

NON STOPPING TIMES AND STOPPING THEOREMS

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ABSTRACT. Given a random time, we give some characterizations of the set of martingales for which the stopping theorems still hold. We also investigate how the stopping theorems are modified when we consider arbitrary random times. To this end, we introduce some families of martingales with remarkable properties.

1. INTRODUCTION

The role of stopping times in martingale theory is fundamental. In particular, there are myriads of applications of Doob's optional stopping theorem:

- If (M_t) is a uniformly integrable martingale, and T is a stopping time (both with respect to the filtration (\mathcal{F}_t) which is assumed to satisfy the usual hypotheses under a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$), then:

$$\mathbb{E}[M_T] = \mathbb{E}[M_\infty] = \mathbb{E}[M_0] \quad (1.1)$$

and, in fact:

$$\mathbb{E}[M_\infty | \mathcal{F}_T] = M_T \quad (1.2)$$

In this paper, we would like to discuss in some depth the following question, which arises very naturally:

- What happens to (1.1) and (1.2) when T is replaced by a random time ρ , and \mathcal{F}_T by $\mathcal{F}_\rho = \sigma\{H_\rho; H \text{ is } (\mathcal{F}_t) \text{ optional}\}$?

Some partial answers to this general question have been given by D. Williams [33] on one hand, and Knight-Maisonneuve [20] on the other hand:

- (1) There exist “non-stopping” times ρ , which we have called pseudo-stopping times in [24] such that, for every bounded martingale M ,

$$\mathbb{E}[M_\rho] = \mathbb{E}[M_0] \quad (1.3)$$

- (2) If for every bounded martingale M , one has:

$$\mathbb{E}[M_\infty | \mathcal{F}_\rho] = M_\rho, \quad (1.4)$$

then ρ is a stopping time ([20]).

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Besides the fact that it is mathematically an interesting question to understand how and why the usual results fail to hold when stopping times are replaced with arbitrary random times, it should be noticed that random times that are not stopping times play a key role in various contexts, such as in the modeling of default times in mathematical finance (see [15]), in Markov Processes theory (see [14]), in the characterization of the set of zeros of continuous martingales ([8]), in path decomposition of some diffusions (see [16] or [25]), in the study of Strong Brownian Filtrations (see [10]), etc.

The most studied family of random times, after stopping times, are ends of optional sets, also named honest times (such the last zero of the standard Brownian Motion before a fixed time). A very powerful, but not so well known, technique to study such random times, is the progressive expansions or enlargements of filtrations. The theory of progressive enlargements of filtrations was introduced independently by Barlow ([9]) and Yor ([34]), and further developed by Jeulin and Yor ([18, 17, 16, 35]). The reader can find many applications of this theory in the cited references and in [24] and [25]. The concept of dual projections also play an important role in the study of arbitrary random times (see [13] or [32]).

The main idea in the progressive enlargements setting is to consider the larger filtration (\mathcal{F}_t^ρ) , which is the smallest right continuous filtration which contains (\mathcal{F}_t) and which makes ρ a stopping time, and then to see how martingales of the smaller filtration are changed when considered as processes of the larger one. In [4], the authors used these ideas to give a solution to equation (1.4) in a Brownian setting, using a predictable representation property for martingales in the larger filtration (\mathcal{F}_t^ρ) . In this paper, we shall solve equation (1.3) for arbitrary random times and equation (1.4) for honest times. In this latter case, we propose two different approaches and our characterizations ((Theorem 4.5 and Proposition 4.11)) of the set of martingales which satisfy (1.4) will be different (but not necessarily more handy) from the one in [4], in that our solution is based only on quantities relative to the filtration (\mathcal{F}_t) , which moreover is not assumed to be a Brownian filtration. More precisely, the organization of the paper is as follows:

In Section 2, we recall some basic facts about progressive enlargements of filtrations and arbitrary random times.

In Section 3, we solve equation (1.3) for arbitrary random times, using elementary properties of dual projections and Laguerre polynomials .

In Section 4, we provide two different approaches to solve (1.4) for honest times (Theorem 4.5 and Proposition 4.11). In particular, we shall see how to obtain a large class of solutions to (1.4), by considering martingales which vanish at L . We illustrate these facts in the celebrated special case when L is the last time before a fixed (or a stopping) time when a standard Brownian Motion vanishes.

In Section 5, we introduce a family of test martingales, with interesting and universal properties (in a sense that will be clear), to understand how the equalities (1.3) and (1.4) may fail to hold for honest times.

2. BASIC FACTS ABOUT PROGRESSIVE ENLARGEMENTS OF FILTRATIONS AND RANDOM TIMES

In this Section, we recall some results (which may not be so well known) that we shall use in this paper and fix the notations once and for all. Throughout this article, we assume for simplicity that ρ is a random time such that $\mathbb{P}[\rho = 0] = \mathbb{P}[\rho = \infty] = 0$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ a filtered probability space, and $\rho : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ be a random time. We enlarge the initial filtration (\mathcal{F}_t) with the process $(\rho \wedge t)_{t \geq 0}$, so that the new enlarged filtration $(\mathcal{F}_t^\rho)_{t \geq 0}$ is the smallest filtration containing (\mathcal{F}_t) and making ρ a stopping time. A few processes will play a crucial role in our discussion:

- the (\mathcal{F}_t) -supermartingale

$$Z_t^\rho = \mathbb{P}[\rho > t \mid \mathcal{F}_t] \quad (2.1)$$

chosen to be càdlàg, associated to ρ by Azéma (see [16] for detailed references);

- the (\mathcal{F}_t) dual optional projection of the process $1_{\{\rho \leq t\}}$, denoted by A_t^ρ ;
- the càdlàg martingale

$$\mu_t^\rho = \mathbb{E}[A_\infty^\rho \mid \mathcal{F}_t] = A_t^\rho + Z_t^\rho$$

which is in $\text{BMO}(\mathcal{F}_t)$ (see [14] or [35]).

Every (\mathcal{F}_t) local martingale (M_t) , stopped at ρ , is a (\mathcal{F}_t^ρ) semimartingale, with canonical decomposition:

$$M_{t \wedge \rho} = \widetilde{M}_t + \int_0^{t \wedge \rho} \frac{d\langle M, \mu^\rho \rangle_s}{Z_{s-}^\rho} \quad (2.2)$$

where (\widetilde{M}_t) is an (\mathcal{F}_t^ρ) -local martingale.

