A READING GUIDE FOR LAST PASSAGE TIMES WITH FINANCIAL APPLICATIONS IN VIEW

ASHKAN NIKEGHBALI AND ECKHARD PLATEN

ABSTRACT. In this survey on last passage times we propose a new viewpoint which provides a unified approach to many different results which appear in the mathematical finance literature and in the theory of stochastic processes. In particular we are able to improve the assumptions under which some well known results are usually stated. Moreover we give some new and detailed calculations for the computation of the distribution of some large classes of last passage times. We have kept in this survey only the aspects of the theory which we expect potentially to be relevant for financial applications.

NOTATION

In this paper, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ will denote a filtered probability space. $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ is the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ is the space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}$. If Y is a random variable, we denote indifferently by $\mathbb{P}[Y]$ or by $\mathbb{E}_{\mathbb{P}}[Y]$ the expectation of X with respect to $\mathbb{P}$. If $(A_t)_{t\geq 0}$ is an increasing process, as usual, the increasing limit of A_t, when $t \to \infty$, is denoted A_∞.

If $(X_t)_{t\geq 0}$ is a stochastic process, then $\bar{X}_t$ denotes the running maximum $\sup_{u\leq t} X_u$. We also recall that a stochastic process $(X_t)_{t\geq 0}$ is said to be of class (D) if the family of random variables $\{|X_T| \mathbb{1}_{T<\infty} : T \text{ a stopping time } \}$ is uniformly integrable.

We say that a nonnegative local martingale $(M_t)_{t\geq 0}$ belongs to $\mathcal{M}_0$ if it satisfies the following: $(\bar{M}_t)_{t\geq 0}$ is continuous and $M_0 = 1$, $\lim_{t\to\infty} M_t = 0$.

We say that g is the end of a predictable or an optional set if

$$g = \sup \{t : \ (t, \omega) \in \Gamma\},$$

where Γ is a predictable or an optional set.

1. INTRODUCTION

Ends of optional sets, most commonly called "last passage times", are random times which are not stopping times. For, instance if $(W_t)_{t\geq 0}$ is a standard Brownian motion, then

$$g = \sup\{t \leq 1 : W_t = 0\}$$

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is such a time.The distribution of this time was already studied by Paul Lévy in [19]. Another interesting example, which will play an important role in our discussion, can be constructed as follows: let $(M_t)_{t\geq 0}$ be a continuous nonnegative local martingale, with $M_0 = 1$ and $\lim_{t\to\infty} M_t = 0$. Then

$$g = \sup\{t \geq 0 : M_t = \bar{M}_t\}, \tag{1.1}$$

is another such time. As illustrated with the above two examples, last passage times look into the future and that feature makes their analysis more delicate. The standard theorems from martingale theory such as Doob's optional stopping theorem do not apply to them. They have nevertheless received some attention in stochastic analysis (see e.g.the papers by Chung [8] and Azéma [1]). They play an important role in the theory of enlargements of filtrations (see e.g. the works by Barlow [4], Jeulin [16], Yor [40], Nikeghbali and Yor [30] or the monograph [11] for a survey), in the characterizations of strong Brownian filtrations (see the monograph by Mansuy and Yor [23] for details and references) and in path decompositions of diffusions (e.g. [16] and [28]). One should mention that last passage times have also inspired some fear because the standard martingale techniques do not apply to them, as noticed by Kai Lai Chung ([8]):

> For some reason the notion of a last exit time, which is manifestly involved in the arguments, would not be dealt with openly and directly. This may be partially due to the fact that such a time is not an "optional" (or "stopping") time, does not belong to the standard equipment, and so must be evaded at all costs.

Last passage times have also played an increasing role in financial modeling in recent years, thus outlining the needs for a framework which would allow a systematic study of them. They appear for instance in the seminal paper of Elliott, Jeanblanc and Yor [13] on models of default risk (see also the recent book [15] for more references towards this direction) or in the work by Imkeller [14] on insider trading. In these works, examples of last passage times such as the last time some transient diffusion hits some fixed level or the last time the standard Brownian motion hits zero before some fixed time are considered; they do not contain statements which would hold for any such time. On the other hand, in a series of very recent papers (which have grown into the book [37]), Madan, Roynette and Yor ([21],[22]), have discovered that the price of European put and call options for asset prices which can be modeled by nonnegative and continuous martingales that vanish at infinity, can be expressed in terms of the probability distributions of some last passage times. Their formulae are very general and exhibit the striking feature of being model independent. More precisely, let $(M_t)_{t\geq 0}$ be a continuous and nonnegative local martingale, with $M_0 = 1$ and $\lim_{t\to\infty} M_t = 0$. Then Madan, Roynette and Yor prove that the price of a

European put option for a risky asset modeled by (M_t), with strike $K \geq 0$, is given by

$$\mathbb{E}[(K - M_t)_+] = K\,\mathbb{P}(g_K \leq t) \tag{1.2}$$

where

$$g_K = \sup\{s : M_s = K\}. \tag{1.3}$$

The above representation holds for instance if the underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies the usual assumptions, which are standard in the mathematical finance literature. A formula similar to (1.2) holds as well for call options but one needs to make some extra assumptions ([37], chapter 2 p. 21 and p.24-25):

- the martingale $(M_t)_{t\geq 0}$ must be a true martingale, i.e. $\mathbb{E}[M_t] = 1$ for all $t \geq 0$;
- the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies $\mathcal{F}_t = \sigma(M_s, s \leq t)$ and $\mathcal{F} = \mathcal{F}_\infty = \sigma(M_s, s \geq 0)$.

If the above conditions hold, then it is shown in [37] that the price of the European call option associated to $(M_t)_{t\geq 0}$ and $K \geq 0$ is given by

$$\mathbb{E}[(M_t - K)_+] = \mathbb{P}^{(M)}(g_K \leq t), \tag{1.4}$$

where g_K is defined by (1.3) and $\mathbb{P}^{(M)}$ is the unique probability measure on $(\Omega, \mathcal{F}_\infty)$ satisfying

$$\mathbb{P}^{(M)}_{|\mathcal{F}_t} = M_t . \mathbb{P}_{|\mathcal{F}_t} . \tag{1.5}$$

In fact, formula (1.4) suggests a natural approach to understand last passage times, but it also reveals all the difficulties that can be attached to such times. Indeed in the framework suggested in [37] and recalled above, formula (1.4) is not rigorously correct. There are two issues that can fortunately be fixed:

- The measure $\mathbb{P}^{(M)}$ does not always exist; one has to make extra assumptions of topological nature on the filtration $(\mathcal{F}_t)_{t\geq 0}$. These conditions were introduced by Parthasarathy in his book [31] and we shall call them (P). We have included them in the Appendix in order to concentrate exclusively on last passage times in the main body of the paper. An important fact for now is that the canonical path spaces $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ and $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$, endowed with the filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$ and $\mathcal{F}_\infty = \sigma(X_s, s \geq 0)$, where $(X_t)_{t\geq 0}$ is the coordinate process satisfy condition (P) and consequently the measure $\mathbb{P}^{(M)}$ exists.
- The other problem is caused by the fact that if one takes the usual augmentation of the Wiener space with respect to the Wiener measure, then the measure $\mathbb{P}^{(M)}$ does not exist again (see the introduction of [24]). This is due to the fact that since $\lim_{t\to\infty} M_t = 0$, the measure $\mathbb{P}^{(M)}$ is locally absolutely continuous with respect to the Wiener measure but is globally singular with respect to it, and putting all the negligible sets in $\mathcal{F}_0$ prevents such a measure

from existing. On the other hand, without the usual assumptions, many results from the theory of stochastic processes do not hold or hold only almost surely at best (e.g. the existence of an adapted, right continuous with left limit version for martingales, the Doob-Meyer decomposition, the different projections, stochastic integrals, etc.). For instance, it is shown in [24] that without the usual augmentation on the Wiener space, there does not exist an adapted and continuous version of the local time which is defined everywhere. More generally it is well known that one needs some sort of augmentation to have well defined versions of continuous and increasing processes which are finite for finite times and which are adapted. To overcome this difficulty which can become very annoying, it is proposed in [24] to consider a new kind of augmentation of filtrations, called the *natural augmentation* (in fact this augmentation has already been discovered earlier by Bichteler in [5]). We have also stated a few definitions and results related to this augmentation in the appendix. It is enough for our purpose to note that a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies the natural conditions if the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and for all $t \geq 0$ and for every $\mathbb{P}$-negligible set $A \in \mathcal{F}_t$, all the subsets of A are contained in $\mathcal{F}_0$. This definition excludes events which are negligible in $\mathcal{F}_\infty$ but not negligible for any $\mathcal{F}_t$ (for instance $g = \sup\{t : W_t = 0\}$ where W is the standard Brownian motion). Moreover most important results of the theory of stochastic processes which are proved under the usual augmentation also hold under the natural augmentation. The remarkable feature of this augmentation is that it combines very well with changes of probability measures which are only locally absolutely continuous. More precisely a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is said to satisfy the property (NP) if and only if it is the natural enlargement of a filtered probability space $(\Omega, \mathcal{F}^0, (\mathcal{F}^0_t)_{t\geq 0}, \mathbb{P}^0)$ such that the filtered measurable space $(\Omega, \mathcal{F}^0, (\mathcal{F}^0_t)_{t\geq 0})$ enjoys property (P). The measure $\mathbb{P}^{(M)}$ always exists under the conditions (NP) which are hence the optimal conditions to establish formula (1.4).

