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THE CANONICAL BV LAPLACIAN ON HALF-DENSITIES

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ABSTRACT. This is a didactical review on the canonical BV Laplacian on half-densities.

CONTENTS

1. INTRODUCTION

The BV Laplacian is a central ingredient of the Batalin–Vilkovisky (BV) formalism. Its global definition is not canonical, yet a remarkable fact observed by Khudaverdian [7, 8] is that its extension to half-densities is so, and there are various ways to show it. In this review we give a short, self-contained presentation which requires just a few simple computations. It gives a special emphasis on the fact that half-densities have a natural inner product.

This note has no pretence of originality. In Appendix C, we give a short historical overview with the relevant references.

In Appendix A, we collect some technical background material. In Appendix B, we recall why the BV formalism is important in applications.

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1.1. **Overview.** We recall definitions and properties of BV Laplacians (on functions and densities) on an oriented odd symplectic manifold with the goal of proving that on half-densities it is canonically defined (whereas on other kinds of densities, including functions, it requires the choice of a compatible density). This goes as follows:

(1) On functions we define $\Delta_\mu f = \frac{1}{2}\mathrm{div}_\mu X_f$, where μ is an even, nowhere vanishing[1] density, and X_f denotes the hamiltonian vector field of f.
(2) On half-densities we define $\Delta_\mu^{(\frac{1}{2})}\sigma = \Delta_\mu(\sigma/\mu^{1/2})\mu^{1/2}$.
(3) We check that the leading term is invariant under symplectomorphisms and does not depend on the choice of μ.
(4) We check that $\Delta_\mu^{(\frac{1}{2})}$ is a symmetric operator. This implies that in any Darboux chart only the zeroth-order term may depend on μ.
(5) We show that for a compatible density μ, i.e., for μ satisfying $\Delta_\mu^2 = 0$, the zeroth-order term of the previous point must vanish. This shows that $\Delta_\mu^{(\frac{1}{2})}$ does not depend on the choice of the compatible density μ.
(6) We show, via the presentation of the given odd symplectic manifold as a symplectomorphic odd cotangent bundle, that compatible densities always exist.
(7) We conclude that the BV Laplacian on half-densities, with the property of squaring to zero, is canonically defined.
(8) As an aside, we also show how to define this canonical operator, denoted by Δ, directly on Darboux charts without any choice involved. Moreover, we prove that the compatibility condition for a density μ can also be expressed as $\Delta\mu^{\frac{1}{2}} = 0$ and that the BV Laplacian on functions can now be recovered as $\Delta_\mu = \Delta(f\mu^{\frac{1}{2}})/\mu^{\frac{1}{2}}$.

We go through details, also checking signs carefully, to make this presentation useful also as a reference. However, it should be observed that, roughly, the above flow can be easily checked without going into sign details—the only crucial point being (4), which requires some care.

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2. The standard BV Laplacian

On functions of odd variables $p_1, \dots, p_n$ and even variables $q^1, \dots, q^n$, the standard BV Laplacian[2] (a.k.a. BV Laplace operator or BV operator) is defined as

$$\triangle := \partial_i \partial^i,$$

where we use Einstein's summation convention and the shorthand notations

$$\partial_i = \frac{\partial}{\partial q^i} \quad \text{and} \quad \partial^i = \frac{\partial}{\partial p_i}.$$

[1] Throughout the paper, nowhere vanishing means nonvanishing at every point of the body after the nilpotents are set to zero. An even, nowhere vanishing density is the same as a basis for the module of densities.

[2] This operator was introduced to show that certain integrals are invariant under deformations of the integration domain; see Appendix B.

This odd operator has the following properties which are easily proved by direct computation:

$$\triangle^2 = 0, \tag{1a}$$

$$\triangle(fg) = (\triangle f)g + (-1)^f f\triangle g - (-1)^f (f,g), \tag{1b}$$

$$\triangle \mathrm{e}^S = \left(\triangle S - \frac{1}{2}(S,S)\right)\mathrm{e}^S. \tag{1c}$$

Here f and g are functions of homogeneous degree, and when we put a function (or any other object) as an exponent of (-1) we mean its degree.[3] The function S here is assumed to be even. Finally, $(\ ,\ )$ denotes the BV bracket. As an exercise, just using (1a) and (1b), you can also show that

$$\triangle(f,g) = (\triangle f, g) - (-1)^f (f, \triangle g). \tag{2}$$

For reference, we spell out our conventions:

$$(f,g) := f\overleftarrow{\partial^i}\,\overrightarrow{\partial_i} g - f\overleftarrow{\partial_i}\,\overrightarrow{\partial^i} g, \tag{3}$$

where the arrows denote the directions the derivatives are applied from. Explicitly, $\overrightarrow{\partial_i} f = \partial_i f$, $\overrightarrow{\partial^i} f = -\partial^i f$, and

$$f\overleftarrow{\partial_i} = \partial_i f,$$
$$f\overleftarrow{\partial^i} = -(-1)^f \partial^i f.$$

The geometric interpretation is that the p_is and q^is are Darboux coordinates for the odd symplectic form $\omega = \mathrm{d}p_i \mathrm{d}q^i$ with $(\ ,\ )$ its associated odd Poisson bracket.[4] The main problem with this, however, is that the standard BV Laplacian does not transform well under symplectomorphism, so it cannot be used as such to define an operator on functions on an odd symplectic manifold. In the next sections, we will see how to obviate this problem.

3. The BV bracket

Let $(\mathcal{M}, \omega)$ be an odd symplectic manifold. Given a function f, we denote by X_f its hamiltonian vector field defined via

$$\iota_{X_f}\omega = \mathrm{d}f.$$

Given a second function g, we define the BV bracket as

$$(f,g) := (-1)^{f+1} X_f(g).$$

This is an odd Poisson bracket, which in local Darboux coordinates agrees with the one of the previous section:[5]

Lemma 3.1. *In Darboux coordinates, $\omega = \mathrm{d}p_i \mathrm{d}q^i$, the BV bracket agrees with the standard one defined in* (3)*.*

[3] A more precise but more cumbersome notation would be $(-1)^{|f|}$, where $|f|$ denotes the degree of f.

[4] If the q^is are coordinates on an open subset U of $\mathbb{R}^n$, the functions of the p and q variables can be identified with multivector fields on U. The BV bracket gets then identified with the Schouten–Nijenhuis bracket. We will return to this, with a general basis, in Section 6; see footnote 13 on page 11.

[5] This is a simple computation which however requires checking signs. It is just needed to make sure that the conventions we choose are compatible.

Proof. Writing $X = X^i\partial_i + X_i\partial^i$, we get[6]

$$\iota_X\omega = \mathrm{d}p_i\, X^i + X_i\,\mathrm{d}q^i = \mathrm{d}p_i\, X^i - (-1)^X \mathrm{d}q^i\, X_i.$$

From

$$\mathrm{d}f = \mathrm{d}p_i\, \partial^i f + \mathrm{d}q^i\, \partial_i f$$

we then get

$$(X_f)^i = \partial^i f = (-1)^{f+1} f \overleftarrow{\partial^i} \tag{4a}$$

and

$$(X_f)_i = (-1)^f \partial_i f = (-1)^f f \overleftarrow{\partial_i}, \tag{4b}$$

so $(-1)^{f+1}X_f(g)$ is given by (3). □

The BV bracket satisfies several properties.

