

# RANDOM PERMUTATION MATRICES UNDER THE GENERALIZED EWENS MEASURE

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ABSTRACT. We consider a generalisation of the Ewens measure for the symmetric group, calculating moments of the characteristic polynomial and similar multiplicative statistics. In addition we study the asymptotic behaviour of linear statistics (such as the trace of a permutation matrix or of a wreath product) under this new measure.

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## 1. INTRODUCTION

The trace of a matrix is one of the most natural additive class functions associated to the spectra of a matrix. Traces of unitary matrices chosen randomly with Haar measure have been much studied, for example by Diaconis-Shahshahani [8], Diaconis-Evans [6] and Rains [24] using methods from representation theory; by Diaconis and Gamburd [7] using combinatorics; and using methods from mathematical physics by Haake et al. [11].

Another natural class function, this time multiplicative, is the characteristic polynomial, and the distribution of characteristic polynomials of random unitary matrices has been studied by many authors, including Keating-Snaith [17] and Hughes, Keating and O’Connell [13].

From the characteristic polynomial one can find the number of eigenvalues lying in a certain arc (since the underlying matrix is unitary, all the eigenvalues lie on the unit circle). The problem of studying the number of eigenvalues lying in an arc was studied by Rains [24] and Wieand [26], who found a very interesting correlation structure when multiple arcs were considered, and Hughes, Keating and O’Connell [13] who made the connection with characteristic polynomials.

One of the reasons for such an extensive study into random unitary matrices and their spectra is that the statistical distribution of the eigenvalues is expected to have the same behaviour as the zeros of the Riemann zeta function (see Montgomery [21] and Keating-Snaith [17]).

The statistics of the distribution of spectra of infinite subgroups of the unitary group, such as the symplectic and orthogonal groups are also expected to model families of other  $L$ -functions (Keating-Snaith [16]), and the distribution of traces for these groups have been studied, frequently in the same papers.

However, the statistics of the spectra of finite subgroups of the unitary group, such as the permutation group, is not so well studied, though there are many results known.

Wieand [25] studied the number of eigenvalues of a uniformly random permutation matrix lying in a fixed arc, and Hambly, Keevash, O’Connell and Stark [12] found corresponding results for the characteristic polynomial, making the same connection between the characteristic polynomial and the counting function of eigenvalues.

In all these cases, the permutation matrices were chosen with uniform measure and the results were similar to those found for the full unitary group with Haar measure. However, there were some significant differences primarily stemming from the fact that the full unitary group is rotation invariant, so the characteristic polynomial is isotropic. The group of permutation matrix is clearly not rotation invariant, and the distribution of the characteristic

polynomial depends weakly on the angle of the parameter. Most results require the angle to have the form  $2\pi\tau$  with  $\tau$  irrational and of finite type. (It is worth pointing out that those angles which are not of this type, have Hausdorff dimension zero). More recently Ben Arous and Dang ([2]) have extended some of the results of Wieand to more general measures together with new observations very specific to permutation matrices. In particular they prove that the fluctuations of smooth linear statistics (with bounded variance) of random permutation matrices sampled under the Ewens measure are asymptotically non-Gaussian but infinitely divisible.

A permutation matrix has the advantage that the eigenvalues are determined by the cycle-type  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  of the corresponding permutation. This allows to write down in many situations explicit expression for the studied object. For example, the characteristic polynomial of an  $N \times N$  permutation matrix  $M$  is

$$Z_N(x) = \det(I - xM) = \prod_{j=1}^N (1 - xe^{i\alpha_j}) = \prod_{m=1}^{\ell(\lambda)} (1 - x^{\lambda_m}) = \prod_{k=1}^N (1 - x^k)^{C_k} \quad (1.1)$$

where  $e^{i\alpha_1}, \dots, e^{i\alpha_N}$  are the eigenvalues of  $M$  and  $C_k$  is the number of cycles of length  $k$  (i.e. the number of  $j$  such that  $\lambda_j = k$ ).

Equation (1.1) has been used by Dehaye and Zeindler [5] to introduce multiplicative class functions (i.e. invariant under conjugation) associated to function  $f$ . These functions have the form

$$W^N(f)(x) = \prod_{k=1}^N f(x^k)^{C_k} \quad (1.2)$$

and generalise the characteristic polynomial, which is the case  $f(x) = (1-x)$ . This nice generalisation is not possible for the unitary group because the eigenvalues lack the structure of eigenvalues of permutation matrices that allows (1.1) to hold. The most natural analogue for unitary matrices is Heine's identity which connects Haar averages of multiplicative class functions with determinants of Toeplitz matrices.

Equation (1.1) can also be used to express linear statistics or traces of functions of random permutation matrices in terms of the cycle counts. More precisely, if we identify a random permutation matrix  $M$  with the permutation  $\sigma$  it represents, we have:

**Definition 1.1.** *Let  $F : S^1 \rightarrow \mathbb{C}$  be given. We then define the trace of  $F$  to be the function  $\text{Tr}(F) : \mathfrak{S}_N \rightarrow \mathbb{C}$  with*

$$\text{Tr}(F)(\sigma) := \sum_{k=1}^N F(\omega_k) \quad (1.3)$$

where  $(\omega_k)_{k=1}^N$  are the eigenvalues of  $\sigma$  with multiplicity.

Observe that when  $F(x) = x^d$ , we have  $\text{Tr}(F)(\sigma) = \text{Tr}(\sigma^d)$ , and this justifies the use of the terminology trace. The trace of a function is also referred to as a linear statistic on  $\mathfrak{S}_N$ .

Now it follows immediately from equation (1.1) that:

**Lemma 1.2.** *Let  $F : S^1 \rightarrow \mathbb{C}$  and  $\sigma \in \mathfrak{S}_N$  with cycle type  $\lambda$  be given. We then have*

$$\text{Tr}(F)(\sigma) = \sum_{k=1}^N k C_k \Delta_k(F) \quad (1.4)$$

with  $\Delta_k(F) := \frac{1}{k} \sum_{m=1}^k F(e^{2\pi i m/k})$ .

The reason why the expression in (1.1) and (1.2) are useful is that many things are known about the cycle counts  $C_k$ . For example, the cycle counts  $C_k$  converge weakly to independent Poisson random variables  $P_k$ , with mean  $1/k$ . Also useful in this context is the Feller-coupling. This allows in many situations to replace  $C_k$  by  $P_k$ . Several details on the cycle counts and the Feller coupling can be found in the book by Arratia, Barbour and Tavaré [1].

These results all concern the uniform measure, where each permutation has weight  $1/N!$ , or the Ewens measure, where the probability is proportional to the total number of cycles,  $\theta^{\ell(\lambda)}/(N!h_N)$ , with  $h_N$  the required normalisation constant (the case  $\theta = 1$  corresponding to the uniform measure). A common ingredient in all the above cited works is the use of the Feller coupling (details can again be found in the book by Arratia, Barbour and Tavaré [1]) and some improvements on the known bounds for the approximation given by this coupling (see [2], Section 4).

In recent years, there have been many works in random matrix theory aiming at understanding how much the spectral properties of random matrices depend on the probability distributions of its entries. Here a similar question translates into how are the linear and multiplicative (i.e. multiplicative class functions) statistics affected if one considers more general probability distributions than the Ewens measure on the symmetric group? The Ewens measure can be naturally generalised to a weighted probability measure which assigns to the permutation matrix  $M$  (i.e. to the associated permutation) the weight

$$\frac{1}{N!h_N} \prod_{k=1}^N \theta_k^{C_k}$$

where  $h_N$  is a normalisation constant. The Ewens measure corresponds to the special case where  $\theta_k = \theta$  is a constant. This measure has recently appeared in mathematical physics models (see e.g. [3] and [9]) and one has only recently started to gain insight into the cycle structures of such random permutations. One major obstacle with such measures is that there exists nothing such as the Feller coupling and therefore the classical probabilistic

arguments do not apply here. In a recent work, Nikeghbali and Zeindler [23] propose a new approach based on combinatorial arguments and singularity analysis of generating functions to obtain asymptotic expansions for the characteristic functions of the  $C_k$ 's as well as the total number of cycles, thus extending the classical limit theorems (and some distributional approximations) for the cycle structures of random permutations under the Ewens measure. In this paper we shall use the methods introduced in [23], namely some combinatorial lemmas, generating series and singularity analysis, to study linear and multiplicative statistics for random permutation matrices under the general weighted probability measure. In fact we shall consider the more general random matrix model obtained from the wreath product  $S^1 \wr \mathfrak{S}_N$  (see for instance [27]): this amounts to replacing the 1's in the permutation matrices by independent random variables taking values in the unit circle  $S^1$ . It should be noted that many groups closely related to  $\mathfrak{S}_N$  exhibit such matrices, for instance the Weyl group of  $SO(2N)$ .

More precisely this paper is organised as follows:

In Section 2 we fix some notation and terminology, recall some useful combinatorial lemmas together with some results of Hwang (and some slight extensions) on singularity analysis of generating functions. In particular we shall introduce two relevant classes of generating functions according to their behaviour near singularities on the circle of convergence. In this article, we shall state our theorems for random matrices under the generalised Ewens measures for which the generating series of  $(\theta_k)_{k \geq 1}$  is in one of these two classes.

In Section 3 we study the multiplicative class functions associated to a function  $f$  and obtain the asymptotic behaviour of the joint moments. In particular we extend earlier results of [5], [28] and of [12] on the characteristic polynomial of uniformly chosen random permutation matrices.

In Section 4 we focus both on the traces of powers and powers of traces which are classical statistics in random matrix theory. In fact we prove more generally that the fluctuations of the linear statistics for Laurent polynomials are asymptotically infinitely divisible (they converge in law to an infinite weighted sum of independent Poisson variables). We also establish the convergence of the integer moments of linear statistics for functions of bounded variation together with the rate of convergence.

In Section 5 we consider the more general model consisting of the wreath product  $S^1 \wr \mathfrak{S}_N$  and study the linear statistics for general functions  $F$  in (1.3). In such models, the 1's in the permutation matrix are replaced with  $(z_j)_{1 \leq j \leq N}$  which are i.i.d random variables taking their values on the unit circle  $S^1$ . In this framework, Lemma 1.2 can be naturally extended (see

Lemma 5.1) and the quantity

$$\Delta_k(F, z) := \frac{1}{k} \sum_{\omega^k=z} F(\omega). \quad (1.5)$$

naturally appears in our technical conditions. Under some conditions on rate of convergence to 0 of the  $L^1$ -norm of  $\Delta_k(F, z)$ , and some assumptions on the singularities of the generating series of  $(\theta_k)_{k \geq 1}$ , we are able to compute the asymptotics of the characteristic function of  $\text{Tr}(F)$  with a good error term. From these asymptotics we are able to compute the fluctuations of  $\text{Tr}(F)$ . We also translate our conditions in terms of the Fourier coefficients of  $F$ , where  $F$  has to be in some Sobolev space  $H^s$ .

In Section 6 we still work within the framework of the wreath product  $S^1 \wr \mathfrak{S}_N$  and consider the case where the variance of  $\text{Tr}(F)$  is diverging. This time we restrict ourselves to the Ewens measure since our methods do not seem to apply in this situation. We hence go back to probabilistic arguments (i.e. use the Feller coupling) to prove that under some technical conditions on  $F$ , the fluctuations are Gaussian. In fact we essentially adapt the proof by Ben Arous and Dang ([2]) to this more general situations. Nonetheless our theorem is also slightly more general in that it applies to a larger class of functions  $F$ .

#### ACKNOWLEDGEMENTS

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## 2. THE GENERALISED EWENS MEASURE, GENERATING SERIES AND SINGULARITY ANALYSIS

In this section we fix notation and we recall some facts about the symmetric group and generating functions, as well as the main results from singular analysis (we also provide some variants and extensions for the purpose of this paper). Our presentation follows closely [23].

**2.1. Some combinatorial lemmas and the generalised Ewens measure.** We present in this section some basic facts about  $\mathfrak{S}_N$  and then define the generalised Ewens measure. We give here only a very short overview and refer to [1] and [20] for more details.

**2.1.1. Conjugation classes and functions on  $\mathfrak{S}_N$ .** We first take a closer look at the conjugation classes of the symmetric group  $\mathfrak{S}_N$  (the group of all permutations of a set of  $N$  objects). We only need to consider the conjugation classes since all probability measures and functions considered in this paper are invariant under conjugation (i.e. they are class functions). It is

well-known that the conjugation classes of  $\mathfrak{S}_N$  can be parameterised with partitions of  $N$ .

**Definition 2.1.** A partition  $\lambda$  is a sequence of non-negative integers  $\lambda_1 \geq \lambda_2 \geq \dots$  eventually trailing to 0's, usually omitted. The size of the partition is  $|\lambda| := \sum_m \lambda_m$ . We call  $\lambda$  a partition of  $N$  if  $|\lambda| = N$ , and this will be denoted by  $\lambda \vdash N$ . The length of  $\lambda$  is the largest  $\ell$  such that  $\lambda_\ell \neq 0$ .