Both (1.2) and (1.4) suggest that the processes $(K - M_t)_+$ and $(M_t - K)_+$ are uniquely characterized by a pair $(\mathcal{Q}, g)$ where $\mathcal{Q}$ is a measure and g is the last zero of the process. In the first case $\mathcal{Q} = K.\mathbb{P}$ which is equivalent to $\mathbb{P}$ and in the second case $\mathcal{Q} = \mathbb{P}^{(M)}$ is singular with respect to $\mathbb{P}$. We shall present an approach to last passage times based on this remark: first introduce the class of processes $(X_t)_{t\geq 0}$ which are uniquely characterized by a pair $(\mathcal{Q}, g)$ where $\mathcal{Q}$ is a sigma-finite measure and where g is the last zero of X. We shall then see how with this approach one recovers at once formulae (1.2) and (1.4) and other formulas that appear in [37]. We also give a few extra formulae that can be interesting for applications; additionally we fix a few small inaccuracies which appear in the literature by providing the correct assumptions on the underlying filtered probability space. This approach seems to be the most natural one to understand the role of last passage times. Our paper is organized as follows:

In Section 2, we introduce a natural class of stochastic processes (which may have jumps), called of class (Σ), to study last last passage times. We show that they are uniquely characterized by a pair $(\mathcal{Q}, g)$ where $\mathcal{Q}$ is a measure and g is the last zero of the process. This result which solves a conjecture of Madan, Roynette and Yor ([21]), and which unifies the framework of the generalized Black-Scholes formulae and some problems of penalization of the Wiener measure by Najnudel, Roynette and Yor ([26]), was obtained by Najnudel and Nikeghbali in [25]. We also state and prove a special case of this result which was obtained by Cheridito, Nikeghbali and Platen in [7]. It is very useful in applications, and strong enough to derive almost all our formulas (except those involving the European call option).

In Section 3 we show how to recover naturally from our approach the multiplicative characterization of last passage times obtained by Nikeghbali and Yor in [30] which underlies some of the important results by Madan, Roynette and Yor or presented in the monograph [37]. This derivation is not so surprising but is new. In particular, we shall see that any end of a predictable set that avoids stopping times can be written as the last time when a nonnegative local martingale whose supremum process is continuous, starting at one and vanishing at infinity, reaches its maximum. Moreover there exist formulas for the conditional distribution of this random time that looks into the future and for the conditional distribution of the global maximum over the whole trajectory. Very remarkably, these formulas are universal and do not depend on any Markov assumption. We believe that these results are of their own interest. Indeed nonnegative local martingales with no positive jumps (hence the supremum process is continuous) which vanish at infinity occur in different situations in financial modeling: they often model stock prices under a risk neutral probability measure or they can model benchmarked derivative prices (the growth optimal portfolio is used as a numeraire, see [36] for more details and references). Consequently, any information (such as the conditional distribution) on the time when such a process is at its highest or on the value of the global maximum can be valuable.

In Section 4, following Profeta, Roynette and Yor ([37]), we give some examples of explicit computations for the distribution of last passage times of the form g_K in (1.3). We also give in this section some new formulas for the distribution of last passage times, together with some examples, based on the multiplicative approach of Nikeghbali and Yor in [30].

2. Last passage times and processes of the class Σ

2.1. **A remarkable measure.** The relevant class of stochastic processes consists of submartingales called of class (Σ); they were first introduced by Yor in [41] and some of its main properties were further studied by Nikeghbali in [27]. Let us recall its definition.

Definition 2.1 (([27, 41])). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. A nonnegative submartingale (resp. local submartingale) $(X_t)_{t\geq 0}$ is of class (Σ), iff it can be decomposed as $X_t = N_t + A_t$, where $(N_t)_{t\geq 0}$ and $(A_t)_{t\geq 0}$ are $(\mathcal{F}_t)_{t\geq 0}$-adapted processes satisfying the following assumptions:

- $(N_t)_{t\geq 0}$ is a càdlàg martingale (resp. local martingale);
- $(A_t)_{t\geq 0}$ is a continuous increasing process, with $A_0 = 0$;
- The measure (dA_t) is carried by the set $\{t \geq 0, X_t = 0\}$.

We shall say that $(X_t)_{t\geq 0}$ is of class (ΣD) if X is of class (Σ) and of class (D).

One notes that a process of class (Σ) is "almost" a martingale: outside the zeros of X, the process A does not increase. In fact many processes one often encounters fall into this class, e.g. $X_t = |M_t|$, where (M_t) is a continuous martingale; $X_t = (M_t - K)_+$ when (M_t) is a càdlàg local martingale with only positive jumps and $K \in \mathbb{R}$ is a constant, $X_t = \bar{M}_t - M_t$ where M is a local martingale with only negative jumps. Other remarkable families of examples consist of a large class of recurrent diffusions on natural scale (such as some powers of Bessel processes of dimension $\delta \in (0, 2)$, see [25]) or of a function of a symmetric Lévy process (in these cases, A_t is the local time of the diffusion process or of the Lévy process), or the age process of the standard Brownian motion W_t in the filtration of the zeros of the Brownian motion, namely $\sqrt{t - g_t}$, where $g_t = \sup\{u \leq t : W_u = 0\}$ (for more example see [27]).

Before giving our characterization results, we state a simple but useful lemma:

Lemma 2.2 ([27]). *Let $(X_t)_{t\geq 0}$ be a process of class (Σ) and let f be a locally bounded and nonnegative Borel function. Define $F(x) = \int_0^x f(y)\mathrm{d}y$. Then $f(A_t)X_t$ is again of class (Σ) and decomposes as*

$$f(A_t)X_t = \int_0^t f(A_u)\mathrm{d}N_u + F(A_t). \tag{2.1}$$

Proof. If f is $\mathcal{C}^1$, then an integration by parts yields:

$$\begin{aligned} f(A_t)\,X_t &= \int_0^t f(A_u)\,dX_u + \int_0^t f'(A_u)\,X_u dA_u \\ &= \int_0^t f(A_u)\,dN_u + \int_0^t f(A_u)\,dA_u + \int_0^t f'(A_u)\,X_u dA_u. \end{aligned}$$

Since (dA_t) is carried by the set $\{t: X_t = 0\}$, we have $\int_0^t f'(A_u) X_u dA_u = 0$. As $\int_0^t f(A_u)\, dA_u = F(A_t)$, we have thus obtained that:

$$F(A_t) - f(A_t) X_t = -\int_0^t f(A_u)\, dN_u. \tag{2.2}$$

The general case follows from a density and monotone class arguments. □

We now show that any process of class (Σ) is uniquely characterized by its last zero and a remarkable measure. We start with an elementary case that we can shortly prove in this survey and which is very useful in applications. This result is in fact an extension of a result by Azéma and Yor [3] and Azéma, Meyer and Yor [2] (Indeed, in the case when X is of class (ΣD), one could deduce it from part 1 of Theorem 8.1 in [2]).

Theorem 2.3 ([7]). *Assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies the usual assumptions or the natural conditions. Let $(X_t)_{t\geq 0}$ be a process of class (Σ) such that $\lim_{t\to\infty} X_t = X_\infty$ exists a.s. and is finite (in particular N_∞ and A_∞ exist and are a.s. finite). Let*

$$g := \sup\{t : X_t = 0\} \quad \textit{with the convention } \sup\emptyset = 0.$$

1. *If $(X_t)_{t\geq 0}$ is of class (D), then*
$$X_T = \mathbb{E}[X_\infty 1_{\{g\leq T\}} \mid \mathcal{F}_T] \quad \textit{for every stopping time } T. \tag{2.3}$$
2. *More generally, if there exists a strictly positive Borel function f such that $(f(A_t)X_t)_{t\geq 0}$ is of class (D), then (2.3) holds.*
3. *If $(N_t^+)_{t\geq 0}$ is of class (D), then (2.3) holds.*

Proof. (1) For a given stopping time T, denote

$$d_T = \inf\{t > T : X_t = 0\} \quad \text{with the convention } \inf\emptyset = \infty.$$

d_T is a stopping time. Since $X_\infty 1_{\{g\leq T\}} = X_{d_T}$ and $A_T = A_{d_T}$, it follows from Doob's optional stopping theorem that

$$\mathbb{E}[X_\infty 1_{\{g\leq T\}} \mid \mathcal{F}_T] = \mathbb{E}[N_{d_T} + A_{d_T} \mid \mathcal{F}_T] = \mathbb{E}[N_{d_T} + A_T \mid \mathcal{F}_T] = N_T + A_T = X_T.$$

(2) Assume that there exists a strictly positive Borel function such that $(f(A_t)X_t)_{t\geq 0}$ is of class (D). This property is preserved if one replaces f by a smaller strictly positive Borel function, hence, one can suppose that f is locally bounded. Then $(f(A_t)X_t)_{t\geq 0}$ is of class (ΣD), and from part (1) of the theorem, we have:

$$f(A_T)X_T = \mathbb{E}[f(A_\infty)X_\infty 1_{\{g\leq T\}} \mid \mathcal{F}_T].$$

But on the set $\{g \leq T\}$, we have $A_\infty = A_T$, and consequently

$$f(A_T)X_T = f(A_T)\mathbb{E}[X_\infty 1_{\{g\leq T\}} \mid \mathcal{F}_T].$$

The result follows by dividing both sides by $f(A_T)$ which is strictly positive.

