

ENLARGEMENTS OF FILTRATIONS AND PATH DECOMPOSITIONS AT NON STOPPING TIMES

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ABSTRACT. Azéma associated with an honest time L the supermartingale $Z_t^L = \mathbb{P}[L > t | \mathcal{F}_t]$ and established some of its important properties. This supermartingale plays a central role in the general theory of stochastic processes and in particular in the theory of progressive enlargements of filtrations. In this paper, we shall give an additive characterization for these supermartingales, which in turn will naturally provide many examples of enlargements of filtrations. We combine this characterization with some arguments from both initial and progressive enlargements of filtrations to establish some path decomposition results, closely related to or reminiscent of Williams' path decomposition results. In particular, some of the fragments of the paths in our decompositions end or start with a new family of random times which are not stopping times, nor honest times.

1. INTRODUCTION

Other than stopping times, the most commonly studied random times occur as the ends of optional sets, or honest times. We shall denote the class of such times by L . A very powerful technique for studying such random times is that of progressive enlargement of filtrations. The theory of progressive enlargements of filtrations was introduced independently by Barlow ([4]) and Yor ([25]), and further developed by Jeulin and Yor ([9, 8, 7, 26]). It has many applications in various parts of Probability Theory: path decompositions for some diffusion processes ([7], [16]), mathematical models of default times and insider trading in mathematical finance ([6]), probabilistic inequalities ([9],[17]), or new proofs of well known results, such as Pitman's theorem (see [26], chapter XII).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, satisfying the usual assumptions, and L the end of an (\mathcal{F}_t) optional set Γ , i.e:

$$L = \sup \{t : (t, \omega) \in \Gamma\}.$$

The main idea is to consider the larger filtration

$$\mathcal{F}_t^L = \mathcal{G}_{t+}, \quad \text{with} \quad \mathcal{G}_t \equiv \mathcal{F}_t \vee \sigma \{L \wedge t, \},$$

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which is the smallest right continuous filtration which contains (\mathcal{F}_t) and which makes L a stopping time, and then to see how martingales of the smaller filtration are changed when considered as stochastic processes of the larger one. One process plays an essential role in this theory, namely the supermartingale:

$$Z_t^L = \mathbb{P}(L > t | \mathcal{F}_t), \quad (1.1)$$

associated with L by Azéma in [1], and chosen to be càdlàg. An (\mathcal{F}_t) local martingale (M_t) , is a semimartingale in the larger filtration (\mathcal{F}_t^L) and decomposes as:

$$M_t = \widetilde{M}_t + \int_0^{t \wedge L} \frac{d\langle M, Z^L \rangle_s}{Z_{s-}^L} - \int_L^t \frac{d\langle M, Z^L \rangle_s}{1 - Z_{s-}^L}, \quad (1.2)$$

where $(\widetilde{M}_t)_{t \geq 0}$ denotes a $((\mathcal{F}_t^L), \mathbb{P})$ local martingale. One limitation of this formula is that it may not be easy to compute the supermartingale Z^L for a given L ; in fact only a few examples are known (see [7, 26]). Hence, it would be useful to obtain a characterization result which would help produce examples of honest times and their associated supermartingales.

For simplicity, we make the following assumptions throughout this paper, which we call the **(CA)** conditions:

- (1) all (\mathcal{F}_t) -martingales are continuous (e.g: the Brownian filtration).
- (2) the random time L avoids every (\mathcal{F}_t) -stopping time T , i.e. $\mathbb{P}[L = T] = 0$.

Remark 1.1. Under the conditions **(CA)**, the optional and the predictable sigma fields (with respect to (\mathcal{F}_t)) are equal and the supermartingale (Z_t^L) is continuous.

One of the aims of this paper is to characterize the supermartingales (Z_t^L) . In [16], we gave a multiplicative characterization for the supermartingales (Z_t^L) , while here we shall adopt an additive approach (Doob-Meyer decomposition). The paper is organized as follows:

In Section 2, we prove the characterization result for Azéma's supermartingales. To state it, we need to define a special class of submartingales, whose definition goes back to Yor [24], and which was also studied in [13] (under more general conditions):

Definition 1.2. Let (X_t) be a positive local submartingale, which decomposes as:

$$X_t = N_t + A_t.$$

We say that (X_t) is of class (Σ) if:

- (1) (N_t) is a continuous local martingale, with $N_0 = 0$;
- (2) (A_t) is a continuous increasing process, with $A_0 = 0$;
- (3) the measure (dA_t) is carried by the set $\{t : X_t = 0\}$.

If additionally, (X_t) is of class (D) , we shall say that (X_t) is of class (ΣD) .

Now, consider the Doob-Meyer decomposition of Z_t^L :

$$Z_t^L \equiv 1 + \mu_t^L - A_t^L.$$

We prove that if (Z_t) is a nonnegative supermartingale, with $Z_\infty = 0$, then, Z may be represented as $\mathbb{P}(L > t | \mathcal{F}_t)$, for some honest time L which avoids stopping times, if and only if $(X_t \equiv 1 - Z_t)$ is a submartingale of the class (Σ) , with the limit condition:

$$\lim_{t \rightarrow \infty} X_t = 1.$$

Section 3 contains our main results: we apply the results of Section 2 to obtain decompositions analogous to Williams' path decomposition result for the supermartingale (Z_t^L) . We also establish some path decomposition results for certain classes of diffusion processes which play an important role in applications. In particular, we shall see that the pseudo-stopping times, introduced in [15], play an important role in path decompositions, exhibiting thus a new family of random times enjoying nice properties with respect to path decomposition.

