

# A NOTE ON GLUING VIA FIBER PRODUCTS IN THE (CLASSICAL) BV-BFV FORMALISM

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ABSTRACT. In classical field theory, gluing spacetime manifolds along boundary corresponds to taking a fiber product of the corresponding spaces of fields (as differential graded Fréchet manifolds) up to homotopy. We construct this homotopy explicitly in several examples in the setting of BV-BFV formalism (Batalin–Vilkovisky formalism with cutting–gluing).

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A. S. C. acknowledges partial support of SNF Grant No. 200020 192080 and of the Simons Collaboration on Global Categorical Symmetries. This research was (partly) supported by the NCCR SwissMAP, funded by the Swiss National Science Foundation.

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**Acknowledgments.** We thank Eugene Rabinovich for sharing some ideas related to this work. We are also very much indebted to Domenico Fiorenza and Bruno Vallette for helpful comments and for suggesting (independently) the construction sketched in Section 2.3.

## 1. PRELIMINARY DEFINITIONS: WEAK EQUIVALENCES OF DG SYMPLECTIC MANIFOLDS

**1.1. Differential geometric setting.** We start by fixing some definitions. First we recall a (standard) notion of a dg symplectic manifold (everywhere in the paper “dg” stands for “differential graded”). The subsequent definition of weak equivalence is not (to our knowledge) a part of the standard lore of dg geometry but has been used in the BV-BFV formalism [2, Definition 2.6.3].

**Definition 1.1.** A  $k$ -symplectic manifold<sup>1</sup> (a more full name is “degree  $k$  dg symplectic manifold”) is a triple  $(\mathcal{F}, Q, \omega)$  consisting of a  $\mathbb{Z}$ -graded supermanifold  $\mathcal{F}$  equipped with a degree  $+1$  cohomological vector field  $Q$  and a degree  $k$  symplectic form  $\omega \in \Omega^2(\mathcal{F})_k$  satisfying the compatibility condition

$$(1) \quad \mathcal{L}_Q \omega = 0.$$

If a  $k$ -symplectic manifold  $(\mathcal{F}, Q, \omega)$  is additionally equipped with a Hamiltonian function  $S \in C^\infty(\mathcal{F})_{k+1}$  for  $Q$ , i.e., one has  $\iota_Q \omega = \delta S$ ,<sup>2</sup> we call the quadruple  $(\mathcal{F}, Q, \omega, S)$  a  $k$ -Hamiltonian manifold.

The case relevant for the rest of the paper will be  $k = -1$  – the case of the Batalin–Vilkovisky formalism, where  $\mathcal{F}$  will be the space of fields of a field theory,  $Q$  the BRST operator and  $S$  the action functional.

A note on terminology:  $(-1)$ -Hamiltonian manifolds are also known as BV manifolds and 0-Hamiltonian manifolds – as BFV manifolds (they arise in Batalin–Fradkin–Vilkovisky construction for homological

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<sup>1</sup>A word of warning: in the literature, the term “ $k$ -symplectic” is sometimes used to indicate that instead of a symplectic 2-form, one has a closed  $k$ -form subject to suitable nondegeneracy condition. Here we do not mean that:  $k$  for us is the intrinsic (cohomological) degree of the nondegenerate closed 2-form.

<sup>2</sup>We use the notation  $\delta$  for de Rham differential on  $\mathcal{F}$  (which will later be the space of fields in a field theory), and we reserve  $d$  for the de Rham differential on the underlying spacetime manifold  $M$  of that field theory.

resolution of coisotropic reductions). Furthermore,  $(k-1)$ -Hamiltonian manifolds are elsewhere also called  $\text{BF}^k\text{V}$  manifolds.

**Definition 1.2.** A weak equivalence of two  $k$ -symplectic manifolds  $(\mathcal{F}, Q, \omega)$  and  $(\tilde{\mathcal{F}}, \tilde{Q}, \tilde{\omega})$  is a pair of dg maps  $f: \mathcal{F} \rightarrow \tilde{\mathcal{F}}$ ,  $g: \tilde{\mathcal{F}} \rightarrow \mathcal{F}$  such that

- (a) The pullbacks  $f^*: C^\infty(\tilde{\mathcal{F}}) \rightarrow C^\infty(\mathcal{F})$  and  $g^*: C^\infty(\mathcal{F}) \rightarrow C^\infty(\tilde{\mathcal{F}})$  are quasi-isomorphisms of chain complexes, inducing mutually inverse maps in cohomology.
- (b) One has  $f^*\tilde{\omega} = \omega + \mathcal{L}_Q\beta$  and  $g^*\omega = \tilde{\omega} + \mathcal{L}_{\tilde{Q}}\tilde{\beta}$  for some closed 2-forms  $\beta \in \Omega^2(\mathcal{F})_{k-1}$ ,  $\tilde{\beta} \in \Omega^2(\tilde{\mathcal{F}})_{k-1}$ .

**Lemma 1.3.** *If two  $k$ -Hamiltonian manifolds  $(\mathcal{F}, Q, \omega, S)$  and  $(\tilde{\mathcal{F}}, \tilde{Q}, \tilde{\omega}, \tilde{S})$  are weakly equivalent as  $k$ -symplectic manifolds, then the corresponding Hamiltonian functions are related by*

$$(2) \quad f^*\tilde{S} = S - \frac{1}{2}\iota_Q\iota_Q\beta,$$

$$(3) \quad g^*S = \tilde{S} - \frac{1}{2}\iota_{\tilde{Q}}\iota_{\tilde{Q}}\tilde{\beta},$$

where the equalities are modulo locally constant functions if  $k = -1$  and on the nose for  $k \neq -1$ .

Lemma 1.3 is essentially Proposition 2.6.5 of [2] in the non lax case.

*Proof.* Consider the expression  $\iota_Q f^*\tilde{\omega}$ . On the one hand, using chain map property of  $f^*$  (or dg map property of  $f$ ), we have

$$(4) \quad \iota_Q f^*\tilde{\omega} = f^*\iota_{\tilde{Q}}\tilde{\omega} = f^*\delta\tilde{S} = \delta f^*\tilde{S}.$$

On the other hand, we have

$$(5) \quad \iota_Q f^*\tilde{\omega} = \iota_Q(\omega + \mathcal{L}_Q\beta).$$

Using the fact that  $\beta$  is a closed 2-form, we have  $\iota_Q\mathcal{L}_Q\beta = -\iota_Q\delta\iota_Q\beta$  which, compared with  $\iota_Q\mathcal{L}_Q\beta = \mathcal{L}_Q\iota_Q\beta = \iota_Q\delta\iota_Q\beta - \delta\iota_Q\iota_Q\beta$ , implies  $\iota_Q\mathcal{L}_Q\beta = -\frac{1}{2}\delta\iota_Q\iota_Q\beta$ . Thus, continuing (5), we have

$$(6) \quad \iota_Q f^*\tilde{\omega} = \delta(S - \frac{1}{2}\iota_Q\iota_Q\beta).$$

Compared with (4), this implies that functions  $f^*\tilde{S}$  and  $S - \frac{1}{2}\iota_Q\iota_Q\beta$  on  $\mathcal{F}$  have the same de Rham differential. Thus, they coincide up to a shift by a locally constant function. Since constants are concentrated in degree zero, we have equality (2) on the nose for  $k \neq -1$  and up to a constant for  $k = -1$ . The proof of (3) is similar.  $\square$

**1.2. Relative version.** The following relative version of Definitions 1.1, 1.2 will be useful for us.

**Definition 1.4.** Let  $(\mathcal{F}', Q', \omega')$  be a  $(k+1)$ -symplectic manifold and let  $(\mathcal{F}, Q)$  be a dg manifold,  $\pi: \mathcal{F} \rightarrow \mathcal{F}'$  a dg map and  $\omega$  a degree  $k$  symplectic form on  $\mathcal{F}$  which instead of (1) satisfies

$$\mathcal{L}_Q \omega = \pi^* \omega'.$$

Then we say that  $(\mathcal{F}, Q, \omega, \pi)$  is a  $k$ -symplectic manifold *relative* to the  $(k+1)$ -symplectic manifold  $(\mathcal{F}', Q', \omega')$ .

If additionally one has  $\omega' = \delta\alpha'$  (the symplectic form on  $\mathcal{F}'$  is exact),  $Q'$  has a Hamiltonian function  $S' \in C^\infty(\mathcal{F}')_{k+2}$  and  $\mathcal{F}$  is equipped with a function  $S \in C^\infty(\mathcal{F})_{k+1}$  satisfying

$$(7) \quad \iota_Q \omega = \delta S - \pi^* \alpha',$$

then we say that  $(\mathcal{F}, Q, \omega, S, \pi)$  is a  $k$ -Hamiltonian manifold relative to the exact  $(k+1)$ -Hamiltonian manifold  $(\mathcal{F}', Q', \omega' = \delta\alpha', S')$ .

**Definition 1.5.** Let  $(\mathcal{F}, Q, \omega, \pi)$  and  $(\tilde{\mathcal{F}}, \tilde{Q}, \tilde{\omega}, \tilde{\pi})$  be two  $k$ -symplectic manifolds, both relative to the same  $(k+1)$ -symplectic manifold  $(\mathcal{F}', Q', \omega')$ . We define the weak equivalence between  $(\mathcal{F}, Q, \omega, \pi)$  and  $(\tilde{\mathcal{F}}, \tilde{Q}, \tilde{\omega}, \tilde{\pi})$  as in the nonrelative case (Definition 1.2), requiring additionally that  $\tilde{\pi} \circ f = \pi$  and  $\pi \circ g = \tilde{\pi}$ .

We remark that Lemma 1.3 works in the relative case without any change.

**1.3. Linear algebra setting.** Here is a version Definitions 1.1, 1.2 adapted to the linear algebra case.

**Definition 1.6.** We call a *degree  $k$  Poincaré cochain complex* (or just  *$k$ -Poincaré complex*) a cochain complex  $\mathcal{F}^\bullet$  over  $\mathbb{R}$  with differential  $d_Q$ , equipped with a graded skew-symmetric non-degenerate pairing  $\omega: \bigoplus_i \mathcal{F}^i \otimes \mathcal{F}^{-i-k} \rightarrow \mathbb{R}$  satisfying

$$(8) \quad \omega(d_Q x, y) = (-1)^{|x|} \omega(x, d_Q y)$$

for any homogeneous elements  $x, y \in \mathcal{F}$ .