(3) Since $X \geq 0$ and since $(N_t^+)_{t\geq 0}$ is of class (D), we note that $(\exp(-A_t)X_t)_{t\geq 0}$ is of class (D) and the result follows from (2). $\square$

Remark 2.4. With the notation of the introduction, the measure $\mathcal{Q}$ is given by $\mathcal{Q} = X_\infty . \mathbb{P}$.

An interesting case is constructed as follows: let $(M_t)_{t\geq 0} \in \mathcal{M}_0$. Then $X_t = \bar{M}_t - M_t$ is of class (Σ) but not of class (D). However, with the notation of the Theorem, $N_t = -M_t$ satisfies $N_t^+ = 0$ and hence $\mathcal{Q} = \bar{M}_\infty . \mathbb{P}$. In this example, the process X_t is called the drawdown process and is extensively studied in [7]; the interested reader can find there more results related to drawdown and relative drawdown processes and applications to some options.

We now see how the results of Madan, Roynette and Yor and Profeta, Roynette and Yor on the price of put options would follow from Theorem 2.3.

Corollary 2.5 (Madan-Roynette-Yor [21])**.** *Let K be a constant and (M_t) a local martingale with no positive jumps such that (M_t^-) is of class (D). Denote $g_K = \sup\{t \geq 0 : M_t = K\}$. Then*

$$(K - M_T)^+ = \mathbb{E}[(K - M_\infty)^+ \mathbb{1}_{\{g_K \leq T\}} \mid \mathcal{F}_T], \tag{2.4}$$

for every stopping time T. In particular, if $M_\infty = 0$, then

$$(K - M_T)^+ = K\, \mathbb{P}[g_K \leq T \mid \mathcal{F}_T].$$

Proof. $K - M_t$ is a local martingale with no negative jumps. It follows that $(K - M_t)^+$ is a local submartingale of class (Σ) (see for instance [7]). Since (M_t^-) is of class (Σ), $(K - M_t)^+$ is of class (ΣD) and the result follows from Theorem 2.3 by noting that $g_K = \sup\{t : (K - M_t)^+ = 0\}$. $\square$

The following extension of Corollary 2.5 has been proved by Profeta, Roynette and Yor [37] with methods from the theory of enlargement of filtrations. We can deduce it under slightly weaker assumptions from Theorem 2.3.

Corollary 2.6 (Profeta, Roynette and Yor [37])**.** *Let $K^1, \ldots, K^n$ be positive constants and $(M_t^1), \ldots, (M_t^n)$ nonnegative local martingales that have no positive jumps. Assume $[M^i, M^j]_t = 0$ for $i \neq j$ and denote $g^i = \sup\{t : M_t^i = K^i\}$. Then*

$$\prod_{i=1}^n (K^i - M_T^i)^+ = \mathbb{E}[\prod_{i=1}^n (K^i - M_\infty^i)^+ \mathbb{1}_{\{g^i \leq T\}} \mid \mathcal{F}_T], \tag{2.5}$$

for every stopping time T. In particular, if $M_\infty^i = 0$ for all $i = 1, \ldots, n$, then

$$\prod_{i=1}^n (K^i - M_T^i)^+ = \prod_{i=1}^n K^i\, \mathbb{P}\left[\bigvee_{i=1}^n g^i \leq T \mid F_T\right]$$

Proof. $X_t^i = (K^i - M_t^i)^+$ are local submartingales of class (Σ) such that $[X^i, X^j]_t = 0$ for $i \neq j$. A simple induction shows that $\prod_{i=1}^n X_t^i$ is again of class (Σ) and is bounded. Now (2.5) is a consequence of Theorem 2.3. □

We now state a more general and subtle result from [25] linking a process X of class (Σ) and a pair $(\mathcal{Q}, g)$ consisting of the last zero of X and a remarkable sigma-finite measure $\mathcal{Q}$. This theorem unifies the results of Madan, Profeta, Roynette and Yor on prices of European put and call options and the results of Najnudel, Roynette and Yor on penalizations of the Wiener measure. It is also the right framework to study call options in terms of last passage times. This new framework is also much more general since it contains processes with jumps. For stating this general case, we need to be very careful: the theorem below would be wrong under the usual assumptions and it would also be wrong if the filtration $(\mathcal{F}_t)$ does not allow the extension of coherent probability measures. The natural assumptions are here to ensure, in particular, that there exists a continuous and adapted version of the process A which is defined everywhere (think of A_t being for example the local time at the level 0 of the Wiener process). According to Section 1, typical probability spaces where the following theorem holds are $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ and $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ endowed with the filtration generated by the coordinate process.

Theorem 2.7 (Najnudel-Nikeghbali [25]). *Let $(X_t)_{t\geq 0}$ be a true submartingale of the class (Σ): its local martingale part $(N_t)_{t\geq 0}$ is a true martingale, and X_t is integrable for all $t \geq 0$. We suppose that $(X_t)_{t\geq 0}$ is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ satisfying the property (NP), in particular, this space satisfies the natural conditions and $\mathcal{F}$ is the σ-algebra generated by $\mathcal{F}_t$ for $t \geq 0$. Then, there exists a unique σ-finite measure $\mathcal{Q}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that for $g := \sup\{t \geq 0, X_t = 0\}$:*

- $\mathcal{Q}[g = \infty] = 0$;
- *For all $t \geq 0$, and for all $\mathcal{F}_t$-measurable, bounded random variables Γ_t,*

$$\mathcal{Q}\left[\Gamma_t \mathbb{1}_{g\leq t}\right] = \mathbb{P}\left[\Gamma_t X_t\right]. \tag{2.6}$$

Remark 2.8. For example, the theorem applies when X_t is the absolute value of the standard Brownian motion, or a Bessel process of dimension $d \in (0, 2)$. It also applies when $X_t = |Y_t|^{\alpha-1}$, where Y is a symmetric Lévy stable process of index $\alpha \in (1, 2)$.

In [25], the measure $\mathcal{Q}$ is explicitly constructed in the following way (with a slightly different notation). Let f be a Borel, integrable, strictly positive and bounded function from $\mathbb{R}$ to $\mathbb{R}$, and let us define the function G by the formula:

$$G(x) = \int_x^\infty f(y)\, dy.$$

One can prove that the process

$$\left(M_t^f := G(A_t) - \mathbb{E}_{\mathbb{P}}[G(A_\infty)|\mathcal{F}_t] + f(A_t)X_t\right)_{t\geq 0}, \tag{2.7}$$

is a martingale with respect to $\mathbb{P}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$. Since $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ satisfies the natural conditions and since $G(A_t) \geq G(A_\infty)$, one can suppose that this martingale is nonnegative and càdlàg, by choosing carefully the version of $\mathbb{E}_{\mathbb{P}}[G(A_\infty)|\mathcal{F}_t]$. In this case, since $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ satisfies the property (NP), there exists a unique finite measure $\mathcal{M}^f$ such that for all $t \geq 0$, and for all bounded, $\mathcal{F}_t$-measurable functionals Γ_t:

$$\mathcal{M}^f[\Gamma_t] = \mathbb{E}_{\mathbb{P}}[\Gamma_t M_t^f].$$

Now, since f is strictly positive, one can define a σ-finite measure $\mathcal{Q}^f$ by:

$$\mathcal{Q}^f := \frac{1}{f(A_\infty)} . \mathcal{M}^f.$$

It is proved in [25] that if the function G/f is uniformly bounded (this condition is, for example, satisfied for $f(x) = e^{-x}$), then $\mathcal{Q}^f$ satisfies the conditions defining $\mathcal{Q}$ in Theorem 2.7, which implies the existence part of this result. The uniqueness part is proved just after in a very easy way; one remarkable consequence of it is the fact that $\mathcal{Q}^f$ does not depend on the particular choice of f.

Remark 2.9. If $\mathcal{Q}$ is a probability measure, then (2.6) can be written as

$$X_t = \mathcal{Q}[g \leq t|\mathcal{F}_t].$$

Now we are able to state a rigorous and more general version for the price of a European call option when $(M_t)_{t\geq 0}$ is a true martingale.