The most interesting case in the theory of progressive enlargements of filtrations is when ρ is an honest time; we will always denote honest times by L instead of ρ . Indeed, if L is an honest time, then every (\mathcal{F}_t) local martingale (M_t) , is an (\mathcal{F}_t^ρ) semimartingale, with canonical decomposition:

$$M_t = \widetilde{M}_t + \int_0^{t \wedge L} \frac{d\langle M, \mu^L \rangle_s}{Z_{s-}^L} - \int_L^t \frac{d\langle M, \mu^L \rangle_s}{1 - Z_{s-}^L}. \quad (2.3)$$

We shall often need to make one (or sometimes both) of the following assumptions:

- Assumption **(C)**: all (\mathcal{F}_t) -martingales are continuous (e.g: the Brownian filtration).
- Assumption **(A)**: the random time ρ avoids every (\mathcal{F}_t) -stopping time T , i.e. $\mathbb{P}[L = T] = 0$.

When we refer to assumptions **(CA)**, this will mean that both the conditions **(C)** and **(A)** hold. Under conditions **(C)** or **(A)**, A_t^ρ is also the dual predictable projection of $1_{\{\rho \leq t\}}$; moreover under **(A)**, A_t^ρ is continuous.

Now, we give the definitions of some sigma fields associated with arbitrary random times, following Chung and Doob ([12]):

Definition 2.1. Three classical σ -fields associated with a filtration (\mathcal{F}_t) and any random time ρ are:

$$\begin{cases} \mathcal{F}_{\rho+} &= \sigma \{z_\rho, (z_t) \text{ any } (\mathcal{F}_t) \text{ progressively measurable process}\}; \\ \mathcal{F}_\rho &= \sigma \{z_\rho, (z_t) \text{ any } (\mathcal{F}_t) \text{ optional process}\}; \\ \mathcal{F}_{\rho-} &= \sigma \{z_\rho, (z_t) \text{ any } (\mathcal{F}_t) \text{ predictable process}\}; \end{cases}$$

Under condition **(A)**, we have: $\mathcal{F}_\rho = \mathcal{F}_{\rho-}$.

We conclude this section by giving two results (due to Azéma [2]) which will play an important role in the next sections. The reader can also refer to the book [14] for a very nice introduction to the results of Azéma and the theory of progressive enlargements of filtrations.

Lemma 2.2 (Azéma [2]). *Let L be an honest time; then under **(A)**, A_∞^L follows the exponential law with parameter 1 and the measure dA_t^L is carried by the set $\{t : Z_t^L = 1\}$. Moreover, A^L does not increase after L , i.e. $A_t^L = A_\infty^L$.*

Lemma 2.3 (Azéma [2]). *Let L be an honest time and assume **(A)** holds. Then,*

$$L = \sup \{t : 1 - Z_t^L = 0\}.$$

In particular, $1 - Z_L^L = 0$.

3. A RESOLUTION OF THE EQUATION $\mathbb{E}(M_\rho) = \mathbb{E}(M_0)$

We wish to solve equation (1.3), where ρ is given and the unknown are all bounded (\mathcal{F}_t) martingales for which (1.3) hold. We consider the class of bounded martingales because we want to make sure that $\mathbb{E}(M_\rho)$ exists; in fact, we can look for solutions to equation (1.3) in the space of \mathcal{H}^1 martingales (see [19] or [22]). We recall that the space \mathcal{H}^1 is the Banach space of (càdlàg) (\mathcal{F}_t) martingales (M_t) such that

$$\|M\|_{\mathcal{H}^1} = \mathbb{E} \left[\sup_{t \geq 0} |M_t| \right] < \infty.$$

Definition 3.1. We call \mathcal{S}_1 the set of solutions of equation (1.3), i.e.

$$\mathcal{S}_1 \equiv \{M \in \mathcal{H}^1 : \mathbb{E}(M_\rho) = \mathbb{E}(M_\infty)\}.$$

Theorem 3.2. *The map*

$$T(M) = \mathbb{E}[\langle M, \mu^\rho \rangle_\infty],$$

defines a continuous linear form on the Banach space \mathcal{H}^1 , and we have the following characterizations for \mathcal{S}_1 :

(1)

$$\mathcal{S}_1 = \ker T,$$

or in other words,

$$\mathcal{S}_1 = \{M \in \mathcal{H}^1 : \mathbb{E}[\langle M, \mu^\rho \rangle_\infty] = 0\}. \quad (3.1)$$

(2)

$$\mathcal{S}_1 = \{M \in \mathcal{H}^1 : \mathbb{E}[M_\infty (\mu_\infty^\rho - 1)] = 0\}. \quad (3.2)$$

(3)

$$\mathcal{S}_1 = \{M \in \mathcal{H}^1 : \mathbb{E}[M_\infty (A_\infty^\rho - 1)] = 0\}. \quad (3.3)$$

Consequently, \mathcal{S}_1 is a closed linear subspace of \mathcal{H}^1 .

Proof. The fact that T defines a linear form is a consequence of the well known duality between \mathcal{H}^1 and BMO (see [19] or [22] for details and references). Now, let ρ be a random time and let M be a martingale in \mathcal{H}^1 . We have (see [13] or [28] where dual projections and their properties applications are discussed):

$$\mathbb{E}[M_\rho] = \mathbb{E}\left[\int_0^\infty M_s dA_s^\rho\right] = \mathbb{E}[M_\infty A_\infty^\rho]. \quad (3.4)$$

Hence,

$$\mathbb{E}[M_\rho] = \mathbb{E}[M_\infty] \Leftrightarrow \mathbb{E}[M_\infty (A_\infty^\rho - 1)] = 0,$$

and this establishes (3).

But as $\mathbb{P}[\rho = \infty] = 0$, $Z_\infty^\rho = 0$, and $\mu_\infty^\rho = A_\infty^\rho$. We thus have:

$$\mathbb{E}[M_\rho] = \mathbb{E}[M_\infty \mu_\infty^\rho] = \mathbb{E}[M_\infty] + \mathbb{E}[\langle M, \mu^\rho \rangle_\infty],$$

and (1) and (2) follow easily. \square

Remark 3.3. As will be shown later, one must not confuse (3.1) with the stronger condition $\langle M, \mu^\rho \rangle_t = 0$ for every t .

Theorem 3.2 shows that the set of solutions of equation (1.3) is a linear space of codimension 1 in \mathcal{H}^1 if the linear form T is not null. The case when this form is null corresponds to the remarkable class of random times called pseudo-stopping times, defined and studied in [24].

Proposition 3.4 ([24]). *The following are equivalent:*

- (1) (1.3) holds for every martingale in \mathcal{H}^1 ;
- (2) $A_\infty^\rho = \mu_\infty^\rho = 1$ a.s.;
- (3) If (M_t) is an (\mathcal{F}_t) local martingale, then $(M_{t \wedge \rho})$ is an (\mathcal{F}_t^ρ) local martingale.