A  $k$ -Poincaré complex can be seen as a  $k$ -symplectic manifold with  $Q = (d_Q)^*$  – the pullback by  $d_Q$  (in particular,  $Q$  is a *linear* cohomological vector field on  $\mathcal{F}^\bullet$ ) and with a constant  $k$ -symplectic form  $\omega$ . It also has a quadratic Hamiltonian function for  $Q$ ,

$$(9) \quad S(x) = \frac{1}{2} \omega(d_Q x, x).$$

**Definition 1.7.** A weak equivalence between two  $k$ -Poincaré complexes  $(\mathcal{F}, d_Q, \omega)$ ,  $(\tilde{\mathcal{F}}, \tilde{d}_Q, \tilde{\omega})$  is the following set of data:

(i) A pair of chain maps of complexes

$$(10) \quad f: \mathcal{F} \rightarrow \tilde{\mathcal{F}}, \quad g: \tilde{\mathcal{F}} \rightarrow \mathcal{F}$$

(ii) A pair of maps  $H: \mathcal{F}^\bullet \rightarrow \mathcal{F}^{\bullet-1}$ ,  $\tilde{H}: \tilde{\mathcal{F}}^\bullet \rightarrow \tilde{\mathcal{F}}^{\bullet-1}$  satisfying the chain homotopy property:

$$d_Q H + H d_Q = \text{id} - g f, \quad \tilde{d}_Q \tilde{H} + \tilde{H} \tilde{d}_Q = \text{id} - f g$$

(thus,  $f, g$  are quasi-isomorphisms inducing mutually inverse maps in cohomology).

(iii) A degree  $k-1$  skew-symmetric bilinear form  $\beta$  on  $\mathcal{F}$  and a degree  $k-1$  skew-symmetric bilinear form  $\tilde{\beta}$  on  $\tilde{\mathcal{F}}$  such that

$$(11) \quad \tilde{\omega}(f(x), f(y)) = \omega(x, y) + \underbrace{(-1)^k (\beta(d_Q x, y) + (-1)^{|x|+1} \beta(x, d_Q y))}_{(\mathcal{L}_Q \beta)(x, y)},$$

$$(12) \quad \omega(g(\tilde{x}), g(\tilde{y})) = \tilde{\omega}(\tilde{x}, \tilde{y}) + \underbrace{(-1)^k (\tilde{\beta}(\tilde{d}_Q \tilde{x}, \tilde{y}) + (-1)^{|\tilde{x}|+1} \tilde{\beta}(\tilde{x}, \tilde{d}_Q \tilde{y}))}_{(\mathcal{L}_{\tilde{Q}} \tilde{\beta})(\tilde{x}, \tilde{y})}.$$

Here  $x, y$  are test elements in  $\mathcal{F}$  and  $\tilde{x}, \tilde{y}$  are in  $\tilde{\mathcal{F}}$ .

**Lemma 1.8.** Let  $(\mathcal{F}, d_Q, \omega)$ ,  $(\tilde{\mathcal{F}}, \tilde{d}_Q, \tilde{\omega})$  be two  $k$ -Poincaré complexes.

Assume that we have a pair of quasi-inverse quasi-isomorphisms  $\mathcal{F} \xrightleftharpoons[g]{f} \tilde{\mathcal{F}}$

with chain homotopies  $H, \tilde{H}$  and assume additionally that  $f^* \tilde{\omega} = \omega$ . Then  $(\mathcal{F}, d_Q, \omega)$  and  $(\tilde{\mathcal{F}}, \tilde{d}_Q, \tilde{\omega})$  are weakly equivalent Poincaré complexes with  $\beta = 0$  and with  $\tilde{\beta}$  defined in terms of the data  $(\tilde{\omega}, \tilde{d}_Q, \tilde{H})$  by

$$(13) \quad \tilde{\beta}(\tilde{x}, \tilde{y}) = (-1)^{k+1} \left( \tilde{\omega}(\tilde{H}\tilde{x}, \tilde{y}) + (-1)^{|\tilde{x}|+1} \tilde{\omega}(\tilde{x}, \tilde{H}\tilde{y}) \right) + \frac{1}{2} \left( -\tilde{\omega}(\tilde{H}\tilde{x}, \tilde{H}\tilde{d}_Q \tilde{y}) + (-1)^{|\tilde{x}|} \tilde{\omega}(\tilde{H}\tilde{d}_Q \tilde{x}, \tilde{H}\tilde{y}) - \tilde{\omega}(\tilde{H}\tilde{x}, \tilde{d}_Q \tilde{H}\tilde{y}) \right).$$

*Proof.* Property (11) holds with  $\beta = 0$  by the assumption  $f^*\tilde{\omega} = \omega$ . Let us check 12. We have

$$\begin{aligned}
(14) \quad & \omega(g(\tilde{x}), g(\tilde{y})) - \tilde{\omega}(\tilde{x}, \tilde{y}) \underset{\text{using } f^*\tilde{\omega}=\omega}{=} \tilde{\omega}(fg(\tilde{x}), fg(\tilde{y})) - \tilde{\omega}(\tilde{x}, \tilde{y}) \\
& = \omega\left((\text{id} - \tilde{d}_Q\tilde{H} - \tilde{H}\tilde{d}_Q)(\tilde{x}), (\text{id} - \tilde{d}_Q\tilde{H} - \tilde{H}\tilde{d}_Q)(\tilde{y})\right) - \tilde{\omega}(\tilde{x}, \tilde{y}) \\
& = \underbrace{-\tilde{\omega}(\tilde{d}_Q\tilde{H}\tilde{x}, \tilde{y})}_a - \underbrace{\tilde{\omega}(\tilde{H}\tilde{d}_Q\tilde{x}, \tilde{y})}_b - \underbrace{\tilde{\omega}(\tilde{x}, \tilde{d}_Q\tilde{H}\tilde{y})}_c - \underbrace{\tilde{\omega}(\tilde{x}, \tilde{H}\tilde{d}_Q\tilde{y})}_d \\
& + \underbrace{\tilde{\omega}(\tilde{d}_Q\tilde{H}\tilde{x}, \tilde{d}_Q\tilde{H}\tilde{y})}_0 + \underbrace{\tilde{\omega}(\tilde{d}_Q\tilde{H}\tilde{x}, \tilde{H}\tilde{d}_Q\tilde{y})}_e + \underbrace{\tilde{\omega}(\tilde{H}\tilde{d}_Q\tilde{x}, \tilde{H}\tilde{d}_Q\tilde{y})}_f + \underbrace{\tilde{\omega}(\tilde{H}\tilde{d}_Q\tilde{x}, \tilde{d}_Q\tilde{H}\tilde{y})}_g.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(15) \quad & (-1)^k(\tilde{\beta}(\tilde{d}_Q\tilde{x}, \tilde{y}) + (-1)^{|\tilde{x}|+1}\tilde{\beta}(\tilde{x}, \tilde{d}_Q\tilde{y})) \underset{(13)}{=} \\
& = \underbrace{-\tilde{\omega}(\tilde{H}\tilde{d}_Q\tilde{x}, \tilde{y})}_b + \underbrace{(-1)^{|\tilde{x}|+1}\tilde{\omega}(\tilde{d}_Q\tilde{x}, \tilde{H}\tilde{y})}_c \\
& + \underbrace{\frac{1}{2}\tilde{\omega}(\tilde{H}\tilde{d}_Q\tilde{x}, \tilde{H}\tilde{d}_Q\tilde{y})}_{\frac{1}{2}f} + \underbrace{(-1)^{|\tilde{x}|}\frac{1}{2}\tilde{\omega}(\tilde{H}(\tilde{d}_Q)^2\tilde{x}, \tilde{H}\tilde{y})}_0 + \underbrace{\tilde{\omega}(\tilde{H}\tilde{d}_Q\tilde{x}, \tilde{d}_Q\tilde{H}\tilde{y})}_g \\
& + \underbrace{(-1)^{|x'|}\omega'(H'x', d'_Qy')}_a - \underbrace{\tilde{\omega}(\tilde{x}, \tilde{H}\tilde{d}_Q\tilde{y})}_d \\
& + \underbrace{(-1)^{|\tilde{x}|+1}\frac{1}{2}\tilde{\omega}(\tilde{H}\tilde{x}, \tilde{H}(\tilde{d}_Q)^2\tilde{y})}_0 + \underbrace{\frac{1}{2}\tilde{\omega}(\tilde{H}\tilde{d}_Q\tilde{x}, \tilde{H}\tilde{d}_Q\tilde{y})}_{\frac{1}{2}f} + \underbrace{(-1)^{|\tilde{x}|+1}(\tilde{H}\tilde{x}, \tilde{d}_Q\tilde{H}\tilde{d}_Q\tilde{y})}_e.
\end{aligned}$$

Comparing the terms in (14) and (15) and using  $\tilde{\omega}(\tilde{d}_Q\tilde{x}, \tilde{y}) = (-1)^{|\tilde{x}|}\tilde{\omega}(\tilde{x}, \tilde{d}_Q\tilde{y})$  we see that the expressions are equal and thus (12) holds.  $\square$

#### 1.4. Relative version in the linear case.

**Definition 1.9.** Let  $(\mathcal{F}', d'_Q, \omega')$  be a  $(k+1)$ -Poincaré complex and let  $(\mathcal{F}, d_Q)$  be a cochain complex equipped with a chain map  $\pi: \mathcal{F} \rightarrow \mathcal{F}'$  equipped with a degree  $k$  constant symplectic structure  $\omega: \wedge^2 \mathcal{F} \rightarrow \mathbb{R}$  satisfying

$$\omega(d_Qx, y) + (-1)^{|x|+1}\omega(x, d_Qy) = (-1)^{k+1}\omega'(\pi(x), \pi(y))$$

instead of (8). Then we say that  $(\mathcal{F}, d_Q, \omega, \pi)$  is a  $k$ -Poincaré complex *relative* to the  $(k+1)$ -Poincaré complex  $(\mathcal{F}', d'_Q, \omega')$ .

**Definition 1.10.** Let  $(\mathcal{F}, d_Q, \omega, \pi)$  and  $(\tilde{\mathcal{F}}, \tilde{d}_Q, \tilde{\omega}, \tilde{\pi})$  be two  $k$ -Poincaré complexes relative to the same  $(k+1)$ -Poincaré complex  $(\mathcal{F}', d'_Q, \omega')$ .

We define a weak equivalence between  $(\mathcal{F}, d_Q, \omega, \pi)$  and  $(\tilde{\mathcal{F}}, \tilde{d}_Q, \tilde{\omega}, \tilde{\pi})$  as in Definition 1.7 where additionally we require the properties

$$\tilde{\pi} \circ f = \pi, \quad \pi \circ g = \tilde{\pi}, \quad \pi \circ H = 0, \quad \tilde{\pi} \circ \tilde{H} = 0.$$

We note that Lemma 1.8 works in the relative case with no changes.

## 2. GLUING VIA FIBER PRODUCTS IN A CLASSICAL BV-BFV THEORY

**2.1. Classical BV-BFV theories: a reminder.** In [1] the authors of the present paper and N. Reshetikhin introduced a refinement of Batalin–Vilkovisky formalism for gauge theories on manifolds with boundary, compatible with cutting–gluing, dubbed the “BV-BFV formalism.” For reader’s convenience we recap the main definition.<sup>3</sup>

**Definition 2.1.** An  $n$ -dimensional classical BV-BFV theory  $\mathcal{T}$  assigns to a closed  $(n - 1)$ -manifold  $\Sigma$  a “BFV phase space” – a quadruple  $(\Phi_\Sigma, Q_\Sigma, \omega_\Sigma = \delta\beta_\Sigma, S_\Sigma)$  consisting of:

- A graded Fréchet manifold  $\Phi_\Sigma$  of smooth sections of a graded vector bundle  $E^\partial \rightarrow \Sigma$ .
- A degree +1 local<sup>4</sup> cohomological vector field  $Q_\Sigma$  on  $\Phi_\Sigma$ .
- An exact 0-symplectic<sup>5</sup> form  $\omega_\Sigma = \delta\alpha_\Sigma$ , where  $\alpha_\Sigma = \int_\Sigma \underline{\alpha}_\Sigma$  with  $\underline{\alpha}_\Sigma \in \Omega_{\text{loc}}^{n-1,1}(\Sigma \times \Phi_\Sigma)$ .<sup>6</sup>
- A degree +1 function  $S_\Sigma = \int_\Sigma L_\Sigma$  – a Hamiltonian for  $Q_\Sigma$ , with  $L_\Sigma^\partial \in \Omega_{\text{loc}}^{n-1,0}(\Sigma \times \Phi_\Sigma)$ .

