate,
 - $\overline{\psi}$ is its Dirac conjugate,
 - ψ^+ is its associated antifield.

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2. CLASSICAL BV FORMALISM

The path integral formalism for quantum field theory relies on the possibility of integrating out the quadratic terms in the Lagrangian density defining the action, which is achieved through a generalisation to field theory of the saddle point method—known as *Feynman–Laplace method*—requiring the critical points of the action to form a finite subset of its support. However, precisely because of the “continuity”—as opposed to “discreteness”—of topological groups, in theories described by a Lagrangian with gauge freedom one can smoothly deform a critical point into another critical point, resulting in critical loci that are themselves submanifolds. This spoils the applicability of the aforementioned method, making manifest the need for machinery that selects discrete subsets of the critical locus of an action. This is precisely the issue that both the BRST and the BV formalisms address.

We choose to employ the latter because it has a greater range of applicability than the former, and also because of its relation to the Batalin–Fradkin–Vilkovisky (BFV) formalism, which allows for a perturbative quantisation of field theories on the possible boundary of a manifold. Despite all this, though, we will not be concerned with any quantisation in this document, so our treatment of the theory will be minimal; an exhaustive approach is found e.g. in [Mne19], and a review of the BFV formalism is e.g. in [CM20]. Let us then summarise the BV formalism.

Definition 2.1. Given a graded manifold \mathcal{M} , a **cohomological vector field** Q is an element $Q \in \mathfrak{X}(\mathcal{M})$ such that

$$(2.1) \quad Q^2 = 0, \quad |Q| = 1, \quad \text{gh } Q = 1,$$

where one understands Q^2 as $Q \circ Q$, being a map $C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$. A manifold endowed with such a vector field is a **dg manifold**.⁽ⁱⁱⁱ⁾

Remark 2.2. Given that Q is odd, saying that $Q^2 = 0$ is equivalent to saying that $[Q, Q] = 0$.

Example 2.3. A paradigmatic example of dg manifold is given by the odd tangent bundle $T[1]M$ of any non-graded manifold M . If (x^i) are coordinates on M and (θ^i) the coordinates of ghost number 1 on its fibre, then a dg structure is given by

$$(2.2) \quad Q = \theta^i \frac{\partial}{\partial x^i}.$$

Definition 2.4. Given a vector field $X \in \mathfrak{X}(\mathcal{M})$ and a form $\omega \in \Omega(\mathcal{M})$, we say that ω is **X-invariant** if

$$(2.3) \quad \mathcal{L}_X \omega = 0.$$

Definition 2.5. A **dg-symplectic manifold** is a graded symplectic manifold with a cohomological vector field Q under which the symplectic form ω is invariant. That is,

$$(2.4) \quad \mathcal{L}_Q \omega = 0.$$

Definition 2.6. Given a graded symplectic manifold \mathcal{M} and a function $f \in C^\infty(\mathcal{M})$, the **Hamiltonian vector field** associated to f is the field $\vec{f} \in \mathfrak{X}(\mathcal{M})$ such that

$$(2.5) \quad \vec{f} = (-1)^{|f|+1} \{f, \bullet\}.$$

Definition 2.7. A **dg-Hamiltonian manifold of degree k** $(\mathcal{M}, H, Q, \omega)$ is a dg-symplectic manifold \mathcal{M} where the symplectic form ω has ghost number k and the cohomological vector field Q is given as the Hamiltonian vector field of some degree $k+1$ function H , called its **Hamiltonian function**:

$$(2.6) \quad Q = (-1)^k \{H, \bullet\}.$$

Definition 2.8. A **classical BV theory** is a dg-Hamiltonian manifold of degree -1 , that is, a tuple $(\mathcal{F}, S, Q, \omega)$ such that

1. \mathcal{F} is a graded symplectic manifold, called the **field space**.
2. S is an even function over \mathcal{F} of degree 0, called the **BV action**.
3. Q is the cohomological, Hamiltonian vector field of S .
4. ω is a Q -invariant, odd symplectic form over \mathcal{F} of degree -1 .

⁽ⁱⁱⁱ⁾ Here *dg* stands for *differential graded*.

Given that \mathcal{F} is odd symplectic, we can locally associate each field $\phi \in \mathcal{F}$ to another field $\phi^+ \in \mathcal{F}$, known as its **antifield**, that by construction satisfies

$$(2.7) \quad \deg_{\Omega} \phi^+ = n - \deg_{\Omega} \phi, \quad \text{gh } \phi^+ = -\text{gh } \phi - 1.$$

Definition 2.9. A BV theory $(\mathcal{F}, S, Q, \omega)$ is a **BV extension** of a classical theory described by a space of fields \mathcal{F}_{cl} and an action S_{cl} , if the ghost number zero part of \mathcal{F} and of S correspond to \mathcal{F}_{cl} and to S_{cl} , and if the restriction $Q|_{\mathcal{F}_{\text{cl}}}$ yields the gauge symmetries.

Definition 2.10. Two BV theories are **weakly equivalent** if both of them are BV extensions of a same classical theory.

These theories will be **strongly equivalent** if there is a graded symplectomorphism $\Phi : \mathcal{F} \rightarrow \mathcal{F}'$ between their respective field spaces that pulls the action of one theory back to the action of the other:

$$(2.8) \quad \phi^* S' = S.$$

Such a symplectomorphism is known as a **canonical transformation**.

Definition 2.11. Let (q, p) and (q', p') be the respective even-odd coordinates of two different BV field spaces \mathcal{F} and \mathcal{F}' . A **graded generating function of type j** , for $j \in \llbracket 1, 4 \rrbracket$, is a graded function G_j that we use to define two coordinates among q, p, q', p' as a function of the remaining two, in one of the four following ways:

$$(2.9a) \quad p = (-1)^{|q|+1} \frac{\partial G_1(q, q')}{\partial q}, \quad p' = (-1)^{|q'|} \frac{\partial G_1(q, q')}{\partial q'},$$

$$(2.9b) \quad p = (-1)^{|q|+1} \frac{\partial G_2(q, p')}{\partial q}, \quad q' = (-1)^{|p'|} \frac{\partial G_2(q, p')}{\partial p'},$$

$$(2.9c) \quad q = (-1)^{|p|+1} \frac{\partial G_3(p, q')}{\partial p}, \quad p' = (-1)^{|q'|} \frac{\partial G_3(p, q')}{\partial q'},$$

$$(2.9d) \quad q = (-1)^{|p|+1} \frac{\partial G_4(p, p')}{\partial p}, \quad q' = (-1)^{|p'|} \frac{\partial G_4(p, p')}{\partial p'}.$$

Remark 2.12. By design, generating functions have top cohomological degree and ghost number -1 .