Proposition 3.2. *For all functions f, g, and h, we have*

$$\begin{aligned}
[X_f, X_g] &= X_{(f,g)},\\
(f, gh) &= (f,g)\,h + (-1)^{(f+1)g} g\,(f,h),\\
(g,f) &= -(-1)^{(f+1)(g+1)}(f,g),\\
(f,(g,h)) &= ((f,g),h) + (-1)^{(f+1)(g+1)}(g,(f,h)).
\end{aligned}$$

The last three properties say that $(\ ,\ )$ is an odd Poisson bracket,[7] whereas the first says the the map $f \mapsto X_f$ is a morphism of super Lie algebras.

Proof. We prove only the first identity. The others are also easily obtained from the definitions (or recovered from the identical identities for the standard BV bracket which one obtains in each Darboux chart). We have

$$\begin{aligned}
\iota_{[X_f,X_g]}\omega &= [\mathsf{L}_{X_f}, \iota_{X_g}]\omega = \mathsf{L}_{X_f}\iota_{X_g}\omega = \mathsf{L}_{X_f}\mathrm{d}g\\
&= (-1)^{f+1}\mathrm{d}\mathsf{L}_{X_f} g = (-1)^{f+1}\mathrm{d}X_f(g) = \mathrm{d}(f,g) = \iota_{X_{(f,g)}}\omega,
\end{aligned}$$

where L_{X_f} denotes the Lie derivative with respect to X_f. □

4. The BV Laplacian on functions

Let $(\mathcal{M},\omega)$ be an odd symplectic manifold and μ an even, nowhere vanishing density on $\mathcal{M}$, which we assume to be orientable. We then define the μ-dependent BV Laplacian as

$$\Delta_\mu f := \frac{1}{2}\mathrm{div}_\mu X_f. \tag{5}$$

Recall (or see Appendix A.2) that the divergence operator of a vector field X with respect to an even, nowhere vanishing density μ is defined via

$$\mathrm{div}_\mu X\,\mu = \mathsf{L}_X\mu,$$

where L_X denotes the Lie derivative.

[6] We use the convention of total degree, i.e., internal degree plus parity of the form degree. This means that d is odd and that ι_X has parity opposite to that of X. Moreover, X^i has the same parity as X, whereas X_i has opposite parity. Finally, f and X_f have opposite parity to each other.

[7] In particular this means that $(\ ,\ )$ is a super Lie bracket with respect to the opposite parity of its arguments.

Again we have to make sure that this definition of the BV Laplacian agrees with the standard one in the appropriate case:

Proposition 4.1. *In Darboux coordinates,* $\omega = \mathrm{d}p_i \mathrm{d}q^i$*, with standard density* $\mu_{stand} \coloneqq \mathrm{d}^n q\, \mathrm{d}^n p$*, we have*

$$\Delta_{\mu_{stand}} = \triangle.$$

Proof. With the standard density the divergence of a vector field $X = X^i \partial_i + X_i \partial^i$ is given by (14a), i.e.,

$$\mathrm{div}_{\mu_{\mathrm{stand}}} X = \partial_i X^i - (-1)^X \partial^i X_i.$$

If we now insert X_f as calculated in (4), we get

$$\mathrm{div}_{\mu_{\mathrm{stand}}} X_f = 2\partial_i \partial^i f.$$

□

Next we check how the operator depends on the choice of density. Note that, given two even, nowhere vanishing densities μ and $\widetilde{\mu}$, there is a unique even, nowhere vanishing function h such that $\widetilde{\mu} = h\mu$.

Proposition 4.2. *We have*

$$\Delta_{h\mu} = \Delta_\mu + \frac{1}{2}\frac{1}{h} X_h.$$

Proof. The divergence operator depends on the choice of density as

$$\mathrm{div}_{h\mu} X_f = \mathrm{div}_\mu X_f + \frac{1}{h} X_f(h)$$

(see point (2) in Proposition A.4). However, $X_f(h) = (-1)^{f+1}(f,h) = -(h,f) = X_h(f)$, since h is even. □

We can always write $h = \pm \mathrm{e}^S$ with S an even function. This way the formula simplifies to

$$\Delta_{h\mu} = \Delta_\mu + \frac{1}{2} X_S,$$

since $X_{\mathrm{e}^S} = \mathrm{e}^S X_S$. In particular, we have the

Lemma 4.3. *In Darboux coordinates with* $\mu = \pm \mathrm{e}^S \mu_{stand}$*, we have*

$$\Delta_\mu = \triangle + \frac{1}{2} X_S = \triangle - \frac{1}{2}(S,\ ). \tag{6}$$

This, together with Proposition 4.2, gives the

Proposition 4.4. *The BV Laplacian on functions is a second-order differential operator, and its leading term is independent of the choice of density.*

Remark 4.5. Under the assumptions of Lemma 4.3, from (6), also using (2) and the Jacobi identity, we get

$$\Delta_\mu^2 f = -\frac{1}{2}(F_S, f)$$

with

$$F_S \coloneqq \triangle S - \frac{1}{4}(S,S). \tag{7}$$

We then see that $\Delta_\mu^2 = 0$ if and only if F_S is constant. As F_S is odd, this happens if and only if $F_S = 0$.[8] Therefore, in general the property of equation (1a) does not extend to the global BV Laplacian.

Definition 4.6. We say that an even, nowhere vanishing density μ is compatible with the odd symplectic structure if $\Delta_\mu^2 = 0$.

Remark 4.7. We will see that every odd symplectic manifold admits a compatible density.

Remark 4.8. As we have seen in Remark 4.5, the property of equation (1a) does not extend in general. One can try to obviate it by modifying the definition of the BV operator. For example, in Darboux coordinates with $\mu = \pm e^S \mu_{\text{stand}}$, we could define

$$\widetilde{\Delta}_\mu \coloneqq \Delta_\mu + F_S = \triangle + \frac{1}{2} X_S + \frac{1}{2} F_S$$

with F_S as in (7). It is an easy exercise (show first that $\Delta_\mu F_S = 0$ for every S) to show that $\widetilde{\Delta}_\mu^2 = 0$ for every μ. However, since $\widetilde{\Delta}_\mu$ is not of the form of Proposition 4.2 (unless, of course, $F_S = 0$), it is not a BV Laplacian.

If $\Psi\colon \mathcal{M} \to \mathcal{N}$ is a diffeomorphism, we have the pushforward

$$\Psi_*\colon \operatorname{End}(C^\infty(\mathcal{M})) \to \operatorname{End}(C^\infty(\mathcal{N}))$$

defined by $\Psi_* P = \Psi_* \circ P \circ \Psi_*^{-1}$, where on the right-hand side Ψ_* denotes the pushforward of functions. The BV Laplacian transforms nicely under symplectomorphisms.

Proposition 4.9. *For every symplectomorphism Ψ,*

$$\Psi_* \Delta_\mu = \Delta_{\Psi_* \mu}.$$

Proof. The divergence operator changes under a diffeomorphism as

$$\Psi_* \operatorname{div}_\mu X_f = \operatorname{div}_{\Psi_*\mu} \Psi_* X_f$$

(see point (1) in Proposition A.4). If Ψ is a symplectomorphism, we also have $\Psi_* X_f = X_{\Psi_* f}$. □

Remark 4.10. In conjunction with Proposition 4.4, this implies that under a symplectomorphism $\Psi\colon \mathcal{M} \to \mathcal{M}$, the leading term of the BV Laplacian on functions is invariant under symplectomorphisms.

Remark 4.11. The proposition also implies that a symplectomorphism sends compatible densities to compatible densities.