Theorem 2.10. *Let $(M_t)_{t\geq 0}$ be a (true) nonnegative and continuous martingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ satisfying the property (NP). Without loss of generality we can assume that $M_0 = 1$. Then there exists a probability measure $\mathbb{P}^{(M)}$ such that for all $K > 0$ and for all $\mathcal{F}_t$-measurable, bounded (or nonnegative) random variables Γ_t,*

$$\mathbb{P}\left[\Gamma_t (M_t - K)^+\right] = \gamma\, \mathbb{P}^{(M)}[\Gamma_t \, \mathbb{1}_{g_K \leq t}], \tag{2.8}$$

where

$$\gamma = 1 - K + \mathbb{E}[(K - M_\infty)^+] \quad \textit{and} \quad g_K = \sup\{t \geq 0 : M_t = K\}.$$

In particular,

$$\mathbb{E}[(M_t - K)^+] = \gamma\, \mathbb{P}^{(M)}[g_K \leq t], \tag{2.9}$$

The probability measure $\mathbb{P}^{(M)}$ is given by

$$\mathbb{P}^{(M)}_{|\mathcal{F}_t} = M_t . \mathbb{P}_{|\mathcal{F}_t} .$$

When $\lim_{t\to\infty} M_t = 0$*, then* $\gamma = 1$ *and*

$$\mathbb{E}[(M_t - K)^+] = \mathbb{P}^{(M)}[g_K \le t]$$

which is the formula obtained by Profeta, Roynette and Yor ([37]). Moreover, in this case, we have the remarkable identity

$$\mathbb{P}^{(M)}[g_K > t] = K\,\mathbb{P}[g_K > t].$$

Proof. Since $(M_t)_{t\ge 0}$ is a true martingale, $X_t = (M_t - K)^+$ is a true submartingale and hence Theorem 2.7 applies and gives the existence of a σ-finite measure $\mathcal{Q}$ such that for $g := \sup\{t \ge 0, X_t = 0\} \equiv g_K$:

- $\mathcal{Q}[g_K = \infty] = 0$;
- for all $t \ge 0$, and for all $\mathcal{F}_t$-measurable, bounded random variables Γ_t,

$$\mathcal{Q}\left[\Gamma_t\, \mathbb{1}_{g_K \le t}\right] = \mathbb{P}\left[\Gamma_t (M_t - K)^+\right]. \tag{2.10}$$

Now we note that since $(M_t - K)^+ = M_t - K + (K - M_t)^+$, we have $\mathbb{E}[(M_t - K)^+] = 1 - K + \mathbb{E}[(K - M_t)^+]$. An application of Lebesgue's dominated convergence theorem yields: $\lim_{t\to\infty} \mathbb{E}[(M_t - K)^+] = 1 - K + \mathbb{E}[(K - M_\infty)^+]$. We call this limit γ. Now, taking $\Gamma_t \equiv 1$ in (2.6), and then letting $t \to \infty$ we obtain that $\gamma = \mathcal{Q}[\mathbb{1}_{g_K<\infty}]$. But since $\mathcal{Q}[g_K = \infty] = 0$, we have that $\mathcal{Q}$ is a finite measure and with total mass γ. Consequently $\mathbb{P}^{(M)} \equiv \dfrac{1}{\gamma}\mathcal{Q}$ is a probability measure that satisfies (2.9). Now it remains to check that $\mathbb{P}^{(M)}_{|\mathcal{F}_t} = M_t.\,\mathbb{P}_{|\mathcal{F}_t}$. Since the measure $\mathcal{Q}$ is unique, one can directly check that it satisfies the requested properties (see [37], p. 24–26). □

Remark 2.11. The case when (M_t) is a strict local martingale is much more involved: it was dealt with in a special case by Yen and Yor in [39] and was solved in a very general setup by Kardaras, Kreher and Nikeghbali in [17]. The formula involves, in addition to the last passage time, the explosion time of M as well (see [17] for more details).

2.2. **Some examples of Azéma's supermartingales and some identities in law.** When one studies random times which are not stopping times, there is a supermartingale that plays a crucial role, namely the Azéma supermartingale. More precisely, if ρ is a measurable nonnegative random variable, its Azéma's supermartingale is the càdlàg version of $\mathbb{P}[\rho >| \mathcal{F}_t]$. This supermartingale is the key process in the theory of progressive enlargements of filtrations (see e.g. [16] or the survey [29]). It also plays an important role in the modeling of default times in credit risk models (see e.g. [13] and [15]). Given a random time which is not a stopping time, it is in general not possible to compute its Azéma supermartingale. We now give a Corollary of Theorem 2.3 which will allow us to compute explicitly the Azéma supermartingale of many last passage times of interest in applications (this method was first used in [28]).

Corollary 2.12. *Let (X_t) be a process of the class (ΣD), such that $X_\infty = 1$ almost surely. Let $g \equiv \sup\{t : X_t = 0\}$. Then*

$$X_t = \mathbb{P}[g \leq t \mid \mathcal{F}_t]. \tag{2.11}$$

Remark 2.13. The proof is straightforward by taking $X_\infty = 1$ in Theorem 2.3. In particular, we are able to get rid of the assumption that the filtration $(\mathcal{F}_t)$ should be a Brownian type filtration with only continuous martingales, which is a commonly imposed in the literature. Moreover, this result can be viewed as a reciprocal to an important result by Azéma ([1]) which states that one minus the Azéma supermartingale of any end of a predictable set which avoids stopping times is of class (ΣD) and satisfies $X_\infty = 1$.

We now show how to use the above Corollary to compute some Azéma supermartingales.

Example 2.14. Let (M_t) be a continuous martingale such that $\langle M, M\rangle_\infty = \infty$, and let $T_1 = \inf\{t \geq 0 : \ M_t = 1\}$. Let $g = \sup\{t < T_1 : \ M_t = 0\}$. Then $M^+_{t\wedge T_1}$ satisfies the conditions of Corollary 2.12, and hence:

$$\mathbb{P}\left(g \leq t|\mathcal{F}_t\right) = M^+_{t\wedge T_1} = \int_0^{t\wedge T_1} \mathbf{1}_{M_u>0}\, dM_u + \frac{1}{2} L_{t\wedge T_1},$$

where (L_t) is the local time of M at 0. When $M = W$ is the standard Brownian motion, this example plays an important role in the celebrated Williams' path decomposition for the standard Brownian Motion on $[0, T_1]$.

One can also consider $T_{\pm 1} = \inf\{t \geq 0 : \ |M_t| = 1)$ and $\tau = \sup\{t < T_{\pm 1} : \ |M_t| = 0\}$. $|M_{t\wedge T_{\pm 1}}|$ satisfies the conditions of Corollary 2.12, hence hence:

$$\mathbb{P}\left(\tau \leq t|\mathcal{F}_t\right) = |M_{t\wedge T_{\pm 1}}| = \int_0^{t\wedge T_{\pm 1}} sgn\left(M_u\right) dM_u + \ell_{t\wedge T_{\pm 1}}.$$

Example 2.15. Let (Y_t) be a real continuous recurrent diffusion process, with $Y_0 = 0$. Then from the general theory of diffusion processes, there exists a unique continuous and strictly increasing function s, with $s(0) = 0$, $\lim_{x\to+\infty} s(x) = +\infty$, $\lim_{x\to-\infty} s(x) = -\infty$, such that $s(Y_t)$ is a continuous local martingale. Let

$$T_1 \equiv \inf\{t \geq 0 : \ Y_t = 1\}.$$

Now, if we define

$$X_t \equiv \frac{s\left(Y_{t\wedge T_1}\right)^+}{s(1)},$$

we easily note that X is a local submartingale of the class (Σ) which satisfies the hypotheses of Corollary 2.12. Consequently for

$$g = \sup\{t < T_1 : \ Y_t = 0\},$$

we have:

$$\mathbb{P}(g \leq t|\mathcal{F}_t) = \frac{s(Y_{t\wedge T_1})^+}{s(1)}.$$

Example 2.16. Now let (M_t) be a positive local martingale, such that: $M_0 = x, \ x > 0$ and $\lim_{t\to\infty} M_t = 0$. Then, Tanaka's formula shows us that $\left(1 - \frac{M_t}{y} \wedge 1\right)$, for $0 \leq y \leq x$, is a local submartingale of the class (Σ) satisfying the assumptions of Corollary 2.12. Hence with

$$g = \sup\{t : \ M_t = y\},$$

we have:

$$\mathbb{P}(g > t|\mathcal{F}_t) = \frac{M_t}{y} \wedge 1 = 1 + \frac{1}{y}\int_0^t \mathbf{1}_{(M_u<y)}\, dM_u - \frac{1}{2y} L_t^y,$$

where (L_t^y) is the local time of M at y.

Example 2.17. As an illustration of the previous example, consider (R_t), a transient diffusion with values in $[0, \infty)$, which has $\{0\}$ as entrance boundary. Let s be a scale function for R, which we can choose such that:

$$s(0) = -\infty, \text{ and } s(\infty) = 0.$$

Then, under the law $\mathbb{P}_x$, for any $x > 0$, the local martingale $(M_t = -s(R_t))$ satisfies the conditions of the previous example and for $x, y \geq 0$, we have:

$$\mathbb{P}_x(g_y > t|\mathcal{F}_t) = \frac{s(R_t)}{s(y)} \wedge 1 = 1 + \frac{1}{s(y)}\int_0^t \mathbf{1}_{(R_u>y)}\, d(s(R_u)) + \frac{1}{2s(y)} L_t^{s(y)},$$

where $\left(L_t^{s(y)}\right)$ is the local time of $s(R)$ at $s(y)$, and where

$$g_y = \sup\{t : \ R_t = y\}.$$

This last formula was the key point for deriving the distribution of g_y in [32], Theorem 6.1, p.326.