Definition 2.13. Given two graded manifolds \mathcal{M}, \mathcal{N} , and letting $\text{Mor}(\mathcal{M}, \mathcal{N})$ be the manifold of grade-preserving morphisms $\mathcal{M} \rightarrow \mathcal{N}$ in the category of graded manifolds, the **mapping space** $\text{Map}(\mathcal{M}, \mathcal{N})$ is the extension of $\text{Mor}(\mathcal{M}, \mathcal{N})$ that includes grade-shifting maps.

Remark 2.14. If \mathcal{N} is a graded vector space then

$$(2.10) \quad \text{Map}(\mathcal{M}, \mathcal{N}) = C^{\infty}(\mathcal{M}) \otimes \mathcal{N},$$

so locally the mapping space will have the form of such a tensor product of graded spaces. Details of this definition can be found in [CMR14].

Definition 2.15. An **AKSZ theory**^(iv) in n dimensions $(M, \mathcal{N}, H, Q, \alpha)$ is the combination of two things:

1. A **source** consisting of a closed and oriented n -manifold M .
2. A **target** consisting of a dg-Hamiltonian manifold $(\mathcal{N}, H, Q, \omega)$ of degree $n - 1$ whose symplectic form ω is exact:

$$(2.11) \quad \omega = d_{\mathcal{N}}\alpha$$

for $d_{\mathcal{N}}$ the exterior derivative on \mathcal{N} .

Definition 2.16. Given an AKSZ theory $(M, \mathcal{N}, H_{\mathcal{N}}, Q_{\mathcal{N}}, \alpha_{\mathcal{N}})$ in n dimensions, we define the **AKSZ fields space**

$$(2.12) \quad \mathcal{F} = \text{Map}(T[1]M, \mathcal{N}).$$

Employing the notation in Remark 2.17, we take the evaluation map $\text{ev} : T[1]M \times \mathcal{F} \rightarrow \mathcal{N}$ and define, for all $k \in \mathbb{N}$ and for coordinates ξ on \mathcal{N} and X on \mathcal{F} , its pullback as

$$(2.13) \quad \text{ev}^* : \Omega^k(\mathcal{N}) \rightarrow \Omega(M) \otimes \Omega^k(\mathcal{F}) : \beta(\xi) \mapsto \widehat{\beta}(X),$$

and we define the pushed forward projection

$$(2.14) \quad \begin{aligned} \pi_* : \Omega(M) \otimes \Omega^k(\mathcal{F}) &\rightarrow \Omega^k(\mathcal{F}), \\ \varphi \otimes \Phi &\mapsto \int_M \varphi^{\text{top}} \otimes \Phi \quad \forall \varphi \in \Omega(M), \Phi \in \Omega^k(\mathcal{F}). \end{aligned}$$

We construct coordinates (X^i) on \mathcal{F} —the so called **AKSZ superfields**—associated to the coordinates (x^i) on \mathcal{N} as

$$(2.15) \quad X^i = \text{ev}^* x^i,$$

and compose π_* with ev^* to produce the **transgression map**

$$(2.16) \quad \mathcal{T} = \pi_* \text{ev}^*.$$

Letting d_M be the exterior derivative on M , and further letting $\widetilde{d}_M, \widetilde{Q}_{\mathcal{N}} \in \mathfrak{X}(\mathcal{F})$ be the respective lifts to \mathcal{F} of d_M and of $Q_{\mathcal{N}}$, we finally define the **AKSZ construction** associated to this theory as the tuple $(\mathcal{F}, S, Q, \omega)$ for

1. the **AKSZ action**

$$(2.17) \quad S = \iota_{\widetilde{d}_M} \mathcal{T} \alpha_{\mathcal{N}} + \mathcal{T} H_{\mathcal{N}},$$

2. the **AKSZ vector field**

$$(2.18) \quad Q = \widetilde{d}_M + \widetilde{Q}_{\mathcal{N}},$$

^(iv) Here we focus on this special case of the more general AKSZ method.

3. and the **AKSZ symplectic form**

$$(2.19) \quad \omega = \mathcal{T}\omega_{\mathcal{N}}.$$

Remark 2.17. Given a form $\beta \in \Omega(\mathcal{N})$, and coordinates x over \mathcal{N} and X over \mathcal{F} , by $\widehat{\beta}(X)$ one understands the coordinates expression for $\beta(x)$, but symbolically replacing x by X .

Given any manifold M and form $\varphi \in \Omega(M)$, φ^{top} denotes the components of φ with top cohomological degree, that is, those components such that $\deg_{\Omega(M)} \varphi^{\text{top}} = \dim M$.

Remark 2.18. The maps π_* and \mathcal{T} are graded:

$$(2.20) \quad |\pi_*| = |\mathcal{T}| = -\dim M.$$

Remark 2.19. By Definition 2.13, locally

$$(2.21) \quad \mathcal{F} \cong \Omega(M) \otimes \mathcal{N},$$

so in practice we can write S , Q and ω explicitly:

$$(2.22) \quad S = \int_M \left((\widehat{\alpha}_{\mathcal{N}})_i(X) d_M X^i + \widehat{H}_{\mathcal{N}}(X) \right),$$

$$(2.23) \quad Q = \int_M \left(d_M X^i + \widehat{Q}_{\mathcal{N}}^i(X) \right) \frac{\partial}{\partial X^i},$$

$$(2.24) \quad \omega = (-1)^n \int_M (\widehat{\omega}_{\mathcal{N}})_{ij}(X) d_{\mathcal{N}} X^i \wedge d_{\mathcal{N}} X^j.$$

Given this, together with the fact that each superfield X^i can be decomposed over summands $\{X_{(j)}^i\}_j$ of definite cohomological degree $j \in \llbracket 0, n \rrbracket$, at the end of the day the AKSZ action is the action that we would obtain by

1. symbolically replacing the original fields with their associated superfields,
2. expanding those in components of definite cohomological degree,
3. keeping only the terms which have top cohomological degree.

An example of such construction is given in the next section, so let us proceed without further ado.

3. 3D GRAVITY IN VACUUM AND BF THEORY

Definition 3.1. An n -dimensional **spacetime** is a closed n -manifold with a **mostly positive** metric, that is, of signature $(p = n - 1, n = 1)$.

Definition 3.2. Given a principal $\mathrm{SO}(n-1, 1)$ -bundle P over an n -dimensional manifold, its **Minkowski bundle** is the associated vector bundle (\mathcal{V}, η) with typical fibre \mathbb{R}^n , endowed with the **Minkowski metric**

$$(3.1) \quad \eta := (-\mathbb{I}_1) \oplus \mathbb{I}_{(n-1)},$$

for \mathbb{I}_k the identity matrix in k dimensions.

