A DEFINITION AND SOME CHARACTERISTIC PROPERTIES OF PSEUDO-STOPPING TIMES

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Abstract. Recently, D. Williams [19] gave an explicit example of a random time ρ associated with Brownian motion such that ρ is not a stopping time but $\mathbb{E}M_\rho = \mathbb{E}M_0$ for every bounded martingale M. The aim of this paper is to give some characterizations for such random times, which we call pseudo-stopping times, and to construct further examples, using techniques of progressive enlargements of filtrations.

1. Introduction

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space, and $\rho : (\Omega, \mathcal{F}) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ be a random time. We recall that the space $\mathcal{H}^1$ is the Banach space of (càdlàg) $(\mathcal{F}_t)$-martingales (M_t) such that

$$\|M\|_{\mathcal{H}^1} = \mathbb{E}\left[\sup_{t\geq 0} |M_t|\right] < \infty.$$

Definition 1. *We say that ρ is a $(\mathcal{F}_t)$ pseudo-stopping time if for every $(\mathcal{F}_t)$-martingale (M_t) in $\mathcal{H}^1$, we have*

$$\mathbb{E}M_\rho = \mathbb{E}M_0. \tag{1.1}$$

Remark 1. *It is equivalent to assume that (1.1) holds for bounded martingales, since these are dense in $\mathcal{H}^1$.*

We indicate immediately that a class of pseudo-stopping times with respect to a filtration $(\mathcal{F}_t)$ which are not in general $(\mathcal{F}_t)$ stopping times may be obtained by considering stopping times with respect to a larger filtration $(\mathcal{G}_t)$ such that $(\mathcal{F}_t)$ is immersed in $(\mathcal{G}_t)$, i.e: every $(\mathcal{F}_t)$ martingale is a $(\mathcal{G}_t)$ martingale. This situation is described in ([3]) and refered to there as the (H) hypothesis. We shall discuss this situation in more details in Section 3. For now, we give an example. Let $B_t = \left(B_t^1, \ldots, B_t^d\right)$ be a d-dimensional Brownian motion, and $R_t = |B_t|$, $t \geq 0$, its radial part; it is well known that

$$(\mathcal{R}_t \equiv \sigma\left\{R_s,\ s \leq t\right\},\ t \geq 0),$$

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the natural filtration of R, is immersed in $(\mathcal{B}_t \equiv \sigma\{B_s,\ s \leq t\},\ t \geq 0)$, the natural filtration of B. Thus an example of $(\mathcal{R}_t)$ pseudo-stopping time is:

$$T_a^{(1)} = \inf\left\{t,\ B_t^1 > a\right\}.$$

Recently, D. Williams [19] showed that with respect to the filtration $(\mathcal{F}_t)$ generated by a one dimensional Brownian motion $(B_t)_{t\geq 0}$, there exist pseudo-stopping times ρ which are not $(\mathcal{F}_t)$ stopping times. D. Williams' example is the following: let

$$T_1 = \inf\{t:\ B_t = 1\},\ \sigma = \sup\{t < T_1:\ B_t = 0\};$$

then

$$\rho = \sup\{s < \sigma:\ B_s = S_s\},\ \text{where } S_s = \sup_{u \leq s} B_u$$

is a $(\mathcal{F}_t)$ pseudo-stopping time. This paper has two main aims:

- to understand better the nature of pseudo-stopping times;
- to construct further examples of pseudo-stopping times;

In Section 2, with the help of the theory of progressive enlargements of filtrations, we give some equivalent properties for ρ to be a pseudo-stopping time. We also comment there on the difference between (1.1) and the property

$$\mathbb{E}\left[M_\infty \mid \mathcal{F}_\rho\right] = M_\rho \tag{1.2}$$

for every uniformly integrable $(\mathcal{F}_t)$-martingale (M_t), which was shown by Knight and Maisonneuve [12] to be equivalent to ρ being a $(\mathcal{F}_t)$-stopping time.

In Section 3, we give some other examples of pseudo-stopping times. We associate with the end L of a given $(\mathcal{F}_t)$ predictable set Γ, i.e

$$L = \sup\{t : (t,\omega) \in \Gamma\},$$

a pseudo-stopping time $\rho < L$ in a manner which generalizes D. Williams' example. We also link the pseudo-stopping times with randomized stopping times.

In Section 4, we give a discrete time analogue of the Williams random time ρ. This approach is based on the analogue of Williams' path decomposition obtained by Le Gall for the standard random walk [13].

2. Some characteristic properties of pseudo-stopping times

2.1. **Basic facts about progressive enlargements.** We recall here some basic results about the progressive enlargement of a filtration $(\mathcal{F}_t)$ by a random time ρ. All these results may be found in [11], [9], [20], [4] or [16].

We enlarge the initial filtration $(\mathcal{F}_t)$ with the process $(\rho \wedge t)_{t\geq 0}$, so that the new enlarged filtration $(\mathcal{F}_t^\rho)_{t\geq 0}$ is the smallest filtration containing $(\mathcal{F}_t)$

and making ρ a stopping time. A few processes play a crucial role in our discussion:

- the $(\mathcal{F}_t)$-supermartingale
$$Z_t^\rho = \mathbb{P}\left[\rho > t \mid \mathcal{F}_t\right] \tag{2.1}$$
chosen to be càdlàg, associated to ρ by Azéma (see [9] for detailed references);
- the $(\mathcal{F}_t)$-dual optional and predictable projections of the process $1_{\{\rho \le t\}}$, denoted respectively by A_t^ρ and a_t^ρ;
- the càdlàg martingale
$$\mu_t^\rho = \mathbb{E}\left[A_\infty^\rho \mid \mathcal{F}_t\right] = A_t^\rho + Z_t^\rho$$
which is in BMO$(\mathcal{F}_t)$ (see [4] or [20]). We recall that the space of BMO martingales (see [6] for more details and references) is the Banach space of (càdlàg) square integrable $(\mathcal{F}_t)$-martingales (Y_t) which satisfy
$$\|Y\|_{BMO}^2 = \text{essup}_T \mathbb{E}\left[(Y_\infty - Y_{T-})^2 \mid \mathcal{F}_T\right] < \infty$$
where T ranges over all $(\mathcal{F}_t)$-stopping times.

We also consider the Doob-Meyer decomposition of (2.1):
$$Z_t^\rho = m_t^\rho - a_t^\rho$$
If ρ avoids any $(\mathcal{F}_t)$-stopping time, that is to say $P\left[\rho = T > 0\right] = 0$ for any stopping time T, then $A_t^\rho = a_t^\rho$ is continuous.

Finally, we recall that every $(\mathcal{F}_t)$-local martingale (M_t), stopped at ρ, is a $(\mathcal{F}_t^\rho)$ semimartingale, with canonical decomposition:
$$M_{t\wedge\rho} = \widetilde{M}_t + \int_0^{t\wedge\rho} \frac{d < M, \mu^\rho >_s}{Z_{s-}^\rho} \tag{2.2}$$
where $\left(\widetilde{M}_t\right)$ is an $(\mathcal{F}_t^\rho)$-local martingale.