The structure of the class (Σ), together with the examples above suggest that the increasing process (A_t) should have some interesting properties. The examples we have considered so far show that A_t can be the local time of a martingale, of a diffusion process or a Lévy process. It can also be the supremum process of a local martingale with only negative jumps. Hence it would be interesting to have some information about the distribution of A_∞ when it is finite. This is clearly the case for the examples above and more generally for processes of the class (ΣD). In many situations, by stopping, one is in the situation where $X_\infty = \varphi(A_\infty)$: this occurs in the resolution of the Skorokhod problem by Azéma and Yor (see e.g. [38]), or in the resolution of the Skorokhod problem for processes of the class (Σ) (see [27]); it also appears naturally around problems related to the drawdown or relative drawdown processes (see [7]). All these situations are covered by the following theorem:

Theorem 2.18 ([7])**.** *Let $(X_t)_{t\geq 0}$ be a process of the class (Σ) and assume that there exists a locally bounded Borel function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that the process $f(A_t)X_t$ is of class (ΣD) and $\lim_{t\to\infty} f(A_t)X_t = 1$ almost surely. Denote $F(x) = \int_0^x f(y)\mathrm{d}y$ and assume that $F(\infty) = \infty$. Then for every stopping time T and for all Borel functions $h : [0, a) \to \mathbb{R}$ satisfying*

$$\int_0^\infty |h(y)|e^{-F(y)} dF(y) < \infty,$$

one has

$$\begin{aligned}\mathbb{E}[h(A_\infty) \mid \mathcal{F}_T] &= h(0)f(0)X_0 + h^F(0)(1 - f(0)X_0) + \int_0^T (h - h^F)(A_u)f(A_u)dN_u \\ &= h(A_T)f(A_T)X_T + h^F(A_T)(1 - f(A_T)X_T)\end{aligned}$$

where

$$h^F(x) = e^{F(x)} \int_x^\infty h(y)e^{-F(y)}dF(y), x \geq 0.$$

In particular, conditioned on F_T, the law of A_∞ is given by

$$\mathbb{P}[A_\infty > x \mid \mathcal{F}_T] = \mathbb{1}_{\{A_T > x\}} + \mathbb{1}_{\{A_T \leq x\}}(1 - f(A_T)X_T)e^{F(A_T)-F(x)}, \quad x \geq 0.$$

We postpone an important application of the above theorem to the next section.

3. The multiplicative approach

There also exists a simple multiplicative approach to study ends of predictable sets which was developed by Nikeghbali and Yor in [30] and which underlies some of the techniques used by Madan, Profeta, Roynette and Yor in their studies of the generalized Black-Scholes formulae ([21],[22],[37]). This multiplicative approach has also been used by Coculescu and Nikeghbali in [9] to obtain general formulas when the default time in credit risk models is modeled by the end of a predictable set. It was also more recently used by Li and Rutkowski in [20]. The idea in this approach is to consider a multiplicative decomposition for the Azéma supermartingale of the end of a predictable set rather than the Doob-Meyer decomposition. As a consequence, every end of a predictable set has an intuitive representation in terms of some simple and intuitive last passage time. We deduce these results from those of the previous section. This is new and as a consequence we avoid the assumption that the filtration $(\mathcal{F}_t)_{t\geq 0}$ covers only continuous martingales.

Proposition 3.1. *Let* $(M_t)_{t\geq 0} \in \mathcal{M}_0$*. Consider*

$$g = \sup\{t \geq 0 : \ M_t = \bar{M}_t\} = \sup\{t \geq 0 : \ M_t = \bar{M}_\infty\}. \tag{3.1}$$

Then the Azéma supermartingale of g *is given by*

$$Z_t \equiv \mathbb{P}[g > t \mid \mathcal{F}_t] = \frac{M_t}{\bar{M}_t} = 1 + \int_0^t \frac{\mathrm{d}M_s}{\bar{M}_s} - \log \bar{M}_t. \tag{3.2}$$

Proof. The result follows directly from Theorem 2.3 when applied to the process $X_t = 1 - \dfrac{M_t}{\bar{M}_t}$ which is of class (ΣD). □

The remarkable feature of the above family of examples is that it actually covers all cases of ends of optional sets which avoid stopping times:

Theorem 3.2. *Let* g *be the end of an optional set which avoids stopping times, that is* $\mathbb{P}[g = T > 0] = 0$ *for all* $(\mathcal{F}_t)$*-stopping times* T*. Then there exists a unique local martingale* $(M_t) \in \mathcal{M}_0$ *such that*

$$g = \sup\{t \geq 0 : \ M_t = \bar{M}_t\} \quad \text{and } \mathbb{P}[g > t \mid \mathcal{F}_t] = \frac{M_t}{\bar{M}_t}.$$

Proof. It is a well known result of Azéma ([1]) that $1 - \mathbb{P}[g > t \mid \mathcal{F}_t] \equiv X_t$ is of class (Σ). Let us denote it $X_t = N_t + A_t$. Now it follows from (2.1) that

$$\exp(A_t)X_t = \int_0^t \exp(A_u)\mathrm{d}N_u + \exp(A_t) - 1,$$

or equivalently

$$\exp(A_t)(1 - X_t) = 1 - \int_0^t \exp(A_u)\mathrm{d}N_u. \tag{3.3}$$

Let us write $M_t = 1 - \int_0^t \exp(A_u)\mathrm{d}N_u$. This is a local martingale starting from 1. Using the above representation $M_t = \exp(A_t)(1 - X_t)$, we see that $\bar{M}_t = \exp(A_t)$ since (A_t) is continuous, increasing, and increases only on the zeros of X (hence $(\bar{M}_t)$ is continuous). Hence it follows from (3.3) that

$$Z_t = 1 - X_t = \frac{M_t}{\bar{M}_t}.$$

It also follows from the above representation that $\lim_{t\to\infty} M_t = 0$, which completes the proof. □

Remark 3.3. In modeling the default time in credit risk modeling, it is important to know when the hazard process and the martingale hazard process are different. In [9], Theorem 3.2 is used to show that under the assumption that the end of an optional set g avoids stopping times, the martingale hazard process and the hazard process are always different and this difference is explicitly computed (in fact in [9], it is assumed that the filtration $(\mathcal{F}_t)$ is such that all martingales are continuous; our slightly improved version shows that one can avoid this assumption). Theorem 3.2 is also used in [9] to compute the price of defaultable claims.

In [30], Doob's maximal identity (which state that for a local martingale in $\mathcal{M}_0$ the distribution of $\bar{M}_\infty$ is the same as the inverse of a uniform random variable on $(0,1)$) is used to derive the conditional distribution of $\bar{M}_\infty$, which is a crucial step in some problems of enlargements of filtrations. Here we shall deduce it from the more general Theorem 2.18. This result seems to be important enough on its own to be included in this survey.It is also valuable in the context where prices under the risk neutral probability measure as well as benchmarked portfolios are local martingales in $\mathcal{M}_0$, to have information on the conditional distribution of the maximum of martingales in $\mathcal{M}_0$.

Proposition 3.4 ([30]). *Let $(M_t)_{t\geq 0} \in \mathcal{M}_0$. For any Borel bounded or positive function f, we have:*

$$\begin{aligned} \mathbb{E}\left(f\left(\bar{M}_\infty\right)|\mathcal{F}_t\right) &= f\left(\bar{M}_t\right)\left(1-\frac{M_t}{\bar{M}_t}\right)+\int_0^{M_t/\bar{M}_t} dx f\left(\frac{M_t}{x}\right) \qquad (3.4)\\ &= f\left(\bar{M}_t\right)\left(1-\frac{M_t}{\bar{M}_t}\right)+M_t\int_{\bar{M}_t}^{\infty} dx\frac{f(x)}{x^2}. \end{aligned}$$

Moreover we have the following representation of $\mathbb{E}\left(f\left(\bar{M}_\infty\right)|\mathcal{F}_t\right)$ as a stochastic integral:

$$\mathbb{E}\left(f\left(\bar{M}_\infty\right)|\mathcal{F}_t\right)=\mathbb{E}\left(f\left(\bar{M}_\infty\right)\right)+\int_0^t g\left(\bar{M}_s\right)dM_s, \qquad (3.5)$$

where h is given by $g(x)=\int_x^\infty \frac{\mathrm{d}y}{y^2}(f(y)-f(x))$.