Let  $\sigma \in \mathfrak{S}_N$  be arbitrary. We can write  $\sigma = \sigma_1 \cdots \sigma_\ell$  with  $\sigma_m$ ,  $1 \leq m \leq \ell$ , disjoint cycles of length  $\lambda_m$ . Since disjoint cycles commute, we can assume that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell$ . We call the partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$  the *cycle-type* of  $\sigma$ . We write  $\mathcal{C}_\lambda$  for the set of all  $\sigma \in \mathfrak{S}_N$  with cycle type  $\lambda$ . One can now show that two elements  $\sigma, \tau \in \mathfrak{S}_N$  are conjugate if and only if  $\sigma$  and  $\tau$  have the same cycle-type and that  $\mathcal{C}_\lambda$  are the conjugation classes of  $\mathfrak{S}_N$ . Since this is well-known, we omit the proof and refer to [20] for more details.

**Definition 2.2.** Let  $\sigma \in \mathfrak{S}_N$  be given with cycle-type  $\lambda$ . The cycle numbers  $C_k$  are defined as

$$C_k = C_k(\sigma) := \# \{m : \lambda_m = k\} \quad (2.1)$$

and the total number of cycles  $T(\sigma)$  is

$$T(\sigma) := \sum_{k=1}^N C_k. \quad (2.2)$$

The functions  $C_k(\sigma)$  and  $T(\sigma)$  depend only on the cycle type of  $\sigma$  are thus class functions. Clearly  $T(\sigma)$  equals  $\ell(\lambda)$ , the length of the partition corresponding to  $\sigma$ .

All expectations in this paper have the form  $\frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} u(\sigma)$  for a class function  $u$ . Since  $u$  is constant on conjugation classes, it is more natural to sum over all conjugation classes. We thus need to know the size of each conjugation class.

**Lemma 2.3.** We have

$$|\mathcal{C}_\lambda| = \frac{|\mathfrak{S}_N|}{z_\lambda} \text{ with } z_\lambda := \prod_{k=1}^N k^{C_k} C_k! \quad (2.3)$$

with  $C_k$  defined in (2.1), and

$$\frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} u(\sigma) = \sum_{\lambda \vdash N} \frac{1}{z_\lambda} u(\mathcal{C}_\lambda) \quad (2.4)$$

for a class function  $u : \mathfrak{S}_N \rightarrow \mathbb{C}$ .

*Proof.* The first part can be found in [20] or in [4, chapter 39]. The second part follow immediately from the first part.  $\square$

2.1.2. *Definition of the generalised Ewens measures.* We now define the generalised Ewens measures.

**Definition 2.4.** Let  $\Theta = (\theta_k)_{k=1}^\infty$  be a sequence of strictly positive numbers. We define for  $\sigma \in \mathfrak{S}_N$  with cycle-type  $\lambda$

$$\mathbb{P}_\Theta[\sigma] := \frac{1}{h_N N!} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} = \frac{1}{h_N N!} \prod_{k=1}^N \theta_k^{C_k(\sigma)} \quad (2.5)$$

with  $h_N = h_N(\Theta)$  a normalisation constant and  $h_0 := 1$ .

The second equality in (2.5) follows immediately from the definition of  $C_k$  (Definition 2.2). The uniform measure and the Ewens measure are special cases, with  $\theta_k \equiv 1$  and  $\theta_k \equiv \theta$  a constant respectively.

We now introduce two generating functions closely related to  $\mathbb{P}_\Theta$ .

$$g_\Theta(t) := \sum_{k=1}^{\infty} \frac{\theta_k t^k}{k} \quad \text{and} \quad G_\Theta(t) := \exp\left(\sum_{k=1}^{\infty} \frac{\theta_k t^k}{k}\right) \quad (2.6)$$

At the moment,  $g_\Theta(t)$  and  $G_\Theta(t)$  are just formal power series, however we will see in Section 2.2 that

$$G_\Theta(t) = \sum_{N=0}^{\infty} h_N t^N. \quad (2.7)$$

where the  $h_N$  are given in Definition 2.4.

**2.2. Generating functions and singularity analysis.** The idea of generating functions is to encode information of a sequence into a formal power series.

**Definition 2.5.** Let  $(g_N)_{N \in \mathbb{N}}$  be a sequence of complex numbers and define the (ordinary) generating function of the sequence as the formal power series

$$G(t) = \sum_{N=0}^{\infty} g_N t^N. \quad (2.8)$$

We define  $[t^N][G]$  to be the coefficient of  $t^N$  in  $G(t)$ , i.e.  $[t^N][G] := g_N$ ,

The reason why generating functions are useful is that it is often possible to compute the generating function without knowing  $g_N$  explicitly.

The main tool in this paper to calculate generating functions is:

**Lemma 2.6.** Let  $(a_m)_{m \in \mathbb{N}}$  be a sequence of complex numbers. Then

$$\sum_{\lambda} \frac{1}{z_\lambda} \left( \prod_{m=1}^{\ell(\lambda)} a_{\lambda_m} \right) t^{|\lambda|} = \sum_{\lambda} \frac{1}{z_\lambda} \left( \prod_{k=1}^{\infty} (a_k t^k)^{C_k} \right) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} a_k t^k\right) \quad (2.9)$$

with the same  $z_\lambda$  as in Lemma 2.3.



If any one of the sums in (2.9) is absolutely convergent then so are the others.

*Proof.* The first equality follows immediately from the definition of  $C_k$ . The proof of the second equality in (2.9) can be found in [20] or can be directly verified using the definitions of  $z_\lambda$  and the exponential function. The last statement follows with dominated convergence.  $\square$

We now use this lemma to prove the identity given in (2.7). The constant  $h_N$  in (2.5) is chosen so that  $\mathbb{P}_\Theta[\sigma]$  is a probability measure on  $\mathfrak{S}_N$ . It thus follows that

$$h_N = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \prod_{k=1}^N \theta_k^{C_k} = \sum_{\lambda \vdash N} \frac{1}{z_\lambda} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m}. \quad (2.10)$$

It now follows with Lemma 2.6 that

$$\sum_{N=0}^{\infty} h_N t^N = \sum_{\lambda} \frac{1}{z_\lambda} t^{|\lambda|} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} = \exp\left(\sum_{k=1}^{\infty} \frac{\theta_k}{k} t^k\right) = G_\Theta(t), \quad (2.11)$$

which proves (2.7).

**Corollary 2.7.** *In the special case of the Ewens measure, when  $\theta_k$  is a constant  $\theta$ , say, we have*

$$G_\Theta(t) = \sum_{N=0}^{\infty} h_N t^N = (1-t)^{-\theta}. \quad (2.12)$$

From this it immediately follows that  $h_N = (-1)^N \binom{-\theta}{N} = \binom{N+\theta-1}{N}$ .

Given a generating function  $G(t)$ , a natural question is: what is the coefficient of  $t^N$  and what is the asymptotic behaviour of this coefficient as  $N \rightarrow \infty$ ? If  $G(t)$  is holomorphic near 0 then one can use Cauchy's integral formula to do this. But it turns out that it is often difficult to compute the integral exactly, but we will now see that one can nevertheless extract the asymptotic behaviour of the coefficient when  $G(t)$  has a special form.

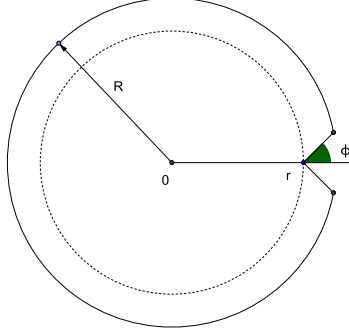
**Definition 2.8.** *Given  $R > r$  and  $0 < \phi < \frac{\pi}{2}$ , let*

$$\Delta_0 = \Delta_0(r, R, \phi) = \{z \in \mathbb{C} : |z| < R, z \neq r, |\arg(z-r)| > \phi\} \quad (2.13)$$

**Definition 2.9.** *Let  $r > 0$ ,  $\vartheta \geq 0$  and a complex constant  $K$  be given. We say that a function  $g(t)$  is in  $\mathcal{F}(r, \vartheta, K)$  if there exists  $R > r$  and  $0 < \phi < \frac{\pi}{2}$  such that  $g(t)$  is holomorphic in  $\Delta_0(r, R, \phi)$ , and*

$$g(t) = \vartheta \log\left(\frac{1}{1-t/r}\right) + K + O(t-r) \quad (2.14)$$

as  $t \rightarrow r$  with  $t \in \Delta_0(r, R, \phi)$ .

FIGURE 1. Illustration of  $\Delta_0(r, R, \phi)$ 

The following theorem, proven by Hwang in [14], gives the asymptotic behaviour of the coefficient of  $t^N$  for certain special generating functions

**Theorem 2.10** (Hwang [14]). *Let  $g(t) \in \mathcal{F}(r, \vartheta, K)$ , and let  $S(t)$  be holomorphic in  $|t| \leq r$ . Set  $G(t, w) = e^{wg(t)}S(t)$ , then*

$$[t^N][G(t, w)] = \frac{e^{Kw} N^{w\vartheta-1}}{r^N} \left( \frac{S(r)}{\Gamma(\vartheta w)} + O\left(\frac{1}{N}\right) \right) \quad (2.15)$$

uniformly for bounded complex  $w$ .

*Remark.* The idea of the proof is to take a suitable Hankel contour and to estimate the integral over each piece. The details can be found in [14, chapter 5].

*Remark.* One can compute lower order error terms if one has more terms in the expansion of  $g(t)$  near  $r$ .

As a first simple application of this result, we compute the asymptotic behaviour of  $h_N$  for the generalised Ewens measure if  $g_\Theta(t)$ , as defined in (2.6), is in  $\mathcal{F}(r, \vartheta, K)$ .

**Lemma 2.11.** *Let  $g_\Theta(t) \in \mathcal{F}(r, \vartheta, K)$ . We then have*

$$h_N = \frac{e^{Kw} N^{\vartheta-1}}{r^N} \left( \frac{1}{\Gamma(\vartheta)} + O\left(\frac{1}{N}\right) \right) \quad (2.16)$$

*Proof.* We have proven in (2.11) that  $\sum_{N=0}^{\infty} h_N t^N = \exp(g_\Theta(t))$ . We thus can apply Theorem 2.10 with  $g(t) = g_\Theta(t)$ ,  $w = 1$  and  $S(t) \equiv 1$ .  $\square$

*Remark.* For the Ewens measure, when  $\Theta$  is the constant sequence  $(\theta)_{k=1}^{\infty}$ , we have  $g_\Theta(t) \in \mathcal{F}(1, \theta, 0)$  and thus  $h_N = \frac{N^{\theta-1}}{\Gamma(\theta)} (1 + O(\frac{1}{N}))$ . However, in this special case, one can do much more since  $h_N$  is known to equal  $\binom{N+\theta-1}{N}$ .

Essentially, one can think of Hwang's result as concerning functions with a solitary singularity at  $t = r$ . In Section 3.3 we will need a version of this theorem with multiple singularities that we now state.

**Definition 2.12.** Let  $\boldsymbol{\xi} = (\xi_i)_{i=1}^d$  with  $\xi_i \neq \xi_j$  for  $i \neq j$ , and with  $|\xi_i| = r$  (so the  $\xi_i$  are distinct points lying on the circle of radius  $r$ ). Let  $R > r$  and let  $0 < \phi < \frac{\pi}{2}$ , then set

$$\Delta_d(r, R, \phi, \boldsymbol{\xi}) := \bigcap_{i=1}^d \{z \in \mathbb{C} : |z| < R, z \neq \xi_i, |\arg(z - \xi_i) - \arg(\xi_i)| > \phi\} \quad (2.17)$$

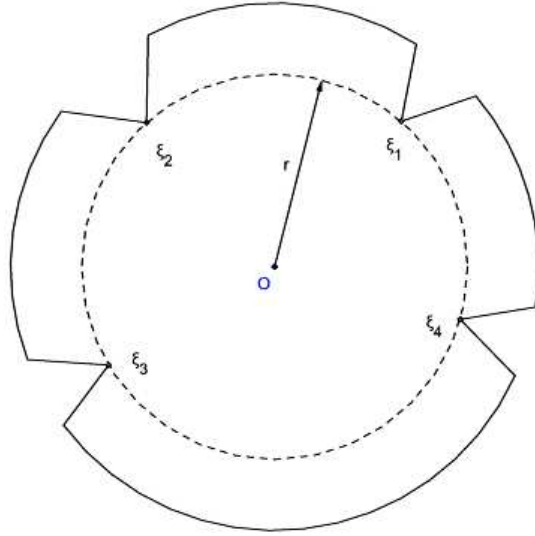


FIGURE 2. Illustration of the domain  $\Delta_d(r, R, \phi, \boldsymbol{\xi})$

**Definition 2.13.** Let  $\boldsymbol{\vartheta} = (\vartheta_i)_{i=1}^d$  and  $\mathbf{K} = (K_i)_{i=1}^d$  be two sequences of complex numbers, and let  $r > 0$ . We say a function  $g(t)$  is in  $\mathcal{F}(r, \boldsymbol{\vartheta}, \mathbf{K})$  if there exists  $R > r$  and  $0 < \phi < \frac{\pi}{2}$  such that  $g(t)$  is holomorphic in  $\Delta_d(r, R, \phi, \boldsymbol{\xi})$ , and for each  $i = 1, \dots, d$ ,

$$g(t) = \vartheta_i \log \left( \frac{1}{1 - t/\xi_i} \right) + K_i + O(t - \xi_i) \quad (2.18)$$

as  $t \rightarrow \xi_i$  with  $t \in \Delta_d(r, R, \phi, \boldsymbol{\xi})$ .

