

SYMPLECTIC MICROGEOMETRY IV: QUANTIZATION

ALBERTO S. CATTANEO, BENOIT DHERIN, AND ALAN WEINSTEIN

ABSTRACT. We construct a special class of semiclassical Fourier integral operators whose wave fronts are the symplectic micromorphisms of [8]. These operators have very good properties: they form a category on which the wave front map becomes a functor into the cotangent microbundle category, and they admit a total symbol calculus in terms of symplectic micromorphisms enhanced with half-density germs. This new operator category encompasses the semi-classical pseudo-differential calculus and offers a functorial framework for the semi-classical analysis of the Schrödinger equation. We also comment on applications to classical and quantum mechanics as well as to a functorial and geometrical approach to the quantization of Poisson manifolds.

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1. INTRODUCTION

From an analytical point of view, symplectic geometry is the geometry underlying the calculus of Fourier integral operators (FIO's) [2, 15, 16, 18, 27]. The present article is concerned with developing a similar calculus in the context of microsymplectic geometry [8]. Since the latter enjoys much better functorial properties than its macro analog, it is not too surprising that these good properties persist in the associated operator calculus.

The geometry of canonical relation composition in the symplectic “category” is central to symplectic geometry itself in many ways [32]. This “geometric calculus” also plays an important role in the calculus of FIO's through a canonical relation associated to each FIO: its wave front. The geometry of the wave front usually contains a lot of information about the class of FIO's associated to it. The reader will find

detailed treatments of this relationship in the standard literature on FIO's [15, 16, 18, 19, 27]). Here, we are mostly concerned with functorial and categorical aspects of this relationship between the FIO's and their wave fronts. Namely, just as for canonical relations, the composition of two FIO's usually fails to be a FIO. Now, a well-behaved composition of the wave front canonical relations, if some restriction is imposed on them, can guarantee a well-behaved composition of the operators in the corresponding FIO classes. In this sense, the symplectic geometry of the wave fronts controls the categorical and functorial aspects of the FIO calculus.

In [8], we constructed a category of particularly well-behaved germs of canonical relations: the symplectic micromorphisms. This article studies the corresponding class of (semi-classical) FIO's, which inherits, for the most part, the good properties of the symplectic micromorphism composition. It turns out that this class of operators encompasses and extends the whole calculus of pseudo-differential operators in the semi-classical limit. Moreover, the hamiltonian flows of classical dynamics can be described, for asymptotically small times, in terms of some symplectic micromorphisms satisfying purely algebraic relations mimicking time translations at a purely categorical level. Upon quantization, we recover the Schrödinger flow of quantum mechanics in the semi-classical limit in terms of our semi-classical FIO's. This approach to quantization is very related to the considerations surrounding the notion of "thick morphism" developed by Voronov in [28, 29, 30] and by Khudaverdian and Voronov in [21]. Their work, done in the context of supermanifolds, was originally aimed at an understanding of L_∞ morphisms between homotopy structures, but many of their results are parallel to ours when restricted to the case of ordinary manifolds. In addition, many of our results have also been used in [11, 12, 13] by Menicattini and one of the authors to quantize momentum maps as well as the underlying group actions. In [14], it has been used by Wagemann and one of the authors to quantize Leibniz algebras.

One defining trait of Fourier integral operators is that they admit explicit representations in terms of oscillatory integrals, once given a generating family for their wave fronts. Symplectic micromorphisms possess canonical generating families (up to a choice of an exponential map germ on the smooth manifolds) with very good properties (stated in paragraph 2.4.2) which allow one to define a total symbol calculus for their corresponding semi-classical FIO's. These canonical generating families are obtained as deformations of generating families for

cotangent lifts. Locally, such a semiclassical FIO

$$Q_{\hbar}(a, f, \phi) : L_{\hbar}^2(\mathbb{R}^m) \longrightarrow L_{\hbar}^2(\mathbb{R}^n)$$

may be written in integral form similar to that for pseudo-differential operators

$$(1.1) \quad (Q_{\hbar}u)(y) := (2\pi\hbar)^{-\frac{m+n}{2}} \int a(p, y) e^{\frac{i}{\hbar}(\langle p, \phi(y) - x \rangle + f(p, y))} u(x) dp dx,$$

where $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a smooth map, a is an \hbar -dependent smooth function (on the pullback bundle $\phi^*(T^*\mathbb{R}^m)$) called the symbol of the operator, and the phase is the generating function of a symplectic micromorphism from $T^*\mathbb{R}^m$ to $T^*\mathbb{R}^n$ with core map ϕ . (See section 2.5.1 for a precise definition of the semiclassical intrinsic Hilbert space $L_{\hbar}^2(\mathbb{R}^n)$.) We recover pseudo-differential operators when $m = n$, $\phi = \text{id}$ and $f = 0$, in which case the corresponding symplectic micromorphism is the identity.

In this paper, we will mostly focus on the semi-classical limit of such operators, using the integral symbol "s.c. \int " instead of " \int " to remind us of this fact. The semi-classical limit is concerned with equivalence classes of such operators that have the same asymptotic behavior when the parameter \hbar in the phase of the oscillatory integral goes to zero. Depending on the problem at hand, one may be interested in asymptotics of order \hbar^N for some fixed N ; here, we will be interested in asymptotics modulo \hbar^∞ , which roughly means that we consider two FIO's equivalent if they have the same complete asymptotic expansions in \hbar as $\hbar \rightarrow 0$.

An application of the stationary phase principle, Theorem (1.1) shows that the semi-classical limit of an FIO is controlled by the germ of its wave front $\text{WF}(Q_{\hbar})$ and of its symbol around $\text{gr } \phi$ (the graph of the symplectic micromorphism core map). The symplectic microgeometry terminology introduced in [8] is a very convenient language to deal with the semi-classical limit of these FIO's.

Let us recall some basic definitions of microgeometry as introduced in [8]. A **microfold** $[M, A]$ (called a "local manifold pair" in [31]) is an equivalence class of manifold pairs (M, A) , where A is a closed submanifold of M such that two pairs (M_1, A) and (M_2, A) are equivalent if there exists a third one (M_3, A) for which M_3 is simultaneously an open submanifold of both M_1 and M_2 . A **symplectic microfold** is a microfold $[M, A]$ such that M is a symplectic manifold and A is a lagrangian submanifold; a **lagrangian submicrofold** $[L, S]$ of $[M, A]$ is a submicrofold (i.e. a microfold such that $L \subset M$ and $S \subset A$) such that L is a lagrangian submanifold.

This paper initiates the study of Fourier integral operators for which the semi-classical limit is controlled by a lagrangian submicrofold of its wave front that satisfies the transversality condition in [8], or, in other words, by a symplectic micromorphism.

Outline of the paper. In Section 2, we give a brief introduction to the calculus of semi-classical Fourier integral operators. In particular, we review the central notion of generating families for lagrangian submanifolds. Example 2 recalls a standard construction for generating families of conormal bundles which depends only on a choice of a tubular neighborhood. This is the central construction on which we are going to build on in this paper. We also stress the functorial aspects of the FIO calculus, and we introduce the notion of a “Fourier category” associated to any always-well-composing collection of wave fronts. In paragraph 2.4.2, we isolate special conditions for the wave fronts in terms of their generating families guaranteeing a very convenient integral representation for their associated FIO’s.

In Section 3, we focus on semi-classical FIO’s whose wave fronts are cotangent lifts of smooth maps. These canonical relations form a category, and so do their associated FIO’s. We show how to use the standard generating family construction for conormal bundles in the context of cotangent lifts using exponential map germs to construct the tubular neighborhood required by the construction. We show that the domain of such a generating family admits a canonical vertical half-density. This fact is very important to us, since it reduces the usual ambiguity of the FIO integral representation to only one thing: the choice of exponential germs on the underlying manifolds.

In Section 4, we extend the class of allowed wave fronts from cotangent lifts to all symplectic micromorphisms. To do so, we first introduce a notion of deformation for conormal bundles and their generating families. Then, we show that symplectic micromorphisms are in one-to-one correspondence with deformations of cotangent lifts generating families once an exponential map germ has been fixed. This allows us to extend the integral representation of Section 3 to FIO’s whose wave fronts are general symplectic micromorphisms. We study the local theory of these operators, with several examples; in particular, we show that semi-classical pseudo-differential operators fall into our class. Moreover, we give explicit formulas for the operator composition in this local setting; this involves a composition formula for the wave front generating families as well as for the total symbols of these operators.

The presence of a canonical exponential in the local setting allows us to identify our category of FIO's with the category of enhanced symplectic micromorphisms between cotangent bundles of \mathbb{R}^n for $n = 1, 2, \dots$, where an enhanced symplectic micromorphism is a symplectic micromorphism carrying near its core a half-density germ corresponding to the total symbol of the operator. Namely, in this context, we have two inverse functors: the quantization functor Q_h that associates to an enhanced symplectic micromorphism a semi-classical FIO through the integral representation as in (1.1) and the total symbol functor σ that associates to the operator (1.1) its wave front (in the form of its generating function f) enhanced with the total symbol a (identified in the local case with a smooth function germ on $\phi^*(T^*\mathbb{R}^m)$ around the zero section).

In Section 5, we comment on applications and further directions. In particular, we explain how to extend the quantization and total symbol functors, which are present in the local setting, to a semi-classical FIO calculus over any smooth manifold category enriched with some additional geometric structures sufficient to allow the construction of exponential map germs on the manifolds in a canonical way. We continue by relating the calculus developed here to the quantization of Poisson manifolds via oscillatory integrals and symplectic groupoids [20, 33, 34]. (In particular the generating function of the formal symplectic groupoid integrating a Poisson structure obtained in [7] can be understood as the jet of the generating function of the quantizing FIO in this setting.) Finally, we show how the small-time asymptotics of classical hamiltonian mechanics can be expressed in a purely categorical way in the framework of symplectic microgeometry. In particular, we show how classical flows can be modeled as the action of a special monoid in the microsymplectic category, the energy monoid, on the hamiltonian system's phase-space; enhancement and quantization of the action symplectic micromorphism recovers the usual unitary Schrödinger flows of quantum mechanics. We also explain how symmetries can be modeled very naturally in this framework, both at the classical and quantum level.

2. CATEGORIES OF FOURIER INTEGRAL OPERATORS

In this section, we give a brief presentation of the theory of Fourier integral operators. We focus mostly on the categorical and geometrical aspects of this calculus since they are the main concern for us in this paper, referring the reader to standard texts [15, 16, 18, 19, 27] on the subject for the more analytical aspects.

2.1. Categories of canonical relations.

2.1.1. *The symplectic “category”.* Let M_i , $i = 1, 2, 3$, be three symplectic manifolds, and let L_i ($i = 1, 2$) be a canonical relation from M_i to M_{i+1} , i.e., a closed lagrangian submanifold of the symplectic manifold product $\overline{M}_i \times M_{i+1}$, where \overline{M} denotes the symplectic manifold with opposite symplectic form $-\omega_M$. One can compose L_1 with L_2 as binary relations yielding the subset $L_2 \circ L_1$ of $\overline{M}_1 \times M_3$. This composition fails in general to be a lagrangian submanifold, or even a submanifold; however, there are many examples for which it does. For instance, we have the well know proposition:

Proposition 1. *A sufficient condition for the set-theoretic composition of the canonical relations L_1 and L_2 to be a canonical relation is that the intersection in $\overline{M}_1 \times M_2 \times \overline{M}_2 \times M_3$ of $L_1 \times L_2$ with $M_1 \times \Delta_{M_2} \times M_3$ (where Δ_M denotes the diagonal in $M \times M$) is transversal and properly embedded in $M_1 \times M_3$ via the canonical factor projection. In this case, we say that the canonical relations have **strongly tranverse composition**.*

We denote by **Syml^{ext}** the (extended) symplectic “category” whose objects are cotangent bundles and whose morphisms are taken to be canonical relations between them. The quotation marks are there to stress that this is not a category in the usual sense since composition is not always defined. However, this “category” contains a honest subcategory formed by the cotangent lifts as described below.

2.1.2. *Schwartz transform and cotangent lifts.* The canonical relations from T^*M to T^*N can be put in one-to-one correspondence with the lagrangian submanifolds of $T^*(M \times N)$ via the Schwartz transform (see [2]),

$$\mathcal{S} : \overline{T^*M} \times T^*N \longrightarrow T^*(M \times N),$$

which is the symplectomorphism that sends $((p_1, x_1), (p_2, x_2))$ to $(-p_1, p_2, x_1, x_2)$.

Now, to any smooth map $M \xleftarrow{\phi} N$, we can associate a special canonical relation, its cotangent lift, which we will consider going in the opposite direction to ϕ ,

$$T^*\phi : T^*M \longrightarrow T^*N,$$

by pulling back the conormal bundle¹ of $\text{gr } \phi$, seen as a submanifold of $T^*(M \times N)$, via the Schwartz transform:

$$T^*\phi := \left\{ \left((p_1, \phi(x_2)), ((T_{x_2}\phi)^*p_1, x_2) \right) : (p_1, x_2) \in \phi^*(T^*M) \right\}.$$

The collection \mathcal{C} of all cotangent lifts is a subcategory of $\mathbf{Symp}^{\text{ext}}$ which is a true category; namely, we always have that

$$T^*\phi_2 \circ T^*\phi_1 = T^*(\phi_1 \circ \phi_2).$$

2.2. Generating families.

2.2.1. *Generating functions.* An **exact lagrangian embedding** $\lambda : \Sigma \hookrightarrow T^*X$ is a lagrangian embedding for which $\lambda^*\theta = dS$ for some $S \in C^\infty(\Sigma)$, where θ is the Liouville 1-form on T^*X .

Composing λ with the bundle projection $\pi : T^*X \rightarrow X$, we obtain a map π_Σ from Σ to X . **Ante-caustic points** are elements of Σ at which $T\pi_\Sigma$ is not an isomorphism.²

When π_Σ is a diffeomorphism, we say that the lagrangian submanifold $\lambda(\Sigma)$ is **projectable**, in which case the differential $dS_\Sigma : X \rightarrow T^*X$ of $S_\Sigma := S \circ \pi_\Sigma^{-1}$ parametrizes the lagrangian submanifold $\lambda(\Sigma)$. The function S is called a **generating function** of the lagrangian submanifold. (It is well-defined up to a constant on each component of Σ .) Conversely, to any function $S \in C^\infty(X)$, we can associate the projectable lagrangian submanifold $\text{Im } dS \subset T^*X$ that has S as a generating function; explicitly,

$$\text{Im } dS := \{(dS(x), x) : x \in X\}.$$

There are, however, many interesting non-projectable lagrangian submanifolds. For instance, the conormal bundle N^*C of a (non-open) submanifold $C \subset X$ is highly non-projectable in the sense that all points are ante-caustic points: if we consider the lagrangian embedding given by the inclusion

$$\iota_C : N^*C \longrightarrow T^*X,$$

we see that the preimage of $\pi \circ \iota_C$ at any point $c \in C$ consists of the whole fiber N_c^*C . As a consequence cotangent lifts are also non-projectable, since they are conormal bundles to the graph of the underlying map.

