THE CHARACTERISTIC POLYNOMIAL ON COMPACT GROUPS WITH HAAR MEASURE : SOME EQUALITIES IN LAW

P. BOURGADE, A. NIKEGHBALI, AND A. ROUAULT

Presented by Marc Yor.

Abstract. This note presents some equalities in law for $Z_N := \det(\mathrm{Id}-G)$, where G is an element of a subgroup of the set of unitary matrices of size N, endowed with its unique probability Haar measure. Indeed, under some general conditions, Z_N can be decomposed as a product of independent random variables, whose laws are explicitly known. Our results can be obtained in two ways : either by a recursive decomposition of the Haar measure (Section 1) or by previous results by Killip and Nenciu ([3]) on orthogonal polynomials with respect to some measure on the unit circle (Section 2). This latter method leads naturally to a study of determinants of a class of principal submatrices.

Résumé. Cette note présente quelques égalités en loi pour $Z_N := \det(\mathrm{Id}-G)$, où G est un sous-groupe de l'ensemble des matrices unitaires de taille N, muni de son unique mesure de Haar normalisée. En effet, sous des conditions assez générales, Z_N peut être décomposé comme le produit de variables aléatoires indépendantes, dont on connait la loi explicitement. Notre résultat peut être obtenu de deux manières : soit par une décomposition récursive de la mesure de Haar (Partie 1) soit en utilisant un résultat de Killip et Nenciu ([3]) à propos des polynômes orthogonaux relativement à une certaine mesure sur le cercle unité (Partie 2). Cette dernière méthode nous conduit naturellement à l'étude des déterminants de certaines sous-matrices.

In this note, $\langle a, b\rangle$ denotes the Hermitian product of two elements a and b in $\mathbb{C}^N$ (the dimension is implicit).

1. A recursive decomposition, consequences

1.1. **The general equality in law.** Let $\mathcal{G}$ be a subgroup of $U(N)$, the group of unitary matrices of size N. Let $(e_1, \dots, e_N)$ be an orthonormal basis of $\mathbb{C}^N$ and $\mathcal{H} := \{H \in \mathcal{G} \mid H(e_1) = e_1\}$, the subgroup of $\mathcal{G}$ which

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stabilizes e_1. For a generic compact group $\mathcal{A}$, we write $\mu_{\mathcal{A}}$ for the unique Haar probability measure on $\mathcal{A}$. Then we have the following Theorem.

Theorem 1.1. *Let M and H be independent matrices, $M \in \mathcal{G}$ and $H \in \mathcal{H}$ with distribution $\mu_{\mathcal{H}}$. Then $MH \sim \mu_{\mathcal{G}}$ if and only if $M(e_1) \sim f(\mu_{\mathcal{G}})$, where f is the map $f : G \mapsto G(e_1)$.*

Let $\mathcal{M}$ be the set of elements of $\mathcal{G}$ which are reflections with respect to a hyperplane of $\mathbb{C}^N$. Define also

$$g : \left\{ \begin{array}{ccc} \mathcal{H} & \to & U(N-1) \\ H & \mapsto & H_{\mathrm{span}(e_2,\dots,e_N)} \end{array} \right. ,$$

where $H_{\mathrm{span}(e_2,\dots,e_N)}$ is the restriction of H to $\mathrm{span}(e_2,\dots,e_N)$. Now suppose that $\{G(e) \mid G \in \mathcal{G}\} = \{M(e) \mid M \in \mathcal{M}\}$. Under this additional condition the following Theorem can be proven, using Theorem 1.1 and elementary manipulations of determinants.

Theorem 1.2. *Let $G \sim \mu_{\mathcal{G}}$, $G' \sim \mu_{\mathcal{G}}$ and $H \sim g(\mu_{\mathcal{H}})$ be independent. Then*

$$\det(\mathrm{Id}_N - G) \overset{\mathrm{law}}{=} (1 - \langle e_1, G'(e_1)\rangle) \det(\mathrm{Id}_{N-1} - H).$$

1.2. **Examples : the unitary group, the group of permutations.** Take $G = U(N)$. As all reflections with respect to a hyperplane of $\mathbb{C}^N$ are elements of G, one can apply Theorem 1.2. The corresponding measures are the following.