In particular,  $(\Phi_\Sigma, Q_\Sigma, \omega_\Sigma, S_\Sigma)$  is a 0-Hamiltonian manifold.

To an  $n$ -manifold  $M$  with boundary  $\Sigma = \partial M$ , the theory  $\mathcal{T}$  assigns a quintuple  $(\mathcal{F}_M, Q_M, \omega_M, S_M, \pi_{M,\Sigma})$  consisting of:

- A graded Fréchet manifold  $\mathcal{F}_M$  (“space of BV fields”) of smooth sections of a graded vector bundle  $E \rightarrow M$ .
- A degree +1 local cohomological vector field  $Q_M$  on  $\mathcal{F}_M$ .

<sup>3</sup>We are following the version of the definition from [9] in the non-extended case.

<sup>4</sup>Here “local” means that the result of acting by  $Q_\Sigma$  on a field at  $x$  depends only on the jet of the field at  $x$ .

<sup>5</sup>In the context of Fréchet manifolds, we understand a symplectic structure as a *weakly* nondegenerate closed 2-form, where “weakly nondegenerate” means that the induced sharp map  $\omega^\#: T\Phi \rightarrow T^*\Phi$  is injective.

<sup>6</sup>The subscript “loc” stands for locality: the evaluation of the form at  $x \in \Sigma$  depends only on the jet of the field at  $x$ . The superscript  $n - 1, 1$  is the de Rham bi-degree – along  $\Sigma$  and along  $\Phi_\Sigma$ . Equivalently, one can say that  $\underline{\alpha}_\Sigma$  is a form on the jet bundle of  $E^\partial$  of horizontal de Rham degree  $n - 1$  and vertical de Rham degree 1.

- A  $(-1)$ -symplectic form  $\omega_M = \int_M \underline{\omega}_M$  on  $\mathcal{F}_M$ , with  $\underline{\omega}_M \in \Omega_{\text{loc}}^{n,2}(M \times \mathcal{F}_M)$ .
- A degree 0 function  $S_M = \int_M L_M$  on  $\mathcal{F}_M$  (the BV action) of the form, with  $L_M \in \Omega_{\text{loc}}^{n,0}(M \times \mathcal{F}_M)$ .
- A surjective submersion  $\pi_{M,\Sigma}: \mathcal{F}_M \rightarrow \Phi_\Sigma$  satisfying the dg map property  $Q_M \pi_{M,\Sigma}^* = \pi_{M,\Sigma}^* Q_\Sigma$ .

Instead of being a Hamiltonian for  $Q_M$ , the function  $S_M$  is assumed to satisfy the structure relation

$$(16) \quad \iota_{Q_M} \omega_M = \delta S_M - \pi_{M,\Sigma}^* \alpha_\Sigma,$$

linking the bulk data on  $M$  and boundary data on  $\Sigma$ .

In particular  $(\mathcal{F}_M, Q_M, \omega_M, S_M, \pi_{M,\Sigma})$  is a  $(-1)$ -Hamiltonian manifold relative to the exact 0-Hamiltonian manifold  $(\Phi_\Sigma, Q_\Sigma, \omega_\Sigma = \delta \alpha_\Sigma, S_\Sigma)$ , in the sense of Definition 1.4.

**2.2. Gluing via fiber products.** Assume that we have an  $n$ -dimensional classical BV-BFV theory  $\mathcal{T}$  on a closed  $n$ -manifold  $M$ .<sup>7</sup> Let  $M$  be cut by a closed  $(n-1)$ -submanifold  $\Sigma$  into two  $n$ -manifolds with boundary,  $M_1$  and  $M_2$ . Let  $\mathcal{F}_M$  be the space of fields on  $M$  and let

$$(17) \quad \begin{aligned} \tilde{\mathcal{F}}_{M,\Sigma} &= \mathcal{F}_{M_1} \times_{\Phi_\Sigma} \mathcal{F}_{M_2} \\ &= \{(\phi_1, \phi_2) \in \mathcal{F}_{M_1} \times \mathcal{F}_{M_2} \text{ s.t. } \pi_1(\phi_1) = \pi_2(\phi_2)\} \end{aligned}$$

be the fiber product (in the category of dg manifolds<sup>8</sup>) of the spaces of fields on  $M_1$  and  $M_2$  over the phase space for the interface  $\Sigma$ . Here  $\phi_{1,2}$  are fields on  $M_1$  and  $M_2$  and  $\pi_i = \pi_{M_i,\Sigma}: \mathcal{F}_{M_i} \rightarrow \Phi_\Sigma$  with  $i = 1, 2$  is the restriction of fields to the boundary (a structure map of the BV-BFV

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<sup>7</sup> The assumption that  $M$  is closed is made for simplicity of exposition and is not essential. If  $M$  has boundary, we need to replace  $(-1)$ -symplectic manifolds/Poincaré complexes below with their relative versions, relative to the boundary BFV data assigned by  $\mathcal{T}$  to  $\partial M$ .

<sup>8</sup>Fiber products in the category of dg manifolds are not always well-defined, but in the case of (17) it is well-defined, since  $\pi_{1,2}$  are surjective submersions. In the context of dg geometry, one usually gets to this nice case by replacing the dg manifolds appropriately with equivalent ones. A related point: it would be natural to speak in terms of *homotopy* fiber products in present context, but due to the fact that  $\pi_{1,2}$  are surjective submersions, the strict fiber product provides a particular model for the homotopy fiber product.



package). Put another way,  $\tilde{\mathcal{F}}_{M,\Sigma}$  fits into the fiber product diagram

$$\begin{array}{ccc} \tilde{\mathcal{F}}_{M,\Sigma} & \xrightarrow{p_2} & \mathcal{F}_{M_2} \\ p_1 \downarrow & & \pi_2 \downarrow \\ \mathcal{F}_{M_1} & \xrightarrow{\pi_1} & \Phi_\Sigma \end{array}$$

One has a natural inclusion<sup>9</sup>

$$(18) \quad i: \mathcal{F}_M \hookrightarrow \tilde{\mathcal{F}}_{M,\Sigma}.$$

Let  $\omega_M$  be the BV  $(-1)$ -symplectic structure on  $\mathcal{F}_M$  and let

$$(19) \quad \tilde{\omega}_{M,\Sigma} = p_1^* \omega_{M_1} + p_2^* \omega_{M_2}$$

be the  $(-1)$ -symplectic structure on  $\tilde{\mathcal{F}}_{M,\Sigma}$  arising from  $\omega_{M_1}$  and  $\omega_{M_2}$  via fiber product construction.

**Conjecture 2.2.** The inclusion (18) can be extended to a weak equivalence of  $(-1)$ -symplectic dg manifolds, in the sense of Definition 1.2.

A version adapted for a free (quadratic) BV-BFV theory is the following.

**Conjecture 2.3.** If  $\mathcal{T}$  is a free BV-BFV theory, the inclusion (18) can be extended to a weak equivalence of degree  $-1$  Poincaré cochain complexes, in the sense of Definition 1.7.

The main result of this note is the proof of this conjecture in a collection of cases. More precisely, we prove the following.

**Theorem 2.4.** *Let  $\mathcal{T}$  be one of the following:*

- *abelian Chern–Simons theory,*
- *abelian BF theory,*
- *$\mathfrak{p}$ -form electrodynamics in the first- or second-order formalism (including massless scalar field and usual electrodynamics as  $\mathfrak{p} = 0$  and  $\mathfrak{p} = 1$  cases).*

*Then, given any open neighborhood  $U \subset M$  of  $\Sigma$ , one can extend the inclusion (18) to a package of maps  $(i, p, H, \tilde{H})$ :*

$$(20) \quad H \hookrightarrow \mathcal{F}_M \begin{array}{c} \xleftarrow{i} \\ \xrightarrow{p} \end{array} \tilde{\mathcal{F}}_{M,\Sigma} \hookrightarrow \tilde{H}$$

*such that:*

---

<sup>9</sup>Note that  $i$  is not the identity: the right hand side of (18) is strictly larger than the left hand side. Indeed,  $\mathcal{F}_M$  consists of smooth fields, whereas  $\tilde{\mathcal{F}}_{M,\Sigma}$  consists of fields smooth away from  $\Sigma$  and with a special type of singularity allowed on  $\Sigma$ .

(a)  $i$  and  $p$  are chain maps w.r.t. differentials  $d_Q$  on  $\mathcal{F}_M$  and  $\widetilde{d}_Q$  on  $\widetilde{\mathcal{F}}_{M,\Sigma}$ :

$$(21) \quad \widetilde{d}_Q i = i d_Q,$$

$$(22) \quad d_Q p = p \widetilde{d}_Q.$$

(b)  $H$  is the chain homotopy between identity and  $p i$  on  $\mathcal{F}_M$  and  $\widetilde{H}$  is the chain homotopy between identity and  $i p$  on  $\widetilde{\mathcal{F}}_{M,\Sigma}$ :

$$(23) \quad d_Q H + H d_Q = \text{id} - p i,$$

$$(24) \quad \widetilde{d}_Q \widetilde{H} + \widetilde{H} \widetilde{d}_Q = \text{id} - i p.$$

(c) The package of maps  $(i, p, H, \widetilde{H})$  is local near  $\Sigma$ :  $p$  is smoothing in the neighborhood  $U$  of  $\Sigma$  and is identity outside  $U$ .<sup>10</sup> The operators  $H, \widetilde{H}$  vanish on fields supported outside  $U$ .

(d) The  $(-1)$ -symplectic structures on  $\mathcal{F}_M$  and  $\widetilde{\mathcal{F}}_{M,\Sigma}$  are related by

$$(25) \quad i^* \widetilde{\omega} = \omega,$$

$$(26) \quad p^* \omega = \widetilde{\omega} + \mathcal{L}_{\widetilde{Q}} \widetilde{\beta},$$

with  $\widetilde{\beta}$  given by (13).

In particular, relations (21)–(24) imply that  $i$  and  $p$  are quasi-isomorphisms (mutually inverse in cohomology).

The proof of this theorem (and the construction of the corresponding maps) is given in the subsequent sections: the case of abelian Chern–Simons and abelian  $BF$  is covered in Section 4. The case of  $\mathfrak{p}$ -form theory in the first-order formalism—in Section 6.2 and the second-order formalism—in Section 6.3 (“local-near- $\Sigma$  version”).