¹Recall that the conormal bundle N^*S of a submanifold $S \subset X$ is the lagrangian submanifold of T^*X consisting of the covectors to X based along S and vanishing on TS .

²We have chosen this term because the images of these points under π_Σ are known as caustic points. (Although the ante-caustic points play a key role in symplectic geometry, we have not found another concise term for them in the literature.)

For these lagrangian submanifolds with antecaustic points, there is still a notion of generating function. The price to pay, however, is the introduction of additional variables for the generating functions through a fibration $p : B \rightarrow X$ that “unfolds” the lagrangian submanifold at antecaustic points. This leads to the notion of generating family.

2.2.2. Generating families. A submersion $p : B \rightarrow X$ together with a smooth function $S \in C^\infty(B)$ defines two lagrangian submanifolds: the lagrangian submanifold $\text{Im } dS$ in T^*B whose generating function is S (we regard it as a canonical relation from the point to T^*B) and the cotangent lift T^*p , which we can see as a canonical relation from T^*B to T^*X . If these lagrangian submanifolds have a strongly transversal composition³ their composition is a lagrangian submanifold of T^*X . A **generating family** for a lagrangian submanifold L in T^*X is a triple (B, p, S) as above such that

$$L = T^*p \circ \text{Im } dS.$$

Given a function $S \in C^\infty(B)$ and a fibration $p : B \rightarrow X$, the canonical relations $\text{Im } dS$ and T^*p have a strongly transversal composition if and only if the first factor projection π_B of T^*p on T^*B is transversal to $\text{Im } dS$ and the second factor projection π_X of T^*p on T^*X becomes a proper embedding when restricted to the points $y \in T^*p$ such that $\pi_B(y) \in \text{Im } dS$. Observe that the intersection $\text{Im } \pi_B \cap \text{Im } dS$ consists of the points such that

$$(2.1) \quad (dS(b), b) = (T_b^*p(\eta), b),$$

for some $\eta \in T_{\pi(b)}^*X$. Since $\text{Im } \pi_B$ is the annihilator of the vertical bundle of the fibration p (i.e. all covectors vanishing on the subbundle $\ker p_*$ of TB), a point of $(dS(b), b)$ is in $\text{Im } \pi_B$ iff the vertical part of dS vanishes⁴ at b . In the case of strongly transversal composition, the set Σ of all points in B where this happens is a submanifold, called the **fiber critical submanifold** of the generating family. The smooth

³Recall that the composition is called **transversal** if the product $\text{Im } dS \times T^*p$ of these canonical relations has transversal intersection with the diagonal $\Delta_{T^*B \times T^*X}$ in $T^*B \times \overline{T^*B} \times T^*X$ (we ignore the point in the definition of $\text{Im } dS$), and **strongly transversal** if the image of this transversal intersection embeds properly in T^*X under the natural projection. When the composition is transversal, S is also called a **Morse family** of functions over X .

⁴ $i^* \circ dS(b) = 0$, where i^* is the dual of the inclusion i of the vertical subbundle $\ker p_*$ into TB .

map

$$\begin{aligned} \lambda : \Sigma &\longrightarrow T^*X \\ b &\longmapsto (\eta, p(b)) \end{aligned}$$

with η defined by equation (2.1) is a lagrangian embedding whose image is exactly $T^*p \circ \text{Im } dS$.

Since the standard construction of a generating family for a conormal bundle N^*C out of a tubular neighborhood of $C \subset X$, is central for us, we spell it out in the following example ([2]).

Example 2. To begin, we fix a **tubular neighborhood** of C , that is, the data (V, Ψ) of a neighborhood V of C in X equipped with a diffeomorphism Ψ from a neighborhood U of the zero section of the normal bundle NC into V that maps the zero section identically to C . We denote by U_c the restriction of U to the fiber N_cC and, correspondingly, by Ψ_c the restriction of Ψ to U_c . This allows us to map a neighborhood of the zero section of the bundle $N^*C \oplus NC$ over C diffeomorphically into an open submanifold of $N^*C \times X$ as follows:

$$(p, v, c) \longmapsto (p, \Psi_c(v)),$$

where $p \in N_c^*C$ and $v \in N_cC$. This gives us a generating family (B_C^Ψ, p_C, S_C) for N^*C , where B_C^Ψ is defined as the image of this mapping. We will denote by (p, x, c) points in B_C^Ψ . The fibration $p_C : B_C^\Psi \rightarrow V$ is the projection $(p, x, c) \mapsto x$, and the generating function is given by the canonical pairing

$$S_C(p, x, c) = \langle p, \Psi_c^{-1}(x) \rangle.$$

The critical submanifold Σ_C of S_C then consists of the points of the form (p, c, c) where $c \in C$ and $p \in N_cC$. This yields an embedding

$$\tau_C : N^*C \longrightarrow B_C^\Psi,$$

whose image is exactly Σ_C , which has an obvious retraction $r_C(p, x, c) = (p, c)$. Composing τ_C with the lagrangian embedding

$$\lambda_C : \Sigma_C \longrightarrow T^*X,$$

generated by the generating family, we obtain the canonical inclusion ι_C of N^*C into T^*X .

2.3. Half-densities.

2.3.1. *α -densities and the intrinsic Hilbert space.* An α -density for $\alpha \in \mathbb{C}$ on a n -dimensional real vector space V is a map ρ from the space of frames $\mathbf{F}(V)$ (a frame is an ordered basis $\mathbf{e} = (e_1, \dots, e_n)$) to \mathbb{C} such that

$$\rho(\mathbf{e} \cdot A) = |\det A|^\alpha \rho(\mathbf{e})$$

for any matrix $A \in GL(n)$. We denote by $|V|^\alpha$ the one-dimensional complex vector space of α -densities on V .

Given a finite dimensional vector bundle E over a smooth manifold X , we denote by $|E|^\alpha$ the complex line bundle over M whose fiber at $x \in X$ is $|E_x|^\alpha$. We will reserve the notation $|\Omega|^\alpha(X)$ for the space of smooth sections of $|TM|^\alpha$; its subspace of compactly supported sections will be denoted by $|\Omega|_c^\alpha(X)$. For general density bundles $|E|^\alpha \rightarrow X$, we will use the standard notation $\Gamma(X, |E|^\alpha)$ to denote the section space.

An exact sequence of vector bundles

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

over a manifold X induces a canonical isomorphism between the density bundles $|B|^\alpha \simeq |A|^\alpha \otimes |C|^\alpha$ as well as their corresponding section spaces. In particular, we can identify $|\Omega|^{\frac{1}{2}}(M) \otimes |\Omega|^{\frac{1}{2}}(M)$ with the space $|\Omega|^1(M)$ of 1-density bundle sections on M . These sections can be integrated, which allows us to give to the vector space $|\Omega|_c^{\frac{1}{2}}(M)$ of compactly supported half-densities the structure of a pre-Hilbert space with the symmetric bilinear form

$$\langle \mu, \nu \rangle := \int_M \bar{\mu} \nu,$$

where $\bar{\mu}$ is the complex conjugate of the half-density μ . The completion of this pre-Hilbert space is usually called the intrinsic Hilbert space of M (see [2] for more details), and we will denote it by $\mathcal{H}(M)$.

2.3.2. *Integrating half-densities over fibrations.* Let $p : B \rightarrow X$ be a fibration. There is a notion of ‘‘pushforward’’ from half-densities on B to half-densities on X that requires some extra data in the form of a section of $|\ker p_*|^{\frac{1}{2}} \rightarrow X$, where $\ker p_*$ the vertical bundle of the fibration. It goes as follows: First observe that the exact sequence

$$0 \longrightarrow \ker p_* \longrightarrow TB \longrightarrow p^*(TX) \longrightarrow 0,$$

of vector bundles over B induces the canonical isomorphism

$$(2.2) \quad |\Omega|^{\frac{1}{2}}(B) \simeq \Gamma(B, |\ker p_*|^{\frac{1}{2}}) \otimes \Gamma(B, |p^*(TX)|^{\frac{1}{2}}).$$

Now, suppose that we are given a smooth family of half-densities $\rho(x) \in |\Omega|^{\frac{1}{2}}(p^{-1}(x))$ compactly supported on the fibers of p . The data

of ρ allows us to map half-densities on B to half-densities of M by integrating them on the fibers of p . Namely, ρ can be regarded as a section of the half density bundle $|\ker p_*|^{1/2} \rightarrow X$, and, in the light of (2.2), we can regard the tensor product $\mu \otimes \rho$ for any half-density μ on B as living in

$$\Gamma(B, |\ker p_*|^1) \otimes \Gamma(B, |p^*(TX)|^{1/2}).$$

The restriction of $\mu \otimes \rho$ to the fibers of p gives thus a family of densities on the fiber $p^{-1}(x)$ with values in the fixed vector space $|T_x X|^{1/2}$. Therefore, we can integrate $\mu \otimes \rho$ on this fibers and obtain

$$\theta(x) := \int_{p^{-1}(x)} \mu \otimes \rho \in |T_x X|^{1/2},$$

which is a half-density on X .

2.4. Fourier integral operators. We describe here special classes of FIO that are of concern for us. For a more general presentation, we refer the reader to the standard references [16, 15, 19]. We begin by outlining the main ingredients out of which FIO's are made and by commenting on the general problem of FIO composition, which parallels the ill-defined composition of canonical relations.

2.4.1. The FIO “category”. Given a canonical relation L from T^*X to T^*Y , one can associate a class of operators $\mathbf{Four}_{\hbar}(L; X, Y)$, the Fourier integral operators with **wave front** L , from the intrinsic Hilbert space $\mathcal{H}(X)$ to the intrinsic Hilbert space $\mathcal{H}(Y)$. More precisely, an operator Q in this class is a family $Q = \{Q_{\hbar} : \hbar \in (0, 1]\}$ of operators depending smoothly on a parameter \hbar in a way that will be made clear later on. We will be mostly concerned with the asymptotics of these operators in the **semiclassical limit**, that is, when $\hbar \rightarrow 0$. For us, a **semiclassical FIO** will mean an equivalence class of FIO's that have the same expansion in \hbar at all orders. The space of all FIO's between X and Y will be denoted by $\mathbf{Four}_{\hbar}(X, Y)$ and the space of semiclassical FIO's by $\mathbf{SFour}(X, Y)$. We will come back to the latter at the end of this section.

An FIO can be defined explicitly in terms of an oscillatory integral representation. For this, we need to fix a generating family for the wave front as well as some “vertical” half-density on the total space of the generating family. We refer the reader to [18] for the general description of these representations.

In general, the composition of two FIO's fails to be a FIO and, therefore, the collection \mathbf{Four}_{\hbar} of all FIO's only form a “category” in

the same sense as $\mathbf{Simpl}^{\text{ext}}$ does. Actually, this is more than an mere analogy. Namely, we can define the map

$$WF : \mathbf{Four}_\hbar(X, Y) \longrightarrow \mathbf{Simpl}^{\text{ext}}(T^*X, T^*Y)$$

that associates to an FIO T its wave front $WF(T)$. Now, it is a well know-result ([18]) that when the wave fronts of two FIO's T_1 and T_2 have strongly transversal composition, then the composition of the operators themselves yields a FIO whose wave front is given by the composition of the wave fronts:

$$WF(T_2 \circ T_1) = WF(T_2) \circ WF(T_1).$$

Moreover, the wave front generating families also compose: if (B_i, p_i, S_i) is a wave front generating family for $T_i \in \mathbf{Four}_\hbar(M_i, M_{i+1})$ ($i = 1, 2$), then the fibration

$$B_1 \times_{M_2} B_2 \longrightarrow M_1 \times M_3$$

together with the generating function

$$(S_1 + S_2)(b_1, b_2) = S_1(b_1) + S_2(b_2)$$

is the generating family for $WF(T_2 \circ T_1)$. Therefore, whenever we can find a subcategory \mathcal{L} of canonical relations in $\mathbf{Simpl}^{\text{ext}}$ with strongly transversal composition, we obtain an honest category $\mathbf{Four}_\hbar^{\mathcal{L}}$ of operators formed by the FIO's having canonical relations in \mathcal{L} as wave front. In this case the wave front map becomes a functor

$$WF : \mathbf{Four}_\hbar^{\mathcal{L}} \longrightarrow \mathcal{L}$$

between the corresponding categories.

Example 3. For instance, the category \mathcal{C} of cotangent lifts has an associated category of FIO's, for which the "kernel" of the wave front functor WF assigns to each identity morphism $T^* \text{id}_X$ in \mathcal{C} the algebra of pseudodifferential operators on X .

2.4.2. *Integral representation.* We now give a more descriptive presentation of a FIO class whose wave fronts have generating families with very good properties. In particular, this class encompasses the FIO's on cotangent lifts as we will see in details in the next section.

Assumption. *Let (B, p, S) be a generating family for a canonical relation from T^*X to T^*Y . We denote by p_X and p_Y the compositions of the generating family fibration $p : B \rightarrow X \times Y$ with the canonical projections on the corresponding factors. From now on, we will assume that:*

(1) The manifold B is fibered over its critical submanifold $\pi_\Sigma : B \rightarrow \Sigma$, and any fiber $\pi_\Sigma^{-1}(s)$ can be identified with a neighborhood of $p_X(s)$ in X .

(2) The generating function $S : B \rightarrow \mathbb{R}$ has a unique non-degenerate critical point on the fiber $p_Y^{-1}(y)$ for each $y \in Y$. The set Z of all these critical points is a submanifold of Σ .

This assumption implies in particular that a half-density Ψ on X induces a family $\Psi(s) \in |\Omega|^{\frac{1}{2}}(\pi_\Sigma^{-1}(s))$ by considering the restrictions of Ψ to neighborhood of $p_X(s)$ in X . Thus, given another half-density

$$a \in |\Omega|^{\frac{1}{2}}(L)$$

on the canonical relation L generated by (B, p, S) , that we transport with the lagrangian embedding λ to the critical submanifold Σ itself, we can identify the tensor product $\Psi \otimes \lambda^*a$ with a half-density on B . Now, suppose we are given a section ρ of the half-density vertical bundle $|\ker(p_Y)_*|^{\frac{1}{2}} \rightarrow B$. With this extra data, we can define an operator

$$Q_h(a, B) : |\Omega|^{\frac{1}{2}}(X) \rightarrow |\Omega|^{\frac{1}{2}}(Y),$$

by fiber integration as explained in paragraph 2.3.2:

$$(2.3) \quad (Q_h(a, B)\Psi)(y) := \int_{p_Y^{-1}(y)} \lambda^*a \otimes \Psi \otimes \rho e^{\frac{i}{\hbar}S}.$$

This integral representation can be taken as a definition for the FIO's in $\mathbf{Four}_h(L; X, Y)$ provided that the wave front L has a generating family with the properties as above. Note that any other choice of ρ would yield the same class of FIO.