(1) The distribution $g(\mu_{\mathcal{H}})$ is clearly $\mu_{U(N-1)}$.
(2) $\langle e_1, G(e_1)\rangle$ is distributed as the first coordinate of a vector of the $N-$dimensional unit complex sphere with uniform measure : $\langle e_1, G(e_1)\rangle \sim e^{i\theta_N}\sqrt{\beta_{1,N-1}}$ with θ_n uniform on $(0, 2\pi)$ and independent of $\beta_{1,N-1}$, a beta variable with parameters 1 and $N-1$.

Thus iterations of Theorem 1.2 lead to the following Corollary.

Corollary 1.3. *([2]) Let $G \in U(N)$ be $\mu_{U(N)}$ distributed. Then*

$$\det(\mathrm{Id}_N - G) \overset{\mathrm{law}}{=} \prod_{k=1}^{N} \left(1 - e^{\mathrm{i}\theta_k}\sqrt{\beta_{1,k-1}}\right),$$

with $\theta_1, \dots, \theta_N, \beta_{1,0}, \dots, \beta_{1,N-1}$ independent random variables, the θ_k's uniformly distributed on $(0, 2\pi)$ and the $\beta_{1,j}$'s ($0 \le j \le N-1$) being beta distributed with parameters 1 and j (by convention, $\beta_{1,0}$ is the Dirac distribution at 1).

The group $\mathcal{S}_N$ of permutations of size N gives another possible application. Identify an element $\sigma \in \mathcal{S}_N$ with the matrix $(\delta^j_{\sigma(i)})_{1\le i,j\le N}$ (δ is Kronecker's symbol). As $\det(\mathrm{Id}_N - \sigma)$ is equal to 0, we prefer to deal with the group $\tilde{\mathcal{S}}_N$ of matrices $(e^{\mathrm{i}\theta_j}\delta^j_{\sigma(i)})_{1\le i,j\le N}$, with $\sigma \in \mathcal{S}_N$ and $\theta_1, \dots, \theta_N$ independent uniform random variables on $(0, 2\pi)$. Then the measures corresponding to Theorem 1.2 are the following.

(1) The distribution $g(\mu_{\tilde{\mathcal{S}}_N})$ is $\mu_{\tilde{\mathcal{S}}_{N-1}}$.
(2) $\langle e_1, G(e_1)\rangle$ is 0 with probability $1-1/N$ and $e^{\mathrm{i}\theta}$ (θ uniform on $(0,2\pi)$) with probability $1/N$.

As previously, iterations of Theorem 1.2 give the following result.

Corollary 1.4. *Let $S_N \in \tilde{\mathcal{S}}_N$ be $\mu_{\tilde{\mathcal{S}}_N}$ distributed. Then*

$$\det(\mathrm{Id}_N - S_N) \overset{\text{law}}{=} \prod_{k=1}^{N} \left(1 - e^{\mathrm{i}\theta_k} X_k\right),$$

with $\theta_1, \dots, \theta_N, X_1, \dots, X_N$ independent random variables, the θ_k's uniformly distributed on $(0,2\pi)$ and the X_k's Bernoulli variables : $\mathbb{P}(X_k = 1) = 1/k$, $\mathbb{P}(X_k = 0) = 1 - 1/k$.

Remark. Let k_σ be the number of cycles of a random permutation of size N, with respect to the (probability) Haar measure. Corollary 1.4 allows us to recover the following celebrated result about the law of k_σ :

$$k_\sigma \overset{\text{law}}{=} X_1 + \cdots + X_N,$$

with the previous notations. Indeed, if a permutation $\sigma \in \mathcal{S}_N$ has k_σ cycles with lengths $l_1, \dots, l_{k_\sigma}$ ($\sum_k l_k = N$), then it is easy to see that under the Haar measure

$$\det(x\mathrm{Id} - \tilde{\mathcal{S}}_N) \overset{\text{law}}{=} \prod_{k=1}^{k_\sigma} (x^{l_k} - e^{\mathrm{i}\alpha_k})$$

with the α_k's independent and uniform on $(0,2\pi)$. Using the previous relation and the result of Corollary 1.4 we get

$$\prod_{k=1}^{N} \left(1 - e^{\mathrm{i}\theta_k} X_k\right) \overset{\text{law}}{=} \prod_{k=1}^{k_\sigma} (1 - e^{\mathrm{i}\alpha_k}).$$

The equality of the Mellin transforms of the modulus of the above members easily implies the expected result : $k_\sigma \overset{\text{law}}{=} X_1 + \cdots + X_N$. Our discussion on the permutation group is closely related to the so-called Chinese restaurant process and the Feller decomposition of the symmetric group (see, e.g. [1]).