Theorem 2.10 generalises to

**Theorem 2.14.** Let  $g \in \mathcal{F}(r, \boldsymbol{\vartheta}, \mathbf{K})$ , and let  $S(t)$  be holomorphic in  $t$  for  $|t| \leq r$ . Set  $G(t, w) = e^{wg(t)} S(t)$ . We have

$$[t^N] [G(t, w)] = \sum_{i=1}^d \frac{e^{K_i w} N^{w\vartheta_i - 1}}{\xi_i^N} \left( \frac{S(\xi_i)}{\Gamma(\vartheta_i w)} + O\left(\frac{1}{N}\right) \right) \quad (2.19)$$

uniformly for bounded  $w$ .

*Sketch of the proof.* The proof is combination of the proof of a multiple singularities theorem in [10, Section VI.5] and the proof of Theorem 2.10. More precisely, we apply Cauchy's integral formula with the curve  $C$  illustrated in Figure 3, where the radius  $R$  of the great circle is chosen fix with  $R > r$ , while the radii of the small circles are  $1/n$ .

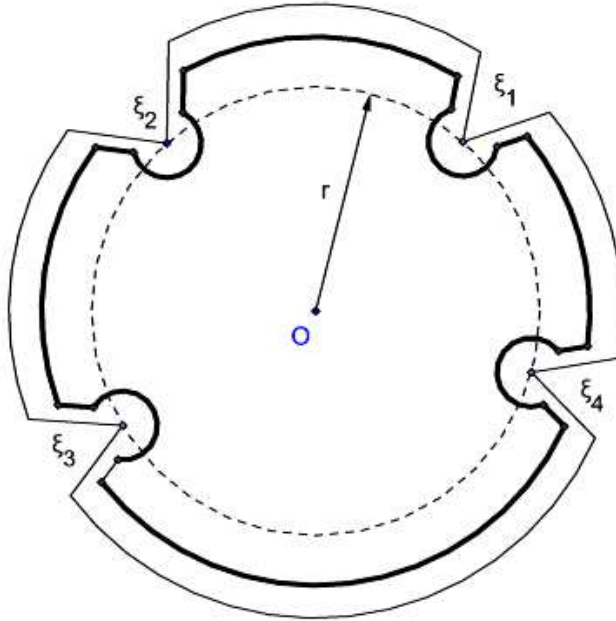


FIGURE 3. The curve  $C$

A straightforward computation then shows that the integral over this curve gives (2.19) and that the error terms are uniform for bounded  $w$ .  $\square$

In practice the computation of the asymptotic behaviour near the singularity is often very difficult, and it is not easy to prove whether a function  $g(t)$  is in  $\mathcal{F}(r, \vartheta, K)$  or not. An alternate approach is to combine singularity analysis with more elementary methods. The idea is to write  $G = G_1 G_2$  in a way that we can apply singularity analysis on  $G_1$  and can estimate the growth rate of  $[t^N][G_2]$ . One then can compute the coefficient  $[t^N][G]$  directly and apply elementary analysis on it. This method is called the convolution method.

**Definition 2.15.** Let  $\vartheta \geq 0, r > 0, 0 < \gamma \leq 1$  be given. We say  $g(t)$  is in  $e\mathcal{F}(r, \vartheta, \gamma)$  if  $g(t)$  is holomorphic in  $|t| < r$  with

$$g(t) = \vartheta \log \left( \frac{1}{1 - t/r} \right) + g_0(t) \quad (2.20)$$

and

$$[t^N][g_0] = O(r^{-N}N^{-1-\gamma}) \quad (2.21)$$

as  $N \rightarrow \infty$ .

**Theorem 2.16** (Hwang [15]). *Let  $g(t) \in e\mathcal{F}(r, \vartheta, \gamma)$ , and let  $S(t)$  be holomorphic in  $|t| \leq r$ . Set  $G(t, w) = e^{wg(t)}S(t)$ , then*

$$[t^N][G(t, w)] = \frac{e^{wg_0(r)}N^{w\vartheta-1}}{r^N} \frac{S(r)}{\Gamma(\vartheta w)} + R_N(w) \quad (2.22)$$

with

$$R_N(w) = \begin{cases} O\left(\frac{N^{\vartheta \operatorname{Re}(w)-1-\gamma \log(N)}}{r^N}\right) & \text{if } \operatorname{Re}(w) \geq 0 \\ O\left(\frac{N^{-1-\gamma}}{r^N}\right) & \text{if } \operatorname{Re}(w) < 0 \end{cases} \quad (2.23)$$

uniformly for bounded  $w$ .

This theorem is more general than Theorem 2.10, but the error terms are worse. As in lemma 2.11, we can compute the asymptotic behaviour of  $h_N$  in the case of the generalised Ewens measures when  $g_\Theta(t) \in e\mathcal{F}(r, \vartheta, \gamma)$ .

**Lemma 2.17.** *Assume that  $g_\Theta(t) \in e\mathcal{F}(r, \vartheta, \gamma)$ . We then have*

$$h_N = \frac{e^{g_0(r)}N^{\vartheta-1}}{r^N\Gamma(\vartheta)} + O\left(\frac{N^{\vartheta-1-\gamma \log N}}{r^N}\right) \quad (2.24)$$

### 3. MOMENTS OF MULTIPLICATIVE CLASS FUNCTIONS

We extend in this section the results of [12], [28] and [5] to the generalised Ewens measure  $\mathbb{P}_\Theta$ . More precisely, we compute the asymptotic behaviour of the moments of the characteristic polynomial  $Z_N(x)$  and of multiplicative class functions  $W^N(P)$  with respect to  $\mathbb{P}_\Theta$  using the methods of generating functions and singularity analysis introduced in the previous section.

**3.1. Multiplicative class functions.** It is well known that  $\mathfrak{S}_N$  can be identified with the group of permutation matrices via

$$\sigma \mapsto (\delta_{i, \sigma(j)})_{1 \leq i, j \leq N}. \quad (3.1)$$

It is easy to see that this map is an injective group homomorphism. We thus do not distinguish between  $\mathfrak{S}_N$  and the group of permutation matrices and use for both the notation  $\mathfrak{S}_N$ . It will be always clear from the context if it is necessary to consider  $\sigma \in \mathfrak{S}_N$  as a matrix.

**Definition 3.1.** *Let  $x \in \mathbb{C}$  and  $\sigma \in \mathfrak{S}_N$ . The characteristic polynomial of  $\sigma$  is*

$$Z_N(x) = Z_N(x)(\sigma) := \det(I_N - x\sigma). \quad (3.2)$$

It is a standard fact that the characteristic polynomial can be written in terms of the cycle type of  $\sigma$ .

**Lemma 3.2.** *Let  $\sigma \in \mathfrak{S}_N$  be given with cycle type  $\lambda$ ; then*

$$Z_N(x) = \prod_{m=1}^{\ell(\lambda)} (1 - x^{\lambda_m}), \quad (3.3)$$

with  $\ell(\lambda)$  the length of the partition  $\lambda$ , which is the same as the number of cycles  $T(\sigma)$ .

*Proof.* Since any permutation matrix is conjugate to a block matrix with each block corresponding to one of the cycles, and the characteristic polynomial factors over the blocks, it is sufficient to prove this result in the simple case of a one-cycle permutation, where it follows from a simple calculation. More explicit details can be found for instance in [28, Chapter 2.2].  $\square$

Equation (3.3) shows that the spectrum of permutation matrix is uniquely determined by the cycle type. We use this as motivation to define multiplicative class functions on  $\mathfrak{S}_N$ .

**Definition 3.3.** *Let  $P(x)$  be a polynomial in  $x$ . We then define the multiplicative class function associated to the polynomial  $P$  as*

$$W^N(P)(x) = W^N(P)(x)(\sigma) := \prod_{m=1}^{\ell(\lambda)} P(x^{\lambda_m}). \quad (3.4)$$

For brevity, we simply call this a multiplicative class function.

It follows immediately that the characteristic polynomial is the multiplicative class function associated to the polynomial  $P(x) = 1 - x$ . The main difference between  $Z_N(x)$  and  $W^N(P)$  is that  $W^N(P)$  is independent of the interpretation of  $\mathfrak{S}_N$  as matrices.

We now wish to obtain the asymptotic behaviour of the moments

$$\mathbb{E}_{\Theta} \left[ (W^N(P_1)(x_1))^{k_1} (W^N(P_2)(x_2))^{k_2} \right]$$

for  $x_1 \neq x_2$ . The easiest way to achieve this is to extend the definition of  $W^N(P)$ .

**Definition 3.4.** *Let  $P(x_1, x_2)$  be a polynomial in the two variables  $x_1, x_2$ . For  $\sigma \in \mathfrak{S}_N$  with cycle type  $\lambda$ , we set*

$$W^N(P)(x_1, x_2) = W^N(P)(x_1, x_2)(\sigma) := \prod_{m=1}^{\ell(\lambda)} P(x_1^{\lambda_m}, x_2^{\lambda_m}). \quad (3.5)$$

A simple computation using the definitions above shows

$$\begin{aligned} (W^N(P_1)(x_1))^{k_1} (W^N(P_2)(x_2))^{k_2} &= W^N(P_1^{k_1})(x_1) W^N(P_2^{k_2})(x_2) \\ &= W^N(P)(x_1, x_2) \end{aligned}$$

with  $P(x_1, x_2) = P_1^{k_1}(x_1) P_2^{k_2}(x_2)$ .

This shows that it is enough to consider  $\mathbb{E}_\Theta [W^N(P)(x_1, x_2)]$ .

*Remark.* There is no restriction to the number of variables. All arguments used here work also for more than two variables. We restrict ourself to two variables since this is enough to illustrate the general case.

**3.2. Generating functions for  $W^N$ .** We compute in this section the generating functions of the moments of multiplicative class functions.

**Theorem 3.5.** *Let  $P$  be a complex polynomial with*

$$P(x_1, x_2) = \sum_{k_1, k_2=0}^{\infty} b_{k_1, k_2} x_1^{k_1} x_2^{k_2}. \quad (3.6)$$

*We have as formal power series*

$$\sum_{N=0}^{\infty} t^N h_N \mathbb{E}_\Theta [W^N(P)(x_1, x_2)] = \prod_{k_1, k_2=0}^{\infty} (G_\Theta(x_1^{k_1} x_2^{k_2} t))^{b_{k_1, k_2}} \quad (3.7)$$

where  $G_\Theta(t)^b = \exp(b \cdot g_\Theta(t))$ , and  $G_\Theta$  and  $g_\Theta$  are defined in (2.6).

*Proof.* From (3.5) and (2.5) we have

$$\mathbb{E}_\Theta [W^N(P)(x_1, x_2)] = \frac{1}{h_N N!} \sum_{\sigma \in \mathfrak{S}_N} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} P(x_1^{\lambda_m}, x_2^{\lambda_m}) \quad (3.8)$$

and since  $W^N(P)$  is a class function, we may use (2.4) to obtain

$$\mathbb{E}_\Theta [W^N(P)(x_1, x_2)] = \frac{1}{h_N} \sum_{\lambda \vdash N} \frac{1}{z_\lambda} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} P(x_1^{\lambda_m}, x_2^{\lambda_m}) \quad (3.9)$$

We now compute the generating function of  $h_N \mathbb{E}_\Theta [W^N(P)(x_1, x_2)]$  with the help of Lemma 2.6

$$\begin{aligned} \sum_{N=0}^{\infty} t^N h_N \mathbb{E} [W^N(P)(x_1, x_2)] &= \sum_{N=0}^{\infty} t^N \sum_{\lambda \vdash N} \frac{1}{z_\lambda} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} P(x_1^{\lambda_m}, x_2^{\lambda_m}) \\ &= \sum_{\lambda} \frac{1}{z_\lambda} t^{|\lambda|} \prod_{m=1}^{\ell(\lambda)} (\theta_{\lambda_m} P(x_1^{\lambda_m}, x_2^{\lambda_m})) \\ &= \exp \left( \sum_{m=1}^{\infty} \frac{\theta_m}{m} t^m P(x_1^m, x_2^m) \right) \\ &= \exp \left( \sum_{k_1, k_2=0}^{\infty} b_{k_1, k_2} \sum_{m=1}^{\infty} \frac{\theta_m}{m} (x_1^{k_1} x_2^{k_2} t)^m \right) \end{aligned}$$

Note that

$$\sum_{m=1}^{\infty} \frac{\theta_m}{m} (x_1^{k_1} x_2^{k_2} t)^m = g_{\Theta} \left( x_1^{k_1} x_2^{k_2} t \right) \quad (3.10)$$

where  $g_{\Theta}$  is defined in (2.6), and hence

$$\sum_{N=0}^{\infty} t^N h_N \mathbb{E} [W^N(P)(x_1, x_2)] = \prod_{k_1, k_2=0}^{\infty} (G_{\Theta}(x_1^{k_1} x_2^{k_2} t))^{b_{k_1, k_2}}. \quad (3.11)$$

and this proves (3.7), as required.  $\square$

*Remark.* The requirement for  $P$  to be a polynomial is there to ensure absolute convergence, and clearly this condition can be considerably weakened, see [5, Sections 5,6].