This integral representation makes it clear that, once a section ρ has been fixed, we have a map, called the **total symbol map**,

$$\sigma_{B,\rho} : \mathbf{Four}_h(L; X, Y) \rightarrow |\Omega|^{\frac{1}{2}}(L),$$

that associates to the operator T its **total symbol**, that is, the half-density a on its wave front L .

Remark 4. Note that the total symbol is not an invariant of the FIO since it depends on a choice of a generating family as well as on the choice of the vertical half-density section ρ .

2.5. The semiclassical limit.

2.5.1. *The semiclassical intrinsic Hilbert space.* We say that a smooth map $\hbar \mapsto f_\hbar$ from the parameter space $(0, 1]$ to a normed vector space V is of order \hbar^∞ (and we will write $f_\hbar \in \mathcal{O}(\hbar^\infty)$) if, for each positive integer N , there is a real positive constant C_N such that $\|f(\hbar)\| \leq$

$C_N \hbar^N$. Two paths are equivalent if their difference is of order \hbar^∞ . We denote by V_\hbar the quotient space. Whenever V is finite dimensional, the Borel summation Theorem allows us to identify V_\hbar with the space $V[[\hbar]]$ of formal power series in \hbar with coefficients in V by taking the Taylor series of f_\hbar at 0. In the particular cases when $V = \mathbb{C}$ or \mathbb{R} , we will mostly prefer the interpretation of the asymptotic spaces \mathbb{C}_\hbar and \mathbb{R}_\hbar in terms of the corresponding rings of formal power series $\mathbb{R}[[\hbar]]$ and $\mathbb{C}[[\hbar]]$.

Now let a_\hbar and b_\hbar be sections of a finite dimensional vector bundle $E \rightarrow X$ depending smoothly on a parameter $\hbar \in (0, 1]$. We will say that a_\hbar and b_\hbar are equivalent modulo \hbar^∞ if the difference between their local representations at any point as well as those of all their derivatives are of order \hbar^∞ . In the case of the line bundle of α -densities on X , we will write $|\Omega_\hbar|^\alpha(X)$ for the resulting quotient space.

The inner product on $|\Omega|^\frac{1}{2}(X)$ yields an inner product on $|\Omega_\hbar|^\frac{1}{2}(X)$ with values in $\mathbb{C}[[\hbar]]$ in the the sense of [5]. We will call the resulting inner product space the semiclassical Hilbert space of the manifold X , and we will denote it by $\mathcal{H}_\hbar(X)$, though of course it is not an Hilbert space in the usual sense. Elements in $\mathcal{H}_\hbar(X)$ can be seen as classes of \hbar -dependent half-densities on X that have the same semiclassical limit, that is, the same $\mathcal{O}(\hbar^\infty)$ asymptotics when $\hbar \rightarrow 0$. Upon Taylor expansion, we can also regard $\mathcal{H}_\hbar(X)$ as the space $|\Omega|^\frac{1}{2}(X)[[\hbar]]$ of formal power series in \hbar with coefficients in the half-densities.

2.5.2. *Oscillatory integrals on microfolds.* Let $\mu_\hbar \in |\Omega|^1(B)$ be a compactly supported density on B depending smoothly on $\hbar \in (0, 1]$, and let $S : B \rightarrow \mathbb{R}$ be a smooth function whose critical points form a submanifold Z of B . The stationary phase theorem tells us that the asymptotics modulo $\mathcal{O}(\hbar^\infty)$ of the oscillatory integral

$$Q_\hbar(\mu, S) = \int_B \mu_\hbar e^{\frac{i}{\hbar}S},$$

depends only on the behavior of μ_\hbar and S in a neighborhood of Z . To be more precise, we introduce the following definition:

Definition 5. Let B be a manifold and $Z \subset B$ a submanifold. A cut-off function $\chi : B \rightarrow \mathbb{R}$ for Z is a smooth function such that there exist neighborhoods U and V of Z in B such that $\bar{U} \subset V$ and such that $\chi|_U \equiv 1$ and $\chi|_{V^c} \equiv 0$ where V^c is the complement of V in B .

The stationary phase theorem tells us that $Q_\hbar(\mu\chi)$ and $Q_\hbar(\mu\chi')$ are equivalent modulo \hbar^∞ for any two cut-off functions $\chi, \chi' : B \rightarrow \mathbb{R}$ for Z . For this reason, any two densities on X having the same germ on

Z will have equal oscillatory integrals modulo \hbar^∞ . This fact allows us to define semi-classical integrals on manifold germs or microfolds (see the definition in the Introduction).

Definition 6. An α -density on $[B, Z]$ is an α -**density germ** $[\mu]$ on B around Z . We will use the notation $|\Omega|^\alpha([B, Z])$. Let $[\mu] \in |\Omega|^1([B, Z])$ be a density germ around Z and let $S : B \rightarrow \mathbb{R}$ be as above. We define the semiclassical oscillatory integral

$$\text{s.c. } \int \mu e^{\frac{i}{\hbar} S}$$

to be the equivalence class modulo \hbar^∞ of $Q_\hbar(\chi\mu)$, where $\mu \in [\mu]$ and χ is a cut-off function for Z in B .

2.5.3. *Semiclassical Fourier integral operators.* We are now interested in the semiclassical limit of FIO's whose defining generating families satisfy the assumptions in paragraph 2.4.2. Therefore, instead of considering $Q_\hbar(a, B)$ in (2.3) as a collection of operators indexed by $\hbar \in (0, 1]$, we will see it as single operator

$$Q_\hbar(a, B) : \mathcal{H}_\hbar(X) \longrightarrow \mathcal{H}_\hbar(Y)$$

acting on the semiclassical intrinsic Hilbert spaces. Moreover, because of our assumption that S has a single critical point on each of the fibers $p_Y^{-1}(y)$ and that these critical points form a submanifold Z of B , we see, from the last paragraph, that generating families (B, p, S) having the same germ around at Z and half-densities a having the same germ around $\lambda(Z) \subset L$ will yield operators $Q_\hbar(a, B)$ with the same asymptotics modulo $\mathcal{O}(\hbar^\infty)$. In other words, these operators will coincide when looked upon as acting on the *semiclassical* intrinsic Hilbert spaces.

From the considerations above, it makes sense to introduce germs of lagrangian submanifolds and generating families to start with. This motivates the following definition:

Definition 7. A generating family for a lagrangian submicrofold $[L, C]$ of a cotangent bundle T^*X is a triple $([B, Z], [S], [p])$ for which there is a representative (B, S, p) that is a generating family for a representative L of the lagrangian submicrofold and such that

- (1) the critical submanifold Σ contains Z ,
- (2) the lagrangian embedding $\lambda : \Sigma \rightarrow T^*X$ maps Z diffeomorphically onto C .

Example 8. As an example of this last definition, let us consider the case of the conormal bundle N^*C seen as the lagrangian submicrofold

$[N^*C, C]$ of $[T^*X, X]$.⁵ The generating family (B_C^Ψ, p_C, S_C) of Example 4.1 induces a generating family for the microbundle N^*C by taking germs appropriately. Namely, one easily sees that the information needed to generate the lagrangian embedding germ

$$[\iota_C] : N^*C \longrightarrow [T^*X, C]$$

is contained in the microfold $[B_C, Z_C]$, where Z_C is the submanifold of points of the form $(0, c, c)$ with $c \in C$, along with the germs

$$\begin{aligned} [p_C] : [B_C, Z_C] &\longrightarrow [T^*X, C], \\ [S_C] : [B_C, Z_C] &\longrightarrow [\mathbb{R}, 0]. \end{aligned}$$

Since Z_C is contained in Σ_C , we can take the germ $[\Sigma_C, Z_C]$, which we will call the **critical submicrofold** of the generating function germ $[S_C]$. As before, we have the embedding

$$[\tau_C] : N_*C \longrightarrow [B_C, Z_C],$$

whose image is $[\Sigma_C, Z_C]$. This data produces the lagrangian embedding germs

$$\begin{aligned} [\lambda_C] : [\Sigma_C, Z_C] &\longrightarrow [T^*X, C], \\ [\iota_C] : N^*C &\longrightarrow [T^*X, C], \end{aligned}$$

where $[\iota_C] = [\lambda_C \circ \tau_C]$ as before.

Forgetting now everything but the $\mathcal{O}(\hbar^\infty)$ asymptotics of our FIO's, we can directly start from the data of a generating family $([B, Z], [p], [S])$ of a lagrangian submicrofold $[L, C]$ in $T^*(X \times Y)$ for which a representative satisfies the assumptions in paragraph (2.4.2) and half-density germs $[a]$ on the lagrangian submicrofold $[L, C]$. In this way, we obtain operators $Q_\hbar([a], [B, Z])$ from $\mathcal{H}_\hbar(X)$ to $\mathcal{H}_\hbar(Y)$ by replacing the integral in (2.3) by its semiclassical version introduced in Definition 6. We call these operators the **semiclassical Fourier integral operators**, and we denote their space by **SFour** (X, Y) . Note that the wave front map WF now takes its values in lagrangian submicrofolds and the total symbol map in half-density germs.

3. QUANTIZATION OF COTANGENT LIFTS

In this section, we study the semiclassical FIO's associated to the category \mathcal{C} of cotangent lifts. For this, we need to introduce the notion of micro (and local) exponential:

⁵From now on, we will write simply T^*X for $[T^*X, X]$ when it is clear from the context that we are interested in the microfold

Definition 9. Let M be a smooth manifold. A **micro exponential** is a diffeomorphism germ $[\Psi] : [TM, Z_M] \rightarrow M$ that sends the zero section Z_M to M and such that, for each x , $T_0\Psi_x = \text{id}$, where Ψ_x is the restriction of Ψ to T_xM . A **local exponential** is a representative $\Psi \in [\Psi]$ of a micro exponential. We will usually denote the domain of Ψ_x by U_x and its range by V_x .

We next show that cotangent lifts are very special in the following sense:

Proposition 10. *Let $T^*\phi : T^*M \rightarrow T^*N$ be a cotangent lift to a smooth map ϕ from N to M and let $\Psi : U \subset TM \rightarrow M$ be a local exponential on M as in Definition 9. Then, there is a canonical generating family $(B_\phi^\Psi, p_\phi, S_\phi)$, depending only on the choice of the local exponential, such that*

- (1) *the assumptions in paragraph 2.4.2 hold;*
- (2) *the image of the critical points Z_ϕ of S_ϕ on the p_N -fibers by the lagrangian embedding is the graph of ϕ seen as a submanifold of $T^*\phi$;*
- (3) *there is a canonical section $\rho_B : N \rightarrow |\ker(p_N)_*|^{\frac{1}{2}}$*

Let us see some immediate consequences of these good properties.

First of all, the $\mathcal{O}(\hbar^\infty)$ asymptotics of FIO's from $\mathcal{H}_\hbar(M)$ to $\mathcal{H}_\hbar(N)$ whose wave front are the cotangent lifts and with integral representation

$$(3.1) \quad \left(Q_\hbar(a, T^*\phi)u \right)(x_2) := \text{s.c.} \int_{p_N^{-1}(x_2)} u \otimes a \otimes \rho_B e^{\frac{i}{\hbar}S_\phi},$$

where $u \in \mathcal{H}_\hbar(M)$ are completely determined by (1) the lagrangian submicrofolds $[T^*\phi, \text{gr } \phi]$, (2) the corresponding generating family germ

$$\left([B_\phi^\Psi, Z_\phi], [p_\phi], [S_\phi] \right),$$

and (3) the half-density germs $[a] \in |\Omega_\hbar|^{\frac{1}{2}}([T^*\phi, \text{gr } \phi])$. Moreover, the whole calculus depends only on the germs of the local exponentials around the zero sections, that is, the micro exponentials

$$[\Psi] : [T^*M, Z_M] \rightarrow M$$

of Definition 9.

Second of all, the class of semiclassical FIO's on cotangent lifts has very good functorial properties. Since the cotangent lifts form a category \mathcal{C} , their associated semiclassical FIO's are always composable when sources and targets are compatible, and so the collection $\mathbf{SFour}_\hbar^{\mathcal{C}}$ of these FIO's also forms a category, and we have a wave front functor

$$\text{WF} : \mathbf{SFour}_\hbar^{\mathcal{C}} \longrightarrow \mathcal{C},$$

as explained in the previous section.

The rest of this section is devoted to the proof of Proposition 10. Throughout, M and N will be two smooth manifolds as above whose points will be denoted by x_1 and x_2 respectively. Correspondingly, we will write (p_1, x_1) and (v_1, x_1) to denote points in T^*M and TM and (p_2, x_2) and (v_2, x_2) for points in T^*N and TN .

3.1. Canonical identifications. Let $\phi : N \rightarrow M$ be a smooth map. The conormal bundle of its graph $N^*(\text{gr } \phi)$ can be identified with the pullback bundle $\phi^*(T^*M)$ via the the lagrangian embedding

$$\iota_\phi : \phi^*(T^*M) \longrightarrow T^*(M \times N),$$

given by

$$(3.2) \quad \iota_\phi(p_1, x_2) = \left((p_1, -d\phi^*p_1), (\phi(x_2), x_2) \right).$$

Let us list here several obvious but crucial identifications, which will be used extensively in what follows:

$$(3.3) \quad \text{gr } \phi \simeq N,$$

$$(3.4) \quad N^* \text{gr } \phi \simeq \phi^*(T^*M),$$

$$(3.5) \quad N \text{ gr } \phi \simeq \phi^*(TM),$$

and, therefore, the vector bundle $N^* \text{gr } \phi \oplus N \text{ gr } \phi$ over $\text{gr } \phi$ can be identified with the pullback vector bundle $\phi^*(T^*M \oplus TM)$ over N . From now on, we will not make a strict distinction between $N^* \text{gr } \phi$ and the cotangent lift $T^*\phi$, and similarly for the other identifications.

3.2. Generating families, tubular neighborhoods and micro exponentials. Example 2 shows how to construct a generating family for a general conormal bundle once a tubular neighborhood for the submanifold has been given. Since cotangent lifts are essentially conormal bundles, this construction works also for them. However, we want to discuss a special class of tubular neighborhoods in the context of cotangent lifts, which will prove to be very convenient for us.

Remark 11. A connection ∇ on M gives rise to a micro exponential by taking the germ of the connection's exponential map \exp^∇ . If we replace germs by jets in the previous definition, we obtain what is called a **formal exponential** in [17] in the context of Fedosov star-products. A micro exponential produces a ‘‘micro linearization’’ of the manifold M , that is, a germ

$$[l] : [M \times M, \Delta_M] \longrightarrow [T^*M, Z_M],$$

given in representatives by $l(x, y) = v$, where $\Psi_x(v) = y$. The notion of manifold linearization has been introduced in [4] in order to extend the pseudo-differential operator calculus on \mathbb{R}^n to general manifolds. Micro exponentials are also related to Milnor's construction of tangent microbundles [24].