We note that (25) holds trivially by construction (19), while (26) follows from items (a), (b), (c) and Lemma 1.8.

*Remark 2.5.* In Theorem 2.4 we *do not* require the property  $p i = \text{id}$  (in fact, it doesn’t hold in the examples discussed in this paper); similarly, we do not require that  $i p$  is an idempotent.

*Remark 2.6.* We want to emphasize the significance of item (c) in Theorem 2.4: it implies functoriality of the weak equivalence with respect to Segal’s functorial picture of QFT. Put another way, the data of the weak equivalence is localized near the cut  $\Sigma$  and does not interact with cutting–gluing away from  $\Sigma$ .

<sup>10</sup>Put another way,  $p$  is an integral operator with distributional kernel which is smooth in  $U \times U$  and the Dirac delta-form on the diagonal outside  $U \times U$ .

A related point is that although in the beginning of this section we asked  $M$  to be closed (for simplicity and so that we can use the convenient language of Poincaré complexes), we can without problem allow  $M$  to have boundary, disjoint from the hypersurface  $\Sigma$ , cf. footnote 7.

In a future paper we plan to study examples of Conjecture 2.2 coming from interacting field theories (in the spirit of Theorem 2.4, with complexes of fields enhanced to  $L_\infty$  algebras of fields).

**2.3. Sketch of proof of Conjecture 2.2 by factoring through cohomology (and ignoring locality near  $\Sigma$ ).** The following construction was independently suggested to us by Domenico Fiorenza and Bruno Vallette.

Consider the setup of Conjecture 2.2. We will assume that the BV-BFV theory  $\mathcal{T}$  is a perturbation of a free theory  $\mathcal{T}_0$  in which Conjecture 2.3 holds.<sup>11</sup> In particular,  $\mathcal{F}_M$  is a graded vector space with cohomological vector field  $Q_M$  vanishing at the origin and corresponding to an  $L_\infty$  algebra structure on  $\mathcal{F}_M[-1]$ ;<sup>12</sup>  $\omega_M$ —a constant  $(-1)$ -symplectic form—corresponds to a degree  $-3$  inner product on  $\mathcal{F}_M[-1]$  with respect to which the  $L_\infty$  operations have cyclic property.

We have a strict (i.e. having only linear component)  $L_\infty$  map (18) and we want to promote it to a weak equivalence in the sense of Definition 1.2.

We employ the construction of [5, Section 10.4.6] adapted to our setting: we consider the cohomology<sup>13</sup>  $\mathbf{H}(\mathcal{F})$  of  $\mathcal{F}$  with respect to  $d_Q$  (the linear part of the  $Q$ ) and the cohomology  $\mathbf{H}(\tilde{\mathcal{F}})$  of  $\tilde{\mathcal{F}}$  with respect to the linear part  $\tilde{d}_Q$  of  $\tilde{Q}$ . We choose contractions

$$(27) \quad h \underset{j}{\overset{r}{\rightleftarrows}} (\mathcal{F}, d_Q) \mathbf{H}(\mathcal{F}), \quad \tilde{h} \underset{\tilde{j}}{\overset{\tilde{r}}{\rightleftarrows}} (\tilde{\mathcal{F}}, \tilde{d}_Q) \mathbf{H}(\tilde{\mathcal{F}})$$

such that

$$(28) \quad i \circ j = \tilde{j} \circ i_*, \quad i_* \circ r = \tilde{r} \circ i, \quad i \circ h = \tilde{h} \circ i,$$

<sup>11</sup>We understand that  $\mathcal{T}$  is obtained from  $\mathcal{T}_0$  by adding terms of polynomial degree  $\geq 3$  in fields to the quadratic action of  $\mathcal{T}_0$  and, respectively, perturbing the linear cohomological vector field  $Q_0 = d_Q$  of  $\mathcal{T}_0$  by terms of polynomial degree  $\geq 2$  in fields. For simplicity we assume that the  $(-1)$ -symplectic form does not get deformed.

<sup>12</sup>For the sake of convenience, below in this section we will omit the degree shift  $[-1]$  in notations, and speak of  $\mathcal{F}$  as an  $L_\infty$  algebra. We will also use the terms “dg map” and “ $L_\infty$  map” interchangeably.

<sup>13</sup>We use the font to distinguish cohomology  $\mathbf{H}$  from chain homotopies  $H$ .

where

$$(29) \quad i_*: \mathbf{H}(\mathcal{F}) \rightarrow \mathbf{H}(\tilde{\mathcal{F}})$$

is the linear isomorphism in cohomology induced from  $i$  ( $i_*$  is an isomorphism by assumption that the free theory  $\mathcal{T}_0$  satisfies Conjecture 2.3). The homological perturbation lemma<sup>14</sup> applied at the level of symmetric coalgebras deforms the maps (27) to

$$(30) \quad \mathcal{H} \wr (\mathcal{F}, Q) \begin{array}{c} \xrightarrow{\mathcal{R}} \\ \xleftarrow{\mathcal{J}} \end{array} (\mathbf{H}(\mathcal{F}), Q_{\mathbf{H}}), \quad \tilde{\mathcal{H}} \wr (\tilde{\mathcal{F}}, \tilde{Q}) \begin{array}{c} \xrightarrow{\tilde{\mathcal{R}}} \\ \xleftarrow{\tilde{\mathcal{J}}} \end{array} (\mathbf{H}(\tilde{\mathcal{F}}), \tilde{Q}_{\mathbf{H}}).$$

Here  $\mathcal{J}, \mathcal{R}, \tilde{\mathcal{J}}, \tilde{\mathcal{R}}$  are  $L_\infty$  maps,  $\mathcal{H}, \tilde{\mathcal{H}}$  –  $L_\infty$  homotopies and  $Q_{\mathbf{H}}, \tilde{Q}_{\mathbf{H}}$  – induced minimal  $L_\infty$  algebra structures on cohomology.

By the fiber product construction (17),  $i$  intertwines the  $L_\infty$  structures  $i \circ Q = \tilde{Q} \circ i: \text{Sym}_{\text{co}} \mathcal{F} \rightarrow \text{Sym}_{\text{co}} \tilde{\mathcal{F}}$ . This implies that the map (29) is a strict  $L_\infty$  isomorphism of minimal  $L_\infty$  algebras  $\mathbf{H}(\mathcal{F}), \mathbf{H}(\tilde{\mathcal{F}})$ . Also, relations (28) become

$$(31) \quad i \circ \mathcal{J} = \tilde{\mathcal{J}} \circ i_*, \quad i_* \circ \mathcal{R} = \tilde{\mathcal{R}} \circ i, \quad i \circ \mathcal{H} = \tilde{\mathcal{H}} \circ i.$$

Next—this is the key step borrowed from [5, Section 10.4.6]—we construct the  $L_\infty$  map

$$(32) \quad p = \mathcal{J} \circ (i_*)^{-1} \circ \tilde{\mathcal{R}}: \tilde{\mathcal{F}} \rightarrow \mathcal{F}.$$

It is an  $L_\infty$  quasi-inverse of the canonical inclusion (18). More precisely, one has the diagram of maps

$$(33) \quad \mathcal{H} \wr (\mathcal{F}, Q) \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{p} \end{array} (\tilde{\mathcal{F}}, \tilde{Q}) \wr \tilde{\mathcal{H}},$$

where  $i$  and  $p$  are mutually quasi-inverse  $L_\infty$  quasi-isomorphisms and  $\mathcal{H}, \tilde{\mathcal{H}}$  are the respective  $L_\infty$  homotopies, i.e., one has

$$(34) \quad \text{id} - p \circ i = Q\mathcal{H} + \mathcal{H}Q,$$

$$(35) \quad \text{id} - i \circ p = \tilde{Q}\tilde{\mathcal{H}} + \tilde{\mathcal{H}}\tilde{Q}.$$

These relations follow immediately from the construction (32) and from (31).

The package of maps (33), upon dualization and passing to the completion of symmetric algebras to algebras of smooth functions, gives rise to a chain equivalence of chain complexes

$$(36) \quad \mathcal{H} \wr (C^\infty(\mathcal{F}), Q) \begin{array}{c} \xrightarrow{i^*} \\ \xleftarrow{p^*} \end{array} (C^\infty(\tilde{\mathcal{F}}), \tilde{Q}) \wr \tilde{\mathcal{H}}.$$

<sup>14</sup>See [4, Section 6.4], [7] and [5, Section 10.3] on the homotopy transfer of  $\infty$ -algebras.

Thus, we have (a) of Definition 1.2 of weak equivalence between  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$ .

For (b) of Definition 1.2, we note that, by a straightforward extension from functions to differential forms<sup>15</sup> (a.k.a. tangent lift from  $\mathcal{F}, \tilde{\mathcal{F}}$  to the shifted tangent bundles  $T[1]\mathcal{F}, T[1]\tilde{\mathcal{F}}$ ), one gets a chain equivalence

$$(37) \quad \tilde{\mathcal{H}}^{\text{lifted}} \hookleftarrow (\Omega^\bullet(\mathcal{F}), \mathcal{L}_Q) \begin{array}{c} \xrightarrow{i^*} \\ \xleftarrow{p^*} \end{array} (\Omega^\bullet(\tilde{\mathcal{F}}), \mathcal{L}_{\tilde{Q}}) \hookrightarrow \tilde{\mathcal{H}}^{\text{lifted}}.$$

We know that, by fiber product construction, we have  $i^*\tilde{\omega} = \omega$ . We also have

$$(38) \quad p^*\omega = p^*i^*\tilde{\omega} = (\text{id} - \mathcal{L}_{\tilde{Q}}\tilde{\mathcal{H}}^{\text{lifted}} - \tilde{\mathcal{H}}^{\text{lifted}}\mathcal{L}_{\tilde{Q}})\tilde{\omega} = \tilde{\omega} + \mathcal{L}_{\tilde{Q}}\tilde{\beta}$$

with  $\tilde{\beta} = -\tilde{\mathcal{H}}^{\text{lifted}}\tilde{\omega}$ . Here we used  $\mathcal{L}_{\tilde{Q}}\tilde{\omega} = 0$  and (the tangent lift of) the equation (35). Note that  $\tilde{\beta}$  is a  $\delta$ -closed 2-form, since  $\delta\tilde{\omega} = 0$  and since  $\tilde{\mathcal{H}}^{\text{lifted}}$  commutes with  $\delta$  (cf. footnote 15). Thus, (b) of Definition 1.2 holds with  $\beta = 0$  and  $\tilde{\beta} = -\tilde{\mathcal{H}}^{\text{lifted}}\tilde{\omega}$ .

*Remark 2.7.* The construction given in this section is not compatible with locality near the hypersurface  $\Sigma$  in the sense of (c) of Theorem 2.4. We hope to find a proof of Conjecture 2.2 by presenting a weak equivalence that is local near  $\Sigma$ .

