THE SIX-FUNCTOR FORMALISM FOR RIGID ANALYTIC MOTIVES

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Abstract. We offer a systematic study of rigid analytic motives over general rigid analytic spaces, and we develop their six-functor formalism. A key ingredient is an extended proper base change theorem that we are able to justify by reducing to the case of algebraic motives. In fact, more generally, we develop a powerful technique for reducing questions about rigid analytic motives to questions about algebraic motives, which is likely to be useful in other contexts as well. We pay special attention to establishing our results without noetherianity assumptions on rigid analytic spaces. This is indeed possible using Raynaud's approach to rigid analytic geometry.

Contents

Key words and phrases. Motives (algebraic, formal and rigid analytic), six-functor formalism, proper base change theorem.

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Introduction

In this paper, we study rigid analytic motives over general rigid analytic spaces and we develop a six-functor formalism for them. We have tried to free our treatment from unnecessary hypotheses, and many of our main results hold in great generality, with the notable exception of Theorems 3.3.3(2) and 3.8.1 where we impose étale descent. (This is necessary for the former but might be superfluous for the latter.) In this introduction, we restrict to étale rigid analytic motives with rational coefficients, for which our results are the most complete.[1]

The six-functor formalism.

Rigid analytic motives were introduced in [Ayo15] as a natural extension of the notion of a motive associated to a scheme. Given a rigid analytic space S, we denote by $\mathbf{RigDA}_{\text{ét}}(S;\mathbb{Q})$ the ∞-category of étale rigid analytic motives over S with rational coefficients. By construction, it is naturally equipped with the structure of a symmetric monoidal ∞-category (see Definition 2.1.15).

Given a morphism of rigid analytic spaces $f : T \to S$, the functoriality of the construction yields an adjunction

$$f^* : \mathbf{RigDA}_{\text{ét}}(S;\mathbb{Q}) \rightleftarrows \mathbf{RigDA}_{\text{ét}}(T;\mathbb{Q}) : f_*. \tag{0.1}$$

When f is locally of finite type, we construct in this paper another adjunction (see Definition 4.3.4)

$$f_! : \mathbf{RigDA}_{\text{ét}}(T;\mathbb{Q}) \rightleftarrows \mathbf{RigDA}_{\text{ét}}(S;\mathbb{Q}) : f^!, \tag{0.2}$$

i.e., we define the "exceptional direct image" and the "exceptional inverse image" functors associated to f. Our main goal in this paper is to show the following result.

Scholium. *The functors f^*, f_*, $f_!$, $f^!$, $\otimes$ and $\underline{\mathrm{Hom}}$ satisfy the usual properties of a six-functor formalism. These include:*

[1] In fact, everything we say here holds more generally with coefficients in an arbitrary ring when the class of rigid analytic spaces is accordingly restricted. For instance, if one is only considering rigid analytic spaces over $\mathbb{Q}_p$ of finite étale cohomological dimension, the results discussed in the introduction are valid with $\mathbb{Z}[p^{-1}]$-coefficients.

- *the compatibility with composition of morphisms (see Proposition 2.1.21 and Corollary 4.3.18);*
- *the localization formula (see Proposition 2.2.3(2));*
- *the base change theorems (see Proposition 2.2.1(3) for the smooth base change, Theorem 2.7.1 for the quasi-compact base change, Theorem 4.1.4 for the extended proper base change, and Proposition 4.4.26 for the exchange between the "ordinary inverse image" and the "exceptional direct image" functors);*
- *the canonical equivalences $f_! \simeq f_*$, when f is proper (see Example 4.3.6), and the equivalence $f^! \simeq f^*(d)[2d]$,[2] when f smooth of pure relative dimension d (see Theorem 4.4.29);*
- *the compatibility with tensor product, the projection formula and duality (see Proposition 2.1.21, and Corollaries 4.1.8, 4.5.3 and 4.5.4).*

Of course, our six-functor formalism matches the one developed by Huber [Hub96] for the étale cohomology of adic spaces. (Similar formalisms for étale cohomology were also developed by Berkovich [Ber93] and de Jong–van der Put [dJvdP96].)

A partial six-functor formalism for rigid analytic motives was obtained in [Ayo15, §1.4] at a minimal cost as an application of the theory developed in [Ayo07a, Chapitre 1]. Given a non-Archimedean field K and a classical affinoid K-algebra A, the assignment sending a finite type A-scheme X to the ∞-category $\mathbf{RigDA}_{\text{ét}}(X^{\text{an}};\mathbb{Q})$ gives rise to a stable homotopical functor in the sense of [Ayo07a, Définition 1.4.1]. (Here X^{an} is the analytification of X.) Applying [Ayo07a, Scholie 1.4.2], we have in particular an adjunction as in (0.2) under the assumption that f is algebraizable, i.e., that f is the analytification of a morphism between finite type A-schemes, for some unspecified classical affinoid K-algebra A. Clearly, it is unnatural and unsatisfactory to restrict to algebraizable morphisms, and it is our objective in this paper to remove this restriction. The key ingredient for doing so is Theorem 4.1.4 which we may consider as an extended proper base change theorem for commuting direct images along proper morphisms and extension by zero along open immersions. It is worth noting that in the algebraic setting, the extended proper base change theorem is essentially a reformulation of the usual one, but this is far from true in the rigid analytic setting. In fact, the usual proper base change theorem in rigid analytic geometry is a particular case of the so-called quasi-compact base change theorem (see Theorem 2.7.1) which is an easier property.

The extended proper base change theorem is already known if one restricts to projective morphisms in which case it can be deduced from the partial six-functor formalism developed in [Ayo15, §1.4]. However, in the rigid analytic setting, it is not possible to deduce the general case of proper morphisms from the special case of projective morphisms. Indeed, the classical argument used in [SGAIV3, Exposé XII] for reducing the proper case to the projective case relies on Chow's lemma for which there is no analogue in rigid analytic geometry. (For instance, there are proper rigid analytic tori which are not algebraizable [FvdP04, §6.6].) Therefore, a new approach was necessary for proving Theorem 4.1.4 in general.

Rigid motives as modules in algebraic motives.

[2]Strictly speaking, Theorem 4.4.29 only gives that $f^!$ is equivalent to the twist of f^* by the Thom space associated to Ω_f. However, in the case of $\mathbf{RigDA}_{\text{ét}}(-;\mathbb{Q})$, Thom spaces are globally given by Tate twists. Indeed, arguing as in [Ayo14a, Remarque 11.3], this would follow from the property that the mapping space $\mathrm{Map}_{\mathbf{RigDA}_{\text{ét}}(S;\mathbb{Q})}(\mathbb{Q},\mathbb{Q}) \simeq \mathbb{Q}^{\pi_0(S)}$ is coconnective. The latter property can be established using Theorem 3.8.1 and [Ayo14a, Proposition 11.1]. We leave the details for the interested reader.

Our approach is based on another contact point with algebraic geometry: instead of using the analytification functor from schemes to rigid analytic spaces, we go backward and associate to a rigid analytic space X the pro-scheme consisting of the special fibers of the different formal models of X. We now sketch the main idea of our construction, which is detailed in Section 3.4.

Let $\mathcal{S}$ be a formal scheme. We may associate to it the ∞-category of formal motives $\mathbf{FDA}_{\text{ét}}(\mathcal{S};\mathbb{Q})$ which is canonically equivalent to the ∞-category of (algebraic) motives $\mathbf{DA}_{\text{ét}}(\mathcal{S}_\sigma;\mathbb{Q})$ over the special fiber $\mathcal{S}_\sigma$ (see Theorem 3.1.10). The "generic fiber" functor induces a functor

$$\xi_{\mathcal{S}} : \mathbf{DA}_{\text{ét}}(\mathcal{S}_\sigma;\mathbb{Q}) \simeq \mathbf{FDA}_{\text{ét}}(\mathcal{S};\mathbb{Q}) \to \mathbf{RigDA}_{\text{ét}}(\mathcal{S}^{\text{rig}};\mathbb{Q})$$

where $\mathcal{S}^{\text{rig}}$ is the rigid analytic space associated to $\mathcal{S}$. It is immediate to see that $\xi_{\mathcal{S}}$ is monoidal and has a right adjoint $\chi_{\mathcal{S}}$ sending the unit object of $\mathbf{RigDA}_{\text{ét}}(\mathcal{S}^{\text{rig}};\mathbb{Q})$ to a commutative algebra object $\chi_{\mathcal{S}}\mathbb{Q}$ of $\mathbf{DA}_{\text{ét}}(\mathcal{S}_\sigma;\mathbb{Q})$. Moreover, the functor $\chi_{\mathcal{S}}$ admits a factorization

$$\mathbf{RigDA}_{\text{ét}}(\mathcal{S}^{\text{rig}};\mathbb{Q}) \xrightarrow{\widetilde{\chi}_{\mathcal{S}}} \mathbf{DA}_{\text{ét}}(\mathcal{S}_\sigma;\chi_{\mathcal{S}}\mathbb{Q}) \xrightarrow{\text{ff}} \mathbf{DA}_{\text{ét}}(\mathcal{S}_\sigma;\mathbb{Q})$$

where $\mathbf{DA}_{\text{ét}}(\mathcal{S}_\sigma;\chi_{\mathcal{S}}\mathbb{Q})$ denotes the ∞-category of $\chi_{\mathcal{S}}\mathbb{Q}$-modules in $\mathbf{DA}_{\text{ét}}(\mathcal{S}_\sigma;\mathbb{Q})$ and where ff is the forgetful functor. Also, the functor $\widetilde{\chi}_{\mathcal{S}}$ admits a left adjoint

$$\begin{array}{rcl} \widetilde{\xi}_{\mathcal{S}} : \mathbf{DA}_{\text{ét}}(\mathcal{S}_\sigma;\chi_{\mathcal{S}}\mathbb{Q}) & \to & \mathbf{RigDA}_{\text{ét}}(\mathcal{S}^{\text{rig}};\mathbb{Q}) \\ M & \mapsto & \xi_{\mathcal{S}}(M) \otimes_{\xi_{\mathcal{S}}\chi_{\mathcal{S}}\mathbb{Q}} \mathbb{Q}. \end{array}$$

Now, if S is a quasi-compact and quasi-separated rigid analytic space, we may consider the cofiltered category $\text{Mdl}(S)$ of formal models of S (see Notation 1.1.9). The above construction yields a functor

$$\widetilde{\xi}_S : \operatorname*{colim}_{\mathcal{S}\in\text{Mdl}(S)} \mathbf{DA}_{\text{ét}}(\mathcal{S}_\sigma;\chi_{\mathcal{S}}\mathbb{Q}) \to \mathbf{RigDA}_{\text{ét}}(S;\mathbb{Q}).$$

One of our main results is the following (see Theorem 3.3.3 and Remark 3.3.5).

Theorem. *Restrict to rigid analytic spaces which are quasi-compact, quasi-separated and having finite Krull dimension. The natural transformation $\widetilde{\xi}$ exhibits the functor $S \mapsto \mathbf{RigDA}_{\text{ét}}(S;\mathbb{Q})$ as the étale sheafification of the functor $S \mapsto \operatorname{colim}_{\mathcal{S}\in\text{Mdl}(S)} \mathbf{DA}_{\text{ét}}(\mathcal{S}_\sigma;\chi_{\mathcal{S}}\mathbb{Q})$ viewed as a presheaf valued in presentable ∞-categories.*

We use the above description of the ∞-categories $\mathbf{RigDA}_{\text{ét}}(S;\mathbb{Q})$ to deduce the extended proper base change theorem in rigid analytic geometry (i.e., Theorem 4.1.4) from its algebraic analogue. In fact, it turns out that we only need a formal consequence of this description which happens to be also a key ingredient in its proof, namely Theorem 3.6.1 (see also Theorem 4.1.3). Once Theorem 4.1.4 is proven, it is easy to construct the adjunction (0.2).

We also point out that the commutative algebras $\chi_{\mathcal{S}}\mathbb{Q}$ admit a concrete description. For precise statements, see Theorem 3.8.1 and Remark 3.8.2. Moreover, the special case of the above theorem where we take for $S = \text{Spf}(k[[\pi]])^{\text{rig}}$, with k a field of characteristic zero, is tightly connected to the main result of [Ayo15, Chapitre 1]. This will be explained in Remark 3.8.3.

Further results and applications.

Besides the six-functor formalism, the paper contains several foundational results on motives of rigid analytic spaces which are of independent interest. In particular, we study the descent property of the ∞-categories $\mathbf{RigDA}_{\text{ét}}(S;\mathbb{Q})$ for the étate topology; see Theorem 2.3.4.

Another notable result is Theorem 2.5.1 which, roughly speaking, asserts that $\mathbf{RigDA}_{\text{ét}}(-;\mathbb{Q})$ transforms limits of certain rigid analytic pro-spaces into colimits of presentable ∞-categories. A similar property is also true for $\mathbf{DA}_{\text{ét}}(-;\mathbb{Q})$ but the proof in the rigid analytic setting is much more involved and relies on approximation techniques as those used in the proof of [Vez19, Proposition 4.5]. We also like to mention that this continuity property for $\mathbf{RigDA}_{\text{ét}}(-;\mathbb{Q})$ plays a crucial role (along with many of the results described above) in the recent paper [LBV21] where a new relative cohomology theory for rigid analytic varieties over a positive characteristic perfectoid space P is defined and studied. Interestingly, this relative cohomology theory takes values in solid quasi-coherent sheaves over the relative Fargues–Fontaine curve associated to P.

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Notation and conventions.

∞-Categories. We use the language of ∞-categories following Lurie's books [Lur09] and [Lur17]. The reader familiar with the content of these books will have no problem understanding our notation pertaining to higher category theory and higher algebra which are often very close to those in loc. cit. Nevertheless, we list below some of these notational conventions which we use frequently.

As usual, we employ the device of Grothendieck universes, and we denote by Cat_∞ the ∞-category of small ∞-categories and CAT_∞ the ∞-category of locally small, but possibly large ∞-categories. We denote by $\mathrm{CAT}_\infty^{\mathrm{L}}$ (resp. $\mathrm{CAT}_\infty^{\mathrm{R}}$) the wide sub-$\infty$-category of CAT_∞ spanned by functors which are left (resp. right) adjoints. Similarly, we denote by Pr^{L} (resp. Pr^{R}) the ∞-category of presentable ∞-categories and left adjoint (resp. right adjoint) functors. We denote by $\mathrm{Pr}^{\mathrm{L}}_\omega \subset \mathrm{Pr}^{\mathrm{L}}$ (resp. $\mathrm{Pr}^{\mathrm{R}}_\omega \subset \mathrm{Pr}^{\mathrm{R}}$) the sub-$\infty$-category of compactly generated ∞-categories and compact-preserving functors (resp. functors commuting with filtered colimits).

1-Categories are typically referred to as just "categories" and viewed as ∞-categories via the nerve construction. For emphasis, we sometimes call them "ordinary categories". We denote by $\mathcal{S}$ the ∞-category of small spaces, by $\mathcal{S}\mathrm{p}$ the ∞-category of small spectra and by $\mathcal{S}\mathrm{p}_{\geq 0} \subset \mathcal{S}\mathrm{p}$ its full sub-∞-category of connective spectra.

Given an ∞-category $\mathcal{C}$, we denote by $\mathrm{Map}_{\mathcal{C}}(x, y)$ the mapping space between two objects x and y in $\mathcal{C}$. Given another ∞-category $\mathcal{D}$, we denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$. If $\mathcal{C}$ is small, we denote by $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ the ∞-category of presheaves on $\mathcal{C}$ and by $\mathrm{y} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ the Yoneda embedding.

Monoidal structures. By "monoidal ∞-category" we always mean "symmetric monoidal ∞-category", i.e., a coCartesian fibration $\mathcal{C}^\otimes \to \mathrm{Fin}_*$ such that the induced functor $(\rho^i_!)_i : \mathcal{C}_{\langle n\rangle} \to \prod_{1\leq i\leq n} \mathcal{C}_{\langle 1\rangle}$ is an equivalence for all $n \geq 0$. (Recall that Fin_* is the category of finite pointed sets, $\langle n\rangle = \{1, \ldots, n\} \cup \{*\}$ and $\rho^i : \langle n\rangle \to \langle 1\rangle$ is the unique map such that $(\rho^i)^{-1}(1) = \{i\}$.) If $\mathcal{C}^\otimes$ is a monoidal ∞-category, we denote by $\mathrm{CAlg}(\mathcal{C})$ the ∞-category of commutative algebras in $\mathcal{C}$. If $A \in \mathrm{CAlg}(\mathcal{C})$, we denote by $\mathrm{Mod}_A(\mathcal{C})$ the ∞-category of A-modules. Using Lurie's straightening construction, a monoidal category can be considered as an object of $\mathrm{CAlg}(\mathrm{CAT}_\infty)$, i.e., as a commutative algebra in $\mathrm{CAT}_\infty^\times$.

The ∞-categories Pr^{L} and $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ underly monoidal ∞-categories $\mathrm{Pr}^{\mathrm{L},\otimes}$ and $\mathrm{Pr}^{\mathrm{L},\otimes}_{\omega}$. A monoidal ∞-category is said to be presentable (resp. compactly generated) if it belongs to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ (resp. $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$).

Sites and topoi. If $\mathcal{C}$ is an ∞-category endowed with a topology τ, we denote by $(\mathcal{C},\tau)$ the corresponding site. We denote by $\mathrm{Shv}_{\tau}(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ the full sub-∞-category of τ-sheaves and by $\mathrm{Shv}^{\wedge}_{\tau}(\mathcal{C}) \subset \mathrm{Shv}_{\tau}(\mathcal{C})$ its full sub-∞-category of τ-hypersheaves. We use L_{τ} to denote the τ-sheafification functor as well as the τ-hypersheafification functor. (The context will make it clear which of the two we mean.) Morphisms of sites $(\mathcal{C},\tau) \to (\mathcal{C}',\tau')$ are underlain by functors in the opposite direction $\mathcal{C}' \to \mathcal{C}$. In particular, a cofiltered inverse system of sites $(\mathcal{C}_{\alpha},\tau_{\alpha})_{\alpha}$ is underlain by a filtered direct system of ∞-categories, and we write $\lim_{\alpha}(\mathcal{C}_{\alpha},\tau_{\alpha})$ for the site $(\mathrm{colim}_{\alpha}\mathcal{C}_{\alpha},\tau)$ where τ is the topology generated by the τ_{α}'s in the obvious way.

In the cases of most interest to us, the sites are underlain by ordinary categories. In these cases, we follow the classical terminology and say that a morphism of sites is an equivalence of sites if it induces an equivalence on the associated ordinary topoi. (This will be repeated each time, to avoid any possible confusion.)

Formal and rigid analytic geometries. We use Raynaud's approach to rigid analytic geometry [Ray74] which is systematically developed in the books of Abbes [Abb10] and Fujiwara–Kato [FK18]. In fact, we mainly use [FK18] where rigid analytic spaces are introduced without noetherianness assumptions.

We denote formal schemes by calligraphic letters $\mathcal{X}$, $\mathcal{Y}$, etc., and rigid analytic spaces by roman letters X, Y, etc. Formal schemes are always assumed adic of finite ideal type in the sense of [FK18, Chapter I, Definitions 1.1.14 & 1.1.16]. Morphisms of formal schemes are always assumed adic in the sense of [FK18, Chapter I, Definition 1.3.1]. Given a formal scheme $\mathcal{X}$, we denote by $\mathcal{X}^{\mathrm{rig}}$ its associated rigid analytic space which we call the Raynaud generic fiber (or simply the generic fiber) of $\mathcal{X}$. Recall that $\mathcal{X}^{\mathrm{rig}}$ is simply $\mathcal{X}$ considered in the localisation of the category of formal schemes with respect to admissible blowups. A general rigid analytic space is locally isomorphic to the generic fiber of a formal scheme. As we show in Corollary 1.2.7, the category of stably uniform adic spaces (see [BV18]) embeds fully faithfully in the category of rigid analytic spaces.

Given a rigid analytic space X, we denote by $|X|$ the associated topological space (see Notation 1.1.11). This is constructed in [FK18, Chapter II, §3.1] where it is called the Zariski–Riemann space of X. The space $|X|$ is endowed with a sheaf of rings $\mathcal{O}_X$, called the structure sheaf, and a subsheaf of rings $\mathcal{O}^{+}_X \subset \mathcal{O}_X$, called the integral structure sheaf. (In [FK18, Chapter II, §3.2], the integral structure sheaf is denoted by $\mathcal{O}^{\mathrm{int}}_X$, but we prefer to follow Huber's notation in [Hub96].)

Motives (algebraic, formal and rigid analytic). We fix a commutative ring spectrum Λ, i.e., an object of $\mathrm{CAlg}(\mathcal{S}\mathrm{p})$ which we assume to be connective for simplicity.

Given a scheme S, we denote by $\mathbf{SH}_{\tau}(S;\Lambda)$ the Morel–Voevodsky ∞-category of τ-motives on S with coefficients in Λ (see, for example, [Jar00]). Here $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$ is either the Nisnevich or the étale topology. When τ is the Nisnevich topology, we sometimes omit the subscript "nis" and speak simply of motives over S. If Λ is the Eilenberg–Mac Lane spectrum associated to a commutative dg-ring (also denoted by Λ), we usually write $\mathbf{DA}_{\tau}(S;\Lambda)$ instead of $\mathbf{SH}_{\tau}(S;\Lambda)$.

Given a formal scheme $\mathcal{S}$, we denote by $\mathbf{FSH}_{\tau}(\mathcal{S};\Lambda)$ the ∞-category of formal τ-motives on $\mathcal{S}$ with coefficients in Λ (see Definition 3.1.1). Similarly, given a rigid analytic space S, we denote by $\mathbf{RigSH}_\tau(S\,;\Lambda)$ the ∞-category of rigid analytic τ-motives on S with coefficients in Λ (see Definition 2.1.11). Here again, $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$ is either the Nisnevich or the étale topology, and when τ is the Nisnevich topology we sometimes omit the subscript "nis". If Λ is the Eilenberg–Mac Lane spectrum associated to a commutative dg-ring (also denoted by Λ), we usually write $\mathbf{FDA}_\tau(\mathcal{S};\Lambda)$ and $\mathbf{RigDA}_\tau(S\,;\Lambda)$ instead of $\mathbf{FSH}_\tau(\mathcal{S};\Lambda)$ and $\mathbf{RigSH}_\tau(S\,;\Lambda)$.

We also consider the unstable (aka., effective) and/or hypercomplete variants of these motivic ∞-categories, which we refer to using superscripts "eff" and/or "$\wedge$". For example, $\mathbf{SH}^\wedge_\tau(S\,;\Lambda)$ is the Morel–Voevodsky ∞-category of hypercomplete τ-motives and $\mathbf{SH}^{\mathrm{eff},\wedge}_\tau(S\,;\Lambda)$ is its effective version. If a statement is equally valid for the T-stable and the effective motivic ∞-categories, we use the superscript "(eff)". For example, the sentence "the ∞-category $\mathbf{RigDA}^{(\mathrm{eff})}_\tau(S\,;\Lambda)$ is presentable" means that both ∞-categories $\mathbf{RigDA}^{\mathrm{eff}}_\tau(S\,;\Lambda)$ and $\mathbf{RigDA}_\tau(S\,;\Lambda)$ are presentable. We use the superscripts "$(\wedge)$", "(eff, $\wedge$)" in a similar way. For example, the sentence "$S \mapsto \mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(S\,;\Lambda)$ is a Pr^{L}-valued τ-(hyper)sheaf" means that we have two τ-sheaves, namely $\mathbf{SH}^{\mathrm{eff}}_\tau(-;\Lambda)$ and $\mathbf{SH}_\tau(-;\Lambda)$, and two τ-hypersheaves, namely $\mathbf{SH}^{\mathrm{eff},\wedge}_\tau(-;\Lambda)$ and $\mathbf{SH}^\wedge_\tau(-;\Lambda)$.

1. Formal and rigid analytic geometry

In this section, we gather a few results in rigid analytic geometry which we need later in the paper. We use Raynaud's approach [Ray74] which can be summarised roughly as follows: the category of rigid analytic spaces is the localisation of the category of formal schemes with respect to admissible blowups. This is correct up to imposing the right conditions on formal schemes and slightly enlarging the localised category to allow gluing along open immersions. Raynaud's approach has been systematically developed by Abbes [Abb10] and Fujiwara–Kato [FK18]. We will mainly follow the book [FK18] where rigid analytic spaces are introduced without noetherianness assumptions. Indeed, one of the aims of the paper is to show that there are reasonable ∞-categories of rigid analytic motives over general rigid analytic spaces. We warn the readers that many results in [FK18] require noetherianness assumptions, especially when it comes to the study of quasi-coherent sheaves. However, the theory of quasi-coherent sheaves is largely irrelevant for what we do in this paper.

The reader who is only interested in motives of classical rigid analytic varieties in the sense of Tate and who is accustomed with Raynaud's notion of formal models, may skip this section and refer back to it when needed.

1.1. Recollections.

Unless otherwise stated, adic rings are always assumed to be complete of finite ideal type in the sense of [FK18, Chapter I, Definitions 1.1.3 & 1.1.6]. (This is also the convention of [Abb10, Définition 1.8.4] and [Hub93, Section 1].) Thus, an adic ring A is a complete linearly topologized ring whose topology is I-adic for some ideal $I \subset A$ of finite type. Morphisms between adic rings are always assumed to be adic in the sense of [FK18, Chapter I, Definition 1.1.15]. Thus, a morphism of adic rings $A \to B$ is a ring homomorphism such that IB is an ideal of definition of B for one (and hence every) ideal of definition I of A.

A useful basic fact when dealing with adic rings is the existence of I-adic completions in the sense of [FK18, Chapter 0, Definition 7.2.6].

Lemma 1.1.1. *Let A be a ring, $I \subset A$ a finitely generated ideal and M an A-module. The Hausdorff completion $\widehat{M} = \lim_{n\in\mathbb{N}} M/I^nM$ of the A-module M endowed with the I-adic topology is itself an I-adic topological A-module. More precisely, for $m \geq 0$ we have:*

- *$I^m\widehat{M}$ is closed in $\widehat{M}$ and coincides with $\widehat{I^m M} = \lim_{n\in\mathbb{N}} I^m M/I^{m+n}M$, which is the Hausdorff completion of $I^m M$;*
- *$M/I^m M \to \widehat{M}/I^m\widehat{M}$ is an isomorphism.*

Proof. This follows from [Bou98, Chapter III, §2 n° 11, Proposition 14 & Corollary 1] when M is finitely generated. See [FK18, Chapter 0, Corollary 7.2.9 & Propositions 7.2.15 & 7.2.16] for general M. □

Notation 1.1.2. If A is an adic ring and $T = (T_i)_i$ is a family of indeterminates, we denote by $A\langle T\rangle$ the algebra of restricted power series in T with coefficients in A, i.e., the I-adic completion of $A[T]$ for an ideal of definition $I \subset A$. Unless otherwise stated, given an ideal $J \subset A\langle T\rangle$, we denote by $A\langle T\rangle/J$ the I-adically complete quotient, i.e., the quotient of $A\langle T\rangle$ by the closure of the ideal J.

Unless otherwise stated, formal schemes are always assumed to be adic of finite ideal type in the sense of [FK18, Chapter I, Definitions 1.1.14 & 1.1.16]. Thus, a formal scheme $\mathcal{X} = (|\mathcal{X}|, \mathcal{O}_{\mathcal{X}})$ is a ringed space with is locally isomorphic to $\mathrm{Spf}(A)$, where A is an adic ring (of finite ideal type, as always). Morphisms of formal schemes are assumed to be adic, i.e., are locally of the form $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$, with $A \to B$ an adic morphism.

Let $\mathcal{X}$ be a formal scheme. An ideal $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ is said to be an ideal of definition if locally it is of the form $I\mathcal{O}_{\mathrm{Spf}(A)}$ where A is an adic ring and $I \subset A$ an ideal of definition. In this case, the ringed space $(|\mathcal{X}|, \mathcal{O}_{\mathcal{X}}/\mathcal{I})$ is an ordinary scheme which we simply denote by $\mathcal{X}/\mathcal{I}$. By [FK18, Chapter I, Corollary 3.7.12], every quasi-compact and quasi-separated formal scheme admits an ideal of definition which we may assume to be finitely generated.

Definition 1.1.3. Let A be an adic ring. We say that A is of principal ideal type if it admits an ideal of definition which is principal (i.e., generated by a nonzero divisor). We will say that A is of monogenic ideal type if it admits an ideal of definition which is monogenic (i.e., generated by one element). Similarly, we say that a formal scheme is of principal ideal type (resp. of monogenic ideal type) if it admits an ideal of definition which is principal (resp. monogenic). There are also obvious local versions of these notions where we only require that an ideal of definition of a specific type exists locally.

Remark 1.1.4. Let A be an adic ring of monogenic ideal type and $\pi \in A$ a generator of an ideal of definition of A. Then A is of principal ideal type if and only if A is π-torsion-free.

Notation 1.1.5. We denote by FSch the category of formal schemes and by $\mathrm{FSch}^{\mathrm{qcqs}}$ its full subcategory spanned by quasi-compact and quasi-separated formal schemes (in the sense of [FK18, Chapter I, Definitions 1.6.1 & 1.6.5]). Note that the category Sch (resp. $\mathrm{Sch}^{\mathrm{qcqs}}$) of schemes (resp. of quasi-compact and quasi-separated schemes) can be identified with the full subcategory of FSch (resp. $\mathrm{FSch}^{\mathrm{qcqs}}$) spanned by those formal schemes for which (0) is an ideal of definition.

Notation 1.1.6. The inclusion of the category of reduced schemes into FSch admits a right adjoint which we denote by $\mathcal{X} \mapsto \mathcal{X}_\sigma$. It commutes with gluing along open immersions and satisfies $\mathcal{X}_\sigma = (\mathcal{X}/\mathcal{I})_{\mathrm{red}}$ whenever $\mathcal{X}$ admits an ideal of definition $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$. The scheme $\mathcal{X}_\sigma$ is called the special fiber of $\mathcal{X}$.

The following notions agree with the ones introduced in [FK18, Chapter I, Definitions 4.2.2 & 4.3.4 & 4.7.1 & 4.8.12 & 5.3.10 & 5.3.16].

Definition 1.1.7. Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of formal schemes.

(1) We say that f is a closed immersion (resp. finite, proper) if locally on $\mathcal{X}$ there is an ideal of definition $\mathcal{I} \subset \mathcal{O}_X$ such that the induced morphism of schemes $\mathcal{Y}/\mathcal{I} \to \mathcal{X}/\mathcal{I}$ is a closed immersion (resp. finite, proper).
(2) We say that f is an open immersion (resp. adically flat, étale, smooth) if locally on $\mathcal{X}$ there is an ideal of definition $\mathcal{I} \subset \mathcal{O}_X$ such that the induced morphism of schemes $\mathcal{Y}/\mathcal{I}^n \to \mathcal{X}/\mathcal{I}^n$ is an open immersion (resp. flat, étale, smooth) for every $n \in \mathbb{N}$.

Let $\mathcal{X}$ be a formal scheme. An ideal $\mathcal{J} \subset \mathcal{O}_{\mathcal{X}}$ is said to be admissible if, locally on $\mathcal{X}$, it is finitely generated and contains an ideal of definition. An admissible blowup of $\mathcal{X}$ is the blowup of an admissible ideal. For more details, see [FK18, Chapter II, §1.1]. We recall here that the composition $\mathcal{X}'' \to \mathcal{X}$ of two admissible blowups $\mathcal{X}'' \to \mathcal{X}'$ and $\mathcal{X}' \to \mathcal{X}$ is itself an admissible blowup if $\mathcal{X}$ is quasi-compact and quasi-separated. (This is [FK18, Chapter II, Proposition 1.1.10].) We denote by $\mathfrak{B}(\mathcal{X})$ the category of admissible blowups and morphisms of formal $\mathcal{X}$-schemes. If $\mathcal{X}$ is quasi-compact and quasi-separated, then $\mathfrak{B}(\mathcal{X})$ is cofiltered (by [FK18, Chapter II, Proposition 1.3.1]) and if $\mathcal{U} \to \mathcal{X}$ is a quasi-compact open immersion, then the obvious functor $\mathfrak{B}(\mathcal{X}) \to \mathfrak{B}(\mathcal{U})$ is surjective (by [FK18, Chapter II, Proposition 1.1.9]).

Notation 1.1.8. (See [FK18, Chapter II, §2]) We denote by $\mathrm{RigSpc}^{\mathrm{qcqs}}$ the 1-categorical localisation of the category $\mathrm{FSch}^{\mathrm{qcqs}}$ with respect to admissible blowups. More concretely, there is a functor $(-)^{\mathrm{rig}} : \mathrm{FSch}^{\mathrm{qcqs}} \to \mathrm{RigSpc}^{\mathrm{qcqs}}$ which is a bijection on objects and, given two quasi-compact and quasi-separated formal schemes $\mathcal{X}$ and $\mathcal{Y}$, we have

$$\mathrm{Hom}_{\mathrm{RigSpc}^{\mathrm{qcqs}}}(\mathcal{Y}^{\mathrm{rig}}, \mathcal{X}^{\mathrm{rig}}) = \operatorname*{colim}_{\mathcal{Y}' \to \mathcal{Y} \in \mathfrak{B}(\mathcal{Y})} \mathrm{Hom}_{\mathrm{FSch}^{\mathrm{qcqs}}}(\mathcal{Y}', \mathcal{X}). \tag{1.1}$$

The objects of $\mathrm{RigSpc}^{\mathrm{qcqs}}$ are the quasi-compact and quasi-separated rigid analytic spaces (according to [FK18, Chapter II, Definitions 2.1.1 & 2.1.2]). If $\mathcal{X}$ is a quasi-compact and quasi-separated formal scheme, $\mathcal{X}^{\mathrm{rig}}$ is called the Raynaud generic fiber (or simply the generic fiber) of $\mathcal{X}$. For this reason, we sometimes write "$\mathcal{X}_\eta$" instead of "$\mathcal{X}^{\mathrm{rig}}$". A map in $\mathrm{RigSpc}^{\mathrm{qcqs}}$ is an open immersion if it is isomorphic to the generic fiber of an open immersion in $\mathrm{FSch}^{\mathrm{qcqs}}$. General rigid analytic spaces are obtained by gluing along open immersions from objects in $\mathrm{RigSpc}^{\mathrm{qcqs}}$ as in [FK18, Chapter II, §2.2.(c)]. The resulting category is denoted by RigSpc and its objects are the rigid analytic spaces. There is also a generic fiber functor $(-)^{\mathrm{rig}} : \mathrm{FSch} \to \mathrm{RigSpc}$ extending the one on quasi-compact and quasi-separated formal schemes.

Notation 1.1.9. Let X be a rigid analytic space. A formal model for X is a formal scheme $\mathcal{X}$ endowed with an isomorphism $X \simeq \mathcal{X}^{\mathrm{rig}}$ (see [FK18, Chapter II, Definition 2.1.7]). Formal models of X form a category which we denote by $\mathrm{Mdl}(X)$. When X is quasi-compact and quasi-separated, $\mathrm{Mdl}(X)$ is cofiltered by [FK18, Chapter II, Proposition 2.1.10]. Similarly, given a morphism $f : Y \to X$ of rigid analytic spaces, we have a category $\mathrm{Mdl}(f)$ of formal models of f whose objects are morphisms of formal schemes $\phi : \mathcal{Y} \to \mathcal{X}$ together with an isomorphism $f \simeq \phi^{\mathrm{rig}}$ in $\mathrm{RigSpc}^{\Delta^1}$. When X and Y are quasi-compact and quasi-separated, the category $\mathrm{Mdl}(f)$ is cofiltered.

Remark 1.1.10. If $\mathcal{X}$ is a formal scheme and $\mathcal{I}$ is an ideal of definition of $\mathcal{X}$, then the admissible blowup of $\mathcal{I}$ is locally of principal ideal type (in the sense of Definition 1.1.3). Therefore, every quasi-compact and quasi-separated rigid analytic space X admits formal models which are locally of principal ideal type and these form a cofinal subcategory of $\mathrm{Mdl}(X)$ which we denote by $\mathrm{Mdl}'(X)$.

Notation 1.1.11. Let X be a quasi-compact and quasi-separated rigid analytic space. We define a locally ringed space $(|X|, \mathcal{O}_X^+)$ by

$$(|X|, \mathcal{O}_X^+) = \lim_{\mathfrak{X} \in \mathrm{Mdl}(X)} (|\mathfrak{X}|, \mathcal{O}_{\mathfrak{X}}).$$

If $\mathfrak{X}_0$ is a formal model of X and $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}_0}$ is an ideal of definition, then $\mathcal{I}\mathcal{O}_X^+$ is an invertible ideal in $\mathcal{O}_X^+$. We set $\mathcal{O}_X = \bigcup_{n \geq 0} (\mathcal{I}\mathcal{O}_X^+)^{-n}$. Then $\mathcal{O}_X$ is a sheaf of rings which does not depend on $\mathcal{I}$ and which contains $\mathcal{O}_X^+$. By gluing along open immersions, the assignment $X \mapsto (|X|, \mathcal{O}_X, \mathcal{O}_X^+)$ can be extended to any rigid analytic space X. For more details, we refer the reader to [FK18, Chapter II, §3]. We say that $|X|$ is the topological space associated to X, that $\mathcal{O}_X$ is the structure sheaf of X, and that $\mathcal{O}_X^+$ is the integral structure sheaf of X.

Remark 1.1.12. Let X be a rigid analytic space. The topological space $|X|$ is valuative, in the sense of [FK18, Chapter 0, Definition 2.3.1], and spectral if X is quasi-compact and quasi-separated. The Krull dimension (or simply the dimension) of X is defined to be the Krull dimension of $|X|$, i.e., the supremum of the lengths of chains of irreducible closed subsets of $|X|$.

Notation 1.1.13. Let X be a rigid analytic space and $x \in |X|$ a point. By [FK18, Chapter II, Proposition 3.2.6], the local ring $\mathcal{O}_{X,x}^+$ is a prevaluative ring. (Here we use the terminology of [Abb10, Définition 1.9.1].) More precisely, there is a nonzero divisor $a \in \mathcal{O}_{X,x}^+$ with the following properties:

- every finitely generated ideal of $\mathcal{O}_{X,x}^+$ containing a power of a is principal;
- $\mathcal{O}_{X,x}^+[a^{-1}] = \mathcal{O}_{X,x}$;
- $\mathfrak{m}_{X,x} = \bigcap_{n \in \mathbb{N}} a^n \mathcal{O}_{X,x}^+$ where $\mathfrak{m}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$;
- $\mathcal{O}_{X,x}^+/\mathfrak{m}_x$ is a valuation ring of the residue field $\mathcal{O}_{X,x}/\mathfrak{m}_x$. We denote by Γ_x its value group (denoted multiplicatively).

We let $\kappa^+(x)$ be the a-adic completion of $\mathcal{O}_{X,x}^+$, $\kappa(x)$ its fraction field and $\widetilde{\kappa}(x)$ the residue field of $\kappa^+(x)$. We also let $\kappa^\circ(x) \subset \kappa(x)$ be the subring of power bounded elements. Then $\kappa^\circ(x)$ is the unique height 1 valuation ring containing $\kappa^+(x)$. Moreover, $\kappa(x)$ is a non-Archimedean complete field for the norm induced by $\kappa^\circ(x)$.

Definition 1.1.14. Let $f : Y \to X$ be a morphism of rigid analytic spaces.

(1) We say that f is a closed immersion (resp. finite, proper) if, locally on X, f admits a formal model which is a closed immersion (resp. finite, proper).
(2) We say that f is a locally closed immersion if it can be written as the composition of a closed immersion $Y \to U$ followed by an open immersion $U \to X$.
(3) We say that f is étale (resp. smooth) with good reduction if, locally on X, f admits a formal model which is étale (resp. smooth).

We next discuss the analytification functor following [FK18, Chapter II, §9.1].

Construction 1.1.15. Let A be an adic ring, $I \subset A$ an ideal of definition, $U = \mathrm{Spec}(A) \smallsetminus \mathrm{Spec}(A/I)$ and $S = \mathrm{Spf}(A)^{\mathrm{rig}}$. There exists an analytification functor

$$(-)^{\mathrm{an}} : \mathrm{Sch}^{\mathrm{lft}}/U \to \mathrm{RigSpc}/S,$$

where $\mathrm{Sch}^{\mathrm{lft}}/U$ is the category of U-schemes locally of finite type. This functor is uniquely determined by the following two properties.

(1) It is compatible with gluing along open immersions.

(2) For a separated finite type U-scheme X with an open immersion $X \to \overline{X}$ into a proper A-scheme, and complement $Y = \overline{X} \setminus X$, we have

$$X^{\mathrm{an}} = (\widehat{\overline{X}})^{\mathrm{rig}} \setminus (\widehat{Y})^{\mathrm{rig}} \tag{1.2}$$

where, for an A-scheme W, $\widehat{W} = \operatorname{colim}_n W \otimes_A A/I^n$ is the I-adic completion of W.

In the second property, one may replace Y with the closure in $\overline{X}$ of $\overline{X} \times_A U \setminus X$. That (1.2) is independent of the choice of the compactification, follows from [FK18, Chapter II, Propositions 9.1.5 & 9.1.9].[3]

1.2. **Relation with adic spaces.**

Recall from [Hub96, page 37] that a Tate ring is a topological ring A admitting a topologically nilpotent unit and an open subring $A_0 \subset A$ which is adic. (Here, by convention, Tate rings are assumed complete.) The ring A_0 is called a ring of definition. If $\pi \in A$ is a topologically nilpotent unit contained in A_0, then the topology of A_0 is π-adic, i.e., the $\pi^n A_0$ form a fundamental system of open neighbourhoods of 0. A morphism of Tate rings $f : A \to B$ is a continuous morphism of rings for which there exists rings of definitions $A_0 \subset A$ and $B_0 \subset B$ with $f(A_0) \subset B_0$.

Notation 1.2.1. Given a Tate ring A, we denote by $A^\circ \subset A$ the subring of power bounded elements and $A^{\circ\circ} \subset A^\circ$ the ideal of topologically nilpotent elements. We say that A is uniform if A° is bounded (which is equivalent to ask that A° is a ring of definition).

A Tate affinoid ring A is a pair $(A^\pm, A^+)$ where $A^\pm$ is a Tate ring and A^+ is an integrally closed open subring of $A^\pm$ contained in $(A^\pm)^\circ$.

Construction 1.2.2.

(1) Let A be an adic ring of principal ideal type and $\pi \in A$ a generator of an ideal of definition. We associate to A a Tate affinoid ring $A^\natural = (A^{\natural\pm}, A^{\natural+})$ where $A^{\natural\pm} = A[\pi^{-1}]$ and $A^{\natural+}$ is the integral closure of A in $A[\pi^{-1}]$.

(2) The functor $A \mapsto A^\natural$, from adic rings of principal ideal type to Tate affinoid rings, admits a ind-right adjoint. The latter associates to a Tate affinoid ring $R = (R^\pm, R^+)$ the ind-adic ring $R_\natural$ consisting of those rings of definition of $R^\pm$ contained in R^+.

Remark 1.2.3. When the Tate affinoid ring R is uniform, then the associated ind-adic ring $R_\natural$ is isomorphic to an adic ring. In fact, we have $R_\natural = R^+$.

Lemma 1.2.4. *The functor $R \mapsto R_\natural$, from the category of Tate affinoid rings to the category of ind-adic rings of principal ideal type, is fully faithful.*

Proof. Indeed, let R and R' be two Tate affinoid rings and $f : R_\natural \to R'_\natural$ a morphism of ind-adic rings. There exists rings of definition $R_0 \subset R^\pm$ and $R'_0 \subset R'^\pm$ contained in R^+ and R'^+ such that f restricts to a morphism of adic rings $f_0 : R_0 \to R'_0$. Then f_0 induces a morphism of Tate rings $f^\pm : R^\pm \to R'^\pm$. Since f_0 is the restriction of f, for every ring of definition $R_1 \subset R^\pm$ containing R_0 and contained in R^+, there exists a ring of definition $R'_1 \subset R'^\pm$ contained in R'^+ and a morphism $f_1 : R_1 \to R'_1$ extending f_0. This shows that $f^\pm$ maps R^+ into R'^+ as needed. □

[3]Proposition 9.1.9 of loc. cit. is stated under the assumption that A is topologically universally rigid-noetherian, but this assumption is unnecessary.

Given a Tate affinoid ring $A = (A^{\pm}, A^{+})$, we denote by $\mathrm{Spa}(A) = (|\mathrm{Spa}(A)|, \mathcal{O}_{\mathrm{Spa}(A)}, \mathcal{O}^{+}_{\mathrm{Spa}(A)})$ the preadic space associated to A as in [Hub96, pages 38–39]. In general, $\mathcal{O}^{+} \subset \mathcal{O}$ are presheaves of rings on the topological space $|\mathrm{Spa}(A)|$ which might fail to be sheaves.

Proposition 1.2.5.

(1) Let A be an adic ring of principal ideal type. There is a homeomorphism $|\mathrm{Spf}(A)^{\mathrm{rig}}| \simeq |\mathrm{Spa}(A^{\natural})|$ modulo which $\mathcal{O}^{+}_{\mathrm{Spf}(A)^{\mathrm{rig}}}$ (resp. $\mathcal{O}_{\mathrm{Spf}(A)^{\mathrm{rig}}}$) is isomorphic to the sheafification of $\mathcal{O}^{+}_{\mathrm{Spa}(A^{\natural})}$ (resp. $\mathcal{O}_{\mathrm{Spa}(A^{\natural})}$).

(2) Let R be an affinoid ring. There exists a homeomorphism $|\mathrm{Spa}(R)| \simeq \lim |\mathrm{Spf}(R_{\natural})^{\mathrm{rig}}|$ modulo which $\mathcal{O}^{+}_{\lim \mathrm{Spf}(R_{\natural})^{\mathrm{rig}}}$ (resp. $\mathcal{O}_{\lim \mathrm{Spf}(R_{\natural})^{\mathrm{rig}}}$) is isomorphic to the sheafification of $\mathcal{O}^{+}_{\mathrm{Spa}(R)}$ (resp. $\mathcal{O}_{\mathrm{Spa}(R)}$).

Proof. A point $x \in |\mathrm{Spf}(A)^{\mathrm{rig}}|$ determines a morphism of adic rings $A \to \kappa^{+}(x)$, and hence a continuous valuation $v_x : A \to \Gamma_x \cup \{0\}$ landing in $\Gamma_x^{+} \cup \{0\}$. (Here $\Gamma_x^{+} \subset \Gamma$ denotes the submonoid defined by the inequality ≤ 1.) Since the image of π in $\kappa^{+}(x)$ is nonzero, v_x extends uniquely to a continuous valuation $v_x : A^{\natural\pm} \to \Gamma_x \cup \{0\}$. Moreover, v_x maps $A^{\natural+}$ into $\Gamma_x^{+} \cup \{0\}$ since $A^{\natural+}$ is integral over A. Therefore, v_x belongs to $\mathrm{Spa}(A^{\natural})$. It is easy to see that $x \mapsto v_x$ is a bijection, which is continuous and open. More precisely, given elements $a_0, \ldots, a_n$ in A generating an admissible ideal of A, the open subset $|\mathrm{Spf}(A\langle \frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\rangle)^{\mathrm{rig}}| \subset |\mathrm{Spf}(A)^{\mathrm{rig}}|$ is mapped bijectively to the rational subset $|\mathrm{Spa}(A^{\natural}\langle \frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\rangle)| \subset |\mathrm{Spa}(A^{\natural})|$. This also shows that $\mathcal{O}_{\mathrm{Spf}(A)^{\mathrm{rig}}}$ is the sheafification of $\mathcal{O}_{\mathrm{Spa}(A^{\natural})}$.

Assertion (2) can be deduced from assertion (1) and the fact that the counit map $(R_{\natural})^{\natural} \to R$ identifies the Tate affinoid ring R with the colimit of the ind-Tate affinoid ring $(R_{\natural})^{\natural}$. □

Definition 1.2.6. A uniform adic space is a triple $X = (|X|, \mathcal{O}_X, \mathcal{O}_X^{+})$, consisting of a topological space $|X|$ and sheaves of rings $\mathcal{O}_X^{+} \subset \mathcal{O}_X$, which is locally isomorphic to $\mathrm{Spa}(A)$, where A is a stably uniform Tate affinoid ring in the sense of [BV18, pages 30–31]. (This is reasonable since by [BV18, Theorem 7] every stably uniform Tate affinoid ring is sheafy.)

Corollary 1.2.7. *Let* Adic *be the category of uniform adic spaces. Then there exists a fully faithful embedding* Adic $\to$ RigSpc *which is compatible with gluing along open immersions and which sends* $\mathrm{Spa}(R)$ *to* $\mathrm{Spf}(R^{+})^{\mathrm{rig}}$.

Proof. It suffices to treat the affinoid case; the general case follows then by gluing along open immersions. Given two stably uniform Tate affinoid rings A and B, the fact that A is sheafy implies that there is a bijection $\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(\mathrm{Spa}(B), \mathrm{Spa}(A))$. It follows from Remark 1.2.3 that there is a functor $\mathrm{Spa}(A) \mapsto \mathrm{Spf}(A^{+})^{\mathrm{rig}}$, from affinoid uniform adic spaces to rigid analytic spaces, and it remains to show that the map

$$\mathrm{Hom}(A^{+}, B^{+}) \to \mathrm{Hom}(\mathrm{Spf}(B^{+})^{\mathrm{rig}}, \mathrm{Spf}(A^{+})^{\mathrm{rig}}),$$

with A and B as above, is a bijection. An element of the right-hand side can be represented by a morphism $\mathcal{Y} \to \mathrm{Spf}(A^{+})$, where $\mathcal{Y} \to \mathrm{Spf}(B^{+})$ is an admissible blowup. We may assume that $\mathcal{O}_{\mathcal{Y}}$ is π-torsion-free, with π a generator of an ideal of definition in B^{+}. We claim that $\mathcal{O}(\mathcal{Y}) = B^{+}$ which implies that $\mathcal{Y} \to \mathrm{Spf}(A^{+})$ factors uniquely through $\mathrm{Spf}(B^{+})$, finishing the proof.

Let $(\mathcal{Y}_i)_i$ be an affine open covering of $\mathcal{Y}$ and set $\mathcal{Y}_{ij} = \mathcal{Y}_i \cap \mathcal{Y}_j$. Let B_i and B_{ij} be the Tate affinoid rings associated to the adic rings $\mathcal{O}(\mathcal{Y}_i)$ and $\mathcal{O}(\mathcal{Y}_{ij})$ respectively. Then $(\mathrm{Spa}(B_i))_i$ is an open covering of $\mathrm{Spa}(B)$, and $\mathrm{Spa}(B_{ij}) = \mathrm{Spa}(B_i) \cap \mathrm{Spa}(B_j)$. Since B is sheafy, we deduce that B^{+} is the equaliser of the usual pair of arrows $\prod_i B_i^{+} \rightrightarrows \prod_{ij} B_{ij}^{+}$. Since $\mathcal{O}_{\mathcal{Y}}$ is π-torsion-free, we have inclusions $\mathcal{O}(\mathcal{Y}_i) \subset B_i^+$ and $\mathcal{O}(\mathcal{Y}_{ij}) \subset B_{ij}^+$. This proves that $\mathcal{O}(\mathcal{Y})$, which is the equaliser of $\prod_i \mathcal{O}(\mathcal{Y}_i) \rightrightarrows \prod_{ij} \mathcal{O}(\mathcal{Y}_{ij})$, is contained in B^+ as needed. □

1.3. **Étale and smooth morphisms.**

In Definition 1.1.14 we introduced the classes of étale and smooth morphisms with good reduction. These classes are too small, and we need to enlarge them to get the correct notions of étaleness and smoothness in rigid analytic geometry. First, we introduce a notation.

Notation 1.3.1. Let A be an adic ring and $J \subset A$ an ideal. We denote by J^{sat} the ideal of A consisting of those elements $a \in A$ for which there exists an ideal of definition $I \subset A$ such that $aI \subset J$. The ideal J^{sat} is called the saturation of J.

We say that J is saturated if $J = J^{\mathrm{sat}}$. The saturation of an ideal is a saturated ideal.

Remark 1.3.2. If A is an adic ring of principal ideal type and $J \subset A$ a saturated ideal, then J is closed and A/J is also of principal ideal type. Moreover, for a closed ideal $J \subset A$, the quotient A/J is of principal ideal type if and only if J is saturated.

Our definition of étaleness uses the Jacobian matrix. Compare with [Fuj95, Definition 1.3.1].

Definition 1.3.3.

(1) Let A be an adic ring and B an adic A-algebra. We say that B is rig-étale over A if there exists a presentation $B \simeq A\langle t_1, \dots, t_n\rangle/J$ and elements $f_1, \dots, f_n \in J$ such that $(f_1, \dots, f_n)^{\mathrm{sat}} = J^{\mathrm{sat}}$ and the determinant of the Jacobian matrix $\det(\partial f_i/\partial t_j)$ generates an open ideal in B.

(2) A morphism $\mathcal{Y} \to \mathcal{X}$ of formal schemes is said to be rig-étale if, locally for the rig topology on $\mathcal{X}$ and $\mathcal{Y}$ (see Definition 1.4.10 below), it is isomorphic to $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ with B rig-étale over A. (When $\mathcal{X}$ and $\mathcal{Y}$ are quasi-compact, this simply means that after replacing $\mathcal{X}$ and $\mathcal{Y}$ by admissible blowups, the resulting morphism is locally isomorphic to $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ with B rig-étale over A.)

(3) A morphism of rigid analytic spaces $Y \to X$ is said to be étale if, locally on X and Y, it admits formal models which are rig-étale.

Remark 1.3.4. If the rigid analytic space X is assumed to be universally noetherian (in the sense of [FK18, Chapter II, Definition 2.2.23]), then a morphism $f : Y \to X$ is étale if and only if it is flat and neat (i.e., $\Omega_f = 0$). This follows from [Hub96, Propositions 1.7.1 and 1.7.5] together with [FK18, Chapter II, Theorem A.5.2]. See also [Fuj95, Proposition 5.1.6] which is proven under more restrictive assumptions.

Remark 1.3.5. Let A be an adic ring and B a rig-étale adic A-algebra given by $A\langle t_1, \dots, t_n\rangle/J$ with J containing $f_1, \dots, f_n$ as in Definition 1.3.3. Consider the adic A-algebras

$$B' = A\langle t_1, \dots, t_n\rangle/(f_1, \dots, f_n) \quad \text{and} \quad B'' = A\langle t_1, \dots, t_n\rangle/(f_1, \dots, f_n)^{\mathrm{sat}}.$$

We have surjective maps $B' \to B \to B''$ inducing isomorphisms $\mathrm{Spf}(B'')^{\mathrm{rig}} \simeq \mathrm{Spf}(B)^{\mathrm{rig}} \simeq \mathrm{Spf}(B')^{\mathrm{rig}}$. Moreover, B' and B'' are rig-étale over A. The case of B'' is clear. For B', we need to prove the following statement. Let C be an adic ring and $c \in C$ an element. Then c generates an open ideal in C if and only if it generates an open ideal in $C/(0)^{\mathrm{sat}}$. Indeed, let I be an ideal of definition and assume that $I \subset (c) + (0)^{\mathrm{sat}}$. We need to show that a power of I is contained in (c). Since I is finitely generated, we may find elements $v_1, \dots, v_m$ in $(0)^{\mathrm{sat}}$ such that $I \subset (c) + (v_1, \dots, v_m)$. Let r be an integer such that $v_i I^r = 0$ for all $1 \leq i \leq m$. Then clearly $I^{r+1} \subset cI^r \subset (c)$ as needed.

Lemma 1.3.6. *Let A be an adic ring of monogenic ideal type and $\pi \in A$ a generator of an ideal of definition of A. Let B be a rig-étale A-algebra. Then there exists an integer $N \in \mathbb{N}$ such that for every π-torsion-free adic A-algebra C, the map $\mathrm{Hom}_A(B, C) \to \mathrm{Hom}_{A/\pi^N}(B/\pi^N, C/\pi^N)$ is injective.*

Proof. The proof of [Fuj95, Proposition 2.1.1] can be easily adapted to the situation considered in the statement. For the reader's convenience we recall the argument.

For $m \in \mathbb{N}$, we set $A_m = A/\pi^m$, $B_m = B/\pi^m$ and $C_m = C/\pi^m$. Since B is rig-étale over A, there exists an integer c such that $\Omega^1_{B_m/A_m}$ is annihilated by π^c independently of m. (Indeed, if B is given as in Definition 1.3.3, it suffices to take c so that π^c belongs to the ideal generated by $\det(\partial f_i/\partial t_j)$.) Now let $f, f' : B \to C$ be two morphisms of A-algebras inducing the same morphism $f_m : B_m \to C_m$ for some $m \geq c+1$. We will show that $f_{m+1} = f'_{m+1}$, which suffices to conclude using induction.

We may consider f_{2m} and f'_{2m} as deformations of f_m. The difference between these deformations is classified by an element $\epsilon \in \mathrm{Hom}(C_m \otimes_{B_m} \Omega^1_{B_m/A_m}, \pi^m C/\pi^{2m} C)$. Since π is a nonzero divisor of C and $\Omega^1_{B_m/A_m}$ annihilated by π^c, the image of any C-linear morphism $C_m \otimes_{B_m} \Omega^1_{B_m/A_m} \to \pi^m C/\pi^{2m} C$ is contained in $\pi^{2m-c} C/\pi^{2m} C$. In particular, the map

$$\mathrm{Hom}(C_m \otimes_{B_m} \Omega^1_{B_m/A_m}, \pi^m C/\pi^{2m} C) \to \mathrm{Hom}(C_m \otimes_{B_m} \Omega^1_{B_m/A_m}, \pi^m C/\pi^{m+1} C)$$

is identically zero. Since the image of ϵ by this map classifies the difference between f_{m+1} and f'_{m+1}, we get the equality $f_{m+1} = f'_{m+1}$. $\square$

Proposition 1.3.7. *Let A be an adic ring of monogenic ideal type and $\pi \in A$ a generator of an ideal of definition of A. Let $t = (t_1, \ldots, t_n)$ be a system of coordinates and $f = (f_1, \ldots, f_n)$ an n-tuple in $A\langle t\rangle$. Let $J \subset A\langle t\rangle$ be an ideal such that $(f) \subset J \subset (f)^{\mathrm{sat}}$ and set $B = A\langle t\rangle/J$. Assume that $\det(\partial f_i/\partial t_j)$ generates an open ideal in B, so that B is a rig-étale adic A-algebra. Then, there exists a positive integer N such that for every π-torsion-free adic A-algebra C and every integer $e \geq N$, the map*

$$\mathrm{Hom}_A(B, C) \to \mathrm{im}\left\{\mathrm{Hom}_{A/\pi^{2e}}(B/\pi^{2e}, C/\pi^{2e}) \to \mathrm{Hom}_{A/\pi^e}(B/\pi^e, C/\pi^e)\right\} \tag{1.3}$$

is bijective. Moreover, the integer N depends continuously on f, i.e., we may find one which works for every n-tuple $f' = (f'_1, \ldots, f'_n)$ in $A\langle t\rangle$ which is π-adically sufficiently close to f.