4.1. **Digression: further properties of the BV Laplacian.** Even though we are not going to use this in the following, it is good to know that the properties stated in equations (1b) and (1c) also hold for the global BV Laplacian.

Proposition 4.12. *For every even, nowhere vanishing density μ, every functions f and g, and every even function S, we have*

$$\Delta_\mu(fg) = (\Delta_\mu f)g + (-1)^f f \Delta_\mu g - (-1)^f (f, g), \tag{8a}$$

$$\Delta_\mu e^S = \left(\Delta_\mu S - \frac{1}{2}(S, S)\right) e^S. \tag{8b}$$

[8] Note that this condition can also be rephrased as $\triangle e^{\frac{1}{2}S} = 0$. The appearance of the factor $\frac{1}{2}$ in the exponent will become clear when we study half-densities (see Lemma 5.13).

Proof. The identities may be proved by observing, thanks to (6), that in every Darboux chart the BV Laplacian differs from $\triangle$ by a vector field.

They may also be proved directly using properties of the divergence operator (see point (3) in Proposition A.4) and of hamiltonian vector fields. To illustrate this, we prove the second identity. Since $X_{\mathrm{e}^S} = \mathrm{e}^S X_S$, we have

$$\mathrm{div}_\mu X_{\mathrm{e}^S} = \mathrm{div}_\mu(\mathrm{e}^S X_S) = \mathrm{e}^S \mathrm{div}_\mu X_S + X_S(\mathrm{e}^S),$$

which proves the identity once we observe that $X_S(\mathrm{e}^S) = -\mathrm{e}^S(S,S)$. □

Remark 4.13. As we have seen in Remark 4.8, it is possible to define an operator $\widetilde{\Delta}_\mu$ that squares to zero, no matter what μ is, so as to satisfy the extension of the property of equation (1a). However, since this is obtained by adding to Δ_μ a multiplication operator, we see that now (8a) is no longer satisfied (unless, of course, $F_S = 0$).

5. The BV Laplacian on densities

A differential operator P defined on functions on a (super)manifold $\mathcal{M}$ can be extended to sections of a trivial line bundle L over $\mathcal{M}$ once an even, nowhere vanishing section λ of L has been chosen. For $\sigma \in \Gamma(L)$ there is a uniquely determined function f such that $\sigma = f\lambda$, and we set

$$P^{(\lambda)}\sigma := P(f)\lambda.$$

Lemma 5.1. *The leading term of the differential operator $P^{(\lambda)}$ does not depend on the choice of λ.*

Proof. If $\widetilde{\lambda}$ is also a nowhere vanishing section, then there is a uniquely determined nowhere vanishing function h with $\widetilde{\lambda} = h\lambda$. For $\sigma = f\widetilde{\lambda} = fh\lambda$, we have

$$P^{(\widetilde{\lambda})}\sigma = P(f)\widetilde{\lambda} = P(f)h\lambda.$$

On the other hand, we have

$$P^{(\lambda)}\sigma = P(fh)\lambda = ((P_{\mathrm{leading}}f)\,h + \cdots)\lambda,$$

where the dots denote terms where less than the maximum number of derivatives hit f. □

We now want to extend the BV Laplacian Δ_μ to s-densities (see Appendix A.2 for a review). Since we already have an even, nowhere vanishing section, μ, of the density bundle, we can use it to get our reference section, μ^s, of the s-density bundle.[9] That is,

Definition 5.2. For an s-density σ, which we can uniquely write as $\sigma = f\mu^s$, we set[10]

$$\Delta_\mu^{(s)}\sigma := \Delta_\mu(f)\mu^s.$$

Note that $\Delta_\mu^{(0)}$ is the same as Δ_μ.

[9]Since we are interested also in nonintegral s, in particular $s = \frac{1}{2}$, it is essential to require that $\mathcal{M}$ is oriented.

[10]We avoid the more pedantic notation $\Delta_\mu^{(\mu^s)}$ on the left-hand side.

Remark 5.3. An immediate consequence of this definition is that $\Delta_\mu^{(s)}$ squares to zero on s-densities if and only if Δ_μ does so on functions (i.e., μ is compatible with the odd symplectic structure).

Proposition 4.4 and Lemma 5.1 immediately imply

Proposition 5.4. *The BV Laplacian on s-densities is a second-order differential operator, and its leading term is independent of the choice of reference density.*

Proposition 4.9 immediately implies

Proposition 5.5. *For every symplectomorphism Ψ and for every s,*

$$\Psi_*\Delta_\mu^{(s)} = \Delta_{\Psi_*\mu}^{(s)}.$$

Remark 5.6. In particular, this implies that, for every s, under a symplectomorphism $\Psi\colon \mathcal{M} \to \mathcal{M}$, the leading term of the BV Laplacian on s-densities is invariant under symplectomorphisms.

Note that the product of an s-density σ and a $(1-s)$-density τ yields a density, which can be integrated (if it has compact support). Given a differential operator P on s-densities, we define its transpose P^t as the differential operator on $(1-s)$-densities that satisfies

$$\int_{\mathcal{M}} P\sigma\,\tau = (-1)^{P\,\sigma} \int_{\mathcal{M}} \sigma\, P^t\tau$$

for all s-densities σ with compact support (the upper index "$P\,\sigma$" denotes the product of the degrees; see footnote 3 on page 3). The case $s = \frac{1}{2}$ is special, since transposition becomes an endomorphism on the space of differential operators.

Proposition 5.7. *We have*

$$(\Delta_\mu^{(s)})^t = \Delta_\mu^{(1-s)}$$

In particular, $\Delta_\mu^{(\frac{1}{2})}$ is symmetric.

Proof. Write $\sigma = f\mu^s$ and $\tau = g\mu^{1-s}$. Then

$$\Delta_\mu^{(s)}\sigma\,\tau = \Delta_\mu f\, g\, \mu.$$

But

$$\begin{aligned} 2\Delta_\mu f\, g &= \mathrm{div}_\mu X_f\, g = (-1)^{(f+1)g} g\, \mathrm{div}_\mu X_f \\ &= (-1)^{(f+1)g}(\mathrm{div}_\mu(gX_f) - (-1)^{(f+1)g} X_f(g)) \\ &= (-1)^{(f+1)g}\mathrm{div}_\mu(gX_f) + (-1)^f (f,g), \end{aligned}$$

where we have used part (3) of Proposition A.4. Similarly,

$$\sigma\,\Delta_\mu^{(1-s)}\tau = f\,\Delta_\mu g\,\mu,$$

and

$$\begin{aligned} 2f\,\Delta_\mu g &= f\,\mathrm{div}_\mu X_g = \mathrm{div}_\mu(fX_g) - (-1)^{f(g+1)} X_g(f) \\ &= \mathrm{div}_\mu(fX_g) + (-1)^{fg+f+g}(g,f) = \mathrm{div}_\mu(fX_g) + (f,g). \end{aligned}$$

Therefore,

$$2\Delta_\mu f\, g - (-1)^f 2f\,\Delta_\mu g = (-1)^{(f+1)g}\mathrm{div}_\mu(gX_f) - (-1)^f \mathrm{div}_\mu(fX_g),$$

which implies, thanks to part (4) of Proposition A.4, that[11]

$$\int_{\mathcal{M}} (\Delta_\mu^{(s)}\sigma\,\tau - (-1)^\sigma \sigma\,\Delta_\mu^{(1-s)}\tau) = 0.$$

□

Remark 5.8. By inspection in the proof one sees that it is essential that we have used the same density μ to define the BV Laplacian on functions and to extend it to s-densities.