As an immediate consequence we have:

**Corollary 3.6.** *Let  $s_1, s_2 \in \mathbb{N}$  be given. Then*

$$\sum_{N=0}^{\infty} t^N h_N \mathbb{E}_{\Theta} [Z_N(x_1)^{s_1} Z_N(x_2)^{s_2}] = \prod_{k_1=0}^{s_1} \prod_{k_2=0}^{s_2} (G_{\Theta}(x_1^{k_1} x_2^{k_2} t))^{(-1)^{k_1+k_2} \binom{s_1}{k_1} \binom{s_2}{k_2}}. \quad (3.12)$$

*Proof.* We have

$$\begin{aligned} Z_N(x_1)^{s_1} Z_N(x_2)^{s_2} &= (W^N(1-x_1))^{s_1} (W^N(1-x_2))^{s_2} \\ &= W^N((1-x_1)^{s_1} (1-x_2)^{s_2}). \end{aligned}$$

The corollary now follows immediately by calculating the Taylor expansion of  $(1-x_1)^{s_1} (1-x_2)^{s_2}$  near 0.  $\square$

**3.3. Asymptotic behaviour of the moments.** Combining the generating functions in Theorem 3.5 with the singularity analysis developed in Section 2.2, we compute the asymptotic behaviour of  $\mathbb{E}_{\Theta} [W^N(P)]$  as  $N \rightarrow \infty$ .

We have to distinguish between the cases  $|x_i| < 1$  and  $|x_i| = 1$ . We consider here only  $g_{\Theta}(t) \in \mathcal{F}(r, \vartheta, K)$ . The results and computations for  $g_{\Theta}(t) \in e\mathcal{F}(r, \vartheta, \gamma)$  are similar, with only minor differences in the error terms.

We first look at the asymptotic behaviour inside the unit disc. We have

**Theorem 3.7.** *Let  $P$  be as in Theorem 3.5, and let  $x_1, x_2 \in \mathbb{C}$  be given with  $\max\{|x_1|, |x_2|\} < 1$ . Assume that  $g_{\Theta}(t) \in \mathcal{F}(r, \vartheta, K)$ , then*

$$\mathbb{E} [W^N(P)] = N^{\vartheta(b_{0,0}-1)} e^{K(b_{0,0}-1)} \left( E_1 + O\left(\frac{1}{N}\right) \right), \quad (3.13)$$

with

$$E_1 = E_1(x_1, x_2) = \frac{\Gamma(\vartheta)}{\Gamma(\vartheta b_{0,0})} \prod_{(k_1, k_2) \neq (0,0)} (G_{\Theta}(r x_1^{k_1} x_2^{k_2}))^{b_{k_1, k_2}}. \quad (3.14)$$



*Proof.* Set

$$S(t) := \prod_{(k_1, k_2) \neq (0,0)} (G_\Theta(x_1^{k_1} x_2^{k_2} t))^{b_{k_1, k_2}}. \quad (3.15)$$

Since  $P$  is polynomial, the product is finite and there is no problem with convergence. The domain of holomorphicity of  $S$  is thus the intersection of the domains of holomorphicity of each factor. This shows that the function  $S(t)$  is holomorphic for  $|t| < r + \epsilon$  for an  $\epsilon > 0$  since  $\max\{|x_1|, |x_2|\} < 1$  and  $G_\Theta(t)$  is holomorphic for  $|t| < r$ .

Separating the  $k_1 = k_2 = 0$  term in (3.7) from the rest, we can write the generating function as

$$\sum_{N=0}^{\infty} t^N h_N \mathbb{E}_\Theta [W^N(P)(x_1, x_2)] = \exp(b_{0,0} \cdot g_\Theta(t)) S(t). \quad (3.16)$$

Applying Theorem 2.10, we get

$$h_N \mathbb{E} [W^N(P)] = N^{\vartheta b_{0,0} - 1} e^{K b_{0,0}} \frac{1}{r^N} \left( \frac{S(r)}{\Gamma(\vartheta b_{0,0})} + O\left(\frac{1}{N}\right) \right).$$

Comparing  $S(r)$  with  $E_1$ , and using Lemma 2.11 to find the asymptotic behaviour of  $h_N$  proves the theorem.  $\square$

As a special case we get the asymptotic behaviour of  $\mathbb{E}_\Theta [Z_N^{s_1}(x_1) Z_N^{s_2}(x_2)]$  with respect to  $\mathbb{P}_\Theta$  inside the unit disc.

**Corollary 3.8.** *Let  $x_1, x_2 \in \mathbb{C}$  be given with  $\max\{|x_1|, |x_2|\} < 1$  and let  $s_1, s_2 \in \mathbb{N}$ . We then have*

$$\mathbb{E}_\Theta [Z_N^{s_1}(x_1) Z_N^{s_2}(x_2)] = \prod_{(k_1, k_2) \neq (0,0)} (G_\Theta(r x_1^{k_1} x_2^{k_2}))^{(-1)^{k_1+k_2} \binom{s_1}{k_1} \binom{s_2}{k_2}} + O\left(\frac{1}{N}\right) \quad (3.17)$$

*Proof.* This follows immediately from the fact that  $Z_N(x) = W^N(1-x)(x)$  and that  $(1-x_1)^{s_1} (1-x_2)^{s_2}$  evaluated at  $x_1 = x_2 = 0$  is 1.  $\square$

In particular, for the uniform measure ( $\theta_k \equiv 1$  for all  $k$ ) Corollary 2.7 gives  $G_\Theta(t) = (1-t)^{-1}$  in which case we have

$$\mathbb{E} [Z_N^{s_1}(x_1) Z_N^{s_2}(x_2)] = \prod_{(k_1, k_2) \neq (0,0)} (1 - x_1^{k_1} x_2^{k_2})^{-\binom{s_1}{k_1} \binom{s_2}{k_2} (-1)^{k_1+k_2}} + O\left(\frac{1}{N}\right). \quad (3.18)$$

This shows that Corollary 3.8 agrees with [28, Theorem 2.13] in the uniform case.

The behaviour on the unit disc is more complicated. The reason is that the generating function can have (for fixed  $x_1, x_2$ ) more than one singularity on the circle of radius  $r$ . Another point that makes this case more laborious is

the requirement to check whether some of the singularities of the factors on the RHS of (3.7) are equal. For simplicity, we assume that all singularities are distinct.

**Theorem 3.9.** *Let  $P$  be as in Theorem 3.5, and let  $x_1, x_2 \in \mathbb{C}$  be given with  $|x_1| = |x_2| = 1$  and  $x_1^{k_1} x_2^{k_2} \neq 1$  for all  $(k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ . Assume that  $g_\Theta(t) \in \mathcal{F}(r, \vartheta, K)$ , then*

$$\begin{aligned} \mathbb{E}_\Theta [W^N(P)] &= \sum_{\substack{k_1, k_2 \\ b_{k_1, k_2} \neq 0}} E_2(k_1, k_2) N^{\vartheta(b_{k_1, k_2} - 1)} x_1^{Nk_1} x_2^{Nk_2} \left( \frac{\Gamma(\vartheta)}{\Gamma(\vartheta b_{k_1, k_2})} + O\left(\frac{1}{N}\right) \right) \end{aligned}$$

with

$$E_2(k_1, k_2) = e^{K(b_{k_1, k_2} - 1)} \prod_{(m_1, m_2) \neq (k_1, k_2)} (G_\Theta(r x_1^{m_1 - k_1} x_2^{m_2 - k_2}))^{b_{m_1, m_2}} \quad (3.19)$$

*Proof.* We define

$$F(t) = \sum_{k_1, k_2} b_{k_1, k_2} g_\Theta(x_1^{k_1} x_2^{k_2} t). \quad (3.20)$$

By (3.7) we see that  $\exp(F(t))$  is the generating function of  $h_N \mathbb{E}_\Theta [W^N(P)]$ . We first take a look at the domain of holomorphicity of  $F(t)$ . We have by assumption that  $g_\Theta(t)$  is holomorphic in  $\Delta_0(r, R, \phi)$  for a  $R > r$  and  $0 < \phi < \frac{\pi}{2}$ . This shows that  $g_\Theta(x_1^{k_1} x_2^{k_2} t)$  is holomorphic for  $t$  in the domain  $\Delta_1(r, R, \phi, r x_1^{-k_1} x_2^{-k_2})$  with  $\Delta_1$  as in Definition 2.12, and that  $F$  is holomorphic in

$$D := \bigcap_{\substack{k_1, k_2 \\ b_{k_1, k_2} \neq 0}} \Delta_1(r, R, \phi, r x_1^{-k_1} x_2^{-k_2}) = \Delta_d(r, R, \phi, \boldsymbol{\xi}) \quad (3.21)$$

where  $\boldsymbol{\xi}$  the finite sequence of all  $r x_1^{-k_1} x_2^{-k_2}$  with  $b_{k_1, k_2} \neq 0$  (in any order). Notice that this is only a finite intersection since  $P$  is a polynomial. Since  $|x_1| = |x_2| = 1$ , we see that  $D$  has a shape as in figure 2 and that  $F$  has singularities at  $t = r x_1^{-k_1} x_2^{-k_2}$ . We thus may use Theorem 2.14 and therefore need to take a look at the behaviour of  $F$  near each singularity. We assumed that  $x_1^{k_1} x_2^{k_2} \neq x_1^{m_1} x_2^{m_2}$  for  $(m_1, m_2) \neq (k_1, k_2)$ , which implies that the singularities are distinct, and thus near the point  $r x_1^{-k_1} x_2^{-k_2}$ ,  $F(t)$  has the expansion

$$\begin{aligned} F(t) &= b_{k_1, k_2} \vartheta \log \left( \frac{1}{1 - t x_1^{k_1} x_2^{k_2} / r} \right) + b_{k_1, k_2} K \\ &+ \sum_{(m_1, m_2) \neq (k_1, k_2)} b_{m_1, m_2} g_\Theta(r x_1^{m_1 - k_1} x_2^{m_2 - k_2}) + O\left(t - r x_1^{-k_1} x_2^{-k_2}\right) \end{aligned} \quad (3.22)$$

for  $t \rightarrow rx_1^{-k_1}x_2^{-k_2}$ . This shows that we can apply Theorem 2.14. Combining this together with Lemma 2.11 proves the theorem.  $\square$

*Remark.* For simplicity we have assumed that all the singularities are distinct. The modification required to cope with the case when  $x_1^{k_1}x_2^{k_2} = x_1^{m_1}x_2^{m_2}$  for some  $(m_1, m_2) \neq (k_1, k_2)$  would appear in (3.22), but technically there is no restriction. Such a situation, with all the details written out explicitly, appears in [5].

To illustrate this theorem, we will calculate the autocorrelation of two characteristic polynomials at distinct points  $x_1, x_2$  on the unit circle subject to  $x_1^{k_1} \neq x_2^{k_2}$  for all  $\{k_1, k_2\} \neq \{0, 0\}$ .

The four coefficients of  $Z_N(x_1)Z_N(x_2)$  are easy to calculate, being  $b_{0,0} = b_{1,1} = 1$  and  $b_{1,0} = b_{0,1} = -1$  and this enables an immediate simplification to occur by observing that only the terms with  $b_{k_1, k_2}$  maximal contribute; the others are of lower order, in this case being  $O(N^{-2\vartheta})$ .

Substituting these values into the theorem we have

$$\mathbb{E}_\Theta [Z_N(x_1)Z_N(x_2)] = E_2(0, 0) + E_2(1, 1)x_1^N x_2^N + O\left(\frac{1}{N}\right) + O\left(\frac{1}{N^{2\vartheta}}\right) \quad (3.23)$$

with

$$E_2(0, 0) = \frac{G_\Theta(rx_1x_2)}{G_\Theta(rx_1)G_\Theta(rx_2)} \quad (3.24)$$

and

$$E_2(1, 1) = \frac{G_\Theta(rx_1^{-1}x_2^{-1})}{G_\Theta(rx_1^{-1})G_\Theta(rx_2^{-1})} \quad (3.25)$$

## 4. TRACES

**4.1. Traces of permutation matrices.** We consider in this section the asymptotic behaviour of traces of permutation matrices. Powers of traces and traces of powers have received much attention in the random matrix literature (see for instance [6], [7],[8]). More specifically, we first look at  $\text{Tr}(\sigma^d)$  for fixed  $d \in \mathbb{Z}$ . Since the embedding of  $\mathfrak{S}_N$  into the unitary group in (3.1) is a group homomorphism, we can interpret  $\sigma^d$  as  $d$ -fold matrix multiplication and as the matrix corresponding to  $\underbrace{\sigma \circ \cdots \circ \sigma}_{d \text{ times}}$ .

We first recall a well known explicit expression for  $\text{Tr}(\sigma^d)$  that we shortly prove for completeness.