Proof. For N sufficiently large, the injectivity of (1.3) follows from Lemma 1.3.6. The fact that there is an N which works for all f' close enough to f follows from the proof of Lemma 1.3.6. (Indeed, the N depends only on the ideal generated by $\det(\partial f_i/\partial t_j)$.)

For the surjectivity of (1.3), it is enough to solve the following problem: given an n-tuple $c_0 = (c_{0,1}, \ldots, c_{0,n})$ in C such that the components of $f(c_0)$ belong to $\pi^{2e} C$, find an n-tuple $c = (c_1, \ldots, c_n)$ in C such that $f(c) = 0$ and the components of $c - c_0$ belong to $\pi^e C$. (Indeed, since C is π-torsion-free an n-tuple c such that $f(c) = 0$ determines an A-morphism $B \to C$.)

This problem can be solved using the Newton method as in the first step of the proof of [Ayo15, Lemme 1.1.52]. In fact, one can also remark that the argument in loc. cit. is valid more generally for non-Archimedean Banach rings, i.e., complete normed rings with a non-Archimedean norm. In particular, it applies with "A", "C" and "R" in loc. cit. replaced with $A[\pi^{-1}]$, $B[\pi^{-1}]$ and $C[\pi^{-1}]$ endowed with the natural norms for which $A/(0)^{\mathrm{sat}}$, $B/(0)^{\mathrm{sat}}$ and $C = C/(0)^{\mathrm{sat}}$ are the unit balls. (More precisely, for $a \in A[\pi^{-1}]$, we set $\|a\| = e^{-v(a)}$ where $v(a)$ is the largest integer such that $a \in \pi^{v(a)}(A/(0)^{\mathrm{sat}})$, and similarly for B and C.) Since π is a nonzero divisor of C, a solution $c = (c_1, \ldots, c_n)$ in $(C[\pi^{-1}])^n$ of the system of equations $f = 0$, close enough to c_0, determines a solution in C^n. We may take for N an integer which is larger than $\ln(2M^2)$ with M as in [Ayo15, page 46].[4] It is clear that N depends π-adically continuously on f. $\square$

Proposition 1.3.8. *Let A be an adic ring of monogenic ideal type and $\pi \in A$ a generator of an ideal of definition of A. Let B be a rig-étale adic A-algebra admitting a presentation $B = A\langle t\rangle/(f)^{\mathrm{sat}}$, with $t = (t_1, \dots, t_n)$ a system of coordinates and $f = (f_1, \dots, f_n)$ an n-tuple in $A\langle t\rangle$ such that $\det(\partial f_i/\partial t_j)$ generates an open ideal in B. Then there exists an integer N such that the following holds. For every n-tuple $f' = (f'_1, \dots, f'_n)$ in $A\langle t\rangle$ such that $f' - f$ belongs $(\pi^N A\langle t\rangle)^n$, the adic A-algebra $B' = A\langle t\rangle/(f')^{\mathrm{sat}}$ is isomorphic to B. Moreover, there is an isomorphism $B \simeq B'$ induced by n-tuple $g = (g_1, \dots, g_n)$ in $A\langle t\rangle$ such that $g - t$ belongs to $(\pi A\langle t\rangle)^n$.*

Proof. This follows by applying Proposition 1.3.7 to the rig-étale adic A-algebras B and B'. $\square$

Notation 1.3.9. Let A be an adic ring. We denote by $\mathcal{E}_A$ the category of rig-étale A-algebras and $\mathcal{E}'_A$ its full subcategory spanned by those adic A-algebras whose zero ideal is saturated. (Thus, every object $B \in \mathcal{E}'_A$ admits a presentation $B \simeq A\langle t\rangle/(f)^{\mathrm{sat}}$ with $t = (t_1, \dots, t_n)$ and $f = (f_1, \dots, f_n)$ such that $\det(\partial f_i/\partial t_j)$ generates an open ideal in B.) The inclusion $\mathcal{E}'_A \to \mathcal{E}_A$ admits a left adjoint given by $B \mapsto B/(0)^{\mathrm{sat}}$. Given a morphism of adic rings $A_1 \to A_2$, there is are induced functors $\mathcal{E}_{A_1} \to \mathcal{E}_{A_2}$ and $\mathcal{E}'_{A_1} \to \mathcal{E}'_{A_2}$ given by $B \mapsto A_2 \widehat{\otimes}_{A_1} B$ and $B \mapsto (A_2 \widehat{\otimes}_{A_1} B)/(0)^{\mathrm{sat}}$ respectively.

Corollary 1.3.10. *Let $(A_\alpha)_\alpha$ be a filtered inductive system of adic rings of monogenic ideal type with colimit A (in the category of adic rings). Then the obvious functor*

$$\operatorname*{colim}_{\alpha} \mathcal{E}'_{A_\alpha} \to \mathcal{E}'_A \tag{1.4}$$

is an equivalence of categories.

Proof. Let R be the colimit of $(A_\alpha)_\alpha$ taken in the category of discrete rings. We may assume that there is a smallest index o and we fix $\pi \in A_o$ generating an ideal of definition of A_o. Then $A = \lim_{n\in\mathbb{N}} R/\pi^n R$, and there is a map of rings $R \to A$ with kernel $J = \bigcap_n \pi^n R$ and with dense image $\widetilde{R} \subset A$. We split the proof into two steps.

Step 1. First, we prove that (1.4) is essentially surjective. By Proposition 1.3.8, an object $B \in \mathcal{E}'_A$ admits a presentation of the form $B = A\langle t\rangle/(\widetilde{f})^{\mathrm{sat}}$ where $t = (t_1, \dots, t_n)$ is a system of coordinates and $\widetilde{f} = (\widetilde{f}_1, \dots, \widetilde{f}_n)$ an n-tuple in $\widetilde{R}[t]$ such that $\widetilde{g} = \det(\partial \widetilde{f}_i/\partial t_j)$ generates an open ideal in B. Using Remark 1.3.5, we can find an integer N and an element $\widetilde{h} \in A\langle t\rangle$ such that $\pi^N - \widetilde{h}\widetilde{g}$ belongs to the closure of the ideal $(\widetilde{f}) \subset \widetilde{R}[t]$ in $A\langle t\rangle$. In particular, we may write

$$\pi^N - \widetilde{h}\widetilde{g} = \sum_{i=1}^{n} \widetilde{a}_i \widetilde{f}_i + \widetilde{v}\pi^{N+1}$$

with $\widetilde{v} \in A\langle t\rangle$ and $\widetilde{a}_1, \dots, \widetilde{a}_n \in \widetilde{R}[t]$. Write $\widetilde{h} = \widetilde{h}_0 + \widetilde{h}_1\pi^{N+1}$ with $\widetilde{h}_0 \in \widetilde{R}[t]$ and $\widetilde{h}_1 \in A\langle t\rangle$. Replacing $\widetilde{h}$ by $\widetilde{h}_0$ and $\widetilde{v}$ by $\widetilde{v} + \widetilde{h}_1$, we may assume that $\widetilde{h}$ belongs to $\widetilde{R}[t]$. It follows from Lemma 1.1.1 that the expression $\pi^N - \widetilde{h}\widetilde{g} - \sum_{i=1}^n \widetilde{a}_i \widetilde{f}_i \in \widetilde{R}[t]$ belongs to $\pi^{N+1}\widetilde{R}[t]$. Said differently, we may also assume that $\widetilde{v} \in \widetilde{R}[t]$. We now choose a lift $f = (f_1, \dots, f_n)$ of $\widetilde{f}$ to an n-tuple in $R[t]$ and set $g = \det(\partial f_i/\partial t_j)$.

[4] Here and below, the page references to [Ayo15] correspond to the published version.

We also choose lifts $h, a_1, \dots, a_n \in R[t]$ of $\widetilde{h}, \widetilde{a}_1, \dots, \widetilde{a}_n$. Since the elements of J are divisible by any power of π, we may also find a lift $v \in R[t]$ of $\widetilde{v}$ such that

$$\pi^N - hg = \sum_{i=1}^n a_i f_i + v\pi^{N+1}.$$

For α sufficiently big, the previous equality can be lifted to an equality

$$\pi^N - h_\alpha g_\alpha = \sum_{i=1}^n a_{\alpha,i} f_{\alpha,i} + v_\alpha \pi^{N+1}$$

in $A_\alpha[t]$ with the property that $g_\alpha = \det(\partial f_{\alpha,i}/\partial t_j)$. Since $1 - v_\alpha\pi$ is invertible in $A_\alpha\langle t\rangle$, it follows that $B_\alpha = A_\alpha\langle t\rangle/(f_\alpha)^{\mathrm{sat}}$ is a rig-étale A_α-algebra. Clearly, the functor (1.4) sends B_α to B.

Step 2. We now prove that (1.4) is fully faithful. We fix two objects $B_o, C_o \in \mathcal{E}'_{A_o}$. For an index α, we set $B_\alpha = (B_o \widehat{\otimes}_{A_o} A_\alpha)/(0)^{\mathrm{sat}}$ and define C_α similarly. We also set $B = (B_o \widehat{\otimes}_{A_o} A)/(0)^{\mathrm{sat}}$ and define C similarly. We want to show that

$$\operatorname*{colim}_\alpha \mathrm{Hom}_{A_\alpha}(B_\alpha, C_\alpha) \to \mathrm{Hom}_A(B, C)$$

is a bijection. (This is enough since we are free to change the smallest index o. We also used that the colimit in (1.4) is filtered in order to describe the hom-set in the domain.) The above map can be rewritten as

$$\operatorname*{colim}_\alpha \mathrm{Hom}_{A_o}(B_o, C_\alpha) \to \mathrm{Hom}_{A_o}(B_o, C).$$

Since C and the C_α's are π-torsion-free, we may replace B_o by any rig-étale A_o-algebra B'_o such that $B_o \simeq B'_o/(0)^{\mathrm{sat}}$. By Remark 1.3.5, we may choose B'_o topologically finitely presented. We now apply Proposition 1.3.7: there exists an integer N such that the maps

$$\mathrm{Hom}_{A_o}(B'_o, C_\alpha) \to \mathrm{im}\left\{\mathrm{Hom}_{A_o/\pi^{2N}}(B'_o/\pi^{2N}, C_\alpha/\pi^{2N}) \to \mathrm{Hom}_{A_o/\pi^N}(B'_o/\pi^N, C_\alpha/\pi^N)\right\}$$

are bijections and similarly for C (instead of C_α). Since filtered colimits commute with taking images, we are left to show that

$$\operatorname*{colim}_\alpha \mathrm{Hom}_{A_o/\pi^e}(B'_o/\pi^e, C_\alpha/\pi^e) \to \mathrm{Hom}_{A_o/\pi^e}(B'_o/\pi^e, C/\pi^e)$$

is a bijection for any positive integer e. This is clear since B'_o/π^e is a finitely presented A_o/π^e-algebra and C/π^e is the colimit of the filtered system $(C_\alpha/\pi^e)_\alpha$. □

For later use, we record the following two results.

Lemma 1.3.11. *Let $e : X' \to X$ be an étale morphism of rigid analytic spaces, and let $s : X \to X'$ be a section of e. Then s is an open immersion.*

Proof. The question is local on X and around $s(X)$. Thus, we may assume that $X = \mathrm{Spf}(A)^{\mathrm{rig}}$ with A an adic ring of principal ideal type, that $X' = \mathrm{Spf}(A')^{\mathrm{rig}}$ with A' a rig-étale adic A-algebra, and that s is induced by a morphism of A-algebras $h : A' \to A$. Fix a generator π of an ideal of definition of A. By Proposition 1.3.8, we may assume that $A' = A\langle t\rangle/(f)^{\mathrm{sat}}$ with $t = (t_1, \dots, t_n)$ a system of coordinates and $f = (f_1, \dots, f_n)$ an n-tuple in $A[t]$ such that $\det(\partial f_i/\partial t_j)$ generates an open ideal in A'. Consider the A-algebra $C = A[t]/(f)$. Then, $C[\pi^{-1}]$ is étale over $A[\pi^{-1}]$ and h induces a morphism of $A[\pi^{-1}]$-algebras $C[\pi^{-1}] \to A[\pi^{-1}]$. From standard properties of ordinary étale algebras, we deduce that $\mathrm{Spec}(A[\pi^{-1}]) \to \mathrm{Spec}(C[\pi^{-1}])$ is a clopen immersion. Passing to the analytification over A in the sense of Construction 1.1.15, we deduce a clopen immersion $\mathrm{Spf}(A)^{\mathrm{rig}} \to \mathrm{Spec}(C[\pi^{-1}])^{\mathrm{an}}$. But the latter factors as follows:

$$\mathrm{Spf}(A)^{\mathrm{rig}} \xrightarrow{s} \mathrm{Spf}(A')^{\mathrm{rig}} \to \mathrm{Spec}(C[\pi^{-1}])^{\mathrm{an}},$$

where the second map is an open immersion. This finishes the proof. □

Proposition 1.3.12. *Let $i : Z \to X$ be a closed immersion of rigid analytic spaces. Let X' be an étale rigid analytic X-space and $s : Z \to X'$ a partial section. Then, locally on X, s extends to a section $\tilde{s} : U \to X'$ defined on an open neighbourhood U of Z. Moreover, $\tilde{s}$ is an open immersion.*

Proof. The question being local on X, we may assume that $X = \mathrm{Spf}(A)^{\mathrm{rig}}$ with A an adic ring of principal ideal type, and $Z = \mathrm{Spf}(B)^{\mathrm{rig}}$ with B a quotient of A by a closed ideal $I \subset A$. We may also assume that $X' = \mathrm{Spf}(A')^{\mathrm{rig}}$ with A' a rig-étale A-algebra, and that the section s is induced by a morphism of A-adic rings $h : A' \to B$. Let $\pi \in A$ be a generator of an ideal of definition. Without loss of generality, we may assume that B and A' are π-torsion-free.

For $N \in \mathbb{N}$ and $J \subset I$ a finitely generated ideal, consider the adic A-algebra $C_{J,N} = A\langle J/\pi^N \rangle$ given as the π-adic completion of the sub-A-algebra $A[J/\pi^N] \subset A[\pi^{-1}]$ generated by fractions a/π^N with $a \in J$. Then B is the filtered colimit in the category of adic rings of the $C_{J,N}$'s when N and J vary. Applying Corollary 1.3.10 to this inductive system, we can find J and N such that the image of $\mathrm{Hom}_A(A', C_{J,N}) \to \mathrm{Hom}_A(A', B)$ contains h. This means that the section s extends to an X-morphism $\mathrm{Spf}(C_{J,N})^{\mathrm{rig}} \to \mathrm{Spf}(A')^{\mathrm{rig}}$. Since $\mathrm{Spf}(C_{J,N})^{\mathrm{rig}}$ is an open subspace of X, this proves the existence of $\tilde{s}$ as in the proposition. That $\tilde{s}$ is an open immersion follows from Lemma 1.3.11. □

Definition 1.3.13.

(1) Let A be an adic ring and B an adic A-algebra. We say that B is rig-smooth over A if, locally on B, there exists a rig-étale morphism of adic A-algebras $A\langle t_1, \ldots, t_m \rangle \to B$.
(2) A morphism $\mathcal{Y} \to \mathcal{X}$ of formal schemes is said to be rig-smooth if, locally for the rig topology on $\mathcal{X}$ and $\mathcal{Y}$ (see Definition 1.4.10 below), it is isomorphic to $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ with B rig-smooth over A.
(3) A morphism of rigid analytic spaces $Y \to X$ is said to be smooth if, locally on X and Y, it admits a formal model which is rig-smooth.

Remark 1.3.14. By [Hub96, Corollary 1.6.10 & Proposition 1.7.1], we see that, via the embedding of Corollary 1.2.7, a map of uniform adic spaces is smooth (resp. étale) if and only if the associated map of rigid analytic spaces is.

The next proposition is similar to [Elk73, page 582, Théorème 7], but we do not assume the adic ring A to be noetherian.

Proposition 1.3.15. *Let A be an adic ring of monogenic ideal type and $\pi \in A$ a generator of an ideal of definition of A. Let B be a rig-étale (resp. rig-smooth) adic A-algebra, and assume that B is π-torsion-free. Then, locally on B, there exists a finitely generated π-torsion-free A-algebra P such that $P[\pi^{-1}]$ is étale (resp. smooth) over $A[\pi^{-1}]$ and its π-adic completion $\widehat{P} = \lim_{n \in \mathbb{N}} P/\pi^n$ is isomorphic to B.*

Proof. According to [Elk73, pages 588–589], the proof of [Elk73, page 582, Théorème 7] can be adapted to cover the above statement. Alternatively, one can use Proposition 1.3.8 as follows. By this proposition, we may assume that the adic A-algebra B is of the form

$$B = A\langle t_1, \ldots, t_m, s_1, \ldots, s_n \rangle / (f_1, \ldots, f_n)^{\mathrm{sat}},$$

with $f_1, \dots, f_n \in A[t_1, \dots, t_m, s_1, \dots, s_n]$, and such that $\det(\partial f_i/\partial s_j)$ generates an open ideal in B. (The rig-étale case corresponds to $m = 0$.) Consider the A-algebra

$$P' = A[t_1, \dots, t_m, s_1, \dots, s_n]/(f_1, \dots, f_n)^{\mathrm{sat}}$$

whose π-adic completion is B. Let $e \in P'$ be the image of $\det(\partial f_i/\partial s_j)$ in P'. By assumption, a power of π is a multiple of e in the π-adic completion of P'. Thus, there are elements $b, c \in B$ and an integer N such that $\pi^N = e \cdot b + c\pi^{N+1}$. The A-algebra $P = P'[(1 - c\pi)^{-1}]$ satisfies the properties required in the statement. □

The following is a variant of Proposition 1.3.12 for smooth morphisms. It will play a crucial role in the proof of the localization property for rigid analytic motives (see Proposition 2.2.3).

Proposition 1.3.16. *Let $Z \to X$ be a closed immersion of rigid analytic spaces. Let X' be a smooth rigid analytic X-space and $s : Z \to X'$ a partial section. Then, locally on X, we may find an open neighbourhood $U \subset X$ of Z, an open neighbourhood $U' \subset X'$ of $s(Z)$ and an isomorphism $U' \simeq \mathbb{B}^m_U$, for some integer $m \geq 0$, modulo which $s : Z \to U'$ is the zero section over Z.*

Proof. The problem being local on X and around $s(Z)$, we may assume that X' is étale over $\mathbb{B}^m_X$ and, by change of coordinates, that the composition

$$Z \xrightarrow{s} X' \to \mathbb{B}^m_X$$

is the zero section over Z. Applying Proposition 1.3.12 to the étale morphism $X' \to \mathbb{B}^m_X$ and the closed immersion $Z \to \mathbb{B}^m_X$ given by the zero section over Z, we find locally an open neighbourhood $U' \subset X'$ of $s(Z)$ such that $U' \to \mathbb{B}^m_X$ is also an open immersion. Letting U be the inverse image of U' by the zero section $X \to \mathbb{B}^m_X$ and replacing U' by $U' \times_X U$, we may assume that U' is an open neighbourhood of the zero section of $\mathbb{B}^n_U$. Since the zero section of $\mathbb{B}^n_U$ admits a system of fundamental neighbourhoods which are m-dimensional relative balls, we may also assume that U' is isomorphic to $\mathbb{B}^m_U$ as needed. □

We end this subsection with the following result.

Proposition 1.3.17. *Let $f : Y \to X$ be a smooth morphism of rigid analytic spaces. Then the induced map $|f| : |Y| \to |X|$ is open.*

Proof. It is enough to show that $f(|Y|)$ is open in $|X|$. The question is local on X and Y. By Proposition 1.3.15 we may assume that $X = \mathrm{Spf}(A)^{\mathrm{rig}}$, with A an adic ring of principal ideal type, and $Y = \mathrm{Spf}(B)^{\mathrm{rig}}$, with $B = \widehat{P}$ the π-adic completion of a finitely presented A-algebra P such that $P[\pi^{-1}]$ is smooth over $A[\pi^{-1}]$. (As usual, π is a generator of an ideal of definition of A. Also, note that finite presentation in Proposition 1.3.15 can be assumed if we don't insist on π-torsion-freeness.) By the Raynaud–Gruson platification theorem [RG71, Theorem 5.2.2], and working locally over X, we may further assume that P is flat over A. By [Gro66, Chapitre IV, Théorème 2.4.6], the morphism $\mathrm{Spec}(P) \to \mathrm{Spec}(A)$ is then open, and we denote by $U \subset \mathrm{Spec}(A)$ its image. Let $(a_i)_i$ be a family in A generating the ideal defining the complement of U in $\mathrm{Spec}(A)$. Let A_i be the π-adic completion of $A[a_i^{-1}]$ and B_i the π-adic completion of $P_i = P \otimes_A A_i$. Set $X_i = \mathrm{Spf}(A_i)^{\mathrm{rig}}$ and $Y_i = \mathrm{Spf}(B_i)^{\mathrm{rig}}$. By construction, $(Y_i)_i$ is an open covering of Y and it is enough to show that $f(Y_i)$ is open in X. We will show more precisely that $f(Y_i) = X_i$, i.e., that $Y_i \to X_i$ is surjective.

Replacing X and Y by X_i and Y_i, we are reduced to showing that $f : Y \to X$ is surjective, for $X = \mathrm{Spf}(A)^{\mathrm{rig}}$ and $Y = \mathrm{Spf}(B)^{\mathrm{rig}} \simeq \mathrm{Spf}(\widehat{P})^{\mathrm{rig}}$ as above, assuming furthermore that the A-algebra P is faithfully flat. To do so, it will be enough to show the following assertion. If $\mathcal{X}' \to \mathrm{Spf}(A)$ is an admissible blowup and $\mathcal{Y}' = (\mathcal{X}' \otimes_A B)/(0)^{\mathrm{sat}}$, the induced map $\mathcal{Y}'_\sigma \to \mathcal{X}'_\sigma$ is surjective. (Indeed, by [FK18, Chapter III, Proposition 3.1.5], the obvious map $|Y| \to |\mathcal{Y}'_\sigma|$ is surjective.) Since P is flat over A, the formal scheme $\mathcal{X}' \times_{\mathrm{Spf}(A)} \mathrm{Spf}(B)$ is already saturated and we have an isomorphism $\mathcal{Y}'/\pi \simeq \mathcal{X}'/\pi \otimes_A P$. In particular, we see that the map $\mathcal{Y}'/\pi \to \mathcal{X}'/\pi$ is faithfully flat, and hence surjective as needed. □

1.4. **Topologies.**

Open covers define the Zariski topologies on schemes and formal schemes, and the analytic topology on rigid analytic spaces. In this subsection, we introduce various finer Grothendieck topologies which we use when discussing motives. On schemes, we mainly consider the étale and Nisnevich topologies. These topologies extend naturally to formal schemes: a family $(\mathcal{Y}_i \to \mathcal{X})_i$ consisting of étale morphisms is an étale (resp. a Nisnevich) cover if $(\mathcal{Y}_{i,\sigma} \to \mathcal{X}_\sigma)_i$ is an étale (resp. a Nisnevich) cover.

Notation 1.4.1. Given a scheme S, we denote by $\text{Ét}/S$ the category of étale S-schemes. Similarly, given a formal scheme $\mathcal{S}$, we denote by $\text{Ét}/\mathcal{S}$ the category of étale formal $\mathcal{S}$-schemes.

Lemma 1.4.2. *Let $\mathcal{S}$ be a formal scheme. The functor $\mathcal{X} \mapsto \mathcal{X}_\sigma$ induces an équivalence of categories $\text{Ét}/\mathcal{S} \to \text{Ét}/\mathcal{S}_\sigma$ respecting the étale and Nisnevich topologies.*

Proof. This follows immediately from [Gro67, Chapitre IV, Théorème 18.1.2]. □

Notation 1.4.3. Given a rigid analytic space S, we denote by $\text{Ét}/S$ the category of étale rigid analytic S-spaces (in the sense of Definition 1.3.3). We denote by $\text{Ét}^{\mathrm{gr}}/S$ the full subcategory of $\text{Ét}/S$ spanned by those étale rigid analytic S-spaces with good reduction (in the sense of Definition 1.1.14).

Definition 1.4.4. Let $(Y_i \to X)_i$ be a family of étale morphisms of rigid analytic spaces. We say that this family is a Nisnevich cover if, locally on X and after refinement, it admits a formal model $(\mathcal{Y}_i \to \mathcal{X})_i$ which is a Nisnevich cover. Nisnevich covers generate a topology on rigid analytic spaces which we call the Nisnevich topology.

Definition 1.4.5. Let $(f_i : Y_i \to X)_i$ be a family of étale morphisms of rigid analytic spaces. We say that this family is an étale cover if it is jointly surjective, i.e., $|X| = \bigcup_i f_i(|Y_i|)$. Étale covers generate the étale topology on rigid analytic spaces.

Remark 1.4.6. By means of Proposition 1.2.5 and Remark 1.3.14, we see that the above definition of étale covers agrees with the one for uniform adic spaces in [Hub96, Section 2.1]. Also, note that if X is quasi-compact, then every étale cover of X can be refined by a finite subfamily. This follows from Proposition 1.3.17.

Notation 1.4.7. The étale topology is generally denoted by "ét" and the Nisnevich topology is denoted by "nis". Also, the Zariski topology is generally denoted by "zar" and the analytic topology is denoted by "an".

Remark 1.4.8. If S is a scheme and $\tau \in \{\mathrm{nis}, \text{ét}\}$, we call $(\text{Ét}/S, \tau)$ the small τ-site of S, and similarly for a formal scheme. If S is a rigid analytic space, we call $(\text{Ét}^{\mathrm{gr}}/S, \mathrm{nis})$ the small Nisnevich site of S and $(\text{Ét}/S, \text{ét})$ the small étale site of S.

The big smooth sites introduced below are used for constructing the categories of motives.

Notation 1.4.9.

(1) If S is a scheme, we denote by Sch/S the overcategory of S-schemes and Sm/S its full subcategory consisting of smooth objects. For $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$, we call $(\mathrm{Sm}/S, \tau)$ the big smooth site of S.
(2) If $\mathcal{S}$ is a formal scheme, we denote by $\mathrm{FSch}/\mathcal{S}$ the overcategory of formal $\mathcal{S}$-schemes and $\mathrm{FSm}/\mathcal{S}$ its full subcategory consisting of smooth objects. For $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$, we call $(\mathrm{FSm}/S, \tau)$ the big smooth site of $\mathcal{S}$.
(3) If S is a rigid analytic space, we denote by RigSpc/S the overcategory of rigid analytic S-spaces and RigSm/S its full subcategory consisting of smooth objects (in the sense of Definition 1.3.13). For $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$, we call $(\mathrm{RigSm}/S, \tau)$ the big smooth site of S.

We next discuss the class of rig topologies on formal schemes.

Definition 1.4.10. Let $(\mathcal{Y}_i \to \mathcal{X})_i$ be a family of morphisms of formal schemes. We say that this family is a rig cover if the induced family $(\mathcal{Y}_i^{\mathrm{rig}} \to \mathcal{X}^{\mathrm{rig}})_i$ is an open cover. The topology generated by rig covers is called the rig topology and it is denoted by "rig".

Remark 1.4.11. Let $\mathcal{X}$ be a quasi-compact and quasi-separated formal scheme. Then every rig cover of $\mathcal{X}$ can be refined by the composition of an admissible blowup $\mathcal{X}' \to \mathcal{X}$ and a Zariski cover of $\mathcal{X}'$.

By "equivalence of sites" we mean a continuous functor inducing an equivalence between the associated ordinary topoi.

Lemma 1.4.12. *Consider full subcategories* $\underline{\mathcal{V}} \subset \mathrm{FSch}$ *(resp.* $\underline{\mathcal{V}} \subset \mathrm{FSch}/\mathcal{S}$ *for a formal scheme* $\mathcal{S}$*) and* $\underline{V} \subset \mathrm{RigSpc}$ *(resp.* $\underline{V} \subset \mathrm{RigSpc}/S$ *with* $S = \mathcal{S}^{\mathrm{rig}}$*) such that:*

- $\underline{\mathcal{V}}$ *is stable by admissible blowups and quasi-compact open formal subschemes;*
- $\underline{V}$ *contains* $\mathcal{X}^{\mathrm{rig}}$ *for every* $\mathcal{X} \in \underline{\mathcal{V}}$*, and every object of* $\underline{V}$ *is locally of this form.*

Then the functor $(-)^{\mathrm{rig}} : \underline{\mathcal{V}} \to \underline{V}$ *defines an equivalence of sites* $(\underline{V}, \mathrm{an}) \xrightarrow{\sim} (\underline{\mathcal{V}}, \mathrm{rig})$. *In particular, we have an equivalence of sites* $(\mathrm{RigSpc}, \mathrm{an}) \xrightarrow{\sim} (\mathrm{FSch}, \mathrm{rig})$ *(resp.* $(\mathrm{RigSpc}/S, \mathrm{an}) \xrightarrow{\sim} (\mathrm{FSch}/\mathcal{S}, \mathrm{rig})$*).*

Proof. The statement would have been a particular case of [Hub96, Corollary A.4], except that we don't know a priori that the continuous functor $(-)^{\mathrm{rig}}$ defines a morphism of sites and that we do not assume that our categories have finite limits. (In fact, we are particularly interested in the case where $\underline{\mathcal{V}}$ is the category of rig-smooth formal $\mathcal{S}$-schemes, which does not admit finite limits.) Instead of trying to modify the proof of [Hub96, Corollary A.4], we present an independent argument. We only treat the absolute case since the relative case is similar.

By [SGAIV1, Exposé III, Théorème 4.1], we may assume that $\underline{\mathcal{V}} \subset \mathrm{FSch}^{\mathrm{qcqs}}$ and that $\underline{V}$ is the full subcategory of $\mathrm{RigSpc}^{\mathrm{qcqs}}$ spanned by objects of the form $\mathcal{X}^{\mathrm{rig}}$ for $\mathcal{X} \in \underline{\mathcal{V}}$. The rig topology on $\underline{\mathcal{V}}$ is not subcanonical (except for very special choices of $\underline{\mathcal{V}}$). We denote by $\underline{\mathcal{V}}'$ the full subcategory of the category of sheaves of sets on $(\underline{\mathcal{V}}, \mathrm{rig})$ spanned by sheafifications of representable presheaves. The obvious functor $a : \underline{\mathcal{V}} \to \underline{\mathcal{V}}'$, sending a formal scheme $\mathcal{X}$ to the sheaf associated of the presheaf represented by $\mathcal{X}$, induces an equivalence of sites $(\underline{\mathcal{V}}', \mathrm{rig}) \simeq (\underline{\mathcal{V}}, \mathrm{rig})$, where the topology of $(\underline{\mathcal{V}}', \mathrm{rig})$ is the one induced from the canonical topology on the topos of sheaves on $(\underline{\mathcal{V}}, \mathrm{rig})$. (This is a well-known fact which follows, for example, from [SGAIV1, Exposé IV, Corollaire 1.2.1]; see also [Ayo07b, Corollaire 4.4.52].) To prove the lemma, we remark that there is an equivalence of categories $\underline{\mathcal{V}}' \simeq \underline{V}$ which identifies the rig topology on $\underline{\mathcal{V}}'$ with the analytic topology on $\underline{V}$. Indeed, for an admissible blowup $\mathcal{Y}' \to \mathcal{Y}$ in $\underline{\mathcal{V}}$, the diagonal map $\mathcal{Y}' \to \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{Y}'$ is a rig cover, which implies that $a\mathcal{Y}' \to a\mathcal{Y}$ is an isomorphism. Using that the Zariski topology is subcanonical on $\underline{\mathcal{V}}$, we deduce that

$$\mathrm{Hom}_{\underline{\mathcal{V}}'}(a\mathcal{Y}, a\mathcal{X}) = \operatorname*{colim}_{\mathcal{Y}'\to\mathcal{Y}\in\mathfrak{B}(\mathcal{Y})} \mathrm{Hom}_{\underline{\mathcal{V}}}(\mathcal{Y}', \mathcal{X})$$

for any $\mathcal{X}, \mathcal{Y} \in \underline{\mathcal{V}}$. The result follows then by comparison with (1.1). □

Corollary 1.4.13. *Let $\tau \in \{\mathrm{nis}, \text{ét}\}$ be one of the topologies introduced above on rigid analytic spaces. Consider full subcategories $\underline{\mathcal{V}} \subset \mathrm{FSch}$ (resp. $\underline{\mathcal{V}} \subset \mathrm{FSch}/\mathcal{S}$ for a formal scheme $\mathcal{S}$) and $\underline{V} \subset \mathrm{RigSpc}$ (resp. $\underline{V} \subset \mathrm{RigSpc}/S$ with $S = \mathcal{S}^{\mathrm{rig}}$) satisfying the following conditions.*

- *If $\tau = \mathrm{nis}$, then $\underline{\mathcal{V}}$ is stable by admissible blowups and every étale morphism whose target is in $\underline{\mathcal{V}}$ lies entirely in $\underline{\mathcal{V}}$.*
- *If $\tau = \text{ét}$, then every rig-étale morphism whose target is in $\underline{\mathcal{V}}$ lies entirely in $\underline{\mathcal{V}}$.*
- *$\underline{V}$ contains $\mathcal{X}^{\mathrm{rig}}$ for every $\mathcal{X} \in \underline{\mathcal{V}}$, and every object of $\underline{V}$ is locally of this form.*

Then there exists a unique topology $\text{rig-}\tau$ on $\underline{\mathcal{V}}$ such that the functor $(-)^{\mathrm{rig}} : \underline{\mathcal{V}} \to \underline{V}$ defines an equivalence of sites $(\underline{V}, \tau) \xrightarrow{\sim} (\underline{\mathcal{V}}, \text{rig-}\tau)$. In particular, we have an equivalence of sites $(\mathrm{RigSpc}, \tau) \xrightarrow{\sim} (\mathrm{FSch}, \text{rig-}\tau)$ (resp. $(\mathrm{RigSpc}/S, \tau) \xrightarrow{\sim} (\mathrm{FSch}/\mathcal{S}, \text{rig-}\tau)$).

Remark 1.4.14. Corollary 1.4.13 gives us two more topologies on formal schemes: the rig-Nisnevich topology (denoted by "rignis") and the rig-étale topology (denoted by "rigét"). These topologies can be described more directly by their corresponding notions of covers. A family $(\mathcal{Y}_i \to \mathcal{X})_i$ of morphisms of formal schemes is a rig-Nisnevich cover if the induced family $(\mathcal{Y}_i^{\mathrm{rig}} \to \mathcal{X}^{\mathrm{rig}})_i$ is a Nisnevich cover. In particular, if $\mathcal{X}$ is a quasi-compact and quasi-separated formal scheme, then every rig-Nisnevich cover of $\mathcal{X}$ can be refined by the composition of an admissible blowup $\mathcal{X}' \to \mathcal{X}$ and a Nisnevich cover of $\mathcal{X}'$. Proposition 1.4.19 below gives an analogous result for rig-étale covers.

Remark 1.4.15. Summarizing, we have a diagram of morphisms of sites:

$$\begin{tikzcd}
(\mathrm{FSch}, \text{ét}) & (\mathrm{FSch}, \text{rigét}) \arrow[l] \arrow[d] & (\mathrm{RigSpc}, \text{ét}) \arrow[l, "\sim"'] \arrow[d] \\
(\mathrm{FSch}, \mathrm{nis}) \arrow[u, leftarrow] \arrow[d] & (\mathrm{FSch}, \mathrm{rignis}) \arrow[l] \arrow[d] & (\mathrm{RigSpc}, \mathrm{nis}) \arrow[l, "\sim"'] \arrow[d] \\
(\mathrm{FSch}, \mathrm{zar}) & (\mathrm{FSch}, \mathrm{rig}) \arrow[l] & (\mathrm{RigSpc}, \mathrm{an}). \arrow[l, "\sim"']
\end{tikzcd}$$

Definition 1.4.16.

(1) Let A be an adic ring and B an adic A-algebra. We say that B is finite rig-étale if B is finite over A and étale over $\mathrm{Spec}(A) \smallsetminus \mathrm{Spec}(A/I)$ for an ideal of definition I of A.
(2) A morphism of formal schemes $\mathcal{Y} \to \mathcal{X}$ is said to be finite rig-étale if it is affine and, locally over $\mathcal{X}$, isomorphic to $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ with B a finite rig-étale adic A-algebra.
(3) A morphism of formal schemes $\mathcal{Y} \to \mathcal{X}$ is said to be a finite rig-étale covering if it is finite rig-étale and the induced morphism $|\mathcal{Y}^{\mathrm{rig}}| \to |\mathcal{X}^{\mathrm{rig}}|$ is surjective.

Lemma 1.4.17. *Let A be an adic ring and B a finite adic A-algebra. Then $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is a finite rig-étale covering if and only if*

$$\mathrm{Spec}(B) \smallsetminus \mathrm{Spec}(B/IB) \to \mathrm{Spec}(A) \smallsetminus \mathrm{Spec}(A/I) \tag{1.5}$$

is a finite étale covering, when I is an ideal of definition of A.

Proof. The morphism (1.5) is finite étale if and only if $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is finite rig-étale. So we need to show that (1.5) is surjective if and only if $|\mathrm{Spf}(B)^{\mathrm{rig}}| \to |\mathrm{Spf}(A)^{\mathrm{rig}}|$ is surjective. This follows easily from the description of $|\mathrm{Spf}(A)^{\mathrm{rig}}|$ in terms of valuation rings of residue fields of points of $\mathrm{Spec}(A) \smallsetminus \mathrm{Spec}(A/I)$ and [Bou98, Chapter VI, §8, n° 6, Proposition 6]. □

Remark 1.4.18. Using the embedding of Corollary 1.2.7, it follows from Lemma 1.4.17 that a map of uniform adic spaces is finite étale (as in [Hub96, Example 1.6.6.(ii)]) if and only if it has a finite rig-étale formal model.

Proposition 1.4.19. *Let $\mathcal{X}$ be a quasi-compact and quasi-separated formal scheme. Then every rig-étale cover of $\mathcal{X}$ can be refined by the composition of an admissible blowup $\mathcal{X}' \to \mathcal{X}$, a Nisnevich cover $(\mathcal{Y}'_i \to \mathcal{X}')_i$, and finite rig-étale coverings $\mathcal{Z}'_i \to \mathcal{Y}'_i$.*

Proof. Let $(\mathcal{U}_j \to \mathcal{X})_{j\in J}$ be a rig-étale cover. We may assume that J is finite (see Remark 1.4.6) and that $\mathcal{X} = \mathrm{Spf}(A)$ is affine with A an adic ring of principal ideal type. We fix a generator $\pi \in A$ of an ideal of definition of A. By Proposition 1.3.15, we may refine the rig-étale cover and assume that each $\mathcal{U}_j$ is the adic completion of a finite presentation A-scheme U_j which is étale over $A[\pi^{-1}]$. (Note that finite presentation in Proposition 1.3.15 can be assumed if we don't insist on π-torsion-freeness.) By the Raynaud–Gruson platification theorem [RG71, Theorem 5.2.2], there exists an admissible blowup $X' \to X = \mathrm{Spec}(A)$ such that the strict transform $U'_j \to X'$ of $U_j \to X$ is flat for every j. In particular, the morphism $U'_j \to X'$ is also quasi-finite.

Let $\mathcal{U}'_j$ and $\mathcal{X}'$ be the adic completions of U'_j and X'. By construction, we have $\mathcal{X}'^{\mathrm{rig}} \simeq \mathcal{X}^{\mathrm{rig}}$ and $\mathcal{U}'^{\mathrm{rig}}_j \simeq \mathcal{U}^{\mathrm{rig}}_j$. Thus, $(\mathcal{U}'_j \to \mathcal{X}')_j$ is also a rig-étale cover. Since $\mathcal{O}_{\mathcal{X}'}$ and the $\mathcal{O}_{\mathcal{U}'_j}$'s are π-torsion-free, we deduce that the family $(\mathcal{U}'_j \to \mathcal{X}')_j$ is jointly surjective. Equivalently, the family of quasi-finite morphisms $(U'_j/\pi \to X'/\pi)_j$ is jointly surjective. Using standard properties of the Nisnevich topology, we can find a family of étale morphisms $(Y'_i \to X')_i$ such that:

(1) $(Y'_i/\pi \to X'/\pi)_i$ is a Nisnevich cover of X'/π;
(2) for every index i there is a index j and a clopen immersion $Z'_i \to U'_j \times_{X'} Y'_i$ such that $Z'_i \to Y'_i$ is finite and $Z'_i/\pi \to Y'_i/\pi$ is surjective.

In addition to being finite, the morphism $Z'_i \to Y'_i$ is flat and étale over $Y'_i[\pi^{-1}]$. Since $Z'_i/\pi \to Y'_i/\pi$ is surjective, we may replace Y'_i by an open neighbourhood of Y'_i/π and assume that $Z'_i \to Y'_i$ is also surjective. In particular, we see that $Z'_i[\pi^{-1}] \to Y'_i[\pi^{-1}]$ is a finite étale covering. If $\mathcal{Y}'_i$ and $\mathcal{Z}'_i$ denote the adic completions of Y'_i and Z_i, Lemma 1.4.17 implies that the morphisms $\mathcal{Z}'_i \to \mathcal{Y}'_i$ are finite rig-étale coverings. Moreover, the family $(\mathcal{Y}'_i \to \mathcal{X}')_i$ is a Nisnevich cover by point (1) above. Finally, the family $(\mathcal{Z}'_i \to \mathcal{X})_i$ refines the initial rig-étale cover as needed. □

Corollary 1.4.20. *Let $(\mathcal{S}_\alpha)_\alpha$ be a cofiltered inverse system of quasi-compact and quasi-separated formal schemes with affine transition maps, and let $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$ be the limit of this system. We set $S_\alpha = \mathcal{S}^{\mathrm{rig}}_\alpha$ and $S = \mathcal{S}^{\mathrm{rig}}$. Then, there is an equivalence of sites* $(\mathrm{\acute{E}t}/S, \mathrm{\acute{e}t}) \simeq \lim_\alpha(\mathrm{\acute{E}t}/S_\alpha, \mathrm{\acute{e}t})$.

Proof. Without loss of generality, we may assume that the indexing category of the inverse system $(\mathcal{S}_\alpha)_\alpha$ admits a final object o. We may replace $\mathcal{S}_o$ by the blowup of a finitely generated ideal of definition and each $\mathcal{S}_\alpha$ by its strict transform, and assume that the $\mathcal{S}_\alpha$'s are locally of principal ideal type. The question being local for the Zariski topology on $\mathcal{S}_o$, we may assume that the formal schemes $\mathcal{S}_\alpha$'s are affine of principal ideal type. We set $A_\alpha = \mathcal{O}(\mathcal{S}_\alpha)$ and $A = \mathcal{O}(S)$, and we employ Notation 1.3.9. Using Corollary 1.4.13, it is enough to show that the morphism of sites

$$(\mathcal{E}'_A, \mathrm{rig\acute{e}t}) \to \lim_\alpha(\mathcal{E}'_{A_\alpha}, \mathrm{rig\acute{e}t})$$

is an equivalence. Corollary 1.3.10 gives an equivalence on the underlying categories and it remains to show that the topologies match. For this, we need to show that every rig-étale cover in $\mathcal{E}'_A$ can be refined by the image of a rig-étale cover in $\mathcal{E}'_{A_\alpha}$ for α small enough. This follows readily from Proposition 1.4.19. □

Remark 1.4.21. Keeping the notation of Corollary 1.4.20, we similarly have an equivalence of sites $(\acute{\mathrm{E}}\mathrm{t}^{\mathrm{gr}}/S, \mathrm{nis}) \simeq \lim_\alpha(\acute{\mathrm{E}}\mathrm{t}^{\mathrm{gr}}/S_\alpha, \mathrm{nis})$. This is easier to prove: one reduces to the analogous statement for the small Nisnevich sites of formal schemes, and then further to the analogous statement for the small Nisnevich sites of ordinary schemes using Lemma 1.4.2.

We end this subsection with a short discussion of points in the rigid analytic setting.

Definition 1.4.22. A rigid point s is a rigid analytic space of the form $\mathrm{Spf}(V)^{\mathrm{rig}}$ where V is an adic valuation ring of principal ideal type; compare with [FK18, Chapter II, Definition 8.2.1]. We also write s for the unique closed point of $|s|$. Using Notation 1.1.13, we then have $V = \kappa^+(s)$. Also, $\kappa(s)$ is the fraction field of V, $\widetilde{\kappa}(s)$ is the residue field of V and $\kappa^\circ(s)$ is the localisation of V at its height 1 prime ideal. A morphism of rigid points $s' \to s$ is a morphism of rigid analytic spaces sending the closed point of $|s'|$ to the closed point of $|s|$. Said differently, the induced morphism $\kappa^+(s) \to \kappa^+(s')$ is local.

Remark 1.4.23. A morphism of rigid points $\overline{s} \to s$ is said to be algebraic if the complete field $\kappa(\overline{s})$ contains a dense separable extension of $\kappa(s)$. Algebraic rigid points over s are all obtained by the following recipe. Start with a separable extension $L/\kappa(s)$ and choose a valuation ring $V \subset L$ such that $V \cap \kappa(s) = \kappa^+(s)$. (By [Bou98, Chapter VI, §8, n° 6, Proposition 6 & Corollary 1] such valuation rings exist, and they are conjugate under the automorphism group of the extension $L/\kappa(s)$ if the latter is Galois.) Then define a rigid point $\overline{s}$ by taking $\kappa^+(\overline{s})$ to be the adic completion of V (considered as a $\kappa^+(s)$-algebra). By [BGR84, Proposition 3.4.1/6], if L is a separable closure of $\kappa(s)$, then $\kappa(\overline{s})$ is algebraically closed (and not only separably closed).

Definition 1.4.24. Let $\overline{s}$ be a rigid point.

(1) We say that $\overline{s}$ is nis-geometric if the valuation ring $\kappa^+(\overline{s})$ is Henselian.
(2) We say that $\overline{s}$ is ét-geometric (or, simply, geometric) if the field $\kappa(\overline{s})$ is algebraically closed.

Remark 1.4.25. Let S be a rigid analytic space.

(1) A point $s \in S$ determines a rigid point, which we denote again by s, given by $\mathrm{Spf}(\kappa^+(s))^{\mathrm{rig}}$. Moreover, we have an obvious morphism of rigid analytic spaces $s \to S$ sending the closed point of $|s|$ to $s \in |S|$.
(2) A morphism of rigid analytic spaces $\overline{s} \to S$ from a rigid point $\overline{s}$ is called a rigid point of S. It factors uniquely as $\overline{s} \to s \to S$, where $s \in |S|$ is the image of the closed point of $|\overline{s}|$. By abuse of language, we say that "s is the image of $\overline{s} \to S$" or that "$\overline{s}$ is over s". We say that a rigid point $\overline{s} \to S$ of S is algebraic if the morphism of rigid points $\overline{s} \to s$ is algebraic. (See Remark 1.4.23.)

Lemma 1.4.26. *Let $\mathcal{S}$ be a formal scheme and set $S = \mathcal{S}^{\mathrm{rig}}$.*

(1) Given a point $s \in S$, there is a canonical isomorphism

$$\mathrm{Spf}(\kappa^+(s)) \simeq \lim_{\mathrm{Spf}(\kappa^+(s)) \to \mathcal{U} \to \mathcal{S}} \mathcal{U},$$

where the limit is over factorizations of $\mathrm{Spf}(\kappa^+(s)) \to \mathcal{S}$ *with* $\mathcal{U}$ *affine and such that* $\mathcal{U}^{\mathrm{rig}}$ *is an open neighbourhood of* s *in* S.

(2) Given an algebraic rigid point $\overline{s} \to S$*, there is a canonical isomorphism*

$$\mathrm{Spf}(\kappa^+(\overline{s})) \simeq \lim_{\mathrm{Spf}(\kappa^+(\overline{s}))\to\mathcal{U}\to\mathcal{S}} \mathcal{U},$$

where the limit is over factorizations of $\mathrm{Spf}(\kappa^+(\overline{s})) \to \mathcal{S}$ *with* $\mathcal{U}$ *affine and rig-étale over* $\mathcal{S}$.

Proof. Assertion (1) follows immediately from [FK18, Chapter II, Proposition 3.2.6] and the definition of $\kappa^+(s)$; see Notation 1.1.13. To prove assertion (2), we may assume that $\mathcal{S} = \mathrm{Spf}(A)$ is affine and prove that the A-algebra $\kappa^+(\overline{s})$ is a filtered colimit of rig-étale adic A-algebras in the category of adic rings. Let $s \in S$ be the image of $\overline{s}$. Using assertion (1), we may write

$$\kappa^+(s) = \operatorname*{colim}_{\alpha} A_\alpha,$$

in the category of adic rings, where A_α are adic A-algebras such that the $\mathrm{Spf}(A_\alpha)^{\mathrm{rig}}$ are open neighbourhoods of s in $S = \mathrm{Spf}(A)^{\mathrm{rig}}$. Applying Corollary 1.3.10 to the inductive system $(A_\alpha)_\alpha$, we see that every rig-étale $\kappa^+(s)$-algebra whose zero ideal is saturated is a filtered colimit in the category of adic rings of rig-étale adic A-algebras. Thus, it is enough to show that $\kappa^+(\overline{s})$ is a filtered colimit of adic rig-étale $\kappa^+(s)$-algebras. This follows immediately from Remark 1.4.23 and the following fact. If $L/\kappa(s)$ is a finite separable extension and $R \subset L$ is a sub-$\kappa^+(s)$-algebra of finite type with fraction field L, then R is a rig-étale $\kappa^+(s)$-algebra. (We leave it to the reader to find a presentation of R as in Definition 1.3.3(1).) □

Construction 1.4.27. Let $\tau \in \{\mathrm{nis}, \mathrm{ét}\}$. Let S be a rigid analytic space and let $s \in S$ be a point. We may construct an algebraic τ-geometric rigid point $\overline{s} \to S$ over s as follows.

(1) (The case $\tau = \mathrm{nis}$) Let $\widetilde{\kappa}(\overline{s})/\widetilde{\kappa}(s)$ be a separable extension and denote by $\overline{\kappa}^+(s)$ the Henselisation of $\kappa^+(s)$ at the point $\mathrm{Spec}(\widetilde{\kappa}(\overline{s})) \to \mathrm{Spec}(\kappa^+(s))$. Then $\overline{\kappa}^+(s)$ is again a valuation ring. (This follows from [Bou98, Chapter VI, §8, n° 6, Proposition 6].) We denote by $\kappa^+(\overline{s})$ the adic completion of $\overline{\kappa}^+(s)$ and set $\overline{s} = \mathrm{Spf}(\kappa^+(\overline{s}))^{\mathrm{rig}}$. We have an obvious map $\overline{s} \to S$, which factors through $s \to S$. The map $\overline{s} \to S$ is a nis-geometric rigid point of S.

(2) (The case $\tau = \mathrm{ét}$) Let $\overline{\kappa}(s)$ be a separably closed algebraic extension of $\kappa(s)$. (We do not require this extension to be separable.) Let $\overline{\kappa}^+(s) \subset \overline{\kappa}(s)$ be a valuation ring which extends $\kappa^+(s) \subset \kappa(s)$. We denote by $\kappa^+(\overline{s})$ the adic completion of $\overline{\kappa}^+(s)$ and set $\overline{s} = \mathrm{Spf}(\kappa^+(\overline{s}))^{\mathrm{rig}}$. (As mentioned above, by [BGR84, Proposition 3.4.1/6], the fraction field $\kappa(\overline{s})$ of $\kappa^+(\overline{s})$ is always algebraically closed.) We have an obvious map $\overline{s} \to S$ which factors through $s \to S$. The map $\overline{s} \to S$ is an étale geometric rigid point of S.

In the situation of (1) (resp. (2)), given a presheaf $\mathcal{F}$ on $\mathrm{Ét}^{\mathrm{gr}}/S$ (resp. $\mathrm{Ét}/S$) with values in an ∞-category admitting filtered colimits, we set:

$$\mathcal{F}_{\overline{s}} = \operatorname*{colim}_{\overline{s}\to U\to S} \mathcal{F}(U),$$

where the colimit is over the étale neighbourhoods with good reduction (resp. étale neighbourhoods) of $\overline{s}$ in S. The object $\mathcal{F}_{\overline{s}}$ is called the stalk of $\mathcal{F}$ at $\overline{s}$.

Remark 1.4.28. The functors $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ introduced in Construction 1.4.27 admit a more basic version for the analytic topology, given by $\mathcal{F} \mapsto \mathcal{F}_s = \mathrm{colim}_{s\in U\subset X}\, \mathcal{F}(U)$, where the colimit is over the open neighbourhoods of s in S.

Proposition 1.4.29. *Let* S *be a rigid analytic space.*

(1) The site $(\mathrm{\acute{E}t}^{\mathrm{gr}}/S, \mathrm{nis})$ *admits a conservative family of points given by* $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$*, where* $\overline{s} \to S$ *run over the* nis*-geometric rigid points as in Construction 1.4.27(1).*
(2) The site $(\mathrm{\acute{E}t}/S, \mathrm{\acute{e}t})$ *admits a conservative family of points given by* $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$*, where* $\overline{s} \to S$ *run over the geometric rigid points as in Construction 1.4.27(2).*

Proof. We only treat the second part. By a standard argument, one reduces to prove the following two assertions.

(1) Every étale cover of a geometric rigid point $\overline{s}$ splits.
(2) A family $(Y_i \to X)_i$ in $\mathrm{\acute{E}t}/S$ is an étale cover if, for every geometric rigid point $\overline{s} \to S$ and every S-morphism $\overline{s} \to X$, there exists i and an X-morphism $\overline{s} \to Y_i$.

The first assertion follows from Proposition 1.4.19 (and Corollary 1.4.13). The second assertion follows from Definition 1.4.5. □

Corollary 1.4.30. *Let S be a rigid analytic space and $U \subset S$ a nonempty open subspace. Assume that U and S are quasi-compact. Then, every étale cover of U can be refined by the base change of an étale cover of S.*

Proof. Fix an étale cover $(U_i \to U)_i$ of U with U_i quasi-compact and quasi-separated. Given an algebraic geometric rigid point $\overline{s} \to S$, we consider $\overline{u} = \overline{s} \times_S U$. This is a quasi-compact open rigid analytic subspace of $\overline{s}$. Thus, $\overline{u}$ is either empty or $\overline{u} \to U$ is an algebraic geometric rigid point of U. In both cases, the morphism $\overline{u} \to U$ factors through U_i for some i. Using Corollary 1.4.20 and Lemma 1.4.26, there exists an étale neighbourhood $V_{\overline{s}} \to S$ of $\overline{s}$ such that $V_{\overline{s}} \times_S U$ factors through U_i. This shows that the base change of the étale cover $(V_{\overline{s}} \to S)_{\overline{s}}$ refines $(U_i \to U)_i$ as needed. □

2. Rigid analytic motives

In this section, we recall the construction of rigid analytic motives following [Ayo15] and prove some of their basic properties. In particular, we prove in Subsection 2.3 that the functor $\mathbf{RigSH}_\tau(-;\Lambda)$, sending a rigid analytic space S to the ∞-category of rigid analytic motives over S, is a τ-sheaf with values in Pr^{L}. An important result obtained in this section is Theorem 2.5.1 asserting that this sheaf transforms certain limits of rigid analytic spaces into colimits of presentable ∞-categories. This result plays an important role at several places in the paper, notably for constructing direct images with compact support in Subsection 4.3. In Subsection 2.8, we use this result for computing the stalks of $\mathbf{RigSH}_\tau(-;\Lambda)$.

2.1. The construction.

From now on, we fix a connective commutative ring spectrum $\Lambda \in \mathrm{CAlg}(\mathcal{S}\mathrm{p}_{\geq 0})$ and denote by Mod_Λ the ∞-category of Λ-modules. Connectivity of Λ is assumed here for convenience. It implies that Mod_Λ admits a t-structure whose heart is the ordinary category of $\pi_0\Lambda$-modules. Examples of Λ include localisations of the sphere spectrum at various primes and Eilenberg–Mac Lane spectra of ordinary rings such as $\mathbb{Z}$, $\mathbb{Z}/n$, $\mathbb{Q}$, etc.

Notation 2.1.1. Given an ∞-category $\mathcal{C}$, we denote by $\mathcal{P}(\mathcal{C})$ the ∞-category of presheaves on $\mathcal{C}$ with values in the ∞-category $\mathcal{S}$ of Kan complexes. If $\mathcal{C}$ is endowed with a Grothendieck topology τ, we denote by $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C})$ the full sub-∞-category of $\mathcal{P}(\mathcal{C})$ spanned by the τ-(hyper)sheaves. Thus, $\mathrm{Shv}_\tau(\mathcal{C})$ is the ∞-topos associated to the site $(\mathcal{C}, \tau)$ as in [Lur09, Definition 6.2.2.6] and $\mathrm{Shv}_\tau^\wedge(\mathcal{C})$ is its hypercompletion in the sense of [Lur09, §6.5.2].

Notation 2.1.2. Given an ∞-category $\mathcal{C}$, we denote by $\mathrm{PSh}(\mathcal{C};\Lambda)$ the ∞-category of presheaves of Λ-modules on $\mathcal{C}$, i.e., contravariant functors from $\mathcal{C}$ to Mod_Λ. If $\mathcal{C}$ is endowed with a Grothendieck topology τ, we denote by $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C};\Lambda)$ the full sub-∞-category of $\mathrm{PSh}(\mathcal{C};\Lambda)$ spanned by the τ-(hyper)sheaves. (For the precise meaning, see Definition 2.3.1 below.) We denote by

$$\mathrm{L}_\tau : \mathrm{PSh}(\mathcal{C};\Lambda) \to \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C};\Lambda) \tag{2.1}$$

the left adjoint to the obvious inclusion. This functor is called τ-(hyper)sheafification. We also denote by

$$(-)^\wedge : \mathrm{Shv}_\tau(\mathcal{C};\Lambda) \to \mathrm{Shv}_\tau^\wedge(\mathcal{C};\Lambda) \tag{2.2}$$

the left adjoint to the obvious inclusion. This functor is called hypercompletion.