We now come to the fundamental consequence of this proposition.

Lemma 5.9. *In each Darboux chart, there is an odd function G_μ, depending on μ, such that*

$$\Delta_\mu^{(\frac{1}{2})}\sigma = (\triangle f + G_\mu f)\,\mu_{stand}^{\frac{1}{2}},$$

where we have written $\sigma = f\,\mu_{stand}^{\frac{1}{2}}$.

Proof. Since $\Delta_\mu^{(\frac{1}{2})}$ is a second-order differential operator and its leading term is independent of μ, we have

$$\Delta_\mu^{(\frac{1}{2})}\sigma = (\triangle f + Y_\mu(f) + G_\mu\, f)\,\mu_{\rm stand}^{\frac{1}{2}},$$

for some vector field Y_μ and some function G_μ. Since $\triangle$ has constant coefficients, it is symmetric. Moroever, since Y_μ is a vector field, we have $Y_\mu^t = -Y_\mu$+some function. Since $\Delta_\mu^{(\frac{1}{2})}$ is symmetric, we get $Y_\mu = 0$. □

This implies the following

Corollary 5.10. *For every compatible μ, the BV Laplacian on half-densities has a canonical form in every Darboux chart:*

$$\Delta_\mu^{(\frac{1}{2})}\sigma = \triangle f\,\mu_{stand}^{\frac{1}{2}},$$

where we have written $\sigma = f\,\mu_{stand}^{\frac{1}{2}}$. In particular, $\Delta_\mu^{(\frac{1}{2})}$ is independent of the choice of the compatible density μ.

Proof. Using $\triangle^2 = 0$, $G_\mu^2 = 0$, and (1b), we get

$$0 = (\Delta_\mu^{(\frac{1}{2})})^2\sigma = (\triangle G_\mu\, f + (G_\mu, f))\mu_{\rm stand}^{\frac{1}{2}}.$$

Therefore, for every function f, we have

$$\triangle G_\mu\, f + (G_\mu, f) = 0.$$

In particular, for $f = 1$, we get $\triangle G_\mu = 0$, so the condition simplifies to $(G_\mu, f) = 0$ for every f. This implies that G_μ is constant, but then it must vanish because it is odd. □

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[11] Note the different sign in (8a), which instead yields

$$\int_{\mathcal{M}} (\Delta_\mu^{(s)}\sigma\,\tau + (-1)^\sigma \sigma\,\Delta_\mu^{(1-s)}\tau) = \int_{\mathcal{M}} (\Delta_\mu f\,g + (-1)^f f\,\Delta_\mu g)\,\mu = (-1)^f \int_{\mathcal{M}} (f, g)\,\mu.$$

Subtracting the two identities yields the interesting formula

$$\int_{\mathcal{M}} (f, g)\,\mu = 2\int_{\mathcal{M}} f\,\Delta_\mu g\,\mu,$$

which in particular shows that the integral of the BV bracket of two functions (at least one of which has compact support) against the density μ vanishes if one of the two functions is in the kernel of Δ_μ.

In every Darboux chart, we can consider the BV Laplacian $\Delta^{(\frac{1}{2})}_{\mu_{\text{stand}}}$, which sends $f\,\mu^{\frac{1}{2}}_{\text{stand}}$ to $\triangle f\,\mu^{\frac{1}{2}}_{\text{stand}}$. If we move from this Darboux chart to another one via the transition map ϕ, the operator goes to $\phi_*\Delta^{(\frac{1}{2})}_{\mu_{\text{stand}}}$, which, by Proposition 5.5, is $\Delta^{(\frac{1}{2})}_{\phi_*\mu_{\text{stand}}}$. However, since μ_{stand} is a compatible density, by the corollary this is again $\Delta^{(\frac{1}{2})}_{\mu_{\text{stand}}}$ (in the new chart). Therefore, we have the

Theorem/Definition 5.11. There is a canonical BV Laplacian Δ acting on half-densities, defined as $\Delta^{(\frac{1}{2})}_{\mu_{\text{stand}}}$ in every Darboux chart.

Note that, by its very definition, the canonical BV Laplacian is invariant under symplectomorphisms.

Corollary 5.10 may now be reformulated as

Theorem 5.12. *For every compatible μ, the BV Laplacian on half-densities is equal to the canonical BV Laplacian:*

$$\Delta^{(\frac{1}{2})}_{\mu} = \Delta.$$

Note that, from its very definition, $\Delta^2 = 0$. On the other hand, we have not proved that Δ has the form $\Delta^{(\frac{1}{2})}_{\mu}$, since we have not shown yet that every odd symplectic manifold admits compatible densities. On the other hand, what is remarkable is that Δ is canonically defined (by Theorem/Definition 5.11) without choosing any density (in particular it would exist even if there were no compatible densities).

The canonical BV Laplacian gives another way of defining the BV Laplacian on functions. Namely, given an even, nowhere vanishing density μ, we define a differential operator $\widehat{\Delta}_\mu$ on functions via

$$\widehat{\Delta}_\mu f\,\mu^{\frac{1}{2}} = \Delta(f\mu^{\frac{1}{2}}). \tag{9}$$

Note that $\widehat{\Delta}_\mu^2 = 0$ for every μ. However, it is not a BV Laplacian in general. We actually have the

Lemma 5.13. *For every μ,*

$$\widehat{\Delta}_\mu = \widetilde{\Delta}_\mu$$

in every Darboux chart, where $\widetilde{\Delta}_\mu$ is the operator defined in Remark 4.8.

Proof. Writing $\mu = \mathrm{e}^S\mu_{\text{stand}}$, we have

$$\widehat{\Delta}_\mu f\,\mathrm{e}^{\frac{S}{2}}\mu^{\frac{1}{2}}_{\text{stand}} = \Delta(f\mathrm{e}^{\frac{S}{2}}\mu^{\frac{1}{2}}_{\text{stand}}) = \triangle(f\mathrm{e}^{\frac{S}{2}})\,\mu^{\frac{1}{2}}_{\text{stand}}.$$

One can easily compute

$$\triangle(f\mathrm{e}^{\frac{S}{2}}) = \left(\triangle f - \frac{1}{2}(S,f) + \frac{1}{2}F_S\right)\mathrm{e}^{\frac{S}{2}} = \widetilde{\Delta}_\mu f\,\mathrm{e}^{\frac{S}{2}},$$

with F_S as in (7). □

This yields the following

Corollary 5.14. *Let μ be an even, nowhere vanishing density. Then*

(1) $\widetilde{\Delta}_\mu$ is globally defined;
(2) μ is compatible if and only if $\widehat{\Delta}_\mu = \Delta_\mu$;
(3) μ is compatible if and only if $\Delta\mu^{\frac{1}{2}} = 0$.