2. A CHARACTERIZATION OF AZÉMA'S SUPERMARTINGALE AND APPLICATIONS

2.1. The characterization of Azéma's supermartingale for honest times. Azéma has studied in depth the supermartingale $Z_t^L = \mathbb{P}(L > t | \mathcal{F}_t)$ associated with an honest time L and has proved many interesting properties. A classical example of such a random time, which has received much attention in the literature ([7, 26], see [14] for a one parameter extension), is:

$$L = \sup \{t \leq 1 : B_t = 0\},$$

where as usual (B_t) denotes the standard Brownian Motion.

Let us briefly recall some results of Azéma. We assume that the conditions **(CA)** hold. We consider the Doob-Meyer decomposition of Z^L :

$$Z_t^L = 1 + \mu_t^L - A_t^L \tag{2.1}$$

The process (A_t^L) , which we shall sometimes denote (A_t) in the sequel, is the dual predictable projection of the increasing process $\mathbf{1}_{\{L \leq t\}}$, and

$$\mu_t^L = \mathbb{E}(A_\infty^L | \mathcal{F}_t) - 1$$

Proposition 2.1 (Azéma [1]). *Let L be the end of an optional set, or an honest time (as was discovered by M.T. Barlow [4], every honest time is the end of some optional set); then*

$$L = \sup \{t : Z_t = 1\},$$

and the measure dA_t is carried by the set $\{t : Z_t = 1\}$. In particular, A does not increase after L , i.e. $A_L = A_\infty$.

To prove our main theorem, we shall need the following very useful lemma, which appears in the papers of Azéma, Meyer and Yor [2] and Azéma and Yor [3].

Lemma 2.2. *Let (X_t) be a submartingale of the class (ΣD) and let*

$$L = \sup \{t : X_t = 0\}.$$

Assume further that:

$$\mathbb{P}(X_\infty = 0) = 0.$$

Then:

$$X_t = \mathbb{E}(X_\infty \mathbf{1}_{\{L \leq t\}} | \mathcal{F}_t). \quad (2.2)$$

Now, we can state our characterization theorem.

Theorem 2.3. *Let (X_t) be a submartingale of the class (ΣD) satisfying: $\lim_{t \rightarrow \infty} X_t = 1$. Let*

$$L = \sup \{t : X_t = 0\}.$$

Then (X_t) is related to Azéma's supermartingale associated with L in the following way:

$$X_t = 1 - Z_t^L = \mathbb{P}(L \leq t | \mathcal{F}_t).$$

Consequently, if (Z_t) is a nonnegative supermartingale, with $Z_0 = 1$, then, Z may be represented as $\mathbf{P}(L > t | \mathcal{F}_t)$, for some honest time L which avoids stopping times, if and only if $(X_t \equiv 1 - Z_t)$ is a submartingale of the class (Σ) , with the limit condition:

$$\lim_{t \rightarrow \infty} X_t = 1.$$

Proof. This is an immediate application of Lemma 2.2, with $X_\infty = 1$ and Proposition 2.1. \square

2.2. Some fundamental examples. In the sequel, we give some explicit (yet generic) computations of Azéma's supermartingales for some honest times associated with some very well known stochastic processes. These computations are the first steps towards the path decompositions proved in the next section.

2.2.1. A Brownian example. First, consider (B_t) , the standard Brownian Motion, and let $T_1 = \inf \{t \geq 0 : B_t = 1\}$. Let $\sigma = \sup \{t < T_1 : B_t = 0\}$. Then $B_{t \wedge T_1}^+$ satisfies the conditions of Theorem 2.3, and hence:

$$\mathbb{P}(\sigma \leq t | \mathcal{F}_t) = B_{t \wedge T_1}^+ = \int_0^{t \wedge T_1} \mathbf{1}_{B_u > 0} dB_u + \frac{1}{2} \ell_{t \wedge T_1},$$

where (ℓ_t) is the local time of B at 0. This example plays an important role in Williams' celebrated path decomposition for the standard Brownian Motion on $[0, T_1]$. This result is usually obtained by exploiting the strong Markov property of the Brownian Motion. Our method allows us to get rid of the Markov property, and to get similar formulae in the more general context of continuous local martingales, as is shown in the next paragraph.