We can also give the following elementary but useful corollary of Theorem 3.2:

Corollary 3.5. *Let M be an L^2 bounded martingale such that $M_\infty \in (L^2(\sigma(A_\infty^\rho)))^\perp$, the orthogonal of $L^2(\sigma(A_\infty^\rho))$. Then $M \in \mathcal{S}_1$.*

Now, one may want to find some L^2 bounded martingales such that $M_\infty \in L^2(\sigma(A_\infty))$. This can be done with the help of orthogonal polynomials if one knows the law of A_∞ . We shall now illustrate this with the important case of honest times, giving a complete description of \mathcal{S}_1 in terms of Laguerre polynomials.

We first introduce some basic facts about Laguerre polynomials ([1]). Let us consider the Hilbert space $L^2(\exp(-x) dx)$. The Laguerre Polynomials, $\tilde{\mathcal{L}}_n(x)$ are the orthogonal polynomials associated with the measure $\exp(-x) dx$. They are given by the formula:

$$\begin{aligned}\tilde{\mathcal{L}}_n(x) &= \sum_{k=0}^n (-1)^k \frac{(n!)^2}{(k!)^2 (n-k)!} x^k \\ &= \exp(x) \frac{d^n}{dx^n} (x^n \exp(-x))\end{aligned}$$

They satisfy the orthogonality relation:

$$\int_0^\infty dx \exp(-x) \tilde{\mathcal{L}}_m(x) \tilde{\mathcal{L}}_n(x) = (n!)^2 \delta_{m,n}$$

We can normalize and take:

$$\mathcal{L}_n(x) = \frac{\tilde{\mathcal{L}}_n(x)}{n!}$$

so that the family $(\mathcal{L}_n(x))$ is an orthonormal basis in $L^2(\exp(-x) dx)$. For example,

$$\begin{aligned}\mathcal{L}_0(x) &= 1 \\ \mathcal{L}_1(x) &= 1 - x \\ \mathcal{L}_2(x) &= \frac{1}{2} (2 - 4x + x^2) \\ \mathcal{L}_3(x) &= \frac{1}{6} (6 - 18x + 9x^2 - x^3) \\ \mathcal{L}_4(x) &= \frac{1}{24} (24 - 96x + 72x^2 - 16x^3 + x^4).\end{aligned}$$

Theorem 3.6. *Let L be an honest time and assume condition **(A)** holds. Let M be an L^2 bounded martingale. Then the following are equivalent:*

- (1) $M \in \mathcal{S}_1$;
- (2) M_∞ may be represented as:

$$M_\infty = X + \varphi(A_\infty), \quad (3.5)$$

where $X \in (L^2(\sigma(A_\infty)))^\perp$ and where $\varphi \in L^2(\sigma(A_\infty))$ admits the following representation:

$$\varphi(A_\infty) = \alpha_0 + \sum_{n=2}^{\infty} \alpha_n \mathcal{L}_n(A_\infty), \quad (3.6)$$

with $\alpha \in \mathbb{R}$ and (α_n) such that $\sum \alpha_n^2 < \infty$, i.e. in the development of φ , the coefficient of \mathcal{L}_1 is $\alpha_1 = 0$.

Proof. We note that:

$$M_\infty = X + \mathbb{E}[M_\infty | A_\infty] \equiv X + \varphi(A_\infty),$$

with $X \in (L^2(\sigma(A_\infty)))^\perp$ and $\varphi(A_\infty) \in L^2(\sigma(A_\infty))$. Now, from (3.3), $M \in \mathcal{S}_1$ if and only if

$$\mathbb{E}[\varphi(A_\infty)(A_\infty - 1)] = 0,$$

or equivalently

$$\mathbb{E}[\varphi(A_\infty) \mathcal{L}_1(A_\infty)] = 0. \quad (3.7)$$

Since the family $(\mathcal{L}_n)_{n \geq 0}$ is total in $L^2(\exp(-x) dx)$, we can represent $\varphi(x)$ as:

$$\varphi(x) = \sum_{n=0}^{\infty} \alpha_n \mathcal{L}_n(x),$$

with $\sum \alpha_n^2 < \infty$. Now putting the series expansion of φ in (3.7) and using the fact that the family $(L_n)_{n \geq 0}$ is orthogonal gives the desired result. \square

Example 3.7. Let the filtration (\mathcal{F}_t) be generated by a one dimensional Brownian motion $(B_t)_{t \geq 0}$, and let

$$T_1 = \inf\{t : B_t = 1\}, \text{ and } L = \sup\{t < T_1 : B_t = 0\}.$$

It is well known that

$$Z_t = \mathbb{P}[L > t | \mathcal{F}_t] = 1 - B_{t \wedge T_1}^+$$

An application of Tanaka's formula yields: $A_\infty = \frac{1}{2} \ell_{T_1}$, where (ℓ_t) is the Brownian local time at zero. Now, from the previous theorem, it is easily seen that any martingale of the form $M_t = \mathbb{E}[\mathcal{L}_n(\ell_{T_1}) | \mathcal{F}_t]$; $n \neq 1$ is in \mathcal{S}_1 .

It is also possible to use the Kunita-Watanabe orthogonal decompositions for square integrable martingales to give a description of \mathcal{S}_1 ; more precisely:

Proposition 3.8. *Let M be an L^2 martingale and let ρ be an arbitrary random time. Then the following are equivalent:*

- (1) $M \in \mathcal{S}_1$;
- (2) (M_t) decomposes as:

$$M_t = N_t + \int_0^t k_s d\mu_s^\rho, \quad (3.8)$$

where N is an L^2 martingale such that

$$\langle N, \mu^\rho \rangle_t = 0, \quad \forall t \geq 0$$

and k is a predictable process such that:

$$\mathbb{E} \left[\int_0^\infty k_s^2 d\langle \mu^\rho \rangle_s \right] < \infty; \quad \mathbb{E} \left[\int_0^\infty k_s d\langle \mu^\rho \rangle_s \right] = 0.$$

Proof. From the Kunita-Watanabe decomposition ([21]), any L^2 martingale M can be decomposed as: $M_t = N_t + \int_0^t k_s d\mu_s^\rho$, where $\langle N, \mu^\rho \rangle_t = 0$, $\forall t \geq 0$, and where k is a predictable process such that $\mathbb{E} [\int_0^\infty k_s^2 d\langle \mu^\rho \rangle_s] < \infty$. Now, from (3.1), it follows that $M \in \mathcal{S}_1$ if and only if

$$\mathbb{E} [\langle M, \mu^\rho \rangle_\infty] = 0,$$

or equivalently

$$\mathbb{E} \left[\int_0^\infty k_s d\langle \mu^\rho \rangle_s \right] = 0,$$

which completes the proof. \square

4. A RESOLUTION OF THE EQUATION $\mathbb{E}(M_\infty | \mathcal{F}_L) = M_L$

In this Section, we shall try to give explicit solutions to equation (1.4), when $\rho \equiv L$ is an honest time satisfying **(A)**.