Proof. The proof follows immediately from Theorem 2.18 applied to the process

$$X_t = 1-\frac{M_t}{\bar{M}_t} = -\int_0^t \frac{\mathrm{d}M_s}{\bar{M}_s}+\log\bar{M}_t.$$

Indeed in this case we take $h(x)=f(\exp(x))$ in Theorem 2.18, with $N_t=-\int_0^t \frac{\mathrm{d}M_s}{\bar{M}_s}$ and $A_t=\log\bar{M}_t$. □

Remark 3.5. By taking $t = 0$ above we obtain the well known identity in law

$$\bar{M}_\infty \overset{\text{law}}{=} \frac{1}{U},$$

where U is a uniform random variable on $(0, 1)$.

Remark 3.6. Formulae (3.4) and (3.5) suggest that there should exist ways of creating new financial products. For instance one could imagine an option with pay-off $f(\bar{M}_\infty)$ in areas where modeling risk over a long period has been of much concern, as it is in pension fund and insurance. In this case formulas (3.4) and (3.5) give the price of the option together with a hedging strategy. The formulas are very robust since they do not depend on the underlying dynamics of the stock price (M_t).

4. Distributions of last passage times

At this point in our discussion, the following questions can be raised:

- the last passage time g_K defined by (1.3) plays a central role in the generalized Black-Scholes formulae; can one compute the law of such times for a wide range of examples?
- Theorem 3.2 gives a simple general representation for the Azéma supermartingale of ends of optional sets that avoid stopping times. Can this be used to obtain a systematic way to compute the law of such a random time?

The next subsections give some answers to the above questions.

4.1. Some examples for distributions of g_K.

In this paragraph, we follow closely the computations in the monograph by Profeta, Roynette and Yor (p. 32–41) to obtain a family of interesting examples.

We assume that $(M_t)_{t \geq 0} \in \mathcal{M}_0$ and is continuous. We make the following additional assumptions:

i) for every $t > 0$, the law of the random variable M_t admits a density $(m_t(x), x \geq 0)$ and $(t, x) \to m_t(x)$ may be chosen continuous on $(0, \infty)^2$;
ii) We assume that the quadratic covariation of $(\langle M, M \rangle_t)_{t \geq 0}$ of M satisfies $\mathrm{d}\langle M, M \rangle_t = \sigma_t^2 \mathrm{d}t$. We further assume that there exists a jointly continuous function:
$$(t, x) \to \theta(x) := \mathbb{E}[\sigma_t^2 \mid M_t = x]$$
on $(0, \infty)^2$.

Theorem 4.1 (Profeta, Roynette and Yor, [37]). *Under the preceding hypotheses, the law of g_K is given by*

$$\mathbb{P}[g_K \in \mathrm{d}t] = (1 - \frac{a}{K})^+ \delta_0(\mathrm{d}t) + \frac{1}{2K} \theta_t(K) m_t(K) \mathbb{1}_{\{t>0\}} \mathrm{d}t \tag{4.1}$$

where $a = M_0$ and δ_0 is the Dirac measure at 0.

Proof. Using Tanaka's formula we have

$$\mathbb{E}[(K - M_t)^+] = (K - a)^+ + \frac{1}{2}\,\mathbb{E}[L_t^K],$$

where $(L_t^K, t \geq 0, K \geq 0)$ denotes the continuous family of local times of the martingale (M_t). Thus from (1.2), it follows that

$$\mathbb{P}[g_K \in \mathrm{d}t] = (1 - \frac{a}{K})^+ \delta_0(\mathrm{d}t) + \frac{\mathbb{1}_{\{t>0\}}}{2K} \mathrm{d}_t\,\mathbb{E}[L_t^K],$$

and it remains to show that

$$\mathrm{d}_t\,\mathbb{E}[L_t^K] = \theta_t(K) m_t(K)\mathrm{d}t.$$

To prove the above formula we note that for every $f : \mathbb{R}_+ \to \mathbb{R}_+$ Borel, the density of occupation formula yields

$$\int_0^t f(M_s)\mathrm{d}\langle M, M\rangle_s = \int_0^\infty f(K) L_t^K \mathrm{d}K.$$

Under the assumption of the theorem the above equality becomes

$$\int_0^t f(M_s)\sigma_s^2 \mathrm{d}s = \int_0^\infty f(K) L_t^K \mathrm{d}K.$$

Taking expectation on both sides we obtain

$$\mathbb{E}[\int_0^t f(M_s)\sigma_s^2 \mathrm{d}s] = \int_0^\infty f(K)\,\mathbb{E}[L_t^K]\mathrm{d}K.$$

But under our assumptions, we can also write:

$$\begin{aligned} \mathbb{E}[\int_0^t f(M_s)\sigma_s^2 \mathrm{d}s] &= \int_0^t \mathbb{E}[f(M_s)\,\mathbb{E}[\sigma_s^2 \mid M_s]]\mathrm{d}s & (4.2) \\ &= \int_0^\infty f(K)\mathrm{d}K \int_0^t \theta_s(K) m_s(K)\mathrm{d}s. & (4.3) \end{aligned}$$

Since our formulas hold for every $f : \mathbb{R}_+ \to \mathbb{R}_+$ Borel, we can thus conclude that

$$\mathbb{E}[L_t^K] = \int_0^t \theta_s(K) m_s(K)\mathrm{d}s,$$

which proves the theorem. □

We now give some examples.

Example 4.2. Consider $M_t = \exp(W_t - \frac{t}{2})$, where (W_t) is the standard Brownian motion. From Itô's formula, we have $M_t = 1 + \int_0^t M_s \mathrm{d}W_s$, thus $\mathrm{d}\langle M, M\rangle_t = M_t^2 \mathrm{d}t$ and we may apply Theorem 4.1 with $\theta_t(x) = x^2$ and

$$m_t(x) = \frac{1}{x\sqrt{2\pi t}} \exp(-\frac{1}{2t}(\log x + \frac{t}{2})^2).$$

We thus obtain

$$\mathbb{P}[g_K \in \mathrm{d}t] = (1 - \frac{1}{K})^+ \delta_0(\mathrm{d}t) + \frac{\mathbb{1}_{\{t>0\}}}{2\sqrt{2\pi t}} \exp(-\frac{1}{2t}(\log K + \frac{t}{2})^2)\mathrm{d}t.$$

Example 4.3. We come back to the case of transient diffusions considered in Example 2.17 (recall that s is the scale function). We assume further that the infinitesimal generator of our diffusion has the form

$$\Gamma = \frac{1}{2}a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}.$$

In this case, if we denote by $p(t, x, y)$ the density of the random variable X_t with respect to the Lebesgue measure $\mathrm{d}y$, then the distribution of g_K is (see [37], p.40):

$$\mathbb{P}[g_K \in \mathrm{d}t] = (1 - \frac{s(x)}{s(K)})^+ \delta_0(\mathrm{d}t) - \frac{s'(K)a(K)}{2s(K)} p(t, x, K)\mathrm{d}t. \tag{4.4}$$

For instance if $(M_t)_{t\geq 0}$ is a Bessel process, i.e. $a = 1$ and $b(x) = \frac{2\nu + 1}{2x}$, with index $\nu > 0$ (i.e. with dimension $d = 2\nu + 2 > 2$), we have $s(x) = -x^{-2\nu}$ and

$$p(t, 0, K) = \frac{2^{-\nu}}{\Gamma(\nu + 1)} t^{-(\nu+1)} K^{2\nu+1} \exp(-\frac{-K^2}{2t}).$$

Hence

$$\begin{aligned}\mathbb{P}_0^{(\nu)}[g_K \in \mathrm{d}t] &= \frac{\nu 2^{-\nu}}{\Gamma(\nu + 1)} \frac{1}{K} \frac{K^{2\nu+1}}{t^{\nu+1}} \exp(-\frac{-K^2}{2t})\mathrm{d}t \\ &= \frac{2^{-\nu}}{\Gamma(\nu)} \frac{K^{2\nu}}{t^{\nu+1}} \exp(-\frac{-K^2}{2t})\mathrm{d}t.\end{aligned}$$

A few more examples based on Theorem 4.1 are computed in [37].

4.2. **The general distribution of last passage times.** We now use Theorem 3.2 to obtain a representation for the law of an arbitrary end of an optional set which avoids stopping times. Indeed, we know from Theorem 3.2 that any such time is the last time when a local martingale in $\mathcal{M}_0$ is equal to its running maximum. We start with a simple lemma:

Lemma 4.4. *Under the assumptions of Theorem 3.2 the law of the last passage time g is given by*

$$\mathbb{P}[g \leq t] = \mathbb{E}[\log \bar{M}_t]. \tag{4.5}$$

Proof. This follows immediately by taking the expectation of Z_t in Theorem 3.2. □

We now state and prove a theorem that can be very useful in practice to compute the law of a last passage time.

Theorem 4.5. *Under the assumptions of Theorem 3.2 define $\tau_a = \inf\{t : M_t > a\}$ for $a \geq 1$. Then for any bounded or positive Borel function f, we have*

$$\mathbb{E}[f(g)] = \int_1^\infty \mathbb{E}[f(\tau_a)\mathbb{1}_{\{\tau_a<\infty\}}]\frac{\mathrm{d}a}{a}. \tag{4.6}$$

In particular, the Laplace transform of the law of g is given by

$$\mathbb{E}[\exp(-\lambda g)] = \int_1^\infty \mathbb{E}[\exp(-\lambda \tau_a)]\frac{\mathrm{d}a}{a}, \tag{4.7}$$

for $\lambda > 0$.