One can also consider $T_{\pm 1} = \inf \{t \geq 0 : |B_t| = 1\}$ and $\tau = \sup \{t < T_{\pm 1} : |B_t| = 0\}$. $|B_{t \wedge T_{\pm 1}}|$ satisfies the conditions of Theorem 2.3, and hence:

$$\mathbb{P}(\tau \leq t | \mathcal{F}_t) = |B_{t \wedge T_{\pm 1}}| = \int_0^{t \wedge T_{\pm 1}} \text{sgn}(B_u) dB_u + \ell_{t \wedge T_{\pm 1}}.$$

2.2.2. Generalization to continuous local martingales. More generally, consider a continuous local martingale (M_t) such that $M_0 = 0$ and $\langle M \rangle_{\infty} = \infty$, a.s.; let $T_1 = \inf \{t \geq 0 : M_t = 1\}$ and $\sigma = \sup \{t < T_1 : M_t = 0\}$. Then, again, an application of Theorem 2.3 gives:

$$\mathbb{P}(\sigma \leq t | \mathcal{F}_t) = M_{t \wedge T_1}^+ = \int_0^{t \wedge T_1} \mathbf{1}_{M_u > 0} dM_u + \frac{1}{2} L_{t \wedge T_1},$$

where (L_t) is the local time of M at 0.

2.2.3. Recurrent diffusions. Let (Y_t) be a real continuous recurrent diffusion process, with $Y_0 = 0$. Then from the general theory of diffusion processes, there exists a unique continuous and strictly increasing function s , with $s(0) = 0$, $\lim_{x \rightarrow +\infty} s(x) = +\infty$, $\lim_{x \rightarrow -\infty} s(x) = -\infty$, such that $s(Y_t)$ is a continuous local martingale. Our aim is to establish some results analogous to those established for the Brownian Motion and recurrent continuous local martingales. Let

$$T_1 \equiv \inf \{t \geq 0 : Y_t = 1\}.$$

Now, if we define

$$X_t \equiv \frac{s(Y_{t \wedge T_1})^+}{s(1)},$$

we easily note that X is a local submartingale of the class (Σ) which satisfies the hypotheses of Theorem 2.3. Consequently, if we note

$$\sigma = \sup \{t < T_1 : Y_t = 0\},$$

we have:

$$\mathbb{P}(\sigma \leq t | \mathcal{F}_t) = \frac{s(Y_{t \wedge T_1})^+}{s(1)}.$$

2.2.4. Nonnegative continuous martingales which vanish at infinity. Now let (M_t) be a positive local martingale, such that: $M_0 = x$, $x > 0$ and $\lim_{t \rightarrow \infty} M_t = 0$. Then, Tanaka's formula shows that $\left(1 - \frac{M_t}{y} \wedge 1\right)$, for $0 \leq y \leq x$, is a local submartingale of the class (Σ) satisfying the assumptions of Theorem 2.3, and hence with

$$g = \sup \{t : M_t = y\},$$

we have:

$$\mathbb{P}(g > t | \mathcal{F}_t) = \frac{M_t}{y} \wedge 1 = 1 + \frac{1}{y} \int_0^t \mathbf{1}_{(M_u < y)} dM_u - \frac{1}{2y} L_t^y,$$

where (L_t^y) is the local time of M at y .

2.2.5. *Transient diffusions.* As an illustration of the previous example, consider (R_t) , a transient diffusion with values in $[0, \infty)$, which has $\{0\}$ as entrance boundary. Let s be a scale function for R , which we can choose such that:

$$s(0) = -\infty, \text{ and } s(\infty) = 0.$$

Then, under the law \mathbb{P}_x , for any $x > 0$, the local martingale $(M_t = -s(R_t))$ satisfies the conditions of the previous example and for $0 \leq x \leq y$, we have:

$$\mathbb{P}_x(g_y > t | \mathcal{F}_t) = \frac{s(R_t)}{s(y)} \wedge 1 = 1 + \frac{1}{s(y)} \int_0^t \mathbf{1}_{(R_u > y)} d(s(R_u)) + \frac{1}{2s(y)} L_t^{s(y)}, \quad (2.3)$$

where $(L_t^{s(y)})$ is the local time of $s(R)$ at $s(y)$, and where

$$g_y = \sup \{t : R_t = y\}.$$

Formula (2.3) was the key point in the derivation of the distribution of g_y in [18], Theorem 6.1, p.326.

3. PATH DECOMPOSITIONS

In this section, inspired by Williams' path decompositions for the standard Brownian Motion and for transient diffusions given their minima, we establish path decompositions for Azéma's supermartingales and some families of recurrent and transient diffusions. What follows is similar in spirit to what we have done in [16], but in an additive setting, and of course the results are different. It is also an opportunity to show that the techniques of progressive and initial enlargements of filtrations can be combined to prove, quite shortly, some non trivial path decomposition results.

Let us recall briefly the random times introduced by D. Williams to study the paths of a standard Brownian Motion B :

$$T_1 = \inf \{t : B_t = 1\}, \quad \sigma = \sup \{t < T_1 : B_t = 0\};$$

and

$$\rho = \sup \{u < \sigma : B_u = S_u\}, \quad \text{where } S_u = \sup_{s \leq u} B_s.$$

D. Williams ([23]) discovered the remarkable fact that although ρ is not a stopping time, it nevertheless satisfies the optional stopping theorem, i.e. for every bounded martingale (M_t) of the filtration (\mathcal{F}_t) , we have:

$$\mathbb{E}M_\rho = \mathbb{E}M_\infty.$$

In [15], we have called such random times pseudo-stopping times and we have characterized them. Before stating and proving our results, we shall first recall in the next subsection some standard facts about pseudo-stopping times and multiple enlargements of filtrations.