Remark 2. *We also recall that in a filtration $(\mathcal{F}_t)$ where all martingales are continuous, $A_t^\rho = a_t^\rho$ since optional processes are predictable (see* [17]*, chapter IV).*

2.2. **A characterization of pseudo-stopping times.** We now discuss some characteristic properties of pseudo-stopping times. We assume throughout that $\mathbb{P}\left[\rho = \infty\right] = 0$.

Theorem 1. *The following four properties are equivalent:*

(1) *ρ is a $(\mathcal{F}_t)$ pseudo-stopping time, i.e (1.1) is satisfied;*
(2) *$\mu_t^\rho \equiv 1$, a.s*
(3) *$A_\infty^\rho \equiv 1$, a.s*

(4) *every* $(\mathcal{F}_t)$ *local martingale* (M_t) *satisfies*

$$(M_{t\wedge\rho})_{t\geq 0} \text{ is a local } (\mathcal{F}_t^\rho) \text{ martingale.}$$

If, furthermore, all $(\mathcal{F}_t)$ *martingales are continuous, then each of the preceding properties is equivalent to*

(5)

$$(Z_t^\rho)_{t\geq 0} \text{ is a decreasing } (\mathcal{F}_t) \text{ predictable process}$$

Proof. (1) $\Longrightarrow$ (2) For every square integrable $(\mathcal{F}_t)$ martingale (M_t), we have

$$\mathbb{E}[M_\rho] = \mathbb{E}\left[\int_0^\infty M_s dA_s^\rho\right] = \mathbb{E}[M_\infty A_\infty^\rho] = \mathbb{E}[M_\infty \mu_\infty^\rho].$$

Since $\mathbb{E}M_\rho = \mathbb{E}M_0 = \mathbb{E}M_\infty$, we have

$$\mathbb{E}[M_\infty] = \mathbb{E}[M_\infty A_\infty^\rho] = \mathbb{E}[M_\infty \mu_\infty^\rho].$$

Consequently, $\mu_\infty^\rho \equiv 1$, *a.s*, hence $\mu_t^\rho \equiv 1$, *a.s* which is equivalent to: $A_\infty^\rho \equiv 1$, *a.s.* Hence, 2. and 3. are equivalent.

(2) $\Longrightarrow$ (4). This is a consequence of the decomposition formula (2.2).

(4) $\Longrightarrow$ (1). It suffices to consider any $\mathcal{H}^1$-martingale (M_t), which, assuming 4., satisfies: $(M_{t\wedge\rho})_{t\geq 0}$ is a martingale in the enlarged filtration, for which ρ is a stopping time. Then as a consequence of the optional stopping theorem applied in $(\mathcal{F}_t^\rho)$ at time ρ, we get

$$\mathbb{E}[M_\rho] = \mathbb{E}[M_0],$$

hence ρ is a pseudo-stopping time.

Finally, in the case where all $(\mathcal{F}_t)$ martingales are continuous, we show:

a) (2) $\Rightarrow$ (5) If ρ is a pseudo-stopping time, then Z_t^ρ decomposes as

$$Z_t^\rho = 1 - A_t^\rho.$$

As all $(\mathcal{F}_t)$ martingales are continuous, optional processes are in fact predictable, and so (Z_t^ρ) is a predictable decreasing process.

b) (5) $\Rightarrow$ (2) Conversely, if (Z_t^ρ) is a predictable decreasing process, then from the unicity in the Doob-Meyer decomposition, the martingale part μ_t^ρ is constant, i.e. $\mu_t^\rho \equiv 1$, *a.s.* Thus, (2) is satisfied. □

In the next proposition, we deal with uniformly integrable martingales (M_t) instead of martingales in $\mathcal{H}^1$ (or $\mathcal{H}^2, \ldots$).

Proposition 1. *The following properties are equivalent:*

(1) ρ *is a* $(\mathcal{F}_t)$ *pseudo-stopping time;*
(2) *for every uniformly integrable martingale*

$$\mathbb{E}[|M_\rho|] \leq \mathbb{E}[|M_\infty|].$$

Remark 3. *In fact, we shall further show in the next proof, that for* ρ *a pseudo-stopping time and for* (M_t) *a uniformly integrable martingale:*

$$\mathbb{E}[|M_\rho|] < \infty, \text{ and } \mathbb{E}[M_\rho] = \mathbb{E}[M_\infty].$$

Proof. (1) $\Rightarrow$ (2) If (M_t) is uniformly integrable, it may be decomposed as:

$$M_t = M_t^{(+)} - M_t^{(-)} \tag{2.3}$$

where

$$M_t^{(+)} = \mathbb{E}\left[M_\infty^+ \mid \mathcal{F}_t\right] \text{ and } M_t^{(-)} = \mathbb{E}\left[M_\infty^- \mid \mathcal{F}_t\right].$$

(Note that $M_\infty^\pm$ indicates the positive and negative parts of M_∞, whereas $\left(M_t^{(\pm)}\right)$ are the martingales with terminal values $M_\infty^\pm$). Thus to prove (2) it suffices to prove

$$\mathbb{E}\left[M_\rho\right] = \mathbb{E}\left[M_\infty\right],$$

under the further assumption that $M \geq 0$. In this latter case, we have $M_t = \mathbb{E}\left[M_\infty \mid \mathcal{F}_t\right]$, with $M_\infty \geq 0$. Now let

$$M_t^{(n)} = \mathbb{E}\left[(M_\infty \wedge n) \mid \mathcal{F}_t\right].$$

$\left(M_t^{(n)}\right)$ is a bounded martingale, hence we have

$$\mathbb{E}\left[M_\infty^{(n)}\right] = \mathbb{E}\left[M_\rho^{(n)}\right].$$

We also have

$$\mathbb{P}\left[\sup_{t\geq 0}\left(M_t - M_t^{(n)}\right) > \varepsilon\right] \leq \frac{1}{\varepsilon}\mathbb{E}\left[M_\infty - M_\infty^{(n)}\right],$$

so that $\left(M_\rho^{(n)}\right)$ converges to (M_ρ) in probability; but the sequence $\left(M_\rho^{(n)}\right)$ is increasing, so it in fact converges almost surely. Hence the monotone convergence theorem yields

$$\mathbb{E}\left[M_\infty\right] = \mathbb{E}\left[M_\rho\right].$$

Finally, going back to (2.3) in the general case, we obtain:

$$\begin{aligned}\mathbb{E}\left[|M_\rho|\right] &\leq \mathbb{E}\left[M_\rho^{(+)} + M_\rho^{(-)}\right] \\ &= \mathbb{E}\left[M_\infty^+ + M_\infty^-\right] \\ &= \mathbb{E}\left[|M_\infty|\right].\end{aligned}$$