Remark 3.3. Here we will focus on the case where $n = 3$, so

$$(3.2) \quad \eta = \mathrm{diag}(-1, 1, 1).$$

Definition 3.4. Given a Minkowski bundle (\mathcal{V}, η) , a **cotriad** or **coframe field** over a 3-spacetime M is a non-degenerate 1-form e over M valued in $\mathcal{V}^{\wedge 1}$. Associated to it there is a **triad** or **frame field** e^{-1} , which is its inverse in the following sense:

$$(3.3) \quad e \in \Omega^1(M, \mathcal{V}^{\wedge 1}), \quad e^{-1} \in \Omega^1(M, \mathcal{V}^{\wedge 1})^*, \quad e^{-1}(e) = 1.$$

Remark 3.5. We talk of $\mathcal{V}^{\wedge 1}$ and not simply of \mathcal{V} because we associate to the (co-)triads a multivector degree

$$(3.4) \quad \mathrm{deg}_{\mathcal{V}} e = -\mathrm{deg}_{\mathcal{V}} e^{-1} = 1.$$

From now on, we will always conceive \mathcal{V} as $\mathcal{V}^{\wedge 1}$.

Definition 3.6. The **Palatini–Cartan formalism** in three dimensions, or simply **3D gravity**, consists of

1. an oriented spacetime M , assumed to have no boundary,
2. a principal $\mathrm{SO}(2, 1)$ -bundle $P \rightarrow M$, with Minkowski bundle \mathcal{V} ,
3. a possibly non-zero **cosmological constant** $\Lambda \in \mathbb{R}$,

together with a field content given by

4. a cotriad $e \in \Omega^1(M, \mathcal{V})$ over M ,
5. a connection 1-form $\Gamma \in \mathrm{Conn}(P) \cong \Omega(M, \mathcal{V}^{\wedge 2})$ over P .

With these we define the classical field space

$$(3.5) \quad \mathcal{F}_{\mathrm{GR}}^0 = \Omega_{\mathrm{nd}}^1(M, \mathcal{V}) \times \mathrm{Conn}(P) \quad \text{(v)}$$

and the classical action

$$(3.6) \quad S_{\mathrm{GR}}^0(\Lambda) = \int_M \left\langle e \wedge F_{\Gamma} + \frac{\Lambda}{6} e^{\wedge 3} \right\rangle,$$

where the angle brackets $\langle \bullet \rangle$ designate the appropriate contraction of any indices other than those over $\Omega(M)$.^(vi)

^(v) Where “nd” stands for *non-degenerate*.

^(vi) The expression within the brackets actually defines a density, which could be integrated over a non-orientable manifold, yet for simplicity we assume orientability.

Remark 3.7. The reason why we can say that connections take values in $\mathcal{V}^{\wedge 2}$ follows from the fact that, since we work in three dimensions, $\mathcal{V}^{\wedge 2} \cong \mathfrak{so}(2, 1)$ as a Lie algebra.

Remark 3.8. For convenience and when it is possible, we often express a particular action S_i as the integral of a top form $\langle \mathcal{L}_i \rangle$:

$$(3.7) \quad S_i = \int_M \langle \mathcal{L}_i \rangle.$$

Remark 3.9. By *appropriate contractions* one understands that ultimately S_{GR}^0 must be a scalar, so any internal index must be contracted. This is to say that we must correctly “trace over” the multivector indices that both e and F_Γ have over \mathcal{V} , leading specifically to

$$(3.8) \quad \langle e \wedge F_\Gamma \rangle = \epsilon_{abc} e^a \wedge F_\Gamma^{bc}, \quad \langle e^{\wedge 3} \rangle = \epsilon_{abc} e^a \wedge e^b \wedge e^c,$$

for the 3-dimensional Levi–Civita symbol ϵ_{abc} and leaving implicit the decomposition over the generators of $\Omega(M)$.

Definition 3.10. Given a connection 1-form Γ and its associated covariant derivative D_Γ , the **Lie covariant derivative** \mathcal{L}_ξ^Γ with respect to a vector field ξ is defined as

$$(3.9) \quad \mathcal{L}_\xi^\Gamma = [\iota_\xi, D_\Gamma].$$

Construction 3.11 (BV extension of 3D gravity). Based on the data from 3D gravity, let \mathcal{F}_{GR} be the field space

$$(3.10) \quad \mathcal{F}_{\text{GR}} = T^*[-1](\mathcal{F}_{\text{GR}}^0 \times \mathfrak{X}(M)[1] \times \Omega^0(M, \text{ad}(P)[1])),$$

where $\mathfrak{X}(M)[1]$ is the space of vector fields over M with shifted degree and $\text{ad}(P)[1]$ stands for the degree-shifted adjoint bundle of P . On the base of this shifted cotangent bundle we denote the fields as

$$(3.11) \quad ((e, \Gamma), \xi, \chi) \in \mathcal{F}_{\text{GR}}^0 \times \mathfrak{X}(M)[1] \times \Omega^0(M, \text{ad}(P)[1]),$$

where ξ encodes the diffeomorphism invariance of general relativity and χ is the ghost field required for the internal gauge transformations. In turn, the fibre includes their associated antifields, and the fact that \mathcal{F}_{GR} is a shifted cotangent bundle allows us to define the canonical symplectic form

$$(3.12) \quad \omega_{\text{GR}} = \int_M \left(\delta e \delta e^+ + \delta \Gamma \delta \Gamma^+ + \delta \xi \delta \xi^+ + \delta \chi \delta \chi^+ \right)$$

Finally, we can define the action $S_{\text{GR}}(\Lambda) := S_{\text{GR}}^0(\Lambda) + S_{\text{GR}}^1$ with

$$(3.13) \quad S_{\text{GR}}^1 = \int_M \left\langle e^+ (\mathcal{L}_\xi^\Gamma e + [\chi, e]) + \Gamma^+ (D_\Gamma \chi + \iota_\xi F_\Gamma) \right. \\ \left. + \frac{1}{2} \iota_{[\xi, \xi]} \xi^+ + \frac{1}{2} \chi^+ ([\chi, \chi] - \iota_\xi^2 F_\Gamma) \right\rangle,$$

and its associated cohomological vector field Q_{GR} , which acts as

$$(3.14a) \quad Q_{\text{GR}}(e) = \mathcal{L}_\xi^\Gamma e + [\chi, e],$$

$$(3.14b) \quad Q_{\text{GR}}(\Gamma) = D_\Gamma \chi + \iota_\xi F_\Gamma,$$

$$(3.14c) \quad Q_{\text{GR}}(\xi) = \frac{1}{2} [\xi, \xi]$$

$$(3.14d) \quad Q_{\text{GR}}(\chi) = \frac{1}{2} ([\chi, \chi] - \iota_\xi^2 F_\Gamma).$$

We then write $\mathcal{T}_{\text{GR}} := (\mathcal{F}_{\text{GR}}, S_{\text{GR}}, Q_{\text{GR}}, \omega_{\text{GR}})$ to refer to this theory.