### 3. “SMEARING” HOMOTOPY IN DE RHAM THEORY

The following lemma is rephrased from Kontsevich-Soibelman, [4, section 6.5, lemma 2].<sup>16</sup>

**Lemma 3.1.** *Let  $N$  be an oriented (not necessarily compact)  $n$ -manifold and  $V \subset N \times N$  a tubular neighborhood of the diagonal  $\text{Diag} \subset N \times N$  with projection  $r: V \rightarrow \text{Diag}$ . Then:*

<sup>15</sup> The tangent lift is constructed as follows: we have “doubled” contractions (27),  $h \oplus h \hookleftarrow (T[1]\mathcal{F}, d_Q \oplus d_Q) \begin{array}{c} \xrightarrow{r \oplus r} \\ \xleftarrow{j \oplus j} \end{array} T[1]\mathbf{H}(\mathcal{F})$  and similar one for tilde-side. We think of  $T[1]\mathcal{F}$  as  $\mathcal{F} \oplus \mathcal{F}[1]$  with an extra differential  $\delta$  mapping between the two copies of  $\mathcal{F}$ . Then we pass to (completed) symmetric algebras of the duals and turn on the perturbation of the differential  $L_{d_Q} \rightarrow L_Q$  via homological perturbation lemma. This yields the contraction  $\mathcal{H}^{\text{lifted}} \hookleftarrow (\Omega^\bullet(\mathcal{F}), \mathcal{L}_Q) \begin{array}{c} \xrightarrow{\mathcal{R}^*} \\ \xleftarrow{\mathcal{J}^*} \end{array} (\Omega^\bullet(\mathbf{H}(\mathcal{F})), \mathcal{L}_{Q_H})$  and similar for tilde-side. Then we repeat the construction (32) at the level of  $T[1]\mathcal{F}, T[1]\tilde{\mathcal{F}}$ . This gives us the maps of the chain equivalence (37). We remark that all maps in (37) commute with  $\delta$  by construction, since the maps of the non-perturbed “doubled” contractions above commute with  $\delta$  and since the differential perturbation commutes with  $\delta$  also.

<sup>16</sup>See also [3].

- (1) *There exists a smooth closed  $n$ -form  $\rho$  on  $N \times N$  satisfying*
- $\text{supp}(\rho) \subset V$ ,
  - $r_*\rho = 1$  (the integral of  $\rho$  over each fiber of  $r$  is 1).<sup>17</sup>
- (2) *Denote  $R: \Omega_{\text{distr}}^\bullet(N) \rightarrow \Omega^\bullet(N)$  the integral operator from distributional to smooth forms determined by the integral kernel  $\rho$ . Then*

- *$R$  is a chain map w.r.t. the de Rham differential.*
- *Let  $\iota: \Omega^\bullet(N) \hookrightarrow \Omega_{\text{distr}}^\bullet(N)$  be the canonical inclusion of smooth forms into distributional forms. Then  $\iota R$  and  $R\iota$  are chain-homotopic to identity. More precisely, there exists a distributional form  $\chi \in \Omega_{\text{distr}}^{n-1}(N \times N)$  supported in  $V$  and smooth away from  $\text{Diag}$  such that  $d\chi = \delta_{\text{Diag}} - \rho$  (as distributional forms). As a consequence, one has*

$$(39) \quad \text{id}_{\Omega_{\text{distr}}^\bullet} - \iota R = d\kappa + \kappa d,$$

$$(40) \quad \text{id}_{\Omega^\bullet} - R\iota = d\kappa|_{\Omega^\bullet} + \kappa|_{\Omega^\bullet} d.$$

Here  $\kappa: \Omega_{\text{distr}}^\bullet(N) \rightarrow \Omega_{\text{distr}}^{\bullet-1}(N)$  is the operator defined by the integral kernel  $\chi$ .

We call the operator  $R$  a “smearing” (or “smoothing,” or “mollifying”) operator.

*Proof.* A form  $\rho$  satisfying conditions (1) can be constructed e.g. by taking a Mathai-Quillen representative  $\tau \in \Omega^n(\text{NDiag})$  of the Thom class of the normal bundle of  $\text{Diag}$  (see [8]) and choosing an embedding  $\phi: \text{NDiag} \hookrightarrow V$ .<sup>18</sup> Then one can define

$$\rho = \begin{cases} (\phi^{-1})^*\tau & \text{in } \phi(\text{NDiag}), \\ 0 & \text{outside } \phi(\text{NDiag}) \end{cases}$$

Likewise, one can construct the distributional form  $\chi$  as follows: first construct  $\bar{\chi} \in \Omega_{\text{distr}}^{n-1}(\text{NDiag})$  as  $\bar{\chi} = \pi_* s^* \tau$  where

$$\begin{array}{ccc} [1, +\infty) \times \text{NDiag} & \xrightarrow{s} & \text{NDiag} \\ \pi \downarrow & & \\ \text{NDiag} & & \end{array}$$

where  $s$  stretches the vector in the fiber of  $\text{NDiag} \rightarrow \text{Diag}$  by a factor in  $[1, +\infty)$ ;  $\pi$  is the projection to the second factor; by Stokes’ theorem one has  $d\bar{\chi} = \delta_{\text{zero section}} - \tau$ . Then one sets  $\chi = \phi_* \bar{\chi}$ .  $\square$

<sup>17</sup>If  $N$  is compact, an equivalent statement is that the cohomology class  $[\rho] \in H_c^n(V)$  is Poincaré dual to the homology class of the diagonal  $[\text{Diag}] \in H_n(V)$ .

<sup>18</sup>We denote the normal bundle by  $\mathbf{N}$  to avoid confusion with the manifold  $N$ .

## 4. ABELIAN CHERN–SIMONS THEORY

Abelian Chern–Simons theory in BV-BFV formalism (we refer to [1] for details) assigns to an oriented compact 3-manifold  $M$  the space of fields

$$\mathcal{F}_M = \Omega^\bullet(M)[1] \quad \ni \mathcal{A}$$

– the degree shifted de Rham complex, equipped with the quadratic BV action

$$S_M = \frac{1}{2} \int_M \mathcal{A} \wedge d\mathcal{A},$$

the linear cohomological vector field  $Q_M = \int_M d\mathcal{A} \frac{\delta}{\delta \mathcal{A}}$  and the constant  $(-1)$ -symplectic form  $\omega_M = -\frac{1}{2} \int_M \delta \mathcal{A} \wedge \delta \mathcal{A}$ .

To a surface  $\Sigma$  cutting  $M$  into  $M_1$  and  $M_2$ , the theory assigns the phase space

$$\Phi_\Sigma = \Omega^\bullet(\Sigma)[1] \quad \ni \mathcal{A}_\Sigma$$

equipped with the symplectic form  $\omega_\Sigma = \delta \left( \frac{1}{2} \int_\Sigma \mathcal{A}_\Sigma \wedge \delta \mathcal{A}_\Sigma \right)$ , the cohomological vector field  $Q_\Sigma = \int_\Sigma d\mathcal{A}_\Sigma \frac{\delta}{\delta \mathcal{A}_\Sigma}$  and the Hamiltonian  $S_\Sigma = \frac{1}{2} \int_\Sigma \mathcal{A}_\Sigma \wedge d\mathcal{A}_\Sigma$ . Projections  $\pi_{1,2}: \mathcal{F}_{M_{1,2}} \rightarrow \Phi_\Sigma$  are given by pulling back a form on  $M_{1,2}$  to  $\Sigma$ .

The canonical inclusion (18) is:

$$\mathcal{F}_M = \Omega^\bullet(M)[1] \quad \xrightarrow{i} \quad \tilde{\mathcal{F}}_{M,\Sigma} = \Omega^\bullet(M_1)[1] \times_{\Omega^\bullet(\Sigma)[1]} \Omega^\bullet(M_2)[1],$$

where left and right side are complexes w.r.t. the de Rham differential on  $M$ .

We construct a chain map  $p: \tilde{\mathcal{F}}_{M,\Sigma} \rightarrow \mathcal{F}_M$  as follows. Fix an embedding  $\psi: \mathbf{N}\Sigma \rightarrow M$  of the normal bundle of  $\Sigma$  into  $M$ , with image  $U = \psi(\mathbf{N}\Sigma)$  an open neighborhood of  $\Sigma$  (which can be made as thin as needed). We apply Lemma 3.1 to  $N = \mathbf{N}\Sigma$  and consider the corresponding forms  $\rho, \chi$ . Define a distributional  $n$ -form  $\hat{\rho}$  on  $M \times M$  by

$$\hat{\rho} = \begin{cases} (\psi^{-1} \times \psi^{-1})^* \rho & \text{in } U \times U, \\ \delta_{\text{Diag}_M} & \text{outside } U \times U \end{cases}$$

and define a distributional  $(n-1)$ -form  $\hat{\chi}$  on  $M \times M$  as

$$\hat{\chi} = \begin{cases} (\psi \times \psi)_* \chi & \text{in } U \times U, \\ 0 & \text{outside } U \times U \end{cases}$$

We define the map  $p: \tilde{\mathcal{F}}_{M,\Sigma} \rightarrow \mathcal{F}_M$  as the integral operator with kernel  $\hat{\rho}$  (thus,  $p$  is smoothing inside the neighborhood  $U$  of  $\Sigma$  and identity outside); since  $\hat{\rho}$  is a closed distributional form on  $M \times M$ ,  $p$  is indeed a chain map. Likewise, we define the chain homotopy  $\tilde{H}: \tilde{\mathcal{F}}_{M,\Sigma} \rightarrow \tilde{\mathcal{F}}_{M,\Sigma}[-1]$  as the integral operator with kernel  $\hat{\chi}$ . Denote

its restriction to  $\mathcal{F}_M$  by  $H$ . The chain homotopy identities (23), (24) are immediate consequences of (39), (40).

Thus we have the quadruple of maps (20) satisfying all points of the Theorem 2.4.

**4.1. Abelian  $BF$  theory.** The case of abelian  $BF$  theory is a trivial modification of the construction for abelian Chern–Simons. Now  $M$  can be of any dimension  $n$ ; the BV action is

$$S_M = \int_M \mathcal{B} \wedge d\mathcal{A}$$

and the space of fields consists of two copies of de Rham complex:

$$\mathcal{F}_M = \Omega^\bullet(M)[\mathfrak{p}] \oplus \Omega^\bullet(M)[n - \mathfrak{p} - 1] \ni (\mathcal{A}, \mathcal{B})$$

(for some fixed shift  $\mathfrak{p} \in \mathbb{Z}$ ) with differential  $d_Q = d \oplus d$  and with constant symplectic form  $\omega = (-1)^n \int_M \delta\mathcal{B} \wedge \delta\mathcal{A}$ .