3.1. Basic facts about pseudo-stopping times and double enlargements of filtrations.

3.1.1. *Pseudo-stopping times.* In [15], following D. Williams, we have proposed the following generalization of stopping times:

Definition 3.1 ([15]). Let $\rho : (\Omega, \mathcal{F}) \rightarrow \mathbf{R}_+$ be a random time; ρ is called a pseudo-stopping time if for every bounded (\mathcal{F}_t) martingale (M_t) we have:

$$\mathbf{E}(M_\rho) = \mathbf{E}(M_0).$$

David Williams ([23]) gave the first example of such a random time and the following systematic construction is established in [15]:

Proposition 3.2 ([15]). *Let L be an honest time. Then, under the conditions (CA),*

$$\rho \equiv \sup \left\{ t < L : Z_t^L = \inf_{u \leq L} Z_u^L \right\},$$

is a pseudo-stopping time, with

$$Z_t^\rho \equiv \mathbf{P}(\rho > t \mid \mathcal{F}_t) = \inf_{u \leq t} Z_u^L,$$

and Z_ρ^ρ follows the uniform distribution on $(0, 1)$.

The following property, also proved in [15], is essential in studying path decompositions:

Proposition 3.3 ([15]). *Let ρ be a pseudo-stopping time and let M_t be an (\mathcal{F}_t) local martingale. Then $(M_{t \wedge \rho})$ is an (\mathcal{F}_t^ρ) local martingale.*

To conclude, let us illustrate Proposition 3.2 with an example. Let Y be a recurrent diffusion; with the notations and assumptions of paragraph 2.2.3,

$$\rho \equiv \sup \left\{ t < \sigma : Y_t = \max_{u \leq \sigma} Y_u \right\},$$

is a pseudo-stopping time.

3.1.2. *Double enlargements of filtrations.* We recall some lesser known results of Jeulin ([7]) about successive progressive enlargements of filtrations. The reader can also refer to [5] for a more recent exposition (summary) of these facts.

Proposition 3.4 (Jeulin [7]). *Let L be an honest time for the filtration (\mathcal{F}_t) and let ρ be an honest time for (\mathcal{F}_t^L) , and define $(\mathcal{F}_t^{L,\rho})$ the filtration obtained by enlarging progressively (\mathcal{F}_t^L) with ρ . Then, any (\mathcal{F}_t) local martingale (M_t) is a semimartingale in $(\mathcal{F}_t^{L,\rho})$ and decomposes as:*

$$M_t = \widetilde{M}_t + \int_0^{t \wedge \rho} \frac{d\langle M, Z^\rho \rangle_u}{Z_{u-}^\rho} + \int_\rho^{t \wedge L} \frac{d\langle M, Z^L - Z^\rho \rangle_u}{Z_{u-}^L - Z_{u-}^\rho} - \int_L^t \frac{d\langle M, Z^L \rangle_u}{1 - Z_{u-}^L}, \quad (3.1)$$

where (\widetilde{M}_t) is an $(\mathcal{F}_t^{L,\rho})$ local martingale.

Honest times enjoy the remarkable property that every (\mathcal{F}_t) semimartingale is an (\mathcal{F}_t^L) semimartingale, or in the jargon of the theory of enlargements of filtrations, the pair of filtrations $((\mathcal{F}_t), (\mathcal{F}_t^L))$ satisfy the (\mathcal{H}') hypothesis. The previous proposition shows that there might be non-honest times which enjoy this property; indeed, the pseudo-stopping times defined in Proposition 3.2 have this property:

Corollary 3.5. *Let us consider the pseudo-stopping time defined in Proposition 3.2. Then, every (\mathcal{F}_t) semimartingale is an (\mathcal{F}_t^ρ) semimartingale, or in other words, the pair of filtrations $((\mathcal{F}_t), (\mathcal{F}_t^\rho))$ satisfy the (\mathcal{H}') hypothesis.*

Proof. It suffices to prove that every $((\mathcal{F}_t))$ local martingale (M_t) is an (\mathcal{F}_t^ρ) semimartingale. From Proposition 3.4, every $((\mathcal{F}_t))$ local martingale is an $(\mathcal{F}_t^{L,\rho})$ semimartingale, and since $\mathcal{F}_t^\rho \subset \mathcal{F}_t^{L,\rho}$ and (M_t) is (\mathcal{F}_t^ρ) adapted, it follows from a well known result of Stricker (see [19]) that (M_t) is also an (\mathcal{F}_t^ρ) semimartingale. \square

We shall also need another result of Jeulin which certainly deserves to be better known: the problem of initial enlargement with A_∞^L .