Definition 4.1. We call \mathcal{S}_2 the set of solutions to equation (1.4):

$$\mathcal{S}_2 = \{M \in \mathcal{H}^1 : \mathbb{E}[M_\infty | \mathcal{F}_L] = M_L\}.$$

Remark 4.2. $\mathcal{S}_2 \subset \mathcal{S}_1$.

We recall here that equation (1.4) was solved, in the case of the Brownian filtration, by Azéma, Knight, Jeulin and Yor in [4]. We propose two other characterizations of the set of solutions to this equation (Theorem 4.5 and Proposition 4.11). From now on, we assume that L is an honest time satisfying **(A)**.

4.1. A general solution related to the enlargements formulae. Recall that under condition **(A)**, $\mathcal{F}_L = \mathcal{F}_{L-}$.

Lemma 4.3. $\mathcal{F}_L = \mathcal{F}_{L-}^L$.

Proof. From results of Jeulin ([16]), every (\mathcal{F}_t^L) predictable process H can be represented as

$$H = J\mathbf{1}_{]0, L]} + K\mathbf{1}_{]L, \infty[},$$

where J and K are (\mathcal{F}_t) predictable processes. Hence, $\mathcal{F}_{L-}^L = \mathcal{F}_{L-}$, and since under **(A)**, $\mathcal{F}_L = \mathcal{F}_{L-}$, the lemma is proved. \square

Remark 4.4. Using the representation of optional (\mathcal{F}_t^L) processes, it is possible to show that $\mathcal{F}_{L+} = \mathcal{F}_L^L$ (see [16]).

Now, we state a general necessary and sufficient condition for M to be in \mathcal{S}_2 .

Theorem 4.5. *Let $M \in \mathcal{H}^1$. The following are equivalent:*

- (1) $M \in \mathcal{S}_2$;

(2)

$$\mathbb{E} \left[\int_0^\infty \frac{d\langle M, \mu \rangle_s}{1 - Z_{s-}} \mid \mathcal{F}_L \right] = \int_0^L \frac{d\langle M, \mu \rangle_s}{1 - Z_{s-}},$$

or equivalently:

$$\mathbb{E} \left[\int_L^\infty \frac{d\langle M, \mu \rangle_s}{1 - Z_{s-}} \mid \mathcal{F}_L \right] = 0.$$

Proof. Let $M \in \mathcal{H}^1$; then from (2.3), there exists an \mathcal{F}_t^L martingale \widetilde{M} such that:

$$M_t = \widetilde{M}_t + \int_0^{t \wedge L} \frac{d\langle M, \mu \rangle_s}{Z_{s-}} - \int_L^t \frac{d\langle M, \mu \rangle_s}{1 - Z_{s-}}.$$

We deduce from this decomposition formula that:

$$M_L = \widetilde{M}_L + \int_0^L \frac{d\langle M, \mu \rangle_s}{Z_{s-}}, \quad (4.1)$$

and

$$\mathbb{E}[M_\infty \mid \mathcal{F}_L] = \mathbb{E}[\widetilde{M}_\infty \mid \mathcal{F}_L] + \int_0^L \frac{d\langle M, \mu \rangle_s}{Z_{s-}} + \mathbb{E} \left[\int_L^\infty \frac{d\langle M, \mu \rangle_s}{1 - Z_{s-}} \mid \mathcal{F}_L \right]. \quad (4.2)$$

Now, from Lemma 4.3,

$$\mathbb{E}[\widetilde{M}_\infty \mid \mathcal{F}_L] = \mathbb{E}[\widetilde{M}_\infty \mid \mathcal{F}_{L-}^L] = \mathbb{E} \left[\mathbb{E}[\widetilde{M}_\infty \mid \mathcal{F}_L^L] \mid \mathcal{F}_{L-}^L \right].$$

But now, from the optional stopping theorem,

$$\mathbb{E}[\widetilde{M}_\infty \mid \mathcal{F}_L^L] = \widetilde{M}_L;$$

moreover, M and \widetilde{M} have the same jumps, and L avoids (\mathcal{F}_t) stopping times, hence $\widetilde{M}_L = \widetilde{M}_{L-}$, *a.s.* Hence,

$$\mathbb{E}[\widetilde{M}_\infty \mid \mathcal{F}_{L-}^L] = \widetilde{M}_L.$$

Now, plugging this into (4.2), and comparing with (4.1), we obtain the equivalence between (1) and (2). \square

4.2. A solution related to Martingales which vanish at L . It is a remarkable fact, discovered by Azéma and Yor ([8]), that a uniformly integrable martingale vanishes at L , if and only if it is a solution to equation (1.4):

Proposition 4.6 (Azéma-Yor [8]). *Let M be an L^2 bounded martingale; then the following are equivalent:*

- (1) $\mathbb{E}[M_\infty \mid \mathcal{F}_L] = 0$, or in other words, $M_\infty \in (L^2(\mathcal{F}_L))^\perp$;
- (2) $M_L = 0$.

If one of the above conditions is satisfied, then $M \in \mathcal{S}_2$.

Remark 4.7. Proposition 4.6 is still true if M is only assumed to be uniformly integrable.

Corollary 4.8. *Let M be an L^2 bounded martingale and let*

$$M^L \equiv \mathbb{E}[M_\infty | \mathcal{F}_L].$$

Then the martingale

$$M_t - \mathbb{E}[M^L | \mathcal{F}_t]$$

belongs to \mathcal{S}_2 .

Theorem 4.6 and the above corollary show that it is enough to solve equation (1.4) when $M \in L^2(\mathcal{F}_L)$. Indeed, M_∞ can be decomposed uniquely as:

$$M_\infty = X_1 + X_2,$$

where $X_1 \equiv M - M^L \in L^2(\mathcal{F}_L)^\perp$ and $X_2 \equiv M^L \in L^2(\mathcal{F}_L)$. Thus, $M_t = M_t^1 + M_t^2$, with $M_t^1 = \mathbb{E}[X_1 | \mathcal{F}_t]$, $M_t^2 = \mathbb{E}[X_2 | \mathcal{F}_t]$, and from Theorem 4.6, $M^1 \in \mathcal{S}_2$. Now, we give a description of L^2 martingales (M_t) such that $M_\infty = x_L$, where (x_t) is a predictable process (recall that we work under condition **(A)**). Indeed, in all generality, for every L^2 martingale, there exists a predictable process x such that: $\mathbb{E}[M_\infty | \mathcal{F}_L] = x_L$.