Remark 2.1.3. The ∞-category $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C};\Lambda)$ is stable and admits a t-structure whose truncation functors are denoted by $\tau_{\geq m}$ and $\tau_{\leq n}$, and whose heart is the category of ordinary sheaves of $\pi_0\Lambda$-modules on the homotopy category of $\mathcal{C}$. An object $\mathcal{F} \in \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C};\Lambda)$ is said to be m-connective (resp. n-coconnective) if the natural map $\tau_{\geq m}\mathcal{F} \to \mathcal{F}$ (resp. $\mathcal{F} \to \tau_{\leq n}\mathcal{F}$) is an equivalence. As usual, when $m = 0$ (resp. $n = 0$) we say that $\mathcal{F}$ is connective (resp. coconnective).

We record the following lemma which we will use at several occasions.

Lemma 2.1.4. *Consider two sites* $(\mathcal{C},\tau)$ *and* $(\mathcal{C}',\tau')$ *where* $\mathcal{C}$ *and* $\mathcal{C}'$ *are ordinary categories, and let* $F : \mathcal{C} \to \mathcal{C}'$ *be a functor. Assume the following conditions.*

(1) The topologies τ *and* τ' *are induced by pretopologies* Cov_τ *and* $\mathrm{Cov}_{\tau'}$ *in the sense of* [SGAIV1, Exposé II, Définition 1.3].

(2) For $X \in \mathcal{C}$*,* F *takes a family in* $\mathrm{Cov}_\tau(X)$ *to a family in* $\mathrm{Cov}_{\tau'}(F(X))$*. Moreover, if* $a : U \to X$ *is an arrow which is a member of a family belonging to* $\mathrm{Cov}_\tau(X)$ *and* $b : V \to X$ *a second arrow in* $\mathcal{C}$*, we have* $F(U \times_X V) \simeq F(U) \times_{F(X)} F(V)$*.*

Then, the inverse image functors on presheaves induce by sheafification the following functors:

$$F^* : \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}) \to \mathrm{Shv}_{\tau'}^{(\wedge)}(\mathcal{C}') \qquad \textit{and} \qquad F^* : \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C};\Lambda) \to \mathrm{Shv}_{\tau'}^{(\wedge)}(\mathcal{C}';\Lambda). \tag{2.3}$$

Moreover, if F *defines an equivalence of sites* $F : (\mathcal{C}',\tau') \xrightarrow{\sim} (\mathcal{C},\tau)$*, i.e., induces an equivalence between the associated ordinary topoi, then the functors* (2.3) *are equivalences of* ∞*-categories.*

Proof. The case of (hyper)sheaves of Λ-modules follows from the case of (hyper)sheaves of Kan complexes using, for example, Remark 2.3.3(2) below. To construct $F^* : \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}) \to \mathrm{Shv}_{\tau'}^{(\wedge)}(\mathcal{C}')$, we need to show that $F^* : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C}')$ takes a τ-(hyper)cover to a τ'-(hyper)cover which follows immediately from conditions (1) and (2).

It remains to prove the last statement. The case of hypersheaves follows from the case of sheaves. Therefore, it is enough to show that $F^* : \mathrm{Shv}_\tau(\mathcal{C}) \to \mathrm{Shv}_{\tau'}(\mathcal{C}')$ is an equivalence. Since $\mathcal{C}$ and $\mathcal{C}'$ are ordinary categories, the Yoneda functors composed with sheafification factorize through the sub-∞-categories $\mathrm{Shv}_\tau(\mathcal{C})_{\leq 0} \subset \mathrm{Shv}_\tau(\mathcal{C})$ and $\mathrm{Shv}_{\tau'}(\mathcal{C}')_{\leq 0} \subset \mathrm{Shv}_{\tau'}(\mathcal{C}')$ of 0-truncated objects. By hypothesis, the functor F^* induces an equivalence of ordinary topoi $\mathrm{Shv}_\tau(\mathcal{C})_{\leq 0} \simeq \mathrm{Shv}_{\tau'}(\mathcal{C}')_{\leq 0}$. Thus, there exists a functor $u : \mathcal{C}' \to \mathrm{Shv}_\tau(\mathcal{C})$ making the triangles

$$\begin{tikzcd} \mathcal{C} \arrow[d, "F"'] \arrow[dr] & \\ \mathcal{C}' \arrow[r, "u"] & \mathrm{Shv}_\tau(\mathcal{C}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}' \arrow[r, "u"] \arrow[dr] & \mathrm{Shv}_\tau(\mathcal{C}) \arrow[d, "F^*"] \\ & \mathrm{Shv}_{\tau'}(\mathcal{C}') \end{tikzcd}$$

commutative. Let $\widetilde{u} : \mathcal{P}(\mathcal{C}') \to \mathrm{Shv}_\tau(\mathcal{C})$ be the left Kan extension of u along the Yoneda embedding $\mathrm{y} : \mathcal{C}' \to \mathcal{P}(\mathcal{C}')$. Given $X' \in \mathcal{C}'$ and a covering sieve $R' \subset \mathrm{y}(X')$ generated by a family $(Y'_i \to X')_i$ in $\mathrm{Cov}_{\tau'}(X')$, the induced map $\widetilde{u}(R') \to \widetilde{u}\mathrm{y}(X') = u(X')$ is an equivalence. Indeed, R' is equivalent to the colimit of the Čech nerve associated to the family $(Y'_i \to X')_i$. It follows that $\widetilde{u}(R')$ is equivalent to the colimit in $\mathrm{Shv}_\tau(\mathcal{C})$ of the Čech nerve in $\mathrm{Shv}_\tau(\mathcal{C})_{\leq 0}$ associated to the family $(u(Y'_i) \to u(X'))_i$. (Here, we use that the functor $u : \mathcal{C}' \to \mathrm{Shv}_\tau(\mathcal{C})_{\leq 0}$ preserves representable fiber products.) The family $(u(Y'_i) \to u(X'))_i$ is jointly effectively epimorphic since its image by the equivalence $\mathrm{Shv}_\tau(\mathcal{C})_{\leq 0} \simeq \mathrm{Shv}_{\tau'}(\mathcal{C}')_{\leq 0}$ is jointly effectively epimorphic. (Here we use [Lur09, Proposition 7.2.1.14] which insures that effective epimorphisms can be detected after 0-truncation.) This proves that $\widetilde{u}(R')$ is equivalent to $u(X')$ as needed.

From the above discussion, we deduce from [Lur09, Proposition 5.5.4.20] that $\widetilde{u}$ factors uniquely through the τ'-sheafification $\mathrm{L}_{\tau'} : \mathcal{P}(\mathcal{C}') \to \mathrm{Shv}_{\tau'}(\mathcal{C}')$ yielding a functor $\mathrm{Shv}_{\tau'}(\mathcal{C}') \to \mathrm{Shv}_\tau(\mathcal{C})$. That the latter is a two-sided inverse to F^* follows from the above two triangles and the universal property of the Yoneda functors $\mathcal{C} \to \mathrm{Shv}_\tau(\mathcal{C})$ and $\mathcal{C}' \to \mathrm{Shv}_\tau(\mathcal{C}')$. □

Below and elsewhere in this paper, "monoidal" always means "symmetric monoidal".

Remark 2.1.5. Recall that Mod_Λ underlies a monoidal ∞-category $\mathrm{Mod}_\Lambda^\otimes$. Applying [Lur09, Proposition 3.1.2.1] to the coCartesian fibration $\mathrm{Mod}_\Lambda^\otimes \to \mathrm{Fin}_*$, we deduce that

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}_\Lambda^\otimes) \times_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fin}_*)} \mathrm{Fin}_* \to \mathrm{Fin}_*$$

defines a monoidal ∞-category $\mathrm{PSh}(\mathcal{C};\Lambda)^\otimes$ whose underlying ∞-category is $\mathrm{PSh}(\mathcal{C};\Lambda)$. By [Lur17, Proposition 2.2.1.9], $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C};\Lambda)$ underlies a unique monoidal ∞-category $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C};\Lambda)^\otimes$ such that (2.1) lifts to a monoidal functor.

Remark 2.1.6. There is a monoidal functor $\Lambda \otimes - : \mathcal{S}^\times \to \mathrm{Mod}_\Lambda^\otimes$ sending a Kan complex to the associated free Λ-module. (More precisely, this is the composition of the infinite suspension functor $\Sigma^\infty : \mathcal{S}^\times \to \mathcal{S}\mathrm{p}^\otimes$ with the change of algebra functor $\Lambda \otimes - : \mathcal{S}\mathrm{p}^\otimes \to \mathrm{Mod}_\Lambda^\otimes$ provided by [Lur17, Theorem 4.5.3.1].) It induces monoidal functors

$$\mathcal{P}(\mathcal{C})^\times \to \mathrm{PSh}(\mathcal{C};\Lambda)^\otimes \qquad \text{and} \qquad \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C})^\times \to \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C};\Lambda)^\otimes.$$

Composing with the Yoneda functors $\mathrm{y} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ and $\mathrm{L}_\tau \circ \mathrm{y} : \mathcal{C} \to \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C})$, we get functors

$$\Lambda(-) : \mathcal{C} \to \mathrm{PSh}(\mathcal{C};\Lambda) \qquad \text{and} \qquad \Lambda_\tau(-) : \mathcal{C} \to \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C};\Lambda).$$

If $\mathcal{C}$ has finite direct products, the above functors lift to monoidal functors from $\mathcal{C}^\times$ to $\mathrm{PSh}(\mathcal{C};\Lambda)^\otimes$ and $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C};\Lambda)^\otimes$. In particular, the monoidal structure on $\mathrm{PSh}(\mathcal{C};\Lambda)$ described in Remark 2.1.5 coincides with the one given by Day convolution according to [Lur17, Corollary 4.8.1.12 & Remark 4.8.1.13].

Definition 2.1.7.

(1) We denote by Pr^{L} (resp. Pr^{R}) the ∞-category of presentable ∞-categories and left adjoint (resp. right adjoint) functors; see [Lur09, Definition 5.5.3.1]. There is an equivalence $\mathrm{Pr}^{\mathrm{R}} \simeq (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}}$ (see [Lur09, Corollary 5.5.3.4]), and both Pr^{L} and Pr^{R} are sub-∞-categories of CAT_∞, the ∞-category of (possibly large) ∞-categories. The ∞-category Pr^{L} underlies a monoidal ∞-category $\mathrm{Pr}^{\mathrm{L},\otimes}$ by [Lur17, Proposition 4.8.1.15].

(2) We also denote by $\mathrm{Pr}^{\mathrm{L}}_\omega$ the ∞-category of compactly generated ∞-categories and left adjoint compact-preserving functors. It is opposite to $\mathrm{Pr}^{\mathrm{R}}_\omega$, the ∞-category of compactly generated ∞-categories and right adjoint functors which commute with filtered colimits. See

[Lur09, Definition 5.5.7.1, & Notations 5.5.7.5 & 5.5.7.7]. By [Lur17, Lemma 5.3.2.11], $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ underlies a monoidal ∞-category $\mathrm{Pr}^{\mathrm{L},\otimes}_{\omega}$ and the inclusion $\mathrm{Pr}^{\mathrm{L}}_{\omega} \to \mathrm{Pr}^{\mathrm{L}}$ lifts to a monoidal functor $\mathrm{Pr}^{\mathrm{L},\otimes}_{\omega} \to \mathrm{Pr}^{\mathrm{L},\otimes}$.

(3) A monoidal ∞-category $\mathcal{M}^{\otimes}$ is said to be presentable (resp. compactly generated) if the underlying ∞-category $\mathcal{M}$ is presentable (resp. compactly generated) and the endofunctor $A \otimes -$ is a left adjoint functor for all $A \in \mathcal{M}$ (resp. is a left adjoint compact-preserving functor for all compact $A \in \mathcal{M}$). This is equivalent to say that $\mathcal{M}^{\otimes}$ belongs to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ (resp. $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$).

Remark 2.1.8. The ∞-categories $\mathrm{PSh}(\mathcal{C};\Lambda)$ and $\mathrm{Shv}^{(\wedge)}_{\tau}(\mathcal{C};\Lambda)$ are presentable (by [Lur09, Proposition 5.5.3.6 & Remark 5.5.1.6]) and they are respectively generated under colimits by the objects $\Lambda(X)$ and $\Lambda_{\tau}(X)$, for $X \in \mathcal{C}$. In fact, the objects $\Lambda(X)$ are compact, so that $\mathrm{PSh}(\mathcal{C};\Lambda)$ is compactly generated. More is true: the monoidal ∞-categories $\mathrm{PSh}(\mathcal{C};\Lambda)^{\otimes}$ and $\mathrm{Shv}^{(\wedge)}_{\tau}(\mathcal{C};\Lambda)^{\otimes}$ are presentable, and, if $\mathcal{C}$ has finite direct products, $\mathrm{PSh}(\mathcal{C};\Lambda)^{\otimes}$ is even compactly generated.

To define the ∞-category of rigid analytic motives over a rigid analytic space S, we consider the case where $(\mathcal{C},\tau)$ is the big smooth site $(\mathrm{RigSm}/S,\tau)$ with $\tau \in \{\mathrm{nis}, \mathrm{ét}\}$. (See Notation 1.4.9(3).) Before proceeding to the definition, we make a remark concerning these sites.

Remark 2.1.9. The category RigSm/S is not small, and some care is needed when speaking about presheaves and τ-(hyper)sheaves on it. In fact, the only problem that one needs to keep in mind is that the ∞-category $\mathrm{PSh}(\mathrm{RigSm}/S;\Lambda)$ is not locally small. However, this problem disappears when passing to the sub-∞-category $\mathrm{Shv}^{(\wedge)}_{\tau}(\mathrm{RigSm}/S;\Lambda)$. Indeed, it is easy to see that this ∞-category is equivalent to $\mathrm{Shv}^{(\wedge)}_{\tau}((\mathrm{RigSm}/S)^{<\alpha};\Lambda)$, where α is an infinite cardinal and $(\mathrm{RigSm}/S)^{<\alpha} \subset \mathrm{RigSm}/S$ is the full subcategory spanned by those rigid analytic S-spaces that can be covered by $< \alpha$ opens which are quasi-compact and quasi-separated. (This uses Lemma 2.1.4.) Clearly, $(\mathrm{RigSm}/S)^{<\alpha}$ is essentially small and thus $\mathrm{Shv}^{(\wedge)}_{\tau}(\mathrm{RigSm}/S;\Lambda)$ is a presentable ∞-category. The same remark applies to other sites such as $(\mathrm{Ét}/S,\tau)$, etc. Below, whenever we need to speak about general presheaves on RigSm/S, $\mathrm{Ét}/S$, etc., we implicitly fix an infinite cardinal α and replace these categories by $(\mathrm{RigSm}/S)^{<\alpha}$, $(\mathrm{Ét}/S)^{<\alpha}$, etc.

We will use the following notation.

Notation 2.1.10.

(1) Let $\mathcal{X}$ be a formal scheme. We denote by $\mathbb{A}^n_{\mathcal{X}}$ the relative n-dimensional affine space given by $\mathrm{Spf}(\mathcal{O}_{\mathcal{X}}\langle t_1,\ldots,t_n\rangle)$. By abuse of notation, we also write "$\mathcal{X} \times \mathbb{A}^n$" instead of "$\mathbb{A}^n_{\mathcal{X}}$" although FSch has no direct products (nor a final object).

(2) Let X be a rigid analytic space. If X admits a formal model $\mathcal{X}$, we set $\mathbb{B}^n_X = (\mathbb{A}^n_{\mathcal{X}})^{\mathrm{rig}}$. This is independent of the choice of $\mathcal{X}$ and, in general, we may define $\mathbb{B}^n_X$ by gluing along open immersions. The rigid analytic X-space $\mathbb{B}^n_X$ is called the relative n-dimensional ball. By abuse of notation, we also write "$X \times \mathbb{B}^n$" instead of "$\mathbb{B}^n_X$" although RigSpc has no direct products (nor a final object).

(3) If X is a rigid analytic space, we denote by $\mathbb{U}^1_X \subset \mathbb{B}^1_X$ the open rigid analytic subspace of $\mathbb{B}^1_X$ which is locally given by $\mathrm{Spf}(\mathcal{O}_{\mathcal{X}}\langle t,t^{-1}\rangle) \subset \mathrm{Spf}(\mathcal{O}_{\mathcal{X}}\langle t\rangle)$. The rigid analytic X-space $\mathbb{U}^1_X$ is called the relative unit circle.[5]

We fix a rigid analytic space S and $\tau \in \{\mathrm{nis}, \mathrm{ét}\}$.

[5] In [Ayo15], the relative unit circle is denoted by $\partial\mathbb{B}^1_X$ and, in other places in the literature, it is denoted by $\mathbb{T}^1_X$.

Definition 2.1.11. Let $\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(S;\Lambda)$ be the full sub-∞-category of $\mathrm{Shv}^{(\wedge)}_{\tau}(\mathrm{RigSm}/S;\Lambda)$ spanned by those objects which are local with respect to the collection of maps of the form $\Lambda_{\tau}(\mathbb{B}^1_X)\to\Lambda_{\tau}(X)$, for $X\in\mathrm{RigSm}/S$, and their desuspensions. Let

$$\mathrm{L}_{\mathbb{B}^1}:\mathrm{Shv}^{(\wedge)}_{\tau}(\mathrm{RigSm}/S;\Lambda)\to\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(S;\Lambda) \tag{2.4}$$

be the left adjoint to the obvious inclusion. This is called the $\mathbb{B}^1$-localisation functor. We also set $\mathrm{L}_{\mathbb{B}^1,\tau}=\mathrm{L}_{\mathbb{B}^1}\circ\mathrm{L}_{\tau}$ with L_{τ} the τ-(hyper)sheafification functor, see (2.1). The functor $\mathrm{L}_{\mathbb{B}^1,\tau}$ is called the $(\mathbb{B}^1,\tau)$-localisation functor. Given a smooth rigid analytic S-space X, we set $\mathrm{M}^{\mathrm{eff}}(X)=\mathrm{L}_{\mathbb{B}^1}(\Lambda_{\tau}(X))$. This is the effective motive of X.

Remark 2.1.12. The defining condition for a τ-(hyper)sheaf of Λ-modules $\mathcal{F}$ to belong to the sub-∞-category $\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(S;\Lambda)$ is equivalent to the condition that $\mathcal{F}$ is $\mathbb{B}^1$-invariant in the following sense: for every $X\in\mathrm{RigSm}/S$, the map of Λ-modules $\mathcal{F}(X)\to\mathcal{F}(\mathbb{B}^1_X)$ is an equivalence. Since $\mathcal{F}$ is a τ-(hyper)sheaf, it is enough to ask this condition for X varying in a subcategory $\mathcal{C}\subset\mathrm{RigSm}/S$ such that every object of RigSm/S admits a τ-hypercover by objects in $\mathcal{C}$ which is moreover truncated in the non-hypercomplete case.

Remark 2.1.13. The ∞-category $\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(S;\Lambda)$ is stable and, by [Lur17, Proposition 2.2.1.9], it underlies a unique monoidal ∞-category $\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(S;\Lambda)^{\otimes}$ such that $\mathrm{L}_{\mathbb{B}^1}$ lifts to a monoidal functor. Moreover, this monoidal ∞-category is presentable, i.e., belongs to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, since we localise with respect to a small set of morphisms.

Remark 2.1.14. There is another site that one can use for constructing $\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(S;\Lambda)$, at least when S admits a formal model $\mathcal{S}$ (e.g., S quasi-compact and quasi-separated). Indeed, by Corollary 1.4.13, the site $(\mathrm{RigSm}/S;\tau)$ is equivalent to the site $(\mathrm{FRigSm}/\mathcal{S};\text{rig-}\tau)$ where $\mathrm{FRigSm}/\mathcal{S}$ denotes the full subcategory of $\mathrm{FSch}/\mathcal{S}$ whose objects are the rig-smooth formal $\mathcal{S}$-schemes. (See Definition 1.3.13 and Remark 1.4.14). Using Lemma 2.1.4, we deduce an equivalence of ∞-categories

$$\mathrm{Shv}^{(\wedge)}_{\text{rig-}\tau}(\mathrm{FRigSm}/\mathcal{S};\Lambda)\simeq\mathrm{Shv}^{(\wedge)}_{\tau}(\mathrm{RigSm}/S;\Lambda)$$

and $\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(S;\Lambda)$ is equivalent to the sub-∞-category of $\mathrm{Shv}^{(\wedge)}_{\text{rig-}\tau}(\mathrm{FRigSm}/\mathcal{S};\Lambda)$ spanned by those objects which are local with respect to the collection of maps $\Lambda_{\text{rig-}\tau}(\mathbb{A}^1_{\mathcal{X}})\to\Lambda_{\text{rig-}\tau}(\mathcal{X})$, with $\mathcal{X}\in\mathrm{FRigSm}/\mathcal{S}$, and their desuspensions.

Definition 2.1.15. Let T_S (or simply T if S is clear from the context) be the image by $\mathrm{L}_{\mathbb{B}^1}$ of the cofiber of the split inclusion $\Lambda_{\tau}(S)\to\Lambda_{\tau}(\mathbb{U}^1_S)$ induced by the unit section. With the notation of [Rob15, Definition 2.6], we set

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)^{\otimes}=\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(S;\Lambda)^{\otimes}[\mathrm{T}_S^{-1}]. \tag{2.5}$$

More precisely, there is a morphism $\Sigma^{\infty}_{\mathrm{T}}:\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(S;\Lambda)^{\otimes}\to\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)^{\otimes}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, sending T_S to a $\otimes$-invertible object, and which is initial for this property. We denote by $\Omega^{\infty}_{\mathrm{T}}:\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)\to\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(S;\Lambda)$ the right adjoint to $\Sigma^{\infty}_{\mathrm{T}}$. Given a smooth rigid analytic S-space X, we set $\mathrm{M}(X)=\Sigma^{\infty}_{\mathrm{T}}\mathrm{M}^{\mathrm{eff}}(X)$. This is the motive of X.

Definition 2.1.16. Objects of $\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)$ are called rigid analytic motives over S. We will denote by Λ (or Λ_S if we need to be more precise) the monoidal unit of $\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)$. For any $n\in\mathbb{N}$, we denote by $\Lambda(n)$ the image of $\mathrm{T}_S^{\otimes n}[-n]$ by $\Sigma^{\infty}_{\mathrm{T}}$, and by $\Lambda(-n)$ the $\otimes$-inverse of $\Lambda(n)$. For $n\in\mathbb{Z}$, we denote by $M\mapsto M(n)$ the Tate twist given by tensoring with $\Lambda(n)$.

Remark 2.1.17. The object T_S is symmetric in the sense of [Rob15, Definition 2.16]. (See, for example, [Jar00, Lemma 3.13] whose proof extends immediately to the rigid analytic setting.) By [Rob15, Corollary 2.22], it follows that the ∞-category $\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)$ underlying (2.5) is equivalent to the colimit in Pr^{L} of the $\mathbb{N}$-diagram whose transition maps are given by tensoring with T_S. Also, by [Rob15, Corollary 2.23], the monoidal ∞-category (2.5) is stable.

Remark 2.1.18. When Λ is the Eilenberg–Mac Lane spectrum associated to an ordinary ring, also denoted by Λ, the ∞-category $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(S;\Lambda)$ is more commonly denoted by $\mathbf{RigDA}^{(\mathrm{eff},\wedge)}_{\tau}(S;\Lambda)$. Also, when τ is the Nisnevich topology, we sometimes drop the subscript "nis".

Remark 2.1.19. There is a more traditional description of the ∞-category $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(S;\Lambda)$ using the language of model categories. This is the approach taken in [Ayo15, §1.4.2].

Assume that Λ is given as a symmetric S^1-spectrum, and denote by $\mathrm{Mod}_{\Delta}(\Lambda)$ the simplicial category of Λ-modules which we endow with the model structure described in [HSS00, Corollary 5.4.2]. Note that the ∞-category Mod_{Λ} is equivalent to the simplicial nerve of the full subcategory of $\mathrm{Mod}_{\Delta}(\Lambda)$ consisting of cofibrant-fibrant objects. Let $\mathrm{PSh}_{\Delta}(\mathrm{RigSm}/S;\Lambda)$ be the simplicial category whose objects are the presheaves on RigSm/S with values in $\mathrm{Mod}_{\Delta}(\Lambda)$, which we endow with its projective global model structure. The projective $(\mathbb{B}^1,\tau)$-local structure on $\mathrm{PSh}_{\Delta}(\mathrm{RigSm}/S;\Lambda)$, also known as the motivic model structure, is obtained from the latter via the Bousfield localization with respect to the union of the following classes of maps:

(1) morphisms of presheaves inducing isomorphisms on the τ-sheaves associated to their homotopy presheaves;
(2) morphisms of the form $\Lambda(\mathbb{B}^1_X)[n] \to \Lambda(X)[n]$ induced by the canonical projection, for $X \in \mathrm{RigSm}/S$ and $n \in \mathbb{Z}$.

The ∞-category $\mathbf{RigSH}^{\mathrm{eff},\wedge}_{\tau}(S;\Lambda)$ is equivalent to the simplicial nerve of the full simplicial subcategory of $\mathrm{PSh}_{\Delta}(\mathrm{RigSm}/S;\Lambda)$ consisting of motivically cofibrant-fibrant objects. This follows from [Lur09, Propositions 4.2.4.4 & A.3.7.8].

To obtain the T-stable version, we form the category $\mathrm{Spt}_T(\mathrm{PSh}_{\Delta}(\mathrm{RigSm}/S;\Lambda))$ of T-spectra of presheaves of Λ-modules on RigSm/S. (Here T is any cofibrant replacement of $\Lambda(\mathbb{U}^1_S)/\Lambda(S)$.) The $(\mathbb{B}^1,\tau)$-local model structure induces the stable $(\mathbb{B}^1,\tau)$-local model structure on T-spectra, which is also known as the motivic model structure. The ∞-category $\mathbf{RigSH}^{\wedge}_{\tau}(S;\Lambda)$ is equivalent to the simplicial nerve of the full simplicial subcategory of $\mathrm{Spt}_T(\mathrm{PSh}_{\Delta}(\mathrm{RigSm}/S;\Lambda))$ consisting of motivically cofibrant-fibrant objects. This follows from [Rob15, Theorem 2.26].

The above discussion can be adapted to the non-hypercomplete case. One only needs to replace the class of maps in (1) above by a smaller one, namely the class of maps of the form $\mathrm{hocolim}_{[n]\in\mathbf{\Delta}}\Lambda(Y_n) \to \Lambda(Y_{-1})$ where $Y_\bullet$ is a truncated τ-hypercover of $Y_{-1} \in \mathrm{RigSm}/S$. In both cases, the weak equivalences of the (stable) $(\mathbb{B}^1,\tau)$-local model structure are called the (stable) $(\mathbb{B}^1,\tau)$-local equivalences.

Lemma 2.1.20. *The monoidal ∞-category $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(S;\Lambda)^{\otimes}$ is presentable and its underlying ∞-category is generated under colimits, and up to desuspension and negative Tate twists when applicable, by the motives $\mathrm{M}^{(\mathrm{eff})}(X)$ with $X \in \mathrm{RigSm}/S$ quasi-compact and quasi-separated.*

Proof. That the monoidal ∞-category of the statement is presentable was mentioned above. The claim about the generators follows from Remark 2.1.8 in the effective case. In the T-stable case, we then use the universal property of $\otimes$-inversion given by [Rob15, Proposition 2.9]. □

Proposition 2.1.21. *The assignment $S \mapsto \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(S;\Lambda)^\otimes$ extends naturally into a functor*

$$\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(-;\Lambda)^\otimes : \mathrm{RigSpc}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}). \tag{2.6}$$

Proof. We refer to [Rob14, §9.1] for the construction of an analogous functor in the algebraic setting. □

Notation 2.1.22. Let $f : Y \to X$ be a morphism of rigid analytic spaces. The image of f by (2.6) is the inverse image functor

$$f^* : \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(Y;\Lambda)$$

which has the structure of a monoidal functor. Its right adjoint f_* is the direct image functor. It has the structure of a right-lax monoidal functor. (See Lemma 3.4.1 below.)

2.2. **Previously available functoriality.**

We gather here part of what is known about the functor $S \mapsto \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(S;\Lambda)$ introduced in Subsection 2.1. The results that we discuss here were obtained in [Ayo15, §1.4] under the assumption that S is of finite type over a non-Archimedean field. However, the proofs apply also to the general case with very little modification.

Proposition 2.2.1. *Let $f : Y \to X$ be a smooth morphism of rigid analytic spaces.*

(1) The functor f^*, as in Notation 2.1.22, admits a left adjoint

$$f_\sharp : \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(Y;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X;\Lambda)$$

sending the motive of a smooth rigid analytic Y-space V to the motive of V considered as a smooth rigid analytic X-space in the obvious way.

(2) (Smooth projection formula) *The canonical map*

$$f_\sharp(f^*M \otimes N) \to M \otimes f_\sharp N$$

is an equivalence for all $M \in \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X;\Lambda)$ and $N \in \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(Y;\Lambda)$.

(3) (Smooth base change) *Let $g : X' \to X$ be a morphism of rigid analytic spaces and form a Cartesian square*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X. \end{array}$$

The natural transformations $f'_\sharp \circ g'^* \to g^* \circ f_\sharp$ and $f^* \circ g_* \to g'_* \circ f'^*$, between functors from $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(Y;\Lambda)$ to $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X';\Lambda)$ and back, are equivalences.

Proof. The functor $f^* : \mathrm{RigSm}/X \to \mathrm{RigSm}/Y$ admits a left adjoint $f_\sharp$ sending a smooth rigid analytic Y-space V to V considered as a smooth rigid analytic X-space. The adjunction $(f_\sharp, f^*)$ induces an adjunction between categories of motives. This is discussed in [Ayo15, Théorèmes 1.4.13 & 1.4.16] using the language of model categories. For the second assertion, we refer to the proof of [Ayo07b, Proposition 4.5.31]. For the third assertion, we refer to the proof of [Ayo15, Lemme 1.4.32]. Both proofs are formal and extend readily to the context we are considering. □

Corollary 2.2.2. *Let $j : U \to X$ be an open immersion of rigid analytic spaces. Then the functors*

$$j_\sharp,\ j_* : \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(U;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X;\Lambda)$$

are fully faithful.

Proof. This follows from Proposition 2.2.1(3) with f and g equal to j. $\square$

Proposition 2.2.3. *Let $i : Z \to X$ be a closed immersion of rigid analytic spaces (as in Definition 1.1.14) and $j : U \to X$ the complementary open immersion (i.e., such that $|U| = |X| \smallsetminus |Z|$).*

(1) The functor $i_* : \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(Z;\Lambda) \to \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(X;\Lambda)$ is fully faithful.

(2) (Localization) *The counit of the adjunction $(j_\sharp, j^*)$ and the unit of the adjunction (i^*, i_*) form a cofiber sequence*

$$j_\sharp j^* \to \mathrm{id} \to i_* i^* \tag{2.7}$$

of endofunctors of $\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(X;\Lambda)$. In particular, the pair (i^*, j^*) is conservative.

(3) (Closed projection formula) *The canonical map*

$$M \otimes i_* N \to i_*(i^* M \otimes N) \tag{2.8}$$

is an equivalence for all $M \in \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(X;\Lambda)$ and $N \in \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(Z;\Lambda)$.

(4) (Closed base change) *Let $g : X' \to X$ be a morphism of rigid analytic spaces and form a Cartesian square*

$$\begin{tikzcd} Z' \arrow[r, "g'"] \arrow[d, "i'"] & Z \arrow[d, "i"] \\ X' \arrow[r, "g"] & X. \end{tikzcd}$$

The natural transformation $g^* \circ i_* \to i'_* \circ g'^*$, between functors from $\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(Z;\Lambda)$ to $\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(X';\Lambda)$, is an equivalence. If moreover g is smooth, then the natural transformation $g_\sharp \circ i'_* \to i_* \circ g'_\sharp$, from $\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(Z';\Lambda)$ to $\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(X;\Lambda)$, is an equivalence.

Proof. Assertion (2) implies all the others. Indeed, applying i^* to the cofiber sequence (2.7) and using that $i^* j_\sharp \simeq 0$ (which follows from Proposition 2.2.1(3)), we deduce that $i^* i_* i^* \to i^*$ is an equivalence. Assertion (1) follows then from Lemma 2.2.5 below. We may check that (2.8) is an equivalence after applying i^* and j^*. Assertion (3) follows then by using that $j^* i_* \simeq 0$ (by Proposition 2.2.1(2)) and $i^* i_* \simeq \mathrm{id}$ (by assertion (1)). Similarly, to prove assertion (4) we use that the pairs (i^*, j^*) and (i'^*, j'^*) are conservative (with $j' : U' \to X'$ the base change of j), and the equivalences $j^* i_* \simeq 0$, $j'^* i'_* \simeq 0$, $i^* i_* \simeq \mathrm{id}$ and $i'^* i'_* \simeq \mathrm{id}$, and smooth base change as in Proposition 2.2.1(3) for the second natural transformation.

We now discuss the proof of assertion (2). When X is of finite type over a non-Archimedean field, assertion (2) can be found in [Ayo15, §1.4.3]. (See [Ayo15, Théorème 1.4.20] for the effective case and the proof of [Ayo15, Corollaire 1.4.28] for the T-stable case.) We claim that the proofs of loc. cit. extend to general rigid analytic spaces.

The key step is to show that [Ayo15, Théorème 1.4.20] is still valid for general rigid analytic spaces, i.e., that assertion (2) holds true in the effective case. This is the statement that for any $\mathcal{F}$ in $\mathbf{RigSH}_\tau^{\mathrm{eff},(\wedge)}(X;\Lambda)$, the square

$$\begin{tikzcd} j_\sharp j^* \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ 0 \arrow[r] & i_* i^* \mathcal{F} \end{tikzcd} \tag{2.9}$$

is coCartesian in $\mathbf{RigSH}_\tau^{\mathrm{eff},(\wedge)}(X;\Lambda)$. Using Lemma 2.1.20 and Lemma 2.2.5 below, we may assume that $\mathcal{F} = \mathrm{L}_{\mathbb{B}^1,\tau}\Lambda(X')$ with $X' \in \mathrm{RigSm}/X$. (See Definition 2.1.11.) Using Lemma 2.2.4

below, we have an equivalence

$$i_*i^*\mathrm{L}_{\mathbb{B}^1,\tau}\Lambda(X') \simeq \mathrm{L}_{\mathbb{B}^1,\tau}i_*\Lambda_{t_\emptyset}(X'_Z)$$

where $X'_Z = X' \times_X Z$ and $t_\emptyset$ the topology on RigSpc generated by one family, namely the empty family considered as a cover of the empty rigid analytic space. Thus, it is enough to show that $\mathrm{L}_{\mathbb{B}^1,\tau}$ transforms the square

$$\begin{tikzcd}
\Lambda_{t_\emptyset}(X'_U) \arrow[r] \arrow[d] & \Lambda_{t_\emptyset}(X') \arrow[d] \\
0 \arrow[r] & i_*\Lambda_{t_\emptyset}(X'_Z)
\end{tikzcd}$$

into a coCartesian one. Using the analogues of [Ayo07b, Corollaire 4.5.40 & Lemme 4.5.41], we reduce to show that [Ayo15, Proposition 1.4.21] is valid for general rigid analytic spaces. More precisely, given a partial section $s : Z \to X'$ defined over Z, we need to show that the morphism $T_{X',s} \otimes \Lambda \to \{*\} \otimes \Lambda$ is a $(\mathbb{B}^1, \tau)$-equivalence (i.e., becomes an equivalence after applying $\mathrm{L}_{\mathbb{B}^1,\tau}$). Here $T_{X',s}$ is the presheaf of sets on RigSm$/X$ given by

$$T_{X',s}(P) = \begin{cases} \mathrm{Hom}_X(P, X') \times_{\mathrm{Hom}_Z(P\times_X Z, X')} \{*\} & \text{if} \quad P \times_X Z \neq \emptyset, \\ \{*\} & \text{if} \quad P \times_X Z = \emptyset. \end{cases}$$

Arguing as in the first and second steps of the proof of [Ayo15, Proposition 1.4.21] one proves that the problem is local on X and around $s(Z)$ for the analytic topology. (In loc. cit., we only consider hypersheaves, but the reader can easily check that hypercompletion is not used in this reduction.) Using Proposition 1.3.16, it is thus enough to treat the case $X' = \mathbb{B}^m_X$ and s the zero section restricted to Z. In this case, we may use an explicit homotopy to conclude as in the third step of the proof of [Ayo07b, Proposition 4.5.42].

Now that assertion (2) is proven in the effective case, we explain how it extends to the T-stable case. Since assertion (2) in the effective case implies assertion (3) in the effective case, the functor

$$i_* : \mathbf{RigSH}^{\mathrm{eff},(\wedge)}_\tau(Z;\Lambda) \to \mathbf{RigSH}^{\mathrm{eff},(\wedge)}_\tau(X;\Lambda)$$

commutes with tensoring with T, i.e., there is an equivalence of functors $\mathrm{T}_X \otimes i_*(-) \simeq i_*(\mathrm{T}_Z \otimes -)$. (See Definition 2.1.15.) Using Remark 2.1.17 and the fact that i_* belongs to Pr^{L} (by Lemma 2.2.5 below), we deduce that i_* commutes with $\Sigma^\infty_{\mathrm{T}}$, i.e., there is an equivalence $\Sigma^\infty_{\mathrm{T}} \circ i_* \simeq i_* \circ \Sigma^\infty_{\mathrm{T}}$. Therefore, applying $\Sigma^\infty_{\mathrm{T}}$ to the coCartesian squares (2.9), we deduce that

$$\begin{tikzcd}
j_\sharp j^*M \arrow[r] \arrow[d] & M \arrow[d] \\
0 \arrow[r] & i_*i^*M
\end{tikzcd}$$

is coCartesian for any M in the image of $\Sigma^\infty_{\mathrm{T}}(-)$ up to a twist. Using Lemma 2.1.20 and Lemma 2.2.5, we deduce that the above square is coCartesian for any $M \in \mathbf{RigSH}^{(\wedge)}_\tau(X;\Lambda)$. □

Lemma 2.2.4. *Let $i : Z \to X$ be a closed immersion of rigid analytic spaces. The functor*

$$i_* : \mathrm{Shv}_{t_\emptyset}(\mathrm{RigSm}/Z;\Lambda) \to \mathrm{Shv}_{t_\emptyset}(\mathrm{RigSm}/X;\Lambda) \tag{2.10}$$

commutes with τ-(hyper)sheafification and the $(\mathbb{B}^1, \tau)$-localisation functor.

Proof. This is a generalisation of [Ayo15, Lemma 1.4.18]. For the proof of loc. cit. to extend to our context, we need to show the following property. Given a smooth rigid analytic X-space X' such that $X'_Z = X' \times_X Z$ is nonempty, every τ-cover of X'_Z can be refined by the inverse image of a τ-cover of X'. To prove this, we may assume that $X' = X$. The question is local on X. Thus, we may assume that $X = \mathrm{Spf}(A)^{\mathrm{rig}}$, with A an adic ring of principal ideal type, and $Z = \mathrm{Spf}(B)^{\mathrm{rig}}$ with B a quotient of A by a saturated closed ideal I. Let π be a generator of an ideal of definition of A. Then B is the filtered colimit in the category of adic rings of $C_{J,N} = A\langle J/\pi^N\rangle$ where $N \in \mathbb{N}$ and $J \subset I$ is a finitely generated ideal. Set $Y_{J,N} = \mathrm{Spf}(C_{J,N})^{\mathrm{rig}}$.

By Corollary 1.4.20 and Remark 1.4.21, every τ-cover $(V_i \to Z)_i$ can be refined by the restriction to Z of a τ-cover $(U_j \to Y_{J,N})_j$ for well chosen J and N. We get a τ-cover of X with the required property by adding to the family $(U_j \to X)_j$ the open inclusion $X \smallsetminus Z \to X$. □

Lemma 2.2.5. *Let $i : Z \to X$ be a closed immersion of rigid analytic spaces.*

(1) The functor $i_* : \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(Z;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X;\Lambda)$ commutes with colimits. Thus, it admits a right adjoint which we denote by $i^!$.

(2) The image of the functor $i^* : \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(Z;\Lambda)$ generates the ∞-category $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(Z;\Lambda)$ by colimits.

Proof. In the effective case, assertion (1) follows from Lemma 2.2.4. Indeed, for a rigid analytic space S, the colimit of a diagram in $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(S;\Lambda)$ is computed by applying $\mathrm{L}_{\mathbb{B}^1,\tau}$ to the colimit of the same diagram in $\mathrm{Shv}_{t_\emptyset}(\mathrm{RigSm}/S;\Lambda)$. So, it is enough to show that (2.10) commutes with colimits, which is obvious. The passage from the effective case to the T-stable case follows from Remark 2.1.17 and the commutation $\mathrm{T}_X \otimes i_*(-) \simeq i_*(\mathrm{T}_Z \otimes -)$. (This relies on assertion (2) of Proposition 2.2.3, but only in the effective case, so there is no vicious circle.)

We now prove assertion (2). By Lemma 2.1.20, it is enough to show that the motive $\mathrm{M}^{(\mathrm{eff})}(V)$ of a smooth rigid analytic Z-space V is a colimit of objects in the image of i^*. The problem is local on X and V, so we may assume that $X = \mathrm{Spf}(A)^{\mathrm{rig}}$, $Z = \mathrm{Spf}(B)^{\mathrm{rig}}$ and $V = \mathrm{Spf}(F)^{\mathrm{rig}}$ where A is an adic ring of principal ideal type, B a quotient of A by a saturated closed ideal and $F \in \mathcal{E}'_{B\langle s\rangle}$ with $s = (s_1,\ldots,s_m)$ a system of coordinates. (For the definition of the category $\mathcal{E}'_{B\langle s\rangle}$, see Notation 1.3.9.) Writing B as the colimit of $C_{J,N}$ as in the proof of Lemma 2.2.4, we may apply Corollary 1.3.10 to find $E \in \mathcal{E}'_{C_{J,N}\langle s\rangle}$, for some J and N, such that $E \widehat{\otimes}_{C_{J,N}} B/(0)^{\mathrm{sat}} \simeq F$. Thus, $U = \mathrm{Spf}(E)^{\mathrm{rig}}$ is a smooth rigid analytic X-space such $U \times_X Z \simeq V$, and we have $i^*\mathrm{M}^{(\mathrm{eff})}(U) \simeq \mathrm{M}^{(\mathrm{eff})}(V)$ as needed. □

One of the aims of this paper is to define the full six-functor formalism for rigid analytic motives. We have seen above that the functors f^*, f_*, $f_\sharp$, $\otimes$ and $\underline{\mathrm{Hom}}$ can be defined with little effort. We now state what was known so far concerning the exceptional functors $f_!$ and $f^!$ following [Ayo15, §1.4.4] (see also [BV21, Theorem 2.9]).

Remark 2.2.6. Let A be an adic ring, $I \subset A$ an ideal of definition, and $U = \mathrm{Spec}(A) \smallsetminus \mathrm{Spec}(A/I)$. Recall from Construction 1.1.15 that there exists an analytification functor

$$(-)^{\mathrm{an}} : \mathrm{Sch}^{\mathrm{lft}}/U \to \mathrm{RigSpc}/U^{\mathrm{an}} \tag{2.11}$$

from the category $\mathrm{Sch}^{\mathrm{lft}}/U$, of U-schemes which are locally of finite type, to the category of rigid analytic U^{an}-spaces. (Note that $U^{\mathrm{an}} = \mathrm{Spf}(A)^{\mathrm{rig}}$.) This functor preserves étale and smooth morphisms, closed immersions and complementary open immersions, as well as proper morphisms.

The following result follows immediately from Propositions 2.2.1 and 2.2.3, and the construction.

Proposition 2.2.7. *Keep the notation as in Remark 2.2.6. The contravariant functor*

$$X \mapsto \mathbf{RigSH}_{\tau}^{(\wedge)}(X^{\mathrm{an}}; \Lambda), \quad f \mapsto f^{\mathrm{an},*}$$

from $\mathrm{Sch}^{\mathrm{lft}}/U$ *to* Pr^{L} *is a stable homotopical functor in the sense that it satisfies the ∞-categorical versions of the properties (1)–(6) listed in* [Ayo07a, §1.4.1].

Remark 2.2.8. The ∞-categorical versions of the properties (1)–(6) listed in [Ayo07a, §1.4.1] can be checked after passing to the homotopy categories. Thus, we may as well reformulate Proposition 2.2.7 by saying that the functor from $\mathrm{Sch}^{\mathrm{lft}}/U$ to the 2-category of triangulated categories, sending X to the homotopy category associated to $\mathbf{RigSH}_{\tau}^{(\wedge)}(X^{\mathrm{an}}; \Lambda)$, is a stable homotopical functor in the sense of [Ayo07a, Définition 1.4.1].

Proposition 2.2.7 gives access to the results developed in [Ayo07a, Ayo07b, Chapitres 1–3] yielding a limited six-functor formalism for rigid analytic motives. We will not list explicitly all the properties that form this formalism since a full six-functor formalism will be obtained later in Section 4. We content ourselves with the following preliminary statement which we actually need in establishing the full six-functor formalism for rigid analytic motives.

Corollary 2.2.9. *Keep the notation as in Remark 2.2.6. Given a morphism $f : Y \to X$ between quasi-projective U-schemes, there is an adjunction*

$$f_!^{\mathrm{an}} : \mathbf{RigSH}_{\tau}^{(\wedge)}(Y^{\mathrm{an}}; \Lambda) \rightleftarrows \mathbf{RigSH}_{\tau}^{(\wedge)}(X^{\mathrm{an}}; \Lambda) : f^{\mathrm{an},!}$$

Moreover, the following properties are satisfied.

(1) *The assignments $f \mapsto f_!^{\mathrm{an}}$ and $f \mapsto f^{\mathrm{an},!}$ are compatible with composition.*[6]

(2) *Given a Cartesian square of quasi-projective U-schemes*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X, \end{tikzcd}$$

there is an equivalence $g^{\mathrm{an},*} \circ f_!^{\mathrm{an}} \simeq f_!'^{\mathrm{an}} \circ g'^{\mathrm{an},*}$.

(3) *There is a natural transformation $f_!^{\mathrm{an}} \to f_*^{\mathrm{an}}$ which is an equivalence if f is projective.*

(4) *If f is smooth, there are equivalences $f^{\mathrm{an},!} \simeq \mathrm{Th}(\Omega_f) \circ f^{\mathrm{an},*}$ and $f_!^{\mathrm{an}} \simeq f_\sharp^{\mathrm{an}} \circ \mathrm{Th}^{-1}(\Omega_f)$ where $\mathrm{Th}(\Omega_f)$ and $\mathrm{Th}^{-1}(\Omega_f)$ are the Thom equivalences associated to Ω_f as in* [Ayo07a, §1.5.3].

Proof. This follows from Proposition 2.2.7 and [Ayo07a, Scholie 1.4.2]. □

Remark 2.2.10. Thom equivalences can be defined for any $\mathcal{O}_X$-module $\mathcal{M}$ which is locally free of finite rank on a rigid analytic space X. Indeed, $\mathcal{M}$ determines a vector bundle $p : M \to X$ whose fiber at a point $x \in X$ is given by $\mathrm{Spec}(\kappa(x)[\mathcal{M}_x])^{\mathrm{an}}$. We set $\mathrm{Th}(\mathcal{M}) = p_\sharp s_*$ and $\mathrm{Th}^{-1}(\mathcal{M}) = s^! p^*$, where $s : X \to M$ is the zero section. If $\mathcal{M}$ is free of rank m, then $\mathrm{Th}(\mathcal{M}) \simeq (-)(m)[2m]$ and $\mathrm{Th}^{-1}(\mathcal{M}) \simeq (-)(-m)[-2m]$. That said, we may write "$\mathrm{Th}(\Omega_{f^{\mathrm{an}}})$" instead of "$\mathrm{Th}(\Omega_f)$" in Corollary 2.2.9(4). (If h is a smooth morphism of rigid analytic spaces, there is an associated $\mathcal{O}$-module Ω_h

[6]Here, we only claim the compatibility with composition up to non-coherent homotopies. A more structured version of this will be obtained later in a more general situation; see Theorem 4.4.2.

which is locally free of finite rank. It can be defined locally as the cokernel of the Jacobian matrix.)

Definition 2.2.11.

(1) If S is a rigid analytic space, we denote by $\mathbb{P}^n_S$ the relative n-dimensional projective space over S. If $S = \mathrm{Spf}(A)^{\mathrm{rig}}$, for an adic ring A, then $\mathbb{P}^n_S = (\mathbb{P}^n_{\mathrm{Spf}(A)})^{\mathrm{rig}}$, and for general S, $\mathbb{P}^n_S$ is defined by gluing. If A and U are as in Remark 2.2.6, we also have $\mathbb{P}^n_{U^{\mathrm{an}}} \simeq (\mathbb{P}^n_U)^{\mathrm{an}}$.

(2) Let $f : Y \to X$ be a morphism of rigid analytic spaces. We say that f is locally projective if, locally on X, f can be factored as a closed immersion followed by a projection of the form $\mathbb{P}^n_X \to X$.

For later use, we also record the following statement.

Proposition 2.2.12. *Let $f : Y \to X$ be a locally projective morphism of rigid analytic spaces.*

(1) (Projective projection formula) *The canonical map $M \otimes f_*N \to f_*(f^*M \otimes N)$ is an equivalence for all $M \in \mathbf{RigSH}^{(\wedge)}_\tau(X;\Lambda)$ and $N \in \mathbf{RigSH}^{(\wedge)}_\tau(Y;\Lambda)$.*

(2) (Projective extended base change) *Let $g : X' \to X$ be a morphism of rigid analytic spaces and form a Cartesian square*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X. \end{array}$$

The natural transformation $g^* \circ f_* \to f'_* \circ g'^*$, between functors from $\mathbf{RigSH}^{(\wedge)}_\tau(Y;\Lambda)$ to $\mathbf{RigSH}^{(\wedge)}_\tau(X';\Lambda)$, is an equivalence. If moreover g is smooth, then the natural transformation $g_\sharp \circ f'_* \to f_* \circ g'_\sharp$, from $\mathbf{RigSH}^{(\wedge)}_\tau(Y';\Lambda)$ to $\mathbf{RigSH}^{(\wedge)}_\tau(X;\Lambda)$, is an equivalence.

Proof. If $f = f_1 \circ f_2$, then the assertions for f follow from their analogues for f_1 and f_2. Also, the assertions can be checked locally on X. Thus, it is enough to treat the case of a closed immersion $i : Z \to X$ and the case of $p : \mathbb{P}^n_X \to X$. The case of a closed immersion follows from Proposition 2.2.3. For $p : \mathbb{P}^n_X \to X$, we use Corollary 2.2.9 which provides us with a canonical equivalence $p_* \simeq p_! = p_\sharp \circ \mathrm{Th}^{-1}(\Omega_p)$. The result follows then from Proposition 2.2.1. □

We now go back to the notation introduced in Remark 2.2.6. Given a U-scheme X which is locally of finite type, the analytification functor (2.11) induces a premorphism of sites

$$\mathrm{An} : (\mathrm{RigSm}/X^{\mathrm{an}}, \tau) \to (\mathrm{Sm}/X, \tau). \tag{2.12}$$

(Indeed, the analytification of an étale cover is an étale cover, and the analytification of a Nisnevich cover can be refined by an open cover; see [Ayo15, Théorème 1.2.39] whose proof can be adapted to our context.) By the functoriality of the construction of the ∞-categories of motives, (2.12) induces a functor

$$\mathrm{An}^* : \mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(X;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X^{\mathrm{an}};\Lambda). \tag{2.13}$$

In [Ayo15], this functor is denoted by Rig^*.

Proposition 2.2.13. *The functors* (2.13) *are part of a morphism of* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$*-valued presheaves*

$$\mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(-;\Lambda)^\otimes \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau((-)^{\mathrm{an}};\Lambda)^\otimes \tag{2.14}$$

on $\mathrm{Sch}^{\mathrm{lft}}/U$*. In particular, the functors* An^* *are monoidal and commute with the inverse image functors. Moreover, if f is a smooth morphism in* $\mathrm{Sch}^{\mathrm{lft}}/U$*, the natural transformation*

$$f_\sharp^{\mathrm{an}} \circ \mathrm{An}^* \to \mathrm{An}^* \circ f_\sharp$$

is an equivalence.

Proof. One argues as in [Rob14, §9.1] for the first assertion. The second assertion is clear. □

Proposition 2.2.14. *Let $f : Y \to X$ be a proper morphism in* $\mathrm{Sch}^{\mathrm{lft}}/U$*. Then, the natural transformation*

$$\mathrm{An}^* \circ f_* \to f_*^{\mathrm{an}} \circ \mathrm{An}^*,$$

between functors from $\mathbf{SH}_\tau^{(\wedge)}(Y;\Lambda)$ *to* $\mathbf{RigSH}_\tau^{(\wedge)}(X^{\mathrm{an}};\Lambda)$*, is invertible.*

Proof. We split the proof into two steps.

Step 1. Here we assume that f is projective. It is enough to prove the claim when f is a closed immersion and when f is the projection $\mathbb{P}^n_X \to X$. In the first case, one uses Proposition 2.2.3 and its algebraic analogue. In the second case, one uses Corollary 2.2.9 and its algebraic analogue to reduce to show that $f_\sharp^{\mathrm{an}} \circ \mathrm{An}^* \simeq \mathrm{An}^* \circ f_\sharp$ which holds by Proposition 2.2.13.

Step 2. Here we deal with the general case. We may assume that X is quasi-compact and quasi-separated. Using Proposition 2.2.13, we reduce easily to show that

$$\mathrm{An}^* \circ f_* \circ j_\sharp \to f_*^{\mathrm{an}} \circ j_\sharp^{\mathrm{an}} \circ \mathrm{An}^*$$

is an equivalence for every open immersion $j : V \to Y$, with V affine. By the refined version of Chow's lemma given in [Con07, Corollary 2.6], there is a blowup $e : Y' \to Y$, with centre disjoint from V, such that $f' : Y' \to X$ is projective. Let $j' : V \to Y'$ be the obvious inclusion. by Proposition 2.2.12(2) and its algebraic version, we have equivalences $e_* \circ j'_\sharp \simeq j_\sharp$ and $e_*^{\mathrm{an}} \circ j'^{\mathrm{an}}_\sharp \simeq j_\sharp^{\mathrm{an}}$. Thus, it is enough to prove the proposition for $f' = f \circ e$. Since this morphism is projective, we may conclude by the first step. □

Remark 2.2.15. The method used in the second step of the proof of Proposition 2.2.14 will be used again in the second part of the proof of Proposition 4.1.1 below to deduce the proper base change theorem for $\mathbf{SH}_\tau^{(\wedge)}(-;\Lambda)$ from its special case for projective morphisms which is covered by [Ayo07a, Corollaire 1.7.18]. (For a slightly different method using the usual version of Chow's lemma but requiring the schemes to be noetherian, see the proof of [CD19, Proposition 2.3.11(2)].) Similarly, this method can be used to generalise Proposition 2.2.12 to the case where f is locally the analytification of a proper morphism of schemes. However, our aim is to prove a more substantial generalisation of that proposition which cannot be reached using this method. This will be achieved in Theorem 4.1.4 below.

2.3. **Descent.**

In this subsection, we prove that the functor $S \mapsto \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(S;\Lambda)$, $f \mapsto f^*$, whose existence is claimed in Proposition 2.1.21, admits (hyper)descent for the topology τ. This can be considered as a folklore theorem, but we reproduce the proof here for completeness. For a comparable result in the algebraic setting, see [Hoy17, Proposition 4.8].

For later use, we recall the precise definition of a (hyper)sheaf valued in a general ∞-category. (Compare with [Dre18, Definition 2.1].)

Definition 2.3.1. Let $(\mathcal{C}, \tau)$ be a site and let $\mathcal{V}$ be an ∞-category admitting all limits. A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ is called a τ-(hyper)sheaf (or is said to satisfy τ-(hyper)descent) if its right Kan extension $\overline{F} : \mathcal{P}(\mathcal{C})^{\mathrm{op}} \to \mathcal{V}$, along the Yoneda embedding, factors through the opposite of the localisation functor $\mathrm{L}_\tau : \mathcal{P}(\mathcal{C}) \to \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C})$. This is equivalent to the condition that $\overline{F}$ induces an equivalence

$$\overline{F}(X_{-1}) \xrightarrow{\sim} \lim_{[n]\in\Delta} \overline{F}(X_n) \tag{2.15}$$

for every effective τ-hypercover $X_\bullet$. (An effective τ-hypercover $X_\bullet$ is an augmented simplicial object of $\mathcal{P}(\mathcal{C})$ such that $\mathrm{L}_\tau(X_\bullet)$ is an effective hypercovering of the ∞-topos $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C})_{/\mathrm{L}_\tau X_{-1}}$, in the sense of [Lur09, Definition 6.5.3.2].) We denote by $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}; \mathcal{V})$ the full sub-∞-category of $\mathrm{PSh}(\mathcal{C}; \mathcal{V}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V})$ spanned by τ-(hyper)sheaves. When $\mathcal{V}$ is the ∞-category $\mathcal{S}$ of spaces, we get back the ∞-topos $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C})$.

We gather a few facts about (hyper)sheaves with values in general ∞-categories. We refer the reader to [Dre18, §2] for proofs and more details.

Remark 2.3.2. Keep the notation as in Definition 2.3.1.

(1) In the hypercomplete case, every τ-hypercover is effective. Therefore, for F to be a τ-hypersheaf, the equivalence (2.15) needs to hold for every τ-hypercover, but see Remark 2.3.3(3) below.

(2) In the non-hypercomplete case, for F to be a τ-sheaf, it is enough that the equivalence (2.15) holds for $X_\bullet$ the Čech nerve associated to a τ-cover in $\mathcal{C}$. It then holds for any truncated τ-hypercover. See [Lur09, Definition 6.2.2.6 & Lemma 6.5.3.9].

Remark 2.3.3. Keep the notation as in Definition 2.3.1.

(1) Let $\phi : \mathcal{V} \to \mathcal{V}'$ be a limit-preserving functor between ∞-categories admitting all limits. Then the induced functor $\Phi : \mathrm{PSh}(\mathcal{C}; \mathcal{V}) \to \mathrm{PSh}(\mathcal{C}; \mathcal{V}')$ preserves τ-(hyper)sheaves. If moreover ϕ detects limits, then Φ detects τ-(hyper)sheaves.