Proposition 3.6. *Let T be a totally inaccessible stopping time, such that $\mathbb{P}(T > 0) = 1$ and let (A_t) be the (\mathcal{F}_t) dual predictable projection of $(\mathbf{1}_{T \leq t})$. Then the following hold:*

- (1) *A is continuous and $T = \inf \{t : A_t = A_T\} = \sup \{t : A_t = A_T\}$ (Azéma [1]);*
- (2) *define $\mathcal{G}_t \equiv \bigcap_{\varepsilon > 0} (\mathcal{F}_{t+\varepsilon} \vee \sigma(A_T))$; then every continuous (\mathcal{F}_t) martingale is a (\mathcal{G}_t) martingale (Jeulin [7]).*

Remark 3.7. In fact, every (\mathcal{F}_t) martingale, which does not jump at T , is a (\mathcal{G}_t) martingale.

Corollary 3.8. (1) *If L is an honest time which avoids stopping times, then every (\mathcal{F}_t) martingale is an $(\mathcal{F}_t^{L,\sigma(A_\infty^L)})$ semimartingale and the decomposition formula is the same as the decomposition formula (1.2):*

$$M_t = \widetilde{M}_t + \int_0^{t \wedge L} \frac{d\langle M, Z^L \rangle_s}{Z_{s-}^L} + \int_L^t \frac{d\langle M, Z^L \rangle_s}{1 - Z_{s-}^L}.$$

- (2) *Similarly, under the assumptions (CA), the pseudo-stopping times of Proposition 3.2 are inaccessible stopping times for the filtration $(\mathcal{F}_t^{L,\rho})$ and $(\log(\frac{1}{Z_{t \wedge \rho}^\rho}))$ is the $(\mathcal{F}_t^{L,\rho})$ dual predictable projection of $(\mathbf{1}_{\rho \leq t})$. Here again, every (\mathcal{F}_t) martingale is an $(\mathcal{F}_t^{L,\sigma(Z_\rho^\rho)})$ semimartingale whose decomposition is given by (3.1):*

$$M_t = \widetilde{M}_t + \int_\rho^{t \wedge L} \frac{d\langle M, Z^L \rangle_u}{Z_u^L - Z_u^\rho} - \int_L^t \frac{d\langle M, Z^L \rangle_u}{1 - Z_u^L}, \quad (3.2)$$

where (\widetilde{M}_t) is an $(\mathcal{F}_t^{L,\rho})$ and $(\mathcal{F}_t^{L,\sigma(Z_\rho^\rho)})$ local martingale, every continuous $(\mathcal{F}_t^{L,\rho})$ martingale being an $(\mathcal{F}_t^{L,\sigma(Z_\rho^\rho)})$ martingale.

Proof. (1). This first point follows from the fact that L is a totally inaccessible stopping time for (\mathcal{F}_t^L) (see [8]), and Proposition 3.6 can be applied with $A_T \equiv A_\infty^L$.

(2). First, we note from Proposition 3.2 that (Z_t^ρ) is a continuous and decreasing process ($Z_t^\rho = 1 - A_t^\rho$). Moreover, from a result of Jeulin and Yor ([8]), $\mathbf{1}_{(\rho \leq t)} - \int_0^{t \wedge \rho} \frac{A_u^\rho}{Z_u^\rho} = \mathbf{1}_{(\rho \leq t)} - \log\left(\frac{1}{Z_{t \wedge \rho}^\rho}\right)$ is an (\mathcal{F}_t^ρ) martingale. It remains a martingale in $(\mathcal{F}_t^{L,\rho})$, since it is of finite variation and $\rho < L$. Consequently, $(\log\left(\frac{1}{Z_{t \wedge \rho}^\rho}\right))$ is also the $(\mathcal{F}_t^{L,\rho})$ dual predictable projection of $(\mathbf{1}_{\rho \leq t})$ and the announced results now easily follow from Propositions 3.4 and 3.6. \square

3.2. An analogue of Williams' path decomposition theorem for $\mathbb{P}(\mathbf{L} \leq \mathbf{t} | \mathcal{F}_t)$. We are going to use techniques from both stochastic calculus and the general theory of stochastic processes (the Dubins-Schwarz theorem and the decomposition formula in the larger filtration) to generalize some fragments of Williams' path decomposition for the standard Brownian to more general processes, namely the submartingale $(\mathbb{P}(L \leq t | \mathcal{F}_t))$, associated with an honest time L , under the conditions **(CA)**.

Theorem 3.9. *Let*

$$X_t = N_t + A_t,$$

be a submartingale of the class (Σ) satisfying

$$\lim_{t \rightarrow \infty} X_t = 1. \tag{3.3}$$

and let $L = \sup\{t : X_t = 0\}$. Recall (Theorem 2.3) that

$$X_t = 1 - Z_t^L = \mathbf{P}(L \leq t | \mathcal{F}_t).$$

Let us also define the random time:

$$\rho = \sup\{t < L : X_t = S_L\},$$

where

$$S_t = \sup_{u \leq t} X_u.$$

Then:

- (1) *the process (X_t) , stopped at ρ is, up to the time change $(\langle N \rangle_t)$, a reflected Brownian Motion started from 0, stopped when it first hits 1.*

(2) The random time ρ is a pseudo-stopping time and

$$\mathbb{P}(\rho > t | \mathcal{F}_t) = 1 - S_t.$$

Moreover, X_ρ is uniformly distributed on $(0, 1)$, and conditionally on $X_\rho = m$, (X_t) is up to the time change $(\langle N \rangle_t)$, a reflected Brownian Motion started from 0 and stopped when it first hits m .