**Lemma 4.1.** *We have for  $d \in \mathbb{Z}$*

$$\text{Tr}(\sigma^d) = \sum_{k=1}^N \mathbb{1}_{k|d} k C_k(\sigma), \quad (4.1)$$

*Proof.* The matrix corresponding to  $\sigma^d$  has the form  $(\delta_{i,\sigma^d(j)})$ . We thus have

$$\mathrm{Tr}(\sigma^d) = \sum_{i=1}^N \delta_{i,\sigma^d(i)} = \# \{i : \sigma^d(i) = i\}. \quad (4.2)$$

Therefore  $\mathrm{Tr}(\sigma^d)$  is the number of 1-cycles of  $\sigma^d$ . A simple computation now shows that the number of 1-cycles of  $\sigma^d$  is indeed  $\sum_{k=1}^N \mathbb{1}_{k|d} k C_k(\sigma)$ .  $\square$

Using this expression and the method of generating functions developed in Section 2.2, we prove a weak convergence result for  $\mathrm{Tr}(\sigma^d)$ .

**Theorem 4.2.** *Let  $d \in \mathbb{N}$  be given. We then have*

$$\sum_{N=0}^{\infty} h_N \mathbb{E}_{\Theta} \left[ e^{is \mathrm{Tr}(\sigma^d)} \right] t^N = \exp \left( \sum_{k|d} \frac{\theta_k}{k} (e^{isk} - 1) t^k \right) G_{\Theta}(t). \quad (4.3)$$

If  $g_{\theta}$  is of class  $\mathcal{F}(\vartheta, r, K)$ , then

$$\mathbb{E}_{\Theta} \left[ e^{is \mathrm{Tr}(\sigma^d)} \right] = \exp \left( \sum_{k|d} \frac{\theta_k}{k} (e^{isk} - 1) r^k \right) + O \left( \frac{1}{N} \right). \quad (4.4)$$

If  $g_{\theta}$  is of class  $e\mathcal{F}(\vartheta, r, \gamma)$ , then

$$\mathbb{E}_{\Theta} \left[ e^{is \mathrm{Tr}(\sigma^d)} \right] = \exp \left( \sum_{k|d} \frac{\theta_k}{k} (e^{isk} - 1) r^k \right) + O \left( \frac{\log(N)}{N^{\gamma}} \right) \quad (4.5)$$

*Proof.* Applying Lemma 4.1, and evaluating the expectation explicitly in terms of partitions using Lemma 2.3, we have

$$\sum_{N=0}^{\infty} h_N \mathbb{E}_{\Theta} \left[ e^{is \mathrm{Tr}(\sigma^d)} \right] t^N = \sum_{\lambda} \frac{1}{z_{\lambda}} \prod_{k=1}^{\infty} \left( \theta_k e^{isk \mathbb{1}_{k|d}} t^k \right)^{C_k} \quad (4.6)$$

The cycle index theorem (Lemma 2.6) yields that this equals

$$\exp \left( \sum_{k=1}^{\infty} \frac{1}{k} \theta_k e^{isk \mathbb{1}_{k|d}} t^k \right) = \exp \left( \sum_{k|d} \frac{\theta_k}{k} (e^{isk} - 1) t^k \right) G_{\Theta}(t) \quad (4.7)$$

where  $G_{\Theta}(t)$  is given in (2.6). This proves equation (4.3).

Applying Theorem 2.10 to this yields equation (4.4), and Theorem 2.16 yields equation (4.5), as required.  $\square$

*Remark.* An alternative way to prove Theorem 4.2 is to use Theorem 3.1 in [23], which computes the generating function of  $h_N \mathbb{E}_{\Theta} \left[ \exp \left( \sum_{k=1}^b is_k C_k \right) \right]$  and its asymptotic behaviour for  $g_{\Theta}(t) \in \mathcal{F}(\vartheta, r, K)$  and  $g_{\Theta}(t) \in e\mathcal{F}(\vartheta, r, \gamma)$ .

We obtain as an immediate corollary:

**Corollary 4.3.** *Let  $d \in \mathbb{Z}$  be fixed and assume that  $g_\Theta$  is in  $\mathcal{F}(\vartheta, r, K)$  or  $e\mathcal{F}(\vartheta, r, \gamma)$ . Then*

$$\mathrm{Tr}(\sigma^d) \xrightarrow{d} \sum_{k|d} kP_k \text{ as } N \rightarrow \infty \quad (4.8)$$

where  $P_k$  are independent Poisson distributed random variables with  $\mathbb{E}[P_k] = \frac{\theta_k}{k} r^k$ .

**4.2. Traces of functions.** Recall from the introduction that if  $M$  is the permutation matrix representing the permutation  $\sigma$ , then for a function  $F : S^1 \rightarrow \mathbb{C}$ , we defined the trace of  $F$  to be the function  $\mathrm{Tr}(F) : \mathfrak{S}_N \rightarrow \mathbb{C}$  with

$$\mathrm{Tr}(F)(\sigma) := \sum_{k=1}^N F(\omega_k) \quad (4.9)$$

where  $(\omega_k)_{k=1}^N$  are the eigenvalues of  $M$  or  $\sigma$  with multiplicity. Moreover,

$$\mathrm{Tr}(F)(\sigma) = \sum_{k=1}^N kC_k \Delta_k(F) \quad (4.10)$$

with

$$\Delta_k(F) := \frac{1}{k} \sum_{m=1}^k F\left(e^{2\pi im/k}\right) \quad (4.11)$$

(compare with Lemma 1.2).

The asymptotic behaviour of  $\mathrm{Tr}(F)$  is not so easy to compute for an arbitrary function defined on the unit circle. This problem will be dealt with more carefully in Sections 5 and 6. However, if  $F$  is a Laurent polynomial, we can use the same method as for  $\mathrm{Tr}(\sigma^d)$ .

**Theorem 4.4.** *Let*

$$F(x) = \sum_d b_d x^d \quad (4.12)$$

be a Laurent polynomial. If  $g_\Theta \in \mathcal{F}(\vartheta, r, K)$  or  $g_\Theta \in e\mathcal{F}(\vartheta, r, \gamma)$ , then

$$\mathrm{Tr}(F)(\sigma) - Nb_0 \xrightarrow{d} \sum_{\substack{d=-\infty \\ d \neq 0}}^{\infty} b_d \sum_{\substack{k \geq 1 \\ k|d}} kP_k \text{ as } N \rightarrow \infty \quad (4.13)$$

where  $P_k$  are independent Poisson distributed random variables with  $\mathbb{E}[P_k] = \frac{\theta_k}{k} r^k$ .

*Proof.* Due to the linearity of  $\text{Tr}(F)$ , we may assume the constant term,  $b_0$ , is zero. As before we apply the cycle index theorem to obtain

$$\sum_{N=0}^{\infty} h_N \mathbb{E}_{\Theta} \left[ e^{is \text{Tr}(F)(\sigma)} \right] t^N = \sum_{\lambda} \frac{1}{z_{\lambda}} \prod_{k=1}^{\infty} \left( \theta_k e^{isk \Delta_k(F)} t^k \right)^{C_k} \quad (4.14)$$

$$= \exp \left( \sum_{k=1}^{\infty} \frac{\theta_k}{k} e^{isk \Delta_k(F)} t^k \right) \quad (4.15)$$

and since  $\Delta_k(x^d) = \mathbb{1}_{k|d}$  and is linear, this equals

$$\begin{aligned} & \exp \left( \sum_{k=1}^{\infty} \frac{\theta_k}{k} \exp \left( is \sum_{d \neq 0} b_d k \mathbb{1}_{k|d} \right) t^k \right) \\ &= \exp \left( \sum_{k=1}^{\infty} \frac{\theta_k}{k} \left( \prod_{\substack{d \neq 0 \\ k|d}} e^{is b_d k} - 1 \right) t^k \right) G_{\Theta}(t) \end{aligned} \quad (4.16)$$

Note the first factor is entire, so Theorem 2.10 (for the case of  $\mathcal{F}(r, \vartheta, K)$ ) and Theorem 2.16 (for the case of  $e\mathcal{F}(r, \vartheta, \gamma)$ ) yields

$$\mathbb{E}_{\Theta} \left[ e^{is \text{Tr}(F)(\sigma)} \right] \rightarrow \exp \left( \sum_{k=1}^{\infty} \frac{\theta_k}{k} \left( \prod_{\substack{d \neq 0 \\ k|d}} e^{is b_d k} - 1 \right) r^k \right) \quad (4.17)$$

The RHS is the characteristic function of the RHS in (4.13). This completes the proof.  $\square$

**Theorem 4.5.** *Let  $F : S^1 \rightarrow \mathbb{C}$  be of bounded variation, and  $d \in \mathbb{N}$  be given. Then*

$$\frac{1}{N^d} \mathbb{E}_{\Theta} \left[ \left( \text{Tr}(F)(\sigma) \right)^d \right] = \left( \int_{S^1} F(\varphi) d\varphi \right)^d + O \left( \frac{\mathbb{E}_{\Theta} [T(\sigma)]}{N} \right). \quad (4.18)$$

where  $T(\sigma)$  is the total number of cycles of  $\sigma$ , and  $d\varphi$  the uniform measure on  $S^1$ .

Moreover if  $g_{\Theta}(t) \in \mathcal{F}(\vartheta, r, K)$  or  $g_{\Theta}(t) \in e\mathcal{F}(\vartheta, r, \gamma)$  then  $\mathbb{E}_{\Theta} [T(\sigma)] \sim \vartheta \log(N)$ , and thus we have a quick convergence of the moments.

*Proof.* Since  $F$  is of bounded variation we can apply Koksma's inequality ([19, Theorem 5.1]) to see that

$$\left| \frac{1}{k} \sum_{m=1}^k F(e^{2\pi i m/k}) - \int_{S^1} F(\varphi) d\varphi \right| \leq 2D_k V(F) \quad (4.19)$$

with  $V(F)$  the variation of  $F$  and  $D_k$  the discrepancy of the sequence  $(e^{2\pi im/k})_{m=1}^k$ . But the discrepancy  $D_k$  is dominated by  $1/k$ . We thus have

$$\Delta_k(F) = \frac{1}{k} \sum_{m=1}^k F(e^{2\pi im/k}) = \int_{S^1} F(\varphi) d\varphi + O\left(\frac{1}{k}\right) \quad (4.20)$$

We now combine (4.10) and (4.20) and get

$$\mathrm{Tr}(F)(\sigma) = \sum_{k=1}^N C_k \left( k \int_{S^1} F(\varphi) d\varphi + O(1) \right) = N \int_{S^1} F(\varphi) d\varphi + O(T(\sigma)), \quad (4.21)$$

where we have used that  $\sum_{k=1}^N kC_k = N$  and  $\sum_{k=1}^N C_k = T(\sigma)$ . Notice that (4.21) is independent of any probability measure on  $\mathfrak{S}_N$ . Using the binomial theorem and the fact that  $0 < T(\sigma)/N \leq 1$  for all  $\sigma$ , we get

$$\frac{1}{N^d} (\mathrm{Tr}(F)(\sigma))^d = \left( \int_{S^1} F(\varphi) d\varphi \right)^d + O_{F,d} \left( \frac{T(\sigma)}{N} \right) \quad (4.22)$$

where the constant implicit in the big-O is independent of  $\sigma$  and  $N$ . We apply  $\mathbb{E}_\Theta[\cdot]$  on both sides, and this proves the first part of the theorem.

The last statement follows from [23, Theorem 4.2], where it is shown that if  $g_\Theta(t) \in \mathcal{F}(\vartheta, r, K)$  or  $g_\Theta(t) \in e\mathcal{F}(\vartheta, r, \gamma)$  then  $\mathbb{E}_\Theta[T(\sigma)] \sim \vartheta \log(N)$ .  $\square$

*Remark.* In fact, for many probability distributions on  $\mathfrak{S}_N$ ,  $\mathbb{E}_\Theta[T(\sigma)] = o(N)$ . The only way for this not to be true is for  $\sigma$  to frequently have only small cycles, which will occur if  $\Theta = (\theta_k)_{k=1}^\infty$  is a sequence tending to zero very rapidly.

## 5. WREATH PRODUCT, TRACES AND THE GENERALISED EWENS MEASURE

We consider in this section the traces of the wreath product  $S^1 \wr \mathfrak{S}_N$  (see for instance [27]). More precisely, we consider random matrices of the form

$$M(\sigma, z_1, \dots, z_N) := \mathrm{diag}(z_1, \dots, z_N) \cdot \sigma$$

where  $\sigma$  is a random permutation of  $\mathfrak{S}_N$ , and  $(z_j)_{j \geq 1}$  is a sequence of i.i.d. random variables with values in  $S^1$  (the complex unit circle), independent of  $\sigma$ . Many groups closely related to  $\mathfrak{S}_N$  give similar matrices, for instance the Weyl group of  $SO(2N)$ .

The trace of a function  $F$  is then extended in the obvious way by

$$\mathrm{Tr}(F) = \mathrm{Tr}(F, z)(\sigma) := \sum_{k=1}^N F(\omega_k) \quad (5.1)$$

where  $(\omega_k)_{k=1}^N$  are the  $N$  eigenvalues of  $M(\sigma, z)$ .