Returning to cotangent lifts, we can now easily construct a tubular neighborhood for the graph of $\phi : N \rightarrow M$ out of the extra data of a micro exponential $[\Psi]$ on M only. For the neighborhood itself, we take

$$V_\phi := \bigcup_{x_2 \in N} \Psi(U_{\phi(x_2)}) \times \phi^{-1}(\Psi(U_{\phi(x_2)})),$$

where U_{x_1} is the domain of Ψ_{x_1} for a fixed representative $\Psi \in [\Psi]$. Now, using the identification of the graph normal bundle with the pull back $\phi^*(TM)$, we obtain the tubular neighborhood diffeomorphism germ from $N \text{ gr } \phi$ to V_ϕ given explicitly by

$$(v_1, x_2) \longmapsto (\Psi_{\phi(x_2)}(v_1), x_2).$$

We will denote this map again by Ψ in order to keep the notation simple and to acknowledge the micro exponential dependence in the tubular neighborhood notation (V_ϕ, Ψ) .

Now, repeating the generating family construction for conormal bundles given in Example 4.1 with the tubular neighborhood (V_ϕ, Ψ) , we obtain a generating family $(B_\phi^\Psi, p_\phi, S_\phi)$ for the cotangent lift $T^*\phi$. In very explicit terms, we have that

$$\begin{aligned} B_\phi^\Psi &= \left\{ (p_1, x_1, x_2) : p_1 \in T_{\phi(x_2)}^*M, x_1 \in \text{Im } \Psi_{\phi(x_2)}, x_2 \in N \right\}, \\ p_\phi(p_1, x_1, x_2) &= (x_1, x_2), \\ S_\phi(p_1, x_1, x_2) &= \langle p_1, \Psi_{\phi(x_2)}^{-1}(x_1) \rangle, \\ \Sigma_\phi &= \left\{ (p_1, \phi(x_2), x_2) : p_1 \in T_{\phi(x_2)}^*M, x_2 \in N \right\}. \end{aligned}$$

Again, we denote by τ_ϕ the embedding of $\phi^*(T^*M)$ into B_ϕ^Ψ and by λ_ϕ the lagrangian embedding of Σ_ϕ into $T^*(M \times N)$. The composition $\iota_\phi := \lambda_\phi \circ \tau_\phi$ yields the usual inclusion (3.2).

Let us check now that this generating family satisfies the assumptions of paragraph 2.4.2. First of all, observe that we have a retraction π_Σ from B_ϕ^Ψ onto the critical submanifold Σ given by

$$\pi_\Sigma : (p_1, x_1, x_2) \mapsto (p_1, \phi(x_2), x_2),$$

the fiber of which is the neighborhood V_{x_1} of x_1 in M determined by the exponential Ψ . This shows the first part of the assumption. Now,

if we consider the restriction of S_ϕ on the fiber

$$p_N^{-1}(x_2) = T_{\phi(x_2)}^* M \times \Psi_{\phi(x_2)}(U_{\phi(x_2)}),$$

and if compute its critical point there, we see that the vanishing of

$$\frac{\partial S_\phi}{\partial p_1}(p_1, x_1, x_2) = \Psi_{\phi(x_2)}^{-1}(x_1)$$

on the fiber implies that $\Psi_{\phi(x_2)}^{-1}(x_1) = 0$, which is equivalent to $x_1 = \phi(x_2)$. In turn, the vanishing of

$$\frac{\partial S_\phi}{\partial x_1}(p_1, x_1, x_2) = \langle p_1, T_{x_1} \Psi_{\phi(x_2)}^{-1} \rangle$$

at $x_1 = \phi(x_2)$ implies that $p_1 = 0$ since the derivative $T_{\phi(x_2)} \Psi_{\phi(x_2)}^{-1}$ is the identity by definition of the exponential. This shows that S_ϕ has a single critical point on $p_N^{-1}(x_2)$ given by $(0, \phi(x_2), x_2)$. We see that the collection of all these critical points is the following submanifold

$$Z_\phi := \left\{ (0, \phi(x_2), x_2) : x_2 \in N \right\},$$

which is contained in the (usual) critical submanifold Σ_ϕ of the generating family. Finally, we see that the lagrangian embedding λ_ϕ carries Z_ϕ to the graph of ϕ , seen as a submanifold of the cotangent lift $T^*\phi$.

We see then that the semiclassical limit of an FIO whose wave front is a cotangent lift is controlled by the lagrangian microfold $[T^*\phi, \text{gr } \phi]$ (which we will continue to denote simply by $T^*\phi$) and its generating family germ

$$(3.6) \quad \left([B_\phi^\Psi, Z_\phi], [p_\phi], [S_\phi] \right).$$

3.3. The canonical half-density ρ_B . The fibration $p_N : B_\phi^\Psi \rightarrow N$ has a section, given by $s_N(x_2) = (0, \phi(x_2), x_2)$, whose image is Z_ϕ . We will exhibit a canonical section ρ_B of the half-density bundle

$$(3.7) \quad |\ker(p_{N^*})|^{\frac{1}{2}} \longrightarrow B_\phi^\Psi$$

associated to the vertical bundle of this projection.

Let us introduce the shorthand $\mathbb{T}M$ for the vector bundle $T^*M \oplus TM$ over M . As a first step, observe that the half-density bundle $|\mathbb{T}M|^{\frac{1}{2}}$ has a canonical section ρ_M obtained fiberwise from the canonical symplectic form on $\mathbb{T}_x M$, seen as a symplectic vector space for each $x \in M$. This section ρ_M produces a new section, which we will continue to denote by ρ_M , on the pullback bundle

$$(\phi \circ p_N)^*(\mathbb{T}M) \longrightarrow B_\phi^\Psi.$$

Our strategy now is to construct a vector bundle isomorphism A between $(\phi \circ p_N)^*(\mathbb{T}M)$ and $\ker(p_N)_*$ and to use it in order to carry ρ_M to a section of (3.7), which will be our canonical half-density ρ_B . In order to produce A , we can first observe that the tangent space of B_ϕ^Ψ at $z = (p_1, x_1, x_2)$ can be decomposed into the direct sum

$$T_z B_\phi^\Psi = T_{\phi(x_2)}^* M \oplus T_{x_1} M \oplus T_{x_2} N.$$

Thus, the vertical bundle fiber at z is

$$\ker_z(p_{N*}) = T_{\phi(x_2)}^* M \oplus T_{x_1} M$$

since it is the tangent space to the fiber $p_N^{-1}(x_2)$ at z . At this point, we can write down the desired vector bundle isomorphism A by using the derivative of a micro exponential. Fiberwise, $A(z)$ is the linear map from $(\phi \circ p_N)^*(\mathbb{T}M)_z = T_{\phi(x_2)}^* M \oplus T_{\phi(x_2)} M$ to $\ker_z(p_{N*}) = T_{\phi(x_2)}^* M \oplus T_{x_1} M$ given in matrix form by

$$A(z) = \begin{pmatrix} \text{id}_{T_{\phi(x_2)}^* M} & 0 \\ 0 & \partial_{v_1} \Psi_{\phi(x_2)}(\Psi_{\phi(x_2)}^{-1}(x_1)) \end{pmatrix}.$$

Finally, we can set

$$\rho_B(z)(\mathbf{e}) := \frac{1}{(2\pi\hbar)^{\frac{m+n}{2}}} \rho_M(z)(\mathbf{e} \cdot A(z)^{-1}),$$

where $m = \dim M$ and $n = \dim N$.

Example 12. Let us compute ρ_B in the case where $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ and $\Psi_x : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a global diffeomorphism of for all $x \in \mathbb{R}^n$. In this case, we have that

$$B_\phi^\Psi = \mathbb{R}_{p_1}^m \times \mathbb{R}_{x_1}^m \times \mathbb{R}_{x_2}^n,$$

and we can identify the fibers of both $(\phi \circ p_N)^*(\mathbb{T}M)$ and $\ker(p_N)_*$ with the vector space $V := (\mathbb{R}^m)^* \oplus \mathbb{R}^m$. The canonical section ρ_M is the constant symplectic half density

$$\rho_M(x_2) = \sqrt{dp_1} \sqrt{dx_1}$$

on the symplectic vector space V . Hence, we obtain that

$$\rho_B((p_1, x_1, x_2)) = \left| \det \partial_{v_1} \Psi_{\phi(x_2)}(\Psi_{\phi(x_2)}^{-1}(x_1)) \right|^{-\frac{1}{2}} \frac{\sqrt{dp_1} \sqrt{dx_1}}{(2\pi\hbar)^m},$$

since $\rho_M(x_2)(\mathbf{e} \cdot A(z)^{-1}) = |\det A(z)^{-1}|^{\frac{1}{2}} \rho_M(x_2)$.

In particular, in the case of the global exponential coming from the canonical affine connection on \mathbb{R}^m , $\Psi_{x_1}(v_1) = x_1 + v_1$, we have that

$$\left| \det \partial_{v_1} \Psi_{\phi(x_2)}(v_1) \right|^{\frac{1}{2}} = 1$$

in the above formula for $\rho_B(z)$.

3.4. The Fourier integral operator $Q_\hbar(a, T^*\phi)$. In this paragraph, we put everything together to construct the semi-classical Fourier integral operator

$$Q_\hbar(a, T^*\phi) : \mathcal{H}_\hbar(M) \longrightarrow \mathcal{H}_\hbar(N)$$

associated to a half-density $a \in |\Omega_\hbar|^{\frac{1}{2}}(\phi^*(T^*M))$. It goes as follows: we start with $u \in \mathcal{H}_\hbar(M)$. Now, the restriction of the half-density product

$$u \otimes a \in |\Omega_\hbar|^{\frac{1}{2}}(M \times \phi^*(T^*M))$$

to the submanifold $B_\phi^\Psi \subset (M \times \phi^*(T^*M))$ produces a half-density on B_ϕ^Ψ , which we will still write as $u \otimes a$. We are now in the case of Section 2.3.2, with a half-density $u \otimes a \otimes e^{\frac{i}{\hbar}S_\phi}$ on B_ϕ^Ψ and the canonical half-density section ρ_B on the fibers of the projection $p_N : B_\phi^\Psi \rightarrow N$. This allows us to define our operator pointwise as the semi-classical oscillatory integral (3.1) since, as already shown, S_ϕ has a unique critical point in the fiber $p_N^{-1}(x_2)$, namely at $s_N(x_2) = (0, \phi(x_2), x_2)$.

Example 13. We continue Example 12, in which $M = \mathbb{R}^m$, $N = \mathbb{R}^n$, the micro exponential is a global diffeomorphism (as for instance the one coming from the canonical affine connection, i.e., $\Psi_{x_1}(v_1) = x_1 + v_1$). In coordinates, we have

$$\begin{aligned} S_\phi(p_1, x_1, x_2) &:= \langle p_1, \Psi_{\phi(x_2)}^{-1}(x_1) \rangle, \\ a &:= a(p_1, x_2) \sqrt{dp_1} \sqrt{dx_2}, \\ u &:= u(x_1) \sqrt{dx_1}. \end{aligned}$$

Using the formula for ρ_B computed in Example 12, we obtain that

$$au\rho_B = (a(p_1, x_2)u(x_1) \left| \det \partial_{v_1} \Psi_{\phi(x_2)}^{-1}(x_1) \right|^{-\frac{1}{2}} \sqrt{dx_2}) \frac{dp_1 dx_1}{(2\pi\hbar)^{ym}},$$

is a density on $p_N^{-1}(x_2)$ with values in $|T_{x_2}^*N|^{\frac{1}{2}}$. Hence, $(Q_\hbar(a, T^*\phi)u)(x_2)$ coincides with

$$\left(\text{s.c.} \int a(p_1, x_2)u(x_1) \left| \det \partial_{v_1} \Psi_{\phi(x_2)} \right|^{-\frac{1}{2}} e^{\frac{i}{\hbar} \langle p_1, \Psi_{\phi(x_2)}^{-1}(x_1) \rangle} \frac{dp_1 dx_1}{(2\pi\hbar)^m} \right) \sqrt{dx_2}.$$

In the case, we take the canonical affine connection $\Psi_{x_1}(v_1) = x_1 + v_1$, we obtain the semi-classical version of pseudo-differential operators.

4. QUANTIZATION OF SYMPLECTIC MICROMORPHISMS

In this section, we extend the semiclassical FIO calculus on cotangent lifts developed in the previous section to a more general class of wave fronts. We do so by deforming the canonical generating families

of cotangent lifts. It turns out that these new wave fronts belong to a special class of canonical relation germs: the class of symplectic micromorphisms which are the morphisms of the cotangent microbundle category $\mathbf{Cot}_{\text{mic}}^{\text{ext}}$ as constructed in [8]. As is the case for cotangent lifts, symplectic micromorphisms always compose well, and we obtain a new category of semiclassical Fourier integral operators together with a wave front functor

$$\mathbf{WF} : \mathbf{SFour}^{\mathbf{Cot}_{\text{mic}}^{\text{ext}}} \rightarrow \mathbf{Cot}_{\text{mic}}^{\text{ext}}.$$

This new category encompasses well known examples of semiclassical FIO's, the main example of which is the class of pseudo-differential operators.

Since our deformations of cotangent lift generating families have a meaning for general conormal bundles, we start with this case.

4.1. Conormal microbundle deformations. Our goal here is to describe a class of lagrangian deformations of conormal bundles along a tubular neighborhood and show how to obtain their generating families as deformations of the standard generating family of Example 2.

A tubular neighborhood (V, Ψ) of C in X fibers the neighborhood V of C into slices $V_c := \Psi(U_c)$ over the points $c \in C$. The conormal bundle N^*V_c of each of these slices is a lagrangian submanifold of T^*X , which is transversal to the conormal bundle of C . Moreover, N^*C and N^*V_c intersect only at c . The distribution Λ over C given by the collection of tangent spaces to N^*V_c for each $c \in C$ is thus a lagrangian distribution over C , which is transversal to N^*C , hence we have the lagrangian splitting

$$T(T^*X) = T(N^*C) \oplus \Lambda$$

of the tangent space to T^*X along C .

Definition 14. A deformation of the lagrangian microfold N^*C along a tubular neighborhood (V, Ψ) of C in M is a lagrangian submicrofold $[L, C]$ of T^*X that is transversal to the lagrangian distribution Λ defined above.