(3) Conditionally on \mathcal{F}_L , the process (X_{L+t}) is, up to the time change $(\langle N \rangle_{L+t} - \langle N \rangle_L)$, a Bessel process of dimension 3, started from 0, and stopped when it first hits 1.

Proof. (1). First, from Skorokhod's reflection lemma (see [20] or [13]), we have:

$$A_t = \sup_{u \leq t} (-N_u).$$

Moreover, there exists a Brownian Motion (β_u) such that:

$$N_t = \beta_{\langle N \rangle_t}.$$

Hence, up to the time change $(\langle N \rangle_t)$, (X_t) has the same decomposition as the absolute value of a Brownian Motion (this is immediate from Tanaka's formula). Thus it is a time changed reflected Brownian Motion.

(2). The first point follows immediately from Proposition 3.2: indeed, ρ is a pseudo-stopping time and X_ρ is equal to Z_ρ^ρ , which is uniformly distributed. The second point follows from a combination of Proposition 3.4 and Corollary 3.8. Indeed, in the filtration $(\mathcal{F}_t^{L, \sigma(X_\rho)})$, obtained by initially enlarging the filtration (\mathcal{F}_t^L) with $\log\left(\frac{1}{Z_{t \wedge \rho}^\rho}\right) = \log\left(\frac{1}{1 - X_\rho}\right)$, we have:

$$X_{t \wedge \rho} = N_{t \wedge \rho} + A_{t \wedge \rho}.$$

(3). We first note that, since $X_L = 0$, we have $N_L = -A_L$, and consequently,

$$X_{L+t} = N_{L+t} - N_L.$$

Now, using the fact that $X_t = 1 - Z_t^L = \mathbf{P}(L \leq t | \mathcal{F}_t)$ the decomposition formula (1.2) yields:

$$X_{L+t} = N_{L+t} - N_L = \tilde{N}_t + \int_0^t \frac{d\langle N \rangle_{L+u}}{X_{L+u}},$$

where \tilde{N} is an (\mathcal{F}_t^L) martingale. Now, the result follows from the fact that the Bessel process of dimension 3 (R_t) can be characterized as the unique solution to the stochastic differential equation:

$$dR_t = dB_t + \frac{dt}{R_t},$$

where (B_t) is a one dimensional Brownian Motion. □

As an illustration of the above theorem, let us consider

$$X_t \equiv \alpha B_t^+ + \beta B_t^-,$$

where B is the standard Brownian Motion and $\alpha > 0, \beta > 0$. Let $T_1 = \inf \{t : X_t = 1\}$. Then, it is easy to check that $(X_{t \wedge T_1})$ satisfies the assumptions of the Theorem 3.9 with the time change

$$\langle N \rangle_t = \alpha^2 \int_0^t \mathbf{1}_{(B_u > 0)} du + \beta^2 \int_0^t \mathbf{1}_{(B_u \leq 0)} du.$$

3.3. Path decompositions for some recurrent diffusions. D. Williams's path decomposition also admits a generalization to the wider class of recurrent diffusions (Y_t) , satisfying the stochastic differential equation:

$$Y_t = B_t + \int_0^t b(Y_u) du, \tag{3.4}$$

where (B_t) is the standard Brownian Motion, and b is a Borel integrable function which allows existence and uniqueness for equation (3.4). We note \mathcal{L} the infinitesimal generator of this diffusion:

$$\mathcal{L} = \frac{1}{2} \frac{d^2}{dx^2} + b(x) \frac{d}{dx}.$$

Let $T_1 \equiv \inf \{t : Y_t = 1\}$, and denote by s the scale function of Y , which is strictly increasing and which vanishes at zero, i.e:

$$s(z) = \int_0^z \exp(-2\widehat{b}(y)) dy,$$

where

$$\widehat{b}(y) = \int_0^y b(u) du.$$

From the results of paragraph (2.2.3), if we define

$$\sigma = \sup \{t < T_1 : Y_t = 0\},$$

we have, with (L_t) the local time at 0 of the local martingale $s(Y)$:

$$\begin{aligned} \mathbb{P}(\sigma \leq t | \mathcal{F}_t) &= \frac{s(Y_{t \wedge T_1})^+}{s(1)} \\ &= \frac{1}{s(1)} \int_0^{t \wedge T_1} \mathbf{1}_{(Y_u > 0)} s'(Y_u) dB_u + \frac{1}{2s(1)} L_{t \wedge T_1}, \end{aligned}$$

where the last equality is obtained by an application of Tanaka's formula. Moreover, from Proposition (3.2),

$$\rho \equiv \sup \left\{ t < \sigma : Y_t = \max_{u \leq \sigma} Y_u \right\}, \tag{3.5}$$

is a pseudo-stopping time.