Proof. The first two properties are now trivial. We prove the third. Setting $f = 1$ in (9) yields $\Delta(\mu^{\frac{1}{2}}) = \widehat{\Delta}_\mu 1\, \mu^{\frac{1}{2}}$. Lemma 5.13 then implies that $\Delta(\mu^{\frac{1}{2}}) = 0$ if and only if in every Darboux chart $F_S = \widetilde{\Delta}_\mu 1 = 0$. By Remark 4.5 this happens if and only if μ is compatible. □

5.1. **Digression: Darboux expression of $\Delta_\mu^{(s)}$.** In a Darboux chart we have $\mu = \mathrm{e}^S \mu_{\text{stand}}$. If $\sigma = f\,\mu_{\text{stand}}^s$, we then have $\sigma = \mathrm{e}^{-sS} f\,\mu^s$, so

$$\Delta_\mu^{(s)}\sigma = \Delta_\mu(\mathrm{e}^{-sS} f)\,\mu^s.$$

By using (6) and the properties of $\triangle$ in (1), we get

$$\Delta_\mu^{(s)}\sigma = \left(\triangle f + \left(s - \frac{1}{2}\right)(S, f) - \left(s\triangle S + \frac{1}{2}s(s-1)(S,S)\right) f\right)\mu_{\text{stand}}^s.$$

One can now explictly check that $(\Delta_\mu^{(s)})^t = \Delta_\mu^{(1-s)}$. Moreover, one explicitly sees that the first-order term vanishes for every S if and only if $s = \frac{1}{2}$. Finally, for μ compatible, i.e., $\Delta S = \frac{1}{4}(S,S)$, the zeroth-order term becomes $\frac{1}{4}(2s-1)s(S,S)$, which vanishes for $s = \frac{1}{2}$ or $s = 0$.

This formula, for $s = \frac{1}{2}$, yields a different way to prove Corollary 5.10 and therefore Theorem/Definition 5.11 and Theorem 5.12. However, proving this formula is a bit more laborious than proving Proposition 5.7.

6. Odd cotangent bundles

We now focus on an odd cotangent bundle ΠT^*M,[12] where M is an ordinary manifold. We will see that it is easy to show the existence of compatible densities in this case. Since every odd symplectic manifold is symplectomorphic to the odd symplectic bundle of its body [13], this implies that every odd symplectic manifold has a compatible density and therefore that the canonical BV Laplacian is indeed a BV Laplacian on half-densities.

The first observation is that functions on ΠT^*M are the same as multivector fields on M: $C^\infty(\Pi T^*M) = \Gamma(\Lambda^\bullet TM)$ as super algebras.[13]

The second observation is about the density bundle on ΠT^*M. If $\phi_{\alpha\beta}$ denotes a transition function on M, then the corresponding transition function on T^*M has a fiber part given by $((\mathrm{d}\phi_{\alpha\beta})^*)^{-1}$. Therefore, the Berezinian of the corresponding transition function for ΠT^*M is $(\mathrm{d}\phi_{\alpha\beta})^2$. This means that densities on ΠT^*M transform like 2-densities on M. Since we assume M to be oriented, we also have that half-densities on ΠT^*M transform like top forms on M. More precisely, we have a canonical isomorphism

$$\Gamma(\mathrm{Dens}^{\frac{1}{2}}(\Pi T^*M)) \simeq \Gamma(\Lambda^\bullet TM) \otimes_{C^\infty(M)} \Omega^n(M),$$

where n is the dimension of M. The next observation is that we have a canonical isomorphism (of $C^\infty(M)$-modules)

$$\begin{array}{rccc} \phi\colon & \Gamma(\Lambda^\bullet TM) \otimes_{C^\infty(M)} \Omega^n(M) & \to & \Omega^{n-\bullet}(M) \\ & X \otimes v & \mapsto & \iota_X v \end{array}$$

[12] Here Π denotes the change-of-parity functor. It simply means that the fiber coordinates are now regarded as odd.

[13] The interested reader might also appreciate that $(C^\infty(\Pi T^*M), (\ ,\ ))$ and $(\Gamma(\Lambda^\bullet TM), [\ ,\ ]_{\mathrm{SN}})$ are naturally isomorphic odd Poisson algebras, where $(\ ,\ )$ is the canonical BV bracket, associated to the canonical symplectic form, and $[\ ,\ ]_{\mathrm{SN}}$ is the Schouten–Nijenhuis bracket.

We can use this isomorphism to define an operator on half-densities:

$$\mathcal{D} := \phi^{-1}\circ \mathrm{d}\circ\phi,$$

where d is the de Rham differential. One immediately sees that $\mathcal{D}^2 = 0$ and that $\mathcal{D}v = 0$ for every top form v on M.

Proposition 6.1. *The canonical operator $\mathcal{D}$ is the same as the canonical BV Laplacian Δ on half-densities.*

Proof. We just have to check how $\mathcal{D}$ acts in Darboux coordinates. Let $q^1,\dots,q^n$ denote coordinates on a chart of M and let $p_1,\dots,p_n$ be the corresponding odd fiber coordinates. By linearity it is enough to consider a half-density $\sigma = g\,\mu_{\text{stand}}$ with $g(p,q)$ of the form $f(q)\,p_{i_1}\cdots p_{i_k}$, with pairwise distinct indices i_j. In the identification described above this corresponds to $\sigma = f(q)\,\partial_{i_1}\wedge\cdots\wedge\partial_{i_k}\otimes \mathrm{d}^n q$, so

$$\phi(\sigma) = f(q)\,\iota_{\partial_{i_1}}\cdots\iota_{\partial_{i_k}}\mathrm{d}^n q.$$

We then have

$$\mathrm{d}\phi(\sigma) = \mathrm{d}f(q)\,\iota_{\partial_{i_1}}\cdots\iota_{\partial_{i_k}}\mathrm{d}^n q = \partial_i f(q)\,\mathrm{d}q^i\wedge\iota_{\partial_{i_1}}\cdots\iota_{\partial_{i_k}}\mathrm{d}^n q,$$

since the form $\iota_{\partial_{i_1}}\cdots\iota_{\partial_{i_k}}\mathrm{d}^n q$ has a constant coefficient, so it is closed. If the index i is different from all the i_js, the corresponding term vanishes, since $\mathrm{d}q^i$ is already contained in $\mathrm{d}^n q$. Otherwise, we have

$$\begin{aligned}\mathrm{d}q^{i_j}\wedge\iota_{\partial_{i_1}}\cdots\iota_{\partial_{i_k}}\mathrm{d}^n q &= (-1)^{j-1}\mathrm{d}q^{i_j}\wedge\iota_{\partial_{i_j}}\iota_{\partial_{i_1}}\cdots\widehat{\iota}_{\partial_{i_j}}\cdots\iota_{\partial_{i_k}}\mathrm{d}^n q\\ &= (-1)^{j-1}\iota_{\partial_{i_1}}\cdots\widehat{\iota}_{\partial_{i_j}}\cdots\iota_{\partial_{i_k}}\mathrm{d}^n q,\end{aligned}$$

where the caret denotes omission. On the other hand,

$$\partial_{i_j}(p_{i_1}\cdots p_{i_k}) = (-1)^{j-1}p_{i_1}\cdots\widehat{p}_{i_j}\cdots p_{i_k}.$$

Therefore,

$$\phi(\triangle g\,\mu_{\text{stand}}) = \mathrm{d}\phi(g\,\mu_{\text{stand}}),$$

which shows that $\Delta = \mathcal{D}$. □

We then have $\Delta v = 0$ for every top form v on M (regarded as a half-density on ΠT^*M). As a notation, we write μ_v for the even, nowhere vanishing density v^2 on ΠT^*M associated to a volume form v on M. We then have

$$\Delta\mu_v^{\frac{1}{2}} = 0,$$

which shows, by point (3) of Corollary 5.14, that μ_v is a compatible density. We then have the

Proposition 6.2. *An odd cotangent bundle possesses compatible densities.*

This implies, by Theorem 5.12, that $\Delta = \Delta_{\mu_v}^{(\frac{1}{2})}$. Given a volume form v, we can now use (9) to get Δ_{μ_v} on functions. It is an easy exercise to see that

$$\Delta_{\mu_v} = \phi_v^{-1}\circ\mathrm{d}\circ\phi_v,$$

where ϕ_v is the (v-dependent!) isomorphism

$$\begin{array}{rccc}\phi_v\colon & C^\infty(\Pi T^*M) = \Gamma(\Lambda^\bullet TM) & \to & \Omega^{n-\bullet}(M)\\ & X & \mapsto & \iota_X v\end{array}$$

Note that, for a vector field X, we have $\Delta_{\mu_v}X = \mathrm{div}_v X$. For this reason, Δ_{μ_v} is interpreted as the extension to multivector fields of the divergence operator with respect to v.