Our goal is to show that all deformations of N^*C along a given tubular neighborhood are obtained from the standard conormal bundle generating family by deforming its generating function of the conormal bundle as follows:

Definition 15. Let $[S_C]$ be the generating function of the conormal bundle to C . We call a deformation of $[S_C]$ a function germ of the form

$[S_C^f] = [S_C] - [r_C^* f]$. where

$$[f] \in C_0^\infty(N^*C, C) \quad \text{s.t.} \quad \partial_p f(0, x) = 0.$$

Proposition 16. *The triple $([B_C^\Psi, Z_C], [p_C], [S_C^f])$ generates a deformation $[L_f^\Psi, C]$ of N^*C along the tubular neighborhood (V, Ψ) . Conversely, all such deformations arise this way. Moreover, the critical submicrofold $[\Sigma_C^f, Z_C]$ is given, in representatives, by all the points $z \in B_C^\Psi$ of the form*

$$z = (p, \Psi_c(\partial_p f(p, c), c),$$

where (p, c) is taken in an appropriate neighborhood of the zero section of N^*C . The corresponding lagrangian embedding germ $[\lambda_C^f] : [\Sigma_C^f, Z_C] \rightarrow [T^*X, C]$ is given explicitly by

$$\lambda_C^f(z) = \left(T_{v_f}^* \Psi_c(p), \Psi_c(c) \right),$$

where $v_f = \partial_p f(p, c)$.

Remark 17. A consequence of Proposition 16 is that the critical submicrofolds for all deformations arise as follows: for each deformation $[f]$, there is a diffeomorphism germ

$$[\chi_f] : [\Sigma_C, Z_C] \longrightarrow [\Sigma_C^f, Z_C],$$

that fixes the core. It is given in representatives by

$$\chi_f(p, c, c) = \left(p, c, \Psi_c(\partial_p f(p, c)) \right).$$

Observe also that we obtain a family of lagrangian embeddings $[\iota_C^f] : N^*C \rightarrow [T^*X, C]$ deforming the canonical inclusion by composition: $[\iota_C^f] = [\lambda_C^f \circ \chi^f \circ \tau_C]$. Explicitly, we have

$$\iota_C^f(p, c) = \left(T_{v_f}^* \Psi_c(p), \Psi_c(c) \right).$$

Proof. Let us first prove that the triple $([B_C^\Psi, Z_C], [p_C], [S_C^f])$ is a generating family. This amounts to showing that there is a representative $S_C^f \in [S_C^f]$ that is non-degenerate. It is enough to work locally. In local coordinates, we have that

$$S_C^f(p, x, c) = \langle p, \Psi_c^{-1}(x) \rangle - f(p, c),$$

and that the critical set Σ_C^f is the locus of points such that $H(p, x, c) = 0$ with

$$H(p, x, c) := \Psi_c^{-1}(x) - \partial_p f(p, c).$$

Observe now that, for all $(0, c, c) \in Z_C$, we have $H(0, c, c) = 0$ and

$$\frac{\partial H}{\partial x}(0, c, c) = \text{id}.$$

Hence, an application of the implicit function theorem tells us that, for each (p, c) with p small enough, there is an unique $x(p, c)$ such that $H(p, x(p, c), c) = 0$, and thus the solutions of this equation form a submanifold Σ_C^f containing Z_C . Taking the germ $[\Sigma_C^f, Z_C]$ yields thus a submicrofold: the critical submicrofold of $[S_C^f]$. Actually, the equation $H(p, x, c) = 0$ is explicitly solvable since f is independent of x : for each (p, c) sufficiently close to the zero section, we can take

$$x(p, c) = \Psi_c(\partial_p f(p, c)),$$

which gives the form of the critical submicrofold in representatives. It follows that $([B_C^\Psi, Z_C], [p_C], [S_C^f])$ is a generating family generating a lagrangian submicrofold, which we denote by $[L_f^\Psi, C]$. A straightforward computation now gives the form of the lagrangian embedding germ λ_c^f .

It remains to see that $[L_f^\Psi, C]$ is a deformation of N^*C and that all deformations arise this way. For this, let us identify the cotangent bundle T^*N^*C with $N^*C \oplus NC$ and introduce the diffeomorphism germ

$$g : [T^*N^*C, C] \longrightarrow [B_C^\Psi, Z_C],$$

that maps (p, v, c) to $(p, \Psi(v), c)$. It produces an equivalent generating family

$$([T^*N^*C, C], p_C \circ g, S_C \circ g)$$

whose critical submicrofold is the conormal microbundle N^*C . The lagrangian embedding germ of this new generating family can be conveniently described as the restriction to the critical submicrofold of a symplectomorphism germ

$$[\chi] : [T^*N^*C, C] \longrightarrow [T^*X, C],$$

that sends the vertical distribution $V(T^*N^*C)$ over C to the lagrangian distribution Λ . (The existence of such a germ $[\chi]$ is guaranteed by Theorem 7.1 in [31].) Now, in this new description, the generating function deformations read

$$(S_C^f \circ g)(p, v, c) = \langle p, v \rangle - f(p, c).$$

The corresponding critical submicrofolds are nothing but the lagrangian submicrofolds $[\text{gr } df, C]$ and the lagrangian embedding germs are given by the restriction of $[\chi]$ to them. Since lagrangian submicrofolds of the form $[\text{gr } df, C]$ are exactly the lagrangian submicrofolds through C that are transversal to the vertical distribution in T^*N^*C , then the

generated lagrangian submicrofolds $[\chi(\text{gr } df), C]$ are exactly all the lagrangian submicrofolds through C that are transversal to Λ , that is, all the deformations of N^*C along the tubular neighborhood. \square

4.2. Symplectic micromorphisms as deformed cotangent lifts.

In micro-geometry, we can define a canonical relation from T^*M to T^*N to be a lagrangian submicrofold

$$[V, \text{gr } \phi] \subset \overline{T^*M} \times T^*N$$

whose core is the graph of a smooth map $\phi : N \rightarrow M$. Although canonical relations do not compose well in general, it is possible to single out a class for which they do. This class is the class of symplectic micromorphisms as introduced in [8]. To distinguish them, we will use the special notation

$$([V], \phi) : T^*M \longrightarrow T^*N,$$

instead of $[V, \text{gr } \phi]$. They can be characterized in several ways, the most useful for us now being the following:

Definition 18. A symplectic micromorphism $([V], \phi)$ from T^*M to T^*N is a canonical relation $[V, \text{gr } \phi]$ that is transversal to the lagrangian distribution

$$(4.1) \quad \Lambda : = TZ_M \oplus V(T^*N).$$

over $\text{gr } \phi$.

Cotangent lifts are symplectic micromorphisms, but not all symplectic micromorphism arise this way. However, any symplectic micromorphism can be realized as a cotangent lift deformation in the sense of paragraph 4.1. More precisely, given a micro exponential $[\Psi]$ on M , one can deform the associated generating family

$$([B_\phi^\Psi, Z_\phi], [p_\phi], [S_\phi])$$

of the cotangent lift $T^*\phi : T^*M \rightarrow T^*N$ by adding a function germ to it:

$$(4.2) \quad S_\phi^f(p_1, x_1, x_2) = \langle p_1, \Psi_{\phi(x_2)}^{-1}(x_1) \rangle - f(p_1, x_2),$$

where $[f]$ is a function germ on the conormal bundle $N^*(\text{gr } \phi)$ (identified, as usual, with the pullback bundle $\phi^*(T^*M)$) that is identically null on the zero section and whose derivative in the fiber direction vanishes. This yields a lagrangian submicrofold $[L_\Psi^f, \text{gr } \phi]$, which is a deformation of $T^*\phi$ along the tubular neighborhood (V_ϕ, Ψ) in the sense of Definition 14.

Now, as explained in paragraph 4.1, $[L_{\Psi}^f, \text{gr } \phi]$ is transversal to the lagrangian distribution Λ^{Ψ} given by tangent spaces at points of $\text{gr } \phi$ to the conormal bundles $N^*V_{(\phi(x), x)}$ of the tubular neighborhood slices

$$V_{(\phi(x), x)} := \Psi_{\phi(x)}(U_{\phi(x)}) \times \{x\}.$$

As is easily checked, the lagrangian distribution Λ^{Ψ} is independent of the micro exponential and coincides with the lagrangian distribution (4.1) defining symplectic micromorphisms. Therefore, an application of Proposition 16 to this case immediately yields:

Proposition 19. *Once a micro exponential $[\Psi]$ on M is fixed, there is a one-to-one correspondence between the symplectic micromorphisms from T^*M to T^*N with core map ϕ and the deformations $[L_{\Psi}^f, \text{gr } \phi]$ of the core map cotangent lift along the tubular neighborhood (V_{ϕ}, Ψ) .*

4.3. Enhancements and quantization. Our goal here is to realize the class of semiclassical FIO's on symplectic micromorphisms in terms of a two-step construction performed on them: enhancement and quantization.

Definition 20. An **enhancement** of a symplectic micromorphism $([V], \phi) : T^*M \rightarrow T^*N$ is a half-density germ

$$[a] \in |\Omega|^{\frac{1}{2}}([V, \text{gr } \phi]).$$

The triple $([a], [V], \phi) : T^*M \rightarrow T^*N$ will be called an **enhanced symplectic micromorphism**.

Let us fix a micro exponential $[\Psi]$ on M and let $([B_{\phi}^{\Psi}, Z_{\phi}], p_{\phi}, S_{\phi}^f)$ be the corresponding generating family of $([V], \phi) : T^*M \rightarrow T^*N$ as in Proposition 19. An enhancement $[a]$ of the symplectic micromorphism $([V], \phi)$ yields a half density germ

$$[a_{\Psi}] := [(\iota_{\phi}^f)^* a] \in |\Omega|^{\frac{1}{2}}(\phi^*(T^*M)),$$

where $[\iota_{\phi}^f] : \phi^*(T^*M) \rightarrow T^*(M \times N)$ is the lagrangian embedding germ given by the generating family. Now, we can associate to the triple $T = ([a], [V], \phi)$ a semiclassical Fourier integral operator

$$Q_{\hbar}(T) : \mathcal{H}_{\hbar}(M) \longrightarrow \mathcal{H}_{\hbar}(N)$$

exactly as we did for cotangent lifts, except that we replace S_{ϕ} by its deformed version S_{ϕ}^f and a by a_{Ψ} in the semiclassical integral (3.1). The crucial point is that both S_{ϕ} and S_{ϕ}^f have a the same unique critical

point in the integration fiber $p_N^{-1}(x_2)$, namely $(0, \phi(x_2), x_2)$. Moreover, the deformed family

$$\left([B_\phi^\Psi, Z_\phi], [p_\phi], [S_\phi^f] \right)$$

continues to satisfy the assumptions of paragraph 2.4.2.

4.4. Local theory.

4.4.1. *Quantization of symplectic micromorphisms.* We are interested here in the enhancement and the quantization of symplectic micromorphisms

$$([L], \phi) : T^*U \longrightarrow T^*V$$

between cotangent bundles of some open subsets $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^l$, whose canonical global coordinates we denote by (p_1, x_1) and (p_2, x_2) respectively. As already noticed in Examples 12 and 13, this case has the special feature of having a global exponential

$$\Psi_x(v) : = x + v$$

for each open subset $U \subset \mathbb{R}^n$. Hence, we can canonically associate a generating family to any symplectic micromorphism $([L], V)$ from T^*U to T^*V . Namely, Proposition 19 shows that there is a unique function germ $[f] \in C_0^\infty(\phi^*(T^*U), Z)$ with $\partial_p f(0, x) = 0$ such that

$$\left([B(U, V), Z_\phi], [p_\phi], [S_\phi^f] \right)$$

is a generating family for $([L], \phi)$, where

$$\begin{aligned} B(U, V) &= (\mathbb{R}^k)^* \times U \times V, \\ Z_\phi &= \{0\} \times \text{gr } \phi, \\ S_\phi^f(p_1, x_1, x_2) &= \langle p_1, \phi(x_2) - x_1 \rangle + f(p_1, x_2). \end{aligned}$$

Remark 21. The generating function S_ϕ^f above differs by a minus sign from our previous definition in (4.2). This sign is irrelevant, and we choose to write the generating function this way here in order to better agree with the usual sign convention for the phase of pseudo-differential operators.

Notation 22. At times, it will prove useful to collect the terms in S_ϕ^f that do not depend on x_1 into the single term

$$(4.3) \quad F(p_1, x_2) : = \langle p_1, \phi(x_2) \rangle + f(p_1, x_2).$$

The use of the upper case letter F in the generating function notation S_ϕ^F instead of the lower case f will mean that we consider the generating function

$$S_\phi^F(p_1, x_1, x_2) = -\langle p_1, x_1 \rangle + F(p_1, x_2),$$

in which the upper case F is related to the lower case f via the relation (4.3). We write S_ϕ in case $F(p_1, x_2) = \langle p_1, \phi(x_2) \rangle$, that is for the generating function of the cotangent lifts. Since, the generating family is canonical in the local setting, we will also use the notation $[S_\phi^F]$ to denote the corresponding symplectic micromorphism $([L^F], \phi)$.

The critical microfold of S_ϕ^F is

$$\Sigma_\phi^F = \left\{ (p_1, \partial_p F(p_1, x_2), x_2) : (p_1, x_2) \in W \right\},$$

where W is a suitable neighborhood of the zero of $\phi^*(T^*U)$. This yields the explicit formula

$$\begin{aligned} i_\phi^F(p_1, x_2) &= \left(\left(-\partial_{x_1} S_\phi^F(z), x_1 \right), \left(\partial_{x_2} S_\phi^F(z), x_2 \right) \right), \\ &= \left(\left(p_1, \partial_p F(p_1, x_2) \right), \left(\partial_x F(p_1, x_2), x_2 \right) \right), \end{aligned}$$

where z is the image of (p_1, x_2) in Σ_ϕ^F , for the embedding of $[\phi^*(T^*U), Z_\phi]$ into $\overline{T^*U} \times T^*V$.

In the local case, an enhancement of $[S_\phi^F]$ can be identified with a germ $[a]$ of semiclassical square integrable function

$$a \in L_\hbar^2((\mathbb{R}^m)^* \times V)$$

around the $\{0\} \times V$, and the semiclassical intrinsic Hilbert space $\mathcal{H}_\hbar(U)$ with $L_\hbar^2(U)$ in the notation of paragraph 2.5.1. The quantization of the enhanced symplectic micromorphism $([a], S_\phi^F)$ is the semiclassical Fourier integral operator from $L_\hbar^2(U)$ to $L_\hbar^2(V)$ given by the formula

$$(4.4) \quad \left(Q_\hbar([a], S_\phi^F) \Psi \right)(x_2) := \text{s.c.} \int_{\mathbb{R}^m \times U} a(p_1, x_2) \Psi(x_1) e^{\frac{i}{\hbar} S_\phi^f(p_1, x_1, x_2)} \frac{dp_1 dx_1}{(2\pi\hbar)^m},$$

where $S_\phi^f(p_1, x_1, x_2) = \langle p_1, \phi(x_2) - x_1 \rangle + f(p_1, x_2)$. Note that these operators comprise the class of semiclassical pseudo-differential operators. Namely, when both U and V are \mathbb{R}^n , the core map ϕ is the identity and $f = 0$, we obtain the well-know integral representation for

semiclassical pseudo-differential operators:

$$(Q_{\hbar}([a], S_{\text{id}})\Psi)(x) = \frac{1}{(2\pi\hbar)^n} \text{ s.c. } \int_{\mathbb{R}^n} a(p, y) f(y) e^{\frac{i}{\hbar}\langle p, x-y \rangle} dp dy,$$

where a is the total symbol of the operator. Hence, this defines a map

$$\text{Op}_{\hbar} : [a] \longmapsto Q_{\hbar}([a], S_{\text{id}}^0)$$

from function germs on $T^*\mathbb{R}^n$ around the the zero section to operators acting on $L_{\hbar}^2(\mathbb{R}^n)$. This is known as the standard quantization of the symbol a , which satisfies the correspondence principle of quantum mechanics:

$$\begin{aligned} \text{Op}_{\hbar}([x])\Psi(x) &= \frac{\hbar}{i} \frac{\partial \Psi}{\partial x}(x), \\ \text{Op}_{\hbar}([p])\Psi(x) &= x\Psi(x), \end{aligned}$$

where $[p]$ denotes the germ of the projection $(p, x) \mapsto p$ and $[x]$ the germ of the projection $(p, x) \mapsto x$ around the zero section.