Proposition 3.10. *Let (Y_t) be a diffusion process satisfying equation (3.4). Define:*

$$\bar{Y}_t = \max_{u \leq t} Y_u.$$

Then:

- The process $(Y_{\sigma+t}, t \leq T_1 - \sigma)$ is an $(\mathcal{F}_{\sigma+t})$ diffusion, starting from 0, considered up to the first time it hits 1, and is independent of \mathcal{F}_σ . Its infinitesimal generator is given by:

$$\mathcal{L} = \frac{1}{2} \frac{d^2}{dx^2} + \left(b(x) + \frac{s'(x)}{s(x)} \right) \frac{d}{dx}.$$

- The random time ρ is a pseudo-stopping time and satisfies:

$$\mathbb{P}(\rho > t | \mathcal{F}_t) = 1 - \frac{s(\bar{Y}_{t \wedge T_1})^+}{s(1)}.$$

Moreover, $Y_\rho = \bar{Y}_\sigma$ follows the same law as $s^{-1}(s(1)U)$, where U is a random variable following the uniform law on $(0, 1)$, and is independent of the whole process $(Y_{\sigma+t}, t \leq T_1 - \sigma)$.

- **Conditionally on $Y_\rho = m$,**
 - (1) the process $(Y_t; t \leq \rho)$ is a diffusion process, considered up to T_m , the first time when it hits m , with the same infinitesimal generator as Y .
 - (2) the process $(Y_{\rho+t}; t \leq \sigma - \rho)$ is a $(\mathcal{F}_{\rho+t})$ diffusion process, started from m , considered up to T_0 , the first time when it hits 0, and is independent of $(Y_t; t \leq \rho)$; its infinitesimal generator is given by:

$$\frac{1}{2} \frac{d^2}{dx^2} + \left(b(x) + \mathbf{1}_{(x>0)} \frac{s'(x)}{s(x) - s(m)} \right) \frac{d}{dx}.$$

Proof. The proof is based on enlargements arguments. First, we note that from Proposition 3.2 $s(Y_\rho)$ is distributed as $s(1)U$, where U follows the uniform law on $(0, 1)$.

Now, let us study the path of Y on $[\sigma, T_1]$. From formula (1.2), the Brownian Motion B is a semimartingale in the filtration (\mathcal{F}_t^σ) and decomposes as:

$$B_t = \tilde{B}_t + \int_0^{t \wedge \sigma} \frac{d \langle B, Z^\sigma \rangle_u}{Z_u^\sigma} + \int_\sigma^{t \wedge T_1} \frac{d \langle B, 1 - Z^\sigma \rangle_u}{1 - Z_u^\sigma},$$

where \tilde{B} is a (\mathcal{F}_t^σ) Brownian Motion (indeed it is a continuous local martingale with bracket t). Consequently, the diffusion Y , which is an (\mathcal{F}_t) semimartingale, remains a semimartingale in (\mathcal{F}_t^σ) and its decomposition is given by:

$$Y_t = \tilde{B}_t + \int_0^t b(Y_u) du - \int_0^{t \wedge \sigma} \mathbf{1}_{Y_u > 0} \frac{s'(Y_u)}{s(1) - s(Y_u)} du + \int_\sigma^{t \wedge T_1} \frac{s'(Y_u)}{s(Y_u)} du.$$

Now, considering $Y_{\sigma+t} - Y_\sigma = Y_{\sigma+t}$, for $t \leq T_1 - \sigma$, we obtain:

$$Y_{\sigma+t} = \left(\tilde{B}_{\sigma+t} - \tilde{B}_\sigma \right) + \int_\sigma^{\sigma+t} b(Y_u) du + \int_\sigma^{(\sigma+t) \wedge T_1} \frac{s'(Y_u)}{s(Y_u)} du.$$

Now, using the fact that σ is a stopping time for the filtration (\mathcal{F}_t^σ) , we have that $(\tilde{B}_{\sigma+t} - \tilde{B}_\sigma)$, which we note (\hat{B}_t) , is a Brownian Motion, which is independent of $\mathcal{F}_\sigma^\sigma \supset \mathcal{F}_\sigma$. Consequently, for $t \leq T_1 - \sigma$, we have:

$$Y_{\sigma+t} = \hat{B}_t + \int_0^t b(Y_{\sigma+u}) du + \int_0^{t \wedge (T_1 - \sigma)} \frac{s'(Y_{\sigma+u})}{s(Y_{\sigma+u})} du,$$

and the result announced for the path on $[\sigma, T_1]$ follows now easily.

Now, let us consider the path of Y on $[0, \rho]$, and $[\rho, \sigma]$. For this, we enlarge initially the filtration (\mathcal{F}_t^σ) with the variable Y_ρ : according to Proposition 3.4 and Corollary 3.8, for $t \leq \sigma$, B decomposes in $(\mathcal{F}_t^{\sigma, \sigma(Y_\rho)})$, which we note $(\mathcal{F}_t^{\sigma, Y_\rho})$ for notational convenience, as:

$$B_t = \tilde{B}_t - \int_\rho^{t \wedge \sigma} \mathbf{1}_{(Y_u > 0)} \frac{s'(Y_u)}{s(Y_\rho) - s(Y_u)} du,$$

where \tilde{B} is an $(\mathcal{F}_t^{\sigma, Y_\rho})$ Brownian Motion which is independent of Y_ρ . Consequently, for $t \leq \rho$, Y decomposes in $(\mathcal{F}_t^{\sigma, Y_\rho})$ as:

$$Y_t = \tilde{B}_t + \int_0^t b(Y_u) du, \text{ for } t \leq \rho, \quad (3.6)$$

and for $t \leq (\sigma - \rho)$, we have:

$$Y_{\rho+t} = Y_\rho + \left(\tilde{B}_{\rho+t} - \tilde{B}_\rho \right) + \int_\rho^{t \wedge (\sigma - \rho)} b(Y_u) du - \int_\rho^{t \wedge (\sigma - \rho)} \frac{s'(Y_u)}{s(Y_\rho) - s(Y_u)} du. \quad (3.7)$$