Hence (2) holds. Further, we may now write:

$$\begin{aligned}\mathbb{E}\left[M_\rho\right] &= \mathbb{E}\left[M_\rho^{(+)} - M_\rho^{(-)}\right] \\ &= \mathbb{E}\left[M_\infty^+ - M_\infty^-\right] \\ &= \mathbb{E}\left[M_\infty\right].\end{aligned}$$

(2) $\Rightarrow$ (1) We need only apply property (2) to any martingale (M_t) taking values in $[0,1]$. Thus:

$$\begin{aligned}\mathbb{E}\left[M_\rho\right] &\leq \mathbb{E}\left[M_\infty\right] \\ \mathbb{E}\left[1 - M_\rho\right] &\leq \mathbb{E}\left[1 - M_\infty\right].\end{aligned}$$

But, since the sums on both sides add up to 1, we must have:

$$\mathbb{E}\left[M_\rho\right] = \mathbb{E}\left[M_\infty\right]$$

Hence, ρ is a $(\mathcal{F}_t)$ pseudo-stopping time. □

As an application of the theorem, we can check that in D. Williams' example, his time ρ associated with a Brownian motion is a pseudo-stopping time. Indeed, the dual predictable (=optional) projection A_t^ρ of $1_{\{\rho \leq t\}}$ is $\max_{s \leq t \wedge T_1} B_s$ ([19], [18]) and $A_\infty^\rho \equiv 1$.

2.3. **Around the result of Knight and Maisonneuve.** We now comment on the statement of the fourth property in Theorem 1.

For the properties of the different sigma fields $\mathcal{F}_\rho$, $\mathcal{F}_{\rho+}$, $\mathcal{F}_{\rho-}$, associated with a general random time ρ, the reader can consult [18] or [20]. Here, we just recall the definitions:

Definition 2. *Three classical σ-fields associated with a filtration $(\mathcal{F}_t)$ and any random time ρ are:*

$$\begin{cases} \mathcal{F}_{\rho+} &= \sigma\left\{z_\rho,\ (z_t)\ \textit{any}\ (\mathcal{F}_t)\ \textit{progressively measurable process}\right\}; \\ \mathcal{F}_{\rho} &= \sigma\left\{z_\rho,\ (z_t)\ \textit{any}\ (\mathcal{F}_t)\ \textit{optional process}\right\}; \\ \mathcal{F}_{\rho-} &= \sigma\left\{z_\rho,\ (z_t)\ \textit{any}\ (\mathcal{F}_t)\ \textit{predictable process}\right\}; \end{cases}$$

The result of Knight and Maisonneuve which was recalled in the introduction may be stated as follows:

Theorem 2. *If for all uniformly integrable $(\mathcal{F}_t)$-martingales (M_t), one has*

$$\mathbb{E}\left[M_\infty \mid \mathcal{F}_\rho\right] = M_\rho, \qquad \textit{on}\ \left\{\rho < \infty\right\},$$

then ρ is a $(\mathcal{F}_t)$-stopping time (the converse is Doob's optional stopping theorem).

Refining slightly the argument in [12], we obtain the following:

Theorem 3. *If for all bounded $(\mathcal{F}_t)$-martingales (M_t), one has*

$$\mathbb{E}\left[M_\infty \mid \sigma\left\{M_\rho, \rho\right\}\right] = M_\rho, \qquad \textit{on}\ \left\{\rho < \infty\right\},$$

then ρ is a $(\mathcal{F}_t)$-stopping time.

Proof. For $t \geq 0$ we have

$$\mathbb{E}\left[M_\infty \mathbf{1}_{(\rho \leq t)}\right] = \mathbb{E}\left[M_\rho 1_{(\rho \leq t)}\right] = \mathbb{E}\left[\int_0^t M_s dA_s^\rho\right] = \mathbb{E}\left[M_\infty A_t^\rho\right].$$

Comparing the two extreme terms, we get

$$\mathbf{1}_{(\rho \leq t)} = A_t^\rho,$$

i.e ρ is a $(\mathcal{F}_t)$-stopping time. □

An interesting open question in view of what has been proved for pseudo-stopping times is whether $\mathbb{E}[M_\infty \mid M_\rho] = M_\rho$, *on* $\{\rho < \infty\}$ is equivalent to ρ being a stopping time.

To illustrate the result of Knight and Maisonneuve, we show explicitly how, in the framework of D. Williams' example, M_ρ and $\mathbb{E}[M_\infty \mid \mathcal{F}_\rho]$ differ, for

$$M_t = \exp\left(\lambda B_{t\wedge T_1} - \frac{\lambda^2}{2}(t\wedge T_1)\right), \qquad \lambda > 0.$$

We write

$$\begin{aligned} M_\infty &= \exp\left(\lambda - \frac{\lambda^2}{2}T_1\right) \\ &= \exp(\lambda)\exp\left(-\frac{\lambda^2}{2}\left(\rho + (\sigma-\rho) + (T_1-\sigma)\right)\right) \end{aligned}$$

and we compute:

$$\mathbb{E}[M_\infty \mid \mathcal{F}_\rho] = \tag{2.4}$$

$$\exp\left(\lambda - \frac{\lambda^2}{2}\rho\right)\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}(\sigma-\rho)\right) \mid \mathcal{F}_\rho\right]\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}(T_1-\sigma)\right)\right], \tag{2.5}$$

since $(T_1 - \sigma)$ is independent from $\mathcal{F}_\sigma$, (and consequently from $\mathcal{F}_\rho$, since $\mathcal{F}_\rho \subset \mathcal{F}_\sigma$).

We now recall D. Williams' path decomposition results for $(B_u)_{u\le T_1}$ on the intervals $(0,\rho)$, (ρ,σ), (σ,T_1):

- $(B_{\sigma+u})_{u\le T_1-\sigma}$ is a BES(3) process, independent of $\mathcal{F}_\sigma$; hence we have
$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}(T_1-\sigma)\right)\right] = \frac{\lambda}{\sinh(\lambda)}.$$
- S_ρ, where $S_s = \sup_{u\le s} B_u$, is uniformly distributed on $(0,1)$;
- Conditionally on $S_\rho = h$, the processes $(B_u)_{u\le\rho}$ and $(B_{\sigma-u})_{u\le\sigma-\rho}$ are two independent Brownian motions considered up to their first hitting time of h. Consequently, we have:
$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}(\sigma-\rho)\right) \mid \mathcal{F}_\rho\right] = \exp(-\lambda S_\rho).$$

  Plugging these informations in (2.4), we obtain:
$$\mathbb{E}[M_\infty \mid \mathcal{F}_\rho] = \exp\left(\lambda(1-B_\rho) - \frac{\lambda^2}{2}\rho\right)\left(\frac{\lambda}{\sinh(\lambda)}\right),$$
  whilst
$$M_\rho = \exp\left(\lambda B_\rho - \frac{\lambda^2}{2}\rho\right) \tag{2.6}$$
  and these two quantities are obviously different.