Here are other examples which are not pseudo-differential operators.

The first example describes the class of semiclassical FIO's whose wave fronts are the symplectic micromorphisms from $T^*\mathbb{R}^n$ to the point.

Example 23. Consider a micromorphism $([L], \phi)$ from $T^*\mathbb{R}^n$ to the cotangent bundle of the point, which we denote by \star . The core map $\phi : \{\star\} \rightarrow \mathbb{R}^n$ is completely determined by the choice of a point $x_0 \in \mathbb{R}^n$, and $[L]$ is a lagrangian submanifold germ through x_0 transversal to the zero section. Since $\phi^*(T^*\mathbb{R}^n)$ is the fiber $T_{x_0}^*\mathbb{R}^n$, the generating function is a function germ $[S_{x_0}^f] : [T_{x_0}^*\mathbb{R}^n, 0] \rightarrow [\mathbb{R}, 0]$ of the form

$$S_{x_0}^f(p) = -\langle p, x_0 \rangle + f(p).$$

A enhancement $[a]$ in this case is also a function germ on $T_{x_0}^*U$ at the origin, and the corresponding quantization

$$Q_{\hbar}([a], S_{x_0}^f) : L_{\hbar}^2(\mathbb{R}^n) \longrightarrow \mathbb{C}[[\hbar]]$$

is a distribution on \mathbb{R}^n . When $f = 0$, we obtain that

$$Q_{\hbar}([a], S_{x_0})\Psi = \left(\mathcal{F}_{\hbar}^{-1}(a\mathcal{F}_{\hbar}(\Psi)) \right)(x_0),$$

where \mathcal{F}_{\hbar} is the asymptotic Fourier transform on \mathbb{R}^n . In particular, this yields the derivatives of the delta function concentrated at x_0 :

$$Q_{\hbar}(p^{\alpha}, S_{x_0}) = (-i\hbar)^{|\alpha|} \partial^{\alpha} \delta(x - x_0),$$

where $\alpha \in \mathbb{N}^n$ is a multi-index.

The second example describes the class of semiclassical FIO's whose wave fronts are the unique symplectic micromorphism from the point to $T^*\mathbb{R}^n$.

Example 24. There is only one symplectic micromorphism e_U from $T^*\mathbf{E}$ (where $\mathbf{E} = \{\star\}$ is the point) to T^*U . Explicitly, e_U is given by $([\{0\} \times U], \text{pr})$, where pr is the projection of U to the unique point of E . The generating function of e_U is the zero function $S_{\text{pr}}^0(x) = 0$. An enhancement of e_U is, therefore, a function $f \in L_h^2(U)$, and its quantization

$$Q_h(f, e_U) : \mathbb{C}[[\hbar]] \longrightarrow L_h^2(U)$$

can be identified with itself: $Q_h(f, e_U)\Psi = f\Psi$.

4.4.2. *Generating function composition formula.* Consider U , V , and W open subsets of euclidean spaces, and let $(B(U, V), S_\phi^F)$ and $(B(V, W), S_\psi^G)$ be the generating families generating the symplectic micromorphisms

$$T^*U \xrightarrow{([L_1], \phi)} T^*V \xrightarrow{([L_2], \psi)} T^*W,$$

and let $(B(U, W), S_{\phi \circ \psi}^{G \circ F})$ be the local generating family of the composition $([L_2 \circ L_1], \phi \circ \psi)$. Obviously, symplectic micromorphism composition induces a composition operation on local generating families

$$S_\psi^G \circ S_\phi^F := S_{\phi \circ \psi}^{G \circ F}.$$

This composition is associative in the sense that it produces a category in which objects are open subsets U of \mathbb{R}^n for some n and a morphism from U to V is a local generating family $(B(U, V), S_\phi^F)$. The identity morphism on U is the local generating family $(B(U, U), S_{\text{id}}^I)$, where the generating function is the usual phase of pseudo-differential operators

$$S_{\text{id}}^I(p_1, x_1, x_2) = I(p_1, x_2) - p_1 x_1,$$

where $I(p_1, x_2) = p_1 x_2$. This category inherits a symmetric monoidal structure from the microsymplectic category. The unit object is the point $\mathbb{R}^0 = \{\star\}$, the tensor product on objects is the usual cartesian product $U \times V$, and the tensor product of local generating families

$$\left(B(U_1, V_1), S_{\phi_1}^{F_1} \right) \otimes \left(B(U_2, V_2), S_{\phi_2}^{F_2} \right)$$

is the local generating family

$$\left(B(U_1 \times U_2, V_1 \times V_2), S_{\phi_1}^{F_1} \otimes S_{\phi_2}^{F_2} \right),$$

where the tensor product of the generating functions is given by

$$(S_\phi^F \otimes S_\psi^G)(p_1, \tilde{p}_2, x_1, x_2, \tilde{x}_2, \tilde{x}_3) := S_\phi^F(p_1, x_1, x_2) + S_\psi^G(\tilde{p}_2, \tilde{x}_2, \tilde{x}_3).$$

The unit object \mathbb{R}^0 is initial: There is only one morphism from \mathbb{R}^0 to any other open subset U . Namely, $B(\mathbb{R}^0, U) = U$ and the only generating function in $G(\mathbb{R}^0, U)$ is the zero function that we denote by $S_U^0(x) = 0$.

We now derive an explicit formula for the generating function $S_{\phi \circ \psi}^{G \circ F}$. We will use two general facts on composition and reduction of generating families:

Composition. Let $p_i : B_i \rightarrow M_i \times M_{i+1}$ be a generating family for the canonical relation $L_i \subset \overline{T^*M_i} \times T^*M_{i+1}$ with generating function S_i for $i = 1, 2$. Suppose that the composition of L_1 and L_2 is transversal. Then, the fibration $B_1 \times_{M_2} B_2 \rightarrow M_1 \times M_3$ together with the generating function $(S_1 + S_2)(b_1, b_2) = S_1(b_1) + S_2(b_2)$ is a generating family for the canonical relation $L_2 \circ L_1$.

Reduction. Let $p : B \rightarrow M$ be a generating family with generating function S . Suppose that we can factor p as a composition $B \xrightarrow{\pi} B' \xrightarrow{p'} M$ of two fibrations such that the restriction of S to each fiber $\pi^{-1}(b')$ has the unique critical point $\gamma(b')$. Then, the two generating families (B, p, S) and $(B', p', S \circ \gamma)$ generate the same Lagrangian submanifold of T^*M .

With this in mind, we obtain that the fibration

$$B(U, V) \times_V B(V, W) \xrightarrow{p_V} U \times W$$

with generating function

$$(S_\phi^F \otimes_V S_\psi^G)(p_1, p_2, x_1, x_2, x_2, x_3) := S_\phi^F(p_1, x_1, x_2) + S_\psi^G(p_2, x_2, x_3),$$

is a generating family for $[L_\psi^G \circ L_\phi^F]$. Now, we can factor the fibration p_V into

$$B(U, V) \times_V B(V, W) \xrightarrow{\pi} B(U, W) \xrightarrow{\pi_U \times \pi_W} U \times W.$$

Assuming that the restriction of $S_\phi^F \otimes_V S_\psi^G$ to each fiber $\pi^{-1}(p_1, x_1, x_3)$ has a unique critical point $\gamma(p_1, x_1, x_3)$, we obtain a formula for the generating function composition

$$S_\psi^G \circ S_\phi^F := (S_\phi^F \otimes_V S_\psi^G) \circ \gamma.$$

The following lemma guarantees the existence of a unique critical point on each fiber.

Lemma 25. *In the notation as above, we have that the restriction of $S_\phi^F \otimes_V S_\psi^G$ to the fiber $\pi^{-1}(p_1, x_1, x_3)$ has a unique non-degenerate critical point*

$$(\bar{p}_2, \bar{x}_2) = \gamma(p_1, x_1, x_3)$$

which is given as the unique solution of system

$$(4.5) \quad \bar{p}_2 = \partial_x F(p_1, \bar{x}_2),$$

$$(4.6) \quad \bar{x}_2 = \partial_p G(\bar{p}_2, x_3).$$

Proof. By definition, we have:

$$(S_\phi^F \otimes_V S_\psi^G)(p_1, x_1, x_2, p_2, x_3) = p_1 x_1 + F(p_1, x_2) + G(p_2, x_3) - p_2 x_2.$$

Now, the critical points along $\pi^{-1}(p_1, x_1, x_3) = (\mathbb{R}_V^l)^* \times V$ are the points (\bar{p}_2, \bar{x}_2) in the fiber satisfying the equation

$$\frac{\partial(S_\phi^F \otimes_V S_\psi^G)}{\partial(p_2, x_2)}(p_1, x_1, \bar{x}_2, \bar{p}_2, x_3) = 0,$$

which is exactly equivalent to the system (4.5)-(4.6), since

$$\frac{\partial(S_\phi^F \otimes_V S_\psi^G)}{\partial(p_2, x_2)}(p_1, x_1, \bar{x}_2, \bar{p}_2, x_3) = \begin{pmatrix} \bar{p}_2 - \partial_{x_2} F(p_1, \bar{x}_2) \\ \bar{x}_2 - \partial_{p_2} G(\bar{p}_2, x_3) \end{pmatrix}.$$

Thanks to the fact that $F(0, x_1) = 0$ and $\partial_{p_2} G(0, x_3) = \psi(x_3)$, we get

$$\frac{\partial(S_\phi^F \otimes_V S_\psi^G)}{\partial(p_2, x_2)}(0, x_1, \psi(x_3), 0, x_3) = 0$$

for all x_1 , meaning that $\gamma(0, x_1, x_3) = (0, \psi(x_3))$. In particular, this is true for $x_1 = \phi \circ \psi(x_3)$. Now, the Hessian at this point is

$$\frac{\partial^2(S_\phi^F \otimes_V S_\psi^G)}{\partial^2(p_2, x_2)}(0, \phi \circ \psi(x_3), \psi(x_3), 0, x_3) = \begin{pmatrix} \text{id} & 0 \\ -\partial_p \partial_x G(0, x_3) & \text{id} \end{pmatrix},$$

which is invertible. Now, for small p_1 , the implicit function theorem gives us a section of critical points

$$[B(V, W), Z_{\phi \circ \psi}] \xrightarrow{\gamma} [(B(U, V) \times_V B(V, W), Z_\phi \times_V Z_\psi)].$$

□

Definition 26. Given a function $f \in C^\infty(\mathbb{R}^k)$ with only one critical point on \mathbb{R}^k , we denote by $\mathbf{Stat}_{(x)}\{f\}$ the value of f at this point x_0 where $\partial_x f(x_0) = 0$. If f also depends on a variable y in \mathbb{R}^l , we denote by $\mathbf{Stat}_{(x)}\{f\}(y)$ the function depending on y defined by $f(x_0(y), y)$ where $x_0(y)$ is the implicit function solution of the equation $\partial_x f(x_0(y), y) = 0$.

We now immediately have the following proposition:

Proposition 27.

$$(G \circ F)(p_1, x_3) = \mathbf{Stat}_{(\bar{p}, \bar{x})} \left\{ F(p_1, \bar{x}) + G(\bar{p}, x_3) - \bar{p} \bar{x} \right\}.$$

Moreover,

$$F = I \circ F = F \circ I,$$

where $I(p, x) = \langle p, x \rangle$.

4.4.3. *Enhancement composition formula.* We denote by $E(K)$ the space of enhancements of a symplectic micromorphism $K : T^*U \rightarrow T^*V$. In the local setting, the micromorphism K is associated to a unique generating family $(B(U, V), S_\phi^F)$. Suppose now that we have another symplectic micromorphism $L : T^*V \rightarrow T^*W$ with generating family $(B(V, W), S_\psi^G)$. We want to define a composition for enhancements

$$\circ : E(K) \times E(L) \longrightarrow E(L \circ K).$$

In the previous paragraph, we have seen that we have a fibration

$$B(U, V) \times_V B(V, W) \xrightarrow{\pi} B(U, W)$$

on whose fiber the generating function $S_\phi^F \otimes_V S_\psi^G$ has a single non degenerate critical point. The evaluation of $S_\phi^F \otimes S_\psi^G$ at this critical point is exactly $S_\psi^G \circ S_\phi^F$. For $[a] \in E(K)$ and $[b] \in E(L)$, we define the enhancement composition as

$$(4.7) \quad (a \circ b)(p_1, x_3) := \text{s.c.} \int_{\pi^{-1}(p_1, \phi \circ \psi(x_3), x_3)} a(p_1, x_2) b(p_2, x_3) e^{\frac{i}{\hbar} \Theta(F, G)} \frac{dp_2 dx_2}{(2\pi\hbar)^l},$$

where $\Theta(F, G)$ is defined as the difference

$$\Theta(F, G) = S_\phi^F \otimes_V S_\psi^G - \pi^*(S_\psi^G \circ S_\phi^F).$$

Remark 28. Observe that the leading term in \hbar of the stationary phase asymptotic expansion of $a \circ b$ is proportional to $a(p_1, \bar{x}_2) b(\bar{p}_2, x_3)$ where (\bar{p}_2, \bar{x}_2) is the critical point of $S_\phi^F \otimes S_\psi^G$ on the fiber $\pi^{-1}(p_1, \psi \circ \phi(x_3), x_3)$. This is the usual composition of half-densities carried by composable canonical relations.

Proposition 29. *In the notation above, we have that*

$$\begin{aligned} ([a], S_\phi^F) &= ([1], S_{\text{id}}^I) \circ ([a], S_\phi^F) \\ &= ([a], S_\phi^F) \circ ([1], S_{\text{id}}^I) \end{aligned}$$

for any enhanced symplectic micromorphism $([a], S_\phi^F)$.