Proof. We differentiate (4.5) to obtain:

$$\begin{aligned} \mathbb{E}[f(g)] &= \mathbb{E}[\int_0^\infty f(s)\frac{\mathrm{d}\bar{M}_s}{\bar{M}_s}] = \mathbb{E}[\int_1^{\bar{M}_\infty} f(\tau_a)\frac{\mathrm{d}a}{a}] &(4.8)\\ &= \int_1^\infty \mathbb{E}[f(\tau_a)\mathbb{1}_{\{\tau_a<\infty\}}]\frac{\mathrm{d}a}{a}. &(4.9) \end{aligned}$$

□

This result allows us to derive some explicit examples for the law of a last passage time when the underlying nonnegative local martingale M belongs to some class of well-known diffusions. We begin with the standard asset price model in finance, the Black-Scholes model. We set

$$M_t = \exp\{2\,\sigma\,W_t - 2\,\sigma^2\,t\}, \tag{4.10}$$

which follows a geometric Brownian motion for $t \geq 0$. Here $W = \{W_t, t \geq 0\}$ denotes a standard Wiener process under the real world probability measure $\mathbf{P}$ and we assume $\sigma > 0$. The last passage time considered here, that is the time of the total maximum of M_t, is then given as

$$g = \sup\left\{t \geq 0 : (W_t - \sigma\,t) = \sup_{s\geq 0}(W_s - \sigma\,s)\right\}.$$

Proposition 4.6. *The law of g is characterized by its Laplace transform*

$$\mathbb{E}[\exp(-\lambda g)] = \frac{2}{1+\sqrt{1+\frac{2\lambda}{\sigma^2}}} \tag{4.11}$$

for $\lambda \geq 0$.

Proof. We can use (4.7) to compute the law of g. For this we will use the Laplace transform of $\tau_a = \inf\{t : M_t > a\}$, which is given for instance in [6]:

$$\mathbb{E}[\exp(-\lambda\tau_a)] = \left(\frac{1}{a}\right)^{\sqrt{\frac{1}{4}+\frac{\lambda}{2\,\sigma^2}}+\frac{1}{2}} \tag{4.12}$$

for $a > 1$ and $\lambda \geq 0$. Substituting formula (4.12) into (4.7) yields

$$\begin{aligned}\mathbb{E}[\exp(-\lambda g)] &= \int_1^\infty \left(\frac{1}{a}\right)^{\sqrt{\frac{1}{4}+\frac{\lambda}{2\sigma^2}}+\frac{1}{2}} \frac{da}{a} = \int_0^\infty e^{-u\left(\sqrt{\frac{1}{4}+\frac{\lambda}{2\sigma^2}}+\frac{1}{2}\right)}\,du \\ &= \frac{2}{1+\sqrt{1+\frac{2\lambda}{\sigma^2}}}.\end{aligned}$$

□

Remark 4.7. It is interesting to note that by (4.11) the time g has the same law as the first hitting time of a level twice the value of an independent exponential random variable $\tilde{e}$ by a Brownian motion with drift, that is,

$$g \overset{\text{law}}{=} \frac{1}{\sigma^2} T_{2\tilde{e}}$$

with $T_a = \inf\{t : \tilde{W}_t + t = a\}$, where $\tilde{W}_t$ follows a standard Brownian motion.

One can sometimes reduce the problem of finding the law of the end of a predictable set to that of a geometric Brownian motion after time change.

Proposition 4.8. *Assume that the hypotheses of Theorem 3.2 hold and assume further that all martingales of the filtration $(\mathcal{F}_t)_{t\geq 0}$ are continuous. Then there exists a unique local martingale $D = \{D_t,\, t \geq 0\}$ with $\langle D\rangle_\infty = \infty$ a.s. and $D_t = \int_0^t \frac{dM_u}{M_u} = W_{\langle D\rangle_t}$, where W is an $(\mathcal{F}_{\langle D\rangle_u^{-1}})$-Brownian motion, such that*

$$g = \sup\left\{t : W_{\langle D\rangle_t} - \frac{1}{2}\langle D\rangle_t = \sup_{s\geq 0}\left(W_{\langle D\rangle_s} - \frac{1}{2}\langle D\rangle_s\right)\right\}.$$

Proof. It is a standard fact of stochastic calculus that there exists a local martingale D such that $\langle D\rangle_\infty = \infty$ and $M_t = \exp\{D_t - \frac{1}{2}\langle D\rangle_t\}$. Moreover, the local martingale D is unique and $D_t = \int_0^t \frac{dM_u}{M_u}$. From the Dubins-Schwarz theorem there exists then an $(\mathcal{F}_{\langle D\rangle_u^{-1}})$-Brownian motion $W = \{W_u,\ u \geq 0\}$ in $\langle D\rangle_t$-time such that $D_t = W_{\langle D\rangle_t}$. If we denote by $\langle D\rangle_u^{-1}$, the generalized inverse of $\langle D\rangle_t$ defined by

$$\langle D\rangle_u^{-1} = \inf\{t \geq 0 : \langle D\rangle_t > u\},$$

then we can define the last passage time

$$L = \sup\left\{t \geq 0 : W_u - \frac{1}{2}u = \sup_{s\geq 0}\left(W_s - \frac{1}{2}s\right)\right\}.$$

Consequently, $g = \langle D\rangle_L^{-1}$ is also given by

$$g = \sup\left\{t : W_{\langle D\rangle_t} - \frac{1}{2}\langle D\rangle_t = \sup_{s\geq 0}\left(W_{\langle D\rangle_s} - \frac{1}{2}\langle D\rangle_s\right)\right\}.$$

□

Squared Bessel processes play an essential role in various financial models. This includes, for instance, the constant elasticity of variance model, see [10]; the affine models, see [12]; and the minimal market model, see [34] [35]. To study last passage times in some of these models let $R^2 = \{R_t^2, t \geq 0\}$ denote a squared Bessel process of dimension $\delta > 2$. In this case R^2 is transient, see [38]. Furthermore, for any squared Bessel process with index $\nu = \frac{\delta}{2} - 1 > 0$ the process $M = \{M_t, t \geq 0\}$ with

$$M_t = \left(\frac{R_0^2}{R_t^2}\right)^\nu \tag{4.13}$$

is a nonnegative, strict local martingale from the class $(\mathcal{M}_0)$. By application of Proposition 3.1 one obtains that for $g = \sup\{t \geq 0 : R_t^2 = I_t\}$ with $I_t = \inf_{s\leq t} R_s^2$ the conditional probability

$$\mathbb{P}(g > t|\mathcal{F}_t) = \left(\frac{I_t}{R_t^2}\right)^\nu, \tag{4.14}$$

for all $t \geq 0$. Moreover it follows from Remark 3.5 that the random limit $1/I_\infty$ is uniformly distributed on $(0, 1)$ for the case of dimension $\delta = 4$. This is an interesting observation for the rather realistic minimal market model, where such dynamics arise.

Proposition 4.9. *The Laplace transform of the last passage time g given in (4.14) is for $\lambda > 0$ of the form*

$$\mathbb{E}[\exp(-\lambda g)] = \frac{2\,\nu\, K_\nu(\sqrt{2\,\lambda\, x})}{(2\,\lambda\, x)^{\frac{\nu}{2}}} \int_0^{\sqrt{2\lambda x}} \frac{u^{\nu-1}}{K_\nu(u)}\, du \tag{4.15}$$

where $R_0^2 = x$ *and* $K_\nu(\cdot)$ *is the modified Bessel function of the second kind, see* [6].

Proof. We first recall the Laplace transform of the random variable $\tau_a = \inf\{t \geq 0 : M_t = a\} = \inf\{t \geq 0 : R_t^2 = \frac{x}{a^{\frac{1}{\nu}}}\}$, $a \geq x^\nu$, from [18] and [6] in the form

$$\mathbb{E}[\exp(-\lambda \tau_a)] = \frac{K_\nu \sqrt{2\,\lambda\,x}}{a\,K_\nu\,\left(\frac{\sqrt{2\,\lambda\,x}}{a^{\frac{1}{\nu}}}\right)}, \tag{4.16}$$

for $\lambda > 0$. A combination of (4.16) and (4.7) gives (4.15). □

In the special case of dimension $\delta = 4$, as it arises for the stylized minimal market model in [34] [35], we have $\nu = 1$ and it follows that

$$\mathbb{E}[\exp(-\lambda g)] = \frac{2\,K_1(\sqrt{2\,\lambda\,x})}{\sqrt{2\,\lambda\,x}} \int_0^{\sqrt{2\lambda x}} \frac{1}{K_1(u)}\,du.$$

Another interesting special case is obtained for the squared Bessel process of dimension $\delta = 3$, where we are able to provide the following explicit formula for the density.