2.4. **Further properties of pseudo-stopping times.** Besides the assumption that ρ is a $(\mathcal{F}_t)$ pseudo-stopping time, we also make the hypothesis that ρ avoids all $(\mathcal{F}_t)$-stopping times. We saw that in this case

$$a_t^\rho = A_t^\rho = 1 - Z_t^\rho$$

is continuous.

For simplicity, we shall write (Z_u) instead of (Z_u^ρ).

Proposition 2. *Under the previous hypotheses, for all uniformly integrable $(\mathcal{F}_t)$ martingales (M_t), and all bounded Borel measurable functions f, one has:*

$$\mathbb{E}\left[M_\rho f\left(Z_\rho\right)\right] = \mathbb{E}\left[M_0\right] \int_0^1 f\left(x\right) dx = \mathbb{E}\left[M_\rho\right] \int_0^1 f\left(x\right) dx.$$

Remark 4. *On the other hand it is not true that*

$$\mathbb{E}\left[M_\infty f\left(Z_\rho\right)\right] = \mathbb{E}\left[M_\rho f\left(Z_\rho\right)\right], \tag{2.7}$$

for every bounded Borel function f. Indeed, from Proposition 2, the right hand side of (2.7) is equal to:

$$\mathbb{E}\left[M_\infty \int_0^1 f\left(x\right) dx\right].$$

Thus, our hypothesis (2.7) would imply the absurd equality between $f\left(Z_\rho\right)$ and $\int_0^1 f\left(x\right) dx$.

Proof. (of Proposition 2) Under our assumptions, we have

$$\begin{aligned}
\mathbb{E}\left[M_\rho f\left(Z_\rho\right)\right] &= \mathbb{E}\left[\int_0^\infty M_u f\left(Z_u\right) dA_u^\rho\right] \\
&= \mathbb{E}\left[\int_0^\infty M_u f\left(1 - A_u^\rho\right) dA_u^\rho\right] \\
&= \mathbb{E}\left[M_\infty \int_0^\infty f\left(1 - A_u^\rho\right) dA_u^\rho\right] \\
&= \mathbb{E}\left[M_\infty \int_0^1 f\left(1 - x\right) dx\right] \\
&= \mathbb{E}\left[M_\infty \int_0^1 f\left(x\right) dx\right].
\end{aligned}$$

□

Taking $M_t \equiv 1$, we find that (Z_ρ) is uniformly distributed on $(0, 1)$, which is already known ([11], [20]) since (recalling that Z_u is decreasing)

$$Z_\rho = \inf_{u \le \rho} Z_u.$$

In fact we have a stronger result: under all changes of probability on $\mathcal{F}_\rho$, of the form

$$d\mathbb{Q} = M_\rho d\mathbb{P}$$

where (M_t) is a positive uniformly integrable $(\mathcal{F}_t)$-martingale such that $\mathbb{E}[M_0] = 1$, the law of Z_ρ (is unchanged and) is uniform.

Corollary 1. *Under the assumptions of Proposition 2, we have*

$$\mathbb{E}[M_\rho \mid Z_\rho] = \mathbb{E}[M_\rho] = \mathbb{E}[M_0]$$

On the other hand, the quantity $\mathbb{E}[M_\infty \mid Z_\rho]$ is not easy to evaluate, as is seen with D. Williams' example, and is different from $\mathbb{E}[M_\rho \mid Z_\rho]$. Indeed, in this framework and with the already used notations:

$$\mathbb{E}[M_\infty \mid Z_\rho] = \exp(\lambda)\,\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}T_1\right) \mid B_\rho\right].$$

Decomposing again T_1 as $T_1 = \rho+(\sigma-\rho)+(T_1-\sigma)$, and using D. Williams" path decomposition, we obtain:

$$\begin{aligned}\mathbb{E}[M_\infty \mid Z_\rho] &= \exp(\lambda)\left(\frac{\lambda}{\sinh(\lambda)}\right)\exp(-\lambda B_\rho)\,\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\rho\right) \mid B_\rho\right] \\ &= \left(\frac{2\lambda}{1-\exp(-2\lambda)}\right)\exp(-2\lambda B_\rho).\end{aligned}$$

Corollary 2. *The family $\{M_\rho;\ M$ uniformly integrable $(\mathcal{F}_t)$-martingale$\}$ is not dense in $L^1(\mathcal{F}_\rho)$.*

Proof. From Proposition 2, the variable $\left(f(Z_\rho) - \int_0^1 f(x)\,dx\right)$ is orthogonal to M_ρ. □

This negative result led us to look for some representation of the generic element of $L^1(\mathcal{F}_\rho)$ in terms of $(\mathcal{F}_t)$-martingales taken at time ρ on one hand, and the variable Z_ρ, on the other hand.

Proposition 3. (i). *Let $K : [0,1]\times\mathbb{R}_+\times\Omega \to \mathbb{R}_+$, be a $\mathcal{B}_{[0,1]}\otimes\mathcal{P}(\mathcal{F}_\bullet)$ measurable process, where $\mathcal{P}(\mathcal{F}_\bullet)$ denotes the $(\mathcal{F}_t)$ predictable σ-field on $\mathbb{R}_+\times\Omega$. Then:*

$$\mathbb{E}[K(1-Z_\rho,\rho)] = \mathbb{E}\left[\int_0^1 dy K(y,\alpha_y)\right] \tag{2.8}$$

where

$$\alpha_y = \inf\{u:\ A_u^\rho > y\}.$$

(ii). *Let $(H_u, u \geq 0)$ be a bounded predictable process. Define a measurable family $(M_t^y)_{t\geq 0}$ of martingales through their terminal values:*

$$M_\infty^y = H_{\alpha_y}.$$

Then

$$H_\rho = M_\rho^{1-Z_\rho},\ a.s.$$

Proof. (i). This follows from the monotone class theorem, once we have shown:

$$\mathbb{E}[f(1-Z_\rho)H_\rho] = \mathbb{E}\left[\int_0^1 dy f(y) H_{\alpha_y}\right] \tag{2.9}$$

for every bounded predictable process H and every Borel bounded function f. But, this identity follows from the fact that: $1 - Z_\rho = A_\rho$; and so:

$$\begin{aligned}\mathbb{E}\left[f\left(A_\rho\right)H_\rho\right] &= \mathbb{E}\left[\int_0^\infty dA_u f\left(A_u\right)H_u\right] \\ &= \mathbb{E}\left[\int_0^1 dy f\left(y\right)H_{\alpha_y}\right].\end{aligned}$$

We shall prove the second statement by showing that for every bounded (k_u) predictable process

$$\mathbb{E}\left[k_\rho H_\rho\right] = \mathbb{E}\left[k_\rho M_\rho^{1-Z_\rho}\right].$$