(2) Assume that $\mathcal{V}$ is presentable. Then the ∞-category $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}; \mathcal{V})$ is an accessible left-exact localization of $\mathrm{PSh}(\mathcal{C}; \mathcal{V})$. In particular, it is also presentable. We denote by

$$\mathrm{L}_\tau : \mathrm{PSh}(\mathcal{C}; \mathcal{V}) \to \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}; \mathcal{V})$$

the τ-(hyper)sheafification functor defined as the left adjoint to the obvious inclusion. (This was introduced in Notation 2.1.2 for $\mathcal{V} = \mathrm{Mod}_\Lambda$.) With respect to the monoidal structure on Pr^{L} of [Lur17, §4.8.1], we have $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}; \mathcal{V}) \simeq \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}) \otimes \mathcal{V}$; see [Dre18, Proposition 2.4(1)] whose proof is also valid in the non-hypercomplete case.

(3) If $(\mathcal{C}, \tau)$ is a Verdier site (in the sense of [DHI04, Definition 9.1]) satisfying the assumptions (1-3) of [DHI04, §10], the condition of F being a τ-(hyper)sheaf can be expressed without recourse to its right Kan extension $\overline{F}$. More precisely, F is a τ-(hyper)sheaf if F transforms representable coproducts in $\mathcal{C}$ into products in $\mathcal{V}$ and if for every internal τ-hypercover $X_\bullet$ (in the sense of [DHI04, Definition 10.1]) which is effective, F induces an equivalence

$$F(X_{-1}) \xrightarrow{\sim} \lim_{[n]\in\Delta} F(X_n).$$

(As explained in Remark 2.3.2, in the hypercomplete case, effectivity is an empty condition and, in the non-hypercomplete case, we may replace it with the condition that $X_\bullet$ is truncated or better with the condition that $X_\bullet$ is the Čech nerve of a basal morphism $X_0 \to X_{-1}$ which is a τ-cover.) This is proven in [Dre18, Proposition 2.7] in the hypercomplete case and is clear in the non-hypercomplete case. It applies to the sites we consider in this paper, such as the big smooth sites of Notation 1.4.9.

The main result of this subsection is the following.

Theorem 2.3.4. *Let* $\tau \in \{\mathrm{nis}, \text{ét}\}$ *be a topology on rigid analytic spaces. The contravariant functor*

$$S \mapsto \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(S;\Lambda), \quad f \mapsto f^*$$

defines a τ*-(hyper)sheaf on* RigSpc *valued in* Pr^{L}.

Remark 2.3.5. The forgetful functor $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$ being limit-preserving and conservative (by [Lur17, Corollary 3.2.2.5 & Lemma 3.2.2.6]), Theorem 2.3.4 and Remark 2.3.3(1) imply that $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)^{\otimes}$ is also a τ-(hyper)sheaf valued in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

Before we can give the proof of Theorem 2.3.4 we need a digression about general (hyper)sheaves on general sites. Let $\mathcal{C}$ be a small ∞-category and X an object of $\mathcal{C}$. Composition with the obvious projection $\mathrm{j}_X : \mathcal{C}_{/X} \to \mathcal{C}$ induces a functor $\mathrm{j}_X^* : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C}_{/X})$ which preserves limits and colimits. We denote by $\mathrm{j}_{X,!}$ the left adjoint of j_X^* and $\mathrm{j}_{X,*}$ its right adjoint. A topology τ on $\mathcal{C}$ induces a topology on $\mathcal{C}_{/X}$ which we also denote by τ. It is easy to see that j_X^* and $\mathrm{j}_{X,*}$ preserve τ-(hyper)sheaves. (For $\mathrm{j}_{X,*}$, note that modulo the equivalence $\mathcal{P}(\mathcal{C}_{/X}) \simeq \mathcal{P}(\mathcal{C})_{/\mathrm{y}(X)}$, the functor $\mathrm{j}_{X,*}$ takes a presheaf F on $\mathcal{C}_{/X}$ to the presheaf $U \mapsto \mathrm{Map}_{\mathcal{P}(\mathcal{C})_{/\mathrm{y}(X)}}(\mathrm{y}(U) \times \mathrm{y}(X), F)$.) We get in this way an adjunction

$$\mathrm{j}_X^* : \mathrm{Shv}^{(\wedge)}_{\tau}(\mathcal{C}) \rightleftarrows \mathrm{Shv}^{(\wedge)}_{\tau}(\mathcal{C}_{/X}) : \mathrm{j}_{X,*}$$

where j_X^* commutes with all limits and colimits. In particular, j_X^* admits a left adjoint (on the level of (hyper)sheaves) which we denote by $\mathrm{j}^{\tau}_{X,!}$. It is related to $\mathrm{j}_{X,!}$ by an equivalence $\mathrm{j}^{\tau}_{X,!} \circ \mathrm{L}_{\tau} \simeq \mathrm{L}_{\tau} \circ (\mathrm{j}_X)_!$. The following lemma is well-known. We include a proof for completeness.

Lemma 2.3.6. *Let* $(\mathcal{C}, \tau)$ *be a site and* $X \in \mathcal{C}$*. The functor* $\mathrm{j}^{\tau}_{X,!}$ *factors through an equivalence*

$$\mathrm{e}_X : \mathrm{Shv}^{(\wedge)}_{\tau}(\mathcal{C}_{/X}) \xrightarrow{\sim} \mathrm{Shv}^{(\wedge)}_{\tau}(\mathcal{C})_{/\mathrm{L}_{\tau}\mathrm{y}(X)}.$$

Proof. The functor $\mathrm{j}^{\tau}_{X,!} : \mathrm{Shv}^{(\wedge)}_{\tau}(\mathcal{C}_{/X}) \to \mathrm{Shv}^{(\wedge)}_{\tau}(\mathcal{C})$ sends the final object $\mathrm{L}_{\tau}\mathrm{y}(\mathrm{id}_X)$ of $\mathrm{Shv}^{(\wedge)}_{\tau}(\mathcal{C}_{/X})$ to $\mathrm{L}_{\tau}\mathrm{y}(X)$. This gives the functor e_X. By construction, we have a commutative square

$$\begin{tikzcd}
\mathcal{P}(\mathcal{C}/X) \arrow[r, "\mathrm{e}'_X"] \arrow[d, "\mathrm{L}_{\tau}"] & \mathcal{P}(\mathcal{C})_{/\mathrm{y}(X)} \arrow[d, "\mathrm{L}'_{\tau}"] \\
\mathrm{Shv}^{(\wedge)}_{\tau}(\mathcal{C}/X) \arrow[r, "\mathrm{e}_X"] & \mathrm{Shv}^{(\wedge)}_{\tau}(\mathcal{C})_{/\mathrm{L}_{\tau}\mathrm{y}(X)}.
\end{tikzcd}$$

By [Lur09, Corollary 5.1.6.12], e'_X is an equivalence. Note that L'_{τ} is essentially surjective on objects. Indeed, given a morphism of τ-(hyper)sheaves $F \to \mathrm{L}_{\tau}\mathrm{y}(X)$, there is an equivalence $F \simeq \mathrm{L}_{\tau}(F \times_{\mathrm{L}_{\tau}\mathrm{y}(X)} \mathrm{y}(X))$ since L_{τ} is exact and idempotent. To finish the proof, it will suffice to show that e_X is fully faithful. Let f_X be a right adjoint to e_X and f'_X a right adjoint to e'_X. We know that the unit $\mathrm{id} \to \mathrm{f}'_X \circ \mathrm{e}'_X$ is an equivalence, and we need to prove that the unit $\mathrm{id} \to \mathrm{f}_X \circ \mathrm{e}_X$ is an equivalence. By [Lur09, Proposition 5.2.5.1], f_X sends a map $F \to \mathrm{L}_{\tau}\mathrm{y}(X)$ to the fiber product $\mathrm{j}_X^* F \times_{\mathrm{j}_X^* \mathrm{L}_{\tau}\mathrm{y}(X)} \{*\}$ and f'_X sends a map $F' \to \mathrm{y}(X)$ to the fiber product $\mathrm{j}_X^* F' \times_{\mathrm{j}_X^* \mathrm{y}(X)} \{*\}$. Since (hyper)sheafification is exact, we deduce that the natural transformation $\mathrm{L}_{\tau} \circ \mathrm{f}'_X \to \mathrm{f}_X \circ \mathrm{L}'_{\tau}$ is an equivalence. Using the commutative square

$$\begin{tikzcd} \mathrm{L}_\tau \arrow[r] \arrow[d, "\sim"] & \mathrm{f}_X \circ \mathrm{e}_X \circ \mathrm{L}_\tau \arrow[d, "\sim"] \\ \mathrm{L}_\tau \circ \mathrm{f}'_X \circ \mathrm{e}'_X \arrow[r, "\sim"] & \mathrm{f}_X \circ \mathrm{L}'_\tau \circ \mathrm{e}'_X, \end{tikzcd}$$

if follows that the natural transformation $\mathrm{L}_\tau \to \mathrm{f}_X \circ \mathrm{e}_X \circ \mathrm{L}_\tau$ is an equivalence, which is enough to conclude since L_τ is essentially surjective. □

We denote by $\mathrm{Top}^{\mathrm{L}}$ the ∞-category of ∞-topoi and exact left adjoint functors, as defined in [Lur09, Definition 6.3.1.5].

Proposition 2.3.7. *Let $(\mathcal{C}, \tau)$ be a site. The functor $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}_{/(-)}) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}$, taking an object X of $\mathcal{C}$ to the ∞-topos $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}_{/X})$ and a morphism f in $\mathcal{C}$ to the functor j_f^*, is a τ-(hyper)sheaf.*

Proof. Every ∞-topos $\mathcal{X}$ determines a $\mathrm{Top}^{\mathrm{L}}$-valued sheaf on itself: by [Lur09, Proposition 6.3.5.14], the functor $\chi : \mathcal{X}^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}$, sending $X \in \mathcal{X}$ to $\mathcal{X}_{/X}$, preserves limits. Take $\mathcal{X} = \mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C})$. Since $\mathrm{L}_\tau : \mathcal{P}(\mathcal{C}) \to \mathcal{X}$ preserves colimits, we deduce that $\chi \circ \mathrm{L}_\tau : \mathcal{P}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}$ preserves limits. It follows that the functor $\chi \circ \mathrm{L}_\tau$ is a right Kan extension of $\chi \circ \mathrm{L}_\tau \circ \mathrm{y} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}$. Since $\chi \circ \mathrm{L}_\tau$ clearly factors through $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C})$, the functor $\chi \circ \mathrm{L}_\tau \circ \mathrm{y}$ is a τ-(hyper)sheaf. Now, by Lemma 2.3.6, the functor $\chi \circ \mathrm{L}_\tau \circ \mathrm{y}$ is equivalent to the one sending $X \in \mathcal{C}$ to $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}_{/X})$. □

Corollary 2.3.8. *Let $(\mathcal{C}, \tau)$ be a site and $\mathcal{V}$ a presentable ∞-category. The functor $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}_{/(-)}; \mathcal{V}) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$, taking an object X of $\mathcal{C}$ to the ∞-category $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}_{/X}; \mathcal{V})$ and a morphism f in $\mathcal{C}$ to the functor j_f^*, is a τ-(hyper)sheaf.*

Proof. By Proposition 2.3.7, the result holds when $\mathcal{V}$ is the ∞-category of spaces $\mathcal{S}$, and we want to reduce to this case. We denote by $\mathcal{X}(-;\mathcal{V}) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$ the functor sending $X \in \mathcal{C}$ to $\mathrm{Shv}_\tau^{(\wedge)}(\mathcal{C}_{/X}; \mathcal{V})$. By Remark 2.3.3, we have an equivalence of functors $\mathcal{X}(-;\mathcal{S}) \otimes \mathcal{V} \xrightarrow{\sim} \mathcal{X}(-;\mathcal{V})$, where the tensor product is taken in Pr^{L} (see [Lur17, §4.8.1]). Moreover, for any $f : Y \to X$ in $\mathcal{C}$, the functor $\mathrm{j}_f^* : \mathcal{X}(X;\mathcal{S}) \to \mathcal{X}(Y;\mathcal{S})$ commutes with all limits. It follows from Lemma 2.3.9 below that there is an equivalence of functors

$$\mathcal{X}(-;\mathcal{S}) \otimes \mathcal{V} \simeq \mathrm{Fun}^{\lim}(\mathcal{V}^{\mathrm{op}}, \mathcal{X}(-;\mathcal{S})).$$

Thus, it is enough to show that $\mathrm{Fun}^{\lim}(\mathcal{V}^{\mathrm{op}}, \mathcal{X}(-;\mathcal{S})) : \mathcal{C}^{\mathrm{op}} \to \mathrm{CAT}_\infty$ is a τ-(hyper)sheaf. This follows from Proposition 2.3.7 since the endofunctor $\mathrm{Fun}^{\lim}(\mathcal{V}^{\mathrm{op}}, -)$ of CAT_∞ preserves limits. □

Lemma 2.3.9. *Let $\mathrm{Pr}^{\mathrm{LR}}$ be the wide sub-∞-category of Pr^{L} where morphisms are the limit-preserving left adjoints. Let $\mathcal{D}$ be a presentable ∞-category. Then the functor $\mathcal{D} \otimes - : \mathrm{Pr}^{\mathrm{LR}} \to \mathrm{CAT}_\infty$, obtained by restriction from the tensor product of Pr^{L}, is equivalent to the functor*

$$\mathrm{Fun}^{\lim}(\mathcal{D}^{\mathrm{op}}; -) : \mathrm{Pr}^{\mathrm{LR}} \to \mathrm{CAT}_\infty,$$

where $\mathrm{Fun}^{\lim}(-,-) \subset \mathrm{Fun}(-,-)$ indicates the full sub-∞-category of limit-preserving functors.

Proof. The endofunctor $\mathcal{D} \otimes -$ of Pr^{L} induces an endofunctor of Pr^{R} given by the composition of

$$\mathrm{Pr}^{\mathrm{R}} \xrightarrow{\sim} (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \xrightarrow{\mathcal{D}\otimes -} (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Pr}^{\mathrm{R}}.$$

By [Lur17, Proposition 4.8.1.17], this coincides with the endofunctor $\mathrm{Fun}^{\lim}(\mathcal{D}^{\mathrm{op}}, -)$ of Pr^{R}. It follows that the endofunctor $\mathcal{D} \otimes -$ of Pr^{L} is given by the composition of

$$\mathrm{Pr}^{\mathrm{L}} \xrightarrow{\sim} (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}} \xrightarrow{\mathrm{Fun}^{\lim}(\mathcal{D}^{\mathrm{op}},-)} (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Pr}^{\mathrm{L}}.$$

It remains to show that the composition of

$$\mathrm{Pr}^{\mathrm{LR}} \to \mathrm{Pr}^{\mathrm{L}} \xrightarrow{\sim} (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}} \xrightarrow{\mathrm{Fun}^{\mathrm{lim}}(\mathcal{D}^{\mathrm{op}},-)} (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Pr}^{\mathrm{L}} \to \mathrm{CAT}_\infty$$

is also given by $\mathrm{Fun}^{\mathrm{lim}}(\mathcal{D}^{\mathrm{op}},-)$. On objects, this is clear. On morphisms, this is also true by the following observation: if $F : \mathcal{E} \to \mathcal{E}'$ is in $\mathrm{Pr}^{\mathrm{LR}}$ with right adjoint G, then $\mathrm{Fun}^{\mathrm{lim}}(\mathcal{D}^{\mathrm{op}}, F)$ is left adjoint to $\mathrm{Fun}^{\mathrm{lim}}(\mathcal{D}^{\mathrm{op}}, G)$. To address higher coherences, we employ the formalism of Cartesian fibrations.

Let S be a simplicial set and $p : \mathcal{M} \to S$ a coCartesian fibration classified by a map $l : S \to \mathrm{Pr}^{\mathrm{LR}}$. Then p is also a Cartesian fibration which is classified by a map $r : S \to (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}}$ equivalent to the composition of

$$S \xrightarrow{l} \mathrm{Pr}^{\mathrm{L}} \xrightarrow{\sim} (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}}.$$

Moreover, p-Cartesian and p-coCartesian edges of $\mathcal{M}$ are preserved by small limits in the following sense. Let $a : s \to s'$ be an edge in S, $\overline{e} : K^{\triangleleft} \to \mathcal{M}_s$ and $\overline{e}' : K^{\triangleleft} \to \mathcal{M}_{s'}$ limit diagrams, and $f : \overline{e} \to \overline{e}'$ an edge in $\mathrm{Fun}(K^{\triangleleft}, \mathcal{M})$ over a. If $f(k)$ is p-coCartesian (resp. p-Cartesian) for every $k \in K$, then the same is true for $f(\infty)$, where $\infty \in K^{\triangleright}$ is the cone point. This is simply a reformulation of the fact that l (resp. r) takes an edge of S to a limit-preserving functor. Consider the simplicial set $\mathcal{N} = \mathcal{M}^{\mathcal{D}^{\mathrm{op}}} \times_{S^{\mathcal{D}^{\mathrm{op}}}} S$ whose n-simplices correspond to pairs consisting of an n-simplex $[n] \to S$ and an S-morphism $[n] \times \mathcal{D}^{\mathrm{op}} \to \mathcal{M}$. Let $\mathcal{N}' \subset \mathcal{N}$ be the largest simplicial subset whose vertices correspond to limit-preserving functors $\mathcal{D}^{\mathrm{op}} \to \mathcal{M}_s$, for some $s \in S$. Let $q : \mathcal{N} \to S$ and $q' : \mathcal{N}' \to S$ be the obvious projections. By [Lur09, Proposition 2.4.2.3(2) & Proposition 3.1.2.1], q is again a coCartesian fibration, classified by $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}},-) \circ l : S \to \mathrm{CAT}_\infty$, and a Cartesian fibration classified by $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}},-) \circ r : S \to (\mathrm{CAT}_\infty)^{\mathrm{op}}$. Since p-coCartesian (resp. p-Cartesian) edges are preserved by small limits, it follows readily that a q-coCartesian (resp. q-Cartesian) edge whose domain (resp. target) belongs to $\mathcal{N}'$ lies entirely in $\mathcal{N}'$. This shows that q' is a coCartesian fibration, classified by $l' = \mathrm{Fun}^{\mathrm{lim}}(\mathcal{D}^{\mathrm{op}},-) \circ l : S \to \mathrm{CAT}_\infty$, and a Cartesian fibration classified by $r' = \mathrm{Fun}^{\mathrm{lim}}(\mathcal{D}^{\mathrm{op}},-) \circ r : S \to (\mathrm{CAT}_\infty)^{\mathrm{op}}$. It follows that l' factors through $\mathrm{CAT}^{\mathrm{L}}_\infty$, r' factors through $\mathrm{CAT}^{\mathrm{R}}_\infty$, and l' coincides with the composition of

$$\mathcal{S}^{\mathrm{op}} \xrightarrow{r'} (\mathrm{CAT}^{\mathrm{R}}_\infty)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{CAT}^{\mathrm{L}}_\infty.$$

Unravelling the definitions, this gives what we want. □

Proof of Theorem 2.3.4. It suffices to prove that for every rigid analytic space S, the functor

$$\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(-;\Lambda) : (\mathrm{\acute{E}t}/S)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}},$$

is a τ-(hyper)sheaf. (When $\tau = \mathrm{nis}$, one can restrict further to $(\mathrm{\acute{E}t}^{\mathrm{gr}}/S)^{\mathrm{op}}$, but this does not change the argument.) This functor transforms coproducts in $\mathrm{\acute{E}t}/S$ into products in Pr^{L}. Thus, it suffices to show that it admits descent with respect to internal hypercovers of $(\mathrm{\acute{E}t}/S, \tau)$, which are truncated in the non-hypercomplete case.

For $U \in \mathrm{\acute{E}t}/S$, we have $(\mathrm{RigSm}/S)/U \simeq \mathrm{RigSm}/U$. Corollary 2.3.8 implies that the functor

$$\mathrm{Shv}^{(\wedge)}_\tau(\mathrm{RigSm}/-;\Lambda) : (\mathrm{\acute{E}t}/S)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$$

is a τ-(hyper)sheaf. Let $U_\bullet$ be an internal hypercover of $(\mathrm{\acute{E}t}/S, \tau)$ which we assume to be truncated in the non-hypercomplete case. For all $n \geq -1$, $\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_\tau(U_n;\Lambda)$ is a full sub-∞-category of

$\mathrm{Shv}^{(\wedge)}_{\tau}(\mathrm{RigSm}/U_n;\Lambda)$. Since limits in CAT_∞ preserve fully faithful embeddings, we deduce that $\lim_{[n]\in\Delta}\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(U_n;\Lambda)$ can be naturally identified with the sub-∞-category of

$$\mathrm{Shv}^{(\wedge)}_{\tau}(\mathrm{RigSm}/U_{-1};\Lambda)\simeq\lim_{[n]\in\Delta}\mathrm{Shv}^{(\wedge)}_{\tau}(\mathrm{RigSm}/U_n;\Lambda)$$

spanned by the objects $\mathcal{F}\in\mathrm{Shv}^{(\wedge)}_{\tau}(\mathrm{RigSm}/U_{-1};\Lambda)$ such that $f^*\mathcal{F}$ belongs to $\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(U_0;\Lambda)$, with $f:U_0\to U_{-1}$. Thus, to prove that $\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(-;\Lambda)$ has descent for the τ-hypercover $U_\bullet$, we need to check the following property: if $\mathcal{F}$ is a τ-(hyper)sheaf on RigSm/S such that $f^*\mathcal{F}$ is $\mathbb{B}^1$-invariant, then so is $\mathcal{F}$. This follows immediately from the equivalence $\underline{\mathrm{Hom}}(\mathbb{B}^1_{U_0},f^*\mathcal{F})\simeq f^*\underline{\mathrm{Hom}}(\mathbb{B}^1_{U_{-1}},\mathcal{F})$ and the fact that f^* is conservative.

We now explain how to deduce the T-stable case from the effective case. We temporarily denote by $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)^*$ (resp. $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)_*$) the presheaf (resp. copresheaf) given informally by $U\mapsto\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)$ and $f\mapsto f^*$ (resp. $f\mapsto f_*$). Recall from Remark 2.1.17 that the presheaf $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*$ can be defined as the colimit in $\mathrm{PSh}(\mathrm{\acute{E}t}/S;\mathrm{Pr}^{\mathrm{L}})$ of the $\mathbb{N}$-diagram of presheaves:

$$\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(-;\Lambda)^*\xrightarrow{\mathrm{T}\otimes-}\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(-;\Lambda)^*\xrightarrow{\mathrm{T}\otimes-}\cdots.$$

It follows from [Lur09, Corollary 5.5.3.4 &Theorem 5.5.3.18] that the copresheaf $\mathbf{RigSH}^{(\wedge)}_{\tau}(U;\Lambda)_*$, can be computed as the limit in $\mathrm{Fun}(\mathrm{\acute{E}t}/S,\mathrm{CAT}_\infty)$ of the $\mathbb{N}^{\mathrm{op}}$-diagram of copresheaves

$$\cdots\xrightarrow{\underline{\mathrm{Hom}}(\mathrm{T},-)}\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(-;\Lambda)_*\xrightarrow{\underline{\mathrm{Hom}}(\mathrm{T},-)}\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(-;\Lambda)_*.$$

Given that the natural transformation $f^*\underline{\mathrm{Hom}}(\mathrm{T},-)\to\underline{\mathrm{Hom}}(\mathrm{T},-)\circ f^*$ is an equivalence for f étale, we deduce that the presheaf $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*$ can also be computed as the limit in $\mathrm{PSh}(\mathrm{\acute{E}t}/S;\mathrm{CAT}_\infty)$ of the $\mathbb{N}^{\mathrm{op}}$-diagram of presheaves

$$\cdots\xrightarrow{\underline{\mathrm{Hom}}(\mathrm{T},-)}\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(-;\Lambda)^*\xrightarrow{\underline{\mathrm{Hom}}(\mathrm{T},-)}\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(-;\Lambda)^*.$$

Since $\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(-;\Lambda)^*$ was proven to be a τ-(hyper)sheaf, this finishes the proof. □

2.4. **Compact generation.**

In this subsection, we formulate conditions (in terms of Λ, S and τ) insuring that the ∞-category $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(S;\Lambda)$ of rigid analytic motives over S is compactly generated. Similar results in the algebraic setting were developed in [Ayo07b, §4.5.5] and [Ayo14a, pages 29–30].

Remark 2.4.1. Let $\mathcal{X}$ be an ∞-topos. An abelian group object of $\mathcal{X}_{\leq 0}$ endowed with the structure of a $\pi_0\Lambda$-module is called a discrete sheaf of $\pi_0\Lambda$-modules on $\mathcal{X}$. The n-th cohomology group of $\mathcal{X}$ with coefficients in a discrete sheaf of $\pi_0\Lambda$-modules $\mathcal{F}$ is defined in [Lur09, Definition 7.2.2.14] and will be denoted by $\mathrm{H}^n(\mathcal{X};\mathcal{F})$.

Recall the following notions. (Compare with [Lur09, Definition 7.2.2.18].)

Definition 2.4.2. Let $\mathcal{X}$ be an ∞-topos.

(1) The Λ-cohomological dimension of an object $X\in\mathcal{X}$ is the smallest $d\in\mathbb{N}\sqcup\{-\infty,\infty\}$ such that for every discrete sheaf of $\pi_0\Lambda$-modules $\mathcal{F}$ on $\mathcal{X}_{/X}$, the cohomology groups $\mathrm{H}^n(\mathcal{X}_{/X};\mathcal{F})$ vanish for $n>d$. The global Λ-cohomological dimension of $\mathcal{X}$ is the Λ-cohomological dimension of a final object of $\mathcal{X}$.

(2) The local Λ-cohomological dimension of $\mathcal{X}$ is the smallest $d \in \mathbb{N} \sqcup \{-\infty, \infty\}$ such that every object $X \in \mathcal{X}$ admits a cover $(Y_i \to X)_i$ such that Y_i is of Λ-cohomological dimension $\leq d$ for all i. (Recall that $(Y_i \to X)_i$ is a cover if $\coprod_i Y_i \to X$ is an effective epimorphism in the sense of [Lur09, §6.2.3].)

Remark 2.4.3. Keep the notation as in Definition 2.4.2. A discrete sheaf of $\pi_0\Lambda$-modules $\mathcal{F}$ on $\mathcal{X}_{/X}$ is a hypersheaf, i.e., belongs to $(\mathcal{X}_{/X})^\wedge \simeq (\mathcal{X}^\wedge)_{/X^\wedge}$. Thus, there are isomorphisms

$$\mathrm{H}^i(\mathcal{X}_{/X}; \mathcal{F}) \simeq \mathrm{H}^i(\mathcal{X}_{/X^\wedge}; \mathcal{F}) \simeq \mathrm{H}^i((\mathcal{X}^\wedge)_{/X^\wedge}; \mathcal{F}).$$

In particular, the Λ-cohomological dimension of an object X is equal to the Λ-cohomological dimension of its hypercompletion $X^\wedge$ considered as an object of $\mathcal{X}$ or $\mathcal{X}^\wedge$. Similarly, the global (resp. local) Λ-cohomological dimensions of $\mathcal{X}$ and $\mathcal{X}^\wedge$ coincide.

Remark 2.4.4. We define the local (resp. global) Λ-cohomological dimension of a site $(\mathcal{C}, \tau)$ to be the local (resp. global) Λ-cohomological dimension of the topos $\mathrm{Shv}_\tau(\mathcal{C})$ (or, equivalently, $\mathrm{Shv}_\tau^\wedge(\mathcal{C})$). Similarly, we define the Λ-cohomological dimension of an object X of a site $(\mathcal{C}, \tau)$ to be the Λ-cohomological dimension of the image of X in $\mathrm{Shv}_\tau(\mathcal{C})$ (or, equivalently, $\mathrm{Shv}_\tau^\wedge(\mathcal{C})$). By Lemma 2.3.6, this coincides with the global Λ-cohomological dimension of the site $(\mathcal{C}_{/X}, \tau)$.

We gather some well-known consequences of the finiteness of the local Λ-cohomological dimension in the following statement. (See Remark 2.1.3.)

Lemma 2.4.5. *Let $(\mathcal{C}, \tau)$ be a site of finite local Λ-cohomological dimension.*

(1) Postnikov towers in $\mathrm{Shv}_\tau^\wedge(\mathcal{C}; \Lambda)$ converge, i.e., the obvious map

$$\mathcal{F} \to \lim_{n \in \mathbb{N}} \tau_{\leq n} \mathcal{F}$$

is an equivalence for every τ-hypersheaf of Λ-modules $\mathcal{F}$ on $\mathcal{C}$.

(2) If $\mathcal{F}$ is a connective τ-hypersheaf of Λ-modules on $\mathcal{C}$ and $X \in \mathcal{C}$ is of Λ-cohomological dimension $\leq d$, then the Λ-module $\mathcal{F}(X)$ is $(-d)$-connective.

(3) Assume that $\mathcal{C}$ is an ordinary category admitting fiber products and that every object of $\mathcal{C}$ is quasi-compact in the sense of [SGAIV2, Exposé VI, Définitions 1.1]. *If $X \in \mathcal{C}$ is of finite Λ-cohomological dimension, then the functor $\mathrm{Shv}_\tau^\wedge(\mathcal{C}; \Lambda) \to \mathrm{Mod}_\Lambda$, $\mathcal{F} \mapsto \mathcal{F}(X)$ commutes with arbitrary colimits. In particular, $\Lambda_\tau(X)$ is a compact object of* $\mathrm{Shv}_\tau^\wedge(\mathcal{C}; \Lambda)$.

Proof. We may replace $(\mathcal{C}, \tau)$ with any site that gives rise to the same hypercomplete topos. Thus, we may assume that every object of $\mathcal{C}$ has Λ-cohomological dimension $\leq d$. Property (2), for every object $X \in \mathcal{C}$, follows from [Ayo07b, Proposition 4.5.58] when $(\mathcal{C}, \tau)$ is an ordinary site and Λ the unit spectrum. However, the proof of loc. cit. can be adapted without difficulty to our setting. That proof gives also property (1). (Note that (1) can be deduced from (2), but usually these two properties are proven together.) Since $\mathcal{F} \mapsto \mathcal{F}(X)$ is an exact functor between stable ∞-categories, it preserves pushouts. By [Lur09, Proposition 4.4.2.7], to prove property (3) it is enough to show that this functor commutes with filtered colimits. This follows from property (2) as in the proof of [Ayo07b, Corollaire 4.5.61]. (The extra conditions on $\mathcal{C}$ are used via [SGAIV2, Exposé VI, Corollaire 5.3] and can be substantially weakened.)

For a modern and more general treatment of this type of question, we refer the reader to [CM21, §2]. In particular, property (1) follows from [CM21, Proposition 2.10] (see also [CM21, Example 2.11]). Property (3) can be deduced from [CM21, Proposition 2.23]. Finally, we mention [Lur09, Proposition 7.2.1.10], which is obviously related to property (1). □

Corollary 2.4.6. *Let $(\mathcal{C}, \tau)$ be a site, and assume the following conditions:*

(1) Λ is eventually coconnective (i.e., its homotopy groups $\pi_i\Lambda$ vanish for i big enough);
(2) $(\mathcal{C}, \tau)$ has finite local Λ-cohomological dimension and $\mathcal{C}$ is an ordinary category with fiber products;
(3) there exists a full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ stable under fiber products, spanned by quasi-compact objects of finite Λ-cohomological dimension, and such that every object of $\mathcal{C}$ admits a τ-cover by objects of $\mathcal{C}_0$.

Then every τ-sheaf of Λ-modules on $\mathcal{C}$ is a τ-hypersheaf, i.e., we have $\mathrm{Shv}^{\wedge}_{\tau}(\mathcal{C}; \Lambda) = \mathrm{Shv}_{\tau}(\mathcal{C}; \Lambda)$.

Proof. By Lemma 2.1.4, we may replace $\mathcal{C}$ with $\mathcal{C}_0$ and assume that every object of $\mathcal{C}$ is quasi-compact, quasi-separated and of finite Λ-cohomological dimension. For $X \in \mathcal{C}$, the τ-sheaf $\Lambda_\tau(X)$ is hypercomplete since Λ is eventually coconnective. Thus, it is enough to show that τ-hypersheaves are stable under colimits in $\mathrm{Shv}_\tau(\mathcal{C}; \Lambda)$. The result then follows from [CM21, Proposition 2.23] but we can also deduce it formally from Lemma 2.4.5 as follows. Indeed, let $p : K \to \mathrm{Shv}^{\wedge}_{\tau}(\mathcal{C}; \Lambda)$ be a diagram of τ-hypersheaves of Λ-modules. The colimit of p in $\mathrm{Shv}_\tau(\mathcal{C}; \Lambda)$ is the τ-sheafification of the colimit of p in $\mathrm{PSh}(\mathcal{C}; \Lambda)$. So it is enough to show that the colimit of p in $\mathrm{PSh}(\mathcal{C}; \Lambda)$ is already a τ-hypersheaf. This follows immediately from Lemma 2.4.5(3). □

We now give some estimates for the local and global Λ-cohomological dimensions of the various small sites associated to a rigid analytic space.

Lemma 2.4.7. *Let X be a rigid analytic space of Krull dimension $\leq d$. The local Λ-cohomological dimension of* $(\mathrm{\acute{E}t}^{\mathrm{gr}}/X, \mathrm{nis})$ *is $\leq d$. If X is quasi-compact and quasi-separated, the same is true for the global Λ-cohomological dimension.*

Proof. Since every object of $\mathrm{\acute{E}t}^{\mathrm{gr}}/X$ can be covered by quasi-compact and quasi-separated rigid analytic spaces of Krull dimension $\leq d$, it is enough to prove the assertion concerning the global Λ-cohomological dimension. In particular, we may assume that X is quasi-compact and quasi-separated. The site $(\mathrm{\acute{E}t}^{\mathrm{gr}}/X, \mathrm{nis})$ is then equivalent to the limit of the Nisnevich sites $(\mathrm{\acute{E}t}/\mathcal{X}_\sigma, \mathrm{nis})$, for $\mathcal{X} \in \mathrm{Mdl}'(X)$ (see Remark 1.1.10). It follows from [SGAIV2, Exposé VII, Théorème 5.7] that the global Λ-cohomological dimension of the site $(\mathrm{\acute{E}t}^{\mathrm{gr}}/X, \mathrm{nis})$ is smaller than the supremum of the global Λ-cohomological dimensions of the sites $(\mathrm{\acute{E}t}/\mathcal{X}_\sigma, \mathrm{nis})$, for $\mathcal{X} \in \mathrm{Mdl}'(X)$. But if $\mathcal{X}$ is a formal model of X belonging to $\mathrm{Mdl}'(X)$, the closed map $|X| \to |\mathcal{X}_\sigma|$ is surjective. Thus, the dimension of $\mathcal{X}_\sigma$ is smaller than the dimension of X, and we conclude using [CM21, Theorem 3.17]. □

Definition 2.4.8. Let G be a profinite group. The Λ-cohomological dimension of G is the smallest $d \in \mathbb{N} \sqcup \{\infty\}$ such that, for every $\pi_0\Lambda$-module M endowed with a continuous action of G, the cohomology groups $\mathrm{H}^i(G; M)$ vanish for $i > d$. The virtual Λ-cohomological dimension of G is the infimum of the Λ-cohomological dimensions of the finite-index subgroups of G. If G admits a finite-index torsion-free subgroup H, then the virtual Λ-cohomological dimension of G is equal to the Λ-cohomological dimension of H. (See [Ser94, Chapitre I, §3.3, Proposition 14′].)

Let k be a field with absolute Galois group G_k. The (virtual) Λ-cohomological dimension of k is defined to be the (virtual) Λ-cohomological dimension of G_k.

Remark 2.4.9. Let k be a field. The following are classical facts about Galois cohomology.

(1) If the Λ-cohomological dimension of k is different from its virtual Λ-cohomological dimension, then k admits a real embedding and 2 is not invertible in $\pi_0\Lambda$.

(2) If k has (virtual) Λ-cohomological dimension $\leq d$ and K/k is an extension of transcendence degree $\leq e$, then K has (virtual) Λ-cohomological dimension $\leq d+e$.
(3) Number fields have virtual Λ-cohomological dimension ≤ 3, and finite fields have Λ-cohomological dimension ≤ 2.

Property (1) follows from [Ser94, Chapitre II, §4.1, Proposition 10′]. Property (2) follows from [Ser94, Chapitre II, §4.2, Proposition 11]. Property (3) follows from [Ser94, Chapitre II, §4.4, Proposition 13].

Definition 2.4.10. Let X be a scheme or a rigid analytic space. We denote by $\mathrm{pvcd}_\Lambda(X) \in \mathbb{N} \sqcup \{-\infty, \infty\}$ the supremum of the virtual Λ-cohomological dimensions of the fields $\kappa(x)$ for $x \in |X|$. This number is called the punctual virtual Λ-cohomological dimension of X.

Lemma 2.4.11. *Let X be a rigid analytic space of Krull dimension $\leq d$ and of punctual virtual Λ-cohomological dimension $\leq e$. Then, the local Λ-cohomological dimension of the site* $(\mathrm{\acute{E}t}/X, \mathrm{\acute{e}t})$ *is $\leq d+e$. The same is true for the global Λ-cohomological dimension if X is quasi-compact and quasi-separated, and if the Λ-cohomological dimension of the residue field of every point of X coincides with the virtual one.*

Proof. Replacing X by a suitable étale cover (e.g., by $X[\frac{1}{2}, \sqrt{-1}] \to X$ and $X[\frac{1}{3}, \sqrt[3]{1}] \to X$), we may assume that the Λ-cohomological dimension of the residue field of each point of X coincides with the virtual one. We may also assume that X is quasi-compact and quasi-separated. Under these conditions, we will show that the global Λ-cohomological dimension of $(\mathrm{\acute{E}t}/X, \mathrm{\acute{e}t})$ is $\leq d+e$, which suffices to conclude.

Denote by $\pi : (\mathrm{\acute{E}t}/X, \mathrm{\acute{e}t}) \to (\mathrm{\acute{E}t}^{\mathrm{gr}}/X, \mathrm{nis})$ the obvious morphism of sites. Given an étale sheaf $\mathcal{F}$ of $\pi_0\Lambda$-modules on $\mathrm{\acute{E}t}/X$, we denote by $\mathrm{R}\pi_*\mathcal{F}$ its (derived) direct image. Using Lemma 2.4.7, we are reduced to showing that $\mathrm{R}\pi_*\mathcal{F}$ is $(-e)$-connective. We check this on stalks at Nisnevich geometric rigid points of X as in Construction 1.4.27. Let $s \in S$ be a point and $t \to S$ a Nisnevich geometric rigid point over s. Thus, $t = \mathrm{Spf}(\kappa^+(t))$ with $\kappa^+(t)$ the adic completion of the Henselisation of $\kappa^+(s)$ at a morphism $\mathrm{Spec}(\widetilde{\kappa}(t)) \to \mathrm{Spec}(\kappa^+(s))$ associated to a separable finite extension $\widetilde{\kappa}(t)/\widetilde{\kappa}(s)$. It follows from Corollary 1.4.20 that $(\mathrm{R}\pi_*\mathcal{F})_t$ is equivalent to $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(t; (t \to S)^*\mathcal{F})$. Thus, it is sufficient to show that the global Λ-cohomological dimension of $(\mathrm{\acute{E}t}/t, \mathrm{\acute{e}t})$ is smaller than e. Since $\kappa^+(t)$ is Henselian, every étale cover of t can be refined by one of the form $\mathrm{Spf}(V)^{\mathrm{rig}} \to t$ where V is the normalisation of $\kappa^+(t)$ in a finite separable extension of $\kappa(t)$. Thus, the global cohomology of $(\mathrm{\acute{E}t}/t, \mathrm{\acute{e}t})$ coincides with the Galois cohomology of $\kappa(t)$. Since the field $\kappa(t)$ is the completion of an algebraic extension of $\kappa(s)$, we deduce that its Λ-cohomological dimension is $\leq e$ as needed. □

The following is a corollary of the proof of Lemma 2.4.11.

Corollary 2.4.12. *Let X be a rigid analytic space, and let $\mathcal{F}$ be a discrete sheaf of $\mathbb{Q}$-vector spaces on* $(\mathrm{\acute{E}t}/X, \mathrm{\acute{e}t})$. *Then the natural map* $\mathrm{H}^*_{\mathrm{nis}}(X; \mathcal{F}) \to \mathrm{H}^*_{\mathrm{\acute{e}t}}(X; \mathcal{F})$ *is an isomorphism.*

Proof. Arguing as in the proof of Lemma 2.4.11, the result follows from the vanishing of the higher Galois cohomology groups with rational coefficients. □

Corollary 2.4.13. *Let X be a rigid analytic space of Krull dimension $\leq d$. If Λ is a $\mathbb{Q}$-algebra, then the local Λ-cohomological dimension of the site* $(\mathrm{\acute{E}t}/X, \mathrm{\acute{e}t})$ *is $\leq d$. If X is quasi-compact and quasi-separated, the same is true for the global Λ-cohomological dimension.*

Definition 2.4.14. Let S be a scheme or a rigid analytic space.

(1) We say that S is $(\Lambda, \text{ét})$-admissible if there exists an open covering $(S_i)_i$ of S such that each S_i has finite Krull dimension and finite punctual virtual Λ-cohomological dimension. For convenience, we also say that S is (Λ, nis)-admissible when S is locally of finite Krull dimension.

(2) If 2 is not invertible in $\pi_0\Lambda$, we say that S is $(\Lambda, \text{ét})$-good if $\mathcal{O}(S)$ contains a primitive n-th root of unity for some $n \geq 3$. For convenience, we agree that S is always (Λ, τ)-good if 2 is invertible in $\pi_0\Lambda$ or if τ is the Nisnevich topology.

Remark 2.4.15. If S is $(\Lambda, \text{ét})$-good, then the Λ-cohomological dimension of the residue field of each of its points coincides with the virtual one. This follows from Remark 2.4.9.

Lemma 2.4.16. *Let $Y \to X$ be a morphism of rigid analytic spaces which is locally of finite type, and let $y \in Y$ be a point with image $x \in X$. If the (virtual) Λ-cohomological dimension of $\kappa(x)$ is finite, then so is the (virtual) Λ-cohomological dimension of $\kappa(y)$.*

Proof. We use the fact that $\kappa(y)/\kappa(x)$ is topologically of finite type, i.e., that $\kappa(y)$ is the completion of a finite type extension of $\kappa(x)$. It follows that the absolute Galois group of $\kappa(y)$ can be identified with a closed subgroup of the absolute Galois group of a finite type extension of $\kappa(y)$. We then conclude using Remark 2.4.9(2). Alternatively, one can deduce the result from [Hub96, Lemma 2.8.4]. □

Corollary 2.4.17. *Let $\tau \in \{\text{nis}, \text{ét}\}$. Let $f : T \to S$ be a morphism of rigid analytic spaces which is locally of finite type. If S is (Λ, τ)-admissible, then so is T.*

Proof. This follows immediately from Lemma 2.4.16. □

Lemma 2.4.18. *Let $\tau \in \{\text{nis}, \text{ét}\}$ and let S be a (Λ, τ)-admissible rigid analytic space.*

(1) (Case $\tau = \text{nis}$) Every Nisnevich sheaf of Λ-modules on $\text{Ét}^{\text{gr}}/S$ is a Nisnevich hypersheaf, i.e., we have

$$\text{Shv}^{\wedge}_{\text{nis}}(\text{Ét}^{\text{gr}}/S; \Lambda) = \text{Shv}_{\text{nis}}(\text{Ét}^{\text{gr}}/S; \Lambda).$$

The same statement is true with "$\text{Ét}^{\text{gr}}/S$" replaced with "$\text{Ét}/S$" or "RigSm/S".

(2) (Case $\tau = \text{ét}$) Assume that Λ is eventually coconnective. Then every étale sheaf of Λ-modules on $\text{Ét}/S$ is an étale hypersheaf, i.e., we have

$$\text{Shv}^{\wedge}_{\text{ét}}(\text{Ét}/S; \Lambda) = \text{Shv}_{\text{ét}}(\text{Ét}/S; \Lambda).$$

The same statement is true with "$\text{Ét}/S$" replaced with "RigSm/S".

Proof. If $\mathcal{F}$ is a τ-sheaf of Λ-modules on RigSm/S whose restriction to $\text{Ét}/X$ (or $\text{Ét}^{\text{gr}}/X$ if applicable) is a τ-hypersheaf for every quasi-compact and quasi-separated $X \in \text{RigSm}/S$, then $\mathcal{F}$ is a τ-hypersheaf. (Indeed, if this holds, the morphism $\mathcal{F} \to \mathcal{F}^{\wedge}$ induces equivalences $\mathcal{F}(X) \simeq \mathcal{F}^{\wedge}(X)$ for every $X \in \text{RigSm}^{\text{qcqs}}/S$, so it is itself an equivalence.) Therefore, using Corollary 2.4.17, it is enough to treat the cases of the small sites of S, with S quasi-compact and quasi-separated. The case of $(\text{Ét}/S, \text{ét})$ follows then from Corollary 2.4.6 and Lemma 2.4.11. The case of $(\text{Ét}^{\text{gr}}/S, \text{nis})$ needs a special treatment. For this, we remark that if $(X_\alpha)_\alpha$ is a cofiltered inverse system of quasi-compact and quasi-separated schemes of dimension $\leq d$ (with d independent of α), then the proof of [CM21, Theorem 3.17] can be adapted to show that the site $\lim_\alpha(\text{Ét}/X_\alpha, \text{nis})$ is locally of homotopy dimension $\leq d$, which implies that the associated topos is hypercomplete by [Lur09, Corollary 7.2.1.12]. Applying this to the inverse system $(\mathcal{S}_\sigma)_{\mathcal{S} \in \text{Mdl}'(S)}$ gives the result. □

Proposition 2.4.19. *Let $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$ and let S be a (Λ, τ)-admissible rigid analytic space. When τ is the étale topology, assume that Λ is eventually coconnective. Then, we have*

$$\mathbf{RigSH}_\tau^{(\mathrm{eff}),\wedge}(S;\Lambda) = \mathbf{RigSH}_\tau^{(\mathrm{eff})}(S;\Lambda).$$

Proof. This follows immediately from Lemma 2.4.18. □

Proposition 2.4.20. *Let $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$ and let S be a rigid analytic space.*

(1) The ∞-category $\mathrm{Shv}_\tau(\mathrm{RigSm}/S;\Lambda)$ is compactly generated if τ is the Nisnevich topology or if Λ is eventually coconnective. A set of compact generators is given, up to desuspension, by the $\Lambda_\tau(X)$ for $X \in \mathrm{RigSm}/S$ quasi-compact, quasi-separated and (Λ, τ)-good.

(2) The ∞-category $\mathrm{Shv}_\tau^\wedge(\mathrm{RigSm}/S;\Lambda)$ is compactly generated if S is (Λ, τ)-admissible. A set of compact generators is given, up to desuspension, by the $\Lambda_\tau(X)$ for $X \in \mathrm{RigSm}/S$ quasi-compact, quasi-separated and (Λ, τ)-good.

The above statements are also true with "RigSm/S*" replaced with "*$\mathrm{\acute{E}t}/S$*" and "*$\mathrm{\acute{E}t}^{\mathrm{gr}}/S$*" when applicable (i.e., when τ is the Nisnevich topology).*

Proof. In each situation, we only need to show that $\Lambda_\tau(X)$ is a compact object assuming that X is quasi-compact and quasi-separated. The problem being local on X, we may actually assume that $X = \mathrm{Spf}(A)^{\mathrm{rig}}$ for an adic ring A of principal ideal type. Saying that $\Lambda_\tau(X)$ is compact is equivalent to saying that the functor $\mathcal{F} \mapsto \mathcal{F}(X)$ commutes with filtered colimits. This can be checked by first restricting to the small site of X. Therefore, we may replace S by X and assume that $S = \mathrm{Spf}(A)^{\mathrm{rig}}$ for an adic ring A. Moreover, it is enough to show the versions of the above statements for $\mathrm{\acute{E}t}/S$, when $\tau = \mathrm{\acute{e}t}$, and for $\mathrm{\acute{E}t}^{\mathrm{gr}}/S$, when $\tau = \mathrm{nis}$. (Here we implicitly rely on Corollary 2.4.17.) We split the proof into two steps. (The reduction to $S = \mathrm{Spf}(A)^{\mathrm{rig}}$ is only needed in the second step.)

Step 1. Here we prove the second statement. We concentrate on the étale topology; the case of the Nisnevich topology is similar. Thus, we need to show that $\Lambda_{\mathrm{\acute{e}t}}(X)$ is a compact object of $\mathrm{Shv}_{\mathrm{\acute{e}t}}^\wedge(\mathrm{\acute{E}t}/S;\Lambda)$ when $X \in \mathrm{\acute{E}t}/S$ is quasi-compact, quasi-separated and $(\Lambda, \mathrm{\acute{e}t})$-good. This follows from combining Lemmas 2.4.5 and 2.4.11, and using Remark 2.4.15.

Step 2. Here we prove the first statement. Let $\pi \in A$ be a generator of an ideal of definition. We may write A as the colimit of a cofiltered inductive system $(A_\alpha)_\alpha$ where each A_α is an adic $\mathbb{Z}[[\pi]]$-algebra which is topologically of finite type. Set $S_\alpha = \mathrm{Spf}(A_\alpha)^{\mathrm{rig}}$. Since the inclusion functor $\mathrm{Pr}^{\mathrm{L}}_\omega \to \mathrm{Pr}^{\mathrm{L}}$ commutes with filtered colimits by [Lur09, Proposition 5.5.7.6], it is enough by Lemma 2.4.21 below to show the first statement for each S_α. Said differently, we may assume that S is of finite type over $\mathrm{Spf}(\mathbb{Z}[[\pi]])^{\mathrm{rig}}$, and hence (Λ, τ)-admissible. Since Λ is eventually coconnective when $\tau = \mathrm{\acute{e}t}$, Lemma 2.4.18 implies that $\mathrm{Shv}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/S;\Lambda)$ is equivalent to $\mathrm{Shv}_{\mathrm{\acute{e}t}}^\wedge(\mathrm{\acute{E}t}/S;\Lambda)$ and similarly for the small Nisnevich site. We may now use the first step to conclude. □

Lemma 2.4.21. *Let $(\mathcal{S}_\alpha)_\alpha$ be a cofiltered inverse system of quasi-compact and quasi-separated formal schemes with affine transition maps, and let $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$ be the limit of this system. We set $S_\alpha = \mathcal{S}_\alpha^{\mathrm{rig}}$ and $S = \mathcal{S}^{\mathrm{rig}}$. Then there is an equivalence*

$$\operatorname*{colim}_\alpha \mathrm{Shv}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/S_\alpha;\Lambda) \simeq \mathrm{Shv}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/S;\Lambda) \tag{2.16}$$

in Pr^{L}, where the colimit is also taken in Pr^{L}. A similar result is also true for the small Nisnevich sites.

Proof. We only discuss the étale case. We have an equivalence of ∞-categories

$$\operatorname*{colim}_{\alpha} \mathrm{PSh}(\mathrm{\acute{E}t}/S_\alpha;\Lambda) \simeq \mathrm{PSh}(\operatorname*{colim}_{\alpha} \mathrm{\acute{E}t}/S_\alpha;\Lambda) \tag{2.17}$$

where the first colimit is taken in Pr^{L}. (This is clear for $\mathcal{P}(-)$ instead of $\mathrm{PSh}(-;\Lambda)$ by the universal property of ∞-categories of presheaves, and we deduce the formula for $\mathrm{PSh}(-;\Lambda)$ using the equivalence $\mathrm{PSh}(-;\Lambda) \simeq \mathcal{P}(-)\otimes\mathrm{Mod}_\Lambda$.) Using Remark 2.3.2, the fact that every cover in $\lim_\alpha(\mathrm{\acute{E}t}/S_\alpha, \mathrm{\acute{e}t})$ is the image of a cover in $(\mathrm{\acute{E}t}/S_\alpha, \mathrm{\acute{e}t})$ for some α, and the universal property of localisation given by [Lur09, Proposition 5.5.4.20], we deduce from (2.17) an equivalence of ∞-categories

$$\operatorname*{colim}_{\alpha} \mathrm{Shv}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/S_\alpha;\Lambda) \simeq \mathrm{Shv}_{\mathrm{\acute{e}t}}(\operatorname*{colim}_{\alpha} \mathrm{\acute{E}t}/S_\alpha;\Lambda) \tag{2.18}$$

where the first colimit is taken in Pr^{L}. On the other hand, by Corollary 1.4.20, we have an equivalence of sites $(\mathrm{\acute{E}t}/S, \mathrm{\acute{e}t}) \simeq \lim_\alpha(\mathrm{\acute{E}t}/S_\alpha, \mathrm{\acute{e}t})$. Applying Lemma 2.1.4 we get an equivalence of ∞-categories

$$\mathrm{Shv}_{\mathrm{\acute{e}t}}(\operatorname*{colim}_{\alpha} \mathrm{\acute{E}t}/S_\alpha;\Lambda) \simeq \mathrm{Shv}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/S;\Lambda). \tag{2.19}$$

We conclude by combining (2.18) and (2.19). □

Proposition 2.4.22. *Let $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$ and let S be a rigid analytic space.*

(1) The ∞-category $\mathbf{RigSH}^{(\mathrm{eff})}_\tau(S;\Lambda)$ is compactly generated if τ is the Nisnevich topology or if Λ is eventually coconnective. A set of compact generators is given, up to desuspension and negative Tate twists when applicable, by the $\mathrm{M}^{(\mathrm{eff})}(X)$ for $X \in \mathrm{RigSm}/S$ quasi-compact, quasi-separated and (Λ,τ)-good.

(2) The ∞-category $\mathbf{RigSH}^{(\mathrm{eff}),\wedge}_\tau(S;\Lambda)$ is compactly generated if S is (Λ,τ)-admissible. A set of compact generators is given, up to desuspension and negative Tate twists when applicable, by the $\mathrm{M}^{(\mathrm{eff})}(X)$ for $X \in \mathrm{RigSm}/S$ quasi-compact, quasi-separated and (Λ,τ)-good.

Moreover, under the stated assumptions, the monoidal ∞-category $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(S;\Lambda)^\otimes$ belongs to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_\omega)$ and, if $f : T \to S$ is a quasi-compact and quasi-separated morphism of rigid analytic spaces with T assumed (Λ,τ)-admissible in the hypercomplete case, the functor $f^* : \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(S;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(T;\Lambda)$ is compact-preserving, i.e., belongs to $\mathrm{Pr}^{\mathrm{L}}_\omega$.

Proof. Using Lemma 2.1.20, we are left to show that the objects $\mathrm{M}^{(\mathrm{eff})}(X)$ are compact, for X as in the statement. In the effective case, this would follow from [Lur09, Corollary 5.5.7.3] and Proposition 2.4.20 if we knew that $\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_\tau(S;\Lambda)$ is stable under filtered colimits in $\mathrm{Shv}^{(\wedge)}_\tau(\mathrm{RigSm}/S;\Lambda)$. But this is indeed the case by Proposition 2.4.20 and Remark 2.1.12. The T-stable case follows from the effective case using Remark 2.1.17 and [Lur09, Proposition 5.5.7.6]. □

Remark 2.4.23. A similar statement with a similar proof is also true for the ∞-category $\mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(S;\Lambda)$ of algebraic motives over a scheme S, generalising [Ayo14a, Proposition 3.19].

2.5. **Continuity, I. A preliminary result.**

The goal of this subsection and the next one is to prove the continuity property for the functor $\mathbf{RigSH}^{(\mathrm{eff})}_\tau(-;\Lambda)$ which, roughly speaking, asserts that this functor transforms limits of certain cofiltered inverse systems of rigid analytic spaces into filtered colimits of presentable ∞-categories. The precise statement is given in Theorem 2.5.1 below. (Note that we do not claim that S is the limit of $(S_\alpha)_\alpha$ in the categorical sense.) Later, in Subsection 2.8, we will generalise Theorem 2.5.1 to include more general inverse systems and a weaker notion of limits; see Theorem 2.8.15 below.

We let $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$ be a topology on rigid analytic spaces.

Theorem 2.5.1. *Let $(\mathcal{S}_\alpha)_\alpha$ be a cofiltered inverse system of quasi-compact and quasi-separated formal schemes with affine transition maps, and let $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$ be the limit of this system. We set $S_\alpha = \mathcal{S}_\alpha^{\mathrm{rig}}$ and $S = \mathcal{S}^{\mathrm{rig}}$. We assume one of the following two alternatives.*

(1) We work in the non-hypercomplete case.

(2) We work in the hypercomplete case, and S and the S_α's are (Λ, τ)-admissible. When τ is the étale topology, we assume furthermore that Λ is eventually coconnective or that the numbers $\mathrm{pvcd}_\Lambda(S_\alpha)$ are bounded independently of α. (See Definition 2.4.10.)

Then the obvious functor

$$\operatorname*{colim}_\alpha \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(S_\alpha;\Lambda) \to \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(S;\Lambda), \tag{2.20}$$

where the colimit is taken in Pr^{L}, is an equivalence.

Remark 2.5.2. Keep the notations and hypotheses as in Theorem 2.5.1. Using [Lur17, Corollary 3.2.3.2], we can upgrade (2.20) into an equivalence

$$\operatorname*{colim}_\alpha \mathbf{RigSH}_\tau^{(\mathrm{eff})}(S_\alpha;\Lambda)^\otimes \simeq \mathbf{RigSH}_\tau^{(\mathrm{eff})}(S;\Lambda)^\otimes \tag{2.21}$$

in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, where the colimit is also taken in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

Remark 2.5.3. The two alternatives considered in the statement of Theorem 2.5.1 have a nontrivial intersection given as follows.

(2′) We work in the hypercomplete case and we assume that the S_α's and S are (Λ,τ)-admissible. When τ is the étale topology, we assume furthermore that Λ is eventually coconnective.

Indeed, by Proposition 2.4.19, we have in this case $\mathbf{RigSH}_\tau^{(\mathrm{eff}),\wedge}(S_\alpha;\Lambda) = \mathbf{RigSH}_\tau^{(\mathrm{eff})}(S_\alpha;\Lambda)$, and similarly for S in place of the S_α's. Said differently, the alternative (1) covers the alternative (2) except when Λ is not eventually coconnective, in which case we need a strong assumption on the punctual virtual Λ-cohomological dimensions of the S_α's.

Remark 2.5.4. Theorem 2.5.1 in the non-hypercomplete case is a motivic version of Lemma 2.4.21. The conclusion of this lemma holds also in the hypercomplete case under the alternative (2) as shown in corollary 2.5.10 below.