Now,

$$\hat{B}_t \equiv \tilde{B}_{\rho+t} - \tilde{B}_\rho$$

is again a standard Brownian Motion, independent of Y_ρ , and hence, conditionally on $Y_\rho = m$, the process $(Y_{\rho+t}; t \leq \sigma - \rho)$ satisfies:

$$Y_{\rho+t} = m + \hat{B}_t + \int_0^{t \wedge (\sigma - \rho)} b(Y_{\rho+u}) du + \int_0^{t \wedge (\sigma - \rho)} \frac{s'(Y_{\rho+u})}{s(Y_{\rho+u}) - s(m)} du.$$

The statement of the Proposition now follows from the last equality and equation (3.6). \square

Remark 3.11. When $b \equiv 0$, we have $s(x) = x$, and we recover D. Williams' path decomposition for the standard Brownian Motion.

3.4. Path decompositions for some transient diffusions. Now, we consider a special subfamily of the transient diffusions of paragraph 2.2.5 which play an important role in the extension of Pitman's theorem (see [26], p.46). More precisely, let (R_t) be any transient diffusion which takes its values in $(0, \infty)$, and satisfies:

$$R_t = x + B_t + \int_0^t c(R_u) du, \quad x > 0, t \geq 0, \quad (3.8)$$

where $c : \mathbb{R}_+ \rightarrow \mathbb{R}$ allows uniqueness in law for this equation. Noting $T_0 = \inf \{t : R_t = 0\}$, we assume that $\mathbb{P}_x(T_0 < \infty) = 0$, so that a scale function s of R may be chosen to satisfy:

$$s(0) = -\infty; s(\infty) = 0; \frac{1}{2}s'' + cs' = 0.$$

We keep the notation of paragraph 2.2.5: for $0 \leq x \leq y$, and

$$g_y = \sup \{t : R_t = y\},$$

we have:

$$\begin{aligned} \mathbb{P}_x(g_y > t | \mathcal{F}_t) &= \frac{s(R_t)}{s(y)} \wedge 1 \\ &= 1 + \frac{1}{s(y)} \int_0^t \mathbf{1}_{R_u > y} s'(R_u) dB_u + \frac{1}{2s(y)} L_t^{s(y)}, \end{aligned}$$

where $L_t^{s(y)}$ is the local time of $s(R)$ at $s(y)$.

From Proposition 3.2, the random time:

$$\rho = \sup \left\{ t < g_y : R_t = \sup_{u \leq g_y} R_u \right\},$$

is a pseudo-stopping time and $\mathbb{P}(\rho > t | \mathcal{F}_t) = \frac{s(\sup_{u \leq t} R_u)}{s(y)} \wedge 1$. Now, likewise Proposition 3.10, the following path decomposition holds for the diffusion R :

Proposition 3.12. *Let (R_t) be a diffusion process satisfying equation (3.8). Then:*

- *The process $(R_{g_y+t}, t \geq 0)$ is an (\mathcal{F}_{g_y+t}) diffusion, starting from y , and is independent of \mathcal{F}_{g_y} . Its infinitesimal generator is given by:*

$$\mathcal{L} = \frac{1}{2} \frac{d^2}{dx^2} + \left(c(x) + \frac{s'(x)}{s(x) - s(y)} \right) \frac{d}{dx}.$$

- *The random time ρ is a pseudo-stopping time and $R_\rho = \bar{R}_{g_y}$ follows the same law as $s^{-1}(s(y)U)$, where U follows the uniform law on $(0, 1)$, and is independent of the whole process $(R_{g_y+t}, t \geq 0)$.*
- **Conditionally on $R_\rho = m$,**

- (1) the process $(R_t; t \leq \rho)$ is a diffusion process, considered up to T_m , the first time when it hits m , with the same infinitesimal generator as R .
- (2) the process $(R_{\rho+t}; t \leq g_y - \rho)$ is a $(\mathcal{F}_{\rho+t})$ diffusion process, started from m , considered up to T_y , the first time when it hits y , and is independent of $(R_t; t \leq \rho)$; its infinitesimal generator is given by:

$$\frac{1}{2} \frac{d^2}{dx^2} + \left(c(x) + \mathbf{1}_{(x>y)} \frac{s'(x)}{s(x) - s(m)} \right) \frac{d}{dx}.$$

Proof. The proof follows exactly the same lines as the proof of Proposition 3.10 and so we will not give it. \square

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