The BFV phase space is

$$\Phi_\Sigma = \Omega^\bullet(\Sigma)[\mathfrak{p}] \oplus \Omega^\bullet(\Sigma)[n - \mathfrak{p} - 1]$$

and the projections  $\pi_{1,2}: \mathcal{F}_{M_{1,2}} \rightarrow \Phi_\Sigma$  are given by the pullback of  $\mathcal{A}$ - and  $\mathcal{B}$ -form from  $M_{1,2}$  to  $\Sigma$ . The canonical inclusion (18) is

$$\mathcal{F}_M = \Omega^\bullet(M)[\mathfrak{p}] \oplus \Omega^\bullet(M)[n - \mathfrak{p} - 1]$$

$$i \downarrow$$

$$\tilde{\mathcal{F}}_{M,\Sigma} = (\Omega^\bullet(M_1) \times_{\Omega^\bullet(\Sigma)} \Omega^\bullet(M_2)) [\mathfrak{p}] \oplus (\Omega^\bullet(M_1) \times_{\Omega^\bullet(\Sigma)} \Omega^\bullet(M_2)) [n - \mathfrak{p} - 1]$$

We proceed as in abelian Chern–Simons and set  $p: \tilde{\mathcal{F}}_{M,\Sigma} \rightarrow \mathcal{F}_M$  to be given by  $p$  from above (the integral operator with kernel  $\hat{\rho}$ ) acting on both  $\mathcal{A}$ - and  $\mathcal{B}$ -forms in  $\tilde{\mathcal{F}}_{M,\Sigma}$ . Likewise, we define the homotopies to be given by  $\tilde{H}$  and  $H$  from above (defined by the integral kernel  $\hat{\chi}$ ) acting diagonally on both components in  $\tilde{\mathcal{F}}_{M,\Sigma}$  or  $\mathcal{F}_M$ , respectively.

## 5. MASSLESS SCALAR FIELD IN THE SECOND-ORDER FORMALISM

As a warm-up for the  $\mathfrak{p}$ -form electrodynamics for general  $\mathfrak{p}$  (coming in Section 6 below), let us first discuss the case  $\mathfrak{p} = 0$ , i.e., massless scalar field.

Massless scalar field on a Riemannian  $n$ -manifold  $M$  is defined by the BV action  $S_M = \int_M \frac{1}{2} d\phi \wedge *d\phi$ . The space of bulk fields is the



Poincaré complex<sup>19</sup>

$$(41) \quad \mathcal{F}_M = \left( \begin{array}{ccc} \Omega^0(M) & \xrightarrow{d*^{-1}d} & \Omega^n(M) \\ \phi & & \phi^+ \end{array} \right)$$

with constant  $(-1)$ -symplectic form  $\omega_M = (-1)^n \int_M \delta\phi^+ \wedge \delta\phi$ . The boundary phase space for a closed  $(n-1)$ -submanifold  $\Sigma \subset M$  (splitting  $M$  into  $M_1$  and  $M_2$ ) is

$$\Phi_\Sigma = \underset{\phi_\Sigma}{\Omega^0(\Sigma)} \oplus \underset{P_\Sigma}{\Omega^{n-1}(\Sigma)}$$

with zero differential and 0-symplectic form  $\omega_\Sigma = - \int_\Sigma \delta\phi_\Sigma \wedge \delta P_\Sigma$ . The projection  $\pi: \mathcal{F}_{M_{1,2}} \rightarrow \Phi_\Sigma$  maps  $(\phi, \phi^+)$  to  $(\phi|_\Sigma, (*^{-1}d\phi)|_\Sigma)$ .

The fiber product (17) is

$$(42) \quad \tilde{\mathcal{F}}_{M,\Sigma} = \left( \begin{array}{ccc} \Omega^0(M_1) & \times & \Omega^0(M_2) \\ \Omega^0(\Sigma) \oplus \Omega^{n-1}(\Sigma) & & \end{array} \xrightarrow{d*^{-1}d} \Omega^n(M_1) \times \Omega^n(M_2) \right).$$

The first term of this complex is the space of functions on  $M$ , smooth on  $M_1$  and  $M_2$ , continuous and differentiable across  $\Sigma$ . The second term consists of  $n$ -forms on  $M$  smooth on  $M_1$  and  $M_2$  and no regularity condition at  $\Sigma$ .

We have, as always, the canonical inclusion  $i: \mathcal{F}_M \hookrightarrow \tilde{\mathcal{F}}_{M,\Sigma}$  of fields on  $M$  into the fiber product.

**5.1. Construction 1 (Hodge-theoretic).** For the map  $p: \tilde{\mathcal{F}}_{M,\Sigma} \rightarrow \mathcal{F}_M$  we take

$$p = e^{-\epsilon\Delta}$$

– the heat flow in time  $\epsilon > 0$ , acting on 0- and  $n$ -forms from the fiber product (42). Here  $\Delta = dd^* + d^*d$  is the Hodge Laplacian acting on forms on  $M$  (in the non-negative convention). We understand  $\epsilon$  as a regulator or smearing parameter.

*Remark 5.1.* Unlike the example of Section 4, here  $p$  is not identity away from a neighborhood of  $\Sigma$  – it is a smearing operator everywhere on  $M$ . Note that we cannot naively use  $p$  of Section 4 since it is not a chain map w.r.t. the differential  $d*d$  in the chain complexes (41,42).

We construct the chain homotopies  $H \hookrightarrow \mathcal{F}_M$  and  $\tilde{H} \hookrightarrow \tilde{\mathcal{F}}_{M,\Sigma}$  as follows:

$$(43) \quad H: \quad \Omega^0(M) \xleftarrow{(-1)^n(\text{id} - e^{-\epsilon\Delta})\Delta^{-1}*} \Omega^n(M)$$

<sup>19</sup>See footnote 23 below for the explanation of the reason why we write the differential as  $d*^{-1}d$  rather than  $d*d$ , and why the projection to boundary field  $\pi$  contains  $*^{-1}d$  rather than  $*d$ .

and

$$(44) \quad \tilde{H}: \quad \tilde{\Omega}^0(M, \Sigma) \xleftarrow{(-1)^n(\text{id} - e^{-\epsilon\Delta})\Delta^{-1}*} \tilde{\Omega}^n(M, \Sigma).$$

Here we denoted the two terms of the fiber product complex (42) by  $\tilde{\Omega}^{0,n}(M, \Sigma)$ . We understand the operator  $\Delta^{-1}$  as zero on harmonic forms and as the inverse of  $\Delta$  on the orthogonal complement of harmonic forms (where  $\Delta$  is strictly positive).

To summarize the result, the package of maps  $(i, p, H, \tilde{H})$  we just constructed satisfies (a) and (b) of Theorem 2.4, but does not satisfy (c).

## 5.2. Construction 2 (local near $\Sigma$ ).

**Lemma 5.2.** *Let  $M$  be a Riemannian  $n$ -manifold cut into  $M_1$  and  $M_2$  by a closed  $(n-1)$ -submanifold  $\Sigma \subset M$ . Let  $U \subset M$  be an open neighborhood of  $\Sigma$  in  $M$ . Then there exists a distributional 0-form  $\chi \in \Omega_{\text{distr}}^0(M \times M)$  such that*

- (a)  $\chi$  vanishes outside  $U \times U$ .
- (b)  $\chi$  is smooth on  $(M \times M) \setminus \text{Diag}_{\bar{U}}$ .
- (c)  $\chi$  satisfies

$$(d *^{-1} d)_1 \chi = \begin{cases} \delta_{\text{Diag}}^{(n,0)} + \text{smooth } (n,0)\text{-form} & \text{in } U \times U, \\ 0 & \text{outside} \end{cases}$$

and

$$(d *^{-1} d)_2 \chi = \begin{cases} \delta_{\text{Diag}}^{(0,n)} + \text{smooth } (0,n)\text{-form} & \text{in } U \times U, \\ 0 & \text{outside} \end{cases}$$

where where the subscript in  $(d *^{-1} d)_{1,2}$  means that the operator acts on the first or second argument of  $\chi$  (first or second copy of  $M$  in  $M \times M$ ). Bi-index  $(i, j)$  refers to the de Rham bi-degree of a form on  $M \times M$ , i.e.,  $\Omega^{i,j}(M \times M) = \Omega^i(M) \hat{\otimes} \Omega^j(M)$ ;  $\delta_{\text{Diag}}^{(i,j)}$  refers to the  $(i, j)$ -component of the delta-form on the diagonal in  $M \times M$ .

*Proof.* Choose a smooth function  $\mu$  on  $M \times M \setminus \text{Diag}_{\partial U}$  (a ‘‘bump function’’) such that

- (1)  $\mu = 0$  outside  $U \times U$ ,
- (2)  $\mu = 1$  in some open neighborhood  $V$  of  $\text{Diag}_U$  contained in  $U \times U$ .

Such a  $\mu$  can be easily constructed using a partition of unity: take some covering of  $M \times M \setminus \text{Diag}_{\partial U}$  by open sets  $\{v_\alpha\}$  such that if  $v_\alpha$  intersects

$\text{Diag}_U$  then it is contained in  $U \times U$  and let  $\{\psi_\alpha\}$  be a partition of unity subordinate to this cover. Then we can define

$$\mu = \sum_{\alpha \text{ s.t. } v_\alpha \cap \text{Diag}_U \neq \emptyset} \psi_\alpha.$$

Let  $G \in C^0(U \times U)$  be the Green function for the Laplacian  $\Delta$  on  $U$ , with Dirichlet boundary condition on  $\partial U$ . Then we define

$$\chi := \begin{cases} (-1)^n \mu \cdot G, & \text{in } U \times U, \\ 0 & \text{outside } U \times U \end{cases}$$

With this definition, properties (a), (b), (c) are obvious.  $\square$

We define the maps

$$H \curvearrowright \mathcal{F}_M \underset{p}{\overset{i}{\rightleftarrows}} \tilde{\mathcal{F}}_{M,\Sigma} \curvearrowleft \tilde{H}$$

as follows (recall that  $i$  is the canonical inclusion).

- We define the chain homotopy  $\tilde{H}: \tilde{\Omega}^n(M, \Sigma) \rightarrow \tilde{\Omega}^0(M, \Sigma)$  as the integral operator with kernel  $\chi$ , and the chain homotopy  $H: \Omega^n(M) \rightarrow \Omega^0(M)$  as the restriction of  $\tilde{H}$  to smooth forms.
- We define the operator  $p: \tilde{\Omega}^n(M, \Sigma) \rightarrow \Omega^n(M)$  as the integral operator with kernel  $\delta_{\text{Diag}}^{(n,0)} - (d *^{-1} d)_1 \chi$ .
- We define  $p: \tilde{\Omega}^0(M, \Sigma) \rightarrow \Omega^0(M)$  as the integral operator with kernel  $\delta_{\text{Diag}}^{(0,n)} - (d *^{-1} d)_2 \chi$ .

With this construction, Theorem 2.4 holds. Indeed, relation (21) is obvious. Relation (22) follows from the identity at the level of integral kernels of left and right side:

$$(d *^{-1} d)_1 (\delta_{\text{Diag}}^{(0,n)} - (d *^{-1} d)_2 \chi) = (d *^{-1} d)_2 (\delta_{\text{Diag}}^{(n,0)} - (d *^{-1} d)_1 \chi).$$

Relations (23), (24) follow from an obvious identity at the level of integral kernels:

$$\begin{aligned} & (d *^{-1} d)_1 \chi + (d *^{-1} d)_2 \chi = \\ & = \left( \delta_{\text{Diag}}^{(n,0)} + \delta_{\text{Diag}}^{(0,n)} \right) - \left( \delta_{\text{Diag}}^{(n,0)} - (d *^{-1} d)_1 \chi \right) - \left( \delta_{\text{Diag}}^{(0,n)} - (d *^{-1} d)_2 \chi \right). \end{aligned}$$

Locality is true by construction.