The proof of Theorem 2.5.1 spans the entire subsection and the next one. In fact, we will obtain this theorem as a combination of two other results, namely Propositions 2.5.8 and 2.5.12, which are both interesting in their own right. The proof of Proposition 2.5.12 will be given in Subsection 2.6.

Notation 2.5.5. Let $(S_\alpha)_\alpha$ be a cofiltered inverse system of quasi-compact and quasi-separated rigid analytic spaces. We define the ∞-category $\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}((S_\alpha)_\alpha;\Lambda)$, of rigid analytic motives over the rigid analytic pro-space $(S_\alpha)_\alpha$, in the usual way from the limit site $\lim_\alpha(\mathrm{RigSm}/S_\alpha,\tau)$, that is, from the ordinary category

$$\mathrm{RigSm}/(S_\alpha)_\alpha = \operatorname*{colim}_\alpha \mathrm{RigSm}/S_\alpha$$

endowed with the limit topology τ. More precisely, one repeats Definitions 2.1.11 and 2.1.15 with "RigSm/S" replaced with "$\mathrm{RigSm}/(S_\alpha)_\alpha$". We denote also by

$$\mathrm{M}^{(\mathrm{eff})} : \mathrm{RigSm}/(S_\alpha)_\alpha \to \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}((S_\alpha)_\alpha;\Lambda)$$

the obvious functor.

Remark 2.5.6. Let Pro(RigSpc) be the category of rigid analytic pro-spaces and consider the over-category $\mathrm{Pro}(\mathrm{RigSpc})/(S_\alpha)_\alpha$ of $(S_\alpha)_\alpha$-objects. There is a fully faithful embedding

$$\mathrm{RigSm}/(S_\alpha)_\alpha \to \mathrm{Pro}(\mathrm{RigSpc})/(S_\alpha)_\alpha$$

and we will identify $\mathrm{RigSm}/(S_\alpha)_\alpha$ with its essential image by this functor. Thus, we may think of an object of $\mathrm{RigSm}/(S_\alpha)_\alpha$ as a pro-object $(X_\alpha)_{\alpha\leq\alpha_0}$, where X_{α_0} is a smooth rigid analytic S_{α_0}-space and, for $\alpha\leq\alpha_0$, $X_\alpha\simeq X_{\alpha_0}\times_{S_{\alpha_0}}S_\alpha$. If $(S_\alpha)_\alpha$ is as in Theorem 2.5.1, given such a pro-object $(X_\alpha)_{\alpha\leq\alpha_0}$, we denote by X the rigid analytic S-space defined as follows. Assume first that there is a formal model $\mathcal{X}_{\alpha_0}$ of X_{α_0} over $\mathcal{S}_{\alpha_0}$. Let $(\mathcal{X}_\alpha)_{\alpha\leq\alpha_0}$ be the formal pro-scheme given by $\mathcal{X}_\alpha=\mathcal{X}_{\alpha_0}\times_{\mathcal{S}_{\alpha_0}}\mathcal{S}_\alpha$. We set $X=\mathcal{X}^{\mathrm{rig}}$ where $\mathcal{X}=\lim_{\alpha\leq\alpha_0}\mathcal{X}_\alpha$. This is independent of the choice of $\mathcal{X}_{\alpha_0}$ and the formation of X is compatible with gluing rigid analytic S_{α_0}-spaces along open immersions. Thus, the construction of X can be extended to the general case where we do not assume the existence of a formal model for X_{α_0}.

Lemma 2.5.7. *Let $(S_\alpha)_\alpha$ and S be as in Theorem 2.5.1 and assume that S is (Λ,τ)-admissible. Then, the ∞-category $\mathrm{Shv}^\wedge_\tau(\mathrm{RigSm}/(S_\alpha)_\alpha;\Lambda)$ is compactly generated, up to desuspension, by the $\Lambda_\tau((X_\alpha)_{\alpha\leq\alpha_0})$ with X_{α_0} quasi-compact, quasi-separated and (Λ,τ)-good.*

Proof. This can be shown by adapting the proof of Proposition 2.4.20(2). The key point is to show that $\lim_{\alpha\leq\alpha_0}(\acute{\mathrm{E}}\mathrm{t}/X_\alpha,\tau)$ has finite local and global Λ-cohomological dimensions. By Corollary 1.4.20, this limit site is equivalent to $(\acute{\mathrm{E}}\mathrm{t}/X,\tau)$. Thus, we may use Lemma 2.4.11 to conclude. □

Proposition 2.5.8. *Let $(S_\alpha)_\alpha$ and S be as in Theorem 2.5.1 and assume one of the alternatives (1) or (2) of that theorem. Then the obvious functor*

$$\operatorname*{colim}_\alpha \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(S_\alpha;\Lambda)\to\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau((S_\alpha)_\alpha;\Lambda),\tag{2.22}$$

where the colimit is taken in Pr^{L}, is an equivalence.

Proof. We first work under the alternative (1), i.e., in the non-hypercomplete case. Here, the result is quite straightforward. Arguing as in the proof of Lemma 2.4.21, we get an equivalence of ∞-categories

$$\operatorname*{colim}_\alpha \mathrm{Shv}_\tau(\mathrm{RigSm}/S_\alpha;\Lambda)\simeq\mathrm{Shv}_\tau(\mathrm{RigSm}/(S_\alpha)_\alpha;\Lambda),\tag{2.23}$$

where the colimit is taken in Pr^{L}. Using the universal property of localisation given by [Lur09, Proposition 5.5.4.20], we deduce from (2.23) that (2.22) is an equivalence in the effective case. We then deduce the T-stable case using Remark 2.1.17 and commutation of colimits with colimits.

Next, we work under the alternative (2). Arguing as before, we see that it is enough to prove the hypercomplete analogue of the equivalence (2.23), i.e., it is enough to show that

$$\operatorname*{colim}_\alpha \mathrm{Shv}^\wedge_\tau(\mathrm{RigSm}/S_\alpha;\Lambda)\to\mathrm{Shv}^\wedge_\tau(\mathrm{RigSm}/(S_\alpha)_\alpha;\Lambda),\tag{2.24}$$

is an equivalence. It follows from Lemma 2.5.7 that the functor (2.24) belongs to $\mathrm{Pr}^{\mathrm{L}}_\omega$ and that it takes a set of compact generators to a set of compact generators. Thus, it remains to show that this functor is fully faithful on compact objects. Explicitly, we need to show the following assertion. Given two compact objects $\mathcal{M}$ and $\mathcal{N}$ in $\mathrm{Shv}^\wedge_\tau(\mathrm{RigSm}/S_{\alpha_0};\Lambda)$, for some index α_0, the natural map

$$\operatorname*{colim}_{\alpha\leq\alpha_0}\mathrm{Map}(f^*_{\alpha\leq\alpha_0}\mathcal{M},f^*_{\alpha\leq\alpha_0}\mathcal{N})\to\mathrm{Map}(f^*_{\alpha_0}\mathcal{M},f^*_{\alpha_0}\mathcal{N})\tag{2.25}$$

is an equivalence. Here $f_{\alpha\leq\alpha_0}:S_\alpha\to S_{\alpha_0}$ and $f_{\alpha_0}:(S_\alpha)_\alpha\to S_{\alpha_0}$ are the obvious morphisms.

Let I be the indexing category of the inverse system $(S_\alpha)_\alpha$. We denote by $\widetilde{S} : I \to \mathrm{RigSpc}$ the diagram of rigid analytic spaces defining the pro-object $(S_\alpha)_\alpha$, i.e., sending α to S_α. We define the site $(\mathrm{RigSm}/\widetilde{S}, \tau)$ in the usual way, i.e., by adapting the beginning of [Ayo07b, §4.5.1]. We have a premorphism of sites (in the sense of [Ayo07b, Définition 4.4.46])

$$\rho : (\mathrm{RigSm}/(S_\alpha)_\alpha, \tau) \to (\mathrm{RigSm}/\widetilde{S}, \tau) \tag{2.26}$$

induced by the functor $\mathrm{RigSm}/\widetilde{S} \to \mathrm{RigSm}/(S_\alpha)_\alpha$ given by $(\beta, X) \mapsto (X \times_{S_\beta} S_\alpha)_{\alpha \leq \beta}$. The inverse image functor ρ^* is given, informally, by $\rho^*(\mathcal{K}) = \mathrm{colim}_\beta\, ((S_\alpha)_\alpha \to S_\beta)^* \mathcal{K}_\beta$, where $\mathcal{K}_\beta$ is the restriction of $\mathcal{K}$ to RigSm/S_β. The inclusion $\mathrm{RigSm}/S_{\alpha_0} \subset \mathrm{RigSm}/\widetilde{S}$ induces a functor

$$\mathrm{Shv}_\tau^{(\wedge)}(\mathrm{RigSm}/S_{\alpha_0}; \Lambda) \to \mathrm{Shv}_\tau^{(\wedge)}(\mathrm{RigSm}/\widetilde{S}; \Lambda). \tag{2.27}$$

We may assume that $\mathcal{M} = \Lambda_\tau(X_{\alpha_0})$ with $X_{\alpha_0} \in \mathrm{RigSm}/S_{\alpha_0}$ quasi-compact, quasi-separated and (Λ, τ)-good. We let $\mathcal{R}$ be the image of $\mathcal{N}$ by the functor (2.27). Arguing as in the proof of [Ayo14a, Proposition 3.20], the assertion that (2.25) is an equivalence would follow if we show that the functor

$$\rho^* : \mathrm{PSh}(\mathrm{RigSm}^{\mathrm{qcqs}}/\widetilde{S}; \Lambda) \to \mathrm{PSh}(\mathrm{RigSm}^{\mathrm{qcqs}}/(S_\alpha)_\alpha; \Lambda)$$

takes $\mathcal{R}$ to a presheaf $\rho^*(\mathcal{R})$ whose restriction to $\mathrm{Ét}^{\mathrm{qcqs}}/(X_{\alpha \leq \alpha_0})_\alpha$ is a τ-hypersheaf. This follows from Lemma 2.5.9 below. (Compare with [Ayo14a, Lemme 3.21].) □

Lemma 2.5.9. *Let $\widetilde{\mathcal{X}} : I \to \mathrm{FSch}$ be a diagram of quasi-compact and quasi-separated formal schemes, with I a cofiltered category, and with affine transition morphisms. Let $(\mathcal{X}_\alpha)_\alpha$ be the associated pro-object and $\mathcal{X}$ its limit. Set $\widetilde{X} = \widetilde{\mathcal{X}}^{\mathrm{rig}}$, $X_\alpha = \mathcal{X}_\alpha^{\mathrm{rig}}$ and $X = \mathcal{X}^{\mathrm{rig}}$. Assume that the alternative (2) in Theorem 2.5.1 is satisfied with "$(X_\alpha)_\alpha$" and "X" instead of "$(S_\alpha)_\alpha$" and "S". Assume also that the X_α's are (Λ, τ)-good. Then the functor*

$$\rho^* : \mathrm{PSh}(\mathrm{Ét}^{\mathrm{qcqs}}/\widetilde{X}; \Lambda) \to \mathrm{PSh}(\mathrm{Ét}^{\mathrm{qcqs}}/(X_\alpha)_\alpha; \Lambda)$$

takes τ-hypersheaves to τ-hypersheaves.

Proof. We split the proof into three steps. Below $\mathcal{K}$ is a τ-hypersheaf of Λ-modules on $\mathrm{Ét}^{\mathrm{qcqs}}/\widetilde{X}$.

Step 1. We first deal with the case where Λ is eventually coconnective. The proof in this case is similar to that of [Ayo14a, Lemme 3.21]. First, one considers the case where $\mathcal{K}$ is discrete, i.e., is the Eilenberg–Mac Lane spectrum associated to an ordinary sheaf of $\pi_0\Lambda$-modules. This case follows from [SGAIV2, Exposé VII, Théorème 5.7]. By induction, one can then treat the case where $\mathcal{K}$ is bounded (i.e., where the discrete sheaves $\pi_i(\mathcal{K})$ vanish for $|i|$ big enough). Finally, we deduce the general case from the bounded case as follows. A general $\mathcal{K}$ can be written as a colimit of objects of the form $\Lambda_\tau(\alpha_0, U)$, for $U \in \mathrm{Ét}^{\mathrm{qcqs}}/X_{\alpha_0}$. Since Λ is eventually coconnective, $\Lambda_\tau(\alpha_0, U)$ is bounded. The result for $\mathcal{K}$ follows then from the bounded case and Lemma 2.4.5(3) which implies that colimits in $\mathrm{PSh}(\mathrm{Ét}^{\mathrm{qcqs}}/(X_\alpha)_\alpha; \Lambda)$ preserve τ-hypersheaves. (Here, we use that the site $(\mathrm{Ét}^{\mathrm{qcqs}}/(X_\alpha)_\alpha, \tau)$ has finite local Λ-cohomological dimension as explained in the proof of Lemma 2.5.7.)

Step 2. We next consider the case of the Nisnevich topology. The site $(\mathrm{Ét}^{\mathrm{qcqs}}/(X_\alpha)_\alpha, \mathrm{nis})$ is equivalent to $(\mathrm{Ét}^{\mathrm{qcqs}}/X, \mathrm{nis})$. Thus, by Lemma 2.4.18(1), every Nisnevich sheaf on $\mathrm{Ét}^{\mathrm{qcqs}}/(X_\alpha)_\alpha$ is a Nisnevich hypersheaf. Thus, to check that $\rho^*\mathcal{K}$ is a Nisnevich hypersheaf, it is enough to prove that $\rho^*\mathcal{K}$ has the Mayer–Vietoris property for the image in $\mathrm{Ét}^{\mathrm{qcqs}}/(X_\alpha)_\alpha$ of a Nisnevich square in $\mathrm{Ét}^{\mathrm{qcqs}}/X_\alpha$, for some α. This is easily checked using exactness of filtered colimits on Mod_Λ and the formula $\rho^*\mathcal{K} = \mathrm{colim}_\beta\, ((X_\alpha)_\alpha \to X_\beta)^*\mathcal{K}_\beta$. The details are left to the reader.

Step 3. We now treat the case of the étale topology assuming that the numbers $\mathrm{pvcd}_\Lambda(X_\alpha)$ are bounded independently of α. In fact, since the X_α's are $(\Lambda, \text{ét})$-good, there is a common bound e for the Λ-cohomological dimensions of the residue fields of all the X_α's.

Denote by π the morphism of sites of the form $(\text{Ét}/(-), \text{ét}) \to (\text{Ét}/(-), \text{nis})$ and by π_* the induced functor on ∞-categories of hypersheaves of Λ-modules. Also, denote by

$$\rho^*_{\text{nis}} : \mathrm{Shv}^\wedge_{\text{nis}}(\text{Ét}^{\text{qcqs}}/\widetilde{X}; \Lambda) \to \mathrm{Shv}^\wedge_{\text{nis}}(\text{Ét}^{\text{qcqs}}/(X_\alpha)_\alpha; \Lambda)$$

the inverse image functor on Nisnevich hypersheaves. By the second step, ρ^*_{nis} coincides with ρ^* on Nisnevich hypersheaves of Λ-modules.

By Lemma 2.4.5(3), the property that $\rho^*\mathcal{K}$ is an étale hypersheaf is stable by colimits in $\mathcal{K}$. Since $\mathcal{K} \simeq \mathrm{colim}_n \tau_{\geq -n}\mathcal{K}$, we may assume that $\mathcal{K}$ is bounded from above, and even connective. By Lemma 2.4.5(1), we have an equivalence $\mathcal{K} \simeq \lim_n \tau_{\leq n}\mathcal{K}$ yielding an equivalence $\pi_*\mathcal{K} \simeq \lim_n \pi_*\tau_{\leq n}\mathcal{K}$. The proof of Lemma 2.4.11 shows that $\pi_*\tau_{\leq n+1}\mathcal{K} \to \pi_*\tau_{\leq n}\mathcal{K}$ induces an isomorphism on homotopy Nisnevich sheaves in degrees $\leq n - e$, and the same is true for $\rho^*_{\text{nis}}\pi_*\tau_{\leq n+1}\mathcal{K} \to \rho^*_{\text{nis}}\pi_*\tau_{\leq n}\mathcal{K}$. Since X and X_α's have finite Krull dimensions, we deduce that the morphisms

$$\lim_n \pi_*\tau_{\leq n}\mathcal{K} \to \pi_*\tau_{\leq m}\mathcal{K} \quad \text{and} \qquad \lim_n \rho_{\text{nis}}\pi_*\tau_{\leq n}\mathcal{K} \to \rho_{\text{nis}}\pi_*\tau_{\leq m}\mathcal{K}$$

induce isomorphisms on homotopy Nisnevich sheaves in degrees $\leq m - e$, for any integer m. It follows that the natural map

$$\rho^*_{\text{nis}}\pi_*\mathcal{K} = \rho^*_{\text{nis}} \lim_n \pi_*\tau_{\leq n}\mathcal{K} \to \lim_n \rho^*_{\text{nis}}\pi_*\tau_{\leq n}\mathcal{K} \tag{2.28}$$

induces isomorphisms on homotopy Nisnevich sheaves. Since both sides are Nisnevich hypersheaves, we deduce that (2.28) is an equivalence. Thus, we are left to show that $\rho^*_{\text{nis}}\pi_*\tau_{\leq n}\mathcal{K}$ is an étale hypersheaf for every n. This follows from the first step since $\tau_{\leq n}\mathcal{K}$ is naturally an étale hypersheaf of $\tau_{\leq n}\Lambda$-modules. □

Lemma 2.5.9 has the following consequence which we state for completeness.

Corollary 2.5.10. *Let $(S_\alpha)_\alpha$ and S be as in Theorem 2.5.1 and assume one of the alternatives (1) or (2) of that theorem. Then the obvious functor*

$$\mathrm{colim}_\alpha \mathrm{Shv}^{(\wedge)}_\tau(\text{Ét}/S_\alpha; \Lambda) \to \mathrm{Shv}^{(\wedge)}_\tau(\text{Ét}/S; \Lambda), \tag{2.29}$$

where the colimit is taken in Pr^{L}*, is an equivalence.*

Proof. The non-hypercomplete case is already stated in Lemma 2.4.21. The hypercomplete case follows from Lemma 2.5.9 by arguing as in the proof of [Ayo14a, Proposition 3.20]. □

The proof of Proposition 2.5.8, adapted to the algebraic setting gives the following generalisation of [Ayo14a, Proposition 3.20] and [Hoy14, Proposition C.12(4)].

Proposition 2.5.11. *Let $(S_\alpha)_\alpha$ be a cofiltered inverse system of quasi-compact and quasi-separated schemes with affine transition maps, and let $S = \lim_\alpha S_\alpha$ be the limit of this system. We assume one of the following two alternatives.*

(1) We work in the non-hypercomplete case.
(2) We work in the hypercomplete case, and S and the S_α's are (Λ, τ)-admissible. When τ is the étale topology, we assume furthermore that Λ is eventually coconnective or that the numbers $\mathrm{pvcd}_\Lambda(S_\alpha)$ are bounded independently of α.

Then the obvious functor

$$\operatorname*{colim}_{\alpha} \mathbf{SH}_{\tau}^{(\mathrm{eff},\wedge)}(S_\alpha;\Lambda) \to \mathbf{SH}_{\tau}^{(\mathrm{eff},\wedge)}(S;\Lambda),$$

where the colimit is taken in Pr^{L}*, is an equivalence.*

Proof. Indeed, in the algebraic setting, $\mathrm{Sm}^{\mathrm{qcqs}}/S$ is equivalent to $\mathrm{colim}_\alpha \mathrm{Sm}^{\mathrm{qcqs}}/S_\alpha$. □

Theorem 2.5.1 follows by combining Proposition 2.5.8 and the next result.

Proposition 2.5.12. *Let* $(\mathcal{S}_\alpha)_\alpha$ *be a cofiltered inverse system of quasi-compact and quasi-separated formal schemes with affine transition maps, and let* $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$ *be the limit of this system. We set* $S_\alpha = \mathcal{S}_\alpha^{\mathrm{rig}}$ *and* $S = \mathcal{S}^{\mathrm{rig}}$*. Then the obvious functor*

$$\mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}((S_\alpha)_\alpha;\Lambda) \to \mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}(S;\Lambda) \tag{2.30}$$

is an equivalence.

The proof of Proposition 2.5.12 is given in the next subsection.

2.6. **Continuity, II. Approximation up to homotopy.**

The goal of this section is to prove Proposition 2.5.12. The proof is similar to that of [Vez19, Proposition 4.5], but some new ingredients are necessary to deal with the generality considered in this paper. We start with some reductions.

Lemma 2.6.1. *It is enough to prove Proposition 2.5.12 in the effective, non-hypercomplete case and for τ the Nisnevich topology. Said differently, it is enough to show that the obvious functor*

$$\mathbf{RigSH}_{\mathrm{nis}}^{\mathrm{eff}}((S_\alpha)_\alpha;\Lambda) \to \mathbf{RigSH}_{\mathrm{nis}}^{\mathrm{eff}}(S;\Lambda) \tag{2.31}$$

is an equivalence.

Proof. The T-stable case follows from the effective case using Remark 2.1.17 and commutation of colimits with colimits. Assume that (2.31) is an equivalence, and let's show that

$$\mathbf{RigSH}_{\tau}^{\mathrm{eff},(\wedge)}((S_\alpha)_\alpha;\Lambda) \to \mathbf{RigSH}_{\tau}^{\mathrm{eff},(\wedge)}(S;\Lambda) \tag{2.32}$$

is also an equivalence for $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$. There are three cases to consider:

1. the Nisnevich topology in the hypercomplete case;
2. the étale topology in the non-hypercomplete case;
3. the étale topology in the hypercomplete case.

In each case, we will prove that the source and the target of (2.32) are obtained from the source and the target of (2.31) by localisation with respect to a set of morphisms and its image by the equivalence (2.31), which suffices to conclude. These sets consist respectively, up to desuspension, of maps of the form $\mathrm{colim}_{[n]\in\mathbf{\Delta}} \mathrm{M}^{\mathrm{eff}}((U_{n,\alpha})_{\alpha\leq\alpha_n}) \to \mathrm{M}^{\mathrm{eff}}((U_{-1,\alpha})_{\alpha\leq\alpha_{-1}})$ where $(U_{\bullet,\alpha})_{\alpha\leq\alpha_\bullet}$ is:

1. a hypercover in the limit site $\lim_{\alpha\leq\alpha_{-1}}(\mathrm{\acute{E}t}^{\mathrm{gr}}/U_{-1,\alpha}, \mathrm{nis})$;
2. a Čech nerve associated to a cover in the limit site $\lim_{\alpha\leq\alpha_{-1}}(\mathrm{\acute{E}t}/U_{-1,\alpha}, \mathrm{\acute{e}t})$;
3. a hypercover in the limit site $\lim_{\alpha\leq\alpha_{-1}}(\mathrm{\acute{E}t}/U_{-1,\alpha}, \mathrm{\acute{e}t})$.

Localising the source of (2.31) by one of these sets yield the source of (2.32) by construction. We now show that localising the target of (2.31) by the image of one of these sets yield the target of (2.32). This relies on the following two facts.

(a) Given an object $(Y_\alpha)_{\alpha\leq\beta}$ in $\mathrm{RigSm}/(S_\alpha)_\alpha$ and defining Y as in Remark 2.5.6, we have an equivalence of sites
$$(\acute{\mathrm{E}}\mathrm{t}/Y, \tau) \simeq \lim_{\alpha\leq\beta}(\acute{\mathrm{E}}\mathrm{t}/Y_\alpha, \tau)$$
and similarly for "$\acute{\mathrm{E}}\mathrm{t}^{\mathrm{gr}}$" instead of "Ét" when applicable.

(b) Every $X \in \mathrm{RigSm}/S$ is locally for the analytic topology in the essential image RigSm'/S of the functor $\mathrm{RigSm}/(S_\alpha)_\alpha \to \mathrm{RigSm}/S$. In particular, we have an equivalence of sites
$$(\mathrm{RigSm}/S, \tau) \simeq (\mathrm{RigSm}'/S, \tau)$$
which is subject to Lemma 2.1.4. Thus, the ∞-category $\mathbf{RigSH}^{\mathrm{eff},(\wedge)}_\tau(S;\Lambda)$ can be defined using the site $(\mathrm{RigSm}'/S, \tau)$.

Property (a) follows from Corollary 1.4.20 and Remark 1.4.21. To prove (b), we may assume that the inverse system $(\mathcal{S}_\alpha)_\alpha$ is affine, induced by an inductive system of adic rings $(A_\alpha)_\alpha$ with colimit A, and that $X = \mathrm{Spf}(B)^{\mathrm{rig}}$ with B a rig-étale adic $A\langle t\rangle$-algebra with $t = (t_1, \dots, t_n)$ a system of coordinates. Then the result follows from Corollary 1.3.10. □

Lemma 2.6.2. *It is enough to prove that* (2.31) *is an equivalence assuming that the formal schemes* $\mathcal{S}_\alpha$ *are affine of principal ideal type.*

Proof. Without loss of generality, we may assume that there is a final object o in the indexing category of the inverse system $(\mathcal{S}_\alpha)_\alpha$. Replacing $\mathcal{S}_o$ by the blowup of an ideal of definition, and the $\mathcal{S}_\alpha$'s by their strict transforms, we may assume that the $\mathcal{S}_\alpha$'s are locally of principal ideal type for every α. The presheaf $\mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(-;\Lambda)$ has descent for the analytic topology by Theorem 2.3.4. Combining this with Proposition 2.5.8 and [Lur17, Proposition 4.7.4.19], we see that the problem is local on $\mathcal{S}_o$, which finishes the proof. (Note that the condition for applying [Lur17, Proposition 4.7.4.19] is indeed satisfied by the base change theorem for open immersions, a special case of the base change theorem for smooth morphisms; see Proposition 2.2.1.) □

We now introduce a notation that we keep using until the end of the proof of Proposition 2.5.12.

Notation 2.6.3. Let $(\mathcal{S}_\alpha)_\alpha$ be a cofiltered inverse system of affine formal schemes, and let $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$. Denote by $\mathcal{S}'_\alpha$ the smallest closed formal subscheme of $\mathcal{S}_\alpha$ containing the image of $\mathcal{S} \to \mathcal{S}_\alpha$. (Said differently, $\mathcal{O}(\mathcal{S}'_\alpha)$ is the quotient of $\mathcal{O}(\mathcal{S}_\alpha)$ by the kernel of $\mathcal{O}(\mathcal{S}_\alpha) \to \mathcal{O}(\mathcal{S})$ which is a closed ideal.) Then, we have a cofiltered inverse system of affine formal schemes $(\mathcal{S}'_\alpha)_\alpha$ and a morphism $(\mathcal{S}'_\alpha)_\alpha \to (\mathcal{S}_\alpha)_\alpha$ of inverse systems given by closed immersions and inducing an isomorphism $\lim_\alpha \mathcal{S}'_\alpha \simeq \lim_\alpha \mathcal{S}_\alpha$ on the limit.

Although, in general, the pro-objects $(\mathcal{S}'_\alpha)_\alpha$ and $(\mathcal{S}_\alpha)_\alpha$ are not isomorphic, we have the following.

Lemma 2.6.4. *Let* $(\mathcal{S}_\alpha)_\alpha$ *be a cofiltered inverse system of affine formal schemes. Let* S_α *and* S'_α *be the rigid analytic spaces associated to* $\mathcal{S}_\alpha$ *and* $\mathcal{S}'_\alpha$. *Then, the obvious functor*
$$\mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}((S_\alpha)_\alpha;\Lambda) \to \mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}((S'_\alpha)_\alpha;\Lambda) \tag{2.33}$$
is an equivalence.

Proof. It will be more convenient to use Proposition 2.5.8 and prove that
$$\operatorname*{colim}_\alpha \mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(S_\alpha;\Lambda) \to \operatorname*{colim}_\alpha \mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(S'_\alpha;\Lambda) \tag{2.34}$$
is an equivalence in Pr^{L}. We set $U_\alpha = S_\alpha \smallsetminus S'_\alpha$ and denote by $j_\alpha : U_\alpha \to S_\alpha$ the obvious inclusion. For each α, $\mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(S_\alpha;\Lambda) \to \mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(S'_\alpha;\Lambda)$ is a localisation functor with respect to the class

of maps of the form $0 \to j_{\alpha,\sharp}M$ where $M \in \mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(U_\alpha;\Lambda)$. This follows from the localisation theorem for rigid analytic motives; see Proposition 2.2.3. Moreover, by Lemma 2.1.20, we may assume that M is, up to desuspension, of the form $\mathrm{M}^{\mathrm{eff}}(X)$ with $X \in \mathrm{RigSm}/U_\alpha$ quasi-compact and quasi-separated.

It follows from the universal property of localisation (given by [Lur09, Proposition 5.5.4.20]) that (2.34) is also a localisation functor with respect to the images of the maps $0 \to j_{\alpha,\sharp}M$, with M as above. Thus, it is enough to show that, for $X \in \mathrm{RigSm}/U_\alpha$ quasi-compact and quasi-separated, there exists $\beta \leq \alpha$ such that $X \times_{S_\alpha} S_\beta = \emptyset$. This follows from the fact that X lies over a quasi-compact open subset $V \subset U_\alpha$ and that, for $\beta \leq \alpha$ small enough, we have $S_\beta \times_{S_\alpha} V = \emptyset$ by, for example, [FK18, Chapter 0, Proposition 2.2.10]. □

Notation 2.6.5. Let $\mathrm{FSch}_{\mathrm{af,pr}}$ be the category of affine formal schemes of principal ideal type, and $\mathrm{Pro}(\mathrm{FSch}_{\mathrm{af,pr}})$ the category of pro-objects in $\mathrm{FSch}_{\mathrm{af,pr}}$. We have an idempotent endofunctor of $\mathrm{Pro}(\mathrm{FSch}_{\mathrm{af,pr}})$ given by $(\mathcal{S}_\alpha)_\alpha \mapsto (\mathcal{S}'_\alpha)_\alpha$. We define a new category $\mathrm{Pro}'(\mathrm{FSch}_{\mathrm{af,pr}})$, having the same objects as $\mathrm{Pro}(\mathrm{FSch}_{\mathrm{af,pr}})$ and where morphisms are given by

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Pro}'(\mathrm{FSch}_{\mathrm{af,pr}})}((\mathcal{T}_\beta)_\beta,(\mathcal{S}_\alpha)_\alpha) &= \mathrm{Hom}_{\mathrm{Pro}(\mathrm{FSch}_{\mathrm{af,pr}})}((\mathcal{T}'_\beta)_\beta,(\mathcal{S}'_\alpha)_\alpha)\\ &\simeq \mathrm{Hom}_{\mathrm{Pro}(\mathrm{FSch}_{\mathrm{af,pr}})}((\mathcal{T}'_\beta)_\beta,(\mathcal{S}_\alpha)_\alpha).\end{aligned}$$

The obvious functor $\mathrm{Pro}(\mathrm{FSch}_{\mathrm{af,pr}})^{\mathrm{op}} \to \mathrm{Pro}'(\mathrm{FSch}_{\mathrm{af,pr}})^{\mathrm{op}}$, given by the identity on objects, is a localisation functor and its right adjoint is given on objects by $(\mathcal{S}_\alpha)_\alpha \mapsto (\mathcal{S}'_\alpha)_\alpha$.

Corollary 2.6.6. *The functor*

$$\mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}((-)^{\mathrm{rig}};\Lambda) : \mathrm{Pro}(\mathrm{FSch}_{\mathrm{af,pr}})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$$

extends uniquely to $\mathrm{Pro}'(\mathrm{FSch}_{\mathrm{af,pr}})^{\mathrm{op}}$.

Proof. Indeed, $\mathrm{Pro}(\mathrm{FSch}_{\mathrm{af,pr}})^{\mathrm{op}} \to \mathrm{Pro}'(\mathrm{FSch}_{\mathrm{af,pr}})^{\mathrm{op}}$ is a localisation functor and $\mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}((-)^{\mathrm{rig}};\Lambda)$ transforms the morphisms $(\mathcal{S}'_\alpha)_\alpha \to (\mathcal{S}_\alpha)_\alpha$ into equivalences by Lemma 2.6.4. Thus, the result follows from [Lur09, Proposition 5.2.7.12]. □

Remark 2.6.7. In the remainder of this subsection, we use the construction of $\mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(S;\Lambda)$ as a localisation of the ∞-category of presheaves of Λ-modules on $\mathrm{FRigSm}/\mathcal{S}$ as explained in Remark 2.1.14. In fact, we will rather use the full subcategory of the latter, denoted by $\mathrm{FRigSm}_{\mathrm{af,pr}}/\mathcal{S}$, spanned by formal $\mathcal{S}$-schemes which are affine and of principal ideal type. (If $\mathcal{S}$ is of principal ideal type and π a generator of an ideal of definition, then the second condition is equivalent to having a π-torsion-free structure sheaf.) We are free to do so since the obvious inclusion induces an equivalence of sites $(\mathrm{FRigSm}/\mathcal{S},\mathrm{rignis}) \simeq (\mathrm{FRigSm}_{\mathrm{af,pr}}/\mathcal{S},\mathrm{rignis})$. We will also need the analogous fact for $\mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}((S_\alpha)_\alpha;\Lambda)$: It can be constructed as a localisation of the ∞-category of presheaves of Λ-modules on

$$\mathrm{FRigSm}_{\mathrm{af,pr}}/(\mathcal{S}_\alpha)_\alpha = \operatorname*{colim}_\alpha \mathrm{FRigSm}_{\mathrm{af,pr}}/\mathcal{S}_\alpha.$$

The above category will be endowed with the limit rig-Nisnevich topology so that the resulting site is equivalent to the one used in Notation 2.5.5 (with $\tau = \mathrm{nis}$). Moreover, (2.31) is induced from the obvious functor

$$\mathrm{FRigSm}_{\mathrm{af,pr}}/(\mathcal{S}_\alpha)_\alpha \to \mathrm{FRigSm}_{\mathrm{af,pr}}/\mathcal{S}$$

by the naturality of the construction of categories of motives.

Remark 2.6.8. (See Remark 2.5.6.) Given a cofiltered inverse system of affine formal schemes of principal ideal type $(\mathcal{S}_\alpha)_\alpha$, we denote by $\mathrm{Pro}(\mathrm{FSch}_{\mathrm{af,\,pr}})/(\mathcal{S}_\alpha)_\alpha$ the overcategory of $(\mathcal{S}_\alpha)_\alpha$-objects. There is a fully faithful embedding

$$\mathrm{FRigSm}_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha \to \mathrm{Pro}(\mathrm{FSch}_{\mathrm{af,\,pr}})/(\mathcal{S}_\alpha)_\alpha \tag{2.35}$$

and we will identify $\mathrm{FRigSm}_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha$ with its essential image by this functor. Thus, we may think of an object of $\mathrm{FRigSm}_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha$ as a pro-object $(\mathcal{X}_\alpha)_{\alpha\leq\alpha_0}$, where $\mathcal{X}_{\alpha_0}$ is a rig-smooth formal $\mathcal{S}_{\alpha_0}$-scheme and, for $\alpha \leq \alpha_0$, $\mathcal{X}_\alpha \simeq \mathcal{X}_{\alpha_0} \times_{\mathcal{S}_{\alpha_0}} \mathcal{S}_\alpha/(0)^{\mathrm{sat}}$. We set $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$, and for an object $(\mathcal{X}_\alpha)_{\alpha\leq\alpha_0}$ as before, we set $\mathcal{X} = \lim_{\alpha\leq\alpha_0} \mathcal{X}_\alpha$.

We now introduce a new category of formal pro-schemes over $(\mathcal{S}_\alpha)_\alpha$ where, roughly speaking, the endofunctor introduced in Notation 2.6.5 becomes an equivalence. We will also consider the ∞-category of motives associated to this new category of formal pro-schemes, and use it to divide the sought after equivalence into two which are easier to establish.

Notation 2.6.9. Keep the assumptions as in Remark 2.6.8. We denote by $\mathrm{FRigSm}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha$ the full subcategory of $\mathrm{Pro}'(\mathrm{FSch}_{\mathrm{af,\,pr}})/(\mathcal{S}_\alpha)_\alpha$ spanned by the objects which belong to the image of (2.35). More concretely, we have a functor

$$\mathrm{FRigSm}_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha \to \mathrm{FRigSm}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha \tag{2.36}$$

which is the identity on objects and such that, in the target, the set of morphisms from $(\mathcal{Y}_\alpha)_{\alpha\leq\beta_0}$ to $(\mathcal{X}_\alpha)_{\alpha\leq\alpha_0}$ is the set of morphisms from $(\mathcal{Y}'_\alpha)_{\alpha\leq\beta_0}$ to $(\mathcal{X}'_\alpha)_{\alpha\leq\alpha_0}$ over $(\mathcal{S}_\alpha)_\alpha$.

Remark 2.6.10. Let $\mathrm{FRig\acute{E}t}_{\mathrm{af,\,pr}}/\mathcal{S}$ be the full subcategory of $\mathrm{FRigSm}_{\mathrm{af,\,pr}}/\mathcal{S}$ spanned by rig-étale formal $\mathcal{S}$-schemes. Similarly, let

$$\mathrm{FRig\acute{E}t}_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha = \operatorname*{colim}_\alpha \mathrm{FRig\acute{E}t}_{\mathrm{af,\,pr}}/\mathcal{S}_\alpha,$$

considered as a full subcategory of $\mathrm{FRigSm}_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha$, and let $\mathrm{FRig\acute{E}t}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha$ be its essential image by the functor (2.36). The obvious functors

$$\mathrm{FRig\acute{E}t}_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha \to \mathrm{FRig\acute{E}t}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha \to \mathrm{FRig\acute{E}t}_{\mathrm{af,\,pr}}/\mathcal{S}$$

are equivalences of categories. Indeed, it is so for their composition by Corollary 1.3.10, and the second functor is faithful. This allows us to define the rig-Nisnevich topology on $\mathrm{FRig\acute{E}t}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha$, and more generally on $\mathrm{FRigSm}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha$ by replacing $(\mathcal{S}_\alpha)_\alpha$ with a general object of the latter category.

Proposition 2.6.11. *Let $(\mathcal{S}_\alpha)_\alpha$ be a cofiltered inverse system of affine formal schemes of principal ideal type. The functor $(\mathcal{X}_\alpha)_{\alpha\leq\alpha_0} \mapsto \mathrm{M}^{\mathrm{eff}}((\mathcal{X}^{\mathrm{rig}}_\alpha)_{\alpha\leq\alpha_0})$ extends naturally to a functor*

$$\mathrm{M}'^{\mathrm{eff}}(-) : \mathrm{FRigSm}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha \to \mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}((S_\alpha)_\alpha;\Lambda).$$

(As usual, we set $S_\alpha = \mathcal{S}^{\mathrm{rig}}_\alpha$.)

Proof. By Corollary 2.6.6, there is a functor

$$\mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}((-)^{\mathrm{rig}};\Lambda) : (\mathrm{FRigSm}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}.$$

For every $(\mathcal{X}_\alpha)_{\alpha\leq\alpha_0}$ in $\mathrm{FRigSm}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha$, with structure morphism $f : (\mathcal{X}_\alpha)_{\alpha\leq\alpha_0} \to (\mathcal{S}_\alpha)_\alpha$, the associated inverse image functor f^* admits a left adjoint $f_\sharp$. Moreover, the motive $\mathrm{M}^{\mathrm{eff}}((\mathcal{X}^{\mathrm{rig}}_\alpha)_{\alpha\leq\alpha_0})$ is equivalent to $f_\sharp f^*\Lambda$. Hence, the result follows by applying Lemma 2.6.12 below. ◻

Lemma 2.6.12. *Let $\mathcal{C}$ be an ∞-category and $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{CAT}_\infty$ a functor. Given a morphism $f : Y \to X$ in $\mathcal{C}$, we denote by $f^* : \mathcal{F}(X) \to \mathcal{F}(Y)$ the induced functor. Assume that $\mathcal{C}$ admits a final object $\star$ and that for every object $X \in \mathcal{C}$, the functor π_X^*, associated to $\pi_X : X \to \star$, admits a left adjoint $\pi_{X,\sharp}$. Then, there is a functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{F}(\star), \mathcal{F}(\star))$ sending $X \in \mathcal{C}$ to the endofunctor $\pi_{X,\sharp}\pi_X^*$ and a morphism $f : Y \to X$ to the composition of*

$$\pi_{Y,\sharp}\pi_Y^* \simeq \pi_{Y,\sharp}f^*\pi_X^* \xrightarrow{\eta} \pi_{Y,\sharp}f^*\pi_X^*\pi_{X,\sharp}\pi_X^* \simeq \pi_{Y,\sharp}\pi_Y^*\pi_{X,\sharp}\pi_X^* \xrightarrow{\delta} \pi_{X,\sharp}\pi_X^*,$$

where η is the unit of the adjunction $(\pi_{X,\sharp}, \pi_X^*)$ and δ is the counit of the adjunction $(\pi_{Y,\sharp}, \pi_Y^*)$.

Proof. Let $p : \mathcal{M} \to \mathcal{C}$ be the Cartesian fibration associated to the functor $\mathcal{F}$ by Lurie's unstraightening construction [Lur09, §3.2]. Since $\star$ is a final object of $\mathcal{C}$, we have a natural transformation $\mathcal{F}(\star)_{\mathrm{cst}} \to \mathcal{F}$, where $\mathcal{F}(\star)_{\mathrm{cst}} : \mathcal{C}^{\mathrm{op}} \to \mathrm{CAT}_\infty$ is the constant functor with value $\mathcal{F}(\star)$. This natural transformation induces a morphism of Cartesian fibrations

$$\begin{tikzcd} \mathcal{F}(\star) \times \mathcal{C} \arrow[rr, "G"] \arrow[dr, "q"'] & & \mathcal{M} \arrow[dl, "p"] \\ & \mathcal{C}. & \end{tikzcd}$$

The fiber of G over an object $X \in \mathcal{C}$ is the functor $\pi_X^* : \mathcal{F}(\star) \to \mathcal{F}(X)$, which admits a left adjoint by assumption. By [Lur17, Proposition 7.3.2.6], the functor G admits a left adjoint F relative to $\mathcal{C}$, in the sense of [Lur17, Definition 7.3.2.2]. Thus, we have a commutative triangle

$$\begin{tikzcd} \mathcal{F}(\star) \times \mathcal{C} \arrow[dr, "q"'] & & \mathcal{M} \arrow[ll, "F"'] \arrow[dl, "p"] \\ & \mathcal{C} & \end{tikzcd}$$

and a natural transformation $\mathrm{id} \to G \circ F$ over $\mathcal{C}$ which is a unit map. Moreover, the fiber of F over an object $X \in \mathcal{C}$ is the functor $\pi_{X,\sharp} : \mathcal{F}(X) \to \mathcal{F}(\star)$.

Composing the endofunctor $F \circ G$ of $\mathcal{F}(\star) \times \mathcal{C}$ with the projection to $\mathcal{F}(\star)$ yields a functor $\mathcal{F}(\star) \times \mathcal{C} \to \mathcal{F}(\star)$ and, by adjunction, a functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{F}(\star), \mathcal{F}(\star))$. We leave it to the reader to check that the latter satisfies the informal description given in the statement. □

Remark 2.6.13. Let $(\mathcal{S}_\alpha)_\alpha$ be a cofiltered inverse system of affine formal schemes of principal ideal type and $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$. We set $S_\alpha = \mathcal{S}_\alpha^{\mathrm{rig}}$ and $S = \mathcal{S}^{\mathrm{rig}}$.

(1) There is a commutative diagram

$$\begin{tikzcd} \mathrm{FRigSm}_{\mathrm{af,pr}}/(\mathcal{S}_\alpha)_\alpha \arrow[r] \arrow[dr, "\mathrm{M}^{\mathrm{eff}}((-)^{\mathrm{rig}})"'] & \mathrm{FRigSm}'_{\mathrm{af,pr}}/(\mathcal{S}_\alpha)_\alpha \arrow[r] \arrow[d, "\mathrm{M}'^{\mathrm{eff}}(-)"] & \mathrm{FRigSm}_{\mathrm{af,pr}}/\mathcal{S} \arrow[d, "\mathrm{M}^{\mathrm{eff}}((-)^{\mathrm{rig}})"] \\ & \mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}((S_\alpha)_\alpha; \Lambda) \arrow[r] & \mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(S; \Lambda). \end{tikzcd}$$

This is not completely obvious. One needs to check that Lemma 2.6.12 applied to the contravariant functor $\mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}((-)^{\mathrm{rig}}; \Lambda)$ defined on $\mathrm{FRigSm}^{\mathrm{af}}/(\mathcal{S}_\alpha)_\alpha$ and $\mathrm{FRigSm}^{\mathrm{af}}/\mathcal{S}$ gives back the functor $\mathrm{M}^{\mathrm{eff}}((-)^{\mathrm{rig}})$. To do so, one reduces to a similar question, but for the contravariant functor $\mathrm{RigSm}/(-)^{\mathrm{rig}}$, which can be easily handled.

(2) It follows from the commutative triangle inside the diagram in (1) that $\mathrm{M}'^{\mathrm{eff}}$ admits descent for the rig-Nisnevich topology, i.e., it takes a truncated hypercover for the rig-Nisnevich topology to a colimit diagram. (See Remark 2.6.10.)
(3) By the universal properties of presheaf categories and localisation, the commutative diagram in (1) gives rise to a commutative diagram in Pr^{L}:

$$\begin{tikzcd} \mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}((S_\alpha)_\alpha;\Lambda) \arrow[r] \arrow[dr, equal] & \mathbf{RigSH}'^{\mathrm{eff}}_{\mathrm{nis}}((S_\alpha)_\alpha;\Lambda) \arrow[d] \arrow[dr] & \\ & \mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}((S_\alpha)_\alpha;\Lambda) \arrow[r] & \mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(S;\Lambda) \end{tikzcd}$$

where $\mathbf{RigSH}'^{\mathrm{eff}}_{\mathrm{nis}}((S_\alpha)_\alpha;\Lambda)$ is defined from the site $(\mathrm{FRigSm}'_{\mathrm{af,pr}}/(\mathcal{S}_\alpha)_\alpha, \mathrm{rignis})$ in the usual way, i.e., by adapting Definition 2.1.11. Thus, to finish the proof of Proposition 2.5.12, it suffices to show Proposition 2.6.14 below.

Proposition 2.6.14. *Let $(\mathcal{S}_\alpha)_\alpha$ be a cofiltered inverse system of affine formal schemes of principal ideal type and $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$. We set $S_\alpha = \mathcal{S}_\alpha^{\mathrm{rig}}$ and $S = \mathcal{S}^{\mathrm{rig}}$. Then the obvious functor*

$$\mathbf{RigSH}'^{\mathrm{eff}}_{\mathrm{nis}}((S_\alpha)_\alpha;\Lambda) \to \mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(S;\Lambda) \tag{2.37}$$

is an equivalence.

Notation 2.6.15. From now on, we fix a cofiltered inverse system $(\mathcal{S}_\alpha)_\alpha$ of affine formal schemes of principal ideal type, and we let $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$. We define $(\mathcal{S}'_\alpha)_\alpha$ as in Notation 2.6.3, and we set $S_\alpha = \mathcal{S}_\alpha^{\mathrm{rig}}$, $S'_\alpha = \mathcal{S}'^{\mathrm{rig}}_\alpha$ and $S = \mathcal{S}^{\mathrm{rig}}$. We set $A_\alpha = \mathcal{O}(\mathcal{S}_\alpha)$, $A'_\alpha = \mathcal{O}(\mathcal{S}'_\alpha)$ and $A = \mathcal{O}(\mathcal{S})$. We identify A'_α with a subring of A and set $A'_\infty = \bigcup_\alpha A'_\alpha$ which is a dense subring of A. We also assume that there is an element π, which "belongs" to all the A_α's and generates an ideal of definition in each A_α. (This is not a restrictive assumption since it is clearly satisfied when the indexing category of $(\mathcal{S}_\alpha)_\alpha$ admits a final object.) Given $(\mathcal{X}_\alpha)_{\alpha\le\alpha_0}$ in $\mathrm{FRigSm}_{\mathrm{af,pr}}/(\mathcal{S}_\alpha)_\alpha$, we use similar notations: $B_\alpha = \mathcal{O}(\mathcal{X}_\alpha)$, $B'_\alpha = \mathcal{O}(\mathcal{X}'_\alpha)$, $B = \mathcal{O}(\mathcal{X})$ and $B'_\infty = \bigcup_{\alpha\le\alpha'} B'_\alpha$ which is a dense subring of B.

Remark 2.6.16. The ∞-category $\mathbf{RigSH}'^{\mathrm{eff}}_{\mathrm{nis}}((S_\alpha)_\alpha;\Lambda)$ is compactly generated, up to desuspension, by $\mathrm{M}'^{\mathrm{eff}}((\mathcal{X}_\alpha)_{\alpha\le\alpha_0})$ where $(\mathcal{X}_\alpha)_{\alpha\le\alpha_0}$ belongs to $\mathrm{FRigSm}'_{\mathrm{af,pr}}/(\mathcal{S}_\alpha)_\alpha$. (This can be proven by adapting the proof of Proposition 2.4.22. The key point is that the small rig-Nisnevich site of $(\mathcal{X}_\alpha)_{\alpha\le\alpha_0}$ is equivalent to the small Nisnevich site of X; see Remark 2.6.10.) Using Proposition 2.4.22, we deduce that the functor (2.37) belongs to $\mathrm{Pr}^{\mathrm{L}}_\omega$. This functor also sends a set of compact generators to a set of compact generators. Indeed, by Proposition 1.3.8, a set of compact generators for $\mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(S;\Lambda)$ is given, up to desuspension, by motives of smooth rigid S-affinoids $X = \mathrm{Spf}(B)^{\mathrm{rig}}$ with B of the form

$$B = A\langle s_1,\ldots,s_m,t_1,\ldots,t_n\rangle/(P_1,\ldots,P_n)^{\mathrm{sat}}$$

with $P_i \in A'_\infty[s_1,\ldots,s_m,t_1,\ldots,t_n]$ such that $\det(\partial P_i/\partial t_j)$ generates an open ideal in B. Clearly, $\mathrm{Spf}(B)$ is in the image of $\mathrm{FRigSm}_{\mathrm{af,pr}}/(\mathcal{S}_\alpha)_\alpha \to \mathrm{FRigSm}_{\mathrm{af,pr}}/\mathcal{S}$. In particular, to prove that the functor (2.6.14) is an equivalence, it remains to show that it is fully faithful.

Before continuing with the proof, we recall the following two statements from [Vez19].

Proposition 2.6.17. *Let R be an adic ring of principal ideal type and $\pi \in R$ a generator of an ideal of definition. Let $s = (s_1,\ldots,s_m)$ and $t = (t_1,\ldots,t_n)$ be two systems of coordinates and let $P = (P_1,\ldots,P_n)$ be an n-tuple of polynomials in $R[s,t]$ with no constant term, i.e., such that $P|_{s=0,\,t=0} = (0,\ldots,0)$. Assume also that $\det(\partial P_i/\partial t_j)|_{s=0,\,t=0}$ generates an open ideal in R. Then,* *there exists a unique n-tuple* $F = (F_1, \dots, F_n)$ *of formal power series in* $(R[\pi^{-1}])[[s]]$ *such that* $P(s, F(s)) = 0$. *Moreover, for N large enough, the* F_i*'s belong to the subring* $R[[\pi^{-N}s]]$.

Proof. This is a slight generalisation of [Vez19, Proposition A.1] and one can easily check that the proof of loc. cit. still works in the present context. More precisely, instead of a Banach K-algebra, with K a complete non-Archimedean field, as in loc. cit., we consider the Banach ring $R[\pi^{-1}]$ endowed with the norm described in the proof of Proposition 1.3.7. (Note that $\det(\partial P_i/\partial t_j)|_{s=0,\,t=0}$ generates an open ideal in R if and only if it is invertible in $R[\pi^{-1}]$.) □

The previous statement has the following generalisation. (See [Vez19, Proposition A.2].)

Corollary 2.6.18. *Let R be an adic ring of principal ideal type and* $\pi \in R$ *a generator of an ideal of definition. Let* $s = (s_1, \dots, s_m)$ *and* $t = (t_1, \dots, t_n)$ *be two systems of coordinates, let* $a = (a_1, \dots, a_m)$ *and* $b = (b_1, \dots, b_n)$ *be two tuples of elements in R, and let* $P = (P_1, \dots, P_n)$ *be an n-tuple of polynomials in* $R[s, t]$ *such that* $P|_{s=a,\,t=b} = (0, \dots, 0)$. *Assume also that* $\det(\partial P_i/\partial t_j)|_{s=a,\,t=b}$ *generates an open ideal in R. Then, there exists a unique n-tuple* $F = (F_1, \dots, F_n)$ *of formal power series in* $(R[\pi^{-1}])[[s-a]]$ *such that* $P(s, F(s)) = 0$. *Moreover, for N large enough, the* F_i*'s belong to the subring* $R[[\pi^{-N}(s-a)]]$.

We introduce some further notations.

Notation 2.6.19. We fix two π-torsion-free rig-smooth adic A_{α_0}-algebras B_{α_0} and C_{α_0}. For $\alpha \le \alpha_0$, we set $B_\alpha = A_\alpha \widehat{\otimes}_{A_{\alpha_0}} B_{\alpha_0}/(0)^{\mathrm{sat}}$, $C_\alpha = A_\alpha \widehat{\otimes}_{A_{\alpha_0}} C_{\alpha_0}/(0)^{\mathrm{sat}}$, $\mathcal{X}_\alpha = \mathrm{Spf}(B_\alpha)$ and $\mathcal{Y}_\alpha = \mathrm{Spf}(C_\alpha)$. Similarly, we set $B = A \widehat{\otimes}_{A_{\alpha_0}} B_{\alpha_0}/(0)^{\mathrm{sat}}$, $C = A \widehat{\otimes}_{A_{\alpha_0}} C_{\alpha_0}/(0)^{\mathrm{sat}}$, $\mathcal{X} = \mathrm{Spf}(B)$ and $\mathcal{Y} = \mathrm{Spf}(C)$. We also denote by B'_α, B'_∞ and $\mathcal{X}'_\alpha$ as in Notation 2.6.15, and we define similarly C'_α, C'_∞ and $\mathcal{Y}'_\alpha$. Moreover, we assume that

$$B_{\alpha_0} = A_{\alpha_0}\langle s, t\rangle/(P)^{\mathrm{sat}}$$

with $s = (s_1, \dots, s_m)$ and $t = (t_1, \dots, t_n)$ two systems of coordinates, and $P = (P_1, \dots, P_n)$ an n-tuple of polynomials in $A_{\alpha_0}[s, t]$ such that $\det(\partial P_i/\partial t_j)$ generates an open ideal of A_{α_0}.

Lemma 2.6.20. *Given a morphism of formal schemes* $f : \mathcal{Y} \to \mathcal{X}$, *there exists an* $\mathbb{A}^1$*-homotopy*

$$H : \mathbb{A}^1_{\mathcal{Y}} = \mathrm{Spf}(C\langle\tau\rangle) \to \mathcal{X}$$

from $f = H \circ i_0$ *to a map* $\tilde{f} = H \circ i_1$ *such that* $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ *descends to a unique map* $\mathcal{Y}'_\alpha \to \mathcal{X}_\alpha$ *for* $\alpha \le \alpha_0$ *small enough.*

Proof. Indeed, suppose that f corresponds to a morphism of adic A-algebras $B \to C$ given by $s_i \mapsto c_i$, for $1 \le i \le m$, and $t_j \mapsto d_j$, for $1 \le j \le n$, where the $c = (c_1, \dots, c_m)$ and $d = (d_1, \dots, d_n)$ are tuples of elements of C satisfying $P(c, d) = 0$. Let $F = (F_1, \dots, F_n)$ be the n-tuple of power series in $C[\pi^{-1}][[s-c]]$ associated by Corollary 2.6.18 to the n-tuple of polynomials $P = (P_1, \dots, P_n)$ (considered with coefficients in C via the map $A_{\alpha_0} \to C$) and their common zero (c, d). By the same corollary, for $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_m)$ an m-tuple of elements in A close enough to c, the expressions $F_i(c + (\tilde{c} - c) \cdot \tau)$ are well-defined elements of $C\langle\tau\rangle$, and the assignment

$$s \mapsto c + (\tilde{c} - c) \cdot \tau, \quad t \mapsto F(c + (\tilde{c} - c) \cdot \tau)$$

gives rise to a map of A-algebras $B \to C\langle\tau\rangle$, and hence to a morphism $H : \mathbb{A}^1_{\mathcal{Y}} \to \mathcal{X}$ of formal schemes. By construction, $H \circ i_0 = f$, and it remains to show that $\tilde{f} = H \circ i_1$ descends to a morphism $\mathcal{Y}'_\alpha \to \mathcal{X}_\alpha$ for a well-chosen m-tuple $\tilde{c}$. (The uniqueness is clear since $C'_\alpha \to C$ is injective.) This is the case when the $\tilde{c}_i$'s belong to the dense subring $C'_\infty = \bigcup_{\alpha \le \alpha_0} C'_\alpha$ of C. Indeed, refining α_0, we may assume that the $\tilde{c}_i$'s belong to C'_{α_0}. Consider the map $\mathcal{Y}'_{\alpha_0} \to \mathcal{S}_{\alpha_0} \times \mathbb{A}^m = \mathrm{Spf}(A_\alpha\langle s\rangle)$ induced by $\tilde{c}$. We have a rig-étale morphism $\mathcal{X}_{\alpha_0} \to \mathcal{S}_{\alpha_0} \times \mathbb{A}^m$ and the morphism $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ gives rise to a section σ of the rig-étale projection $\mathcal{X}_{\alpha_0} \times_{\mathcal{S}_{\alpha_0} \times \mathbb{A}^m, \tilde{c}} \mathcal{Y} \to \mathcal{Y}$. Then $\tilde{f}$ descends to a morphism $\mathcal{Y}'_\alpha \to \mathcal{X}_\alpha$ if and only if the section σ descends to a section of the rig-étale projection $(\mathcal{X}_{\alpha_0} \times_{\mathcal{S}_{\alpha_0} \times \mathbb{A}^m, \tilde{c}} \mathcal{Y}'_\alpha) \to \mathcal{Y}'_\alpha$. That this is true follows from Corollary 1.3.10. □

Corollary 2.6.21. *Keep the notation as above. Fix a system of coordinates $u = (u_1, \dots, u_r)$ for $\mathbb{A}^r$. Given a finite collection $f_1, \dots, f_N$ in $\mathrm{Hom}_{\mathcal{S}}(\mathcal{Y} \times \mathbb{A}^r, \mathcal{X})$ we can find a collection $H_1, \dots, H_N$ in $\mathrm{Hom}_{\mathcal{S}}(\mathcal{Y} \times \mathbb{A}^r \times \mathbb{A}^1, \mathcal{X})$ and some index $\alpha \leq \alpha_0$ such that:*

(1) For all $1 \leq k \leq N$, we have $f_k = H_k \circ i_0$ and the map $\tilde{f}_k = H_k \circ i_1$ descends to a unique map $\mathcal{Y}'_\alpha \times \mathbb{A}^r \to \mathcal{X}_\alpha$ over $\mathcal{S}_\alpha$.