2. Characteristic polynomials as orthogonal polynomials

We now show how Corollary 1.3 can be obtained as a consequence of a result by Killip and Nenciu ([3]).

2.1. A result by Killip and Nenciu.

Let $\mathbb{D}$ be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\partial\mathbb{D}$ the unit circle. Let $(e_1, \dots, e_N)$ be the canonical basis of $\mathbb{C}^N$. If $G \in U(N)$, and if e_1 is cyclic for G, the spectral measure for the pair (G, e_1) is the unique probability ν on $\partial\mathbb{D}$ such that, for every integer $k \geq 0$

$$\langle e_1, G^k e_1\rangle = \int_{\partial\mathbb{D}} z^k d\nu(z). \tag{2.1}$$

In fact, we have the expression

$$\nu = \sum_{j=1}^{N} \pi_j \delta_{e^{i\zeta_j}}$$

where $(e^{i\zeta_j}, j = 1, \dots N)$ are the eigenvalues of G and where $\pi_j = |\langle e_1, \Pi e_j\rangle|^2$ with Π a unitary matrix diagonalizing G.

The relation (2.1) allows to define an isometry from $\mathbb{C}^N$ equipped with the basis $(e_1, Ge_1, \cdots, G^{N-1}e_1)$ into the subspace of $L^2(\partial\mathbb{D}; d\nu)$ spanned by the family $(1, z, \dots, z^N)$. The endomorphism G is then a representation of the multiplication by z.

From the linearly independent family of monomials $\{1, z, z^2, \dots, z^{N-1}\}$ in $L^2(\partial\mathbb{D}, \nu)$, we construct an orthogonal basis $\Phi_0, \dots, \Phi_{N-1}$ of monic polynomials by the Gram-Schmidt procedure. The N^{th} degree polynomial obtained this way is

$$\Phi_N(z) = \prod_{j=1}^{N} (z - e^{i\zeta_j}),$$

i.e. the characteristic polynomial of G. The Φ_k's $(k = 0, \dots, N)$ obey the Szegö recursion relation:

$$\Phi_{j+1}(z) = z\Phi_j(z) - \bar{\alpha}_j \Phi_j^*(z) \tag{2.2}$$

where $\Phi_j^*(z) = z^j \, \overline{\Phi_j(\bar{z}^{-1})}$. The coefficients $\alpha_j' s$ $(j \geq 0)$ are called Schur or Verblunsky coefficients and satisfy the condition $\alpha_0, \cdots, \alpha_{N-2} \in \mathbb{D}$ and $\alpha_{N-1} \in \partial\mathbb{D}$. There is a bijection between this set of coefficients and the set of spectral probability measures ν (Verblunsky's theorem). If $G \sim \mu_{U(N)}$, then we know the exact distribution of the Verblunsky coefficients :

Theorem 2.1. *(Killip and Nenciu* [3]*) Let $G \in U(N)$ be $\mu_{U(N)}$ distributed. The Verblunsky parameters $\alpha_0, \cdots, \alpha_{N-2}, \alpha_{N-1}$ are independent and the density of α_j for $j \leq N-1$ is*

$$\frac{N-j-1}{\pi} \left(1 - |z|^2\right)^{N-j-2} \mathbb{1}_{\mathbb{D}}(z)$$

(for $j = N-1$ by convention this is the uniform measure on the unit circle).

2.2. **Recovering Corollary 1.3.** For $z = 1$, Szegö's recursion (2.2) can be written

$$\Phi_{j+1}(1) = \Phi_j(1) - \overline{\alpha_j}\, \overline{\Phi_j(1)}. \tag{2.3}$$

Under the Haar measure for G, as α_j is independent of $\Phi_j(1)$ and its distribution is invariant by rotation, (2.3) easily yields

$$\Phi_{j+1}(1) \overset{\text{law}}{=} (1 - \alpha_j)\Phi_j(1).$$

In particular, for $j = N-1$ we get by induction

$$\det(\mathrm{Id} - G) \overset{\text{law}}{=} \prod_{k=0}^{N-1} (1 - \alpha_j). \tag{2.4}$$

From the density for α_j given in Theorem 2.1 one can see that this is exactly the same result as Corollary 1.3.