From (2.8), we deduce:

$$\begin{aligned}\mathbb{E}\left[k_\rho M_\rho^{1-Z_\rho}\right] &= \mathbb{E}\left[\int_0^1 dy M_{\alpha_y}^y k_{\alpha_y}\right] \\ &\overset{(a)}{=} \int_0^1 dy \mathbb{E}\left[M_\infty^y k_{\alpha_y}\right] \\ &\overset{(b)}{=} \int_0^1 dy \mathbb{E}\left[H_{\alpha_y} k_{\alpha_y}\right] \\ &\overset{(c)}{=} \mathbb{E}\left[k_\rho H_\rho\right].\end{aligned}$$

((a) follows from the optional stopping theorem for (M_t^y); (b) follows from the definition of M_∞^y; (c) is another consequence of (2.8)). Comparing the extreme terms in the above, we get

$$H_\rho = M_\rho^{1-Z_\rho}.$$

□

3. Some systematic constructions and some examples of pseudo-stopping times

3.1. **First constructions.** Here we discuss some combinations of several pseudo-stopping times which yield a pseudo-stopping time. Here is a first easy result:

Proposition 4. *Let ρ be a $(\mathcal{F}_t)$-pseudo-stopping time and let τ be a $(\mathcal{F}_t^\rho)$-stopping time. Then $\rho \wedge \tau$ is a $(\mathcal{F}_t)$ pseudo-stopping time.*

Proof. Let M be any uniformly integrable $(\mathcal{F}_t)$-martingale. We know that $M_{t\wedge\rho}$ is a uniformly integrable martingale in the enlarged filtration $(\mathcal{F}_t^\rho)$ and ρ is a stopping time in this filtration. If τ is also a $(\mathcal{F}_t^\rho)$-stopping time, then so is $\rho \wedge \tau$. Hence $\mathbb{E}M_{\rho\wedge\tau} = \mathbb{E}M_0$. □

Example 1. *Let ρ be as in D. Williams' example. Let $0 < a < 1$, and $T_a = \inf\{t > 0 : \quad B_t = a\}$. Then*

$$\rho_a = \rho \wedge T_a, \qquad 0 < a < 1,$$

is an increasing family of pseudo-stopping times.

Remark 5. *From the previous proposition, it is easy to see that for any uniformly integrable* $(\mathcal{F}_t)$*-martingale, we have*

$$\mathbb{E}\left[M_{T\wedge\rho}\right] = \mathbb{E}\left[M_0\right]$$

for any $(\mathcal{F}_t)$ *stopping time* T*.*

Remark 6. *As a further comment about Proposition 4, we remark that pseudo-stopping times do not inherit all the "nice" properties of stopping times. As an example, a pseudo-stopping time of a given filtration does not remain in general a pseudo-stopping time in a larger filtration, whereas a stopping time does. Indeed, let us keep the same notation as in section 2.3 and look at the pseudo-stopping time* ρ *in the larger filtration* $(\mathcal{F}_t^\sigma)$*. Using the computations we have already done in section 2.3 and the projections formula (see* [4] *p.186), we get:*

$$\mathbb{P}\left[\rho > t \mid \mathcal{F}_t^\sigma\right] = \frac{1-\max_{s\leq t\wedge T_1} B_s}{1-B^+_{t\wedge T_1}}\mathbf{1}_{\{\sigma>t\}},$$

which is not decreasing. In fact, any end of predictable set that avoids stopping times is not a pseudo-stopping time, as we shall see in the next subsection.

3.2. **A generalization of D. Williams' example.** To keep the discussion as simple as possible, we assume that we are working with an original filtration $(\mathcal{F}_t)$ such that:

- all $(\mathcal{F}_t)$-martingales are continuous (e.g: $(\mathcal{F}_t)$ is the Brownian filtration).
- Moreover, we consider L, the end of a $(\mathcal{F}_t)$ predictable set, such that for every $(\mathcal{F}_t)$ stopping time T, $\mathbb{P}\left[L = T\right] = 0$.

Under these two conditions, the supermartingale $Z_t = P\left[L > t \mid \mathcal{F}_t\right]$ associated with L is a.s. continuous, and satisfies $Z_L = 1$. Then we let,

$$\rho = \sup\left\{t < L : \quad Z_t = \inf_{u\leq L} Z_u\right\}.$$

The following holds:

Proposition 5. (i) $I_L = \inf_{u\leq L} Z_u$ *is uniformly distributed on* $[0,1]$*; (see* [20]*)*

(ii) *The supermartingale* $Z_t^\rho = P\left[\rho > t \mid \mathcal{F}_t\right]$ *associated with* ρ *is given by*

$$Z_t^\rho = \inf_{u\leq t} Z_u.$$

As a consequence, ρ *is a* $(\mathcal{F}_t)$ *pseudo-stopping time.*

Proof. (i) Let

$$T_b = \inf\left\{t, \quad Z_t \leq b\right\}, \qquad 0 < b < 1,$$

then

$$\mathbb{P}\left[I_L \leq b\right] = \mathbb{P}\left[T_b < L\right] = \mathbb{E}\left[Z_{T_b}\right] = b.$$

(ii) Note that for every $(\mathcal{F}_t)$ stopping time T, we have

$$\{T < \rho\} = \left\{T^{'} < L\right\}$$

where

$$T^{'} = \inf\left\{t > T, \quad Z_t \leq \inf_{s \leq T} Z_s\right\}.$$

Consequently, we have

$$\mathbb{E}\left[Z_T^{\rho}\right] = \mathbb{P}\left[T < \rho\right] = \mathbb{P}\left[T^{'} < L\right] = \mathbb{E}\left[Z_{T'}\right] = \mathbb{E}\left[\inf_{u \leq T} Z_u\right].$$

We deduce the desired result from the equality between the two extreme terms for every $(\mathcal{F}_t)$-stopping time T, and the optional section theorem. □

In the literature about enlargements of filtrations ([9], [11], [20], etc.), a number of explicit computations of supermartingales associated to various $L's$ have been given. We shall use some of these computations to produce some examples of pseudo-stopping times, with the help of the proposition.

(1) First let us check again that we recover the example of D. Williams from the Proposition 5. With the notations of the introduction ($L = \sigma$), it is not hard to see that (see [18])

$$Z_t = 1 - B^{+}_{t \wedge T_1}.$$

Hence

$$\rho = \sup\left\{s < \sigma : \ B_s = S_s\right\}.$$

(2) Consider $(R_t)_{t \geq 0}$ a three dimensional Bessel process, starting from zero, its filtration $(\mathcal{F}_t)$, and

$$L = L_1 = \sup\left\{t : \quad R_t = 1\right\}.$$

Then

$$\rho = \sup\left\{t < L : \quad R_t = \sup_{u \leq L} R_u\right\}, \tag{3.1}$$

is a $(\mathcal{F}_t)$ pseudo-stopping time. This follows from the fact that

$$Z_t^L = 1 \wedge \frac{1}{R_t},$$

hence (3.1) is equivalent to:

$$\rho = \sup\left\{t < L : \qquad Z_t^L = \inf_{u \leq L} Z_u^L\right\},$$

and from the above proposition:

$$Z_t^{\rho} = 1 \wedge \left(\frac{1}{\sup\limits_{u \leq t} R_u}\right).$$

We can generalize further this example by noticing that for $n > 2$, we have for $(R_t)_{t\geq 0}$ a BES(n), $Z_t^L = 1 \wedge \left(\frac{1}{R_t}\right)^{n-2}$.