7. Conclusion of the argument

It is a theorem (see [13]) that every odd symplectic manifold $\mathcal{M}$ is, noncanonically, isomorphic to the odd cotangent bundle ΠT^*M of its body M.[14]

On ΠT^*M we have compatible densities (Proposition 6.2), e.g., μ_v for a volume form v on M. If $\Psi\colon \Pi T^*M \to \mathcal{M}$ is a symplectomorphism, which exists by the above mentioned theorem, then $\Psi_*\mu_v$ is a compatible density on $\mathcal{M}$ (Remark 4.11).

We therefore have—also using Theorem 5.12, equation (9), and point (2) of Corollary 5.14—our final result:

Theorem 7.1. *Every odd symplectic manifold possesses compatible densities. For each compatible density μ, we have*

$$\Delta_\mu^{(\frac{1}{2})} = \Delta,$$

with Δ the canonical BV Laplacian on half-densities, and the BV Laplacian Δ_μ on functions may be recovered from

$$\Delta_\mu f\, \mu^{\frac{1}{2}} = \Delta(f\mu^{\frac{1}{2}}).$$

Appendix A. Some background details

A.1. **The Lie derivative.** If Ξ is an object (e.g., function, vector field, density) for which a notion of pushforward Ψ_* under a diffeomorphism Ψ exists, we define the Lie derivative with respect to a vector field X as

$$\mathsf{L}_X\Xi := \left.\frac{\partial}{\partial t}\right|_{t=0} (\Phi^X_{-t})_*\Xi.$$

For an even vector field X, the diffeomorphism Φ^X_t is the flow of X at time t. If X is odd, the variable t is also odd (so the evaluation at zero in the formula is redundant), and we define the morphism $\Phi^X\colon \Pi\mathbb{R}\times\mathcal{M}\to\mathcal{M}$ via[15]

$$\begin{array}{rccc} (\Phi^X)^*\colon & C^\infty(\mathcal{M}) & \to & C^\infty(\Pi\mathbb{R}\times\mathcal{M}) \\ & f & \mapsto & f + tX(f) \end{array}$$

where t is the odd coordinate on $\Pi\mathbb{R}$.[16] Note that in local coordinates z^μ, with $X = X^\mu\frac{\partial}{\partial z^\mu}$, we have in both cases

$$(\Phi^X)^\mu := (\Phi^X)^* z^\mu = z^\mu + tX^\mu + O(t^2), \tag{10}$$

where of course $O(t^2) = 0$ in the odd case. Also note that in both cases we have

$$\Psi_*(\Phi^X)_* = (\Phi^{\Psi_*X})_*\Psi_*$$

for every diffeomorphism Ψ.[17] This implies that

$$\mathsf{L}_{\Psi_*X}\Psi_*\Xi = \Psi_*\mathsf{L}_X\Xi.$$

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[14]This is a global Darboux theorem. If one is familiar with the proof of Darboux's theorem via Moser's trick, the reason why this now works globally is that it turns out that one has to integrate a vector field in the odd directions, which has no problem of definition domain.

[15]This defines the flow of X at odd time t if and only if $[X,X]=0$. Otherwise, the flow property $\Phi^X_{t+s} = \Phi^X_t \circ \Phi^X_s$ is not satisfied, so in general Φ^X is not a flow.

[16]Here Π denotes the change-of-parity functor: $\Pi\mathbb{R}$ is the superdomain with one odd coordinate.

[17]In the even case, one just observes how solutions of ODEs are mapped under diffeomorphisms. In the odd case, it is just an immediate application of the chain rule.

If two objects Ξ_1 and Ξ_2 as above can be multiplied, we immediately get

$$\mathsf{L}_X(\Xi_1\Xi_2) = \mathsf{L}_X\Xi_1\,\Xi_2 + (-1)^{X\Xi_1}\Xi_1\,\mathsf{L}_X\Xi_2. \tag{11}$$

On a function f or on a vector field Y, the Lie derivative is readily computed as

$$\mathsf{L}_X f = X(f) \quad \text{and} \quad \mathsf{L}_X Y = [X, Y].$$

A.2. **Densities.** A density is an object that transforms in such a way that its integration may be defined. For this we have to recall that, if Ψ is a diffeomorphism between two superdomains, then the change-of-variables formula for berezinian integration involves the Berezinian $\mathrm{Ber}(\Psi)$ of the jacobian matrix of Ψ times the sign of the jacobian determinant of the reduction to the body of Ψ. Since we assume throughout that the body of our supermanifold is oriented, we will not have this factor. Therefore, given an oriented atlas for $\mathcal{M}$ that has transition functions $\phi_{\alpha\beta}$, we define the density bundle $\mathrm{Dens}\,\mathcal{M}$ as the line bundle with transition functions $\mathrm{Ber}(\phi_{\alpha\beta})^{-1}$. More generally, for any real number s, we define the line bundle $\mathrm{Dens}^s\,\mathcal{M}$ of s-densities with transition functions $\mathrm{Ber}(\phi_{\alpha\beta})^{-s}$. Note that 1-densities are the same as densities, and 0-densities are the same as functions.

With this definition the integral of a compactly supported[18] density μ over $\mathcal{M}$ is defined as usual (by choosing a partition of unity subordinated to the atlas and by berezinian integration in each chart).

The pushforward of s-densities is also defined. If we go to charts, the pushforward is given by the pushforward of the representing function times the Berezinian of the transformation to the power $-s$.

Armed with the pushforward, we can define the Lie derivative of an s-density. In particular, we need the following

Lemma A.1. *In local coordinates $p_1,\dots,p_m$ and $q^1,\dots,q^n$ (odd and even, respectively), the Lie derivative of the standard density $\mu_{stand} = \mathrm{d}^n q\,\mathrm{d}^m p$ is*

$$\mathsf{L}_X\mu_{stand} = (\partial_i X^i - (-1)^X\partial^i X_i)\,\mu_{stand},$$

where we used the expansion $X = X^i\partial_i + X_i\partial^i$.

Proof. Using (10), we see that[19]

$$\begin{aligned}\mathrm{Ber}(\Phi^X_{-t}) &= \mathrm{Ber}\begin{pmatrix}\delta_i^j - \partial_i(tX^j) & 0\\ 0 & \delta^i_j - \partial^i(tX_j)\end{pmatrix} + O(t^2)\\ &= -\partial_i(tX^i) - \partial^i(tX_i) + O(t^2)\\ &= -t\partial_i(X^i) - (-1)^X t\partial^i(X_i) + O(t^2),\end{aligned}$$

using that the paritiy of t is the same as that of X. Applying the definitions of Lie derivative and of pushforward of a density yields the result. □

Applying (11) to the case of densities, we get

$$\mathsf{L}_X(h\mu) = X(h)\,\mu + (-1)^{Xh}h\,\mathsf{L}_X\mu \tag{12}$$

[18] The support is defined in terms of the coefficient body functions. Namely, an object (function, vector field, density,...) is compactly supported when in its expansion in odd variables all the coefficients are compactly supported functions on the body.