(2) If $f_k \circ d_{i,\epsilon} = f_{k'} \circ d_{i,\epsilon}$ for some $1 \leq k, k' \leq N$ and some $(i, \epsilon) \in \{1, \dots, r\} \times \{0, 1\}$ then $H_k \circ d_{i,\epsilon} = H_{k'} \circ d_{i,\epsilon}$.

(3) If for some $1 \leq k \leq N$ and some $\gamma \leq \alpha_0$ the map $f_k \circ d_{1,1} \in \mathrm{Hom}_{\mathcal{S}}(\mathcal{Y} \times \mathbb{A}^{r-1}, \mathcal{X})$ comes from $\mathrm{Hom}_{\mathcal{S}_\gamma}(\mathcal{Y}'_\gamma \times \mathbb{A}^{r-1}, \mathcal{X}_\gamma)$, then the homotopy $H_k \circ d_{1,1} \in \mathrm{Hom}_{\mathcal{S}}(\mathcal{Y} \times \mathbb{A}^{r-1} \times \mathbb{A}^1, \mathcal{X})$ is constant, i.e., factors through the projection on $\mathcal{Y} \times \mathbb{A}^{r-1}$.

Proof. Suppose that f_k corresponds to a morphism of adic A-algebras $B \to C\langle u\rangle$ given by $(s, t) \mapsto (c_k, d_k)$ where $c_k = (c_{k1}, \dots, c_{km})$ and $d_k = (d_{k1}, \dots, d_{kn})$ are tuples of elements of $C\langle u\rangle$ satisfying $P(c_k, d_k) = 0$. By Lemma 2.6.20, there are n-tuples of formal power series $F_k = (F_{k1}, \dots, F_{kn})$ associated to the f_k's such that

$$(s, t) \mapsto (c_k + (\tilde{c}_k - c) \cdot \tau, F_k(c_k + (\tilde{c}_k - c_k) \cdot \tau))$$

defines a morphism $H_k : \mathcal{Y} \times \mathbb{B}^r \times \mathbb{B}^1 \to \mathcal{X}$ satisfying condition (1), for some $\alpha \leq \alpha_0$, when the $\tilde{c}_{ki}$'s are close enough to the c_{ki}'s and belong to the dense subring $C'_\infty\langle u\rangle = \bigcup_{\alpha \leq \alpha_0} C'_\alpha\langle u\rangle$ of $C\langle u\rangle$.

It remains to explain how to choose the $\tilde{c}_k$'s so that the conditions (2) and (3) above are also satisfied. To do so, we apply [Vez19, Proposition A.5] to the c_{ki}'s. (This result of [Vez19] is stated for Banach algebras over a non-Archimedean field and a sequence of complete subalgebras, but holds more generally for Banach rings and a filtered family of complete subrings; and we apply it here to $C[\pi^{-1}]$ and the family $C'_\alpha[\pi^{-1}]$, for $\alpha \leq \alpha_0$.) Thus we may find elements $\tilde{c}_{ki} \in C'_\infty\langle u\rangle$, which are arbitrary close to the c_{ki}'s, and satisfying the following properties:

(2′) If $c_k|_{u_i=\epsilon} = c_{k'}|_{u_i=\epsilon}$ for some $1 \leq k, k' \leq N$ and some $(i, \epsilon) \in \{1, \dots, r\} \times \{0, 1\}$ then $\tilde{c}_k|_{u_i=\epsilon} = \tilde{c}_{k'}|_{u_i=\epsilon}$.

(3′) If for some $1 \leq k \leq N$ and some $\gamma \leq \alpha_0$, $c_k|_{u_1=1}$ belongs to $C'_\gamma\langle u_2, \dots, u_r\rangle$, then $\tilde{c}_k|_{u_1=1} = c_k|_{u_1=1}$.

With these $\tilde{c}_{ki}$'s, it is easy to see that conditions (2) and (3) are satisfied. Indeed, suppose that $f_k \circ d_{i,\epsilon} = f_{k'} \circ d_{i,\epsilon}$ for some $i \in \{1, \dots, r\}$ and $\epsilon \in \{0, 1\}$. This means that $c_k|_{u_i=\epsilon} = c_{k'}|_{u_i=\epsilon}$ and $d_k|_{u_i=\epsilon} = d_{k'}|_{u_i=\epsilon}$; we denote by $\bar{c}$ and $\bar{d}$ their respective common values. This implies that both $F_k|_{u_i=\epsilon}$ and $F_{k'}|_{u_i=\epsilon}$ are two n-tuples of formal power series $\bar{F}$ with coefficients in $C\langle u_2, \dots, u_r\rangle$ converging around $\bar{c}$ and such that $P(s, \bar{F}(s)) = 0$ and $\bar{F}(\bar{c}) = \bar{d}$. By the uniqueness of such power series stated in Corollary 2.6.18, we conclude that they coincide. Moreover, by property (2′), we have $\tilde{c}_k|_{u_i=\epsilon} = \tilde{c}_{k'}|_{u_i=\epsilon}$; we denote by $\bar{\tilde{c}}$ the common value. It follows that

$$F_k(c_k + (\tilde{c}_k - c_k) \cdot \tau)|_{u_i=\epsilon} = \bar{F}(\bar{c} + (\bar{\tilde{c}} - \bar{c}) \cdot \tau) = F_{k'}(c_{k'} + (\tilde{c}_{k'} - c_{k'}) \cdot \tau)|_{\theta_r=\epsilon}$$

and thus $H_k \circ d_{i,\epsilon} = H_{k'} \circ d_{i,\epsilon}$ proving property (2). Property (3) follows immediately from property (3′) and the definition of H_k. □

Proof of Proposition 2.6.14. We split the argument into two steps.

Step 1. Consider the $\mathbb{A}^1$-localisation functor $\mathrm{L}_{\mathbb{A}^1}$ on the ∞-categories of presheaves of Λ-modules

$$\mathrm{PSh}(\mathrm{FRigSm}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha;\Lambda) \qquad \text{and} \qquad \mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af,\,pr}}/\mathcal{S};\Lambda).$$

For a presheaf $\mathcal{F}$ of Λ-modules, $\mathrm{L}_{\mathbb{A}^1}(\mathcal{F})$ is given by the colimit of the simplicial presheaf $\underline{\mathrm{Hom}}(\Delta^\bullet,\mathcal{F})$ where Δ^r refers to the r-th algebraic simplex and

$$\underline{\mathrm{Hom}}(\Delta^r,\mathcal{F})(-) = \mathcal{F}((-)\langle u_0,\ldots,u_r\rangle/(u_0+\cdots+u_r-1)).$$

Indeed, the map $\mathcal{F}\to\operatorname{colim}\underline{\mathrm{Hom}}(\Delta^\bullet,\mathcal{F})$ is an $\mathbb{A}^1$-equivalence by [MV99, §2.3, Corollary 3.8]. On the other hand, using [MV99, §2.3, Proposition 3.4] and the fact that the endofunctor $\underline{\mathrm{Hom}}(\Delta^1,-)$ preserves colimits, we have equivalences

$$\operatorname{colim}\underline{\mathrm{Hom}}(\Delta^\bullet,\mathcal{F})\simeq\operatorname{colim}\underline{\mathrm{Hom}}(\Delta^\bullet\times\Delta^1,\mathcal{F})\simeq\underline{\mathrm{Hom}}(\Delta^1,\operatorname{colim}\underline{\mathrm{Hom}}(\Delta^\bullet,\mathcal{F}))$$

showing that $\operatorname{colim}\underline{\mathrm{Hom}}(\Delta^\bullet,\mathcal{F})$ is $\mathbb{A}^1$-local.

With $(\mathcal{X}_\alpha)_{\alpha\le\alpha_0}$ and $(\mathcal{Y}_\alpha)_{\alpha\le\alpha_0}$ as in Notation 2.6.19, we claim that the natural map

$$(\mathrm{L}_{\mathbb{A}^1}\Lambda((\mathcal{X}_\alpha)_{\alpha\le\alpha_0}))\,((\mathcal{Y}_\alpha)_{\alpha\le\alpha_0})\to(\mathrm{L}_{\mathbb{A}^1}\Lambda(\mathcal{X}))\,(\mathcal{Y}) \tag{2.38}$$

is an equivalence. By the commutation of colimits with tensor products, it is enough to prove this when Λ is the sphere spectrum. (Here we use the explicit model for the $\mathbb{A}^1$-localisation recalled above.) Similarly, since tensoring with the Eilenberg–Mac Lane spectrum of $\mathbb{Z}$ is conservative on connective spectra, we reduce to prove this when Λ is the (Eilenberg–Mac Lane spectrum associated to the) ring $\mathbb{Z}$. In this case, we may use another model for the $\mathbb{A}^1$-localisation functor $\mathrm{L}_{\mathbb{A}^1}$, namely the one taking $\mathcal{F}$ to the normalised complex associated to the cubical presheaf of complexes of abelian groups $\underline{\mathrm{Hom}}(\mathbb{A}^\bullet,\mathcal{F})$ where, as above, $\underline{\mathrm{Hom}}(\mathbb{A}^r,\mathcal{F})(-)=\mathcal{F}((-)\langle u_1,\ldots,u_r\rangle)$. (This is proven by adapting the method used for the simplicial presheaf $\underline{\mathrm{Hom}}(\Delta^\bullet,\mathcal{F})$. See also [Ayo14b, Théorème 2.23] for a closely related result.) Thus, we are reduced to showing that the morphism of cubical abelian groups

$$\left(\underline{\mathrm{Hom}}(\mathbb{A}^\bullet,\mathbb{Z}((\mathcal{X}_\alpha)_{\alpha\le\alpha_0}))\right)((\mathcal{Y}_\alpha)_{\alpha\le\alpha_0})\to\left(\underline{\mathrm{Hom}}(\mathbb{A}^\bullet,\mathbb{Z}(\mathcal{X}))\right)(\mathcal{Y}) \tag{2.39}$$

induces an isomorphism on the associated normalised complexes. This follows from Corollary 2.6.21 by arguing as in [Vez19, Proposition 4.2]. Note that, since $\mathbb{Z}((\mathcal{X}_\alpha)_{\alpha\le\alpha_0})$ is considered as a presheaf on $\mathrm{FRigSm}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha$, the elements of the left-hand side of (2.39) are linear combinations of $(\mathcal{S}_\alpha)_\alpha$-morphisms of formal pro-schemes from $(\mathcal{Y}'_\alpha\times\mathbb{A}^r)_{\alpha\le\alpha_0}$ to $(\mathcal{X}_\alpha)_{\alpha\le\alpha_0}$.

Step 2. Let $\phi:(\mathrm{FRigSm}_{\mathrm{af,\,pr}}/\mathcal{S},\mathrm{rignis})\to(\mathrm{FRigSm}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha,\mathrm{rignis})$ be the premorphism of sites that gives rise to the adjunction

$$\phi^*_{\mathrm{mot}}:\mathbf{RigSH}'^{\mathrm{eff}}_{\mathrm{nis}}((S_\alpha)_\alpha;\Lambda)\rightleftarrows\mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(S;\Lambda):\phi_{\mathrm{mot},*}.$$

Our goal is to show that ϕ^*_{mot} is an equivalence, and by Remark 2.6.16 it remains to see that ϕ^*_{mot} is fully faithful. We will prove that the unit morphism $\mathrm{id}\to\phi_{\mathrm{mot},*}\phi^*_{\mathrm{mot}}$ is an equivalence. In order to do so, we note that the functor

$$\phi_*:\mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af,\,pr}}/\mathcal{S};\Lambda)\to\mathrm{PSh}(\mathrm{FRigSm}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha;\Lambda)$$

preserves $(\mathbb{A}^1,\mathrm{rignis})$-local equivalences. Preservation of rignis-local equivalences follows immediately from Remark 2.6.10. Preservation of $\mathbb{A}^1$-local equivalences is an easy consequence of the fact that $\mathbb{A}^1$ is an interval. (This is used to construct an explicit $\mathbb{A}^1$-homotopy between the identity of $\phi_*\Lambda((-)\times\mathbb{A}^1)$ and the endomorphism induced by the zero section.) As a consequence, we are left to show that the morphism $\mathcal{F} \to \phi_*\phi^*\mathcal{F}$ is an $(\mathbb{A}^1, \mathrm{rignis})$-local equivalence for all presheaves of Λ-modules $\mathcal{F}$ on $\mathrm{FRigSm}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha$. Since ϕ^* and ϕ_* commute with colimits, and since $(\mathbb{A}^1, \mathrm{rignis})$-local equivalences are preserved by colimits, we may assume that $\mathcal{F} = \Lambda((\mathcal{X}_\alpha)_{\alpha\leq\alpha_0})$ with $(\mathcal{X}_\alpha)_{\alpha\leq\alpha_0}$ as in Notation 2.6.19. In this case, the morphism $\mathcal{F} \to \phi_*\phi^*\mathcal{F}$ can be rewritten as follows:

$$\Lambda((\mathcal{X}_\alpha)_{\alpha\leq\alpha_0}) \to \phi_*\Lambda(\mathcal{X}). \tag{2.40}$$

We claim that this morphism is an $\mathbb{A}^1$-local equivalence. Indeed, if we apply $\mathrm{L}_{\mathbb{A}^1}$ to (2.40) and if we evaluate at an object $(\mathcal{Y}_\alpha)_{\alpha\leq\alpha_0}$ of $\mathrm{FRigSm}'_{\mathrm{af,\,pr}}/(\mathcal{S}_\alpha)_\alpha$, we get precisely the map (2.38) which we know to be an equivalence. □

2.7. **Quasi-compact base change.**

We prove here the so-called quasi-compact base change theorem for rigid analytic motives. This will be obtained as an application of the continuity property for $\mathbf{RigSH}^{(\mathrm{eff},\,\wedge)}_\tau(-;\Lambda)$ proved in Theorem 2.5.1. Our quasi-compact base change theorem can be compared with [Hub96, Proposition 4.4.1] and [dJvdP96, Theorems 5.3.1].

Theorem 2.7.1 (Quasi-compact base change)**.** *Consider a Cartesian square of rigid analytic spaces*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

with f quasi-compact and quasi-separated. Let $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$, and assume one of the following two alternatives.

(1) We work in the non-hypercomplete case. When τ is the étale topology, we assume furthermore that Λ is eventually coconnective.

(2) We work in the hypercomplete case, and X, X', Y and Y' are (Λ, τ)-admissible. When τ is the étale topology, we assume furthermore one of the following conditions:
- *Λ is eventually coconnective;*
- *locally on X and X', one can find formal models $\mathcal{X}$ and $\mathcal{X}'$ such that $\mathcal{X}'$ is a limit of a cofiltered inverse system of finite type formal $\mathcal{X}$-schemes $(\mathcal{X}_\alpha)_\alpha$ with affine transition morphisms and such that the numbers $\mathrm{pvcd}_\Lambda(\mathcal{X}^{\mathrm{rig}}_\alpha)$ are bounded independently of α. (For example, this holds if g is locally of finite type.)*

Then, the commutative square

$$\begin{tikzcd} \mathbf{RigSH}^{(\mathrm{eff},\,\wedge)}_\tau(X;\Lambda) \arrow[r, "f^*"] \arrow[d, "g^*"] & \mathbf{RigSH}^{(\mathrm{eff},\,\wedge)}_\tau(Y;\Lambda) \arrow[d, "g'^*"] \\ \mathbf{RigSH}^{(\mathrm{eff},\,\wedge)}_\tau(X';\Lambda) \arrow[r, "f'^*"] & \mathbf{RigSH}^{(\mathrm{eff},\,\wedge)}_\tau(Y';\Lambda) \end{tikzcd}$$

is right adjointable, i.e., the natural transformation $g^* \circ f_* \to f'_* \circ g'^*$ is an equivalence.

Proof. Using Proposition 2.2.1(3), the problem is local on X and X'. In particular, we may assume that X and X' are quasi-compact and quasi-separated. This implies the same for Y and Y'. We split the proof into two parts. In the first part, we assume that g is of finite type and, in the second part, we explain how to remove this assumption.

Part 1. Here we assume that g is of finite type. Since the problem is local on X and X', we may assume that g factors as a closed immersion followed by a smooth morphism. Using the base change theorem for smooth morphisms of Proposition 2.2.1, we reduce to the case where g is a closed immersion. Thus, we may assume that $X = \mathrm{Spf}(A)^{\mathrm{rig}}$ and $X' = \mathrm{Spf}(A')^{\mathrm{rig}}$ where A is an adic ring of principal ideal type and A' a quotient of A by a closed saturated ideal $I \subset A$. If $\pi \in A$ generates an ideal of definition, then A' is the filtered colimit in the category of adic rings of the A-algebras $A_{J,N} = A\langle J/\pi^N\rangle$ where $N \in \mathbb{N}$ and $J \subset I$ is a finitely generated ideal.

Set $\mathcal{X} = \mathrm{Spf}(A)$ and $\mathcal{X}' = \mathrm{Spf}(A')$. Choose a formal model $\mathcal{Y}$ of Y which is a formal $\mathcal{X}$-scheme and set $\mathcal{Y}' = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$. Let K be the indexing category of the filtered inductive system $(A_{J,N})_{J,N}$, and write "α" instead of "J, N" for the objects of K. We denote by $o \in K$ the initial object (corresponding to $N = 0$ and $J = (0)$). Set $\mathcal{X}_\alpha = \mathrm{Spf}(A_\alpha)$, $\mathcal{Y}_\alpha = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_\alpha$, $X_\alpha = \mathcal{X}_\alpha^{\mathrm{rig}}$ and $Y_\alpha = \mathcal{Y}^\alpha_{\mathrm{rig}}$. For $\alpha \to \beta$ in K, we have Cartesian squares of rigid analytic spaces

$$\begin{tikzcd} Y_\beta \arrow[r, "g'_{\beta\alpha}"] \arrow[d, "f_\beta"] & Y_\alpha \arrow[d, "f_\alpha"] \\ X_\beta \arrow[r, "g_{\beta\alpha}"] & X_\alpha \end{tikzcd}$$

where the horizontal arrows are open immersions. (Note that $f_o = f$.) We deduce commutative squares of ∞-categories

$$\begin{tikzcd} \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X_\alpha;\Lambda) \arrow[r, "f_\alpha^*"] \arrow[d, "g_{\beta\alpha}^*"] & \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(Y_\alpha;\Lambda) \arrow[d, "g'^*_{\beta\alpha}"] \\ \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X_\beta;\Lambda) \arrow[r, "f_\beta^*"] & \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(Y_\beta;\Lambda). \end{tikzcd} \tag{2.41}$$

In fact, we have a functor $K \to \mathrm{Fun}(\Delta^1, \mathrm{Pr}^{\mathrm{L}}_\omega)$ sending $\alpha \in K$ to f_α^* and $\alpha \to \beta$ to the commutative square (2.41). Moreover, since the squares (2.41) are right adjointable by Proposition 2.2.1(3), this functor factors through the sub-∞-category

$$\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Pr}^{\mathrm{L}}_\omega) = \mathrm{Fun}(\Delta^1, \mathrm{Pr}^{\mathrm{L}}_\omega) \cap \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{CAT}_\infty),$$

where $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{CAT}_\infty)$ is the ∞-category introduced in [Lur17, Definition 4.7.4.16].

Consider a colimit diagram $K^\triangleright \to \mathrm{Fun}(\Delta^1, \mathrm{Pr}^{\mathrm{L}}_\omega)$ extending the one described above. Since all the ∞-categories we are considering are stable, Lemma 2.7.2 below implies that this diagram factors also through the sub-∞-category $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Pr}^{\mathrm{L}}_\omega)$. Evaluating the functor $K^\triangleright \to \mathrm{Fun}(\Delta^1, \mathrm{Pr}^{\mathrm{L}}_\omega)$ at the edge $o \to \infty$, where $\infty \in K^\triangleright$ is the cone point, we obtain a commutative square in $\mathrm{Pr}^{\mathrm{L}}_\omega$

$$\begin{tikzcd} \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X_o;\Lambda) \arrow[r, "f_o^*"] \arrow[d, "\mathrm{colim}_\alpha\, g_{\alpha o}^*"] & \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(Y_o;\Lambda) \arrow[d, "\mathrm{colim}_\alpha\, g'^*_{\alpha o}"] \\ \operatorname*{colim}_\alpha \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X_\alpha;\Lambda) \arrow[r, "\mathrm{colim}_\alpha\, f_\alpha^*"] & \operatorname*{colim}_\alpha \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(Y_\alpha;\Lambda) \end{tikzcd}$$

which is right adjointable. By Theorem 2.5.1, this square is equivalent to the one in the statement.

Part 2. We now assume that g is not necessarily of finite type. We may assume that g is induced by a morphism $\mathrm{Spf}(A') \to \mathrm{Spf}(A)$ of affine formal schemes. Set $\mathcal{X} = \mathrm{Spf}(A)$ and $\mathcal{X}' = \mathrm{Spf}(A')$. Let $\mathcal{Y}$ be a quasi-compact and quasi-separated formal $\mathcal{X}$-scheme such that $Y = \mathcal{Y}^{\mathrm{rig}}$, and let $\mathcal{Y}' = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ so that $\mathcal{Y}'^{\mathrm{rig}} = Y'$. Write A' as a filtered colimit $A' = \mathrm{colim}_\alpha A_\alpha$ of finitely generated adic A-algebras A_α. Set also $\mathcal{X}_\alpha = \mathrm{Spf}(A_\alpha)$, $\mathcal{Y}_\alpha = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_\alpha$, $X_\alpha = \mathcal{X}_\alpha^{\mathrm{rig}}$ and $Y_\alpha = \mathcal{Y}_\alpha^{\mathrm{rig}}$. If τ is the étale topology and Λ is not eventually coconnective, we may assume that the numbers $\mathrm{pvcd}_\Lambda(X_\alpha)$ are bounded independently of α.

As in the first part of the proof, we have a diagram $K \to \mathrm{Fun}(\Delta^1, \mathrm{Pr}^{\mathrm{L}}_\omega)$ sending $\alpha \to \beta$ to squares of the form (2.41). Since the morphisms $g_{\beta\alpha} : X_\beta \to X_\alpha$ are of finite type, these squares are right adjointable as shown in the first part of the proof. The result follows again by considering a colimit diagram $K^{\triangleright} \to \mathrm{Fun}(\Delta^1, \mathrm{Pr}^{\mathrm{L}}_\omega)$, and using Lemma 2.7.2 and Theorem 2.5.1. □

The following lemma, which was used in the proof of Theorem 2.7.1, is well-known. We include a proof for completeness. (Recall that we are using the notation $\mathrm{Fun}^{\mathrm{RAd}}$ following [Lur17, Definition 4.7.4.16].)

Lemma 2.7.2. *Let K be a simplicial set. Let $\overline{\mathcal{C}} : K^{\triangleright} \to \mathrm{Fun}(\Delta^1, \mathrm{Pr}^{\mathrm{L}})$ be a colimit diagram and let $\mathcal{C}$ be its restriction to K. Assume the following conditions:*

(1) $\mathcal{C}$ factors through $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Pr}^{\mathrm{L}}) = \mathrm{Fun}(\Delta^1, \mathrm{Pr}^{\mathrm{L}}) \cap \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{CAT}_\infty)$;

(2) for every $s \in K$, the right adjoint to the functor $f_s : \mathcal{C}_0(s) \to \mathcal{C}_1(s)$, associated to s by $\mathcal{C}$, is colimit-preserving.

(Note that the second condition is satisfied if f_s is compact-preserving, and the ∞-categories $\mathcal{C}_0(s)$ and $\mathcal{C}_1(s)$ are stable and compactly generated.) Then, $\overline{\mathcal{C}}$ also factors through $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Pr}^{\mathrm{L}})$. Moreover, the resulting map $K^{\triangleright} \to \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Pr}^{\mathrm{L}})$ is a colimit diagram.

Proof. Using the equivalence $\mathrm{Pr}^{\mathrm{L}} \simeq (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}}$, we deduce a limit diagram

$$\overline{\mathcal{C}}' : (K^{\mathrm{op}})^{\triangleleft} \to \mathrm{Fun}(\Delta^{1,\mathrm{op}}, \mathrm{Pr}^{\mathrm{R}}).$$

We denote by $\mathcal{C}'$ the restriction of $\overline{\mathcal{C}}'$ to K^{op}. Applying $\overline{\mathcal{C}}$ and $\overline{\mathcal{C}}'$ to an edge $e : s \to t$ in $K^{\triangleright}$, we get the following commutative squares of ∞-categories

$$\overline{\mathcal{C}}(e) : \begin{array}{ccc} \mathcal{C}_0(s) & \xrightarrow{f_s} & \mathcal{C}_1(s) \\ \downarrow & & \downarrow \\ \mathcal{C}_0(t) & \xrightarrow{f_t} & \mathcal{C}_1(t) \end{array} \qquad \text{and} \qquad \overline{\mathcal{C}}'(e) : \begin{array}{ccc} \mathcal{C}_0(s) & \xleftarrow{g_s} & \mathcal{C}_1(s) \\ \uparrow & & \uparrow \\ \mathcal{C}_0(t) & \xleftarrow{g_t} & \mathcal{C}_1(t), \end{array}$$

where the functors in the second square are the right adjoints to the functors in the first square. By condition (2) the functors g_s admit right adjoints. Moreover, the first square $\overline{\mathcal{C}}(e)$ is right adjointable if and only if the square $\overline{\mathcal{C}}'(e)$ is right adjointable. We can then reformulate the problem as follows: if $\mathcal{C}'$ factors through

$$\mathrm{Fun}^{\mathrm{RAd}}(\Delta^{1,\mathrm{op}}, \mathrm{Pr}^{\mathrm{R}}) = \mathrm{Fun}(\Delta^{1,\mathrm{op}}, \mathrm{Pr}^{\mathrm{R}}) \cap \mathrm{Fun}^{\mathrm{RAd}}(\Delta^{1,\mathrm{op}}, \mathrm{CAT}_\infty),$$

then the same holds true for $\overline{\mathcal{C}}'$ and the resulting map is a limit diagram. Since limits in Pr^{R} are computed in CAT_∞ (by [Lur09, Theorem 5.5.3.18]), this follows from [Lur17, Corollary 4.7.4.18(2)]. □

Remark 2.7.3. Keep the notations and assumptions of Theorem 2.7.1. The commutative square

$$\begin{tikzcd}
\mathrm{Shv}^{(\wedge)}_\tau(X;\Lambda) \arrow[r, "f^*"] \arrow[d, "g^*"] & \mathrm{Shv}^{(\wedge)}_\tau(Y;\Lambda) \arrow[d, "g'^*"] \\
\mathrm{Shv}^{(\wedge)}_\tau(X';\Lambda) \arrow[r, "f'^*"] & \mathrm{Shv}^{(\wedge)}_\tau(Y';\Lambda)
\end{tikzcd}$$

is also right adjointable. This is proven by the same method: instead of using Theorem 2.5.1, we use the much easier Corollary 2.5.10. There is also an unstable version of this result, asserting that

$$\begin{tikzcd}
\mathrm{Shv}^{(\wedge)}_\tau(X) \arrow[r, "f^*"] \arrow[d, "g^*"] & \mathrm{Shv}^{(\wedge)}_\tau(Y) \arrow[d, "g'^*"] \\
\mathrm{Shv}^{(\wedge)}_\tau(X') \arrow[r, "f'^*"] & \mathrm{Shv}^{(\wedge)}_\tau(Y')
\end{tikzcd}$$

is right adjointable under some assumptions. This holds for instance when τ is the Nisnevich topology, and X, X', Y and Y' locally of finite Krull dimension. When τ is the étale topology, we have a weaker result: under the same assumption on the Krull dimensions, the base change morphism $g^* \circ f_* \to f'_* \circ g'^*$ is an isomorphism when evaluated at truncated étale sheaves and, in particular, at étale sheaves of sets. A proof of this can be obtained by adapting the proof of Theorem 2.7.1. Indeed, Corollary 2.5.10 is still true for the ∞-categories of n-truncated $\mathcal{S}$-valued sheaves $\mathrm{Shv}_\tau(-)_{\leq n}$. (In this case, there is no distinction between sheaves and hypersheaves.) Similarly, if $h : T \to S$ is a quasi-compact morphism between rigid analytic spaces locally of finite Krull dimension, the associated functor $h^* : \mathrm{Shv}_\tau(\acute{\mathrm{E}}\mathrm{t}/S)_{\leq n} \to \mathrm{Shv}_\tau(\acute{\mathrm{E}}\mathrm{t}/T)_{\leq n}$ belongs to $\mathrm{Pr}^{\mathrm{L}}_\omega$.

2.8. **Stalks.**

In this subsection, we determine under some mild hypotheses the stalks of $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(-;\Lambda)$, which is a τ-(hyper)sheaf by Theorem 2.3.4. We then use this to generalise Theorem 2.5.1. We start with a general fact on presheaves with values in a compactly generated ∞-category.

Proposition 2.8.1. *Let $(\mathcal{C},\tau)$ be a site having enough points and let $\mathcal{V}$ be a compactly generated ∞-category. For a morphism $f : \mathcal{F} \to \mathcal{G}$ in $\mathrm{PSh}(\mathcal{C};\mathcal{V})$, the following conditions are equivalent:*

(1) $\mathrm{L}_\tau(f) : \mathrm{L}_\tau(\mathcal{F}) \to \mathrm{L}_\tau(\mathcal{G})$ *is an equivalence in* $\mathrm{Shv}^\wedge_\tau(\mathcal{C};\mathcal{V})$*;*

(2) $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is an equivalence in $\mathcal{V}$ for all x in a conservative family of points of $(\mathcal{C},\tau)$.

Proof. By [Dre18, Proposition 2.5], condition (1) holds if and only if, for all compact objects $A \in \mathcal{V}$, the maps of presheaves of spaces

$$\mathrm{Map}_\mathcal{V}(A, f) : \mathrm{Map}_\mathcal{V}(A,\mathcal{F}) \to \mathrm{Map}_\mathcal{V}(A,\mathcal{G})$$

induce equivalences after τ-hypersheafification. This is the case if and only if for every x as in (2), the induced maps on stalks

$$\mathrm{Map}_\mathcal{V}(A, f)_x : \mathrm{Map}_\mathcal{V}(A,\mathcal{F})_x \to \mathrm{Map}_\mathcal{V}(A,\mathcal{G})_x.$$

are equivalences. Since the A's are compact and stalks are computed by filtered colimits, the above maps are equivalent to

$$\mathrm{Map}_\mathcal{V}(A, f_x) : \mathrm{Map}_\mathcal{V}(A,\mathcal{F}_x) \to \mathrm{Map}_\mathcal{V}(A,\mathcal{G}_x).$$

Since $\mathcal{V}$ is compactly generated and A varies among all compact objects, our condition is equivalent to asking that the maps $f_x : \mathcal{F}_x \to \mathcal{G}_x$ are equivalences as needed. ◻

Later we use Proposition 2.8.1 with $\mathcal{V} = \mathrm{Pr}^{\mathrm{L}}_{\omega}$. This is indeed possible by Proposition 2.8.4 below, whose proof relies on two technical lemmas. The first one is a variant of the characterisation of presentability given in [Lur09, Theorem 5.5.1.1(6)] which is certainly well-known. We provide an argument because we couldn't find a reference.

Lemma 2.8.2. *Let $\mathcal{C}$ be a locally small ∞-category admitting small colimits. Assume that there exists a regular cardinal κ and a set $S \subset \mathcal{C}$ of κ-compact objects such that $\mathcal{C}$ coincides with its smallest full sub-∞-category containing S and stable under colimits. Then $\mathcal{C}$ is κ-compactly generated (in the sense of* [Lur09, Definition 5.5.7.1]*).*

Proof. The difference with [Lur09, Theorem 5.5.1.1(6)] is that we do not assume that every object of $\mathcal{C}$ is a colimit of a diagram with values in the full sub-∞-category spanned by S.

Let $\mathcal{E} \subset \mathcal{C}$ be the smallest sub-∞-category of $\mathcal{C}$ containing S and stable under κ-small colimits. The ∞-category $\mathcal{E}$ can be constructed from S by transfinite induction as follows. Let $\mathcal{E}_0$ be the full sub-∞-category of $\mathcal{C}$ spanned by S and, for an ordinal $\nu > 0$, let $\mathcal{E}_\nu$ be the full sub-∞-category of $\mathcal{C}$ spanned by colimits of κ-small diagrams in $\bigcup_{\mu<\nu} \mathcal{E}_\mu$. Then $\mathcal{E} = \bigcup_{\nu<\kappa} \mathcal{E}_\nu$. This shows that $\mathcal{E}$ is essentially small and that every object of $\mathcal{E}$ is κ-compact (by [Lur09, Corollary 5.3.4.15]). By [Lur09, Proposition 5.3.5.11], the inclusion $\mathcal{E} \to \mathcal{C}$ extends uniquely to a functor $\phi : \mathrm{Ind}_\kappa(\mathcal{E}) \to \mathcal{C}$ preserving κ-filtered colimits, and this functor is fully faithful. In fact, by [Lur09, Proposition 5.3.6.2 and Example 5.3.6.8], $\mathrm{Ind}_\kappa(\mathcal{E})$ admits small colimits and the functor ϕ is colimit-preserving. Using that the essential image of ϕ contains S, we deduce that ϕ is an equivalence of ∞-categories. Since $\mathrm{Ind}_\kappa(\mathcal{E})$ is presentable by [Lur09, Theorem 5.5.1.1], this finishes the proof. (Note that $\mathrm{Ind}_\kappa(\mathcal{E})$ is κ-accessible by definition, see [Lur09, Definition 5.4.2.1].) □

Lemma 2.8.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories such that $\mathcal{C}$ is compactly generated and $\mathcal{D}$ admits small colimits. Assume that there is a functor $G : \mathcal{D} \to \mathcal{C}$ with the following properties:*

(1) it admits a left adjoint;
(2) it is conservative;
(3) it commutes with filtered colimits.

Then $\mathcal{D}$ is compactly generated. Moreover, if F is a left adjoint to G, then F takes a set of compact generators of $\mathcal{C}$ to a set of compact generators of $\mathcal{D}$.

Proof. Since G commutes with filtered colimits, the functor F takes a compact object of $\mathcal{C}$ to a compact object of $\mathcal{D}$. Let $\mathcal{C}_0$ be the full sub-∞-category of $\mathcal{C}$ spanned by compact objects, and let $\mathcal{D}' \subset \mathcal{D}$ be the smallest sub-∞-category containing $F(\mathcal{C}_0)$ and stable under colimits. By Lemma 2.8.2, $\mathcal{D}'$ is compactly generated since $\mathcal{C}_0$ is essentially small. Thus, it suffices to show that the inclusion functor $U : \mathcal{D}' \to \mathcal{D}$ is an equivalence. By [Lur09, Corollary 5.5.2.9 & Remark 5.5.2.10], the functor U admits a right adjoint V and it is enough to show that V is conservative. This follows from the hypothesis that G is conservative. Indeed, we have $G \simeq G' \circ V$ where G' is right adjoint to the functor $F' : \mathcal{C} \to \mathcal{D}'$ induced by F (which exists by [Lur09, Corollary 5.5.2.9]). □

Proposition 2.8.4. *The ∞-category $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ is compactly generated.*

Proof. This is probably well-known, but we couldn't find a reference. We include a proof here for completeness. Denote by $\mathrm{Cat}_\infty^{\mathrm{rex,\,idem}}$ the sub-∞-category of Cat_∞ whose objects are the idempotent complete small ∞-categories admitting finite colimits and whose morphisms are the right exact functors. By [Lur17, Lemma 5.3.2.9(1)], the functor $\mathcal{C} \mapsto \mathrm{Ind}_\omega(\mathcal{C})$ induces an equivalence of ∞-categories between $\mathrm{Cat}_\infty^{\mathrm{rex,\,idem}}$ and $\mathrm{Pr}^{\mathrm{L}}_{\omega}$. Thus, it is enough to show that $\mathrm{Cat}_\infty^{\mathrm{rex,\,idem}}$ is compactly generated. Since $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ admits small colimits by [Lur09, Proposition 5.5.7.6], the same is true for $\mathrm{Cat}_{\infty}^{\mathrm{rex,\,idem}}$ which is moreover obviously locally small.

We will show that $\mathrm{Cat}_{\infty}^{\mathrm{rex,\,idem}}$ is compactly generated by applying Lemma 2.8.3 to the inclusion functor $\mathrm{Cat}_{\infty}^{\mathrm{rex,\,idem}} \to \mathrm{Cat}_{\infty}$. First, note that Cat_{∞} is compactly generated. Indeed, Cat_{∞} is the ∞-category associated to the combinatorial simplicial model category $\mathrm{Set}^{+}_{\Delta}$ of marked simplicial sets where the cofibrations are generated by monomorphisms with compact domain and codomain, and where fibrant objects are stable by filtered colimits. (See [Lur09, Propositions 3.1.3.7 & 3.1.4.1, & Theorem 3.1.5.1].) We now check that the inclusion functor $\mathrm{Cat}_{\infty}^{\mathrm{rex,\,idem}} \to \mathrm{Cat}_{\infty}$ satisfies properties (1)–(3) of Lemma 2.8.3. Property (1) follows from [Lur09, Corollary 5.3.6.10]. Property (2) is obvious: an inverse of a right exact equivalence of ∞-categories is right exact. For property (3), we need to show the following: given a filtered diagram in $\mathrm{Cat}_{\infty}^{\mathrm{rex,\,idem}}$, its colimit computed in Cat_{∞} admits finite colimits and is idempotent complete. The first property follows from [Lur09, Proposition 5.5.7.11]. The second property follows from [Lur09, Corollary 4.4.5.21].[7] □

We record the following lemma for later use.

Lemma 2.8.5. *Let $(\mathcal{C}, \tau)$ be a site and let $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{CAT}_{\infty}$ be a presheaf on $\mathcal{C}$. Set $\mathcal{E} = \lim_{\mathcal{C}^{\mathrm{op}}} \mathcal{F}$. (If $\mathcal{C}$ admits a final object $\star$, then $\mathcal{E} \simeq \mathcal{F}(\star)$.) Given an object $X \in \mathcal{C}$, we denote by $A \mapsto A_X$ the obvious functor $\mathcal{E} \to \mathcal{F}(X)$.*

(1) Assume that $\mathcal{F}$ is a τ-(hyper)sheaf. Then, for $A, B \in \mathcal{E}$, the presheaf on $\mathcal{C}$, given informally by $X \mapsto \mathrm{Map}_{\mathcal{F}(X)}(A_X, B_X)$, is a τ-(hyper)sheaf.

(2) Assume that $\mathcal{F}$ is a τ-hypersheaf and that the limit diagram $(\mathcal{C}^{\triangleright})^{\mathrm{op}} \to \mathrm{CAT}_{\infty}$ extending $\mathcal{F}$ factors through $\mathrm{Pr}^{\mathrm{L}}_{\omega}$. Assume also that $(\mathcal{C}, \tau)$ admits a conservative family of points $(x_i)_i$. Then, the family of functors $(\mathcal{E} \to \mathcal{F}_{x_i})_i$, where the stalks $\mathcal{F}_{x_i}$ are computed in $\mathrm{Pr}^{\mathrm{L}}_{\omega}$, is conservative.

Proof. We denote by $M : (\mathrm{CAT}_{\infty})_{\partial\Delta^1/} \to \mathcal{S}$ the copresheaf corepresented by $\partial\Delta^1 \to \Delta^1$. The functor M commutes with limits and admits the following informal description. It sends an ∞-category $\mathcal{Q}$ together with a functor $q : \partial\Delta^1 \to \mathcal{Q}$ to the mapping space $\mathrm{Map}_{\mathcal{Q}}(q(0), q(1))$. This is indeed a consequence of [DS11, Proposition 1.2].

To give a precise construction of the presheaf described informally in (1), we consider $\mathcal{E}$ as an object of $(\mathrm{CAT}_{\infty})_{\partial\Delta^1/}$ using the functor $e : \partial\Delta^1 \to \mathcal{E}$ mapping 0 to A and 1 to B. By the definition of $\mathcal{E}$, the presheaf $\mathcal{F}$ lifts to a $(\mathrm{CAT}_{\infty})_{\mathcal{E}/}$-valued presheaf $\mathcal{F}'$. The functor e gives rise to a functor

$$(\mathrm{CAT}_{\infty})_{\mathcal{E}/} \to (\mathrm{CAT}_{\infty})_{\partial\Delta^1/}$$

and we denote by $\mathcal{F}''$ the $(\mathrm{CAT}_{\infty})_{\partial\Delta^1/}$-valued presheaf obtained from $\mathcal{F}'$ by composing with this functor. By construction, $\mathcal{F}''$ is a lift of $\mathcal{F}$ admitting the following informal description. It sends an object $X \in \mathcal{C}$ to the ∞-category $\mathcal{F}(X)$ together with the functor $\partial\Delta^1 \to \mathcal{F}(X)$ mapping 0 to A_X and 1 to B_X. The presheaf $X \mapsto \mathrm{Map}_{\mathcal{F}(X)}(A_X, B_X)$ in (1) is then defined to be $M \circ \mathcal{F}''$. That said, the conclusion of assertion (1) is now clear. Indeed, the projection $(\mathrm{CAT}_{\infty})_{\partial\Delta^1/} \to \mathrm{CAT}_{\infty}$ preserves and detects limits by [Lur09, Proposition 1.2.13.8] and, as mentioned above, the functor M is limit-preserving. Thus, the conclusion follows from Remark 2.3.3(1).

Given a point x of $(\mathcal{C}, \tau)$, we denote by $A \mapsto A_x$ the functor $\mathcal{E} \to \mathcal{F}_x$. To prove the second assertion, we fix a morphism $f : A \to B$ in $\mathcal{E}$ inducing equivalences $A_{x_i} \simeq B_{x_i}$ for all i. We need to prove that f is an equivalence. Since $\mathcal{E}$ is compactly generated, it is enough to show that f induces

[7] Corollary 4.4.5.21 can be found in the electronic version of [Lur09] on the author's webpage, but not in the published version.

an equivalence $\mathrm{Map}_{\mathcal{E}}(C, A) \to \mathrm{Map}_{\mathcal{E}}(C, B)$ for every compact object $C \in \mathcal{E}$. The compositions with the f_X's, for $X \in \mathcal{C}$, induce a morphism of presheaves

$$(X \mapsto \mathrm{Map}_{\mathcal{F}(X)}(C_X, A_X)) \to (X \mapsto \mathrm{Map}_{\mathcal{F}(X)}(C_X, B_X)), \tag{2.42}$$

whose construction we leave to the reader. By assertion (1), this is actually a morphism of τ-hypersheaves. Thus, to conclude, it is enough to show that the morphism (2.42) induces equivalences on stalks at x_i for every i. Since C is compact, the stalk at x_i of this morphism is given by the map $\mathrm{Map}_{\mathcal{F}_{x_i}}(C_{x_i}, A_{x_i}) \to \mathrm{Map}_{\mathcal{F}_{x_i}}(C_{x_i}, B_{x_i})$ which is indeed an equivalence since $A_{x_i} \simeq B_{x_i}$. □

By Theorem 2.3.4, the Pr^{L}-valued presheaf $\mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\tau}(-;\Lambda)$ has τ-hyperdescent. Therefore, it is particularly useful to determine its stalks. The next theorem shows that, under some mild hypotheses, these stalks can also be understood as ∞-categories of rigid analytic motives over rigid points (in the sense of Definition 1.4.22).

Theorem 2.8.6. *Let S be a rigid analytic space and let $\overline{s} \to S$ be an algebraic rigid point of S. (See Remark 1.4.25.) Let $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$, and assume one of the following two alternatives.*

(1) We work in the non-hypercomplete case.

(2) We work in the hypercomplete case and S is (Λ, τ)-admissible.

Then there is an equivalence of ∞-categories

$$\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)_{\overline{s}} \simeq \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(\overline{s};\Lambda),$$

where the left-hand side is the stalk of $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)$ at $\overline{s}$, i.e., the colimit, taken in Pr^{L}, of the diagram $(\overline{s} \to U \to S) \mapsto \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(U;\Lambda)$ with $U \in \mathrm{\acute{E}t}/S$.

Proof. We need to show that the obvious functor

$$\operatorname*{colim}_{\overline{s} \to U \to S} \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(U;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(\overline{s};\Lambda)$$

is an equivalence. The question being local on S around the image of $\overline{s}$, we may assume that S is quasi-compact and quasi-separated. In particular, S admits a formal model $\mathcal{S}$. The functor

$$(\mathrm{Spf}(\kappa^+(\overline{s})) \to \mathcal{U} \to \mathcal{S}) \mapsto (\overline{s} \to \mathcal{U}^{\mathrm{rig}} \to S),$$

with $\mathcal{U}$ affine and rig-étale over $\mathcal{S}$, is cofinal. Moreover, by Lemma 1.4.26, we have a canonical isomorphism of formal schemes

$$\mathrm{Spf}(\kappa^+(\overline{s})) \simeq \lim_{\mathrm{Spf}(\kappa^+(\overline{s})) \to \mathcal{U} \to \mathcal{S}} \mathcal{U}.$$

The result follows now from Theorem 2.5.1. Indeed, if S is (Λ, τ)-admissible then so are $\overline{s}$ and every étale rigid analytic S-space U. (For $\overline{s}$, use that the absolute Galois group of $\kappa(\overline{s})$ is a closed subgroup of the absolute Galois group of $\kappa(s)$; for U, use Corollary 2.4.17.) Moreover, by the proof of Lemma 2.4.16, we have the inequality $\mathrm{pvcd}_{\Lambda}(U) \leq \mathrm{pvcd}_{\Lambda}(S)$, and, since S is quasi-compact, the (Λ, τ)-admissibility of S implies that $\mathrm{pvcd}_{\Lambda}(S)$ is finite. □

Remark 2.8.7. Theorem 2.8.6 applies in the case of a rigid point $s \to S$ associated to a point $s \in |S|$. In this case, the stalk $\mathbf{RigSH}^{(\mathrm{eff},\tau)}_{\tau}(-;\Lambda)_s$ has a simpler description: it is the colimit, taken in Pr^{L}, of the diagram $U \mapsto \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(U;\Lambda)$, where U runs over the open neighbourhoods $U \subset S$ of s. Indeed, every étale neighbourhood $s \to T \to S$ of s in S can be refined by an open neighbourhood. (This follows from Corollary 1.3.10 and Lemma 1.4.26(1).) Similarly, if $\overline{s} \to S$ is a nis-geometric rigid point as in Construction 1.4.27(1), we may restrict in the description of the stalk in Theorem 2.8.6 to those étale neighbourhoods U admitting good reduction.

Corollary 2.8.8. *Let S be a rigid analytic space. Assume one of the following two alternatives.*

(1) We work in the non-hypercomplete case, and S is locally of finite Krull dimension. When τ is the étale topology, we assume furthermore that Λ is eventually coconnective

(2) We work in the hypercomplete case, and S is (Λ, τ)-admissible.

Then, the functors

$$\mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}(S;\Lambda) \to \mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}(s;\Lambda),$$

for $s \in S$, are jointly conservative.

Proof. Let Op/S denote the category of open subspaces of S endowed with the analytic topology. By Theorem 2.3.4, $\mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}(-;\Lambda)$ is a hypersheaf on Op/S. (In the non-hypercomplete case, we use [CM21, Theorem 3.12] and [Lur09, Corollary 7.2.1.12] which insure that a sheaf on Op/S is automatically a hypersheaf.) Moreover, by Proposition 2.4.20, this presheaf takes values in $\mathrm{Pr}^{\mathrm{L}}_{\omega}$. The result follows now from Lemma 2.8.5 and Theorem 2.8.6. □

Remark 2.8.9. The algebraic analogue of Corollary 2.8.8 is also true: given a scheme S and assuming one of the alternatives of this corollary, the functors $\mathbf{SH}_{\tau}^{(\mathrm{eff},\wedge)}(S;\Lambda) \to \mathbf{SH}_{\tau}^{(\mathrm{eff},\wedge)}(s;\Lambda)$, for $s \in |S|$, are jointly conservative. This can be deduced from Proposition 2.2.3 by arguing as in the proof of [Hoy18, Corollary 14].

Our next goal is to upgrade Theorem 2.5.1 to a motivic analogue of [Hub96, Proposition 2.4.4]; see Theorem 2.8.15 below. We first introduce, following [Hub96, Definition 2.4.2 & Remark 2.4.5], a notion of weak limit in the category of rigid analytic spaces.

Definition 2.8.10. Let $(S_\alpha)_\alpha$ be a cofiltered inverse system of rigid analytic spaces, with quasi-compact and quasi-separated transition maps. Let S be a rigid analytic space endowed with a map of pro-objects $(f_\alpha)_\alpha : S \to (S_\alpha)_\alpha$, i.e., with an element $(f_\alpha)_\alpha \in \lim_\alpha \mathrm{Hom}(S, S_\alpha)$. We say that S is a weak limit of $(S_\alpha)_\alpha$ and write $S \sim \lim_\alpha S_\alpha$ if the following two conditions are satisfied:

(1) the map $|S| \to \lim_\alpha |S_\alpha|$ is a homeomorphism;

(2) for every $s \in |S|$ with images $s_\alpha \in |S_\alpha|$, the morphism

$$\operatorname*{colim}_{\alpha} \kappa^+(s_\alpha) \to \kappa^+(s),$$

where the colimit is taken in the category of adic rings, is an isomorphism.

Example 2.8.11. Let $(\mathcal{S}_\alpha)_\alpha$ be a cofiltered inverse system of formal schemes with affine transition maps and let $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$ be its limit. Set $S = \mathcal{S}^{\mathrm{rig}}$ and $S_\alpha = \mathcal{S}_\alpha^{\mathrm{rig}}$. Then S is a weak limit of $(S_\alpha)_\alpha$. Indeed, condition (1) follows from commutation of limits with limits, see Notation 1.1.11. The point is that any admissible blowup of $\mathcal{S}$ can be obtained as the strict transform of $\mathcal{S}$ with respect to an admissible blowup of an $\mathcal{S}_\alpha$ for some α. Condition (2) follows from Lemma 1.4.26(1).

Example 2.8.12. Let X be a rigid analytic space and $Z \subset X$ a closed subspace. Let $(U_\alpha)_\alpha$ be an inverse system of open neighbourhoods of Z in X such that, locally at every point of Z, this inverse system is cofinal in the system of all neighbourhoods of Z in X. (When X is quasi-compact, this is equivalent to saying that $(U_\alpha)_\alpha$ is cofinal in the system of all neighbourhoods of Z in X.) Then, Z is a weak limit of $(U_\alpha)_\alpha$. Indeed, condition (2) is obvious and, for condition (1), we need to show that $|Z| = \bigcap_\alpha |U_\alpha|$. This follows easily from the fact that $|X|$ is a valuative topological space (in the sense of [FK18, Chapter 0, Definition 2.3.1]) and that $|Z| \subset |X|$ is stable by generisation.

The following lemma can be compared with [Hub96, Remark 2.4.3(i)]. See also the proof of [Sch12, Proposition 7.16].

Lemma 2.8.13. *Keep the notation as in Definition 2.8.10 and consider the following variants of conditions (1) and (2):*

(1′) the f_α's are quasi-compact and quasi-separated, and the map $|S| \to \lim_\alpha |S_\alpha|$ is a bijection;
(2′) for every $s \in |S|$ with images $s_\alpha \in |S_\alpha|$, the induced morphism of fields

$$\operatorname*{colim}_\alpha \kappa(s_\alpha) \to \kappa(s)$$

has dense image.

Then, conditions (2) and (2′) are equivalent. Moreover, if condition (2) is satisfied, then conditions (1) and (1′) are equivalent.

Proof. We identify $\kappa^+(s_\alpha)$ with a subring of $\kappa^+(s)$ and $\kappa(s_\alpha)$ with a subfield of $\kappa(s)$. We may assume that there is an element $\pi \in \kappa^+(s)$ which belongs to all the $\kappa^+(s_\alpha)$'s and generates an ideal of definition in each one of them. If (2) is satisfied, then $\kappa^+(s)$ is the π-adic completion of $\bigcup_\alpha \kappa^+(s_\alpha)$, which implies that $\bigcup_\alpha \kappa(s_\alpha)$ is dense in $\kappa(s)$. Conversely, if (2′) is satisfied, then $\kappa^+(s)$ is the Hausdorff completion of $\kappa^+(s) \cap \bigcup_\alpha \kappa(s_\alpha)$. Then condition (2) follows from the following equalities $\pi^n\kappa^+(s) \cap \bigcup_\alpha \kappa(s_\alpha) = \bigcup_\alpha \pi^n\kappa^+(s_\alpha)$ which are easily checked using the valuation on $\kappa(s)$.

Clearly, (1) implies (1′). We next assume that (2) is satisfied, and show that (1′) implies (1). Using that the f_α's and the transition morphisms of the inverse system $(S_\alpha)_\alpha$ are quasi-compact and quasi-separated, we may reduce to the case where S and all the S_α's are quasi-compact and quasi-separated. By [Sta20, Lemma 09XU], it is then enough to show that the bijection $|S| \simeq \lim_\alpha |S_\alpha|$ detects generisations. Given $s \in |S|$ with images $s_\alpha \in |S_\alpha|$, the generisations of s are the points of $\mathrm{Spf}(\kappa^+(s))$ while the generisations of $(s_\alpha)_\alpha$ are the points of $\lim_\alpha \mathrm{Spf}(\kappa^+(s_\alpha))$. Thus, condition (2) implies that f induces a bijection between the generisations of s and those of $(s_\alpha)_\alpha$. □

The following can be compared with [Hub96, Remark 2.4.3(ii)] and [Sch12, Proposition 7.16].

Lemma 2.8.14. *Let $(S_\alpha)_\alpha$ be a cofiltered inverse system of rigid analytic spaces, with quasi-compact and quasi-separated transition maps, and admitting a weak limit S. Let X be a rigid analytic S_{α_0}-space for some index α_0. Then $X \times_{S_{\alpha_0}} S$ is a weak limit of $(X \times_{S_{\alpha_0}} S_\alpha)_{\alpha \le \alpha_0}$.*

Proof. We reduce easily to the case where S, the S_α's and X are quasi-compact and quasi-separated. We will check that condition (1′) of Lemma 2.8.13 and condition (2) of Definition 2.8.10 are satisfied by the maps $X \times_{S_{\alpha_0}} S \to X \times_{S_{\alpha_0}} S_\alpha$, for $\alpha \le \alpha_0$. A point of $|X \times_{S_{\alpha_0}} S|$ corresponds to a point $s \in |S|$ and a point of $|X \times_{S_{\alpha_0}} s|$ mapping to the closed point of $|s|$. Using a similar description for the points of the $|X \times_{S_{\alpha_0}} S_\alpha|$'s, condition (1′) and (2) follow from the following assertion: given $s \in |S|$ with images $s_\alpha \in |S_\alpha|$, $X \times_{S_{\alpha_0}} s$ is a weak limit of $(X \times_{S_{\alpha_0}} s)_{\alpha \le \alpha_0}$. To prove this assertion, choose a formal model $\mathcal{X} \to \mathcal{S}_{\alpha_0}$ of $X \to S_{\alpha_0}$ and use Example 2.8.11 and the isomorphism of formal schemes $\mathcal{X} \times_{\mathcal{S}_{\alpha_0}} \mathrm{Spf}(\kappa^+(s)) \simeq \lim_{\alpha \le \alpha_0} \mathcal{X} \times_{\mathcal{S}_{\alpha_0}} \mathrm{Spf}(\kappa^+(s_\alpha))$. □

Theorem 2.8.15. *Let $(S_\alpha)_\alpha$ be a cofiltered inverse system of rigid analytic spaces, with quasi-compact and quasi-separated transition maps, and admitting a weak limit S. Let $\tau \in \{\mathrm{nis}, \text{ét}\}$, and assume one of the following two alternatives.*

(1) We work in the non-hypercomplete case, and S and the S_α's are locally of finite Krull dimension. When τ is the étale topology, we assume furthermore that Λ is eventually coconnective.
(2) We work in the hypercomplete case, and S and the S_α's are (Λ, τ)-admissible (see Definition 2.4.14). When τ is the étale topology, we assume furthermore that Λ is eventually

coconnective or that, for every $s \in |S|$ with images $s_\alpha \in |S_\alpha|$, the Λ-cohomological dimensions of the residue fields $\kappa(s_\alpha)$ are bounded independently of α.

Then, the obvious functor

$$\operatorname*{colim}_{\alpha} \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(S_\alpha;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(S;\Lambda), \tag{2.43}$$

where the colimit is taken in Pr^{L}*, is an equivalence.*

Proof. Let $U_{\alpha_0,\bullet} \to S_{\alpha_0}$ be a hypercover of S_{α_0} in the analytic topology with $U_{\alpha_0,n}$ a disjoint union of a family $(U_{\alpha_0,n,i})_{i\in I_n}$ of open subspaces of S_{α_0}. Set $U_{\alpha,n,i} = U_{\alpha_0,n,i} \times_{S_{\alpha_0}} S_\alpha$ and $U_{n,i} = U_{\alpha_0,n,i} \times_{S_{\alpha_0}} S$. We have hypercovers $U_{\alpha,\bullet} \to S_\alpha$ and $U_\bullet \to S$ with $U_{\alpha,n} = \coprod_{i\in I_n} U_{\alpha,n,i}$ and similarly for U_n. By [Lur17, Proposition 4.7.4.19], there is an equivalence of ∞-categories

$$\operatorname*{colim}_{\alpha} \lim_{[n]\in\mathbf{\Delta}} \prod_{i\in I_n} \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(U_{\alpha,n,i};\Lambda) \simeq \lim_{[n]\in\mathbf{\Delta}} \prod_{i\in I_n} \operatorname*{colim}_{\alpha} \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(U_{\alpha,n,i};\Lambda). \tag{2.44}$$

The right adjointability of the squares that is needed for [Lur17, Proposition 4.7.4.19] holds by the base change theorem for open immersions, which is a special case of Proposition 2.2.1(3). The presheaf $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)$ admits descent for the hypercovers $U_\bullet \to S$ and $U_{\alpha,\bullet} \to S_\alpha$ by Theorem 2.3.4. (In the non-hypercomplete case, we use the assumption that S and the S_α's have locally finite Krull dimension so that descent implies hyperdescent by [CM21, Theorem 3.12] and [Lur09, Corollary 7.2.1.12].) Therefore, the equivalence (2.44) shows that it is enough to prove the theorem for the inverse systems $(U_{\alpha,n,i})_{\alpha\le\alpha_0}$. In particular, we may assume that the S_α's are quasi-compact and quasi-separated.