Corollary 4.10. *For dimension* $\delta = 3$ *the law of the last passage time* g *given in (4.14) has the density*

$$p(t) = \frac{1}{\sqrt{2\pi x t}}\left(1 - \exp\left(\frac{-x}{2t}\right)\right), \tag{4.17}$$

where $R_0^2 = x > 0$ *and* $t \geq 0$.

Proof. For the squared Bessel process of dimension $\delta = 3$ we have $\nu = \frac{1}{2}$ and from (4.16)

$$\mathbb{E}[\exp(-\lambda g)] = \frac{2}{\sqrt{2\,\lambda\,x}} \exp\left\{-\frac{\sqrt{2\,\lambda\,x}}{2}\right\} \sinh\left(\frac{\sqrt{2\,\lambda\,x}}{2}\right).$$

The linearity of the Laplace transform and a close look at a table of inverse Laplace transforms, see for instance [6], then yields (4.17). □

Note that the above density (4.17) of the last passage time is dependent on the initial level of the squared Bessel process. Such dependence was not observed in the case of the geometric Brownian motion.

We end this section by considering again the general case of a transient diffusion $Y = \{Y_t, t \geq 0\}$. Recall that it generates a local martingale M in the class $(\mathcal{M}_0)$ via the ratio $M_t = \frac{s(Y_t)}{s(x)}$, $t \geq 0$, $Y_0 = x > 0$. Here $s(\cdot)$ is the differentiable scale function of Y (see [6]) which we can choose such that $s(0) = -\infty$ and $s(\infty) = 0$. Then we have by Proposition 3.1 again

$$\mathbb{P}[g > t \mid \mathcal{F}_t] = \frac{s(Y_t)}{s(Z_t)},$$

where $Z_t = \inf_{s \le t} Y_s$ and g is defined as $g = \sup\{t \ge 0 : Y_t = Z_t\}$. The law of the last passage time g can then be characterized as follows:

Proposition 4.11. *The Laplace transform of the above last passage time g is for $\lambda > 0$ of the form*

$$\mathbb{E}[\exp(-\lambda g)] = -\int_0^x \frac{s'(u)}{s(u)} \frac{\varphi_\lambda(x)}{\varphi_\lambda(u)} \, du. \tag{4.18}$$

Here $\varphi_\lambda(\cdot)$ is a continuous solution of the equation

$$G\, \varphi_\lambda(y) = \lambda\, \varphi_\lambda(y), \tag{4.19}$$

with G denoting the infinitesimal generator of the diffusion Y.

The function $\varphi_\lambda(\cdot)$ is characterized as the unique (up to a multiplicative constant) solution of (4.19) by demanding that $\varphi_\lambda(\cdot)$ is decreasing and satisfies some appropriate boundary conditions. The reader is referred to [33] for further details on the function $\varphi_\lambda(\cdot)$ and its relation to hitting times, as well as [15], Chapter 5, and more specifically p. 279–281, where examples of diffusions and associated $\varphi_\lambda(\cdot)$ functions are developed.

Proof. Let us consider the hitting time

$$\tau_z = \inf\left\{t \ge 0 : \frac{s(Y_t)}{s(x)} = a\right\} = \inf\{t \ge 0 : Y_t = s^{-1}(a\, s(x))\}$$

for $a \ge 1$ and $z := s^{-1}(a\, s(x)) \le x$. The Laplace transform of τ_z follows by [33] and [6] in the form

$$\mathbb{E}[\exp(-\lambda \tau_z)] = \frac{\varphi_\lambda(x)}{\varphi_\lambda(z)}. \tag{4.20}$$

It suffices to substitute $z = s^{-1}(a\, s(x))$ in (4.20) and then apply the resulting expression in (4.7). $\square$

Appendix

In this appendix we recall a few facts about the *natural conditions* and the Parathasarathy conditions (P). For more details see [24].

Recall that most of the properties which generally hold under the usual conditions remain valid under the natural conditions (for example, existence of càdlàg versions of martingales, the Doob-Meyer decomposition, the début theorem, etc.). Let us recall here the definition.

Definition 4.12. A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, satisfies the natural conditions if and only if the following two assumptions hold:

- The filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous;
- For all $t \geq 0$, and for every $\mathbb{P}$-negligible set $A \in \mathcal{F}_t$, all the subsets of A are contained in $\mathcal{F}_0$.

This definition is slightly different from the definitions given in [5] and [24] but one can easily check that it is equivalent. The natural enlargement of a filtered probability space can be defined by using the following proposition:

Proposition 4.13 ([24])**.** *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. There exists a unique filtered probability space $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}\widetilde{\mathbb{P}})$ (with the same set Ω), such that:*

- *For all $t \geq 0$, $\widetilde{\mathcal{F}}_t$ contains $\mathcal{F}_t$, $\widetilde{\mathcal{F}}$ contains $\mathcal{F}$ and $\widetilde{\mathbb{P}}$ is an extension of $\mathbb{P}$;*
- *The space $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}\widetilde{\mathbb{P}})$ satisfies the natural conditions;*
- *For any filtered probability space $(\Omega, \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, \mathbb{P}')$ satisfying the two items above, $\mathcal{F}'_t$ contains $\widetilde{\mathcal{F}}_t$ for all $t \geq 0$, $\mathcal{F}'$ contains $\widetilde{\mathcal{F}}$ and $\mathbb{P}'$ is an extension of $\widetilde{\mathbb{P}}$.*

The space $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}\widetilde{\mathbb{P}})$ is called the natural enlargement of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Intuitively, the natural enlargement of a filtered probability space is its smallest extension which satisfies the natural conditions. We also introduce a class of filtered measurable spaces $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ such that any compatible family $(\mathbb{Q}_t)_{t\geq 0}$ of probability measures, $\mathbb{Q}_t$ defined on $\mathcal{F}_t$, can be extented to a probability measure $\mathbb{Q}$ defined on $\mathcal{F}$.

Definition 4.14. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered measurable space, such that $\mathcal{F}$ is the σ-algebra generated by $\mathcal{F}_t$, $t \geq 0$: $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$. We say that the property[1] (P) holds if and only if $(\mathcal{F}_t)_{t\geq 0}$ enjoys the following properties:

- for all $t \geq 0$, $\mathcal{F}_t$ is generated by a countable number of sets.
- for all $t \geq 0$, there exists a Polish space Ω_t, and a surjective map π_t from Ω to Ω_t, such that $\mathcal{F}_t$ is the σ-algebra of the inverse images, by π_t, of Borel sets in Ω_t, and such that for all $B \in \mathcal{F}_t$, $\omega \in \Omega$, $\pi_t(\omega) \in \pi_t(B)$ implies $\omega \in B$.
- if $(\omega_n)_{n\geq 0}$ is a sequence of elements of Ω, such that for all $N \geq 0$,

$$\bigcap_{n=0}^{N} A_n(\omega_n) \neq \emptyset,$$

[1] (P) stands for Parthasarathy since such conditions where introduced by him in [31].

where $A_n(\omega_n)$ is the intersection of the sets in $\mathcal{F}_n$ containing ω_n, then:

$$\bigcap_{n=0}^{\infty} A_n(\omega_n) \neq \emptyset.$$

A fundamental example of a filtered measurable space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ satisfying the property (P) can be constructed as follows: we take Ω to be equal to $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$, the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, or $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$, the space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}^d$ (for some $d \geq 1$), and for $t \geq 0$, we define $(\mathcal{F}_t)_{t\geq 0}$ as the natural filtration of the canonical process, and we set

$$\mathcal{F} := \bigvee_{t\geq 0} \mathcal{F}_t.$$

The combination of the property (P) and the natural conditions gives the following definition:

Definition 4.15. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. We say that it satisfies the property (NP) iff it is the natural enlargement of a filtered probability space $(\Omega, \mathcal{F}^0, (\mathcal{F}^0_t)_{t\geq 0}, \mathbb{P}^0)$ such that the filtered measurable space $(\Omega, \mathcal{F}^0, (\mathcal{F}^0_t)_{t\geq 0})$ enjoys property (P).

In [24], the following result, about extensions of probability measures, is proved (in a slightly more general form):

Proposition 4.16. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, satisfying property (NP). Then, the σ-algebra $\mathcal{F}$ is the σ-algebra generated by $(\mathcal{F}_t)_{t\geq 0}$, and for all coherent families of probability measures $(\mathbb{Q}_t)_{t\geq 0}$, such that $\mathbb{Q}_t$ is defined on $\mathcal{F}_t$, and is absolutely continuous with respect to the restriction of $\mathbb{P}$ to $\mathcal{F}_t$, there exists a unique probability measure $\mathbb{Q}$ on $\mathcal{F}$ which coincides with $\mathbb{Q}_t$ on $\mathcal{F}_t$ for all $t \geq 0$.*

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Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, 8057-Zürich, Switzerland & Department of banking and finance, Universität Zürich, Plattenstrasse 14, 8032-Zürich

E-mail address: ashkan.nikeghbali@math.uzh.ch

University of Technology Sydney, School of Finance & Economics and Department of Mathematical Sciences, PO Box 123, Broadway, NSW, 2007, Australia

E-mail address: eckhard.platen@uts.edu.au