[19] By δ^i_j we mean the Kronecker delta, which is equal to 1 for $i = j$ and to 0 otherwise.

for every function h. In particular, we get the general formula for the Lie derivative in local coordinates:

$$\begin{aligned}\mathsf{L}_X\mu &= (X(h) + (-1)^{Xh} h(\partial_i X^i - (-1)^X \partial^i X_i))\,\mu_{\text{stand}}\\ &= (X(h) + (\partial_i X^i - (-1)^X \partial^i X_i)h)\,\mu_{\text{stand}}\\ &= X(h)\mu_{\text{stand}} + (\partial_i X^i - (-1)^X \partial^i X_i)\mu,\end{aligned}$$

with $\mu = h\mu_{\text{stand}}$. This implies the following

Lemma A.2. *For every function f, vector field X, and density μ, we have*

$$\mathsf{L}_{fX}\mu = f\mathsf{L}_X\mu + (-1)^{fX} X(f)\mu.$$

Proof. It is enough to check the formula in local coordinates. We have

$$\begin{aligned}\mathsf{L}_{fX}\mu &= fX(h)\mu_{\text{stand}} + (\partial_i(fX^i) - (-1)^{f+X}\partial^i(fX_i))\mu\\ &= fX(h)\mu_{\text{stand}} + f\,(\partial_i X^i - (-1)^X\partial^i X_i)\mu + (\partial_i f X^i - (-1)^{f+X}\partial^i f X_i)\mu\\ &= f\mathsf{L}_X\mu + ((-1)^{Xf}X^i\partial_i f - (-1)^{f+X}(-1)^{(f+1)(X+1)}X_i\partial^i f)\mu\\ &= f\mathsf{L}_X\mu + (-1)^{fX}X(f)\mu.\end{aligned}$$

□

In the case of densities, the pushforward is related to the change-of-variables formula. Namely, if $\Psi\colon \mathcal{M} \to \mathcal{N}$ is a diffeomorphism (preserving the orientation of the bodies), then

$$\int_{\mathcal{N}} \Psi_*\mu = \int_{\mathcal{M}} \mu \tag{13}$$

for every compactly supported density μ. This leads to

Lemma A.3. *For every vector field X and density μ, one of which is compactly supported, we have*

$$\int_{\mathcal{M}} \mathsf{L}_X\mu = 0.$$

Proof. First consider the case when μ is compactly supported. In this case, $(\Phi^X_{-t})_*\mu$ and hence $\mathsf{L}_X\mu$ are also compactly supported. We then have

$$\int_{\mathcal{M}} \mathsf{L}_X\mu = \int_{\mathcal{M}} \left.\frac{\partial}{\partial t}\right|_{t=0} (\Phi^X_{-t})_*\mu = \left.\frac{\partial}{\partial t}\right|_{t=0} \int_{\mathcal{M}} (\Phi^X_{-t})_*\mu = \left.\frac{\partial}{\partial t}\right|_{t=0} \int_{\mathcal{M}} \mu = 0,$$

where we have also used (13).

If on the other hand μ is not compactly supported, but X is, we replace μ with $\widetilde{\mu} = \rho\mu$, where ρ is a compactly supported bump function which is identically equal to 1 on the support of X. Since $\mathsf{L}_X\mu = \mathsf{L}_X\widetilde{\mu}$, we get the result by the previous case. □

If we have an even, nowhere vanishing density μ, then every other density can be written as $f\mu$ for a uniquely determined function f. Therefore, we can define the divergence operator via the following formula:

$$\operatorname{div}_\mu X\,\mu = \mathsf{L}_X\mu.$$

The properties of the Lie derivative immediately imply properties for the divergence operator:

Proposition A.4. *Let μ be an even, nowhere vanishing density μ and X a vector field. Then*

(1) *for every diffeomorphism Ψ,*
$$\Psi_*\mathrm{div}_\mu X = \mathrm{div}_{\Psi_*\mu}\Psi_* X;$$
(2) *for every even, nowhere vanishing function h,*
$$\mathrm{div}_{h\mu}X = \mathrm{div}_\mu X + \frac{1}{h}X(h);$$
(3) *for every function f,*
$$\mathrm{div}_\mu(fX) = f\mathrm{div}_\mu X + (-1)^{fX}X(f);$$
(4) *under the additional assumption that X is compactly supported,*
$$\int_{\mathcal{M}} \mathrm{div}_\mu X\,\mu = 0.$$

Moreover, from the local coordinate expression for the Lie derivative, we get

$$\mathrm{div}_{\mu_{\mathrm{stand}}}X = \partial_i X^i - (-1)^X\partial^i X_i, \tag{14a}$$

and, rearranging the terms,

$$\mathrm{div}_\mu X = \frac{1}{h}(\partial_i(hX^i) - (-1)^X\partial^i(hX_i)), \tag{14b}$$

for $\mu = h\mu_{\mathrm{stand}}$.

Appendix B. Applications of the BV formalism

We recall here the main reason why the BV formalism was introduced: to study the invariance of certain integrals.

We start considering the case of Section 2, where we have Darboux coordinates $p_1,\dots,p_n$ and $q^1,\dots,q^n$. For the sake of the argument, we rearrange each pair (p_i,q^i) into a new pair (x_i,y^i), with $x_i = p_i$ and $y^i = q^i$ for some is and $x_i = q^i$ and $y^i = p_i$ for the other is. Note that x_i has opposite parity to y^i (but we do not insist on y^i being even). We now use the shorthand notation
$$\partial_i = \frac{\partial}{\partial y^i} \quad\text{and}\quad \partial^i = \frac{\partial}{\partial x_i}.$$
Note that the BV Laplacian still reads $\triangle = \partial_i\partial^i$.

If ψ is an odd function of the variables y, it makes sense, in terms of parity, to set $x_i = (-1)^{x_i}\partial_i\psi$ (we will explain the reason for the signs in a moment). For a function f in the (x,y) variables, we define
$$\int_{\mathcal{L}_\psi} f := \int f|_{x_j=(-1)^{x_j}\partial_j\psi}\,\mathrm{d}^n y,$$
where $|_{x_j=(-1)^{x_j}\partial_j\psi}$ means that we set each x variable to the corresponding right-hand-side expression. The reason for the notation is that $\mathcal{L}_\psi$ is a Lagrangian submanifold determined by ψ. In fact, $\omega = \mathrm{d}x_i\mathrm{d}y^i = \mathrm{d}(x_i\mathrm{d}y^i)$ and
$$(x_i\mathrm{d}y^i)|_{x_i=(-1)^{x_i}\partial_i\psi} = \mathrm{d}\psi.$$
The first result is the BV lemma.