Proposition 4.9. *Let (x_t) be a predictable process such that $\mathbb{E}[|x_L|] < \infty$. Then (for sake of simplicity, we shall next write A instead of A^L):*

$$\mathbb{E}[x_L | \mathcal{F}_t] = x_{L_t} \mathbb{P}(L \leq t | \mathcal{F}_t) + \mathbb{E}\left[\int_t^\infty x_s dA_s | \mathcal{F}_t\right], \quad (4.3)$$

where

$$L_t = \sup\{s < t : 1 - Z_s^L = 0\}.$$

Moreover, the latter martingale can also be written as:

$$\mathbb{E}[x_L | \mathcal{F}_t] = -\int_0^t x_{L_s} d\mu_s^L + \mathbb{E}\left[\int_0^\infty x_s dA_s | \mathcal{F}_t\right],$$

where (μ_s^L) is defined in Section 2 as the martingale part of the supermartingale (Z_t^L) .

Remark 4.10. This proposition will be used in the next section to construct a remarkable family of martingales.

Proof.

$$\begin{aligned} \mathbb{E}[x_L | \mathcal{F}_t] &= \mathbb{E}[x_L \mathbf{1}_{L \leq t} | \mathcal{F}_t] + \mathbb{E}[x_L \mathbf{1}_{L > t} | \mathcal{F}_t] \\ &= x_{L_t} \mathbb{P}(L \leq t | \mathcal{F}_t) + \mathbb{E}[x_L \mathbf{1}_{L > t} | \mathcal{F}_t], \end{aligned}$$

since from lemma 2.3, on the set $\{L \leq t\}$, we have $L_t = L$. Now, let Γ_t be an (\mathcal{F}_t) measurable set;

$$\mathbb{E}[x_L \mathbf{1}_{L > t} \mathbf{1}_{\Gamma_t}] = \mathbb{E}\left[\int_t^\infty x_s dA_s \mathbf{1}_{\Gamma_t}\right];$$

hence

$$\mathbb{E}[x_L \mathbf{1}_{L > t} | \mathcal{F}_t] = \mathbb{E}\left[\int_t^\infty x_s dA_s | \mathcal{F}_t\right],$$

and this completes the proof of the first part of the lemma. The second part follows from balayage arguments (see for example [6], Theorem 6.1); indeed:

$$\begin{aligned} x_{L_t} \mathbb{P}(L \leq t \mid \mathcal{F}_t) &= x_{L_t} (1 - Z_t^L) \\ &= - \int_0^t x_{L_s} d\mu_s^L + \int_0^t x_s dA_s, \end{aligned}$$

where we have used the fact that A lives on the set of times where Z is equal to 1. Now, since $\mathbb{E} \left[\int_t^\infty x_s dA_s \mid \mathcal{F}_t \right] = \mathbb{E} \left[\int_0^\infty x_s dA_s \mid \mathcal{F}_t \right] - \int_0^t x_s dA_s$, we have

$$x_{L_t} \mathbb{P}(L \leq t \mid \mathcal{F}_t) + \mathbb{E} \left[\int_t^\infty x_s dA_s \mid \mathcal{F}_t \right] = - \int_0^t x_{L_s} d\mu_s^L + \mathbb{E} \left[\int_0^\infty x_s dA_s \mid \mathcal{F}_t \right],$$

and the proof of the lemma is now complete. \square

Now, with the help of Proposition 4.9, we can solve equation (1.4) for martingales of the form $M_t \equiv \mathbb{E}[x_L \mid \mathcal{F}_t]$, and hence for any L^2 bounded martingale.

Proposition 4.11. *Let $M_t \equiv \mathbb{E}[x_L \mid \mathcal{F}_t]$ be a uniformly integrable martingale ((x_t) is a predictable process). Then,*

$$\begin{aligned} \mathbb{E}[M_\infty \mid \mathcal{F}_L] &= \mathbb{E} \left[\int_t^\infty x_s dA_s \mid \mathcal{F}_t \right] \Big|_{t=L} \\ &= \mathbb{E} \left[\int_0^\infty x_s dA_s \mid \mathcal{F}_t \right] \Big|_{t=L} - \int_0^\infty x_s dA_s. \end{aligned}$$

Consequently, $\mathbb{E}[M_\infty \mid \mathcal{F}_L] = M_L$ if and only if

$$\mathbb{E} \left[\int_t^\infty x_s dA_s \mid \mathcal{F}_t \right] \Big|_{t=L} = x_L.$$

Remark 4.12. We give in the next section some examples where all the calculations can be done explicitly.

Proof. This proposition is a consequence of Proposition 4.9 and the fact that: $A_\infty = A_L$. \square

4.3. Last zero before a fixed or a random time for the standard Brownian Motion. We shall now use Proposition 4.6 to build martingales which are solutions to equation (1.4) with a Brownian example which has received much attention in the literature (see [35] or [4] for more references). In the sequel, we shall also use some results from [26], where in particular all the following results have been generalized to Bessel processes of dimension $\delta (\equiv 2(1 - \mu)) \in (0, 2)$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq 1}, \mathbf{P})$ be a filtered probability space, where the filtration (\mathcal{F}_t) is generated by a one dimensional Brownian motion $(B_t)_{t \leq 1}$. Let

$$\gamma \equiv \sup \{t \leq 1 : B_t = 0\}.$$

It is well known (see [35, 18]) that:

$$\mathbb{P}[\gamma > t \mid \mathcal{F}_t] = \sqrt{\frac{2}{\pi}} \int_{\frac{|B_t|}{\sqrt{1-t}}}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx, \quad t < 1 \quad (4.4)$$

$$\lambda_t \equiv A_t^\gamma = \sqrt{\frac{2}{\pi}} \int_0^t \frac{d\ell_u}{\sqrt{1-u}}, \quad t < 1. \quad (4.5)$$

Moreover, $m \equiv \frac{1}{\sqrt{1-\gamma}} |B_1|$, $\varepsilon = \text{sgn}(B_1)$ and \mathcal{F}_γ are independent. We also have (as a consequence of Imhof's result, see for example [23] p.55):

$$\mathbb{P}(m \in d\rho) = \rho \exp\left(-\frac{\rho^2}{2}\right) d\rho.$$

Remark 4.13. A generalization of formulae (4.4) and (4.5) (which leads to a multidimensional version of the arc sine law) is proved in [26] for any Bessel process of dimension $\delta \in (0, 2)$.