Denote by $\mathrm{Op}^{\mathrm{qcqs}}/S$ the category of quasi-compact and quasi-separated open subspaces of S, and similarly for other rigid analytic spaces. Given that $\mathrm{Op}^{\mathrm{qcqs}}/S = \operatorname{colim}_\alpha \mathrm{Op}^{\mathrm{qcqs}}/S_\alpha$, there exists a Pr^{L}-valued presheaf $\mathcal{R}$ on $\mathrm{Op}^{\mathrm{qcqs}}/S$ given by

$$\mathcal{R}(U) = \operatorname*{colim}_{\alpha\ge\alpha_0} \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(U_\alpha;\Lambda)$$

for any $U_{\alpha_0} \in \mathrm{Op}^{\mathrm{qcqs}}/S_{\alpha_0}$ such that $U = U_{\alpha_0} \times_{S_{\alpha_0}} S$. (As usual, we set $U_\alpha = U_{\alpha_0} \times_{S_{\alpha_0}} S_\alpha$.) Moreover, we have a morphism of Pr^{L}-valued presheaves

$$\phi : \mathcal{R} \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)$$

on $\mathrm{Op}^{\mathrm{qcqs}}/S$. Since S belongs to $\mathrm{Op}^{\mathrm{qcqs}}/S$, it suffices to show that ϕ is an equivalence of presheaves. We will achieve this by showing the following two properties:

(1) $\mathcal{R}$ and $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)$ are hypersheaves on $\mathrm{Op}^{\mathrm{qcqs}}/S$ for the analytic topology;

(2) ϕ induces an equivalence on stalks for the analytic topology at every point $s \in |S|$.

This suffices indeed by Propositions 2.8.1 and 2.8.4, since the presheaves $\mathcal{R}$ and $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)$ on $\mathrm{Op}^{\mathrm{qcqs}}/S$ take values in $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ by Proposition 2.4.22.

First, we prove (1). That $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)$ is a hypersheaf on $\mathrm{Op}^{\mathrm{qcqs}}/S$ was mentioned above. To handle the case of $\mathcal{R}$, we use again [CM21, Theorem 3.12] and [Lur09, Corollary 7.2.1.12] which insure that a sheaf on $\mathrm{Op}^{\mathrm{qcqs}}/S$ is automatically a hypersheaf. Thus, it is enough to show that $\mathcal{R}$ admits descent for truncated hypercovers $U_\bullet$ in $\mathrm{Op}^{\mathrm{qcqs}}/S$. We may assume that $U_{-1} = S$. Every such hypercover, is the inverse image of a truncated hypercover $U_{\alpha_0,\bullet}$ with $U_{\alpha_0,-1} = S_{\alpha_0}$. We may then use the equivalence (2.44) to conclude.

Next, we prove (2). Fix $s \in |S|$ with images $s_\alpha \in |S_\alpha|$. Since every quasi-compact and quasi-separated open neighbourhood of s is the inverse image of a quasi-compact and quasi-separated open neighbourhood of s_α, for α small enough, the functor ϕ_s can be rewritten as follows:

$$\operatorname*{colim}_{\alpha} \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)_{s_\alpha} \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)_{s}.$$

Using Theorem 2.8.6 (and Remark 2.8.7), this functor is equivalent to

$$\operatorname*{colim}_{\alpha} \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(s_\alpha;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(s;\Lambda).$$

By Theorem 2.5.1, the latter is an equivalence. □

2.9. (Semi-)separatedness.

In this subsection, we discuss two basic properties of the functor $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\Lambda)$, namely semi-separatedness and separatedness.

Definition 2.9.1. Let $e : X' \to X$ be a morphism of rigid analytic spaces.

1. We say that e is radicial if $|e| : |X'| \to |X|$ is injective and, for every $x' \in |X'|$ with image $x \in |X|$, the residue field $\kappa(x')$ contains a dense purely inseparable extension of $\kappa(x)$.
2. We say that e is a universal homeomorphism if it is quasi-compact, quasi-separated, surjective and radicial. (See Remark 2.9.2 below.)

Remark 2.9.2.

1. Radicial morphisms and universal homeomorphisms are stable under base change.
2. If $e : X' \to X$ is a universal homeomorphism, then $|e| : |X'| \to |X|$ is a quasi-compact and quasi-separated bijection which detects generisation. By [Sta20, Lemma 09XU], this implies that $|e| : |X'| \to |X|$ is a homeomorphism of topological spaces. Moreover, by (1), this property is preserved by base change, which explains our terminology.
3. A morphism of schemes $e : X' \to X$ is called a universal homeomorphism if every base change of e induces a homeomorphism on the underlying topological spaces. By [Gro67, Chapitre IV, Corollaire 18.12.13], this is equivalent to saying that e is entire, surjective and radicial.

Lemma 2.9.3. *Let $e : X' \to X$ be a universal homeomorphism of rigid analytic spaces. The induced morphism $e : (\acute{\mathrm{E}}\mathrm{t}/X', \tau) \to (\acute{\mathrm{E}}\mathrm{t}/X, \tau)$ is an equivalence of sites, i.e., induces an equivalence between the associated ordinary topoi, for $\tau \in \{\mathrm{an}, \mathrm{nis}, \acute{\mathrm{e}}\mathrm{t}\}$. In particular, we have an equivalence of ∞-categories $\mathrm{Shv}^{(\wedge)}_{\tau}(\acute{\mathrm{E}}\mathrm{t}/X';\Lambda) \simeq \mathrm{Shv}^{(\wedge)}_{\tau}(\acute{\mathrm{E}}\mathrm{t}/X;\Lambda)$.*

Proof. The second assertion follows from the first one using Lemma 2.1.4. To prove the first assertion, we need to show that the unit $\mathrm{id} \to e_*e^*$ and counit $e^*e_* \to \mathrm{id}$ are equivalences on τ-sheaves of sets (i.e., on discrete τ-sheaves). For $x \in |X|$, we have a morphism of sites $(\acute{\mathrm{E}}\mathrm{t}/x, \tau) \to (\acute{\mathrm{E}}\mathrm{t}/X, \tau)$, and we denote by x^* the associated inverse image functor. Then, the functors x^*, for $x \in |X|$, are jointly conservative on τ-sheaves of sets. The same discussion is equally valid for points of X'. Thus, we are left to show that the natural transformations $x^* \to x^*e_*e^*$ and $x'^*e^*e_* \to x'^*$ are equivalences on τ-sheaves of sets for all $x \in |X|$ and $x' \in |X'|$. Assuming that x is the image of x', these natural transformations are equivalent to $x^* \to e_{x,*}e_x^*x^*$ and $e_x^*e_{x,*}x'^* \to x'^*$, where $e_x : x' \to x$ is the obvious morphism. This follows from Remark 2.7.3 and the fact that the morphism $x' \to X' \times_X x$ identifies x' with $(X' \times_X x)_{\mathrm{red}}$. Thus, we are reduced to prove the lemma for rigid points. Since $\kappa(x')$ contains a dense purely inseparable extension of $\kappa(x)$, the functor $\acute{\mathrm{E}}\mathrm{t}/x \to \acute{\mathrm{E}}\mathrm{t}/x'$ is an equivalence of categories which respects the analytic, Nisnevich and étale topologies. □

Remark 2.9.4. Lemma 2.9.3 admits a variant for universal homeomorphisms of schemes which is well-known, see [SGAIV2, Exposé VIII, Théorème 1.1].

Corollary 2.9.5. *Let $e : S' \to S$ be a universal homeomorphism of rigid analytic spaces. Then, for $\tau \in \{\text{nis}, \text{ét}\}$, we have a coCartesian square in* Pr^{L}

$$\begin{tikzcd} \mathbf{RigSH}^{(\mathrm{eff})}_{\mathrm{nis}}(S;\Lambda) \arrow[r, "e^*"] \arrow[d] & \mathbf{RigSH}^{(\mathrm{eff})}_{\mathrm{nis}}(S';\Lambda) \arrow[d] \\ \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(S;\Lambda) \arrow[r, "e^*"] & \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(S';\Lambda). \end{tikzcd}$$

Said differently, $\mathbf{RigSH}^{(\mathrm{eff})}_{\mathrm{nis}}(S';\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(S';\Lambda)$ is a localisation functor with respect to the image by e^* of morphisms of the form $\mathrm{colim}_{[n]\in\mathbf{\Delta}}\mathrm{M}^{(\mathrm{eff})}(U_\bullet) \to \mathrm{M}^{(\mathrm{eff})}(U_{-1})$, and their desuspensions and negative Tate twists when applicable, with $U_\bullet$ a τ-hypercover in RigSm/S which we assume to be truncated in the non-hypercomplete case.

Proof. Using Remark 2.1.17, one reduces easily to the effective case. From the construction, one sees immediately that $\mathbf{RigSH}^{\mathrm{eff}}_{\mathrm{nis}}(S';\Lambda) \to \mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\tau}(S';\Lambda)$ is the localisation functor with respect to morphisms of the form $\alpha'^*_n\mathcal{F}' \to \alpha'^*_n\mathcal{G}'$ where:

- $\alpha'_n : (\mathrm{RigSm}/S',\tau) \to (\text{Ét}/\mathbb{B}^n_{S'},\tau)$ is the premorphism of sites given by the obvious functor;
- $\mathcal{F}' \to \mathcal{G}'$ is a morphism in $\mathrm{Shv}_{\mathrm{nis}}(\text{Ét}/\mathbb{B}^n_{S'};\Lambda)$ inducing an equivalence in $\mathrm{Shv}^{(\wedge)}_{\tau}(\text{Ét}/\mathbb{B}^n_{S'};\Lambda)$.

For example, $\mathcal{F}' \to \mathcal{G}'$ could be $\mathrm{colim}_{[n]\in\mathbf{\Delta}}\Lambda_{\mathrm{nis}}(U'_\bullet) \to \Lambda_{\mathrm{nis}}(U'_{-1})$ with $U'_\bullet$ a τ-hypercover in $(\text{Ét}/\mathbb{B}^n_{S'},\tau)$ which is truncated in the non-hypercomplete case. The result follows now from the commutative square

$$\begin{tikzcd} \mathrm{Shv}_{\mathrm{nis}}(\text{Ét}/\mathbb{B}^n_S;\Lambda) \arrow[r, "e^*", "\sim"'] \arrow[d, "\alpha^*_n"'] & \mathrm{Shv}_{\mathrm{nis}}(\text{Ét}/\mathbb{B}^n_{S'};\Lambda) \arrow[d, "\alpha'^*_n"] \\ \mathrm{Shv}_{\mathrm{nis}}(\mathrm{RigSm}/S;\Lambda) \arrow[r, "e^*"] & \mathrm{Shv}_{\mathrm{nis}}(\mathrm{RigSm}/S';\Lambda) \end{tikzcd}$$

and Lemma 2.9.3 which insures that the upper horizontal arrow is an equivalence of ∞-categories respecting τ-local equivalences (in both the hypercomplete and non-hypercomplete cases). □

Theorem 2.9.6 (Semi-separatedness)**.** *Let $\tau \in \{\text{nis}, \text{ét}\}$. Let $e : X' \to X$ be a universal homeomorphism of rigid analytic spaces. Assume that X has locally finite Krull dimension. Assume also that every prime number is invertible in either $\mathcal{O}_X$ or $\pi_0\Lambda$. Then the functor*

$$e^* : \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda) \to \mathbf{RigSH}^{(\wedge)}_{\tau}(X';\Lambda)$$

is an equivalence of ∞-categories.

Proof. By Corollary 2.9.5, we may assume that τ is the Nisnevich topology. Since X and X' are locally of finite Krull dimension, we are automatically working in the non-hypercomplete case by Proposition 2.4.19. We need to show that the unit $\mathrm{id} \to e_*e^*$ and the counit $e^*e_* \to \mathrm{id}$ are equivalences. By Corollary 2.8.8, it is enough to show that the natural transformations $x^* \to x^*e_*e^*$ and $x'^*e^*e_* \to x'^*$ are equivalences for all points $x \in |X|$ and $x' \in |X'|$. (Here, we denote by x the morphism of rigid analytic spaces $x \to X$ associated to the point $x \in |X|$, and similarly for x'.) Assuming that x is the image of x', these natural transformations are equivalent to $x^* \to e_{x,*}e^*_x x^*$ and $e^*_x e_{x,*}x'^* \to x'^*$, where $e_x : x' \to x$ is the obvious morphism. This follows from Theorem 2.7.1

and the fact that the morphism $x' \to X' \times_X x$ identifies x' with $(X' \times_X x)_{\mathrm{red}}$. Thus, we are reduced to prove the result for the morphism $e_x : x' \to x$ of rigid points. Moreover, we can write $x' \sim \lim_\alpha x_\alpha$ with $(x_\alpha)_\alpha$ the cofiltered inverse system of rigid analytic x-points such that $\kappa(x_\alpha)$ is a finite purely inseparable extension of $\kappa(x)$ contained in $\kappa(x')$. Using Theorem 2.8.15, we reduce to show that e^* is an equivalence for a morphism of rigid points $e : x' \to x$ such that $\kappa(x')/\kappa(x)$ is a finite purely inseparable extension.

Arguing as in [Ayo14a, Sous-lemme 1.4], we see that $e^* e_* \simeq \mathrm{id}$. Thus, we only need to check that $\mathrm{id} \to e_* e^*$ is an equivalence. Since e^* and e_* commute with colimits (by Proposition 2.4.22), it is enough to show that $\mathrm{id} \to e_* e^*$ is an equivalence when applied to a set of compact generators. Such a set is given, up to desuspension and negative Tate twists, by objects of the form $f_\sharp \Lambda$ with $f : \mathrm{Spf}(A)^{\mathrm{rig}} \to x$ where A a rig-smooth $\kappa^+(x)$-adic algebra. Set $A' = A \widehat{\otimes}_{\kappa^+(x)} \kappa^+(x')$, and let $e' : \mathrm{Spf}(A')^{\mathrm{rig}} \to \mathrm{Spf}(A)^{\mathrm{rig}}$ and $f' : \mathrm{Spf}(A')^{\mathrm{rig}} \to x'$ be the obvious morphisms. Using Propositions 2.2.1 and 2.2.12(2), we have equivalences $e_* e^* f_\sharp \simeq e_* f'_\sharp e'^* \simeq f_\sharp e'_* e'^*$. Thus, to finish the proof, we only need to show that $\Lambda \to e'_* e'^* \Lambda$ is an equivalence in $\mathbf{RigSH}_{\mathrm{nis}}(\mathrm{Spf}(A)^{\mathrm{rig}}; \Lambda)$. Recall that there is a morphism of Pr^{L}-valued presheaves

$$\mathrm{An}^* : \mathbf{SH}_{\mathrm{nis}}(-; \Lambda) \to \mathbf{RigSH}_{\mathrm{nis}}((-)^{\mathrm{an}}; \Lambda)$$

on $\mathrm{Sch}^{\mathrm{lft}}/U$, with $U = \mathrm{Spec}(A[\pi^{-1}])$ where $\pi \in \kappa^+(x)$ a generator of an ideal of definition. Calling $e'' : \mathrm{Spec}(A'[\pi^{-1}]) \to \mathrm{Spec}(A[\pi^{-1}])$ the obvious morphism, we have, by Proposition 2.2.14, equivalences $\mathrm{An}^* e''_* e''^* \simeq e'_* e'^* \mathrm{An}^*$. Thus, it is enough to show that $\Lambda \to e''_* e''^* \Lambda$ is an equivalence in $\mathbf{SH}_{\mathrm{nis}}(\mathrm{Spec}(A[\pi^{-1}]); \Lambda)$. This follows from Theorem 2.9.7 below. □

Theorem 2.9.7. *Let $\tau \in \{\mathrm{nis}, \text{ét}\}$. Let $e : X' \to X$ be a universal homeomorphism of schemes. Assume that every prime number is invertible in either $\mathcal{O}_X$ or $\pi_0\Lambda$. Then the functor*

$$e^* : \mathbf{SH}^{(\wedge)}_\tau(X; \Lambda) \to \mathbf{SH}^{(\wedge)}_\tau(X'; \Lambda)$$

is an equivalence of ∞-categories.

Proof. Using the algebraic analogue of Corollary 2.9.5, we may assume that τ is the Nisnevich topology and we may work in the non-hypercomplete case. Then, the statement is [EK20, Theorem 2.1.1]. Alternatively, we may remark that the proof of [Ayo14a, Théorème 3.9] can be extended easily to the case of $\mathbf{SH}_{\mathrm{nis}}(-; \Lambda)$. We explain this below.

The problem is local on X, so we may assume that X is affine. By [Sta20, Lemma 0EUJ], X' is the limit of a cofiltered inverse system of finitely presented X-schemes $(X'_\alpha)_\alpha$, with $X'_\alpha \to X$ universal homeomorphisms. Using Proposition 2.5.11, we thus reduce to the case where e is assumed to be of finite presentation. In this case, writing X as the limit of a cofiltered inverse system $(X_\alpha)_\alpha$ consisting of $\mathbb{Z}$-schemes which are essentially of finite type, the scheme X' is the limit of $(X'_{\alpha_0} \times_{X_{\alpha_0}} X_\alpha)_{\alpha \leq \alpha_0}$ for a finite universal homeomorphism $X'_{\alpha_0} \to X_{\alpha_0}$. Using Proposition 2.5.11 again and base change for finite morphisms, we reduce to the case where X is of finite type over $\mathbb{Z}$. In conclusion, we may assume that X has finite Krull dimension and that $X' \to X$ is finite.

Arguing as in the beginning of the proof of Theorem 2.9.6, and using Remark 2.8.9 instead of Corollary 2.8.8 and base change for finite morphisms instead of Theorem 2.7.1, we reduce to the case where X is the spectrum of a field K, and X' the spectrum of a finite purely inseparable extension K'/K. If K has characteristic zero, then $K = K'$ and there is nothing left to prove. So, we may assume that K has positive characteristic p. We then write $\mathrm{Spec}(K)$ as the limit of a cofiltered inverse system of finite type $\mathbb{F}_p$-schemes $(X_\alpha)_\alpha$ and $\mathrm{Spec}(K')$ as the limit of $(X'_{\alpha_0} \times_{X_{\alpha_0}} X_\alpha)_{\alpha \leq \alpha_0}$ for a finite universal homeomorphism $X'_{\alpha_0} \to X_{\alpha_0}$. Thus, as before, we are finally reduced to treat the case where X and X' are of finite type over $\mathbb{F}_p$. This case follows from [Ayo14a, Théorème 1.2]. Indeed, the condition ($\mathbf{SS}_p$) of loc. cit. is satisfied for $\mathbf{SH}_{\mathrm{nis}}(-;\Lambda)$, when p is invertible in $\pi_0\Lambda$, as shown in [Ayo14a, Annexe C]. In fact, in loc. cit., this is stated explicitly in [Ayo14a, Théorème C.1] for $\mathbf{DA}^{\wedge}_{\text{ét}}(-;\Lambda)$, but the proofs apply also to $\mathbf{SH}_{\mathrm{nis}}(-;\Lambda)$. Indeed, the main point is to show that elevation to the power p^n on the multiplicative group $\mathbb{G}_{\mathrm{m}}$ induces an autoequivalence of $\mathrm{M}(\mathbb{G}_{\mathrm{m}})$ in $\mathbf{SH}(\mathbb{F}_p;\Lambda)$; see [Ayo14a, Lemme C.4]. This follows from the fact that elevation to the power m on $\mathbb{G}_{\mathrm{m}}$ induces the endomorphism of $\Lambda(1)$ given by multiplication by the element $m_\epsilon = \sum_{i=1}^m \langle(-1)^{i-1}\rangle$ in $\mathrm{K}_0^{\mathrm{MW}}(\mathbb{F}_p)$; see [Mor12, Lemma 3.14]. That this element is invertible in the endomorphism ring of $\Lambda(1)$ when $m = p^n$ is proven in [EK20, Lemma 2.2.8]. ◻

Remark 2.9.8. In the statement of Theorem 2.9.6, we made the assumption that the rigid analytic space X has locally finite Krull dimension, whereas the analogous assumption was not necessary for Theorem 2.9.7. This is because we do not know if the analogue of [Sta20, Lemma 0EUJ] holds for rigid analytic spaces. This is indeed the only obstacle for removing the assumption on the Krull dimension in Theorem 2.9.6. Said differently, semi-separatedness for rigid analytic motives holds for a universal homeomorphism $e : X' \to X$ when, locally on X, this morphism can be obtained as a weak limit of a cofiltered inverse system of universal homeomorphisms $(e_\alpha : X'_\alpha \to X_\alpha)_\alpha$ where the X_α's have finite Krull dimension.

Proposition 2.9.9 (Separatedness). *Let $f : Y \to X$ be a morphism of rigid analytic spaces. Assume that X is $(\Lambda, \text{ét})$-admissible, and that for every point $x \in |X|$, there is a point $y \in |Y|$ mapping to x and such that $\kappa(y)$ contains a dense algebraic extension of $\kappa(x)$. Then the functor*

$$f^* : \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(X;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(Y;\Lambda)$$

is conservative.

Proof. Using Corollary 2.8.8, we reduce to the case of rigid points. More precisely, we need to prove that a morphism $f : y \to x$ of rigid points, with $\kappa(y)$ containing a dense algebraic extension of $\kappa(x)$, induces a conservative functor

$$f^* : \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(x;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(y;\Lambda).$$

To do so, we may obviously replace y by any rigid x-point y' admitting an x-morphism $y' \to y$. Since the completion of a separable closure of $\kappa(x)$ is algebraically closed, we may take for y' a rigid x-point $\overline{x}$ as in Construction 1.4.27(2): $\kappa(\overline{x})$ is the completion of a separable closure $\overline{\kappa}(x)$ of $\kappa(x)$ and $\kappa^+(\overline{x})$ is the completion of a valuation ring $\overline{\kappa}^+(x) \subset \overline{\kappa}(x)$ extending $\kappa^+(x)$. In this case, we have $\overline{x} \sim \lim_\alpha x_\alpha$ where $(x_\alpha)_\alpha$ is the inverse system of rigid x-points such that $\kappa(x_\alpha)$ is a finite subextension of $\overline{\kappa}(x)/\kappa(x)$. By Theorem 2.8.6, we have an equivalence:

$$\mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(-;\Lambda)_{\overline{x}} \simeq \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(\overline{x};\Lambda)$$

where the left-hand side is the stalk of $\mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(-;\Lambda)$ at the point $\overline{x}$ of the site $(\text{Ét}/x, \text{ét})$. Since this point is conservative, we deduce from Lemma 2.8.5(2) that the functor

$$\mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(x;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(\overline{x};\Lambda)$$

is conservative, as needed. ◻

Corollary 2.9.10. *Let X be a $(\Lambda, \text{ét})$-admissible rigid analytic space, and let $f : Y \to X$ be a locally of finite type surjective morphism. Then the functor*

$$f^* : \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(X;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(Y;\Lambda)$$

is conservative.

Proof. For every point $x \in |X|$, we may find a point $y \in |Y|$ mapping to x and such that $\kappa(y)/\kappa(x)$ is a finite extension. (This follows from [FK18, Chapter II, Proposition 8.2.6] by a standard argument.) Thus, the result is a particular case of Proposition 2.9.9. □

Corollary 2.9.11. *Let $e : X' \to X$ be a universal homeomorphism of rigid analytic spaces, and assume that X is $(\Lambda, \text{ét})$-admissible. Then, the functor*

$$e^* : \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(X;\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\text{ét}}(X';\Lambda)$$

is an equivalence of ∞-categories.

Proof. The morphism $(X')_{\mathrm{red}} \to (X' \times_X X')_{\mathrm{red}}$ is a closed immersion and a universal homeomorphism, hence it is an isomorphism. Arguing as in [Ayo14a, Sous-lemme 1.4], we deduce that $e^*e_* \simeq \mathrm{id}$. Since e^* is conservative by Proposition 2.9.9, the result follows. □

Remark 2.9.12. Of course, the T-stable case of Corollary 2.9.11 is already covered by Theorem 2.9.6 under weaker assumptions. The content of this corollary is that semi-separatedness holds also for effective étale motives. It is worth noting that the algebraic analogue of this result is unknown.

Remark 2.9.13. Corollary 2.9.11 can be used to improve on the main result of [Vez17]. Indeed, given a rigid variety B over a non-Archimedean field K, Corollary 2.9.11 implies that $\mathbf{RigDA}^{\mathrm{eff},(\wedge)}_{\text{ét}}(B;\mathbb{Q})$ is equivalent to the ∞-category $\mathbf{RigDA}^{\mathrm{eff},(\wedge)}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}};\mathbb{Q})$ introduced in [Vez17, Definition 3.5]. Thus, assuming that B is normal, [Vez17, Theorem 4.1] can be stated more naturally as an equivalence of ∞-categories

$$\mathbf{RigDA}^{\mathrm{eff},(\wedge)}_{\text{ét}}(B;\mathbb{Q}) \simeq \mathbf{RigDM}^{\mathrm{eff},(\wedge)}_{\text{ét}}(B;\mathbb{Q}).$$

In fact, this equivalence can be obtained more directly by arguing as in the proof of loc. cit., without mentioning the ∞-category $\mathbf{RigDA}^{\mathrm{eff},(\wedge)}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}};\mathbb{Q})$. We leave the details to the interested reader.

2.10. **Rigidity.**

Here, we discuss the rigidity property for rigid analytic motives. Rigidity is the property that the ∞-category of torsion étale motives over a base is equivalent to the ∞-category of torsion étale sheaves on the small étale site of the same base. Rigidity for rigid analytic motives was obtained in [BV21, Theorem 2.1] for $\mathbf{RigDA}^{\wedge}_{\text{ét}}(S;\Lambda)$, with S of finite type over a non-Archimedean field and Λ an ordinary torsion ring. Rigidity in the algebraic setting was obtained in [Ayo14a, Théorème 4.1] for $\mathbf{DA}^{\wedge}_{\text{ét}}(-;\Lambda)$, with Λ an ordinary torsion ring, and in [Bac21a, Theorem 6.6] for $\mathbf{SH}^{\wedge}_{\text{ét}}(-;\Lambda)$, with Λ the sphere spectrum. In the recent preprint [Bac21b], Bachmann proved rigidity for effective motives and removed all finiteness assumptions on the base scheme. We shall revisit these results in this subsection, mainly following [Bac21a, Bac21b].

Notation 2.10.1. Let $\mathcal{C}$ be a stable presentable ∞-category and ℓ a prime number. An object A of $\mathcal{C}$ is said to be ℓ-nilpotent if the zero object of $\mathcal{C}$ is a colimit of the $\mathbb{N}$-diagram

$$A \xrightarrow{\ell\cdot\mathrm{id}} A \xrightarrow{\ell\cdot\mathrm{id}} A \xrightarrow{\ell\cdot\mathrm{id}} \cdots .$$

An object A of $\mathcal{C}$ is said to be ℓ-complete if the zero object of $\mathcal{C}$ is a limit of the $\mathbb{N}^{\mathrm{op}}$-diagram

$$\cdots \xrightarrow{\ell\cdot\mathrm{id}} A \xrightarrow{\ell\cdot\mathrm{id}} A \xrightarrow{\ell\cdot\mathrm{id}} A.$$

We denote by $\mathcal{C}_{\ell\text{-nil}} \subset \mathcal{C}$ and $\mathcal{C}_{\ell\text{-cpl}} \subset \mathcal{C}$ the sub-∞-categories spanned by ℓ-nilpotent and ℓ-complete objects respectively. Given an object A of $\mathcal{C}$, we denote by A/ℓ^n the cofiber of the map

$$\ell^n \cdot \mathrm{id} : A \to A.$$

Since multiplication by ℓ^{2n} is zero on A/ℓ^n, it is both ℓ-nilpotent and ℓ-complete.

We gather a few facts concerning the notions of ℓ-nilpotent and ℓ-complete objects in the following remark. We refer the reader to [Lur18, Part II, Chapter 7] where these notions are developed in greater generality. See also [Bac21a, §2.1].

Remark 2.10.2. Let $\mathcal{C}$ be a stable presentable ∞-category and ℓ a prime number. We denote by $\mathcal{C}[\ell^{-1}]$ the full sub-∞-category of $\mathcal{C}$ spanned by those objects for which multiplication by ℓ is an equivalence.

(1) The ∞-category $\mathcal{C}_{\ell\text{-nil}}$ is stable, presentable and generated under colimits by the objects of the form A/ℓ^n, for $A \in \mathcal{C}$. The inclusion functor $\mathcal{C}_{\ell\text{-nil}} \to \mathcal{C}$ commutes with colimits and finite limits. If $\mathcal{C}$ is compactly generated, then so it is $\mathcal{C}_{\ell\text{-nil}}$.

(2) The ∞-category $\mathcal{C}_{\ell\text{-cpl}}$ is the localisation of $\mathcal{C}$ with respect to the maps $0 \to A$, for $A \in \mathcal{C}[\ell^{-1}]$. We denote by $(-)^\wedge_\ell : \mathcal{C} \to \mathcal{C}_{\ell\text{-cpl}}$ the left adjoint to the inclusion functor. This is called the ℓ-completion functor.

(3) The ℓ-completion functor induces an equivalence of ∞-categories

$$(-)^\wedge_\ell : \mathcal{C}_{\ell\text{-nil}} \xrightarrow{\sim} \mathcal{C}_{\ell\text{-cpl}}.$$

In particular, we see that $\mathcal{C}_{\ell\text{-cpl}}$ is stable, presentable and generated under colimits by the objects of the form A/ℓ^n, for $A \in \mathcal{C}$. If $\mathcal{C}$ is compactly generated, then so is $\mathcal{C}_{\ell\text{-cpl}}$.

(4) If $\mathcal{C}$ underlies a presentable symmetric monoidal ∞-category $\mathcal{C}^\otimes$, then there is an essentially unique morphism $\mathcal{C}^\otimes \to \mathcal{C}^\otimes_{\ell\text{-cpl}}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ whose underlying functor is $(-)^\wedge_\ell : \mathcal{C} \to \mathcal{C}_{\ell\text{-cpl}}$.

(5) Suppose that $\mathcal{C}$ is given as a colimit in Pr^{L} of an inductive system $(\mathcal{C}_\alpha)_\alpha$ of stable presentable ∞-categories. Then $\mathcal{C}[\ell^{-1}]$ is also the colimit of the inductive system $(\mathcal{C}_\alpha[\ell^{-1}])_\alpha$ in Pr^{L}. (This uses the fact that a colimit in Pr^{L} can be computed as a limit in Pr^{R}.) In particular, $\mathcal{C}[\ell^{-1}]$ is generated under colimits by the images of the functors $\mathcal{C}_\alpha[\ell^{-1}] \to \mathcal{C}[\ell^{-1}]$. It follows from (2) and the universal property of localisations (see [Lur09, Proposition 5.5.4.20]) that $\mathcal{C}_{\ell\text{-cpl}}$ is the colimit in Pr^{L} of the inductive system $(\mathcal{C}_{\alpha,\ell\text{-cpl}})_\alpha$. Using (3), we deduce that $\mathcal{C}_{\ell\text{-nil}}$ is also the colimit in Pr^{L} of the inductive system $(\mathcal{C}_{\alpha,\ell\text{-nil}})_\alpha$.

Theorem 2.10.3 (Rigidity). *Let S be a rigid analytic space and ℓ a prime number which is invertible in $\widetilde{\kappa}(s)$ for every $s \in |S|$. Assume one of the following two alternatives.*

(1) We work in the non-hypercomplete case and Λ is eventually coconnective.

(2) We work in the hypercomplete case.

Then the obvious functor

$$\mathrm{Shv}^{(\wedge)}_{\text{ét}}(\text{Ét}/S;\Lambda)_{\ell\text{-cpl}} \to \mathbf{RigSH}^{(\text{eff},\wedge)}_{\text{ét}}(S;\Lambda)_{\ell\text{-cpl}} \tag{2.45}$$

*is an equivalence of ∞-categories. (The same is true with "ℓ-*nil*" instead of "ℓ-*cpl*".)*

We also have the algebraic analogue of Theorem 2.10.3 which can be stated as follows.

Theorem 2.10.4. *Let S be a scheme and ℓ a prime number which is invertible on S. Assume one of the following two alternatives.*

(1) We work in the non-hypercomplete case and Λ is eventually coconnective.

(2) We work in the hypercomplete case.

Then the obvious functor

$$\mathrm{Shv}_{\text{ét}}^{(\wedge)}(\text{Ét}/S\,;\Lambda)_{\ell\text{-cpl}} \to \mathbf{SH}_{\text{ét}}^{(\text{eff},\wedge)}(S\,;\Lambda)_{\ell\text{-cpl}} \tag{2.46}$$

is an equivalence of ∞-categories. (The same is true with "ℓ-nil" instead of "ℓ-cpl".)

Proof. We first consider the alternative (1). We may assume that S is affine and given as the limit of a cofiltered inverse system $(S_\alpha)_\alpha$ of affine schemes of finite type over $\mathbb{Z}$. By the algebraic analogue of Lemma 2.4.21 and Proposition 2.5.11, it is enough to prove the conclusion for the S_α's. Thus, me may assume that S is of finite type over $\mathbb{Z}$ and hence $(\Lambda, \text{ét})$-admissible. By the algebraic analogue of Lemma 2.4.18(2) and Proposition 3.2.2 below, we are then automatically working in the hypercomplete case. This means that we only need to consider the alternative (2). In that case, the result is essentially [Bac21a, Theorem 6.6] improved in [Bac21b, Theorem 3.1]. □

Remark 2.10.5. Arguing as above, we only need to prove Theorem 2.10.3 under the second alternative. Indeed, by Lemma 2.4.21 and Theorem 2.5.1 we may assume that S is $(\Lambda, \text{ét})$-admissible. In this case, there is no distinction between the hypercomplete and the non-hypercomplete cases by Lemma 2.4.18(2) and Proposition 2.4.19.

Our proof of Theorem 2.10.3 follows the arguments in [Bac21a, Bac21b] and relies on some of the key steps in loc. cit. We start with a reduction to the $(\Lambda, \text{ét})$-admissible case.

Lemma 2.10.6. *To prove Theorem 2.10.3, we may work in the hypercomplete case and assume that S is $(\Lambda, \text{ét})$-admissible.*

Proof. We said already that it is enough to work under the second alternative. Assume that Theorem 2.10.3 is known in the hypercomplete case when the base is $(\Lambda, \text{ét})$-admissible. To prove the theorem in general, we argue as in the proof of [Bac21b, Theorem 3.1]. We may assume that $S = \mathcal{S}^{\text{rig}}$ where $\mathcal{S}$ is an affine formal scheme given as the limit of an affine formal pro-scheme $(\mathcal{S}_\alpha)_\alpha$ such that the $\mathcal{S}_\alpha$'s are of finite type over $\mathbb{Z}[\ell^{-1}][[\pi]]$. We set $S_\alpha = \mathcal{S}_\alpha^{\text{rig}}$; these are $(\Lambda, \text{ét})$-admissible rigid analytic spaces. By Lemma 2.4.21, Theorem 2.5.1 and Remark 2.10.2(5), we have a commutative square

$$\begin{tikzcd}
\mathrm{colim}_\alpha \mathrm{Shv}_{\text{nis}}(S_\alpha;\Lambda)_{\ell\text{-cpl}} \arrow[r, "\sim"] \arrow[d] & \mathrm{Shv}_{\text{nis}}(S\,;\Lambda)_{\ell\text{-cpl}} \arrow[d] \\
\mathrm{colim}_\alpha \mathbf{RigSH}_{\text{nis}}^{(\text{eff})}(S_\alpha;\Lambda)_{\ell\text{-cpl}} \arrow[r, "\sim"] & \mathbf{RigSH}_{\text{nis}}^{(\text{eff})}(S\,;\Lambda)_{\ell\text{-cpl}}
\end{tikzcd}$$

where the horizontal arrows are equivalences of ∞-categories. It follows that in the analogous commutative square

$$\begin{tikzcd}
\mathrm{colim}_\alpha \mathrm{Shv}_{\text{ét}}^{\wedge}(S_\alpha;\Lambda)_{\ell\text{-cpl}} \arrow[r] \arrow[d, "\sim"] & \mathrm{Shv}_{\text{ét}}^{\wedge}(S\,;\Lambda)_{\ell\text{-cpl}} \arrow[d] \\
\mathrm{colim}_\alpha \mathbf{RigSH}_{\text{ét}}^{(\text{eff}),\wedge}(S_\alpha;\Lambda)_{\ell\text{-cpl}} \arrow[r] & \mathbf{RigSH}_{\text{ét}}^{(\text{eff}),\wedge}(S\,;\Lambda)_{\ell\text{-cpl}},
\end{tikzcd}$$

the horizontal arrows are localisation functors, whereas, by assumption, the left vertical arrow is an equivalence. This shows that

$$\mathrm{Shv}_{\text{ét}}^{\wedge}(S\,;\Lambda)_{\ell\text{-cpl}} \to \mathbf{RigSH}_{\text{ét}}^{(\text{eff}),\wedge}(S\,;\Lambda)_{\ell\text{-cpl}} \tag{2.47}$$

is a localisation functor. To finish the proof, it remains to see that (2.47) is conservative. Given a geometric rigid point $\overline{s} \to S$, we have a commutative square

$$\begin{tikzcd}
\mathrm{Shv}^{\wedge}_{\acute{e}t}(S\,;\Lambda)_{\ell\text{-cpl}} \arrow[r] \arrow[d] & \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\acute{e}t}(S\,;\Lambda)_{\ell\text{-cpl}} \arrow[d] \\
\mathrm{Shv}^{\wedge}_{\acute{e}t}(\overline{s};\Lambda)_{\ell\text{-cpl}} \arrow[r, "\sim"] & \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\acute{e}t}(\overline{s};\Lambda)_{\ell\text{-cpl}},
\end{tikzcd}$$

and the bottom arrow is an equivalence, again by assumption, since $\overline{s}$ is $(\Lambda, \acute{e}t)$-admissible. This proves that the functor $(-)_{\overline{s}} : \mathrm{Shv}^{\wedge}_{\acute{e}t}(S\,;\Lambda)_{\ell\text{-cpl}} \to (\mathrm{Mod}_{\Lambda})_{\ell\text{-cpl}}$ factors through (2.47). We conclude using Propositions 1.4.29 and 2.8.1. □

We now introduce some notations.

Notation 2.10.7. Let S be a rigid analytic space. The ℓ-completion of the constant étale sheaf $\Lambda \in \mathrm{Shv}^{(\wedge)}_{\acute{e}t}(\acute{E}t/S\,;\Lambda)$ will be denoted simply by Λ_ℓ. This is the unit object of $\mathrm{Shv}^{(\wedge)}_{\acute{e}t}(\acute{E}t/S\,;\Lambda)_{\ell\text{-cpl}}$ endowed with its natural monoidal structure. We denote by

$$\iota^*_S : \mathrm{Shv}^{(\wedge)}_{\acute{e}t}(\acute{E}t/S\,;\Lambda) \to \mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\acute{e}t}(S\,;\Lambda)$$

the obvious functor, and by $\iota_{S,*}$ its right adjoint. Similarly, we denote by

$$\iota^*_{S,\ell} : \mathrm{Shv}^{(\wedge)}_{\acute{e}t}(\acute{E}t/S\,;\Lambda)_{\ell\text{-cpl}} \to \mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\acute{e}t}(S\,;\Lambda)_{\ell\text{-cpl}}$$

the functor induced by ι^*_S on ℓ-completed objects, and by $\iota_{S,\ell,*}$ its right adjoint. We denote by

$$\Sigma^\infty_{\mathrm{T},\ell} : \mathbf{RigSH}^{\mathrm{eff},(\wedge)}_{\acute{e}t}(S\,;\Lambda)_{\ell\text{-cpl}} \to \mathbf{RigSH}^{(\wedge)}_{\acute{e}t}(S\,;\Lambda)_{\ell\text{-cpl}}$$

the functor induced by $\Sigma^\infty_{\mathrm{T}}$ on ℓ-completed objects, and by $\Omega^\infty_{\mathrm{T},\ell}$ its right adjoint. (See Definition 2.1.15.) The functor (2.45) is given by $\iota^*_{S,\ell}$ in the effective case and by $\Sigma^\infty_{\mathrm{T},\ell} \circ \iota^*_{S,\ell}$ in the T-stable case. These notations apply also when S is a scheme.

Recall that $\mathbb{U}^1_S$ is the relative unit sphere over the rigid analytic space S. (See Notation 2.1.10(3).)

Lemma 2.10.8. *Let S be a rigid analytic space and ℓ a prime number which is invertible in $\kappa(s)$ for every $s \in |S|$. There is a $\otimes$-invertible object $\Lambda_\ell(1)$ in $\mathrm{Shv}^{\wedge}_{\acute{e}t}(S\,;\Lambda)_{\ell\text{-cpl}}$ together with a morphism*

$$\sigma : \Lambda_\ell \to \Lambda_\ell(1)[1] \tag{2.48}$$

in $\mathrm{Shv}^{\wedge}_{\acute{e}t}(\acute{E}t/\mathbb{U}^1_S\,;\Lambda)_{\ell\text{-cpl}}$ endowed with a trivialisation (i.e., a homotopy to the null morphism) over the unit section $1_S \subset \mathbb{U}^1_S$. Moreover, the induced morphism $\sigma : \mathrm{T}^{\wedge}_{\ell} \to \iota^*_{S,\ell}(\Lambda_\ell(1)[1])$ is an equivalence in $\mathbf{RigSH}^{\mathrm{eff},\wedge}_{\acute{e}t}(S\,;\Lambda)_{\ell\text{-cpl}}$.

Proof. We may construct $\Lambda_\ell(1)$ and $\sigma : \Lambda_\ell \to \Lambda_\ell(1)[1]$ locally on S provided that the construction is compatible with base change. Assume that $S = \mathrm{Spf}(A)^{\mathrm{rig}}$ with A an adic ring. Let $I \subset A$ be an ideal of definition and set $U = \mathrm{Spec}(A) \smallsetminus \mathrm{Spec}(A/I)$. We denote by $\Lambda_\ell(1) \in \mathrm{Shv}^{\wedge}_{\acute{e}t}(\acute{E}t/U;\Lambda)_{\ell\text{-cpl}}$ the $\otimes$-invertible object obtained from the one introduced in [Bac21a, Definition 3.9] by extension of scalars to Λ. Also, let $\sigma : \Lambda_\ell \to \Lambda_\ell(1)[1]$ be the morphism in $\mathrm{Shv}^{\wedge}_{\acute{e}t}(\acute{E}t/\mathbb{A}^1_U \smallsetminus 0_U;\Lambda)_{\ell\text{-cpl}}$ obtained from the one introduced in [Bac21a, Definition 3.13] by extension of scalars to Λ. As explained in the beginning of [Bac21a, §6], a trivialisation of σ above 1_S gives rise to a morphism $\mathrm{T}^{\wedge}_{\ell} \to \iota^*_{U,\ell}\Lambda_\ell(1)[1]$ in $\mathbf{SH}^{\mathrm{eff},\wedge}_{\acute{e}t}(U;\Lambda)_{\ell\text{-cpl}}$. As explained in the beginning of the proof of [Bac21b, Theorem

3.1], this morphism is an equivalence (see also [Bac21a, Theorem 6.5] in the T-stable case). The lemma follows now from the existence of a commutative square of stable presentable ∞-categories

$$\begin{tikzcd} \mathrm{Shv}^{\wedge}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/U;\Lambda) \arrow[r, "\iota_U^*"] \arrow[d] & \mathbf{SH}^{\mathrm{eff},\wedge}_{\mathrm{\acute{e}t}}(U;\Lambda) \arrow[d] \\ \mathrm{Shv}^{\wedge}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/S;\Lambda) \arrow[r, "\iota_S^*"] & \mathbf{RigSH}^{\mathrm{eff},\wedge}_{\mathrm{\acute{e}t}}(S;\Lambda) \end{tikzcd}$$

where the vertical arrows are induced by the analytification functor. □

Corollary 2.10.9. *Let S be a rigid analytic space and ℓ a prime number which is invertible in $\kappa(s)$ for every $s \in |S|$. Then the obvious functor*

$$\Sigma^{\infty}_{\mathrm{T},\ell} : \mathbf{RigSH}^{\mathrm{eff},\wedge}_{\mathrm{\acute{e}t}}(S;\Lambda)_{\ell\text{-cpl}} \to \mathbf{RigSH}^{\mathrm{eff},\wedge}_{\mathrm{\acute{e}t}}(S;\Lambda)_{\ell\text{-cpl}}$$

is an equivalence of ∞-categories. (The same is true with "ℓ-nil" instead of "ℓ-cpl".)

Proof. Indeed, by Remarks 2.1.17 and 2.10.2(5), $\mathbf{RigSH}^{\wedge}_{\mathrm{\acute{e}t}}(S;\Lambda)_{\ell\text{-cpl}}$ is the colimit in Pr^{L} of the $\mathbb{N}$-diagram whose transition maps are given by tensoring with $\mathrm{T}^{\wedge}_{\ell}$ in $\mathbf{RigSH}^{\mathrm{eff},\wedge}_{\mathrm{\acute{e}t}}(S;\Lambda)_{\ell\text{-cpl}}$. The result follows since $\mathrm{T}^{\wedge}_{\ell}$ is $\otimes$-invertible by Lemma 2.10.8. □

Lemma 2.10.10. *Let S be a $(\Lambda, \mathrm{\acute{e}t})$-admissible rigid analytic space and ℓ a prime number which is invertible in $\widetilde{\kappa}(s)$ for every $s \in |S|$. Then the obvious functor*

$$\mathrm{Shv}^{\wedge}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/S;\Lambda)_{\ell\text{-cpl}} \to \mathbf{RigSH}^{(\mathrm{eff}),\wedge}_{\mathrm{\acute{e}t}}(S;\Lambda)_{\ell\text{-cpl}} \tag{2.49}$$

is fully faithful. (The same is true with "ℓ-nil" instead of "ℓ-cpl".)

Proof. By Corollary 2.10.9, we only need to treat the effective case. The functor

$$\iota_S^* : \mathrm{PSh}(\mathrm{\acute{E}t}/S;\Lambda) \to \mathrm{PSh}(\mathrm{RigSm}/S;\Lambda)$$

is fully faithful and its right adjoint commutes with étale hypersheafification. It follows that the induced functor on étale hypersheaves

$$\iota_S^* : \mathrm{Shv}^{\wedge}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/S;\Lambda) \to \mathrm{Shv}^{\wedge}_{\mathrm{\acute{e}t}}(\mathrm{RigSm}/S;\Lambda)$$

is also fully faithful, and the same is true for the induced functor on ℓ-complete objects

$$\iota_{S,\ell}^* : \mathrm{Shv}^{\wedge}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/S;\Lambda)_{\ell\text{-cpl}} \to \mathrm{Shv}^{\wedge}_{\mathrm{\acute{e}t}}(\mathrm{RigSm}/S;\Lambda)_{\ell\text{-cpl}}.$$

We claim that the functor $\iota_{S,\ell}^*$ takes values in the sub-∞-category $\mathbf{RigSH}^{\mathrm{eff},\wedge}_{\mathrm{\acute{e}t}}(S;\Lambda)_{\ell\text{-cpl}}$ spanned by $\mathbb{B}^1$-local objects; this would finish the proof. Indeed, let $\mathcal{F}$ be an ℓ-complete étale hypersheaf of Λ-modules on $\mathrm{\acute{E}t}/S$. Saying that $\iota_{S,\ell}^*\mathcal{F}$ is $\mathbb{B}^1$-local is equivalent to saying that for every $X \in \mathrm{RigSm}/S$, the map $\Gamma(X;\mathcal{F}|_X) \to \Gamma(\mathbb{B}^1_X;\mathcal{F}|_{\mathbb{B}^1_X})$ is an equivalence. (Here, we denote by $\mathcal{F}|_X$ the ℓ-complete inverse image of $\mathcal{F}$ along the morphism $X \to S$, and similarly for $\mathcal{F}|_{\mathbb{B}^1_X}$.) Since X is $(\Lambda, \mathrm{\acute{e}t})$-admissible, the claim follows from Lemma 2.10.11(1) below (see also [Hub96, Example 0.1.1(2)]). □

Lemma 2.10.11. *Let X be a $(\Lambda, \mathrm{\acute{e}t})$-admissible rigid analytic space and ℓ a prime number which is invertible in $\widetilde{\kappa}(x)$ for every $x \in |X|$. Let $p : \mathbb{B}^1_X \to X$ be the obvious projection and let $\mathcal{F}$ be an ℓ-complete étale hypersheaf on $\mathrm{\acute{E}t}/X$. Then the map $\mathcal{F} \to p_*p^*\mathcal{F}$ is an equivalence.*

Proof. It is enough to prove the results on the stalks for all geometric algebraic rigid points $\overline{x} \to X$. Using Remark 2.7.3, we reduce to show the following. Given a geometric rigid point $s = \mathrm{Spf}(V)^{\mathrm{rig}}$ and an ℓ-complete étale hypersheaf of Λ-modules $\mathcal{F}$ on $\mathrm{Ét}/s$, the map $\mathcal{F}(s) \to \Gamma(\mathbb{B}^1_s; \mathcal{F}|_{\mathbb{B}^1_s})$ is an equivalence. Using Lemmas 2.4.5 and 2.4.11, we reduce to the case where $\mathcal{F}$ is bounded. By an easy induction, we reduce to the case where $\mathcal{F}$ is discrete, and we may then assume that $\mathcal{F}$ is an ordinary étale sheaf of $\mathbb{Z}/\ell^n$-modules. The site $(\mathrm{Ét}/s, \mathrm{ét})$ is equivalent to $(\mathrm{FRigÉt}/\mathrm{Spf}(V), \mathrm{rigét})$ and, since s is geometric, it is also equivalent to $(\mathrm{Ét}/\mathrm{Spec}(V'), \mathrm{ét})$, where $V' = V/\sqrt{(\pi)}$ with π a generator of an ideal of definition of V. Thus, we may consider $\mathcal{F}$ as an ordinary étale sheaf on $\mathrm{FRigÉt}/\mathrm{Spf}(V)$ and on $\mathrm{Ét}/\mathrm{Spec}(V')$. We then have equivalences:

$$\mathrm{R}\Gamma_{\mathrm{ét}}(\mathbb{B}^1_s; \mathcal{F}|_{\mathbb{B}^1_s}) \simeq \mathrm{R}\Gamma_{\mathrm{rigét}}(\mathbb{A}^1_V; \mathcal{F}|_{\mathbb{A}^1_V}) \simeq \mathrm{R}\Gamma_{\mathrm{ét}}(\mathbb{A}^1_{V'}; i^* j_*(\mathbb{Z}/\ell^n) \otimes_{\mathbb{Z}/\ell^n} \mathcal{F}|_{\mathbb{A}^1_{V'}}). \tag{2.50}$$

Here i denotes the closed immersion $\mathrm{Spec}(V') \to \mathrm{Spec}(V)$ and its base changes, and j denotes the open complement of i and its base changes. The second equivalence in (2.50) follows from [Hub96, Corollary 3.5.16]. (More precisely, we reduce to the case where $\mathcal{F}$ is of the form $i'_*\mathbb{Z}/\ell^n$ with $i' : \mathrm{Spec}(V'') \to \mathrm{Spec}(V')$ a closed immersion, and we remark that [Hub96, Corollary 3.5.16] is still valid if we replace the closed point of $\mathrm{Spec}(V)$ by a closed subscheme contained in $\mathrm{Spec}(V')$.) Using the smooth base change theorem in étale cohomology [SGAIV3, Exposé XVI, Théorème 1.1] and the fact that the fraction field of V is algebraically closed, we deduce that $i^* j_*\mathbb{Z}/\ell^n \simeq \mathbb{Z}/\ell^n$ on $\mathbb{A}^1_{V'}$. Thus, the last term in (2.50) is equivalent to $\mathrm{R}\Gamma_{\mathrm{ét}}(\mathbb{A}^1_{V'}; \mathcal{F}|_{\mathbb{A}^1_{V'}})$ which, by homotopy invariance of étale cohomology [SGAIV3, Exposé XV, Corollaire 2.2], is equivalent to $\mathcal{F}(V') \simeq \mathcal{F}(s)$. This proves that $\mathcal{F}(s)$ is indeed equivalent to $\mathrm{R}\Gamma(\mathbb{B}^1_s; \mathcal{F}|_{\mathbb{B}^1_s})$ as needed. □

Proof of Theorem 2.10.3. Using Lemmas 2.10.6 and 2.10.10, it remains to see that the functor (2.49) is essentially surjective (still under the assumption that S is $(\Lambda, \mathrm{ét})$-admissible). Moreover, it is enough to do so in the T-stable case, by Corollary 2.10.9. We follow the argument used in the proof of [BV21, Theorem 2.1].

The question being local on S, we may assume that $S = \mathrm{Spf}(A)^{\mathrm{rig}}$ with A an adic ring of principal ideal type. Let $\pi \in A$ be a generator of an ideal of definition and set $U = \mathrm{Spec}(A[\pi^{-1}])$. It is enough to show that the image of the functor (2.49), in the T-stable case, contains a set of generators of $\mathbf{RigSH}^{\wedge}_{\mathrm{ét}}(S; \Lambda)_{\ell\text{-cpl}}$. Such a set of generators is given, up to shift and Tate twists, by $\mathrm{M}(V)/\ell^n$ where $n \in \mathbb{N}$ and $V = \mathrm{Spf}(B)^{\mathrm{rig}}$ with B a rig-étale adic A-algebra satisfying the conclusion of Proposition 1.3.15. Thus, there exists smooth affine U-scheme X and an open immersion $v : V \to X^{\mathrm{an}}$. Since we are allowed to replace V by the components of an analytic hypercover, we may assume that $\Omega_{X/U}$ is free. Fix a projective compactification $j : X \to P$ over U and denote by $f : X \to U$ and $p : P \to U$ the structural morphisms. Thus, we have a commutative diagram

$$\begin{tikzcd} V \arrow[r, "v"] \arrow[rr, bend left, "\overline{v}"] \arrow[dr, "g"'] & X^{\mathrm{an}} \arrow[r, "j^{\mathrm{an}}"] \arrow[d, "f^{\mathrm{an}}"] & P^{\mathrm{an}} \arrow[dl, "p^{\mathrm{an}}"] \\ & S. & \end{tikzcd}$$

The motive $\mathrm{M}(V)$ is equivalent to $g_\sharp\Lambda \simeq f^{\mathrm{an}}_\sharp v_\sharp\Lambda$. Using Corollary 2.2.9, we see that $\mathrm{M}(V)$ is equivalent, up to shift and Tate twist, to $f^{\mathrm{an}}_! v_\sharp\Lambda \simeq p^{\mathrm{an}}_! j^{\mathrm{an}}_\sharp v_\sharp\Lambda \simeq p_*\overline{v}_\sharp\Lambda$.

Using Lemmas 2.10.8 and 2.10.10, the image of the functor (2.49), in the T-stable case, is closed under shift and Tate twists. Therefore, it remains to see that the latter image contains $p^{\mathrm{an}}_*\overline{v}_\sharp\Lambda/\ell^n$.

Clearly, $\bar{v}_\sharp\Lambda/\ell^n$ belongs to the image of

$$\Sigma^\infty_{\mathrm{T},\ell}\circ\iota^*_{P^{\mathrm{an}},\ell}:\mathrm{Shv}^\wedge_{\text{ét}}(\text{Ét}/P^{\mathrm{an}};\Lambda)_{\ell\text{-cpl}}\to\mathbf{RigSH}^\wedge_{\text{ét}}(P^{\mathrm{an}};\Lambda)_{\ell\text{-cpl}}.$$

Thus, it is enough to show that the natural transformation $\Sigma^\infty_{\mathrm{T},\ell}\circ\iota^*_{S,\ell}\circ p^{\mathrm{an}}_*\to p^{\mathrm{an}}_*\circ\Sigma^\infty_{\mathrm{T},\ell}\circ\iota^*_{P^{\mathrm{an}},\ell}$ is an equivalence. (The first p^{an}_* is the direct image functor on étale hypersheaves, and the second p^{an}_* is the direct image functor on rigid analytic motives.) Using Corollary 2.10.9, it is enough to show that the natural transformation $\iota^*_{S,\ell}\circ p^{\mathrm{an}}_*\to p^{\mathrm{an}}_*\circ\iota^*_{P^{\mathrm{an}},\ell}$ is an equivalence. Given an ℓ-complete étale hypersheaf $\mathcal{F}$ on $\text{Ét}/P^{\mathrm{an}}$, the evaluation of $\iota^*_{S,\ell}p^{\mathrm{an}}_*\mathcal{F}\to p^{\mathrm{an}}_*\iota^*_{P^{\mathrm{an}}}\mathcal{F}$ on a smooth rigid analytic S-space Y is given by

$$\Gamma(Y;g^*p^{\mathrm{an}}_*\mathcal{F})\to\Gamma(Y\times_S P^{\mathrm{an}},g'^*\mathcal{F})=\Gamma(Y;p'_*g'^*\mathcal{F})$$

where p' and g' are as in the Cartesian square

$$\begin{tikzcd}
Y\times_S P^{\mathrm{an}} \arrow[r, "g'"] \arrow[d, "p'"] & P^{\mathrm{an}} \arrow[d, "p^{\mathrm{an}}"] \\
Y \arrow[r, "g"] & S.
\end{tikzcd}$$

The result follows now from the quasi-compact base change theorem, see Remark 2.7.3. □

3. Rigid analytic motives as modules in formal motives

This section contains one of the key results of the paper which, roughly speaking, gives a description of the functor $\mathbf{RigSH}^{(\wedge)}_\tau(-;\Lambda)$ in terms of the functor $\mathbf{SH}^{(\wedge)}_\tau(-;\Lambda)$. This can be considered as a vast generalisation of [Ayo15, Scholie 1.3.26]. In fact, we prefer to work with the functor $\mathbf{FSH}^{(\wedge)}_\tau(-;\Lambda)$, sending a formal scheme to the ∞-category of formal motives, instead of the functor $\mathbf{SH}^{(\wedge)}_\tau(-;\Lambda)$, but this is a merely aesthetic difference, by Theorem 3.1.10. For a precise form of the description alluded to, we refer the reader to Theorems 3.3.3 and 3.8.1.