Proposition 3.12. *The tuple \mathcal{T}_{GR} is a BV extension of 3D gravity, which we shall call 3-dimensional **BV gravity**.*

Definition 3.13. A **BF theory** in $n \geq 2$ dimensions is defined as a pair (\mathcal{F}, S) where, given

1. an n -dimensional oriented manifold M ,
2. a finite dimensional Lie group G and a G -bundle P over M ,
3. a form $B \in \Omega^{n-2}(M, \text{ad}^*(P))$ valued in the coadjoint bundle,
4. a connection form $A \in \text{Conn}(P)$,

one sets the field space to be

$$(3.15) \quad \mathcal{F} = \text{ad}^*(P) \times \text{Conn}(P)$$

and the action to be

$$(3.16) \quad S = \int_M \langle B, F_A \rangle,$$

where F_A is the curvature of A and $\langle \bullet, \bullet \rangle$ is the pairing of dual maps.

Definition 3.14. We define **BF gravity** as the **BF theory** $(\mathcal{F}_{\text{BF}}^0, S_{\text{BF}}^0)$ where $n = 3$, and where P is the $\text{SO}(2, 1)$ -bundle over M with associated Minkowsky bundle \mathcal{V} for some reference Lorentzian metric.

Moreover, for some reference connection A' we interpret B and $A - A'$ as 1-forms valued in \mathcal{V}^* and $\mathcal{V}^{\wedge 2}$ respectively, and further use the internal Minkowski metric to identify \mathcal{V}^* with \mathcal{V} , hence seeing B as taking values in the latter. Thus the field space becomes

$$(3.17) \quad \mathcal{F}_{\text{BF}}^0 \cong \Omega^1(M, \mathcal{V}) \times \Omega^1(M, \mathcal{V}^{\wedge 2}).$$

Finally, we incorporate a cosmological term in the action, resulting in

$$(3.18) \quad S_{BF}^0(\Lambda) = \int_M \langle BF_A + \frac{\Lambda}{6} B^{\wedge 3} \rangle,$$

where $\langle \bullet \rangle$ denotes again the trace over the internal indices.

Proposition 3.15 ([CSS18]). *BF gravity is an AKSZ theory and its AKSZ extension $\mathcal{T}_{BF} := (\mathcal{F}_{BF}, S_{BF}, Q_{BF}, \omega_{BF})$ is strongly equivalent to 3-dimensional BV gravity. A canonical transformation $\Phi_{BF} : S_{GR} \rightarrow S_{BF}$ is provided by the following type 2 generating function:*

$$(3.19) \quad G(q, p')_{BF} = -B^+(e - \iota_\xi \Gamma^+ - \frac{1}{2} \iota_\xi^2 \chi^+) - A\Gamma^+ \\ - \tau^+(-\iota_\xi e + \frac{1}{2} \iota_\xi^2 \Gamma^+ - \frac{1}{3} \iota_\xi^3 \chi^+) - c\chi^+$$

for $q := (e, \Gamma^+, \xi, \chi^+)$ and $p' := (B^+, A, \tau^+, c)$.

Remark 3.16. We recall the decomposition of the fields that results from this proposition, keeping τ^+ implicit to avoid cramping the equations:

$$(3.20a) \quad B = e - \iota_\xi \Gamma^+ - \frac{1}{2} \iota_\xi^2 \chi^+, \quad B^+ = e^+ - \iota_\xi \tau^+,$$

$$(3.20b) \quad A = \Gamma - \iota_\xi e^+ + \frac{1}{2} \iota_\xi^2 \tau^+, \quad A^+ = \Gamma^+,$$

$$(3.20c) \quad \tau = -\iota_\xi e + \frac{1}{2} \iota_\xi^2 \Gamma^+ + \frac{1}{3} \iota_\xi^3 \chi^+, \quad \tau^+ = e^{-1}(\xi^+ - e^+ \Gamma^+ + \iota_\xi e^+ \chi^+),$$

$$(3.20d) \quad c = \chi + \frac{1}{2} \iota_\xi^2 e^+ - \frac{1}{6} \iota_\xi^3 \tau^+, \quad c^+ = \chi^+.$$

4. BV SUPERGRAVITY

Definition 4.1. We define **3D supergravity** and **BF supergravity** by extending respectively \mathcal{F}_{GR}^0 and \mathcal{F}_{BF}^0 to

$$(4.1) \quad \mathcal{F}_{GR\Phi}^0 = \mathcal{F}_{GR}^0 \times \Pi\mathcal{S}(P) \quad \text{and} \quad \mathcal{F}_{BF\Phi}^0 = \mathcal{F}_{BF}^0 \times \Pi\mathcal{S}(P),$$

for $\mathcal{S}(P)$ the space of 1-forms taking values in the spinor bundle associated to the principal bundle P , and then extending their actions by a **Rarita–Schwinger action** term

$$(4.2) \quad S_{GR\Phi}^0(\Lambda) = S_{GR}^0(\Lambda) + \int_M \frac{1}{2} \bar{\psi} D_\Gamma \psi, \\ S_{BF\Phi}^0(\Lambda) = S_{BF}^0(\Lambda) + \int_M \frac{1}{2} \bar{\varphi} D_A \varphi.$$

Here the fields $\psi, \varphi \in \Pi\mathcal{S}(P)$ are spin $\frac{3}{2}$ Majorana spinors.