Proposition 4.14 ([35], chapter XIV). *Let*

$$M_t^f \equiv \mathbb{E}[f(B_1) \mid \mathcal{F}_t] = P_{1-t}f(B_t),$$

with f a Borel function such that: $\mathbb{E}[|f(B_1)|] < \infty$, and (P_t) the semigroup of (B_t) . If f is an odd function, then (M_t^f) is a solution to equation (1.4), or in other words,

$$\mathbb{E}[M_\gamma^f] = \mathbb{E}[M_\infty^f \mid \mathcal{F}_\gamma].$$

Proof. We have:

$$\begin{aligned} \mathbb{E}[f(B_1) \mid \mathcal{F}_\gamma] &= \frac{1}{2} \int_{-\infty}^{\infty} dx |x| \exp\left(-\frac{x^2}{2}\right) f\left(x\sqrt{1-\gamma}\right) \\ &= \frac{1}{2(1-\gamma)} \int_0^{\infty} dy y \exp\left(-\frac{y^2}{2(1-\gamma)}\right) (f(y) + f(-y)), \end{aligned} \quad (4.6)$$

and hence, if f is odd, then $\mathbb{E}[f(B_1) \mid \mathcal{F}_\gamma] = 0$, and from Proposition 4.6, M_t^f is a solution to equation (1.4). \square

Now, let us consider the case when f is an even function such that $\mathbb{E}[|f(B_1)|] < \infty$. From (4.6),

$$\mathbb{E}[f(B_1) \mid \mathcal{F}_\gamma] = \int_0^{\infty} dy \frac{y}{(1-\gamma)} \exp\left(-\frac{y^2}{2(1-\gamma)}\right) f(y).$$

Now, let us define:

$$M^{f,\gamma} \equiv f(B_1) - \int_0^{\infty} dy \frac{y}{(1-\gamma)} \exp\left(-\frac{y^2}{2(1-\gamma)}\right) f(y),$$

then it follows from Corollary 4.8 that the martingale

$$M_t^{f,\perp} \equiv \mathbb{E}[M^{f,\gamma} \mid \mathcal{F}_t], \quad t \leq 1 \quad (4.7)$$

is a solution to equation (1.4). We first note that:

$$M_t^{f,\perp} = P_{1-t}f(B_t) - \int_0^\infty dy y f(y) \mathbb{E} \left[\frac{1}{(1-\gamma)} \exp \left(-\frac{y^2}{2(1-\gamma)} \right) \mid \mathcal{F}_t \right], \quad (4.8)$$

and hence it is enough to have an explicit formula for martingales of the form:

$$\mathbb{E} [h(\gamma) \mid \mathcal{F}_t],$$

where $h : [0, 1] \rightarrow \mathbb{R}$ is a deterministic function. This problem is solved in [26] (it suffices to take $\mu = \frac{1}{2}$ to recover the Brownian setting) and leads to some interesting results for our purpose.

Lemma 4.15 ([26], with $\mu = \frac{1}{2}$). *Let $h : [0, 1] \rightarrow \mathbb{R}_+$, be a Borel function, and let*

$$\gamma(t) \equiv \sup \{u \leq t; B_u = 0\}.$$

Then:

$$\mathbb{E} [h(\gamma) \mid \mathcal{F}_t] = h(\gamma(t)) (1 - Z_t^\gamma) + \mathbb{E} [h(\gamma) \mathbf{1}_{(\gamma > t)} \mid \mathcal{F}_t];$$

with

$$\mathbb{E} [h(\gamma) \mathbf{1}_{(\gamma > t)} \mid \mathcal{F}_t] = \frac{1}{\pi} \int_0^1 dz \frac{h(t + z(1-t))}{\sqrt{z(1-z)}} \exp \left(-\frac{B_t^2}{2z(1-t)} \right). \quad (4.9)$$

Now, with the help of Lemma 4.15, after some elementary calculations, we have the following explicit expression for the family of martingales $(M_t^{f,\perp})$.

Proposition 4.16. *Let f be an even Borel function such that $\mathbb{E} [|f(B_1)|] < \infty$, and let*

$$M_t^{f,\perp} = \mathbb{E} \left[f(B_1) - \int_0^\infty dy \frac{y}{(1-\gamma)} \exp \left(-\frac{y^2}{2(1-\gamma)} \right) f(y) \mid \mathcal{F}_t \right],$$

as in (4.7). Define:

$$\theta(x) \equiv \sqrt{\frac{2}{\pi}} \int_x^\infty dv \exp \left(-\frac{v^2}{2} \right) = \int_0^1 \frac{dv}{\pi \sqrt{v(1-v)}} \exp \left(-\frac{x^2}{2v} \right).$$

Then

$$M_t^{f,\perp} = M_t^{f,1} - M_t^{f,2} - M_t^{f,3},$$

where:

$$M_t^{f,1} = \int_0^\infty \frac{dz}{\sqrt{2\pi(1-t)}} f(z) \left(\exp \left(-\frac{(z+B_t)^2}{2(1-t)} \right) + \exp \left(-\frac{(z-B_t)^2}{2(1-t)} \right) \right),$$

$$M_t^{f,2} = \theta \left(\frac{|B_t|}{\sqrt{1-t}} \right) \int_0^\infty dz z f(z \sqrt{1-\gamma_t}) \exp \left(-\frac{z^2}{2} \right)$$

$$M_t^{f,3} = \int_0^\infty dz z \exp \left(-\frac{z^2}{2} \right) \int_0^1 \frac{dw}{\pi \sqrt{w(1-w)}} f(z \sqrt{1-t} \sqrt{1-w}) \exp \left(-\frac{B_t^2}{2w(1-t)} \right),$$

and $(M_t^{f,\perp})$ is a solution to equation (1.4).

Remark 4.17. The proposition shows that although Proposition 4.6 is elementary, it is in practice difficult to compute the projection of the terminal value of a martingale on the sigma algebra \mathcal{F}_γ .