We now give a more explicit expression of  $\mathrm{Tr}(F)$ .

**Lemma 5.1.**

$$\mathrm{Tr}(F) = \sum_{k=1}^N \sum_{m=1}^{C_k} k \Delta_k(F, Z_{k,m}) \quad (5.2)$$

where  $(Z_{k,m})_{k,m \geq 1}$  is a sequence of independent random variables which is independent of  $(C_k)_{k \geq 1}$  (the sequence of cycle numbers of  $\sigma$ ), with  $Z_{k,m}$  equal in distribution to  $\prod_{j=1}^k z_j$ , and

$$\Delta_k(F, z) := \frac{1}{k} \sum_{\omega^k=z} F(\omega). \quad (5.3)$$

*Proof.* The characteristic polynomial of  $M(\sigma, z_1, \dots, z_N)$  with  $\sigma \in \mathfrak{S}_N$  with cycle type  $\lambda$ , is given by

$$\det(1 - xM(\sigma, z_1, \dots, z_N)) = \prod_{k=1}^N \prod_{m=1}^{C_k} \left( 1 - x^k \prod_{j=1}^k z_j^{k,m} \right) \quad (5.4)$$

where the sequence  $(z_j^{k,m})_{k,m,j}$  is the same sequence as  $(z_j)_{j=1}^N$ , but with a different numeration and ordering. (Note that this is why it is crucial that the  $(z_j)$  are i.i.d.). The proof of (5.4) is similar to the proof of (3.3) and we thus omit the details. The lemma now follows immediately from (5.4).  $\square$

As in the last section, we can compute the generating function of  $\mathrm{Tr}(F)$ .

**Lemma 5.2.** *We define*

$$\chi_k(s) := \mathbb{E} \left[ e^{isk \Delta_k(F, Z_{k,m})} \right]. \quad (5.5)$$

*We then have*

$$\sum_{N=0}^{\infty} h_N \mathbb{E}_{\Theta} \left[ e^{is \mathrm{Tr}(F)} \right] t^N = \exp \left( \sum_{k=1}^{\infty} \frac{\theta_k}{k} \chi_k(s) t^k \right) \quad (5.6)$$

*Remark.* Note that  $\chi_k(s)$  is independent of  $m$  since  $Z_{k,1} \stackrel{d}{=} Z_{k,2} \stackrel{d}{=} \dots \stackrel{d}{=} Z_{k,m}$ .

*Proof.* We compute  $\mathbb{E} [\exp(is \mathrm{Tr}(F))]$ . For this we use the independence of  $C_k$  and  $\Delta_k$  to obtain

$$\begin{aligned} \mathbb{E} \left[ e^{is \mathrm{Tr}(F)} \right] &= \mathbb{E} \left[ \prod_{k=1}^N \prod_{m=1}^{C_k} e^{isk \Delta_k(F, Z_{k,m})} \right] = \mathbb{E} \left[ \prod_{k=1}^N \prod_{m=1}^{C_k} \chi_k(s) \right] \\ &= \mathbb{E} \left[ \prod_{k=1}^N (\chi_k(s))^{C_k} \right]. \end{aligned} \quad (5.7)$$

The theorem now follows immediately from Lemma 2.6.  $\square$



**Definition 5.3.** *Let*

$$g_{\text{Tr}(F)}(t) := \sum_{k=1}^{\infty} \frac{\theta_k}{k} \chi_k(s) t^k \quad (5.8)$$

**Theorem 5.4.** *Assume  $\mathbb{E}[|\Delta_k(F, Z_{k,1})|] = O(k^{-1-\delta})$  for some  $0 < \delta \leq 1$ , and assume  $g_{\Theta}(t)$  is in  $e\mathcal{F}(r, \vartheta, \gamma)$ , where  $e\mathcal{F}(r, \vartheta, \gamma)$  is given in Definition 2.15. Then  $g_{\text{Tr}(F)}(t)$  is in  $e\mathcal{F}(r, \vartheta, \min\{\gamma, \delta\})$  and*

$$\mathbb{E}_{\Theta} \left[ e^{is \text{Tr}(F)} \right] = \exp \left( \sum_{k=1}^{\infty} \frac{\theta_k}{k} (\chi_k(s) - 1) r^k \right) + O(N^{-\min\{\gamma, \delta\}} \log(N)) \quad (5.9)$$

and as  $N$  tends to infinity,  $\text{Tr}(F)$  converges in law to the random variable

$$Y := \sum_{k=1}^{\infty} \sum_{m=1}^{P_k} k \Delta_k(F, Z_{k,m}), \quad (5.10)$$

where  $(P_k)_{k \geq 1}$  is a sequence of independent Poisson random variables, independent of  $(Z_{k,m})_{k,m \geq 1}$ , and such that  $P_k$  has parameter  $\theta_k r^k / k$ . Here, the series defining  $Y$  is a.s. absolutely convergent.

*Remark.* By linearity of trace, if  $F$  is Riemann integrable one can always subtract a suitable constant to make  $\int_{S^1} F(\varphi) d\varphi = 0$ , which ensures  $\Delta_k(F, z) \rightarrow 0$  as  $k \rightarrow \infty$ .

*Remark.* One should compare equation (5.10) with equation (5.2). The replacement of the cycle counts  $C_k$  with  $P_k$  is indicative of Feller-coupling for the generalised Ewens measure.

*Proof.* We have

$$g_{\text{Tr}(F)}(t) = \sum_{k=1}^{\infty} \frac{\theta_k}{k} \chi_k(s) t^k = g_{\Theta}(t) + \sum_{k=1}^{\infty} \frac{\theta_k}{k} (\chi_k(s) - 1) t^k \quad (5.11)$$

We now have

$$|\chi_k(s) - 1| \leq \mathbb{E} \left[ \left| e^{isk \Delta_k(F, Z_{k,1})} - 1 \right| \right] \leq \mathbb{E} [(k |\Delta_k(F, Z_{k,1})|)] = O(k^{-\delta}). \quad (5.12)$$

On the other hand we have  $\theta_k = O(r^{-k})$ . This follows immediately from the fact that  $g_{\Theta}(t)$  is in  $e\mathcal{F}(r, \vartheta, \gamma)$ . We thus have

$$\frac{\theta_k}{k} (\chi_k(s) - 1) = O(r^{-k} k^{-1-\delta}). \quad (5.13)$$

This shows that  $g_{\text{Tr}(F)}(t) \in e\mathcal{F}(r, \vartheta, \min\{\gamma, \delta\})$ .

Since  $g_\Theta(t) \in e\mathcal{F}(r, \vartheta, \gamma)$ , we can write  $g_\Theta(t) = \vartheta \log\left(\frac{1}{1-t/r}\right) + g_0(t)$  with  $g_0(r) < \infty$ . Thus

$$g_{\text{Tr}(F)}(t) = \vartheta \log\left(\frac{1}{1-t/r}\right) + g_0(t) + \sum_{k=1}^{\infty} \frac{\theta_k}{k} (\chi_k(s) - 1)t^k. \quad (5.14)$$

We get with Theorem 2.16 that

$$h_N \mathbb{E}_\Theta \left[ e^{is \text{Tr}(F)} \right] = \frac{N^{\vartheta-1}}{r^N \Gamma(\vartheta)} \exp\left( g_0(r) + \sum_{k=1}^{\infty} \frac{\theta_k}{k} (\chi_k(s) - 1)r^k \right) + R_N \quad (5.15)$$

with

$$R_N = O\left( \frac{N^{\vartheta-1-\min\{\gamma, \delta\}} \log(N)}{r^N} \right). \quad (5.16)$$

Dividing by  $h_N$  proves equation (5.9).

Using the characteristic function of  $\text{Tr}(F)$ , we can deduce its convergence in law to  $Y$ . The absolute convergence of the series in (5.10) comes from

$$\begin{aligned} \mathbb{E} \left[ \sum_{k=1}^{\infty} \sum_{m=1}^{P_k} k |\Delta_k(F, Z_{k,m})| \right] &= \sum_{k=1}^{\infty} k \mathbb{E}[P_k] \mathbb{E}[|\Delta_k(F, Z_{k,1})|] \\ &= \sum_{k \geq 1} \theta_k r^k O(k^{-1-\delta}) \\ &< \infty, \end{aligned} \quad (5.17)$$

since  $\theta_k = O(r^{-k})$ . Now, for  $s \in \mathbb{R}$  and  $k \geq 1$ ,

$$\begin{aligned} \mathbb{E} \left[ e^{isk \sum_{m=1}^{P_k} \Delta_k(F, Z_{k,m})} \right] &= \mathbb{E} \left[ \left( \mathbb{E} \left[ e^{isk \Delta_k(F, Z_{k,1})} \right] \right)^{P_k} \right] = \mathbb{E} \left[ (\chi_k(s))^{P_k} \right] \\ &= \exp\left( \frac{\theta_k}{k} (\chi_k(s) - 1)r^k \right), \end{aligned} \quad (5.18)$$

Hence by absolute convergence,

$$\mathbb{E} [e^{isY}] = \exp\left( \sum_{k=1}^{\infty} \frac{\theta_k}{k} (\chi_k(s) - 1)r^k \right), \quad (5.19)$$

and thus by equation (5.9),  $\mathbb{E}_\Theta [e^{is \text{Tr}(F)}] \rightarrow \mathbb{E} [e^{isY}]$  as  $N \rightarrow \infty$ , and thus  $\text{Tr}(F)$  converges in law to  $Y$ .  $\square$

With a more direct approach one can prove the convergence in law of  $\text{Tr}(F)$  to  $Y$  (albeit without a rate of convergence) under slightly weaker conditions.

**Theorem 5.5.** *Assume that  $g_\Theta(t)$  is in  $e\mathcal{F}(r, \vartheta, \gamma)$  and that*

$$\sum_{k=1}^{\infty} k^{(1-\vartheta)_+} \mathbb{E} [|\Delta_k(F, Z_{k,1})|] < \infty. \quad (5.20)$$

*Then  $\text{Tr}(F)$  converges in law to  $Y$ , where  $Y$  is given by (5.10).*

*Proof.* Under these conditions, the absolute convergence of the series defining  $Y$  is checked as follows:

$$\mathbb{E} \left[ \sum_{k=1}^{\infty} \sum_{m=1}^{P_k} k |\Delta_k(F, Z_{k,m})| \right] = \sum_{k=1}^{\infty} k \mathbb{E} [P_k] \mathbb{E} [|\Delta_k(F, Z_{k,1})|] \quad (5.21)$$

$$= \sum_{k=1}^{\infty} \theta_k r^k \mathbb{E} [|\Delta_k(F, Z_{k,1})|] \quad (5.22)$$

$$= O \left( \sum_{k=1}^{\infty} \mathbb{E} [|\Delta_k(F, Z_{k,1})|] \right) \quad (5.23)$$

and this converges by assumption.

In [23, Corollary 3.1.1] it is proven that for all fixed  $b \geq 1$ ,  $(C_1, C_2, \dots, C_b)$  tends in law to  $(P_1, P_2, \dots, P_b)$  when the dimension  $N$  goes to infinity.

Now let

$$\text{Tr}_b(F) := \sum_{k=1}^b \sum_{m=1}^{C_k} k \Delta_k(F, Z_{k,m}) \quad (5.24)$$

and

$$Y_b := \sum_{k=1}^b \sum_{m=1}^{P_k} k \Delta_k(F, Z_{k,m}) \quad (5.25)$$

The same argumentation as in Theorem 5.4 gives

$$\sum_{N=0}^{\infty} h_N \mathbb{E}_\Theta \left[ e^{is \text{Tr}_b(F)} \right] t^N = \exp \left( \sum_{k=1}^b \frac{\theta_k}{k} (\chi_k(s) - 1) t^k \right) e^{g_\Theta(t)} \quad (5.26)$$

from which follows (again by the same reasoning as in Theorem 5.4)

$$\mathbb{E}_\Theta \left[ e^{is \text{Tr}_b(F)} \right] \rightarrow \mathbb{E} \left[ e^{is Y_b} \right] \quad (5.27)$$

as  $N \rightarrow \infty$ . Here  $b$  is fixed but arbitrary, so this convergence implies (by using the inequality  $|e^{ix} - e^{iy}| \leq |x - y|$ ) that

$$\begin{aligned} \limsup_{N \rightarrow \infty} \left| \mathbb{E}_{\Theta} \left[ e^{is \operatorname{Tr}(F)} \right] - \mathbb{E} \left[ e^{isY} \right] \right| & \\ & \leq |s| \limsup_{N \rightarrow \infty} \sum_{k=b+1}^{\infty} k \mathbb{E} [ |\Delta_k(F, Z_{k,1})| ] \mathbb{E}' [(C_k + P_k)] \\ & \leq |s| \sum_{k=b+1}^{\infty} k \mathbb{E} [ |\Delta_k(F, Z_{k,1})| ] H_k \end{aligned} \quad (5.28)$$

where  $\mathbb{E}'$  is the expectation over the product measure of  $\mathbb{P}_{\Theta}$  and the measures occurring from  $(P_k)_{k \geq 1}$ , and where

$$H_k = \mathbb{E}[P_k] + \sup_{N \geq 1} \mathbb{E}_{\Theta} [C_k]. \quad (5.29)$$