Remark 4.2. The connection forms will act on the spinor fields through the spin $\frac{3}{2}$ real (Majorana) representation of the algebra $\mathfrak{spin}(2, 1)$. This representation will be denoted by $\boldsymbol{\rho}$ and, for the Pauli matrices $\{\sigma^a\}_{a=1}^3$, we will use the shorthand notation $\rho^a := \boldsymbol{\rho}(\sigma^a)$. Letting $\{v_a\}$ be a basis of the sections of the Minkowski bundle \mathcal{V} , we write

$$(4.3) \quad \rho := \rho^a v_a,$$

with which the equations of motion that follow from $S_{BF\Phi}^0$ take the following form:

$$(4.4a) \quad F_A + \frac{\Lambda}{2} B \wedge B = 0,$$

$$(4.4b) \quad D_A \varphi = 0,$$

$$(4.4c) \quad D_A B + \frac{1}{2} \bar{\varphi} \rho \varphi = 0.$$

Meanwhile, the equations of motion issued from $S_{GR\Phi}^0$ are analogous, after replacing

$$(4.5) \quad (B, A, \varphi) \leftrightarrow (e, \Gamma, \psi).$$

Proposition 4.3. *BF supergravity is an AKSZ theory.*

Proof. We take the spacetime M as the source manifold, let V be the Minkowski space that is the typical fibre of the Minkowski bundle \mathcal{V} , and $S(P)$ be the vector space associated to the spinor representation $\boldsymbol{\rho}$. We hence define (b, a, f) as coordinates on the target

$$(4.6) \quad \mathcal{N} \cong V[1] \oplus V^{\wedge 2}[1] \oplus \Pi S(P)[1].$$

We see from this that \mathcal{N} is endowed with the symplectic form

$$(4.7) \quad \omega_{\mathcal{N}} = d_{\mathcal{N}} \alpha_{\mathcal{N}},$$

$$(4.8) \quad \Omega^1(\mathcal{N}) \ni \alpha_{\mathcal{N}} = b d_{\mathcal{N}} a + \frac{1}{2} \bar{f} d_{\mathcal{N}} f,$$

the Hamiltonian function

$$(4.9) \quad H_{\mathcal{N}} = \left\langle \frac{1}{2} b[a, a] + \frac{\Lambda}{6} b^3 \right\rangle + \frac{1}{2} \bar{f} a f,$$

and the cohomological vector field $Q_{\mathcal{N}} = -\{H_{\mathcal{N}}, \bullet\}$ associated to $H_{\mathcal{N}}$ through the Poisson bracket induced by $\omega_{\mathcal{N}}$. \square

Construction 4.4 (AKSZ extension of BF supergravity). Given the target established in the proof of Proposition 4.3, the field space is

$$(4.10) \quad \mathcal{F}_{BF\Phi} \cong \Omega(M, \mathcal{V})[1] \oplus \Omega(M, \mathcal{V}^{\wedge 2})[1] \oplus \Omega(M, \Pi S(P))[1].$$

On it, the coordinates are given by the superfields

$$(4.11a) \quad \tilde{b} = \tau + B + A^+ + c^+,$$

$$(4.11b) \quad \tilde{a} = c + A + B^+ + \tau^+,$$

$$(4.11c) \quad \tilde{f} = \gamma + \varphi + \varphi^+ + \gamma^+,$$

where summands are ordered in increasing cohomological degree, from 0 to 3, and decreasing ghost number, from 1 to -2 . Besides, note that the parity of a field differs by 1 from the parity of its corresponding antifield. Now, replacing the classical fields in $S_{BF\Phi}^0$ by their associated superfield, keeping only those terms of cohomological degree 3 and rearranging them, we find the BV action for BF supergravity:

$$(4.12a) \quad S_{BF\Phi}(\Lambda) = \int_M \left\langle \mathcal{L}_{BF\Phi}^0(\Lambda) + \mathcal{L}_{BF\Phi}^1(\Lambda) + \mathcal{L}_{BF\Phi}^2(\Lambda) \right\rangle,$$

where

$$(4.12b) \quad \mathcal{L}_{BF\Phi}^0(\Lambda) = BF_A + \frac{\Lambda}{6} B^{\wedge 3} + \frac{1}{2} \bar{\varphi} D_A \varphi,$$

$$(4.12c) \quad \mathcal{L}_{BF\Phi}^1(\Lambda) = B^+ ([c, B] + D_A \tau) + A^+ (D_{AC} + \Lambda B \tau) \\ + \frac{1}{2} c^+ ([c, c] + \Lambda \tau \tau) + \tau^+ [c, \tau],$$

$$(4.12d) \quad \mathcal{L}_{BF\Phi}^2 = \bar{\gamma} B^+ \varphi + \frac{1}{2} \bar{\gamma} \tau^+ \gamma + \bar{\varphi}^+ (D_A \gamma + c \varphi) + \bar{\gamma}^+ c \gamma.$$

The symplectic form is given by (4.7) when we replace the coordinates by their corresponding superfields and keep only the terms of ghost number -1 , resulting in

$$(4.13) \quad \omega_{\mathcal{N}} = \int_M \left\langle d_{\mathcal{N}} B d_{\mathcal{N}} B^+ + d_{\mathcal{N}} A d_{\mathcal{N}} A^+ + d_{\mathcal{N}} \bar{\varphi} d_{\mathcal{N}} \varphi^+ \\ + d_{\mathcal{N}} \tau d_{\mathcal{N}} \tau^+ + d_{\mathcal{N}} c d_{\mathcal{N}} c^+ + d_{\mathcal{N}} \bar{\gamma} d_{\mathcal{N}} \gamma^+ \right\rangle.$$

In turn, we can read off the action (4.12) the way in which the cohomological vector field acts on the coordinate fields:

$$(4.14a) \quad Q_{BF\Phi}(B) = [c, B] + D_A \tau + \bar{\gamma} \rho \varphi, \quad Q_{BF\Phi}(\tau) = [c, \tau] + \frac{1}{2} \bar{\gamma} \rho \gamma,$$

$$(4.14b) \quad Q_{BF\Phi}(A) = D_{AC} + \Lambda B \tau, \quad Q_{BF\Phi}(c) = \frac{1}{2} [c, c] + \frac{1}{2} \Lambda \tau \tau,$$

$$(4.14c) \quad Q_{BF\Phi}(\varphi) = D_A \gamma + c \varphi, \quad Q_{BF\Phi}(\gamma) = c \gamma.$$

Thus we conclude the construction collecting everything in a tuple

$$(4.15) \quad \mathcal{J}_{BF\Phi} := (\mathcal{F}_{BF\Phi}, S_{BF\Phi}, Q_{BF\Phi}, \omega_{BF\Phi}).$$

Lemma 4.5. For any degree 1 vector field $\xi \in \mathfrak{X}(M)[1]$, principal connection 1-form $\Gamma \in \text{Conn}(P)$ and field $\varphi \in \Omega(M, \mathcal{V})$ valued in an associated vector bundle \mathcal{V} , the following identity holds:

$$(4.16) \quad [\mathcal{L}_\xi^\Gamma, \iota_\xi] \varphi = \iota_{[\xi, \xi]} \varphi.$$

This is proven in [CS19a, Lemma 9].