(3) Consider $(B_u)_{u\geq 0}$ a one dimensional Brownian motion, $(\mathcal{F}_t)$ its filtration, and

$$g_t = \sup\{s < t : \quad B_s = 0\},$$

then

$$\rho_t = \sup\left\{s < g_t : \quad \frac{|B_s|}{\sqrt{t-s}} = \sup_{u<g_t} \frac{|B_u|}{\sqrt{t-u}}\right\} \tag{3.2}$$

is a $\mathcal{F}_t$ pseudo-stopping time. Again, this follows from the fact that ρ_t is in fact defined from g_t $(= L)$ as in the framework preceding the proposition, since:

$$Z_u^{g_t} \equiv \Phi\left(\frac{|B_u|}{\sqrt{t-u}}\right),$$

with $\Phi(x) = \mathbb{P}(|N| \geq x)$, where N is a standard Gaussian.

(4) We can reinterpret the previous example via a deterministic time-change. We remark that we can write:

$$\frac{B_u}{\sqrt{1-u}} = Y_{\log \frac{1}{1-u}},$$

where $(Y_s)_{s\geq 0}$, is an Ornstein-Uhlenbeck process satisfying

$$Y_s = \beta_s + \frac{1}{2}\int_0^s du Y_u.$$

We then deduce from example 3 that

$$\rho' = \sup\left\{s < L_0' : \qquad |Y_s| = \sup_{u\leq L_0'} |Y_u|\right\}$$

is a $\left(\mathcal{F}_t'\right)$ pseudo-stopping time, where

$$L_0' \equiv \log\left(\frac{1}{1-g_1}\right) = \sup\{s > 0, \qquad Y_s = 0\}$$

and $\left(\mathcal{F}_t'\right)$ is the natural filtration of (Y_t).

(5) Let us consider the case of a transient diffusion X_t. Let s be a scale function such that $s(-\infty) = 0$ and $s(x) > 0$. Let

$$L_a = \sup\{t; \quad X_t = a\},$$

the last passage at the level a. We have (see [15]):

$$Z_t^{L_a} = 1 \wedge \frac{s(X_t)}{s(a)}.$$

Thus

$$\rho_a = \sup\left\{t < L_a : \quad s(X_t) = \inf_{u \le L_a} s(X_u)\right\}$$

is a pseudo-stopping time in the filtration of (X_t). For example, let us consider the case of a brownian motion with a negative drift:

$$X_t \equiv x + \mu t + \sigma B_t, \quad \mu < 0.$$

In this case, the scale function is

$$s(x) = \exp\left(-\frac{2\mu x}{\sigma^2}\right).$$

Hence

$$\rho_a = \sup\left\{t < L_a : \quad \mu t + \sigma B_t = \inf_{u \le L_a} (\mu u + \sigma B_u)\right\}$$

is a pseudo-stopping time in the filtration of (B_t).

As for D. Williams' example, none of these pseudo-stopping times remains a pseudo-stopping time in the larger filtration $\left(\mathcal{F}_t^L\right)$.This is a consequence of a result of Azéma ([1]).

Proposition 6. *Let L be the end of a predictable set such that $\mathbb{P}[L = T] = 0$. Then L is not a pseudo-stopping time.*

Proof. From a result of Azéma ([1]), as $A_t^L = a_t^L$ is continuous, the law of A_∞^L is the exponential law of parameter 1, whilst for pseudo-stopping times, the law of A_∞^L is δ_1, the Dirac mass at one. Hence L cannot be a pseudo-stopping time. $\square$

3.3. **Another generalization.** We now give a generalization of the previous construction. We make the same assumptions about the filtration $(\mathcal{F}_t)$ and the time L, with the extra assumption that $\mathbb{P}[L = \infty] = 0$. Let (Δ_t) be a nonincreasing, continuous and adapted process such that

$$\Delta_0 = 1 \tag{3.3}$$

$$\Delta_\infty = 0. \tag{3.4}$$

Let us define ρ by

$$\rho \equiv \sup\{t < L; \quad Z_t = \Delta_t\}.$$

Again, for every $(\mathcal{F}_t)$ stopping time T, we have

$$\{T < \rho\} = \left\{T' < L\right\}$$

where

$$T' = \inf\{t > T, \quad Z_t \le \Delta_T\}$$

Thus

$$\mathbb{E}\left[Z_T^\rho\right] = \mathbb{P}[T < \rho] = \mathbb{P}\left[T' < L\right] = \mathbb{E}\left[Z_{T'}\right] = \mathbb{E}[\Delta_T],$$

and with the optional section theorem we can conclude that

$$Z_t^\rho = \Delta_t, \qquad t \geq 0.$$

It follows from Theorem 1 that ρ is a pseudo-stopping time. Hence we have proved the following:

Proposition 7. *Let (Δ_t) be a nonincreasing, continuous and adapted process such that:*

$$\Delta_0 = 1$$
$$\Delta_\infty = 0.$$

Then, under the assumptions made above, there always exists a pseudo-stopping time ρ such that $Z_t^\rho = \Delta_t$, for $t \geq 0$.

So we can associate a pseudo-stopping time to any continuous, nonincreasing adapted process (Δ_t) which satisfies (3.3). But there is not uniqueness since we can use different $Z's$ associated to different $L's$ to construct ρ. In other words, every continuous, nonincreasing adapted process (Δ_t) satisfying (3.3) is the dual predictable projection of some $1_{\{\rho \leq t\}}$, where ρ is a pseudo-stopping time.

As an example, we can take

$$\Delta_t = \exp(-S_t)$$

with the already used notations. Then,

$$\rho = \sup \left\{ t < \sigma; \quad 1 - B_t^+ = \exp(-S_t) \right\}$$

is a pseudo-stopping time in the filtration of the Brownian motion (B_t). We could as well take

$$\Delta_t = \exp(-L_t),$$

where L_t is the Brownian local time at level zero. In that case,

$$\rho = \sup \left\{ t < \sigma; \quad 1 - B_t^+ = \exp(-L_t) \right\}$$

is a pseudo-stopping time.

We can also notice that if we take some deterministic Δ, this construction allows us to construct a pseudo-stopping time with a given distribution. For example,

$$\rho = \sup \left\{ t < \sigma; \quad 1 - B_t^+ = \exp(-\lambda t) \right\},$$

where $\lambda > 0$. Then ρ follows an exponential law of parameter λ.