## 6. $\mathfrak{p}$ -FORM ELECTRODYNAMICS IN THE FIRST-ORDER FORMALISM

Consider theory of  $\mathfrak{p}$ -form field, with  $\mathfrak{p} \geq 0$ , defined in the first order formalism by the classical action

$$S_M^{\text{cl}} = \int_M B \wedge dA + \frac{1}{2} B \wedge *B$$

with  $(A, B) \in \Omega^p(M) \oplus \Omega^{n-p-1}(M)$ . In particular, for  $\mathbf{p} = 0$  this is the free massless scalar field; for  $\mathbf{p} = 1$  this is Maxwell's theory of electromagnetic field.

In BV formalism, the BV action is

$$(45) \quad S_M = \int_M B \wedge dA + A^+ \wedge dc_1 + \sum_{k=2}^{\mathbf{p}} c_{k-1}^+ \wedge dc_k + \frac{1}{2} B \wedge *B = \\ = \int_M \mathcal{B} \wedge d\mathcal{A} + \frac{1}{2} B \wedge *B,$$

where

$$(\mathcal{A} = c_{\mathbf{p}} + \cdots + c_1 + A + B^+ \quad , \quad \mathcal{B} = B + A^+ + c_1^+ + \cdots + c_{\mathbf{p}}^+) \in \\ \in \Omega^{0 \cdots \mathbf{p}+1}(M)[\mathbf{p}] \oplus \Omega^{n-\mathbf{p}-1 \cdots n}[n - \mathbf{p} - 1] =: \mathcal{F}_M.$$

Here  $\Omega^{i \cdots j}(M) = \bigoplus_{k=i}^j \Omega^k(M)$  is the truncated de Rham complex.

The action (45) is the free theory action corresponding to the Poincaré cochain complex structure on  $\mathcal{F}_M$  with differential  $d_Q$  given by

$$\begin{array}{ccccccccccc} \Omega^0_{c_{\mathbf{p}}} & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{\mathbf{p}-1}_{c_1} & \xrightarrow{d} & \Omega^{\mathbf{p}}_A & \xrightarrow{d} & \Omega^{\mathbf{p}+1}_{B^+} & & \\ & & & & & & & & & \nearrow * & \\ & & & & & & \Omega^{n-\mathbf{p}-1}_B & \xrightarrow{d} & \Omega^{n-\mathbf{p}}_{A^+} & \xrightarrow{d} & \Omega^{n-\mathbf{p}+1}_{c_1^+} & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n_{c_{\mathbf{p}}^+} \end{array}$$

In particular,  $d_Q(B) = dB + *B$ . Dually, at the level of coordinates on the space of fields, one has a more familiar formula  $Q_M(B^+) = dA + *B$ . The constant  $(-1)$ -symplectic structure on  $\mathcal{F}_M$  is  $\omega_M = (-1)^n \int_M \delta \mathcal{B} \wedge \delta \mathcal{A}$ .

The phase space that BV-BFV formalism assigns to a closed  $(n-1)$ -manifold  $\Sigma$  is

$$(46) \quad \Phi_{\Sigma} = \Omega^{0 \cdots \mathbf{p}}(\Sigma) \oplus \Omega^{n-\mathbf{p}-1 \cdots n-1}(\Sigma) \ni \\ \ni (\mathcal{A}_{\Sigma} = c_{\mathbf{p}} + \cdots + c_1 + A \quad , \quad \mathcal{B}_{\Sigma} = B + A^+ + c_1^+ + \cdots + c_{\mathbf{p}-1}^+)$$

with differential  $d_Q$  just the de Rham differential on both summands and the constant 0-symplectic form  $\int_{\Sigma} \delta \mathcal{B}_{\Sigma} \wedge \delta \mathcal{A}_{\Sigma}$ .

In the setup of Conjecture 2.3, with  $n$ -manifold  $M$  cut into parts  $M_1, M_2$  by a closed  $(n-1)$ -submanifold  $\Sigma$ , we have the canonical inclusion

$$(47) \quad \mathcal{F}_M \xrightarrow{i} \tilde{\mathcal{F}}_{M, \Sigma} = \mathcal{F}_{M_1} \times_{\Phi_{\Sigma}} \mathcal{F}_{M_2}.$$

**6.1. Construction 1: Hodge homotopy.** For the map  $p: \tilde{\mathcal{F}}_{M, \Sigma} \rightarrow \mathcal{F}_M$ , we take

$$(48) \quad p = e^{-\epsilon \Delta},$$

as in Section 5.1, but now acting on all forms of various degrees in the complex  $\tilde{\mathcal{F}}_{M,\Sigma}$ . It is easy to see that  $p$  is a chain map w.r.t.  $d_Q$ .

We define the chain homotopy  $H \zeta \mathcal{F}_M$  as

$$(49) \quad \begin{array}{ccccccc} \Omega^0 & \xleftarrow{\Xi} & \dots & \xleftarrow{\Xi} & \Omega^{p-1} & \xleftarrow{\Xi} & \Omega^p & \xleftarrow{\Xi} & \Omega^{p+1} \\ & & & & & & \swarrow & \searrow & \\ & & & & & & \Phi & & \Psi \\ & & & & & & \swarrow & \searrow & \\ & & & & \Omega^{n-p-1} & \xleftarrow{\Xi} & \Omega^{n-p} & \xleftarrow{\Xi} & \Omega^{n-p+1} & \dots & \xleftarrow{\Xi} & \Omega^n \end{array}$$

where

$$(50) \quad \begin{aligned} \Xi &= (\text{id} - e^{-\epsilon\Delta}) d^* \Delta^{-1}, \\ \Phi &= (-1)^{n+p+1} d *^{-1} \Delta^{-1} (\text{id} - e^{-\epsilon\Delta}) d, \\ \Psi &= (-1)^{n+p+1} (\text{id} - e^{-\epsilon\Delta}) \Delta^{-1} *. \end{aligned}$$

The chain homotopy  $\tilde{H} \zeta \tilde{\mathcal{F}}_{M,\Sigma}$  is defined by the same formulae, where now the operators (50) are understood as acting on forms with admissible singularities at  $\Sigma$  constituting the fiber product  $\tilde{\mathcal{F}}_{M,\Sigma}$ .

**Lemma 6.1.** *Relations (21), (22), (23), (24) hold.*

*Proof.* Relation (21) is obvious. Let us check (23). On  $\Omega^i$  with  $i \leq p-1$  or  $i \geq n-p+1$ , we have

$$d_Q H + H d_Q + p i = d \Xi + \Xi d + e^{-\epsilon\Delta} = (\text{id} - e^{-\epsilon\Delta}) (d d^* + d^* d) \Delta^{-1} + e^{-\epsilon\Delta} = \text{id}.$$

On  $\Omega^p \oplus \Omega^{n-p-1}$ , we have

$$\begin{aligned} (d_Q H + H d_Q + p i)(A, B) &= \\ &= \underbrace{((d \Xi + \Xi d + e^{-\epsilon\Delta}) A)}_{\text{id}} + \underbrace{(\Xi * + \Psi d) B}_0 + \underbrace{(\Xi d + \Phi * + e^{-\epsilon\Delta}) B}_{\text{id}} + \underbrace{\Phi d A}_0 \\ &= (A, B). \end{aligned}$$

On  $\Omega^{p+1} \oplus \Omega^{n-p}$ , we have

$$\begin{aligned} (d_Q H + H d_Q + p i)(B^+, A^+) &= \\ &= \underbrace{((d \Xi + * \Phi + e^{-\epsilon\Delta}) B^+)}_{\text{id}} + \underbrace{(* \Xi + d \Psi) A^+}_0 + \underbrace{(d \Xi + \Xi d + e^{-\epsilon\Delta}) A^+}_{\text{id}} + \underbrace{d \Phi B^+}_0 \\ &= (B^+, A^+). \end{aligned}$$

This finishes the proof of (23). Relation (24) is checked by the same computation (at the level of  $\tilde{\Omega}^\bullet$ ). Chain map property of  $p$ , (22) in fact follows from (24):  $p = \text{id} - [d_Q, \tilde{H}]$  implies (since  $d_Q^2 = 0$ )  $[d_Q, p] = 0$ .  $\square$

*Remark 6.2.* One can construct the chain homotopy (49) together with the map (48) from topological quantum mechanics (TQM) in the sense of [6] with evolution operator

$$\mathcal{U}(t, dt) = e^{-t[d_Q, \mathbb{G}] - dt \mathbb{G}} \in \Omega^\bullet(\mathbb{R}_+) \otimes \text{End}(\mathcal{F})$$

– a  $(d_t + \text{ad}_{d_Q})$ -closed operator-valued form on  $\mathbb{R}_+$ , where the degree  $-1$  operator  $\mathbb{G}$  (the “non-normalized homotopy”) is chosen to be:

$$\mathbb{G}: \begin{array}{ccccccc} \Omega^0 & \xleftarrow{d^*} & \dots & \xleftarrow{d^*} & \Omega^{p-1} & \xleftarrow{d^*} & \Omega^p & \xleftarrow{d^*} & \Omega^{p+1} \\ & & & & & & \swarrow \pm d^* & & \searrow \pm * \\ & & & & & & \Omega^{n-p-1} & \xleftarrow{d^*} & \Omega^{n-p} & \xleftarrow{d^*} & \Omega^{n-p+1} & \dots & \xleftarrow{d^*} & \Omega^n \end{array}$$

Note that with this choice the Hamiltonian of the TQM  $\hat{H}: = [d_Q, \mathbb{G}]$  is just the Laplace operator  $\Delta$ . Then, one has

$$p i = \mathcal{U}|_{t=\epsilon} = e^{-t\hat{H}}$$

and

$$H = - \int_0^\epsilon \mathcal{U} = \mathbb{G} \hat{H}^{-1} (1 - e^{-\epsilon \hat{H}})$$

(where we integrate the 1-form component of  $\mathcal{U}$  along  $\mathbb{R}_+$ ). The relation (23) follows by Stokes’ theorem for the integral over  $[0, \epsilon]$ , using closedness of  $\mathcal{U}$ .

We remark that  $\mathbb{G}$  does not square to zero.

**6.2. Construction 2: homotopy local near the interface.** As in Section 5.2, fix an open neighborhood  $U \subset M$  of  $\Sigma$ . Fix a function  $\mu$  on  $M \times M$  as in the proof of Lemma 5.2. Let  $G \in \bigoplus_{i=0}^n \Omega_{C^0}^{i, n-i}(U \times U)$  be the integral kernel of the inverse of the Laplace operator  $\Delta_U$  acting on forms on  $U$  with Dirichlet boundary conditions at  $\partial U$ ;  $G$  is smooth on  $U \times U$  away from the diagonal, where it is only continuous (hence the subscript  $C^0$ , for continuous forms).