Proposition 4.6. *There is a ghost fermion ε such that on shell*

$$(4.17) \quad Q_{\text{GR}\Phi}(\xi) \equiv Q_{\text{GR}}(\xi) + \frac{1}{2} \bar{\varepsilon} e^{-1}(\rho) \varepsilon.$$

Proof. Let us call $Q' := Q_{\text{GR}\Phi} - Q_{\text{GR}}$ the on-shell extension of Q_{GR} . Now, on shell all antifields are set to zero, so (3.20) reduces to

$$(4.18) \quad (B, A, \tau, c) = (e, \Gamma, -\iota_\xi e, \chi),$$

which we extend additionally with $(\varphi, \gamma) = (\psi, \kappa)$, thus being able to translate on shell the first part of (4.14a) to

$$(4.19) \quad \begin{aligned} Q_{\text{GR}\Phi}(e) &\equiv -D_\Gamma(\iota_\xi e) + [\chi, e] + \bar{\kappa} \rho \psi \\ &= -\iota_\xi D_\Gamma e + \mathcal{L}_\xi^\Gamma e + [\chi, e] + \bar{\kappa} \rho \psi. \end{aligned}$$

Meanwhile, the fact that $|e| = 2$, together with (A.3) and the definition of \mathcal{L}_ξ^Γ , imply that

$$(4.20) \quad \iota_\xi(\mathcal{L}_\xi^\Gamma e) = \frac{1}{2}([\iota_\xi^2, D_\Gamma] - \iota_{[\xi, \xi]})e.$$

Recalling that $Q_{\text{GR}}(\xi) = \frac{1}{2}[\xi, \xi]$, we use all this to further find that

$$(4.21) \quad \begin{aligned} Q_{\text{GR}\Phi}(\iota_\xi e) &= [Q_{\text{GR}\Phi}, \iota_\xi]e + \iota_\xi(Q_{\text{GR}\Phi}e) = \iota_{Q_{\text{GR}\Phi}(\xi)}e + \iota_\xi(Q_{\text{GR}\Phi}e) \\ &\equiv \iota_{Q_{\text{GR}\Phi}(\xi)}e - \iota_\xi^2 D_\Gamma e + \iota_\xi(\mathcal{L}_\xi^\Gamma e) + \iota_\xi[\chi, e] + \bar{\kappa} \rho \iota_\xi \psi \\ &= \iota_{Q'(\xi)}e + \frac{1}{2} \iota_{[\xi, \xi]}e - \iota_\xi^2 D_\Gamma e + \frac{1}{2} \iota_\xi^2 D_\Gamma e - \frac{1}{2} D_\Gamma(\iota_\xi^2 e) \\ &\quad - \frac{1}{2} \iota_{[\xi, \xi]}e + \iota_\xi[\chi, e] + \bar{\kappa} \rho \iota_\xi \psi \\ &= \iota_{Q'(\xi)}e - \frac{1}{2} \iota_\xi^2 D_\Gamma e + \iota_\xi[\chi, e] + \bar{\kappa} \rho \iota_\xi \psi. \end{aligned}$$

Moreover, after adapting to on-shell 3D supergravity both the equation of motion (4.4c) and the second part of (4.14a), from what precedes we deduce that

$$(4.22) \quad \begin{aligned} Q_{\text{GR}\Phi}(\iota_\xi e) &\equiv \iota_{Q'(\xi)}e + \frac{1}{2} \iota_\xi \bar{\psi} \rho \iota_\xi \psi + \iota_\xi[\chi, e] + \bar{\kappa} \rho \iota_\xi \psi \\ &\equiv -Q_{\text{GR}\Phi}(\tau) \equiv \iota_\xi[\chi, e] - \frac{1}{2} \bar{\kappa} \rho \kappa, \end{aligned}$$

which holds if and only if

$$(4.23) \quad \iota_{Q'(\xi)} e \equiv -\frac{1}{2}(\bar{\kappa} + \iota_\xi \bar{\psi}) \rho(\kappa + \iota_\xi \psi).$$

With this we finally conclude that

$$(4.24) \quad \begin{aligned} Q_{\text{GR}\Phi}(\xi) &= Q_{\text{GR}}(\xi) + e^{-1}(e(Q'\xi)) = Q_{\text{GR}}(\xi) - e^{-1}(\iota_{Q'(\xi)} e) \\ &\equiv Q_{\text{GR}}(\xi) + \frac{1}{2}\bar{\varepsilon} e^{-1}(\rho)\varepsilon \end{aligned}$$

for the ghost Majorana fermion $\varepsilon := \kappa + \iota_\xi \psi$. \square

Remark 4.7. This property is expected, since the generators of supersymmetry square to translation generators, and the former are to be encoded by ghost fermions while the latter are realised through ξ .

Remark 4.8. Our current goal being to extend (3.19) as to find a BV theory of 3D supergravity that encodes explicitly both supersymmetry and diffeomorphism invariance, this last proposition will serve us as guiding principle. Indeed, we will be searching for a type 2 generating function $G_{BF\Phi}$ that decomposes as

$$(4.25) \quad G_{BF\Phi} = G_{BF} + G_{BF}^{\text{ext}},$$

and evidently we would like G_{BF}^{ext} to be a minimal extension, that is, as simple as possible without being ineffective. This *without being ineffective* is precisely what Proposition 4.6 addresses: we must ensure that the extended symplectomorphism $\Phi_{BF\Phi} : \mathcal{F}_{\text{GR}\Phi} \rightarrow \mathcal{F}_{BF\Phi}$ leads to a cohomological vector field that on shell is equal to (4.17). Fortunately, finding such extension is eased by the next proposition.

Proposition 4.9. *A minimal extension G_{BF}^{ext} of G_{BF} ensuring that equation (4.17) holds on shell can only depend on spinorial fields or on contractions of those with respect to ξ .*

Proof. As before, we denote by (φ, γ) the spinorial field and ghost in $\mathcal{F}_{BF\Phi}$ and by (ψ, ε) the corresponding pair on $\mathcal{F}_{\text{GR}\Phi}$. Since the extension we are looking for aims at being minimal, all its terms must be spinorial scalars, because if any term in G_{BF}^{ext} did not include spinors, it would effectively amount to a modification of G_{BF} spoiling the known canonical transformation between 3D and BF gravities in the absence of fermions. Consequently, all terms in G_{BF}^{ext} should take the form

$$(4.26) \quad \bar{y}xy'$$

where y and y' are spinors in $\mathcal{F}_{BF\Phi}$ and in $\mathcal{F}_{\text{GR}\Phi}$ respectively, and x is any product of non-spinorial fields—possibly including contractions—in either theory. Of course, not any such combination is valid, given

that every such product must have cohomological degree 3 and ghost number -1 , and every internal index must be contracted. In fact, under these constraints there will be at most two kinds of valid products $\bar{y}xy'$. The first kind will have $x = 1$ and an appropriate distribution of contractions ι_ξ , only consisting—up to redistribution of the ι_ξ —of the following possible pairs (y, y') :