Lemma B.1. *Suppose that $f = \triangle g$ and g is integrable. Then $\int_{\mathcal{L}_\psi} f = 0$ for every ψ.*

Proof. We have

$$\sum_i \partial_i(\partial^i g)|_{x_j=(-1)^{x_j}\partial_j\psi} = \sum_i (\partial_i\partial^i g)|_{x_j=(-1)^{x_j}\partial_j\psi} + \sum_{ik} (\partial_i((-1)^{x_k}\partial_k\psi)\,\partial^k\partial^i g)|_{x_j=(-1)^{x_j}\partial_j\psi}.$$

Since $\sum_{ik}(-1)^{x_k}(\partial_i\partial_k\psi\,\partial^k\partial^i g)=0$, as one can easily see by exchanging the indices, we get

$$(\triangle g)|_{x_j=(-1)^{x_j}\partial_j\psi} = \sum_i \partial_i(\partial^i g)|_{x_j=(-1)^{x_j}\partial_j\psi},$$

so $\int_{\mathcal{L}_\psi} f$ vanishes. □

This leads to the fundamental BV theorem.

Theorem B.2. *Let ψ_t be a family of odd functions in the y variables depending smoothly on the even parameter t. If f is integrable, on every $\mathcal{L}_{\psi_t}$, and $\triangle f = 0$, then*

$$I_t := \int_{\mathcal{L}_{\psi_t}} f$$

is constant.

Proof. We have

$$\dot I_t = \int \sum_i ((-1)^{x_i}\partial_i\dot\psi\,\partial^i f)|_{x_j=(-1)^{x_j}\partial_j\psi}\ \mathrm{d}^n y,$$

where the dot denotes derivative with respect to t. From

$$\triangle(\dot\psi\, f) = \sum_i \partial_i\partial^i(\dot\psi\, f) = \sum_i (-1)^{x_i}\partial_i(\dot\psi\,\partial^i f) = \sum_i (-1)^{x_i}\partial_i\dot\psi\,\partial^i f - \dot\psi\,\triangle f,$$

we get

$$\dot I_t = \int \triangle(\dot\psi\, f)|_{x_j=(-1)^{x_j}\partial_j\psi}\ \mathrm{d}^n y = \int_{\mathcal{L}_{\psi_t}} \triangle(\dot\psi\, f) = 0,$$

where we have used the previous lemma. □

The BV theorem is used in quantum gauge theories,[20] where the choice of ψ (called the gauge-fixing fermion) corresponds to the choice of a gauge fixing and the invariance under deformations of this choice yields the independence of the theory from the gauge fixing, assuming of course that $\triangle f = 0$.

Using the results of Sections 4 and 5, one can extend the BV lemma and the BV theorem globally. The main observation is that the restriction of a half-density on an odd symplectic manifold to a Lagrangian submanifold defines a density there [13].

Theorem B.3. *Let $\mathcal{M}$ be an odd symplectic manifold and σ a half-density. The following hold:*

[20] In this application, the odd symplectic manifold is actually infinite dimensional, so $\triangle$ is not defined. What one does is either to proceed formally or to regularize the theory, e.g., by replacing the space of fields with a finite-dimensional approximation.

(1) If $\sigma = \Delta\tau$ for some τ, then

$$\int_{\mathcal{L}} \sigma = 0$$

for every a Lagrangian submanifold $\mathcal{L}$ of $\mathcal{M}$ on which σ is integrable.

(2) If $\Delta\sigma = 0$ and $\mathcal{L}_t$ is a smooth family of Lagrangian submanifolds of $\mathcal{M}$ on which σ is integrable, then

$$I_t := \int_{\mathcal{L}_t} \sigma$$

is constant.

In the case of an odd cotangent bundle ΠT^*M, one can show that every Lagrangian submanifold is a smooth deformation of an odd conormal bundle ΠN^*C, where C is a submanifold of M. Moreover, one can show that

$$\int_{\Pi N^*C} \sigma = \int_C \phi(\sigma)$$

for every half-density σ. By Stokes' theorem and the characterization of the canonical BV Laplacian as in Proposition 6.1 (i.e., $\phi \circ \Delta = \mathrm{d} \circ \phi$), one gets that

$$\int_{\Pi N^*C_1} \sigma = \int_{\Pi N^*C_2} \sigma$$

if C_1 and C_2 are homologous and $\Delta\sigma = 0$.

By Schwarz' theorem one can then generalize the global BV theorem a bit.

Theorem B.4. *Let $\mathcal{M}$ be an odd symplectic manifold and σ a half-density satisfying $\Delta\sigma = 0$. Then*

$$\int_{\mathcal{L}_1} \sigma = \int_{\mathcal{L}_2} \sigma$$

whenever $\mathcal{L}_2$ can be obtained from $\mathcal{L}_1$ by a combination of smooth deformations and homologous changes of the body, assuming that σ is integrable on each intermediate step.

Appendix C. Historical remarks

The BV formalism was developed by Batalin and Vilkovisky [1, 2] as a generalization of the BRST formalism [3, 15], which in turn put gauge fixing and the Faddeev–Popov determinant [5] into a cohomological setting. They constructud the BV Laplacian (as in our Section 2) and proved their fundamental theorem (in our notes, Theorem B.2). In addition, they showed that, under suitable assumptions, a physical action can be extended to a BV action satisfying the classical master equation.

In [16] Witten recognized the relation between the BV operator on functions on an odd cotangent bundle and the divergence operator on multivector fields on the body (see the last paragraph of Section 6). The relation between the de Rham differential and the divergence operator on multivector fields on an ordinary manifold was already known to Soviet mathematicians in the '80s (see, e.g., [12]). Also note that Bernshtein and Leites [4] had already introduced in 1977 the notion of "integral forms" on supermanifolds—where this notion differs from that of differential forms—and defined the exterior differential for them as the divergence operator.

Khudaverdian [6] was the first to give the global definition of the BV Laplacian on functions, see our equation (5).

Khudaverdian was also the first to observe that an odd symplectic manifold always admits global Darboux coordinates. This was used by Schwarz in [13], where he also observed how to globalize and extend the BV theorem (in our notes, Theorem B.3).

Finally, in [7, 8],[21] Khudaverdian showed the existence of a canonical BV Laplacian on half-densities (in our notes, this corresponds to Theorem/Definition 5.11).

In a series of papers (among others, [9, 10, 11]), Khudaverdian and Voronov further clarified the properties of the canonical BV Laplacian, essentially covering all the constructions we present in these notes (and more). In [9] they extended the construction of the odd Laplacian on half-densities to odd Poisson manifolds. In [10] they discussed the principal and subprincipal symbols of the BV Laplacians and their transposed operators (see our Section 5). In [11] they defined the canonical BV Laplacian on half-densities on an odd cotangent bundle in terms of the de Rham differential of the corresponding differential forms and then showed that this operator is invariant under all symplectomorphisms—not just those coming from the base.

It is worth mentioning that Ševera [14] found a completely different construction for the canonical BV Laplacian in the $\mathbb{Z}$-graded setting. Namely, one assumes that local coordinates on $\mathcal{M}$ have an additional $\mathbb{Z}$-grading for which the odd symplectic form ω has degree -1. The complex of differential forms on $\mathcal{M}$ is then bigraded and has two commuting coboundary operators of total degree $+1$: the de Rham differential d and the operator $\delta \coloneqq \omega\wedge$. It turns out that the associated spectral sequence lives up to the E_2-term. More precisely, Ševera shows that $E_1 = H_\delta(\mathcal{M})$ is canonically isomorphic to $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{M})$ and that the induced coboundary operator d_1 vanishes, which implies that $E_2 = E_1 = \mathrm{Dens}^{\frac{1}{2}}(\mathcal{M})$. Finally, Ševera proves that the canonically induced coboundary operator d_2 is the canonical BV Laplacian and that all higher coboundary operators vanish.

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