Let  $\Gamma \in \bigoplus_{i=0}^n \Omega_{C^0}^{i, n-i}(M \times M)$  be the form on  $M \times M$  defined as

$$\Gamma = \begin{cases} \mu \cdot G & \text{in } U \times U, \\ 0 & \text{outside } U \times U \end{cases}$$

Let  $\Delta_\mu^{-1}: \Omega^\bullet(M) \rightarrow \Omega^\bullet(M)$  be the operator with integral kernel  $\Gamma$ .<sup>20</sup> We remark that  $\Delta_\mu^{-1}$  commutes with the Hodge star (since  $*_1 \Gamma = \mu *_1 G = \mu *_2 G = *_2 \Gamma$  where  $*_1 G = *_2 G$  follows from the fact that  $\Delta_U$

<sup>20</sup>We are thinking of it as a modification of  $\Delta_U^{-1}$  provided by the bump function  $\mu$ , hence the notation; we are not thinking of  $\Delta_\mu^{-1}$  as an inverse of some operator  $\Delta_\mu$ .

– and its inverse – commutes with  $*$ ), but generally does not commute with  $d$ .

We will first define the chain homotopy operators and then, using them, construct the map  $p: \tilde{\mathcal{F}}_{M,\Sigma} \rightarrow \mathcal{F}_M$ .

We define the chain homotopy  $H \subset \mathcal{F}_M$  as

$$(51) \quad \begin{array}{ccccccc} \Omega^0 & \xleftarrow{\Xi} & \dots & \xleftarrow{\Xi} & \Omega^{p-1} & \xleftarrow{\Xi} & \Omega^p & \xleftarrow{\Xi} & \Omega^{p+1} \\ & & & & & & \swarrow \Phi & & \searrow \Psi \\ & & & & & & \Omega^{n-p-1} & \xleftarrow{\Xi'} & \Omega^{n-p} & \xleftarrow{\Xi'} & \Omega^{n-p+1} & \dots & \xleftarrow{\Xi'} & \Omega^n \end{array}$$

where

$$(52) \quad \begin{aligned} \Xi &= \Delta_\mu^{-1} d^*, \quad \Xi' = d^* \Delta_\mu^{-1}, \\ \Phi &= (-1)^{n+p+1} d *^{-1} \Delta_\mu^{-1} d, \quad \Psi = (-1)^{n+p+1} * \Delta_\mu^{-1}. \end{aligned}$$

The chain homotopy  $\tilde{H} \subset \tilde{\mathcal{F}}_{M,\Sigma}$  is defined by the operators with the same integral kernels as in  $H$ , now acting on forms constituting  $\tilde{\mathcal{F}}_{M,\Sigma}$ .

We construct the map  $p: \tilde{\mathcal{F}}_{M,\Sigma} \rightarrow \mathcal{F}_M$  as  $p = \text{id} - d_Q \tilde{H} - \tilde{H} d_Q$  (here the important point is that the image of  $p$  is in smooth forms  $\mathcal{F}_M \subset \tilde{\mathcal{F}}_{M,\Sigma}$ ). Explicitly,  $p$  looks as follows:

$$(53) \quad p = \begin{cases} \text{id} - d \Delta_\mu^{-1} d^* - \Delta_\mu^{-1} d^* d & \text{on } \tilde{\Omega}^i & \text{with } i \leq p, \\ \text{id} - d d^* \Delta_\mu^{-1} - d^* \Delta_\mu^{-1} d & \text{on } \tilde{\Omega}^i & \text{with } i \geq n - p, \\ \text{id} - d \Delta_\mu^{-1} d^* - d^* \Delta_\mu^{-1} d & \text{on } \tilde{\Omega}^i & \text{with } i \in \{p+1, n-p-1\} \end{cases}$$

A note on why  $\Phi$  and  $\Psi$  components of the homotopy are crucial for the construction to work: here for the case  $i = n - p$ , it is important that maps (52) satisfy  $*\Xi' + d\Psi = 0$ ; for  $i = n - p - 1$ , it is important that  $\Xi* + \Psi d = 0$ ; for  $i = p$  and  $i = p + 1$ , it is important that  $d\Phi = \Phi d = 0$ . If not for these properties,  $p$  would not be diagonal w.r.t. components of  $\mathcal{F}_M$ , e.g.,  $p$  acting on  $\tilde{\Omega}^p$  could have a component in  $\tilde{\Omega}^{n-p-1}$ . Even more importantly, these off-diagonal components of  $p$  would fail to map to smooth forms (e.g. if one would try to set  $\Phi$  or  $\Psi$  to zero).

The integral kernel of each component of  $p$  is:

- zero in  $V \subset U \times U$  – a neighborhood of  $\text{Diag}_U$  where  $\mu = 1$  (see the proof of Lemma 5.2),<sup>21</sup>

<sup>21</sup>Indeed, in  $V$  the kernels of  $\Delta_\mu^{-1}$  and of the true Green function  $\Delta_U^{-1}$  coincide and  $\Delta_U^{-1}$  commutes both with  $d$  and with  $*$ , thus, by inspecting (53), the kernel of each component vanishes in  $V$ .

- smooth in  $U \times U$  (smoothness in  $U \times U \setminus V$  is obvious and in  $V$  the kernel of  $p$  is smooth by the previous point),
- $\delta_{\text{Diag}}$  outside  $U \times U$ .

These properties in particular imply the claim made above, that  $p$  takes forms in  $\tilde{\mathcal{F}}_{M,\Sigma}$  in smooth forms in  $\mathcal{F}_M$ .

This finishes the proof of Theorem 2.4 in the present example.

*Remark 6.3.* One can construct an  $(i, p, H, \tilde{H})$ -package for abelian Chern–Simons (or abelian  $BF$ ) – as an alternative to the construction of Section 4 – from the formulae above:

- (Hodge-theoretic version.) One can set  $p = e^{-\epsilon\Delta}$  and homotopies  $H$  and  $\tilde{H}$  to be given by the operator  $\Xi = (\text{id} - e^{-\epsilon\Delta}) d^* \Delta^{-1}$  from (50).
- (Local-near- $\Sigma$  version.) One can set  $H$  and  $\tilde{H}$  to be given by  $\Xi = \Delta_\mu^{-1} d^*$  from (52) and  $p = \text{id} - d\Xi - \Xi d$ .<sup>22</sup>

**6.3. Second-order formalism.** The same model— $\mathfrak{p}$ -form electrodynamics—cast in the second-order formalism has BV action

$$S_M = \int_M \frac{1}{2} dA \wedge *dA + A^+ \wedge dc_1 + \sum_{k=2}^p c_{k-1}^+ \wedge dc_k.$$

The Poincaré complex of fields  $\mathcal{F}_M$  is<sup>23</sup>

$$\Omega_{c_p}^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega_{c_1}^{p-1} \xrightarrow{d} \Omega_A^p \xrightarrow{d *^{-1} d} \Omega_{A^+}^{n-p} \xrightarrow{d} \Omega_{c_1^+}^{n-p+1} \xrightarrow{d} \dots \xrightarrow{d} \Omega_{c_p^+}^n.$$

The phase space  $\Phi_\Sigma$  is the same as above, (46); the projection  $\mathcal{F}_{M_1,2} \rightarrow \Phi_\Sigma$  is the pullback of forms to  $\Sigma$ , except for  $\mathfrak{p}$ -forms where  $\pi(A) = (A|_\Sigma, (*^{-1}dA)|_\Sigma)$ .

We again have the canonical inclusion  $i: \mathcal{F}_M \hookrightarrow \tilde{\mathcal{F}}_{M,\Sigma} = \mathcal{F}_{M_1} \times_{\Phi_\Sigma} \mathcal{F}_{M_2}$  and we again have two variants of the rest of the  $(i, p, H, \tilde{H})$  package.

**Hodge-theoretic version.** We set  $p = e^{-\epsilon\Delta}: \tilde{\mathcal{F}}_{M,\Sigma} \rightarrow \mathcal{F}_M$  and we set the chain homotopy  $H$  to be

$$H: \quad \Omega^0 \xleftarrow{\Xi} \dots \xleftarrow{\Xi} \Omega^{p-1} \xleftarrow{\Xi} \Omega^p \xleftarrow{-\Psi} \Omega^{n-p} \xleftarrow{\Xi} \Omega^{n-p+1} \xleftarrow{\Xi} \dots \xleftarrow{\Xi} \Omega^n,$$

<sup>22</sup>Alternatively, one may replace  $\Xi = \Delta_\mu^{-1} d^*$  with  $\Xi' = d^* \Delta_\mu^{-1}$  from (52) in this construction.

<sup>23</sup> A note on signs: to explain why we write  $d *^{-1} d$  rather than  $d * d$  as a term in  $d_Q$  (and also why  $*^{-1}dA$  appears in  $\pi$  rather than  $*dA$ ), one needs to look at the full BV-BFV package for the second-order  $\mathfrak{p}$ -form theory. With  $S_M$  as above,  $\omega_M = (-1)^n \int_M \delta \bar{\mathcal{B}} \wedge \delta \bar{\mathcal{A}}$ ,  $Q_M = \int_M d \bar{\mathcal{A}} \frac{\delta}{\delta \bar{\mathcal{A}}} + (d \bar{\mathcal{B}} + d *^{-1} dA) \frac{\delta}{\delta \bar{\mathcal{B}}}$  and  $\alpha_{\partial M} = (-1)^{n-1} \int_{\partial M} \bar{\mathcal{B}} \wedge \delta \bar{\mathcal{A}} + (*^{-1}dA) \wedge \delta A$ , one has the main BV-BFV relation (16). Here we denoted  $\bar{\mathcal{A}} = c_p + \dots + c_1 + A$ ,  $\bar{\mathcal{B}} = A^+ + c_1^+ + \dots + c_p^+$  the truncated bulk superfields.



with  $\Psi, \Xi$  as in (50). Likewise, we set  $\tilde{H}$  to be given by the same operators (i.e. defined by the same integral kernels) acting on forms in  $\tilde{\mathcal{F}}_{M,\Sigma}$ .

**Local-near- $\Sigma$  version.** We set  $H$  to be given by

$$H : \quad \Omega^0 \xleftarrow{\Xi} \dots \xleftarrow{\Xi} \Omega^{\mathbf{p}-1} \xleftarrow{\Xi} \Omega^{\mathbf{p}} \xleftarrow{-\Psi} \Omega^{n-\mathbf{p}} \xleftarrow{\Xi'} \Omega^{n-\mathbf{p}+1} \xleftarrow{\Xi'} \dots \xleftarrow{\Xi'} \Omega^n,$$

where now the component operators  $\Psi, \Xi, \Xi'$  are as in (52);  $\tilde{H}$  is again given by the same operators acting on  $\tilde{\Omega}$ -forms. We define  $p$  by formulae (53) where we forget the components in  $\Omega^i$  with  $i = \mathbf{p} + 1, i = n - \mathbf{p} - 1$ , as those are absent (integrated out) in the second-order formulation of the  $\mathbf{p}$ -form theory.

In both versions, items (a), (b) of Theorem 2.4 hold. Second version additionally satisfies item (c) – locality.

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