As a consequence of the explicit form for the martingales $\mathbb{E}[h(\gamma) | \mathcal{F}_t]$, we have the following first interesting result which shows how (1.4) or (1.3) may fail to hold in general:

Proposition 4.18. *Let $h : [0, 1] \rightarrow \mathbb{R}_+$, be a Borel function, and define $N_t^h = \mathbb{E}[h(\gamma) | \mathcal{F}_t]$; then*

$$\mathbb{E}[N_\infty^h | \mathcal{F}_\gamma] = h(\gamma), \quad (4.10)$$

whilst

$$N_\gamma^h = \frac{1}{\pi} \int_0^1 \frac{dv}{\sqrt{v(1-v)}} h(\gamma + v(1-\gamma)). \quad (4.11)$$

The balayage formula can be used to get many solutions to equation (1.4):

Proposition 4.19. *Define:*

$$g_t \equiv \gamma(t) = \sup\{s \leq t : B_s = 0\},$$

and let $T > 0$ be a fixed time (thus with this notation, we have $g_1 = \gamma$). Then, for any bounded predictable process (x_s) ,

$$X_t \equiv x_{g_t \wedge T} B_{t \wedge T}$$

is a uniformly integrable martingale which satisfies (1.4) for $L = g_T$, or more generally for $L = g_t$; $t \leq T$.

Proof. It is a consequence of the balayage formula that $(x_{g_t \wedge T} B_{t \wedge T})$ is a local martingale (see [31], Chapter VI). From our assumptions, we easily obtain that it is a bounded L^2 martingale. Now, $X_{g_t} = 0$ for every $t \leq T$, and hence from Proposition 4.6, X satisfies (1.4). \square

It is also possible to give many examples of honest times such that the standard Brownian Motion, adequately stopped, satisfies (1.4).

Proposition 4.20. *Let T be a stopping time such that $(B_{t \wedge T})$ is a uniformly integrable martingale. Define g_T as above. Then, for every honest time $L \leq g_T$, we have $B_L = 0$ and hence $(B_{t \wedge T})$ satisfies (1.4) for such L 's.*

Proof. As $L \leq g_T$, and both L and g_T are honest, we have $\mathcal{F}_L \subset \mathcal{F}_{g_T}$. Consequently,

$$\mathbb{E}[B_T | \mathcal{F}_L] = \mathbb{E}[\mathbb{E}[B_T | \mathcal{F}_{g_T}] | \mathcal{F}_L] = 0,$$

because $\mathbb{E}[B_T | \mathcal{F}_{g_T}] = 0$ from Proposition 4.6. Now, another application of Proposition 4.6 yields $B_L = 0$ and hence $(B_{t \wedge T})$ satisfies (1.4) with L . \square

Remark 4.21. The last two propositions can be extended to continuous martingales.

5. UNDERSTANDING THE DIFFERENCES WITH A REMARKABLE FAMILY OF MARTINGALES

So far, we have tried to characterize martingales for which, given a random time, (1.3) and (1.4) hold. Now, we try to understand how these equalities may fail. Again, we consider the case of honest times under condition **(A)**. To this end, we introduce a family of uniformly integrable martingales, with some remarkable properties, and which will serve us to test (1.3) and (1.4). From Corollary 3.5 and Proposition 4.6, it follows that interesting examples of families of martingales such that (1.3) and (1.4) may fail to hold, should have the property: M_∞ is $\sigma(A_\infty)$ measurable (indeed, $\sigma(A_\infty) \subset \mathcal{F}_L$ since $A_\infty = A_L$). We should also mention that equation (1.4) has been studied in the special case of David Williams' pseudo-stopping time in [24].

5.1. A remarkable family of martingales. We first prove a useful lemma:

Lemma 5.1. *Let φ be a Borel function such that $\mathbb{E}[|\varphi(A_\infty)|] < \infty$, or equivalently $\int_0^\infty dx |\varphi(x)| \exp(-x) < \infty$, and let $\Phi(x) = \int_0^x dy \varphi(y)$. Then:*

$$\mathbb{E}[\varphi(A_\infty) | \mathcal{F}_t] = \varphi(A_t)(1 - Z_t) - \Phi(A_t) + \mathbb{E}[\Phi(A_\infty) | \mathcal{F}_t]. \quad (5.1)$$

Proof. From Lemma 2.3, $\varphi(A_\infty) = \varphi(A_L)$, and hence: $\mathbb{E}[\varphi(A_\infty) | \mathcal{F}_t] = \mathbb{E}[\varphi(A_L) | \mathcal{F}_t]$. We can thus apply Proposition 4.9 to obtain:

$$\mathbb{E}[\varphi(A_\infty) | \mathcal{F}_t] = \varphi(A_{L_t}) \mathbb{P}(L \leq t | \mathcal{F}_t) + \mathbb{E}\left[\int_t^\infty \varphi(A_s) dA_s | \mathcal{F}_t\right].$$

Now, from Lemma 2.2, $\varphi(A_{L_t}) = \varphi(A_t)$ and moreover, since A is continuous,

$$\int_t^\infty \varphi(A_s) dA_s = \int_{A_t}^{A_\infty} dx \varphi(x) = \Phi(A_\infty) - \Phi(A_t),$$

and the assertion of the lemma follows. \square

Remark 5.2. If f is a function of class \mathcal{C}^1 , then, an application of Lemma 5.1 with $\varphi = f'$ yields:

$$\mathbb{E}[f(A_\infty) - f'(A_\infty) | \mathcal{F}_t] = f(A_t) - f'(A_t)(1 - Z_t).$$

Before introducing our family of martingales, we need to introduce the following transform: we associate with a continuous function φ the function $\widehat{\varphi}$, defined on \mathbb{R}_+ by:

$$\begin{aligned} \widehat{\varphi}(x) &= \exp(x) \int_x^\infty dy \exp(-y) \varphi(y) \\ &= \int_0^\infty dy \exp(-y) \varphi(y+x). \end{aligned}$$

It is easy to see that $\widehat{\varphi}$ is a function of class \mathcal{C}^1 , and:

$$\widehat{\varphi} - \widehat{\varphi}' = \varphi.$$

Proposition 5.3. *Let φ be a continuous function such that $\mathbb{E}[|\varphi(A_\infty)|] < \infty$, or equivalently $\int_0^\infty dx |\varphi(x)| \exp(-x) < \infty$, and let*

$$\widehat{\varphi}(x) = \exp(x) \int_x^\infty dy \exp(-y) \varphi(y).$$

If $\int_0^\infty dx \exp(-x) |\varphi(x)| < \infty$, then:

$$\mathbb{E}[\varphi(A_\infty) | \mathcal{F}_t] = Z_t \widehat{\varphi}(A_t) + (1 - Z_t) \varphi(A_t). \quad (5.2)$$

Proof. It suffices to apply Remark 5.2 to $\widehat{\varphi}$. \square

Remark 5.4. In fact, it can be shown, using monotone class arguments, that formula 5.2 remains valid if φ is only assumed to be a Borel function such that $\int_0^\infty dx |\varphi(x)| \exp(-x) < \infty$. These martingales have already been obtained by different means in [4] and [27] (they are used there in a different framework).