Therefore the theorem is proven if we show that

$$\sum_{k=1}^{\infty} k H_k \mathbb{E} [ |\Delta_k(F, Z_{k,1})| ] < \infty. \quad (5.30)$$

Ercolani and Ueltschi [9, Proposition 2.1(c)] show that

$$\mathbb{E}_{\Theta} [C_k] = \begin{cases} \frac{\theta_k}{k} \frac{h_{N-k}}{h_N} & \text{if } k \leq N \\ 0 & \text{if } k > N \end{cases} \quad (5.31)$$

By Lemma 2.17, we have, for some  $A > 0$  and for  $N$  going to infinity,

$$h_N \sim A(N+1)^{\vartheta-1}/r^N, \quad (5.32)$$

so

$$\frac{h_{N-k}}{h_N} = O \left( r^k \left( 1 - \frac{k}{N+1} \right)^{\vartheta-1} \right). \quad (5.33)$$

Now, for  $k$  fixed,

$$\max_{N \geq k} \left( 1 - \frac{k}{N+1} \right)^{\vartheta-1} = \begin{cases} 1 & \text{if } \vartheta \geq 1 \\ (k+1)^{1-\vartheta} & \text{if } \vartheta < 1 \text{ (attained at } N = k). \end{cases} \quad (5.34)$$

Since  $g_{\Theta}(t) \in e\mathcal{F}(r, \vartheta, \gamma)$  we have  $\theta_k r^k = O(1)$ , and so we deduce that

$$\sup_{N \geq 1} \mathbb{E}_{\Theta} [C_k] = O \left( \frac{\theta_k}{k} r^k (k+1)^{(1-\vartheta)_+} \right) = O(k^{-1+(1-\vartheta)_+}). \quad (5.35)$$

Finally, since

$$\mathbb{E} [P_k] = \theta_k r^k / k = O(1/k) \quad (5.36)$$

we have

$$H_k = O(k^{-1+(1-\vartheta)_+}), \quad (5.37)$$

and so

$$\sum_{k=1}^{\infty} k H_k \mathbb{E} [|\Delta_k(F, Z_{k,1})|] = O\left(\sum_{k=1}^{\infty} k^{(1-\vartheta)_+} \mathbb{E} [|\Delta_k(F, Z_{k,1})|]\right) < \infty \quad (5.38)$$

as required.  $\square$

*Remark.* In the case when  $z_i$  are all equal to 1 almost surely, then  $\Delta_k(F, Z_{k,m}) = \Delta_k(F)$  as given in (4.11), and we are back in the case of permutation matrices. Thus these two theorems fulfill the promise made in Section 4.

The following result gives sufficient conditions, expressed only in terms of the Fourier coefficients of  $F$ , under which we can be assured the conditions of Theorems 5.4 and 5.5 apply.

**Theorem 5.6.** *Let us suppose that  $F$  is continuous and for  $m \in \mathbb{Z}$ , let us define the  $m$ -th Fourier coefficient of  $F$  by*

$$c_m(F) := \frac{1}{2\pi} \int_0^{2\pi} e^{-imx} F(e^{ix}) dx. \quad (5.39)$$

*We assume that the mean value of  $F$  vanishes, i.e.  $c_0(F) = 0$ . If for some  $\delta \in (0, 1]$ ,  $c_m(F) = O(|m|^{-1-\delta})$  when  $|m|$  goes to infinity then*

$$\mathbb{E} [|\Delta_k(F, Z_{k,1})|] = O(k^{-1-\delta}). \quad (5.40)$$

*If there exists  $s > (1 - \vartheta)_+$  such that*

$$\sum_{m \in \mathbb{Z}} |m|^s |c_m(F)| < \infty \quad (5.41)$$

*then*

$$\sum_{k=1}^{\infty} k^{(1-\vartheta)_+} \mathbb{E} [|\Delta_k(F, Z_{k,1})|] < \infty. \quad (5.42)$$

*Remark.* If the assumptions of Theorem 5.6 are satisfied, except that  $c_0(F) = 0$ , then one can still apply the result to the function  $F - c_0(F)$ , and deduce, from Theorem 5.4 or Theorem 5.5, that  $\text{Tr}(F) - N c_0(F)$  converges in law to  $Y$ , where  $Y$  is given by (5.10).

*Proof.* Since  $F$  is continuous, one has, for all  $x \in [0, 2\pi)$ ,

$$F(e^{ix}) = \lim_{n \rightarrow \infty} \sum_{m \in \mathbb{Z}} \frac{(n - |m|)_+}{n} c_m(F) e^{imx}, \quad (5.43)$$

by using the Fejér kernel. Now, by assumption,

$$\sum_{m \in \mathbb{Z}} |c_m(F)| < \infty, \quad (5.44)$$

and hence, by dominated convergence,

$$F(e^{ix}) = \sum_{m \in \mathbb{Z}} c_m(F) e^{imx}, \quad (5.45)$$

where the series is absolutely convergent. Since  $c_0(F) = 0$ , one deduces that for all  $k \geq 1$  and  $x \in [0, 2\pi)$ ,

$$\begin{aligned} \Delta_k(F, e^{ix}) &= \frac{1}{k} \sum_{j=0}^{k-1} F\left(e^{i(x+2j\pi)/k}\right) \\ &= \frac{1}{k} \sum_{m \in \mathbb{Z} \setminus \{0\}} c_m(F) \left( \sum_{j=0}^{k-1} e^{im(x+2j\pi)/k} \right) \\ &= \sum_{\substack{m \in \mathbb{Z} \setminus \{0\}, \\ k|m}} c_m(F) e^{imx/k}. \end{aligned}$$

If  $F$  satisfies the first assumption, that  $c_m(F) = O(|m|^{-1-\delta})$ , then

$$\sup_{x \in [0, 2\pi)} |\Delta_k(F, e^{ix})| \leq \sum_{\substack{m \in \mathbb{Z} \setminus \{0\}, \\ k|m}} |c_m(F)| = O\left( \sum_{\substack{m \in \mathbb{Z} \setminus \{0\}, \\ k|m}} |m|^{-1-\delta} \right) \quad (5.46)$$

$$= O(k^{-1-\delta}), \quad (5.47)$$

for  $k$  going to infinity, which clearly implies  $\mathbb{E}[|\Delta_k(F, Z_{k,1})|] = O(k^{-1-\delta})$ .

If  $F$  satisfies the second assumption, one has

$$\sum_{k=1}^{\infty} k^{(1-\vartheta)_+} \sup_{x \in [0, 2\pi)} |\Delta_k(F, e^{ix})| \leq \sum_{k=1}^{\infty} k^{(1-\vartheta)_+} \sum_{\substack{m \in \mathbb{Z} \setminus \{0\}, \\ k|m}} |c_m(F)| \quad (5.48)$$

$$\leq \sum_{k=1}^{\infty} \sum_{\substack{m \in \mathbb{Z} \setminus \{0\}, \\ k|m}} |m|^{(1-\vartheta)_+} |c_m(F)| \quad (5.49)$$

$$\leq \sum_{m \in \mathbb{Z} \setminus \{0\}} |c_m(F)| |m|^{(1-\vartheta)_+ \tau(|m|)}, \quad (5.50)$$

where  $\tau(|m|)$  denotes the number of divisors of  $|m|$ . Since  $\tau(|m|) = O(|m|^\epsilon)$  for all  $\epsilon > 0$ , one deduces that

$$\sum_{k \geq 1} k^{(1-\vartheta)_+} \mathbb{E}[|\Delta_k(F, Z_{k,1})|] = O\left( \sum_{m \in \mathbb{Z} \setminus \{0\}} |c_m(F)| |m|^s \right) < \infty. \quad (5.51)$$

This completes the proof of the theorem.  $\square$

One deduces the following:

**Corollary 5.7.** *Let  $F$  be a continuous function from  $S^1$  to  $\mathbb{C}$ , contained in a Sobolev space  $H^s$  for some  $s > 1/2 + (1-\vartheta)_+$ . Then, the second condition of Theorem 5.6 is fulfilled, and thus also the conditions of Theorem 5.5.*

*Proof.* By Cauchy-Schwarz inequality, one has, for any  $\alpha \in ((1 - \vartheta)_+, s - 1/2)$ ,

$$\sum_{m \in \mathbb{Z} \setminus \{0\}} |m|^\alpha |c_m(F)| \leq \left( \sum_{m \in \mathbb{Z} \setminus \{0\}} |m|^{2s} |c_m(F)|^2 \right)^{1/2} \left( \sum_{m \in \mathbb{Z} \setminus \{0\}} |m|^{2(\alpha-s)} \right)^{1/2} \quad (5.52)$$

which is finite since  $F \in H^s$  and  $2(\alpha - s) < -1$ .  $\square$

*Remark.* Note that it is not always obvious to estimate directly the Fourier coefficients of a function  $F$ , however standard results from Fourier analysis concerning the differentiability of  $F$  yield sufficient bounds on the decay of the Fourier coefficients of  $F$  for the conditions of Theorem 5.6 to be checked. See, for example, [18, Chapter 9].

## 6. DIVERGING VARIANCE FOR THE CLASSICAL EWENS MEASURE

In the previous two sections, we've been considering the convergence of  $\text{Tr}(F)$  to some limit for random permutation matrices (and their generalisation to wreath products), where the underlying probability space is the generalised Ewens measure. The conditions we've used have all implied that the variance of  $\text{Tr}(F)$  stays bounded as  $N \rightarrow \infty$ .

A recent paper by Ben Arous and Dang [2] dealing with  $\text{Tr}(F)$  for random permutation matrices in the special case of the classical Ewens measure (which is when  $\theta_k = \theta$ , a constant), demonstrates a dichotomy between converging and diverging variance for  $\text{Tr}(F)$  in the classical Ewens distribution. In the former case they also show convergence of  $\text{Tr}(F)$  to an explicit finite limit, and in the latter case they prove the following central limit theorem:

**Theorem 6.1** (Ben Arous and Dang). *Assume that*

$$V_N := \theta \sum_{k=1}^N k \Delta_k(F)^2 \quad (6.1)$$

*tends to infinity as  $N \rightarrow \infty$  and*

$$\max_{1 \leq k \leq N} k |\Delta_k(F)| = o\left(\sqrt{V_N}\right) \quad (6.2)$$

*then*

$$\frac{\text{Tr}(F) - \mathbb{E}[\text{Tr}(F)]}{\sqrt{V_N}} \xrightarrow{d} \mathcal{N}(0, 1). \quad (6.3)$$

In the generalised Ewens measure, we are currently unable to apply the function theoretic methods to prove weak convergence results in the case of diverging variance. However, for the classical Ewens measure we are able to prove a similar central limit theorem the wreath product, with slightly extended application, in the sense that condition (6.2) can be weakened from a sup-norm to a  $p$ -norm.

**Theorem 6.2.** *Assume that*

$$V_N := \theta \sum_{k=1}^N k \mathbb{E} \left[ (\Delta_k(F, Z_{k,1}))^2 \right] \quad (6.4)$$

*tends to infinity as  $N \rightarrow \infty$ . Assume further that there exists a  $p > \max\{\frac{1}{\theta}, 2\}$  such that*

$$\sum_{k=1}^N k^{p-1} \mathbb{E} [|\Delta_k(F, Z_{k,1})|^p] = o\left(V_N^{p/2}\right) \quad (6.5)$$

*with  $\Delta_k(F, z) = \frac{1}{k} \sum_{\omega^k=z} F(\omega)$ . Then*

$$\left( \frac{\text{Tr}(F) - E_N}{\sqrt{V_N}} \right)_{N \geq 1} \quad (6.6)$$

*converges in distribution to a standard gaussian random variable, where*

$$E_N := \theta \sum_{k=1}^N \mathbb{E} [\Delta_k(F, Z_{k,1})]. \quad (6.7)$$

*Remark.* Recall that without loss of generality we may assume  $F$  has mean zero in the sense that  $\int_0^{2\pi} F(e^{ix}) dx = 0$ . We remark that this does not necessarily imply that  $\Delta_k(F, z)$  tends to zero, even though  $\Delta_k(F, z)$  is a discretisation of the integral, without the assumption of further smoothness conditions. Moreover, note that in the framework of the symmetric group (i.e.  $Z_{k,1} = 1$  almost surely), the assumption (6.5) is implied by the condition (6.2) given in [2]. Indeed, if (6.2) is satisfied, then for  $p > 2$ , one has

$$\begin{aligned} \sum_{k=1}^N k^{p-1} |\Delta_k(F)|^p &\leq \left( \max_{1 \leq k \leq N} (k |\Delta_k(F)|) \right)^{p-2} \sum_{k=1}^N k |\Delta_k(F)|^2 \\ &= o(V_N^{(p-2)/2}) O(V_N) = o(V_N^{p/2}). \end{aligned}$$

In the proof of Theorem 6.2, we use Feller coupling, which allows the random variables  $C_k$  and  $P_k$  to be defined on the same space. This coupling allows us to prove the theorem with  $C_k$  replaced by  $P_k$ , and the following lemma allows us to estimate the distance between the two.