Proof. We will show only the first equality, since the second is proven similarly. Recall that $I(p, x) = px$ and that $I \circ F = F = F \circ I$. We have that

$$\begin{aligned} (\Theta(I, F))(z) &= p_1 \phi(x_3) + F(p_1, x_2) + I(p_2, x_3) - p_2 x_2 - p_1 \phi(x_3) - (F \circ I)(p_1, x_3) \\ &= F(p_1, x_2) + p_2(x_3 - x_2) - F(p_1, x_3), \end{aligned}$$

where $z = (p_1, \phi(x_3), x_2, p_2, x_2, x_3)$. Plugging this into the enhancement composition formula (4.7), we obtain

$$\begin{aligned} (1 \circ a)(p_1, x_3) &= \frac{e^{-\frac{i}{\hbar}F(p_1, x_3)}}{(2\pi\hbar)^l} \int dx_2 a(p_1, x_2) e^{\frac{i}{\hbar}F(p_1, x_2)} \int dp_2 e^{\frac{i}{\hbar}p_2(x_3 - x_2)}. \\ &= e^{-\frac{i}{\hbar}F(p_1, x_3)} \int dx_2 a(p_1, x_2) e^{\frac{i}{\hbar}F(p_1, x_2)} \delta_{x_3}(x_2) \\ &= a(p_1, x_3), \end{aligned}$$

where δ_{x_3} is the delta function concentrated at x_3 . \square

Proposition 30. *Let E_1 and E_2 be any two composable enhanced symplectic micromorphisms. Then:*

$$Q_h(E_2) \circ Q_h(E_1) = Q_h(E_2 \circ E_1) \quad \text{mod } \mathcal{O}(\hbar^\infty).$$

Moreover, if $Q_h([a], L) = 0$, then $[a] = 0 \quad \text{mod } \mathcal{O}(\hbar^\infty)$

Proof. Suppose we have enhanced symplectic micromorphisms $E_1 = ([a], S_\phi^F)$ and $E_2 = ([b], S_\psi^G)$ in the situation

$$T^*U \xrightarrow{E_1} T^*V \xrightarrow{E_2} T^*W.$$

A straightforward computation yields

$$\begin{aligned} I &= \left(Q_h(E_2) \circ Q_h(E_1) \Psi \right)(x_3) \\ &= \int_{\mathbb{R}^l \times V} \frac{dp_2 dx_2}{(2\pi\hbar)^l} b(p_2, x_3) (Q_h(E_1) \Psi)(x_2) e^{\frac{i}{\hbar}S_\psi^G(p_2, x_2, x_3)}, \\ &= \int_{\mathbb{R}^l \times V} \frac{dp_2 dx_2}{(2\pi\hbar)^l} \int_{\mathbb{R}^k \times U} \frac{dp_1 dx_1}{(2\pi\hbar)^k} a(p_1, x_2) b(p_2, x_3) \Psi(x_1) e^{\frac{i}{\hbar}(S_\psi^G(p_2, x_2, x_3) + S_\phi^F(p_1, x_2, x_3))}. \end{aligned}$$

Now, interchanging the integrals and writing 1 as

$$1 = e^{\frac{i}{\hbar}(S_\psi^G \circ S_\phi^F)(p_1, \phi \circ \psi(x_3), x_3)} e^{-\frac{i}{\hbar}(S_\psi^G \circ S_\phi^F)(p_1, \phi \circ \psi(x_3), x_3)},$$

we obtain that

$$\begin{aligned} I &= \int_{\mathbb{R}^k \times U} \frac{dp_1 dx_1}{(2\pi\hbar)^k} (a \circ b)(p_1, x_3) \Psi(x_1) e^{\frac{i}{\hbar}(S_\psi^G \circ S_\phi^F)(p_1, \phi \circ \psi(x_3), x_3)}, \\ &= Q_h(E_2 \circ E_1). \end{aligned}$$

Of course, all the computations should be understood modulo $\mathcal{O}(\hbar^\infty)$ and with the appropriate cut-off functions thrown in. The fact that $Q_h([a], L) = 0$ implies that $[a] = 0 \quad \text{mod } \hbar^\infty$ is obvious. \square

Corollary 31. *Enhanced local symplectic micromorphisms form a category.*

Proof. Let $([a_i], T_i)$, $i = 1, 2, 3$, be composable enhanced local symplectic micromorphisms, and set $Q_i := Q_{\hbar}([a_i], T_i)$ be their quantization. We need to prove that

$$[a_3 \circ (a_2 \circ a_1)] = [a_3 \circ (a_2 \circ a_1)].$$

We know that associativity holds at the level of quantization:

$$Q_3(Q_2Q_1) = (Q_3Q_2)Q_1.$$

Now, we also have that

$$\begin{aligned} Q_3(Q_2Q_1) &= Q_{\hbar}\left([a_3 \circ (a_2 \circ a_1)], T_3 \circ T_2 \circ T_1\right), \\ (Q_3Q_2)Q_1 &= Q_{\hbar}\left([(a_3 \circ a_2) \circ a_1], T_3 \circ T_2 \circ T_1\right), \end{aligned}$$

and, therefore, $[a_3 \circ (a_2 \circ a_1)] = [(a_3 \circ a_2) \circ a_1] \pmod{\hbar^\infty}$. \square

Example 32. We have seen that the quantization $Q_{\hbar}([a], S_{\text{id}})$ of the enhanced symplectic micromorphism

$$([a], T^* \text{id}) : T^*\mathbb{R}^n \longrightarrow T^*\mathbb{R}^n$$

coincides with the standard quantization $\text{Op}_{\hbar}(a)$ of a seen as a symbol (germ) in $C^\infty(T^*\mathbb{R}^n)$. Therefore, the enhancement composition degenerates in this case to the star product \star_{qp} defined by the identity

$$\text{Op}_{\hbar}(a) \circ \text{Op}_{\hbar}(b) = \text{Op}_{\hbar}(a \star_{\text{qp}} b),$$

giving the composition of symbols in the qp-ordering for pseudodifferential operators.

The next example shows that one can recover the quantization out of the enhancement composition; so, in our case, enhancing is quantizing.

Example 33. Enhancements (Ψ, e_U) of the unique symplectic micromorphism $e_U : \mathbf{E} \rightarrow T^*U$ can be identified with the space $\Psi \in L_{\hbar}^2(U)$. Let $([a], T)$ be a symplectic micromorphism from T^*U to T^*U . An easy computation shows that composition degenerates into quantization; namely, we have that

$$[a] \circ \Psi = Q_{\hbar}([a], T)\Psi.$$

4.4.4. *States and costates.* We call a **state** on U an enhanced symplectic micromorphism (Ψ, e_U) from the cotangent microbundle of the point \mathbf{E} to T^*U . As is clear from Example 24, the set of states on U can be identified with the Hilbert space $Q_{\hbar}(T^*U) = L_{\hbar}^2(U)$, and we will use the Dirac “ket” notation

$$(\Psi, e_U) \longleftrightarrow |\Psi\rangle,$$

for states. An enhanced symplectic micromorphism $A = ([a], S_\phi^F)$ from T^*U to T^*V induces an operator \hat{A} from the states on U to the states on V by composition

$$\hat{A}|\Psi\rangle := A \circ (\Psi, e_U).$$

The following proposition tells us that the quantization functor stems from the composition of enhancements:

Proposition 34. *Let $A : T^*U \rightarrow T^*V$ be an enhanced symplectic micromorphism, then*

$$\hat{A}|\Psi\rangle = |Q_\hbar(A)\Psi\rangle ,$$

where Ψ is a state on U .

Proof. Set $A = ([a], S_\phi^F)$. Then, we have by definition that $\hat{A}|\Psi\rangle = (a \circ \Psi, e_V)$. Now, a straightforward computation yields that $\Theta(0, F) = S_\phi^F$, which in turns gives us that

$$\begin{aligned} a \circ \Psi(x_2) &= \text{s.c.} \int \Psi(x_1) a(p_1, x_2) e^{\frac{i}{\hbar} S_\phi^F(p_1, x_1, x_2)} \frac{dp_1 dx_1}{(2\pi\hbar)^l} \\ &= Q_\hbar(A)\Psi(x_2). \end{aligned}$$

□

A **costate** on U is an enhanced symplectic micromorphism from T^*U to E . It is completely determined by the data of a point $x \in U$, a germ of a lagrangian submanifold $[L]$ around x in T^*U transversal to the zero section and an enhancement $[a] : T_x^*U \rightarrow \mathbb{R}$ as in Example 23. We will use the Dirac “bra” notation

$$([a], S_x^F) \longleftrightarrow \langle a, F, x |$$

to denote costates, where S_x^F is the generating function of $[V]$. Note that the set of costates does not itself form a vector space; however, one may consider the free vector space generated by costates. Using the composition of enhanced symplectic micromorphisms, we can define a “pairing” between states and costates on U :

$$\langle a, F, x | \Psi \rangle := ([a], S_{x_0}^F) \circ (\Psi, S_{\text{pr}}^0)$$

is a symplectic micromorphism from E to E and, thus, a power series belonging to $\mathbb{C}[[\hbar]]$.

Example 35. If we denote by $\langle x |$ the special costate $(1, 0, x)$ on \mathbb{R} , we obtain that

$$\langle x_0 | \Psi \rangle = \Psi(x_0),$$

and if we denote by \hat{H} the operator on states associated to the enhancement $([H], S_{\text{id}}^0) : \mathbb{R} \rightarrow \mathbb{R}$ of the identity micromorphism, we obtain that

$$\langle x_0 | \hat{H} | \Psi \rangle = \mathcal{F}_h^{-1}(H\mathcal{F}_h(\Psi))(x_0),$$

and in particular

$$\begin{aligned} \langle x_0 | \hat{p} | \Psi \rangle &= -i\hbar \frac{\partial \Psi}{\partial x}(x_0), \\ \langle x_0 | \hat{x} | \Psi \rangle &= x_0 \Psi(x_0). \end{aligned}$$

In view of Example 32, we have that $\hat{f} \circ \hat{g} = \widehat{f \star_M g}$.

5. APPLICATIONS AND FURTHER DIRECTIONS

In this section, we describe in an informal way some of the applications of symplectic micromorphism enhancements and quantization. Roughly, symplectic microgeometry and its quantized version offer a framework in which dynamics can be expressed in a purely categorical way both at the classical and quantum level. There is a special monoid in the microsymplectic category, the energy monoids whose action on cotangent microbundles represent the Hamiltonian flows of classical mechanics. Symmetries of space are implemented by the action of general monoids in the microsymplectic category and they correspond to the presence of a Poisson structure on the core of the monoid. The enhancement and quantization of this picture recovers the semi-classical version of the Schrödinger evolution. Let us start with some consideration on the functoriality of our constructions, very closely related the work of Khudaverdian and Voronov cited in the Introduction.

5.1. Quantization functors. In paragraph 4.4, we restricted the category of manifolds we started with to the category \mathcal{E} formed by the open subsets of the Euclidean spaces \mathbb{R}^n endowed with the canonical Euclidean metric. This extra data of a metric made it possible to produce a canonical (global even!) exponential $\Psi_x(v) = v + x$ for each object in \mathcal{E} since Ψ is exactly the exponential associated to the affine connection generated by the metric. (Note that the only piece of extra data we need for our construction on the top of the manifold is the exponential Ψ ; we do not make direct use of the metric or the affine structure.) This allowed us to identify, in a non ambiguous way, the class of semiclassical Fourier integral operators $\mathbf{SFour}_h(\mathcal{E})$ with wave fronts in the category $\mathbf{Cot}_{\text{mic}}^{\text{ext}}(\mathcal{E})$ formed by the symplectic micromorphisms between the cotangent bundles on the object of \mathcal{E} with the space

of symplectic micromorphism enhancements. In symbols, we had the identification

$$\mathbf{SFour}_h([V], \phi) = |\Omega_h|^{\frac{1}{2}}([V], \phi),$$

thanks to the canonical way we could associate to $([V], \phi)$ a generating function S_ϕ^f and to a semiclassical FIO with wave front $([V], \phi)$ the integral representation (4.4). This allowed us further to endow the corresponding collection $E(\mathcal{E})$ of enhanced symplectic micromorphisms with the structure of a category by pulling back the FIO composition to their total symbols as in paragraph 4.4.3. This situation can be summarized by the following commutative diagram of functors

$$\begin{array}{ccc} \mathbf{SFour}_h(\mathcal{E}) & \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{Q_h} \end{array} & E(\mathcal{E}) \\ & \begin{array}{c} \searrow \text{WF} \\ \swarrow U \end{array} & \\ & & \mathbf{Cot}_{\text{mic}}^{\text{ext}}(\mathcal{E}) \end{array}$$

where σ is the total symbol functor, WF the wave front functor, Q_h the quantization functor, and U the forgetful functor. Observe that σ is an isomorphism of categories with inverse functor given by the quantization functor Q_h .

We conclude this discussion by observing that the functorial constructions above apply in the very exact same way to any other category \mathcal{E} of manifolds carrying some extra data that make it possible to assign in a canonical way a micro exponential to each of the manifolds. For instance, we may take \mathcal{E} to be the category of riemannian manifolds and the canonical micro exponentials given by the germ of the Levi-Civita connection exponential. Note also that the functor Q_h is strictly monoidal with respect to the obvious monoidal structures on the source and target categories.

In the sequel, we will comment on possible applications of our construction, sometimes assuming implicitly a suitable category \mathcal{E} of manifolds endowed with the appropriate extra data.

5.2. The energy monoid. The Lie algebra \mathcal{T} of the time translation group $(\mathbb{R}, +)$ is the abelian Lie algebra on \mathbb{R} . Its dual, which we denote by \mathcal{E} , is thus the Poisson manifold, with zero Poisson structure.

We call \mathcal{T} the Lie algebra of time and \mathcal{E} the Poisson manifold of energy. Its cotangent bundle $T^*\mathcal{E} = \mathcal{T} \times \mathcal{E}$ is a trivial symplectic microgroupoid with source and target coinciding with the bundle projection; the composable pair space is $T^*\mathcal{E} \oplus T^*\mathcal{E}$ and the groupoid product

is just the addition of times in a fiber of constant energy:

$$m_{\mathcal{E}}((t_1, E), (t_2, E)) = (t_1 + t_2, E).$$

One verifies that the graph of the groupoid product is a symplectic micromorphism

$$\mu_{\mathcal{E}} := (\text{gr}[m_{\mathcal{E}}], \Delta_{\mathcal{E}}) : T^*\mathcal{E} \otimes T^*\mathcal{E} \longrightarrow T^*\mathcal{E},$$

where the core map $\Delta_{\mathcal{E}}$ is the diagonal map on \mathcal{E} . It is easy to see that $\mu_{\mathcal{E}}$ satisfies the following associativity and unitality equations

$$\begin{aligned} \mu_{\mathcal{E}} \circ (\mu_{\mathcal{E}} \otimes \text{id}) &= \mu_{\mathcal{E}} \circ (\text{id} \otimes \mu_{\mathcal{E}}), \\ \mu_{\mathcal{E}} \circ (e_{\mathcal{E}} \otimes \text{id}) &= \text{id} = \mu_{\mathcal{E}} \circ (\text{id} \otimes e_{\mathcal{E}}), \end{aligned}$$

where $e_{\mathcal{E}}$ is the unique symplectic micromorphism from the cotangent bundle of the point to $T^*\mathcal{E}$. In other words, $(T^*\mathcal{E}, \mu_{\mathcal{E}})$ is a monoid object in the microsymplectic category (see [10] for a systematic study of these monoids).