One remarkable fact about these martingales, which we shall denote by (M_t^φ) , is that we know their supremum processes when φ is increasing. More precisely, we have:

Proposition 5.5. *Assume that φ is a nonnegative, continuous and increasing function such that the assumptions of Proposition 5.3 are satisfied, and assume further that (M_t^φ) is a continuous martingale. Then the supremum process of $(M_t^\varphi \equiv \mathbb{E}[\varphi(A_\infty) | \mathcal{F}_t])$ is given by:*

$$\sup_{s \leq t} M_s^\varphi = \widehat{\varphi}(A_t).$$

Proof. We have:

$$(\widehat{\varphi}(A_t) - \varphi(A_t))(1 - Z_t) = -M_t^\varphi + \widehat{\varphi}(A_t) \quad (5.3)$$

Moreover,

$$\widehat{\varphi}(A_t) - \varphi(A_t) = \int_0^\infty dx \exp(-x) (\varphi(x + A_t) - \varphi(A_t))$$

and since φ is increasing, we have $(\widehat{\varphi}(A_t) - \varphi(A_t)) \geq 0$. Similarly, we prove that $\widehat{\varphi}$ is increasing. Hence, (5.3) may be considered as a particular case of Skorokhod's reflection equation and thus:

$$\sup_{s \leq t} M_s^\varphi = \widehat{\varphi}(A_t)$$

\square

5.2. Martingales stopped at an honest time. In the sequel, we assume that φ is a continuous function satisfying the following conditions of Proposition 5.3: $\int_0^\infty dx |\varphi(x)| \exp(-x) < \infty$, and $\int_0^\infty dx \exp(-x) |\varphi(x)| x < \infty$.

Proposition 5.6. *Let L be an honest time and $M_t^\varphi = \mathbf{E}[\varphi(A_\infty) | \mathcal{F}_t]$. Then, we have:*

$$\begin{aligned} M_L^\varphi &= \exp(A_L) \int_{A_L}^\infty dx \exp(-x) \varphi(x) \\ &= \int_0^\infty dx \exp(-x) \varphi(A_L + x) = \widehat{\varphi}(A_L) \end{aligned} \quad (5.4)$$

and

$$\mathbf{E}[\varphi(A_\infty) | \mathcal{F}_L] = \varphi(A_L)$$

Proof. The Proposition follows from Proposition 5.3, and the fact that $Z_L = 1$, $A_\infty = A_L$. \square

With Proposition 5.6, it is now clear why (1.4) may fail for honest times. More precisely;

Corollary 5.7.

$$\mathbf{E}[M_\infty^\varphi | \mathcal{F}_L] = M_L^\varphi$$

if and only if φ is constant.

Proof. If $\mathbf{E}[M_\infty^\varphi | \mathcal{F}_L] = M_L^\varphi$, then from Proposition 5.6, φ satisfies:

$$\exp(y) \int_y^\infty dx \exp(-x) \varphi(x) = \varphi(y).$$

The only solutions of this equation are the constant functions. \square

5.3. The expected value of martingales stopped at an honest time.

In the previous section, we saw that $\mathbf{E}[M_\infty^\varphi | \mathcal{F}_L]$ and M_L^φ differ if the function φ is not constant. In this subsection, we shall compare the two quantities $\mathbf{E}[M_\infty^\varphi]$ and $\mathbf{E}[M_L^\varphi]$.

Proposition 5.8. *Let L be an honest time and $M_t^\varphi = \mathbf{E}[\varphi(A_\infty) | \mathcal{F}_t]$. We have:*

$$\begin{aligned} \mathbf{E}[M_L^\varphi] &= \int_0^\infty \int_0^\infty dx dy \exp(-x) \exp(-y) \varphi(y+x) \\ &= \mathbf{E}[\varphi(\mathbf{e}_1 + \mathbf{e}_2)] \\ &= \int_0^\infty dx \exp(-x) x \varphi(x) \end{aligned}$$

where \mathbf{e}_1 and \mathbf{e}_2 are two independent random variables following the standard exponential distribution, whereas

$$\begin{aligned} \mathbf{E}[M_\infty^\varphi] &= \int_0^\infty dx \exp(-x) \varphi(x) \\ &= \mathbf{E}(\varphi(\mathbf{e}_1)) \end{aligned}$$

Proof. This is a consequence of Propositions 5.6 and 5.3. \square

Proposition 5.9. *If φ is a positive increasing function, we have:*

$$\sup_{t \geq 0} M_t^\varphi = M_L^\varphi,$$

and consequently, $(M_t^\varphi) \in \mathcal{H}^1$ and

$$\|M^\varphi\|_{\mathcal{H}^1} = \mathbf{E} [M_L^\varphi].$$

Proof. It is a consequence of Proposition 5.5 and the fact that $A_\infty = A_L$. \square

But unlike the case of equation (1.4), we can find solutions to equation (1.3) among the martingales (M_t^φ) . Recall that (\mathcal{L}_n) is the family of Laguerre polynomials.

Proposition 5.10. *Let L be an honest time. Let φ satisfy the conditions of proposition 5.3 and $\mathbf{E}(\varphi^2(A_\infty)) < \infty$. Set again $M_t^\varphi = \mathbf{E}[\varphi(A_\infty) | \mathcal{F}_t]$; then*

$$\mathbf{E}[M_\infty^\varphi] = \mathbf{E}[M_L^\varphi]$$

if and only if

$$\varphi(x) = \alpha + \sum_{n=2}^{\infty} \alpha_n \mathcal{L}_n(x)$$

where $\alpha \in \mathbb{R}$, and (α_n) are such that $\sum \alpha_n^2 < \infty$, i.e. in the development of φ , the coefficients of \mathcal{L}_1 is $\alpha_1 = 0$. In other words, φ belongs to the orthogonal of \mathcal{L}_1 .

Proof. From Proposition 5.8, $\mathbf{E}[M_\infty^\varphi] = \mathbf{E}[M_L^\varphi]$ if and only if:

$$\int_0^\infty dx \exp(-x) x \varphi(x) = \int_0^\infty dx \exp(-x) \varphi(x)$$

or equivalently:

$$\int_0^\infty dx \exp(-x) \mathcal{L}_1(x) \varphi(x) = 0$$

It now suffices to develop φ in the basis $(\mathcal{L}_n(x))$ to conclude. \square

Proof. It suffices to notice that $(\mathbf{E}[M_\infty^\varphi] - \mathbf{E}[M_L^\varphi])$ is the coefficient of $\mathcal{L}_1(x) = (1-x)$ in the expansion of φ in the basis $(\mathcal{L}_n(x))$. \square

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