In the following section, we will see that we can drop the continuity assumption but we will have to enlarge the initial probability space.

3.4. **Further examples.** In this section, we shall link pseudo-stopping times with other random times that appear in the literature. In particular, we will see that the random times allowing the $(\mathbf{H})$ hypothesis (see [7]) to hold are special cases of pseudo-stopping times.

3.4.1. *The hypothesis* (**H**). First, we give the following obvious result:

Proposition 8. *If ρ is a random time that is independent from $\mathcal{F}_\infty$, then it is a pseudo-stopping time.*

Example 2. *If ρ is an exponential time of parameter λ that is independent from $\mathcal{F}_\infty$, then it is a pseudo-stopping time.*

Example 3. *Another example is given by what D. Williams (*[19]*) calls a "silly" time:*

$$\rho = \frac{1}{1 + |B_2 - B_1|},$$

which is independent from $\mathcal{F}_1$.

Now suppose that our probability space supports a uniform random variable Θ on $(0,1)$ that is independent of the sigma field $\mathcal{F}_\infty$. Assume we are given an $(\mathcal{F}_t)$-adapted increasing and continuous process satisfying $A_0 = 0$ and $A_\infty = 1$. Let us consider the random time defined by:

$$\rho = \inf \{t; \quad A_t > \Theta\}.$$

It is not difficult to check that

$$\mathbb{P}[\rho > t \mid \mathcal{F}_t] = 1 - A_t. \tag{3.5}$$

We have thus constructed a pseudo-stopping time associated with a given continuous process (A_t). This construction is well known, see [8] for more details and references. But this construction is not always possible (for example when $\mathcal{F}_\infty = \mathcal{F}$), which explains why our construction in the previous section is more general.

But the pseudo-stopping times that are constructed in the way of (3.5) enjoy the following noticeable property ([8], [5]):

$$\mathbb{P}[\rho > t \mid \mathcal{F}_t] = \mathbb{P}[\rho > t \mid \mathcal{F}_\infty]. \tag{3.6}$$

Random times with this property are often used in the literature on default modelling (see [8], [7]) and were studied in [5], [3]. There are several equivalent formulations for (3.6). Before we mention them, let us notice that any random time satisfying (3.6) is a pseudo-stopping time. In fact, we have a stronger result: every $(\mathcal{F}_t)$ martingale is an $(\mathcal{F}_t^\rho)$ martingale (see [5]). Thus the fourth statement in Theorem 1 is satisfied.

Now let us consider the (**H**) hypothesis in our framework of progressive enlargement with a random time ρ: every $(\mathcal{F}_t)$-square integrable martingale is an $(\mathcal{F}_t^\rho)$-square integrable martingale. This hypothesis was studied by Dellacherie and Meyer [5], Brémaud and Yor [3]. It is equivalent to one of the following hypothesis (see [7] for more references):

(1) $\forall t$, the σ-algebras $\mathcal{F}_\infty$ and $\mathcal{F}_t^\rho$ are conditionally independent given $\mathcal{F}_t$.

(2) For all bounded $\mathcal{F}_\infty$-measurable random variables $\mathbf{F}$ and all bounded $\mathcal{F}_t^\rho$-measurable random variables $\mathbf{G}_t$, we have

$$\mathbb{E}\left[\mathbf{F}\mathbf{G}_t \mid \mathcal{F}_t\right] = \mathbb{E}\left[\mathbf{F} \mid \mathcal{F}_t\right] \mathbb{E}\left[\mathbf{G}_t \mid \mathcal{F}_t\right].$$

(3) For all bounded $\mathcal{F}_t^\rho$-measurable random variables $\mathbf{G}_t$:

$$\mathbb{E}\left[\mathbf{G}_t \mid \mathcal{F}_\infty\right] = \mathbb{E}\left[\mathbf{G}_t \mid \mathcal{F}_t\right].$$

(4) For all bounded $\mathcal{F}_\infty$-measurable random variables $\mathbf{F}$,

$$\mathbb{E}\left[\mathbf{F} \mid \mathcal{F}_t^\rho\right] = \mathbb{E}\left[\mathbf{F} \mid \mathcal{F}_t\right].$$

(5) For all $s \leq t$,

$$\mathbb{P}\left[\rho \leq s \mid \mathcal{F}_t\right] = \mathbb{P}\left[\rho \leq s \mid \mathcal{F}_\infty\right].$$

Thus, pseudo-stopping times may be considered as a generalized or a weakened form of the $(\mathbf{H})$ hypothesis since then local martingales in the initial filtration remain local martingales in the enlarged one up to time ρ. Moreover, for most of the examples we have considered, such as D. Williams', (3.6) is not satisfied.

3.4.2. *Randomized stopping times and Föllmer measures.* Now we give a relation between pseudo-stopping times and randomized stopping times as presented in [14]. First we give some definitions. We always consider a given probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$.

Definition 3. *A randomized random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability measure μ on $([0,\infty] \times \Omega, \mathcal{B}([0,\infty]) \otimes \mathcal{F})$ such that its projection on Ω is equal to $\mathbb{P}$.*

For example, let ρ be a random time; then μ_ρ defined by

$$\mu_\rho(X) = \mathbb{E}\left[X_\rho\right],$$

for all bounded measurable processes (X_t) is a randomized random variable.

We know from a result of Föllmer (see [6]) that there exists an increasing càdlàg process (A_t) such that $A_0 = 0$ and

$$\mu(X) = \mathbb{E}\left[\int_0^\infty X_s dA_s\right],$$

for all nonnegative process (X_t). The fact that the projection on Ω is equal to $\mathbb{P}$ means that $A_\infty = 1$, *a.s.*

Definition 4. *If the process (A_t) associated with μ on $([0,\infty] \times \Omega, \mathcal{B}([0,\infty]) \otimes \mathcal{F})$ is adapted, then we say that μ is a randomized stopping time.*

By considering the new space $\overline{\Omega} = [0,1] \times \Omega$ endowed with the σ-fields $\overline{\mathcal{F}}=\mathcal{B}([0,1]) \otimes \mathcal{F}$, $\overline{\mathcal{F}}_t=\mathcal{B}([0,1]) \otimes \mathcal{F}_t$ (augmented in the usual way) and the probability measure $\overline{\mathbb{P}} = \lambda \otimes \mathbb{P}$, it is possible to show that for every randomized stopping time μ, there exists a stopping time ρ in this new filtered space such that

$$\mu(X) = \overline{\mathbb{E}}[X_\rho],$$

for all bounded measurable process (X_t) on $([0,\infty] \times \Omega, \mathcal{B}([0,\infty]) \otimes \mathcal{F})$. We take the convention that a random variable H on Ω can be considered as the random variable on $\overline{\Omega}$: $(u,\omega) \to H(\omega)$. Conversely to every stopping time of $\overline{\mathcal{F}}_t$ corresponds a randomized stopping time.