$$(4.27) \quad \begin{aligned} &(\gamma^+, \varepsilon), \quad (\varphi^+, \psi), \quad (\gamma^+, \iota_\xi \psi), \\ &(\iota_\xi \varphi^+, \psi^+), \quad (\iota_\xi^2 \varphi^+, \varepsilon^+), \quad (\iota_\xi^3 \gamma^+, \varepsilon^+), \end{aligned}$$

or the analogous pairings exchanging the roles of the fields in $\mathcal{F}_{\text{GR}\Phi}$ by those in $\mathcal{F}_{\text{BF}\Phi}$ and vice versa. Meanwhile, the second kind of product will have $x \neq 1$, yet it is evident that any product of this type, to be valid, should be obtained from a product of the first kind by replacing any number of contractions with a product x of non-spinorial fields that have the same cohomological degree and ghost number as the power of ι_ξ that they are replacing. In other words,

$$(4.28) \quad \begin{pmatrix} \text{deg}_\Omega x \\ \text{gh } x \end{pmatrix} = \begin{pmatrix} -k \\ k \end{pmatrix}$$

for some $k \in \mathbb{N}$. Now, every non-spinorial field—including the contraction ι_ξ —has its pair $(\text{deg}_\Omega \bullet, \text{gh } \bullet)$ among the following:

$$(4.29) \quad v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad v_5 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

which respectively correspond to the degree pairs of c , A , B^+ , c^+ and ι_ξ , in that order. The question, then, reduces to solving the simple equation

$$(4.30) \quad k^i v_i = 0 \quad \text{for} \quad \{k^i\}_{i=1}^4 \subset \mathbb{N}, \quad k^5 \in \mathbb{Z},$$

which holds if and only if $k^i = 0$ for all i . This is equivalent to saying that any valid product $\bar{y}xy$ is of the first kind, that is, a Dirac product of spinors or of contractions of those with respect to ξ . \square

Remark 4.10. Proposition 4.9 facilitates the labour notably by making G_{BF}^{ext} include at most four terms, which moreover will show a convenient property: they will only fix the spinorial fields and, at most, modify the expression for ξ^+ as a function of the fields in $\mathcal{F}_{BF\Phi}$. Finding an appropriate extension, then, is letting

$$(4.31) \quad G_{BF}^{\text{ext}} = \sum_i k_i \bar{y}_i y_i'$$

and determining the (at most) four parameters k_i that will lead to a $Q_{\text{GR}\Phi}$ that on shell satisfies (4.17) and to an action $S_{\text{GR}\Phi}$ whose classical spinorial part is $\frac{1}{2}\psi D_\Gamma\psi$.

Theorem 4.11. *A BV extension of 3D supergravity is provided by the tuple $\mathcal{I}_{\text{GR}\Phi} := (\mathcal{F}_{\text{GR}\Phi}, S_{\text{GR}\Phi}, Q_{\text{GR}\Phi}, \omega_{\text{GR}\Phi})$ for*

$$(4.32a) \quad \mathcal{F}_{\text{GR}\Phi} = \mathcal{F}_{\text{GR}} \times T^*[-1]\Omega(M, \Pi\mathcal{S}(P)),$$

$$(4.32b) \quad S_{\text{GR}\Phi} = \Phi_{BF\Phi}^* S_{BF\Phi},$$

$$(4.32c) \quad \omega_{\text{GR}\Phi} = \Phi_{BF\Phi}^* \omega_{BF\Phi},$$

$$(4.32d) \quad Q_{\text{GR}\Phi} = \{\bullet, S_{\text{GR}\Phi}\},$$

where the Poisson bracket is defined by $\omega_{\text{GR}\Phi}$ and the canonical transformation $\Phi_{BF\Phi}$ is generated by

$$(4.33) \quad G_{BF\Phi} = G_{BF} + G_{BF}^{\text{ext}},$$

where G_{BF} is the generating function (3.19) and G_{BF}^{ext} is given as

$$(4.34) \quad G_{BF}^{\text{ext}}(q, p') = -\bar{\varphi}^+ \psi - \bar{\gamma}^+ (\varepsilon - \iota_\xi \psi)$$

for $q := (e, \Gamma^+, \xi, \chi^+, \psi, \varepsilon)$ and $p' := (B^+, A, \tau^+, c, \varphi^+, \gamma^+)$.

Proof. This generating function leads to

$$(4.35a) \quad \varphi = \psi, \quad \varphi^+ = \psi^+ - \iota_\xi \varepsilon^+,$$

$$(4.35b) \quad \gamma = \varepsilon - \iota_\xi \psi, \quad \gamma^+ = \varepsilon^+,$$

so following the previous Remark 4.10, we only have to attend some of the terms in $S_{BF\Phi}$ (4.12) to check whether it produces an extension of classical 3D supergravity. Firstly, since the Definition (3.20b) of A in terms of fields in $\mathcal{F}_{\text{GR}\Phi}$ remains unchanged, the expansion of the classical spinorial field gives

$$(4.36) \quad \frac{1}{2}\bar{\varphi} D_A \varphi = \frac{1}{2}\bar{\psi} D_\Gamma \psi - \frac{1}{2}\bar{\psi} (\iota_\xi e^+ - \frac{1}{2}\iota_\xi^2 \tau^+) \psi,$$

so indeed the classical spinorial term is recovered on shell—where, remember, antifields are set to zero. Secondly, since the only terms modifying $Q_{\text{GR}}(\xi)$ are those spinorial terms in $S_{BF\Phi}$ that include a factor of

τ^+ —because only these depend on ξ^+ , specifically through $e^{-1}(\xi^+)$ —we only verify the following terms in $\mathcal{L}_{BF\Phi}$:

$$(4.37) \quad \frac{1}{2}\bar{\psi} A\psi + \bar{\gamma} B^+ \varphi + \frac{1}{2}\bar{\gamma} \tau^+ \gamma + \bar{\varphi}^+ A\gamma + \bar{\varphi}^+ c\varphi + \bar{\gamma}^+ c\gamma.$$

After expansion—which is rendered explicit in Definition 4.13 below—one verifies that

$$(4.38) \quad Q_{\text{GR}\Phi}(\xi) = \frac{1}{2}[\xi, \xi] + \frac{1}{2}\bar{\varepsilon} e^{-1}(\rho)\varepsilon + \dots$$

omitting all terms that contain antifields, so indeed $Q_{\text{GR}\Phi}$ satisfies (4.17). Finally, (4.34) holds necessarily, since (4.32a) merely accounts for the incorporation of fermions, while equations (4.32b) to (4.32d) follow from the definition of a canonical transformation and the fact that $G_{BF\Phi}$ is a generating function. Therefore, we have constructed a theory $\mathcal{T}_{\text{GR}\Phi}$ that is symplectomorphic to BF supergravity, and moreover $\mathcal{T}_{\text{GR}\Phi}$ produces classical 3D supergravity on shell; in other words, $\mathcal{T}_{\text{GR}\Phi}$ is a BV extension of 3D supergravity. \square

Remark 4.12. Due to the fact that it only involves spinors and their contraction, the only equation in (3.20) that the extension G_{BF}^{ext} modifies is the one corresponding to τ^+ , giving

$$(4.39) \quad \tau^+ = e^{-1}(\xi^+ - e^+ \Gamma^+ - \bar{\gamma}^+ \psi + \iota_\xi e^+ \chi^+).$$

Having established this result, we find ourselves in a position to give an explicit form for 3D supergravity.