5.3. Classical flows as symplectic micromorphisms. Consider a classical hamiltonian system $H : T^*Q \rightarrow \mathbb{R}$. The time evolution Ψ_t generated by H produces a lagrangian submanifold

$$W_H := \left\{ \left((t, H(\Psi_t(z))), z, \Psi_t(z) \right) : t \in I, z \in T^*Q \right\},$$

of $\overline{T^*\mathcal{E}} \times \overline{T^*Q} \times T^*Q$, where I is the maximal interval on which the flow Ψ_t is defined. It gives a symplectic micromorphism

$$\rho_H := ([W_H], H|_{Z_Q} \times \text{id}_Q) : T^*\mathcal{E} \otimes T^*Q \longrightarrow T^*Q.$$

Proposition 36. *Let $H \in C^\infty(T^*Q)$ be a hamiltonian. Then that*

$$\begin{aligned} \rho_H \circ (e_{\mathcal{E}} \otimes \text{id}) &= \text{id}, \\ \rho_H \circ (\mu_{\mathcal{E}} \otimes \text{id}) &= \rho_H \circ (\text{id} \otimes \rho_H). \end{aligned}$$

*In other words, this turns T^*Q into a $T^*\mathcal{E}$ -module in the microsymplectic category.*

Proof. A straightforward composition of canonical (micro) relations yields

$$\begin{aligned} \rho_H \circ (\text{id} \otimes \rho_H) &= \left\{ \left(t_2, H(\Psi_{t_1} \circ \Psi_{t_2}(z)), t_1, H(\Psi_{t_1}(z)), z, \Psi_{t_2} \circ \Psi_{t_1}(z) \right) : z, t \right\}, \\ \rho_H \circ (\mu_{\mathcal{E}} \otimes \text{id}) &= \left\{ \left(t_2, H(\Psi_{t_1+t_2}(z)), t_1, H(\Psi_{t_1+t_2}(z)), z, \Psi_{t_1+t_2}(z) \right) : z, t \right\}. \end{aligned}$$

We see that these microfolds coincide for time independent hamiltonians, since $\Psi_{t_1} \circ \Psi_{t_2} = \Psi_{t_1+t_2}$ and $H(\Psi_t(z)) = H(z)$ for all t_1, t_2 and t . One gets the unitality axiom in a similar way. \square

The Hamilton-Jacobi formulation of classical mechanics tells us that, in Darboux coordinates, the hamiltonian flow

$$(p_t, x_t) = \Psi_t^H(p, x)$$

admits, for short times, a generating function S satisfying

$$\begin{aligned} \partial_t S(t, p, x_t) &= H_t(p_t, x_t), \\ \partial_p S(t, p, x_t) &= x, \\ \partial_x S(t, p, x_t) &= p_t, \end{aligned}$$

with the initial condition $S(0, p, x) = \langle p, x \rangle$. This is precisely the generating function of the symplectic micromorphism ρ_H .

5.4. Classical symmetries. It is possible to generalize the previous scheme to a general Hamiltonian action of a Lie group G on T^*P with momentum map $J : T^*P \rightarrow \mathcal{G}^*$, where \mathcal{G} is the Lie algebra of G . In this case, we define the symmetry submanifold to be

$$W_G := \left\{ \left((v, J(\exp(v)z)), z, \exp(v)z \right) : v \in U, z \in T^*Q \right\},$$

where U is the maximal neighborhood of 0 in the Lie algebra \mathcal{G} on which the exponential mapping $\exp : \mathcal{G} \rightarrow G$ is defined. Taking the germ of W_G around the graph of $J|_Q \times \text{id}_Q$, yields a symplectic micromorphism ρ_G from $T^*\mathcal{G}^* \otimes T^*Q$ (where \otimes denotes the tensor product on objects defined by the Cartesian product) to T^*Q . Now, thanks to the exponential mapping, we can define a generating function germ from $T^*\mathcal{G}^* \oplus T^*\mathcal{G}^*$ to \mathbb{R} via the formula

$$S_G(v, w, \mu) := \left\langle \mu, \exp^{-1}(\exp(v)\exp(w)) \right\rangle,$$

where \langle, \rangle is the canonical pairing between the Lie algebra and its dual. This generating function germ defines a symplectic micromorphism μ_G from $T^*\mathcal{G}^* \otimes T^*\mathcal{G}^*$ to $T^*\mathcal{G}^*$. One can show that $(T^*\mathcal{G}^*, \mu_G)$ is a monoid and that (T^*Q, ρ_G) is a $T^*\mathcal{G}^*$ -module.

5.5. Generalized symmetries. We can consider more general symmetries in microgeometry by allowing arbitrary monoids (T^*P, μ) to act on phase spaces T^*Q . It turns out that a general monoid induces a Poisson structure on its core P , and, conversely, any Poisson manifold (P, Π) induces a monoid (T^*P, μ) by considering the local symplectic groupoid integrating P and by taking μ to be the germ of the groupoid product around the graph of the diagonal map on P ([10]). Moreover, one can show that a T^*P -module,

$$\rho : T^*P \otimes T^*Q \longrightarrow T^*Q,$$

induces a momentum map germ, i.e., a Poisson map germ $[J]$ from the cotangent microbundle T^*Q to the Poisson manifold P . Such situations may arise for instance if we start with a Lie groupoid acting in an hamiltonian way on a phase space.

5.6. Quantization of the energy monoid. A straightforward, but lengthy, computation would show that the following enhancement $([1], \mu_{\mathcal{E}})$ of the energy monoid $(T^*\mathcal{E}, \mu_{\mathcal{E}})$ yields a monoid in the category of enhanced symplectic microfolds. More interesting is the quantization of this enhanced monoid: it should produce a associative product on $\mathcal{H}_{\hbar\mathcal{E}}$, that is on $C^\infty(\mathcal{E})[[\hbar]]$. Let us compute it:

$$Q_{\hbar}([1], \mu_{\mathcal{E}})(f \otimes g)(E) = \text{s.c.} \int \hat{f}(t_1)\hat{g}(t_2)e^{-\frac{i}{\hbar}S(t_1, t_2, E)} \frac{dt_1 dt_2}{(2\pi\hbar)},$$

where \hat{f} and \hat{g} are the asymptotic Fourier transforms of f and g , and where S is the generating function of the energy monoid. Since $S = (t_1 + t_2)E$, we see that the quantization of $([1], \mu_{\mathcal{E}})$ yields the extension of the usual product of functions on $C^\infty(\mathcal{E})[[\hbar]]$.

5.7. Quantization of Poisson manifolds. Suppose we have a monoid (T^*P, μ) with induced Poisson structure Π on P . Since the core map of μ is the diagonal map Δ on P , an enhancement of μ is given by any half-density germ

$$[a] \in |\Omega_{\hbar}|^{\frac{1}{2}}(T^*P \oplus T^*P)$$

around the zero-section. Given any such enhancement of μ , we obtain thus an operator

$$\star_{\hbar} := Q_{\hbar}([a], \mu) : \mathcal{H}_{\hbar}(P) \otimes \mathcal{H}_{\hbar}(P) \longrightarrow \mathcal{H}_{\hbar}(P).$$

The functoriality of Q_{\hbar} implies that \star_{\hbar} is an associative product if and only if the enhancement is associative, in which case the associative algebra $(\mathcal{H}_{\hbar}(P), \star_{\hbar})$ should be considered as the quantization of the Poisson manifold (P, Π) . In this context, the quantization of Poisson manifolds becomes equivalent to the existence of associative enhancements. When P is an open subset of \mathbb{R}^n we have the identification of $\mathcal{H}_{\hbar}(P)$ with $C^\infty(P)[[\hbar]]$, and we can write the product as

$$f \star_{\hbar} g(x) = \text{s.c.} \int \hat{f}(p_1)\hat{g}(p_2)a(p_1, p_2, x)e^{\frac{i}{\hbar}S(p_1, p_2, x)} \frac{dp_1 dp_2}{(2\pi)^m},$$

where S is the generating function of μ . Note that when the Poisson structure is the zero Poisson structure, or, equivalently, when μ is the graph of vector bundle addition in T^*P , we have that the generating function is $S = \langle p_1 + p_2, x \rangle$ as in the energy monoid case. The associative enhancement $([1], \mu)$ gives us back the usual product of functions.

General associative enhancements of general monoids (T^*P, μ) corresponding to non-zero Poisson structures should yield star-products on $C^\infty(P)[[\hbar]]$ upon asymptotic expansion. The first non trivial case is given by the integral representation of the Moyal product on \mathbb{R}^{2n} endowed with its canonical symplectic form.

This is to be related to a quantization program for Poisson manifolds relying on asymptotic integrals of the Moyal type and on the symplectic groupoid of the Poisson manifold (see [20, 33, 34]). In this approach, a star-product should be obtained from the asymptotic expansion of a semiclassical Fourier integral operator whose phase is the generating function of the symplectic groupoid. In the special case of symplectic symmetric spaces, similar integrals were worked out in [3, 6, 26, 25]. For the linear Poisson structures, the paper [1] gives an integral version of Kontsevich's star-product, where the generating function is the Baker-Campbell-Hausdorff formula of the associated Lie algebra. In [7], it has been shown that this latter generating function is exactly the generating function of the corresponding symplectic groupoid and that, more generally, generating functions of the local symplectic groupoid for Poisson structures on open subsets of \mathbb{R}^n , can be extracted from the tree-level part of Kontsevich star-product ([22]).

In the same spirit, Wagemann and one of the authors [14] start by showing that the Gutt star-product can be understood as the quantization in the above sense of a symplectic micromorphism, and then they extend this construction to quantize Leibniz algebras. Both in [14] and [13], formal deformation quantizations are obtained from symplectic micromorphism quantization by taking Feynman expansions of the corresponding oscillatory integrals.

5.8. Quantization of symmetries. Suppose now that we have a general symmetry

$$\rho : T^*P \otimes T^*Q \longrightarrow T^*Q,$$

where (T^*P, μ) is a general monoid with induced Poisson structure Π on P , acting on T^*Q . One can show that the core map of ρ is $J|_{Z_Q} \times \text{id}_Q$, where $J : T^*Q \rightarrow P$ is a Poisson map germ around the zero section of T^*Q , which can be considered as the momentum map of the action. An enhancement of ρ is thus a half-density germ

$$[b] \in |\Omega_{\hbar}|^{\frac{1}{2}}(J^*_Q(T^*P) \oplus T^*Q)$$

around the zero section. Quantizing this data, we obtain an operator

$$Q_{\hbar}([b], \rho) : \mathcal{H}_{\hbar P} \otimes \mathcal{H}_{\hbar Q} \longrightarrow \mathcal{H}_{\hbar Q},$$

which is a representation of the quantum algebra $(\mathcal{H}_{\hbar P}, \star_{\hbar})$ on \mathcal{H}_Q provided that

$$[b] \circ ([1] \otimes [b]) = [b] \circ ([a] \otimes [1]).$$

This approach has been used in [13], [11], and [12] to quantize group actions through their cotangent lifts as well as to quantize their momentum maps.

5.9. Quantization of the classical flow. We want to quantize the symplectic micromorphism ρ_H associated to the classical flow Ψ_t^H on $T^*\mathbb{R}^n$ of a hamiltonian $H : T^*\mathbb{R}^n \rightarrow \mathbb{R}$ as described in paragraph 5.3. Since, in general, $Q_{\hbar}(T^*\mathbb{R}^n)$ can be identified with the space of L^2 -functions on \mathbb{R}^n , we obtain, after quantization, a linear operator

$$Q_{\hbar}([a], \rho_H) : L^2(\mathcal{E}) \otimes L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

for any enhancement $a = a(t, p_1, x_2)$ of the symplectic micromorphism. Now, we may define the following the operator on $L^2(\mathbb{R}^n)$ by

$$U_t^a := Q_{\hbar}([a], \rho_H)(u_t \otimes \cdot),$$

where $u_t(E) = e^{-\frac{i}{\hbar}t_0E}$ is the state with “definite time” t_0 on the space of energies. An explicit computation yields

$$(U_t^a \Phi)(x) := \frac{1}{(2\pi\hbar)^n} \text{s.c.} \int \Phi(x_1) a(t_0, p_1, x) e^{\frac{i}{\hbar}p_1x_1 - S(t_0, p_1, x)} dp_1 dx_1,$$

where S is the generating function of phase flow Ψ_H^t solution of the Hamilton-Jacobi equation as explained in paragraph 5.3. Then, a classical result of semi-classical analysis can now be reformulated in terms of symplectic micromorphisms and their enhancements: For any Hamiltonian $H : T^*\mathbb{R}^n \rightarrow \mathbb{R}$, there exists an enhancement of $([a], \rho_H)$ such that U_t^a is the propagator, modulo \hbar^∞ , of the Schrödinger equation with quantum hamiltonian given by the semi-classical pseudo-differential operator

$$\hat{H} = Q_{\hbar}([H], \text{id}_{T^*\mathbb{R}^n}),$$

where the germ $[H]$ is understood as an enhancement of the identity map on $T^*\mathbb{R}^n$ seen as symplectic micromorphism. Moreover, it is easy to show that, $U_0^a = \text{id}$ and $U_{t_2}^a \circ U_{t_1}^a = U_{t_1+t_2}^a$ (when defined) is equivalent to $([a], \rho_H)$ being an action of energy monoid enhanced as in paragraph 5.6 on T^*Q thanks to the fact that

$$u_{t_1+t_2} = Q_{\hbar}([1], \mu_{\mathcal{E}})(u_{t_1} \otimes u_{t_2}),$$

and the module axioms:

$$\begin{aligned} (U_{t_2}^a \circ U_{t_1}^a)\Phi &= \left(Q_{\hbar}([a], \rho_H) \circ (\text{id} \otimes Q_{\hbar}([a], \rho_H)) \right) (u_{t_2} \otimes u_{t_1} \otimes \Phi), \\ &= Q_{\hbar}([a], \rho_H) \circ \left((Q_{\hbar}([1], \mu_{\mathcal{E}})(u_{t_2} \otimes u_{t_1})) \otimes \Phi \right), \\ &= U_{t_1+t_2}^a \Phi. \end{aligned}$$

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INSTITUT FÜR MATHEMATIK, UNIVERSITÄT ZÜRICH, WINTERTHURERSTRASSE
190, CH-8057 ZÜRICH, SWITZERLAND

Email address: cattaneo@math.uzh.ch

GOOGLE, CLANWILLIAM PL, GRAND CANAL DOCK, DUBLIN, IRELAND

Email address: dherin@google.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY
CA 94720 USA AND DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY,
STANFORD, CA 94305 USA

Email address: alanw@math.berkeley.edu