Remark. A similar result holds for $SO(2N)$, and can be shown using either the method of Section 1 or the one in Section 2, with the corresponding result by Killip and Nenciu for the Verblunsky coefficients on the orthogonal group [3].

2.3. **Extension.** We now consider the whole sequence of polynomials $\Phi_j, j \leq N$ for $j \leq N$ as a sequence of characteristic polynomials. For this purpose, we apply the Gram-Schmidt procedure to $1, z, z^{-1}, z^2, \dots, z^{p-1}, z^{1-p}, z^p$ if $N = 2p$ and to $1, z, z^{-1}, z^2, \dots, z^p, z^{-p}$ if $N = 2p+1$ in $L^2(\partial\mathbb{D}); d\nu)$. In the resulting basis, the mapping $f(z) \mapsto zf(z)$ is represented by a so-called CMV matrix ([3] Appendix B, [5]) denoted by $\mathcal{C}_N(G)$. It is five-diagonal and conjugate to G. For $1 \leq j \leq N$ let $\mathcal{C}_N^{(j)}(G)$ the principal submatrix of order j of $\mathcal{C}_N(G)$. It is known (see for instance Proposition 3.1 in [5]) that

$$\Phi_j(z) = \det\left(z\mathrm{Id}_j - \mathcal{C}_N^{(j)}(G)\right).$$

From the recursion (2.3) and looking at the invariance of conditional distributions, we see that

$$\left(\det\left(\mathrm{Id}_j - \mathcal{C}_N^{(j)}(G)\right)\right)_{1\leq j\leq N} = (\Phi_j(1))_{1\leq j\leq N} \overset{\text{law}}{=} \left(\prod_{l=0}^{j}(1-\alpha_l)\right)_{0\leq j\leq N-1}. \tag{2.5}$$

It allows a study of the process $\left(\log\Phi_{\lfloor Nt\rfloor}(1)\ ,\ t\in[0,1]\right)$ as a triangular array of (complex) independent random variables. For $t = 1$ the asymptotic behavior is presented in [2] (see (2.7 below). It is remarkable that for $t < 1$, we do not need any normalization for the CLT.

Theorem 2.2. (1) *As* $N \to \infty$

$$\left(\log\det\left(\mathrm{Id}_j - \mathcal{C}_{\lfloor Nt\rfloor}^{(j)}(G)\right);\ t\in[0,1)\right) \Rightarrow \left(\mathbf{B}_{-\frac{1}{2}\log(1-t)};\ t\in[0,1)\right), \tag{2.6}$$

where $\mathbf{B}$ *is a standard complex Brownian motion and* $\Rightarrow$ *stands for the weak convergence of distributions in the set of càdlàg functions on* $[0,1)$, *starting from* 0, *endowed with the Skorokhod topology.*

(2) *As* $N \to \infty$,

$$\frac{\log\det(\mathrm{Id}_N - G)}{\sqrt{2\log N}} \Rightarrow \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2 \tag{2.7}$$

where $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are independent standard normal and independent of* $\mathbf{B}$, *and* $\Rightarrow$ *stands for the weak convergence of distributions in* $\mathbb{C}$.

This theorem can be proved using the Mellin-Fourier transform of the $1-\alpha_j$'s and independence. This method may also be used to prove large deviations. It is the topic of a companion paper. These results occur in similar way for other random determinants (see [4]).

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Laboratoire de Probabilités et Modéles Aléatoires, Université Pierre et Marie Curie, et C.N.R.S. UMR 7599, 175, rue du Chevaleret, F-75013 Paris, France

E-mail address: bourgade@enst.fr

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

E-mail address: ashkan.nikeghbali@math.unizh.ch

Université Versailles-Saint Quentin, LMV, Bâtiment Fermat, 45 avenue des Etats-Unis, 78035 Versailles Cedex

E-mail address: rouault@fermat.math.uvsq.fr