Definition 4.13. We will call 3-dimensional **BV supergravity** the theory built in Theorem 4.11. Its action is given by

$$(4.40a) \quad S_{\text{GR}\Phi} = S_{\text{GR}} + \int_M \langle \mathcal{L}_{\text{GR}\Phi}^1 \rangle,$$

for the density

$$(4.40b) \quad \mathcal{L}_{\text{GR}\Phi}^1 = \frac{1}{2}\bar{\psi} D_\Gamma \psi + \bar{\psi} e^+ \varepsilon + \frac{1}{2}\bar{\varepsilon} \tau^+ \varepsilon + \bar{\psi}^+ Q_{\text{GR}\Phi}(\psi) + \bar{\varepsilon}^+ Q_{\text{GR}\Phi}(\varepsilon),$$

where τ^+ —as given in (4.39)—is kept implicit for the sake of readability. In turn, the cohomological vector field decomposes as

$$(4.41a) \quad Q_{\text{GR}\Phi} = Q_{\text{GR}} + Q_{\text{GR}}^{\text{ext}},$$

for an extension that acts in the following manner:

$$(4.41b) \quad Q_{\text{GR}}^{\text{ext}}(\Gamma) = Q_{\text{GR}}^{\text{ext}}(\chi) = 0,$$

$$(4.41c) \quad Q_{\text{GR}}^{\text{ext}}(e) = \bar{\psi} \rho \varepsilon - \iota_{\xi} \bar{\psi}^+ \rho \kappa - \frac{1}{2} \iota_{\xi}^2 \bar{\psi}^+ \rho \psi \\ - \iota_{\xi} \bar{\psi}^+ \rho \iota_{\xi} \psi + \frac{1}{2} \iota_{\xi}^2 \bar{\varepsilon}^+ \rho (\varepsilon - 2\iota_{\xi} \psi),$$

$$(4.41d) \quad Q_{\text{GR}}^{\text{ext}}(\xi) = \frac{1}{2} \bar{\varepsilon} \rho \varepsilon + \frac{1}{2} \iota_{\xi}^2 \bar{\psi}^+ \rho \varepsilon - \frac{1}{6} \iota_{\xi}^3 \bar{\varepsilon}^+ \rho (2\varepsilon - 3\iota_{\xi} \psi),$$

$$(4.41e) \quad Q_{\text{GR}\Phi}(\psi) = \chi \psi + D_{\Gamma} \kappa - \iota_{\xi} e^+ \kappa + \frac{1}{2} \iota_{\xi}^2 e^+ \psi + \frac{1}{2} \iota_{\xi}^2 \tau^+ \kappa - \frac{1}{6} \iota_{\xi}^3 \tau^+ \psi,$$

$$(4.41f) \quad Q_{\text{GR}\Phi}(\varepsilon) = \chi \varepsilon + \iota_{\xi} D_{\Gamma} \kappa - \frac{1}{2} \iota_{\xi}^2 e^+ (\varepsilon - 2\iota_{\xi} \psi) + \frac{1}{6} \iota_{\xi}^3 \tau^+ (2\varepsilon - 3\iota_{\xi} \psi),$$

writing $\kappa := (\varepsilon - \iota_{\xi} \psi)$ and $\rho := e^{-1}(\rho)$.

APPENDIX A. SELECTED RESULTS IN GRADED GEOMETRY

Proposition A.1. *As their non-graded counterparts, graded Lie derivatives satisfy*

$$(A.1) \quad [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X, Y]} \quad \forall X, Y \in \mathfrak{X}(\mathcal{M}).$$

Remark A.2. This is a direct consequence of their definition.

Proposition A.3. *Cartan's identity—or Cartan's magic formula—relates the graded interior derivative, the exterior derivative, and the graded Lie derivative:*

$$(A.2) \quad [\iota_X, d] = \mathcal{L}_X.$$

Remark A.4. The proof is verbatim the one used in conventional differential geometry, but keeping track of the gradings.

Proposition A.5. *For any graded vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$ and form $\omega \in \Omega(\mathcal{M})$ over a graded manifold \mathcal{M} we have*

$$(A.3a) \quad [\mathcal{L}_X, \iota_Y] \omega = \iota_{[X, Y]} \omega$$

Proof. The behaviour of the interior and Lie derivatives, as for any derivation, will be fully determined by their action on an arbitrary function $f \in C^\infty(\mathcal{M})$ and on its differential df . Now, on the one hand it is obvious that

$$(A.4) \quad [\mathcal{L}_X, \iota_Y] f = \iota_{[X, Y]} f = 0.$$

On the other hand, recalling that $|\iota_Y| = |Y| - 1$, Cartan's identity and $d^2 = 0$ together imply that

$$(A.5a) \quad \begin{aligned} \iota_Y(\mathcal{L}_X df) &= \iota_Y([\iota_X, d]df) = -(-1)^{|\iota_X|} \iota_Y d(Xf) \\ &= (-1)^{|X|} Y(Xf). \end{aligned}$$

This, jointly with $|\mathcal{L}_X| = |X|$, lets us conclude that

$$(A.5b) \quad \begin{aligned} [\mathcal{L}_X, \iota_Y]df &= \mathcal{L}_X(\iota_Y df) - (-1)^{|\mathcal{L}_X||\iota_Y|} \iota_Y(\mathcal{L}_X df) \\ &= X(Yf) - (-1)^{|X|(|Y|-1)} (-1)^{|X|} Y(Xf) = [X, Y]f \\ &= \iota_{[X, Y]}df, \end{aligned}$$

thus proving the proposition. \square

CONFLICT OF INTEREST AND DATA AVAILABILITY

On behalf of all authors, the corresponding author states that there is no conflict of interest and that the manuscript has no associated data.

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