**Lemma 6.3** (Ben Arous and Dang [2]). *For any  $\theta > 0$  there exists a constant  $K(\theta)$  depending on  $\theta$ , such that for every  $1 \leq m \leq N$ ,*

$$\mathbb{E} [|C_k - P_k|] \leq \frac{K(\theta)}{N} + \frac{\theta}{N} \Psi_N(k), \quad (6.8)$$

*where*

$$\Psi_N(k) := \binom{N-k+\theta-1}{N-k} \binom{N+\theta-1}{N}^{-1}. \quad (6.9)$$



*Proof of Theorem 6.2.* The main idea of the proof is to define the auxiliary random variable

$$Y_N(F) := \sum_{k=1}^N \sum_{m=1}^{P_k} k \Delta_k(F, Z_{k,m}) \quad (6.10)$$

and to show that  $\text{Tr}(F)$  and  $Y_N(F)$  have the same asymptotic behaviour after normalisation, and that (again after normalisation)  $Y_N(F)$  satisfies a central limit theorem.

First we will show that

$$\mathbb{E} [|\text{Tr}(F) - Y_N(F)|] = o\left((V_N)^{\frac{1}{2}}\right) \quad (6.11)$$

We use Lemma 6.3 and that  $Z_{k,m}$  are independent of  $C_k$  and  $P_k$  to get

$$\begin{aligned} \mathbb{E} [|\text{Tr}(F) - Y_N(F)|] &\leq \mathbb{E} \left[ \sum_{k=1}^N |C_k - P_k| k \mathbb{E} [|\Delta_k(F, Z_{k,1})|] \right] \\ &\leq \frac{K(\theta)}{N} \sum_{k=1}^N k \mathbb{E} [|\Delta_k(F, Z_{k,1})|] + \frac{\theta}{N} \sum_{k=1}^N k \mathbb{E} [|\Delta_k(F, Z_{k,1})|] \Psi_N(k) \end{aligned} \quad (6.12)$$

For the first term, we apply Jensen's inequality and condition (6.5) to obtain

$$\frac{1}{N} \sum_{k=1}^N k \mathbb{E} [|\Delta_k(F, Z_{k,1})|] \leq \left( \frac{1}{N} \sum_{k=1}^N k^p \mathbb{E} [|\Delta_k(F, Z_{k,1})|^p] \right)^{1/p} \quad (6.13)$$

$$\leq \left( \sum_{k=1}^N k^{p-1} \mathbb{E} [|\Delta_k(F, Z_{k,1})|^p] \right)^{1/p} \quad (6.14)$$

$$= o\left(V_N^{1/2}\right). \quad (6.15)$$

Now we deal with the second term in (6.12). If  $\theta \geq 1$  then  $\Psi_N(k)$  is bounded by 1, so the same argument as above shows that the second summand is also  $o\left(V_N^{1/2}\right)$  in this case.

If  $0 < \theta < 1$ , we have to be more careful. A simple computation shows that there exists constants  $K_1, K_2$  such that

$$\Psi_N(k) \leq \begin{cases} K_1 \left(1 - \frac{k}{N}\right)^{\theta-1}, & \text{for } k < N, \\ K_2 N^{1-\theta}, & \text{for } k = N \end{cases} \quad (6.16)$$

and so

$$\frac{\theta}{N} \sum_{k=1}^N k \mathbb{E} [|\Delta_k(F, Z_{k,1})|] \Psi_N(k) \quad (6.17)$$

$$\leq \theta K_2 N^{1-\theta} \mathbb{E} [|\Delta_N(F, Z_{N,1})|] + \frac{\theta K_1}{N} \sum_{k=1}^{N-1} k \mathbb{E} [|\Delta_k(F, Z_{k,1})|] \left(1 - \frac{k}{N}\right)^{\theta-1} \quad (6.18)$$

Using the value of  $p$  given in the conditions of the theorem,

$$N^{1-\theta} \mathbb{E} [|\Delta_N(F, Z_{N,1})|] = \left(N^{p-p\theta} \mathbb{E} [|\Delta_N(F, Z_{N,1})|^p]\right)^{1/p} \quad (6.19)$$

$$\leq \left(\sum_{k=1}^N k^{p-p\theta} \mathbb{E} [|\Delta_k(F, Z_{k,1})|^p]\right)^{1/p} \quad (6.20)$$

$$\leq \left(\sum_{k=1}^N k^{p-1} \mathbb{E} [|\Delta_k(F, Z_{k,1})|^p]\right)^{1/p} \quad (6.21)$$

since  $k^{p-p\theta} \leq k^{p-1}$  (since  $p\theta > 1$ ) and  $\mathbb{E} [|\Delta_k(F, Z_{k,1})|^p] \leq \mathbb{E} [|\Delta_k(F, Z_{k,1})|^p]$  (since  $p > 1$ ). Thus, by condition (6.5) this is  $o\left(V_N^{1/2}\right)$ .

Hölder's inequality gives

$$\begin{aligned} & \frac{1}{N} \sum_{k=1}^{N-1} k \mathbb{E} [|\Delta_k(F, Z_{k,1})|] \left(1 - \frac{k}{N}\right)^{\theta-1} \\ & \leq \left(\frac{1}{N} \sum_{k=1}^{N-1} k^p \mathbb{E} [|\Delta_k(F, Z_{k,1})|^p]\right)^{1/p} \left(\frac{1}{N} \sum_{k=1}^{N-1} \left(1 - \frac{k}{N}\right)^{q(\theta-1)}\right)^{1/q} \end{aligned} \quad (6.22)$$

with  $\frac{1}{p} + \frac{1}{q} = 1$ .

After re-ordering the sum, the second factor is  $\left(N^{q(1-\theta)-1} \sum_{j=1}^{N-1} j^{-q(1-\theta)}\right)^{1/q}$ , and if  $q(\theta-1) > -1$  then it is bounded above by a constant. Note that

$$(\theta-1)q > -1 \iff (1-\theta) < \frac{1}{q} \iff (1-\theta) < 1 - \frac{1}{p} \iff \theta > \frac{1}{p}$$

and thus condition (6.5) now ensures the existence of a  $p > \frac{1}{\theta}$  such that the first factor is  $o\left((V_N)^{\frac{1}{2}}\right)$  and the second factor is bounded. This proves (6.11).

Therefore Slutsky's theorem implies that  $Y_N(F)$  and  $\text{Tr}(F)$  have the same asymptotic distribution after scaling. Thus it suffices to show that

$$\frac{Y_N(F) - E_N}{\sqrt{V_n}} \quad (6.23)$$

converges in law to a standard gaussian random variable.

We calculate the mean of  $Y_N(F)$  by first taking expectation with respect to  $Z_{k,m}$  and then with respect to  $P_k$ , to obtain

$$\mathbb{E}[Y_N(F)] = \sum_{k=1}^N \mathbb{E} \left[ \sum_{m=1}^{P_k} k \mathbb{E}[\Delta_k(F, Z_{k,m})] \right] = \sum_{k=1}^N \mathbb{E}[P_k] k \mathbb{E}[\Delta_k(F, Z_{k,1})] \quad (6.24)$$

where we use the fact that  $\mathbb{E}[\Delta_k(F, Z_{k,m})] = \mathbb{E}[\Delta_k(F, Z_{k,1})]$  for all  $m$ . Finally, since  $\mathbb{E}[P_k] = \theta/k$ , we see that  $\mathbb{E}[Y_N(F)] = E_N$  as defined in (6.7).

For the variance, since  $P_k$  and  $Z_{k,m}$  are all independent, one can move the sum outside the variance, to obtain

$$\text{Var}(Y_N(F)) = \sum_{k=1}^N \text{Var} \left( \sum_{m=1}^{P_k} k \Delta_k(F, Z_{k,m}) \right) \quad (6.25)$$

Now, the variance of a sum of random length of i.i.d. random variables is given by the following formula:

$$\text{Var} \left( \sum_{m=1}^P X_m \right) = \text{Var}(X_1) \mathbb{E}[P] + \text{Var}(P) \mathbb{E}[X_1]^2, \quad (6.26)$$

if  $(X_m)_{m \geq 1}$  are i.i.d.,  $L^2$  random variables, and if  $P$  is a  $L^2$  variable, independent of  $(X_m)_{m \geq 1}$  (this result can be proved by a straightforward calculation). Letting  $X_m = k \Delta_k(F, Z_{k,m})$  and  $P = P_k$  and knowing that  $\mathbb{E}[P_k] = \text{Var}(P_k) = \theta/k$ , we deduce that  $\text{Var}(Y_N(F)) = V_N$ .

Finally we apply the Lyapunov central limit theorem since  $Y_N(F)$  is a sum of independent random variables. We will show that

$$\begin{aligned} \sum_{k=1}^N \mathbb{E} \left[ \left| \sum_{m=1}^{P_k} k \Delta_k(F, Z_{k,m}) - \mathbb{E}[k P_k \Delta_k(F, Z_{k,1})] \right|^p \right] \\ \ll \sum_{k=1}^N k^{p-1} \mathbb{E}[|\Delta_k(F, Z_{k,1})|^p] \quad (6.27) \end{aligned}$$

and by Condition (6.5) with  $p > 2$  this is  $o(V_N^{p/2})$  which means  $\frac{Y_N(F) - E_N}{\sqrt{V_N}}$  converges in law to a standard gaussian random variable.

For simplicity, let  $P$  be a Poisson random variable with parameter  $\theta/k$  (we think of  $k$  as being large), and let  $X_m = k \Delta_k(F, Z_{k,1})$  be i.i.d. random

variables with  $\mathbb{E}[|X_m|^p]$  finite. Then

$$\begin{aligned} \mathbb{E} \left[ \left| \sum_{m=1}^P X_m - \mathbb{E}[P] \mathbb{E}[X_1] \right|^p \right] &= \mathbb{E} \left[ \left| \sum_{m=1}^P (X_m - \mathbb{E}[X_1]) + (P - \mathbb{E}[P]) \mathbb{E}[X_1] \right|^p \right] \\ &\leq \left( \mathbb{E} \left[ \left| \sum_{m=1}^P (X_m - \mathbb{E}[X_1]) \right|^p \right]^{1/p} + \mathbb{E}[|P - \mathbb{E}[P]|^p]^{1/p} \mathbb{E}[X_1] \right)^p \end{aligned} \quad (6.28)$$

by the generalised triangle inequality.

Now, for all  $p > 1$ ,

$$\mathbb{E}[|P - \mathbb{E}[P]|^p] \leq \mathbb{E}[P^p] \ll \mathbb{E}[P] \quad (6.29)$$

as  $\mathbb{E}[P] \rightarrow 0$ , and so the second term in (6.28) is

$$\mathbb{E}[|P - \mathbb{E}[P]|^p]^{1/p} \mathbb{E}[X_1] \ll \mathbb{E}[P]^{1/p} \mathbb{E}[X_1] \quad (6.30)$$

$$\ll (\mathbb{E}[P] \mathbb{E}[|X_1|^p])^{1/p} \quad (6.31)$$

by Hölder's inequality, since  $p > 1$ .

To bound the first term in (6.28), let  $q_n = \mathbb{P}[P = n]$ , and note that

$$\mathbb{E} \left[ \left| \sum_{m=1}^P (X_m - \mathbb{E}[X_1]) \right|^p \right] = \sum_{n=0}^{\infty} q_n \mathbb{E} \left[ \left| \sum_{m=1}^n (X_m - \mathbb{E}[X_1]) \right|^p \right] \quad (6.32)$$

$$\leq \sum_{n=0}^{\infty} q_n \left( \sum_{m=1}^n \mathbb{E}[|X_m - \mathbb{E}[X_1]|^p]^{1/p} \right)^p \quad (6.33)$$

$$= \sum_{n=0}^{\infty} q_n n^p \mathbb{E}[|X_1 - \mathbb{E}[X_1]|^p] \quad (6.34)$$

$$= \mathbb{E}[P^p] \mathbb{E}[|X_1 - \mathbb{E}[X_1]|^p] \quad (6.35)$$

$$\ll \mathbb{E}[P] \mathbb{E}[|X_1|^p] \quad (6.36)$$

Thus

$$\begin{aligned} \mathbb{E} \left[ \left| \sum_{m=1}^P X_m - \mathbb{E}[P] \mathbb{E}[X_1] \right|^p \right] &\ll \left( (\mathbb{E}[P] \mathbb{E}[|X_1|^p])^{1/p} + (\mathbb{E}[P] \mathbb{E}[|X_1|^p])^{1/p} \right)^p \\ &\ll \mathbb{E}[P] \mathbb{E}[|X_1|^p] \end{aligned} \quad (6.37)$$

Using  $\mathbb{E}[P] = \theta/k$  and  $\mathbb{E}[|X_1|^p] = k^p \mathbb{E}[|\Delta_k(F, Z_{k,1})|^p]$ , and summing for  $k = 1, \dots, N$  we have proven (6.27). By Condition (6.5), if  $p > 2$  then this is  $o(V_N^{p/2})$  which by Lyapunov's theorem means that  $\frac{Y_N(F) - E_N}{\sqrt{V_N}}$  converges in law to a standard gaussian random variable as required.  $\square$

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