This construction is always carried on the enlarged space $\overline{\Omega}$. The third statement in Theorem 1 allows us to use pseudo-stopping times to construct randomized stopping times without enlarging the initial space.

Proposition 9. *Let ρ be a pseudo-stopping time and A_t^ρ the $(\mathcal{F}_t)$ dual optional projection of the process $1_{\{\rho \leq t\}}$. Then the Föellmer measure μ associated with A_t^ρ is a randomized stopping time. Moreover, for every bounded or nonnegative $(\mathcal{F}_t)$ optional process (X_t):*

$$\mu(X) = \mathbb{E}[X_\rho].$$

3.4.3. *Randomized stopping times and families of stopping times.*

Proposition 10. *Let $(T_u)_{u\geq 0}$ be a family of $(\mathcal{F}_t)$ stopping times and S a positive random variable, independent of the family $(\mathcal{F}_\infty)$. Then*

$$\rho = T_S$$

is a $(\mathcal{F}_t)$ pseudo-stopping time.

Proof. Let (M_t) be a bounded $(\mathcal{F}_t)$ martingale;

$$\begin{aligned}
\mathbb{E}[M_{T_S}] &= \mathbb{E}[\mathbb{E}[M_{T_s} \mid S = s]] \\
&= \mathbb{E}[\mathbb{E}[M_0] \mid S = s] \\
&= \mathbb{E}[M_0].
\end{aligned}$$

□

The previous proposition shows that any independently time changed family of stopping times is a pseudo-stopping time. In fact, this proposition admits a converse: every pseudo-stopping time is, in law, a time changed family of stopping times. More precisely:

Proposition 11. *Let ρ be a $(\mathcal{F}_t)$ pseudo-stopping time, which avoids all $(\mathcal{F}_t)$-stopping times, and $Z_t = \mathbb{P}[\rho > t \mid \mathcal{F}_t]$ its associated supermartingale. Set*

$$\alpha_u \equiv \inf\{t \geq 0, \quad (1 - Z_t) > u, \quad 0 \leq u \leq 1\},$$

the right-continuous generalized inverse of the increasing continuous process $(1-Z_t)$. *Then* $(\alpha_u)_{0\leq u\leq 1}$ *is a family of* $(\mathcal{F}_t)$ *stopping times and*

$$\rho \overset{law}{=} \alpha_U,$$

where U *is a random variable with uniform law, independent of* $(\mathcal{F}_\infty)$.

Proof. The fact α_u is a stopping time, for all u, follows from

$$\{\alpha_u \leq t\} = \{u \leq (1-Z_t)\}, \quad \forall t \geq 0.$$

From (2.9), we also have

$$\mathbb{E}\left[g(\rho)\right] = \mathbb{E}\left[\int_0^1 g(\alpha_u)\, du\right],$$

for all bounded Borel function g. This establishes $\left(\rho \overset{law}{=} \alpha_U\right)$. □

4. A discrete analogue: the coin-tossing case

Let $(X_n)_{n\geq 1}$ be the standard random walk with Bernoulli increments. In his paper [13], Le Gall proved an analogue of Williams' path decomposition for (X_n). To fix ideas, we shall consider the canonical space $\Omega = \mathbb{Z}^N$ endowed with the product σ-field. (X_n) will be the coordinate process and $(\mathbb{P}_x)_{x\in\mathbb{Z}}$ the family of probability laws which make (X_n) the standard random walk with Bernoulli increments. We also denote by $(\mathbb{Q}_x)_{x\in N}$ the unique family of probability measures such that $(X_n, \mathbb{Q}_x)$ is a Markov chain with transition probabilities:

$\mathbb{Q}_0\left[X_1 = 1\right] = 1$

$$if\ x \geq 1, \quad \mathbb{Q}_x\left[X_1 = x+1\right] = \frac{1}{2}\left(1+\frac{1}{x}\right); \quad \mathbb{Q}_x\left[X_1 = x-1\right] = \frac{1}{2}\left(1-\frac{1}{x}\right).$$

Now let $p \geq 1$ and define:

$$\begin{aligned}
\sigma_p &= \inf\{k; \quad X_k = p\},\\
\eta &= \sup\{k \leq \sigma_p : \quad X_k = 0\},\\
m &= \sup\{X_k, \quad k \leq \eta\},\\
\gamma &= \inf\{k \geq 0; \quad X_k = m\}.
\end{aligned}$$

Then, Le Gall's statement is that under $\mathbb{P}_0$:

(1) The processes $(X_k)_{0\leq k\leq \eta}$ and $(X_{\eta+k})_{0\leq k\leq \sigma_p-\eta}$ are independent, with the second being distributed as $(X_k)_{0\leq k\leq\sigma_p}$ under $\mathbb{Q}_0$;
(2) m is uniformly distributed on $\{0,1,\dots,p-1\}$;
(3) Conditionally on $\{m=j\}$, the processes $(X_k)_{0\leq k\leq\gamma}$ and $(X_{\eta-k})_{0\leq k\leq \eta-\gamma}$ are independent, the first being distributed as $(X_k)_{0\leq k\leq\sigma_j}$ under $\mathbb{P}_0$, and the second as $(X_k)_{0\leq k\leq \sigma_{j+1}-1}$ under $\mathbb{Q}_0$.

Proposition 12. *If $(M_n)_{n\in\mathbb{N}}$ is a bounded martingale, then*

$$\mathbb{E}_0\left[M_\gamma\right] = \mathbb{E}_0\left[M_0\right].$$

Thus γ is a pseudo-stopping time.

Proof. The discrete time setup allows us to give an elementary argument, based in part on the fact that M_n, as every $\mathcal{F}_n$ measurable variable, may be written as:

$$M_n = f_n\left(X_1, X_2, \dots, X_n\right),$$

where f_n is a bounded function depending on n variables.

Now, for any bounded function g:

$$\mathbb{E}_0\left[M_\gamma g\left(m\right)\right] = \mathbb{E}_0\left[\mathbb{E}_0\left[M_\gamma \mid m\right] g\left(m\right)\right].$$

But, from (3) in Le Gall's satatement:

$$\begin{aligned} \mathbb{E}_0\left[M_\gamma \mid m = j\right] &= \mathbb{E}_0\left[f_{\sigma_j}\left(X_1, X_2, \dots, X_{\sigma_j}\right)\right] \\ &= \mathbb{E}_0\left[M_{\sigma_j}\right] = \mathbb{E}_0\left[M_0\right]. \end{aligned}$$

Thus, we have obtained:

$$\begin{aligned} \mathbb{E}_0\left[M_\gamma g\left(m\right)\right] &= \mathbb{E}_0\left[M_\gamma\right]\mathbb{E}_0\left[g\left(m\right)\right] \\ &= \mathbb{E}_0\left[M_\infty\right]\mathbb{E}_0\left[g\left(m\right)\right], \end{aligned}$$

which is the discrete analogue of Proposition 2, and shows a fortiori that γ is a pseudo-stopping time. □

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