We start by recalling the definition and the basic properties of the ∞-category $\mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{S};\Lambda)$ of formal motives over a formal scheme $\mathcal{S}$.

3.1. **Formal and algebraic motives.**

Recall that we denote by FSch the category of formal schemes and that, given a formal scheme $\mathcal{S}$, we denote by $\mathrm{FSm}/\mathcal{S}$ the category of smooth formal $\mathcal{S}$-schemes. (Notations 1.1.5 and 1.4.9.) The ∞-category of formal motives over a formal scheme is constructed as in Definitions 2.1.11 and 2.1.15.

We fix a formal scheme $\mathcal{S}$ and $\tau\in\{\mathrm{nis},\text{ét}\}$.

Definition 3.1.1. Let $\mathbf{FSH}^{\mathrm{eff},(\wedge)}_\tau(\mathcal{S};\Lambda)$ be the full sub-∞-category of $\mathrm{Shv}^{(\wedge)}_\tau(\mathrm{FSm}/\mathcal{S};\Lambda)$ spanned by those objects which are local with respect to the collection of maps of the form $\Lambda_\tau(\mathbb{A}^1_\mathcal{X})\to\Lambda_\tau(\mathcal{X})$, for $\mathcal{X}\in\mathrm{FSm}/\mathcal{S}$, and their desuspensions. Let

$$\mathrm{L}_{\mathbb{A}^1}:\mathrm{Shv}^{(\wedge)}_\tau(\mathrm{FSm}/\mathcal{S};\Lambda)\to\mathbf{FSH}^{\mathrm{eff},(\wedge)}_\tau(\mathcal{S};\Lambda)\tag{3.1}$$

be the left adjoint to the obvious inclusion. This is called the $\mathbb{A}^1$-localisation functor. Given a smooth formal $\mathcal{S}$-scheme $\mathcal{X}$, we set $\mathrm{M}^{\mathrm{eff}}(X)=\mathrm{L}_{\mathbb{A}^1}(\Lambda_\tau(\mathcal{X}))$. This is the effective motive of $\mathcal{X}$.

Remark 3.1.2. By [Lur17, Proposition 2.2.1.9], $\mathbf{FSH}^{\mathrm{eff},(\wedge)}_\tau(\mathcal{S};\Lambda)$ underlies a unique monoidal ∞-category $\mathbf{FSH}^{\mathrm{eff},(\wedge)}_\tau(\mathcal{S};\Lambda)^\otimes$ such that $\mathrm{L}_{\mathbb{A}^1}$ lifts to a monoidal functor. Moreover, this monoidal ∞-category is presentable, i.e., belongs to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

Definition 3.1.3. Let $\mathrm{T}_{\mathcal{S}}$ (or simply T if $\mathcal{S}$ is clear from the context) be the image by $\mathrm{L}_{\mathbb{A}^1}$ of the cofiber of the split inclusion $\Lambda_\tau(\mathcal{S}) \to \Lambda_\tau(\mathbb{A}^1_{\mathcal{S}} \smallsetminus 0_{\mathcal{S}})$ induced by the unit section. With the notation of [Rob15, Definition 2.6], we set

$$\mathbf{FSH}^{(\wedge)}_\tau(\mathcal{S};\Lambda)^\otimes = \mathbf{FSH}^{\mathrm{eff},(\wedge)}_\tau(\mathcal{S};\Lambda)^\otimes[\mathrm{T}_{\mathcal{S}}^{-1}]. \tag{3.2}$$

More precisely, there is a morphism $\Sigma^\infty_{\mathrm{T}} : \mathbf{FSH}^{\mathrm{eff},(\wedge)}_\tau(\mathcal{S};\Lambda)^\otimes \to \mathbf{FSH}^{(\wedge)}_\tau(\mathcal{S};\Lambda)^\otimes$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, sending $\mathrm{T}_{\mathcal{S}}$ to a $\otimes$-invertible object, and which is initial for this property. We denote by $\Omega^\infty_{\mathrm{T}}$: $\mathbf{FSH}^{(\wedge)}_\tau(\mathcal{S};\Lambda) \to \mathbf{FSH}^{\mathrm{eff},(\wedge)}_\tau(\mathcal{S};\Lambda)$ the right adjoint to $\Sigma^\infty_{\mathrm{T}}$. Given a smooth formal $\mathcal{S}$-scheme $\mathcal{X}$, we set $\mathrm{M}(\mathcal{X}) = \Sigma^\infty_{\mathrm{T}}\mathrm{M}^{\mathrm{eff}}(\mathcal{X})$. This is the motive of X.

Definition 3.1.4. Objects of $\mathbf{FSH}^{(\wedge)}_\tau(\mathcal{S};\Lambda)$ are called formal motives over $\mathcal{S}$. We will denote by Λ (or $\Lambda_{\mathcal{S}}$ if we need to be more precise) the monoidal unit of $\mathbf{FSH}^{(\wedge)}_\tau(\mathcal{S};\Lambda)$. For any $n \in \mathbb{N}$, we denote by $\Lambda(n)$ the image of $\mathrm{T}_{\mathcal{S}}^{\otimes n}[-n]$ by $\Sigma^\infty_{\mathrm{T}}$, and by $\Lambda(-n)$ the $\otimes$-inverse of $\Lambda(n)$. For $n \in \mathbb{Z}$, we denote by $M \mapsto M(n)$ the Tate twist given by tensoring with $\Lambda(n)$.

Remark 3.1.5.

(1) Remark 2.1.17 applies also in the case of formal motives: the ∞-category $\mathbf{FSH}^{(\wedge)}_\tau(\mathcal{S};\Lambda)$ underlying (3.2) is equivalent to the colimit in Pr^{L} of the $\mathbb{N}$-diagram whose transition maps are given by tensoring with $\mathrm{T}_{\mathcal{S}}$ in $\mathbf{FSH}^{\mathrm{eff},(\wedge)}_\tau(\mathcal{S};\Lambda)$.
(2) When Λ is the Eilenberg–Mac Lane spectrum associated to an ordinary ring, also denoted by Λ, the category $\mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{S};\Lambda)$ is more commonly denoted by $\mathbf{FDA}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{S};\Lambda)$. Also, when τ is the Nisnevich topology, we sometimes drop the subscript "nis".
(3) Just as in Remark 2.1.19, there is a more traditional description of the ∞-category $\mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{S};\Lambda)$ using the language of model categories. This is the approach taken in [Ayo15, §1.4.2].
(4) If S is an ordinary scheme considered as a formal scheme in the obvious way, i.e., such that the zero ideal is an ideal of definition, then the ∞-category $\mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(S;\Lambda)$ is the usual ∞-category $\mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(S;\Lambda)$ of algebraic motives over S. More generally, by Theorem 3.1.10 below, the ∞-categories introduced in Definitions 3.1.1 and 3.1.3 are always equivalent to ∞-categories of algebraic motives.

Lemma 3.1.6. *The monoidal ∞-category $\mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{S};\Lambda)^\otimes$ is presentable and its underlying ∞-category is generated under colimits, and up to desuspension and negative Tate twists when applicable, by the motives $\mathrm{M}^{(\mathrm{eff})}(\mathcal{X})$ with $\mathcal{X} \in \mathrm{FSm}/\mathcal{S}$ quasi-compact and quasi-separated.*

Proof. See the proof of Lemma 2.1.20. □

Proposition 3.1.7. *The assignment $\mathcal{S} \mapsto \mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{S};\Lambda)^\otimes$ extends naturally into a functor*

$$\mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(-;\Lambda)^\otimes : \mathrm{FSch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}). \tag{3.3}$$

Proof. We refer to [Rob14, §9.1] for the construction of an analogous functor in the algebraic setting. □

Notation 3.1.8. Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of formal schemes. The image of f by (3.3) is the inverse image functor

$$f^* : \mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{X};\Lambda) \to \mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{Y};\Lambda)$$

which has the structure of a monoidal functor. Its right adjoint f_* is the direct image functor. It has the structure of a right-lax monoidal functor. (See Lemma 3.4.1 below.)

Notation 3.1.9. Recall that we denote by $\mathcal{X}_\sigma$ the special fiber of a formal scheme $\mathcal{X}$. (See Notation 1.1.6.) The functor $\mathcal{X} \mapsto \mathcal{X}_\sigma$ induces a functor $(-)_\sigma : \mathrm{FSm}/\mathcal{S} \to \mathrm{Sm}/\mathcal{S}_\sigma$ which is continuous for the topology τ. By the functoriality of the construction of ∞-categories of motives, we deduce an adjunction

$$\sigma^* : \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda) \rightleftarrows \mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}_\sigma;\Lambda) : \sigma_*. \tag{3.4}$$

In fact, modulo the identification of Remark 3.1.5(4), σ^* is simply the inverse image functor associated to the morphism of formal schemes $\mathcal{X}_\sigma \to \mathcal{X}$.

Theorem 3.1.10. *The functors σ^* and σ_* in* (3.4) *are equivalences of ∞-categories.*

Proof. This is [Ayo15, Corollaires 1.4.24 & 1.4.29] under the assumption that $\mathcal{S}$ is of finite type over $\mathrm{Spf}(k^\circ)$, with k° a complete valuation ring of height ≤ 1. However, this assumption is not used in the proofs of these results. □

Remark 3.1.11. Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of formal schemes. Modulo the equivalences of Theorem 3.1.10, the operations f^* and f_* coincide with the operations f_σ^* and $f_{\sigma,*}$ associated to the morphism of schemes $f_\sigma : \mathcal{Y}_\sigma \to \mathcal{X}_\sigma$. When f_σ is locally of finite type, we denote by $f_!$ and $f^!$ the operations on formal motives corresponding to the operations $f_{\sigma,!}$ and $f_\sigma^!$ on algebraic motives modulo the equivalences of Theorem 3.1.10 (in the T-stable case). Similarly, if f_σ is smooth, we denote by $f_\sharp$ the operation corresponding to $f_{\sigma,\sharp}$.

Notation 3.1.12. Recall that we denote by $\mathcal{X}^{\mathrm{rig}}$ the generic fiber of a formal scheme $\mathcal{X}$. (See Notation 1.1.8.) The functor $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{rig}}$ induces a functor $(-)^{\mathrm{rig}} : \mathrm{FSm}/\mathcal{S} \to \mathrm{RigSm}/\mathcal{S}^{\mathrm{rig}}$ which is continuous for the topology τ. By the functoriality of the construction of ∞-categories of motives, we deduce an adjunction

$$\xi_{\mathcal{S}} : \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda) \rightleftarrows \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda) : \chi_{\mathcal{S}}. \tag{3.5}$$

Composing with the equivalences of Theorem 3.1.10, we get also an equivalent adjunction

$$\xi_{\mathcal{S}} : \mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}_\sigma;\Lambda) \rightleftarrows \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda) : \chi_{\mathcal{S}}. \tag{3.6}$$

These adjunctions will play an important role in this section.

Proposition 3.1.13. *The functors $\xi_{\mathcal{S}}$, for $\mathcal{S} \in$* FSch*, are part of a morphism of* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$*-valued presheaves*

$$\xi : \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\Lambda)^\otimes \to \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}((-)^{\mathrm{rig}};\Lambda)^\otimes \tag{3.7}$$

on FSch. *In particular, the functors $\xi_{\mathcal{S}}$ are monoidal and commute with the inverse image functors. Moreover, if $f : \mathcal{T} \to \mathcal{S}$ is a smooth morphism in* FSch*, the natural transformation*

$$f_\sharp^{\mathrm{rig}} \circ \xi_{\mathcal{T}} \to \xi_{\mathcal{S}} \circ f_\sharp$$

is an equivalence.

Proof. One argues as in [Rob14, §9.1] for the first assertion. The second assertion is clear. □

In the rest of this subsection we use the above constructions to produce a convenient conservative family of functors for the ∞-category $\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(S;\Lambda)$, for S a rigid analytic space. This family is rather big: it is indexed by formal models of smooth rigid analytic S-spaces. For a better result, we refer the reader to Corollary 3.7.20 below. We start by recording the following general fact.

Proposition 3.1.14. *Let $(F_i : \mathcal{C}_i \to \mathcal{D})_i$ be a small family of functors in* Pr^{L} *having the same target $\mathcal{D}$. Let G_i be the right adjoint of F_i. Then the following conditions are equivalent:*

(1) *the family* $(G_i : \mathcal{D} \to \mathcal{C}_i)_{i\in I}$ *is conservative;*
(2) $\mathcal{D}$ *is generated under colimits by objects of the form* $F_i(A)$, *with* $A \in \mathcal{C}_i$.

Proof. Assume first that (2) is satisfied. Let $f : X \to Y$ be a map in $\mathcal{D}$ such that $G_i(f)$ is an equivalence for every i. We want to show that f is an equivalence. To do so, consider the full sub-∞-category $\mathcal{D}_0 \subset \mathcal{D}$ spanned by objects E such that $\mathrm{Map}_{\mathcal{D}}(E, X) \to \mathrm{Map}_{\mathcal{D}}(E, Y)$ is an equivalence. Clearly, $\mathcal{D}_0$ is stable under arbitrary colimits and contains the images of the F_i's. By (2), it follows that $\mathcal{D}_0 = \mathcal{D}$, and thus f is an equivalence by the Yoneda lemma.

We now assume that (1) is satisfied. Denote by $\mathcal{D}' \subset \mathcal{D}$ the smallest full sub-∞-category containing the images of the F_i's and stable under arbitrary colimits. We need to show that $\mathcal{D}' = \mathcal{D}$. We claim that the ∞-category $\mathcal{D}'$ is presentable. Indeed, as the F_i's are colimit-preserving and the $\mathcal{C}_i$'s are presentable, $\mathcal{D}'$ is the smallest sub-∞-category of $\mathcal{D}$ stable under colimits and containing a certain small set of objects (namely the union of images of sets of generators for the $\mathcal{C}_i$'s). These objects are κ-compact for κ large enough. Thus, our claim follows from Lemma 2.8.2. Using [Lur09, Corollary 5.5.2.9], we may thus consider the right adjoint ρ to the inclusion functor $\mathcal{D}' \to \mathcal{D}$. Fix an object $X \in \mathcal{D}$. We will show that $\rho(X) \to X$ is an equivalence, which will finish the proof. Since the G_i's form a conservative family, it is enough to show that the maps $G_i(\rho(X)) \to G_i(X)$ are equivalences. By the Yoneda lemma, it is enough to show that the maps

$$\mathrm{Map}_{\mathcal{C}_i}(A, G_i(\rho(X))) \to \mathrm{Map}_{\mathcal{C}_i}(A, G_i(X))$$

are equivalences for all $A \in \mathcal{C}_i$. By adjunction, these maps are equivalent to

$$\mathrm{Map}_{\mathcal{D}}(F_i(A), \rho(X)) \to \mathrm{Map}_{\mathcal{D}}(F_i(A), X),$$

which are equivalences since the $F_i(A)$'s belong to $\mathcal{D}'$. □

Proposition 3.1.15. *Let S be a rigid analytic space. For every $U \in \mathrm{RigSm}^{\mathrm{qcqs}}/S$, denote by $f_U : U \to S$ the structural morphism and choose a formal model $\mathcal{U}$ of U. Then, the functors*

$$\chi_{\mathcal{U}} \circ f_U^* : \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(S;\Lambda) \to \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{U};\Lambda),$$

for $U \in \mathrm{RigSm}^{\mathrm{qcqs}}/S$, form a conservative family. In fact, the same is true if we restrict to those U's admitting affine formal models of principal ideal type.

Proof. The functor $\chi_{\mathcal{U}} \circ f_U^*$ has a left adjoint $f_{U,\sharp} \circ \xi_{\mathcal{U}}$ sending the monoidal unit of $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{U};\Lambda)$ to $\mathrm{M}^{(\mathrm{eff})}(U)$. We conclude by Lemma 2.1.20 and Proposition 3.1.14. □

3.2. **Descent, continuity and stalks, I. The case of formal motives.**

In this subsection, we gather a few basic properties of the functor $\mathcal{S} \mapsto \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)$, $f \mapsto f^*$, from Proposition 3.1.7. We fix a topology $\tau \in \{\mathrm{nis}, \text{ét}\}$.

Proposition 3.2.1. *The contravariant functor*

$$\mathcal{S} \mapsto \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda), \quad f \mapsto f^*$$

defines a τ-(hyper)sheaf on FSch *with values in* Pr^{L}.

Proof. The proof is similar to that of Theorem 2.3.4. It suffices to prove that for every formal scheme $\mathcal{S}$, the functor

$$\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\Lambda) : (\text{Ét}/\mathcal{S})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}},$$

is a τ-(hyper)sheaf. One reduces, by an essentially formal argument, to show that the functor

$$\mathrm{Shv}_\tau^{(\wedge)}(\mathrm{FSm}/-;\Lambda) : (\text{Ét}/\mathcal{S})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$$

is a τ-(hyper)sheaf, and this follows from Corollary 2.3.8. The formal argument alluded to can be found in the proof of Theorem 2.3.4, and we will not repeat it here. $\square$

A formal scheme $\mathcal{S}$ is said to be (Λ,τ)-admissible (resp. (Λ,τ)-good) if the scheme $\mathcal{S}_\sigma$ is (Λ,τ)-admissible (resp. (Λ,τ)-good) in the sense of Definition 2.4.14.

Proposition 3.2.2. *Let $\tau \in \{\mathrm{nis}, \mathrm{\acute{e}t}\}$ and let $\mathcal{S}$ be a (Λ,τ)-admissible formal scheme. When τ is the étale topology, assume that Λ is eventually coconnective. Then, we have*

$$\mathbf{FSH}^{(\mathrm{eff}),\wedge}_\tau(\mathcal{S};\Lambda) = \mathbf{FSH}^{(\mathrm{eff})}_\tau(\mathcal{S};\Lambda).$$

Proof. This is proven in the same way as Proposition 2.4.19. $\square$

Proposition 3.2.3. *Let $\mathcal{S}$ be a formal scheme.*

(1) The ∞-category $\mathbf{FSH}^{(\mathrm{eff})}_\tau(\mathcal{S};\Lambda)$ is compactly generated if τ is the Nisnevich topology or if Λ is eventually coconnective. A set of compact generators is given, up to desuspension and negative Tate twists when applicable, by the $\mathrm{M}^{(\mathrm{eff})}(\mathcal{X})$ for $\mathcal{X} \in \mathrm{FSm}/\mathcal{S}$ quasi-compact, quasi-separated and (Λ,τ)-good.

(2) The ∞-category $\mathbf{FSH}^{(\mathrm{eff}),\wedge}_\tau(\mathcal{S};\Lambda)$ is compactly generated if $\mathcal{S}$ is (Λ,τ)-admissible. A set of compact generators is given, up to desuspension and negative Tate twists when applicable, by the $\mathrm{M}^{(\mathrm{eff})}(\mathcal{X})$ for $\mathcal{X} \in \mathrm{FSm}/\mathcal{S}$ quasi-compact, quasi-separated and (Λ,τ)-good.

Moreover, under the stated assumptions, the monoidal ∞-category $\mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{S};\Lambda)^\otimes$ belongs to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_\omega)$ and, if $f : \mathcal{T} \to \mathcal{S}$ is a quasi-compact and quasi-separated morphism of formal schemes with $\mathcal{T}$ assumed (Λ,τ)-admissible in the hypercomplete case, the functor $f^* : \mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{S};\Lambda) \to \mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{T};\Lambda)$ is compact-preserving, i.e., belongs to $\mathrm{Pr}^{\mathrm{L}}_\omega$.

Proof. This is proven in the same way as Proposition 2.4.22. $\square$

Given a formal scheme $\mathcal{S}$, we write "$\mathrm{pvcd}_\Lambda(\mathcal{S})$" instead of "$\mathrm{pvcd}_\Lambda(\mathcal{S}_\sigma)$"; see Definition 2.4.10. Our next statement is an analogue of Theorem 2.5.1 for formal motives.

Proposition 3.2.4. *Let $(\mathcal{S}_\alpha)_\alpha$ be a cofiltered inverse system of quasi-compact and quasi-separated formal schemes with affine transition maps, and let $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$ be the limit of this system. We assume one of the following two alternatives.*

(1) We work in the non-hypercomplete case.

(2) We work in the hypercomplete case, and $\mathcal{S}$ and the $\mathcal{S}_\alpha$'s are (Λ,τ)-admissible. When τ is the étale topology, we assume furthermore that Λ is eventually coconnective or that the numbers $\mathrm{pvcd}_\Lambda(\mathcal{S}_\alpha)$ are bounded independently of α.

Then the obvious functor

$$\operatorname*{colim}_\alpha \mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{S}_\alpha;\Lambda) \to \mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\mathcal{S};\Lambda),$$

where the colimit is taken in Pr^{L}, is an equivalence.

Proof. This follows immediately from Proposition 2.5.11 and Theorem 3.1.10. $\square$

We will use Proposition 3.2.4 to compute the stalks of $\mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(-;\Lambda)$ for the topology rig-τ on FSch. (See Corollary 1.4.13). We first describe a conservative family of points for this topology.

Remark 3.2.5. Let $\mathcal{S}$ be a formal scheme. A rigid point of $\mathcal{S}$ is a morphism $\mathfrak{s} : \mathrm{Spf}(V) \to \mathcal{S}$ where V is an adic valuation ring of principal ideal type. We sometimes also denote by $\mathfrak{s}$ the formal scheme $\mathrm{Spf}(V)$. The assignment $(\mathrm{Spf}(V) \to \mathcal{S}) \mapsto (\mathrm{Spf}(V)^{\mathrm{rig}} \to \mathcal{S}^{\mathrm{rig}})$ is an equivalence of groupoids between rigid points of $\mathcal{S}$ and those of $\mathcal{S}^{\mathrm{rig}}$. (See Remark 1.4.25.) We will say that a rigid point $\mathfrak{s} : \mathrm{Spf}(V) \to \mathcal{S}$ is algebraic (resp. τ-geometric) if the associated rigid point of $\mathcal{S}^{\mathrm{rig}}$ is algebraic (resp. τ-geometric). See Remarks 1.4.23 and 1.4.25, and Definition 1.4.24.

Proposition 3.2.6. *Let $\mathcal{S}$ be a formal scheme. We denote by* $\mathrm{FRig\acute{E}t}/\mathcal{S}$ *the category of rig-étale formal $\mathcal{S}$-schemes. Then, the site* $(\mathrm{FRig\acute{E}t}/\mathcal{S}, \text{rig-}\tau)$ *admits a conservative family of points indexed by τ-geometric algebraic rigid points $\mathfrak{s} = \mathrm{Spf}(V) \to \mathcal{S}$. To such a rigid point $\mathfrak{s}$, the associated topos-theoretic point is given by*

$$\mathcal{F} \mapsto \mathcal{F}_{\mathfrak{s}} = \operatorname*{colim}_{\mathrm{Spf}(V) \to \mathcal{U} \to \mathcal{S}} \mathcal{F}(\mathcal{U})$$

where the colimit is over rig-étale neighbourhoods $\mathcal{U}$ of $\mathfrak{s}$. Moreover, one may restrict to those rigid points of $\mathcal{S}^{\mathrm{rig}}$ as in Construction 1.4.27.

Proof. This follows from Corollary 1.4.13 and Proposition 1.4.29. □

Proposition 3.2.7. *Let $\mathcal{S}$ be a formal scheme and let $\mathfrak{s} \to \mathcal{S}$ be an algebraic rigid point of $\mathcal{S}$. Assume one of the following two alternatives.*

(1) We work in the non-hypercomplete case.
(2) We work in the hypercomplete case, and $\mathcal{S}$ and $\mathcal{S}^{\mathrm{rig}}$ are (Λ, τ)-admissible. When τ is the étale topology, we assume furthermore that Λ is eventually coconnective or that the numbers $\mathrm{pvcd}_\Lambda(\mathcal{S}')$*, for admissible blowups $\mathcal{S}' \to \mathcal{S}$, are bounded independently of $\mathcal{S}'$.*

Then there is an equivalence of ∞-categories

$$\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\Lambda)_{\mathfrak{s}} \simeq \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{s};\Lambda)$$

where the left-hand side is the stalk of $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\Lambda)$ *at $\mathfrak{s}$, i.e., the colimit, taken in* Pr^{L}*, of the diagram* $(\mathfrak{s} \to \mathcal{U} \to \mathcal{S}) \mapsto \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{U};\Lambda)$ *with* $\mathcal{U} \in \mathrm{FRig\acute{E}t}/\mathcal{S}$.

Proof. This follows from Proposition 3.2.4. Indeed, the condition that $\mathcal{S}^{\mathrm{rig}}$ is (Λ, τ)-admissible implies that $\mathfrak{s}$ is (Λ, τ)-admissible. Moreover, if the numbers $\mathrm{pvcd}_\Lambda(\mathcal{S}')$ are bounded independently of $\mathcal{S}'$ for admissible blowups $\mathcal{S}' \to \mathcal{S}$, then the same is true for the numbers $\mathrm{pvcd}_\Lambda(\mathcal{U})$ for the saturated rig-étale neighbourhoods $\mathfrak{s} \to \mathcal{U} \to \mathcal{S}$. □

3.3. **Statement of the main result.**

Let $\mathcal{S}$ be a formal scheme. By Proposition 3.1.13, we have a monoidal functor

$$\xi_{\mathcal{S}}^{\otimes} : \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)^{\otimes} \to \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda)^{\otimes}.$$

From Corollary 3.4.2 below, we deduce that $\chi_{\mathcal{S}}\Lambda$ underlies a commutative algebra in the monoidal ∞-category $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)^{\otimes}$, which we also denote by $\chi_{\mathcal{S}}\Lambda$. Moreover, the functor $\chi_{\mathcal{S}}$ admits a factorization

$$\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda) \xrightarrow{\widetilde{\chi}_{\mathcal{S}}} \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\chi\Lambda) \xrightarrow{\mathrm{ff}} \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda),$$

where $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\chi\Lambda)$ is the ∞-category of $\chi_{\mathcal{S}}\Lambda$-modules in $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)^{\otimes}$ and ff is the forgetful functor. The functor $\widetilde{\chi}_{\mathcal{S}}$ admits a left adjoint

$$\widetilde{\xi}_{\mathcal{S}} : \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\chi\Lambda) \to \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda)$$

that sends a $\chi_{\mathcal{S}}\Lambda$-module M to $\xi_{\mathcal{S}}(M)\otimes_{\xi_{\mathcal{S}}\chi_{\mathcal{S}}\Lambda}\Lambda$. It will be important for us to know that the functors $\widetilde{\xi}_{\mathcal{S}}$, for $\mathcal{S}\in\mathrm{FSch}$, are part of a morphism

$$\widetilde{\xi}^{\otimes}:\mathbf{FSH}_{\tau}^{(\mathrm{eff},\wedge)}(-;\chi\Lambda)^{\otimes}\to\mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}((-)^{\mathrm{rig}};\Lambda)^{\otimes}$$

in the ∞-category $\mathrm{PSh}(\mathrm{FSch};\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))$ of presheaves on FSch valued in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. The construction of $\widetilde{\xi}^{\otimes}$ will be carried in Subsection 3.4 below. Before stating the main result of this section, we introduce the following assumptions.

Assumption 3.3.1. We assume (at least) one of the following four alternatives:

- (i) τ is the Nisnevich topology;
- (ii) $\pi_0\Lambda$ is a $\mathbb{Q}$-algebra;
- (iii) we work in the non-hypercomplete case, Λ is eventually coconnective and every prime number which is not invertible in $\pi_0\Lambda$ is invertible on every formal scheme we consider;
- (iv) we work in the hypercomplete case, every formal scheme we consider is (Λ,τ)-admissible and its generic fiber is also (Λ,τ)-admissible, and every prime number which is not invertible in $\pi_0\Lambda$ is invertible on every formal scheme we consider.

Moreover, under one of the alternatives (iii) or (iv), when we write “FSch”, we actually mean the full subcategory of formal schemes satisfying the properties in (iii) or (iv) respectively.

Assumption 3.3.2. We assume that τ is the étale topology and that one of the two alternatives (iii) or (iv) above is satisfied.

Theorem 3.3.3.

(1) We work under Assumption 3.3.1. Given a formal scheme $\mathcal{S}$, the functor

$$\widetilde{\xi}_{\mathcal{S}}:\mathbf{FSH}_{\tau}^{(\wedge)}(\mathcal{S};\chi\Lambda)\to\mathbf{RigSH}_{\tau}^{(\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda)$$

is fully faithful.

(2) We work under Assumption 3.3.2. The morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$*-valued presheaves*

$$\widetilde{\xi}^{\otimes}:\mathbf{FSH}_{\text{ét}}^{(\wedge)}(-;\chi\Lambda)^{\otimes}\to\mathbf{RigSH}_{\text{ét}}^{(\wedge)}((-)^{\mathrm{rig}};\Lambda)^{\otimes}$$

exhibits $\mathbf{RigSH}_{\text{ét}}^{(\wedge)}((-)^{\mathrm{rig}};\Lambda)^{\otimes}$ *as the rig-étale sheaf associated to* $\mathbf{FSH}_{\text{ét}}^{(\wedge)}(-;\chi\Lambda)^{\otimes}$.

Remark 3.3.4. Our proof of Theorem 3.3.3 relies crucially on T-stability. Therefore, we do not expect this theorem to hold for the effective ∞-categories of motives.

Remark 3.3.5. One can reformulate Theorem 3.3.3(2) as an equivalence between functors defined on rigid analytic spaces. Indeed, by Corollary 1.4.13, we have an equivalence of sites

$$(\mathrm{RigSpc}^{\mathrm{qcqs}},\text{ét})\xrightarrow{\sim}(\mathrm{FSch}^{\mathrm{qcqs}},\text{rigét}).$$

Moreover, the left Kan extension of the $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaf $\mathbf{FSH}_{\text{ét}}^{(\wedge)}(-;\chi\Lambda)^{\otimes}$ along the functor $(-)^{\mathrm{rig}}:\mathrm{FSch}^{\mathrm{qcqs}}\to\mathrm{RigSpc}^{\mathrm{qcqs}}$ is easily seen to be given by

$$S\mapsto\operatorname*{colim}_{\mathcal{S}\in\mathrm{Mdl}(S)}\mathbf{FSH}_{\text{ét}}^{(\wedge)}(\mathcal{S};\chi\Lambda)^{\otimes}.\tag{3.8}$$

(See Notation 1.1.9.) Thus, Theorem 3.3.3(2) implies that the morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves given by

$$\operatorname*{colim}_{\mathcal{S}\in\mathrm{Mdl}(S)}\mathbf{FSH}_{\text{ét}}^{(\wedge)}(\mathcal{S};\chi\Lambda)^{\otimes}\to\mathbf{RigSH}_{\text{ét}}^{(\wedge)}(S;\Lambda)^{\otimes}$$

exhibits $\mathbf{RigSH}_{\text{ét}}^{(\wedge)}(-;\Lambda)^{\otimes}$ as the étale sheafification of the $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaf (3.8).

3.4. Construction of $\widetilde{\xi}^{\otimes}$.

We denote by Fin_* the category of finite pointed sets. Up to isomorphism, the objects of Fin_* are the pointed sets $\langle n\rangle = \{1, \dots, n\} \cup \{*\}$, for $n \in \mathbb{N}$. For $1 \le i \le n$, we denote by $\rho^i : \langle n\rangle \to \langle 1\rangle$ the unique map such that $(\rho^i)^{-1}(1) = \{i\}$. Recall that a symmetric monoidal ∞-category is a coCartesian fibration $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ such that the induced functor $(\rho^i_!)_i : \mathcal{C}_{\langle n\rangle} \to \prod_{1\le i\le n} \mathcal{C}_{\langle 1\rangle}$ is an equivalence for all $n \ge 0$. We usually write "$\mathcal{C}_{\langle n\rangle}$" instead of "$\mathcal{C}^{\otimes}_{\langle n\rangle}$" to denote the fiber of $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ at $\langle n\rangle$. The ∞-category $\mathcal{C}_{\langle 1\rangle}$ is called the underlying ∞-category of $\mathcal{C}^{\otimes}$ and is denoted by $\mathcal{C}$. Recall also that a monoidal functor is a morphism of coCartesian fibrations between symmetric monoidal ∞-categories, i.e., a functor over Fin_* which preserves coCartesian edges.

We remind the reader that "monoidal" always means "symmetric monoidal" in this paper. We denote by $\mathrm{CAlg}(\mathrm{CAT}_\infty)$ the ∞-category of (possibly large) monoidal ∞-categories and monoidal functors between them. The following lemma is well-known.

Lemma 3.4.1. *Let $F^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a monoidal functor between monoidal ∞-categories. Then the following conditions are equivalent.*

(1) The underlying functor F admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$;

(2) The functor $F^{\otimes}$ admits a right adjoint $G^{\otimes}$ making the following triangle commutative

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[dr, "p"'] & & \mathcal{D}^{\otimes} \arrow[ll, "G^{\otimes}"'] \arrow[dl, "q"] \\ & \mathrm{Fin}_* & \end{tikzcd}$$

with p and q the defining coCartesian fibrations.

Moreover, if these conditions are satisfied, we have the following two extra properties.

(a) The natural transformations

$$p \to p \circ G^{\otimes} \circ F^{\otimes} = p \qquad \text{and} \qquad q = q \circ F^{\otimes} \circ G^{\otimes} \to q,$$

induced by the unit and the counit of the adjunction $(F^{\otimes}, G^{\otimes})$, are the identity natural transformations of p and q.

(b) The functor $G^{\otimes}$ is a right-lax monoidal functor (i.e., preserves coCartesian edges over the arrows $\rho^i : \langle n\rangle \to \langle 1\rangle$ for $1 \le i \le n$) and its underlying functor $G_{\langle 1\rangle}$ is equivalent to G.

Proof. This is contained in [Lur17, Propositions 7.3.2.5 & 7.3.2.6, & Corollary 7.3.2.7]. We also remark that property (a) is automatic. In fact, more generally, every invertible natural transformation of p is the identity, and similarly for q. □

Corollary 3.4.2. *Let $F^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a monoidal functor between monoidal ∞-categories, and assume that F admits a right adjoint G. Then the induced functor*

$$\mathrm{CAlg}(F) : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{D})$$

admits also a right adjoint, which is given by $\mathrm{CAlg}(G)$.

Proof. Let $p : \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ and $q : \mathcal{D}^{\otimes} \to \mathrm{Fin}_*$ be the defining coCartesian fibrations. Recall that $\mathrm{CAlg}(\mathcal{C})$ is the full sub-∞-category of $\mathrm{Sect}(p) = \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(\mathrm{Fin}_*, \mathrm{Fin}_*)} \mathrm{id}_{\mathrm{Fin}_*}$ spanned by those sections of p sending the arrows $\rho^i : \langle n\rangle \to \langle 1\rangle$, for $1 \le i \le n$, to coCartesian edges, and similarly for $\mathrm{CAlg}(\mathcal{D})$. It follows that $F^{\otimes}$ and $G^{\otimes}$ induce functors $\mathrm{CAlg}(F)$ and $\mathrm{CAlg}(G)$, and that the unit and counit of the adjunction $(F^{\otimes}, G^{\otimes})$ define natural transformations

$$\mathrm{id} \to \mathrm{CAlg}(G) \circ \mathrm{CAlg}(F) \qquad \text{and} \qquad \mathrm{CAlg}(F) \circ \mathrm{CAlg}(G) \to \mathrm{id}$$

satisfying the usual identities up to homotopy. $\square$

We now start our construction of $\widetilde{\xi}^{\otimes}$. By Proposition 3.1.13, we have a morphism

$$\xi^{\otimes} : \mathbf{FSH}_{\tau}^{(\mathrm{eff},\wedge)}(-;\Lambda)^{\otimes} \to \mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}((-)^{\mathrm{rig}};\Lambda)^{\otimes}$$

in the ∞-category $\mathrm{Fun}(\mathrm{FSch}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{CAT}_{\infty}))$. The formation of ∞-categories of commutative algebras gives a functor $\mathrm{CAlg}(-) : \mathrm{CAlg}(\mathrm{CAT}_{\infty}) \to \mathrm{CAT}_{\infty}$. Applying this functor to $\xi^{\otimes}$ yields a morphism

$$\mathrm{CAlg}(\xi) : \mathrm{CAlg}(\mathbf{FSH}_{\tau}^{(\mathrm{eff},\wedge)}(-;\Lambda)) \to \mathrm{CAlg}(\mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}((-)^{\mathrm{rig}};\Lambda))$$

in the ∞-category $\mathrm{Fun}(\mathrm{FSch}^{\mathrm{op}}, \mathrm{CAT}_{\infty})$. Applying Lurie's unstraightening construction [Lur09, §3.2] to this morphism, we get a commutative triangle

$$\begin{tikzcd}
\Xi_0 \arrow[rr, "F"] \arrow[dr, "p_0"'] & & \Xi_1 \arrow[dl, "p_1"] \\
& \mathrm{FSch}^{\mathrm{op}} &
\end{tikzcd}$$

where p_0 and p_1 are coCartesian fibrations classified by

$$\mathrm{CAlg}(\mathbf{FSH}_{\tau}^{(\mathrm{eff},\wedge)}(-;\Lambda)) \qquad \text{and} \qquad \mathrm{CAlg}(\mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}((-)^{\mathrm{rig}};\Lambda)),$$

and F is the functor induced by $\mathrm{CAlg}(\xi)$. By Corollary 3.4.2, the fibers of F admit right adjoints. More precisely, for $\mathcal{S} \in \mathrm{FSch}$, the functor $F_{\mathcal{S}} = \mathrm{CAlg}(\xi_{\mathcal{S}})$ admits a right adjoint, which is given by $\mathrm{CAlg}(\chi_{\mathcal{S}})$. (Note that $\chi_{\mathcal{S}}^{\otimes}$ is a right-lax monoidal functor.) Applying [Lur17, Proposition 7.3.2.6], we deduce that F admits a right adjoint G making the following triangle

$$\begin{tikzcd}
\Xi_0 \arrow[dr, "p_0"'] & & \Xi_1 \arrow[ll, "G"'] \arrow[dl, "p_1"] \\
& \mathrm{FSch}^{\mathrm{op}} &
\end{tikzcd}$$

commutative and such that, for every $\mathcal{S} \in \mathrm{FSch}$, the functor $G_{\mathcal{S}}$ is equivalent to $\mathrm{CAlg}(\chi_{\mathcal{S}})$.

We now consider the ∞-categories $\mathrm{Sect}(p_0)$ and $\mathrm{Sect}(p_1)$ of sections of p_0 and p_1. The functor G induces a functor $G' : \mathrm{Sect}(p_1) \to \mathrm{Sect}(p_0)$. We have an obvious object $\mathbf{1} \in \mathrm{Sect}(p_1)$, such that $\mathbf{1}_{\mathcal{S}} \in \mathrm{CAlg}(\mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda))$ is the initial algebra for every $\mathcal{S} \in \mathrm{FSch}$. We set:

$$\mathcal{A} = G'(\mathbf{1}).$$

By construction, $\mathcal{A}$ is a section of the coCartesian fibration p_0 such that $\mathcal{A}_{\mathcal{S}}$ is equivalent to $\chi_{\mathcal{S}}\Lambda$ considered as an object of $\mathrm{CAlg}(\mathbf{FSH}_{\tau}^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda))$. For a morphism $f : \mathcal{T} \to \mathcal{S}$ of formal schemes, the induced morphism $\mathcal{A}_{\mathcal{S}} \to \mathcal{A}_{\mathcal{T}}$ in Ξ_0 corresponds to a morphism $f^*\mathcal{A}_{\mathcal{S}} \to \mathcal{A}_{\mathcal{T}}$. This is the morphism induced by the natural transformation $f^* \circ \chi_{\mathcal{S}} \to \chi_{\mathcal{T}} \circ f^{\mathrm{rig},*}$ which one obtains by adjunction from the equivalence $f^{\mathrm{rig},*} \circ \xi_S \simeq \xi_T \circ f^*$. The following fact, which we record for later use, follows easily from this description.

Lemma 3.4.3. *Let $f : \mathcal{T} \to \mathcal{S}$ be a morphism of formal schemes. For f to be sent to a p_0-coCartesian edge by $\mathcal{A}$, it suffices that the commutative square*

$$\begin{tikzcd} \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda) \arrow[r, "\xi_\mathcal{S}"] \arrow[d, "f^*"] & \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda) \arrow[d, "f^{\mathrm{rig},*}"] \\ \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{T};\Lambda) \arrow[r, "\xi_\mathcal{T}"] & \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{T}^{\mathrm{rig}};\Lambda) \end{tikzcd}$$

is right adjointable. This happens when f is smooth.

Proof. Only the last assertion requires a proof. If f is smooth, then there is a commutative square

$$\begin{tikzcd} \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{T};\Lambda) \arrow[r, "\xi_\mathcal{T}"] \arrow[d, "f_\sharp"] & \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{T}^{\mathrm{rig}};\Lambda) \arrow[d, "f_\sharp^{\mathrm{rig}}"] \\ \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda) \arrow[r, "\xi_\mathcal{S}"] & \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda) \end{tikzcd}$$

by Proposition 3.1.13. The natural transformation $f^* \circ \chi_\mathcal{S} \to \chi_\mathcal{T} \circ f^{\mathrm{rig},*}$ deduced from the square of the statement via the adjunctions $(\xi_\mathcal{S}, \chi_\mathcal{S})$ and $(\xi_\mathcal{T}, \chi_\mathcal{T})$ coincides with the natural equivalence deduced from the above square via the adjunctions $(\xi_\mathcal{S} \circ f_\sharp, f^* \circ \chi_\mathcal{S})$ and $(f_\sharp^{\mathrm{rig}} \circ \xi_\mathcal{T}, \chi_\mathcal{T} \circ f^{\mathrm{rig},*})$. □

Before going further, we need a small digression about algebras and modules in general monoidal ∞-categories. Let $\mathcal{C}^\otimes$ be a monoidal ∞-category and $p : \mathcal{C}^\otimes \to \mathrm{Fin}_*$ the defining coCartesian fibration. By [Lur17, §3.3.3], we may associate to $\mathcal{C}^\otimes$ a functor

$$f : \mathrm{Mod}(\mathcal{C})^\otimes \to \mathrm{Fin}_* \times \mathrm{CAlg}(\mathcal{C}) \tag{3.9}$$

such that, for each commutative algebra A of $\mathcal{C}^\otimes$, the induced functor

$$f_A : \mathrm{Mod}_A(\mathcal{C})^\otimes = \mathrm{Mod}(\mathcal{C})^\otimes \times_{\mathrm{CAlg}(\mathcal{C})} \{A\} \to \mathrm{Fin}_* \tag{3.10}$$

makes $\mathrm{Mod}_A(\mathcal{C})^\otimes$ into an ∞-operad. This is the ∞-operad of A-modules, which is a monoidal ∞-category whenever $\mathcal{C}$ admits enough colimits, and these colimits are compatible with the monoidal structure. We recall below the construction of the simplicial set $\mathrm{Mod}(\mathcal{C})^\otimes$ which is a particular case of [Lur17, Construction 3.3.3.1].

Construction 3.4.4. Recall that a map $\gamma : \langle m\rangle \to \langle n\rangle$ is said to be inert (resp. semi-inert) if the induced map $\gamma^{-1}(\{1,\dots,n\}) \to \{1,\dots,n\}$ is a bijection (resp. an injection). The map γ is said to be null if its image is the base-point of $\langle n\rangle$. Let $\mathrm{K} \subset \mathrm{Fun}(\Delta^1, \mathrm{Fin}_*)$ be the full subcategory spanned by the semi-inert maps. We have two obvious functors $e_0, e_1 : \mathrm{K} \to \mathrm{Fin}_*$ induced by the inclusions $\{0\},\{1\} \subset \Delta^1$. Given $\langle m\rangle \in \mathrm{Fin}_*$, a morphism δ in the fiber $e_0^{-1}(\langle m\rangle)$ of e_0 at $\langle m\rangle$ is said to be inert if the map $e_1(\delta)$, which belongs to Fin_*, is inert.

We define a simplicial set $\mathrm{Mod}(\mathcal{C})^\otimes$ as follows. Giving a map $\Delta^n \to \mathrm{Mod}(\mathcal{C})^\otimes$ is equivalent to giving a map $\Delta^n \to \mathrm{Fin}_*$, and a functor $\Delta^n \times_{\mathrm{Fin}_*, e_0} \mathrm{K} \to \mathcal{C}^\otimes$ making the triangle

$$\begin{tikzcd} \Delta^n \times_{\mathrm{Fin}_*, e_0} \mathrm{K} \arrow[r] \arrow[dr, "e_1 \circ \mathrm{pr}_\mathrm{K}"'] & \mathcal{C}^\otimes \arrow[d, "p"] \\ & \mathrm{Fin}_* \end{tikzcd}$$

commutative and such that the following condition is satisfied. For every vertex $\{i\} \subset \Delta^n$, the induced functor $\{i\} \times_{\mathrm{Fin}_*, e_0} \mathrm{K} \to \mathcal{C}^\otimes$ takes an inert map to a p-coCartesian morphism.

There is a full inclusion $\mathrm{Fin}_* \times \mathrm{Fin}_* \to \mathrm{K}$, sending a pair of objects to the null morphism between them, which is a section to (e_0, e_1). This induces the functor (3.9). That the functor (3.10) defines an ∞-operad is a particular case of [Lur17, Theorem 3.3.3.9]. According to [Lur17, Theorem 4.5.3.1], the functor (3.9) is a coCartersian fibration when $\mathcal{C}$ admits geometric realisations which are moreover compatible with the monoidal structure. In this case, the functor (3.10) is also a coCartesian fibration and thus the ∞-operad $\mathrm{Mod}_A(\mathcal{C})^{\otimes}$ is a monoidal ∞-category. (This is also stated explicitly in [Lur17, Theorems 4.5.2.1].)

Remark 3.4.5. It follows from Construction 3.4.4 that $\mathrm{Mod}(-)^{\otimes}$ defines a functor from $\mathrm{CAlg}(\mathrm{CAT}_\infty)$ to CAT_∞ endowed with a natural transformation $f : \mathrm{Mod}(-)^{\otimes} \to \mathrm{Fin}_* \times \mathrm{CAlg}(-)$. In fact, Construction 3.4.4 shows more: $\mathrm{Mod}(-)^{\otimes}$ and f naturally extend to a larger ∞-category of monoidal ∞-categories where the morphisms are given by right-lax monoidal functors.

Now, we go back to the situation we are interested in. We start again with our morphism $\xi^{\otimes}$ in $\mathrm{Fun}(\mathrm{FSch}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{CAT}_\infty))$. Applying the functors $\mathrm{Mod}(-)^{\otimes}$ and $\mathrm{CAlg}(-)$, we obtain a commutative square in $\mathrm{Fun}(\mathrm{FSch}^{\mathrm{op}}, \mathrm{CAT}_\infty)$:

$$\begin{tikzcd}
\mathrm{Mod}(\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\Lambda))^{\otimes} \arrow[r, "\mathrm{Mod}(\xi)^{\otimes}"] \arrow[d, "f_0"'] & \mathrm{Mod}(\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}((-)^{\mathrm{rig}};\Lambda))^{\otimes} \arrow[d, "f_1"] \\
\mathrm{Fin}_* \times \mathrm{CAlg}(\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\Lambda)) \arrow[r, "\mathrm{CAlg}(\xi)"] & \mathrm{Fin}_* \times \mathrm{CAlg}(\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}((-)^{\mathrm{rig}};\Lambda)).
\end{tikzcd}$$

Applying Lurie's unstraightening construction [Lur09, §3.2], we get a commutative diagram

$$\begin{tikzcd}
\mathfrak{M}_0^{\otimes} \arrow[rr, "H^{\otimes}"] \arrow[d, "q_0"'] & & \mathfrak{M}_1^{\otimes} \arrow[d, "q_1"] \\
\mathrm{Fin}_* \times \Xi_0 \arrow[rr, "F"] \arrow[dr, "p_0"'] & & \mathrm{Fin}_* \times \Xi_1 \arrow[dl, "p_1"] \\
& \mathrm{Fin}_* \times \mathrm{FSch}^{\mathrm{op}}. &
\end{tikzcd}$$

The functors p_0, p_1, q_0, q_1, $p_0 \circ q_0$ and $p_1 \circ q_1$ are coCartesian fibrations. Indeed, for p_0 and p_1, this is by construction. For the remaining functors, this follows from the Lemma 3.4.6 below and [Lur09, Proposition 2.4.2.3(3)].

Lemma 3.4.6. *Let $\mathcal{C}$ be an ∞-category and $\mathcal{E}^{\otimes} : \mathcal{C} \to \mathrm{CAlg}(\mathrm{CAT}_\infty)$ a functor. Consider the commutative triangle*

$$\begin{tikzcd}
\mathcal{M}^{\otimes} \arrow[rr, "r"] \arrow[dr] & & \mathrm{Fin}_* \times \mathcal{D} \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd}$$

obtained by applying Lurie's unstraightening construction [Lur09, §3.2] *to the morphism*

$$\mathrm{Mod}(\mathcal{E}(-))^{\otimes} \to \mathrm{Fin}_* \times \mathrm{CAlg}(\mathcal{E}(-))$$

in $\mathrm{Fun}(\mathcal{C}, \mathrm{CAT}_\infty)$. *We assume the following conditions:*

- *for every $X \in \mathcal{C}$, the ∞-category $\mathcal{E}(X)$ admits geometric realisations and these are compatible with the monoidal structure;*

- *for every morphism $f : X \to Y$, the induced functor $\mathcal{E}(f)$ commutes with geometric realisations.*

Then r is a coCartesian fibration.

Proof. By [Lur17, Theorem 4.5.3.1], the morphism $r_X : \mathcal{M}_X^{\otimes} \to \mathrm{Fin}_* \times \mathcal{D}_X$ is a coCartesian fibration for every $X \in \mathcal{C}$. Using [Lur09, Proposition 2.4.2.11], we deduce that r is a locally coCartesian fibration. By [Lur09, Proposition 2.4.2.8], it remains to check that locally r-coCartesian morphisms are stable under composition. Consider a commutative triangle in $\mathrm{Fin}_* \times \mathcal{D}$ that we depict informally as

$$\begin{tikzcd} (\langle n_0\rangle, X_0, R_0) \arrow[rr, "(\gamma_{02},f_{02},\phi_{02})"] \arrow[dr, "(\gamma_{01},f_{01},\phi_{01})"'] & & (\langle n_2\rangle, X_2, R_2) \\ & (\langle n_1\rangle, X_1, R_1). \arrow[ur, "(\gamma_{12},f_{12},\phi_{12})"'] & \end{tikzcd}$$

Here X_i, for $0 \leq i \leq 2$, are objects of $\mathcal{C}$ and $f_{ij} : X_i \to X_j$, for $0 \leq i < j \leq 2$, are morphisms of $\mathcal{C}$, each R_i is a commutative algebra in $\mathcal{E}(X_i)$ and each $\phi_{ij} : \mathcal{E}(f_{ij})(R_i) \to R_j$ is a morphism of commutative algebras in $\mathcal{E}(X_j)$, and the γ_{ij}'s are maps in Fin_*. From this triangle, we deduce a triangle of ∞-categories

$$\begin{tikzcd} \mathrm{Mod}_{R_0}(\mathcal{E}(X_0))_{\langle n_0\rangle} \arrow[rr, "(\gamma_{02},f_{02},\phi_{02})_!"] \arrow[dr, "(\gamma_{01},f_{01},\phi_{01})_!"'] & & \mathrm{Mod}_{R_2}(\mathcal{E}(X_2))_{\langle n_2\rangle} \\ & \mathrm{Mod}_{R_1}(\mathcal{E}(X_1))_{\langle n_1\rangle} \arrow[ur, "(\gamma_{12},f_{12},\phi_{12})_!"'] & \end{tikzcd}$$

and we need to show that this triangle commutes up to equivalence. Using that the $\mathcal{E}(f_{ij})$'s commute with the tensor product of modules, one reduces easily to the case where $n_0 = n_1 = n_2 = 1$ and γ_{ij} are the identity maps. We are then left to check that

$$\mathcal{E}(f_{12})(\mathcal{E}(f_{01})(-) \otimes_{\mathcal{E}(f_{01})(R_0)} R_1) \otimes_{\mathcal{E}(f_{12})(R_1)} R_2 \simeq \mathcal{E}(f_{02})(-) \otimes_{\mathcal{E}(f_{02})(R_0)} R_2,$$

which follows again from the fact that the $\mathcal{E}(f_{ij})$'s commute with the tensor product of modules. $\square$

Recall that we have constructed a section $\mathcal{A} : \mathrm{FSch}^{\mathrm{op}} \to \Xi_0$ together with a morphism $F\mathcal{A} \to \mathbf{1}$. Using Lemma 3.4.6 and [Lur09, Proposition 2.4.2.3(2)], we get coCartesian fibrations

$$\begin{aligned} \Phi_0 &= \mathfrak{M}_0^{\otimes} \times_{\Xi_0, \mathbf{1} \to \mathcal{A}} (\Delta^1 \times \mathrm{FSch}^{\mathrm{op}}) \to \Delta^1 \times \mathrm{Fin}_* \times \mathrm{FSch}^{\mathrm{op}}, \\ \Phi_1 &= \mathfrak{M}_1^{\otimes} \times_{\Xi_1, \mathbf{1} \to F\mathcal{A} \to \mathbf{1}} (\Delta^2 \times \mathrm{FSch}^{\mathrm{op}}) \to \Delta^2 \times \mathrm{Fin}_* \times \mathrm{FSch}^{\mathrm{op}}, \end{aligned}$$

and a morphism $\Phi_0 \to \Phi_1 \times_{\Delta^2} \Delta^{\{0,1\}}$ induced by $H^{\otimes}$. Let us pause and describe informally what we have constructed. For $\mathcal{S} \in \mathrm{FSch}$, the coCartesian fibration $(\Phi_0)_{\mathcal{S}} \to \Delta^1 \times \mathrm{Fin}_*$ is classified by the monoidal functor $- \otimes_{\Lambda} \chi\Lambda : \mathbf{FSH}_{\tau}^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)^{\otimes} \to \mathbf{FSH}_{\tau}^{(\mathrm{eff},\wedge)}(\mathcal{S};\chi\Lambda)^{\otimes}$. Similarly, the coCartesian fibration $(\Phi_1)_{\mathcal{S}} \to \Delta^2 \times \mathrm{Fin}_*$ is classified by the commutative triangle

$$\begin{tikzcd} \mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda)^{\otimes} \arrow[r, "-\otimes_{\Lambda}\xi\chi\Lambda"] \arrow[dr, equal] & \mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\xi\chi\Lambda)^{\otimes} \arrow[d, "-\otimes_{\xi\chi\Lambda}\Lambda"] \\ & \mathbf{RigSH}_{\tau}^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda)^{\otimes}. \end{tikzcd}$$

Finally, applying Lurie's straightening construction [Lur09, §3.2], we get the following commutative diagram in the ∞-category $\mathrm{Fun}(\mathrm{FSch}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{CAT}_\infty))$:

$$\begin{tikzcd}
\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\Lambda)^\otimes \arrow[r, "-\otimes_\Lambda \chi\Lambda"] \arrow[d, "\xi^\otimes"] & \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\chi\Lambda)^\otimes \arrow[d, "\xi^\otimes"] & \\
\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(-;\Lambda)^\otimes \arrow[r, "-\otimes_\Lambda \xi\chi\Lambda"] \arrow[rr, bend right, equal] & \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(-;\xi\chi\Lambda)^\otimes \arrow[r, "-\otimes_{\xi\chi\Lambda}\Lambda"] & \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(-;\Lambda)^\otimes.
\end{tikzcd}$$

The morphism $\widetilde{\xi}^\otimes$ is then defined as the composition of

$$\widetilde{\xi}^\otimes : \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\chi\Lambda)^\otimes \xrightarrow{\xi^\otimes} \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(-;\xi\chi\Lambda)^\otimes \xrightarrow{-\otimes_{\xi\chi\Lambda}\Lambda} \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(-;\Lambda)^\otimes.$$

3.5. **Descent, continuity and stalks, II. The case of $\chi\Lambda$-modules.**

We gather here a few basic properties of the functor $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\chi\Lambda)^\otimes$ and the natural transformation $\widetilde{\xi}^\otimes$ constructed in Subsection 3.4.

Proposition 3.5.1. *The contravariant functor*

$$\mathcal{S} \mapsto \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\chi\Lambda), \quad f \mapsto f^*$$

defines a τ-(hyper)sheaf on FSch *with values in* Pr^{L}.

Proof. Fix an internal hypercover $\mathcal{U}_\bullet$ in the site (FSch, τ), with $\mathcal{U}_n \to \mathcal{U}_{-1}$ étale for every $n \in \mathbb{N}$, and which we assume to be truncated in the non-hypercomplete case. We need to show that

$$\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{U}_\bullet;\chi\Lambda) : \mathbf{\Delta}_+ = \mathbf{\Delta}^{\triangleleft} \to \mathrm{CAT}_\infty$$

is a limit diagram. To do so, we use the fact that $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{U}_\bullet;\Lambda)$ is a limit diagram (by Proposition 3.2.1) and exhibit a natural transformation

$$\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{U}_\bullet;\chi\Lambda) \to \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{U}_\bullet;\Lambda) \tag{3.11}$$

satisfying the hypotheses of [Lur17, Corollary 5.2.2.37]. To do so, we start with the obvious natural transformation

$$-\otimes_\Lambda \chi\Lambda : \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\Lambda) \to \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\chi\Lambda),$$

that we restrict to $\mathrm{\acute{E}t}/\mathcal{U}_{-1}$, and consider the morphism of coCartesian fibrations

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{G} \arrow[dl, "q"] \\
& \mathrm{\acute{E}t}/\mathcal{U}_{-1} &
\end{tikzcd}$$

associated to this natural transformation by Lurie's unstraightening construction [Lur09, §3.2]. Fiberwise, F admits right adjoints. By [Lur17, Proposition 7.3.2.6], we deduce that F admits a right adjoint $G : \mathcal{G} \to \mathcal{F}$ making the triangle

$$\begin{tikzcd}
\mathcal{F} \arrow[dr, "p"'] & & \mathcal{G} \arrow[ll, "G"'] \arrow[dl, "q"] \\
& \mathrm{\acute{E}t}/\mathcal{U}_{-1} &
\end{tikzcd}$$

commutative and which is fiberwise given by the forgetful functor. We claim that G is in fact a morphism of coCartesian fibrations, i.e., takes a q-coCartesian edge to a p-coCartesian edge, and thus determines a natural transformation

$$\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\chi\Lambda) \to \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\Lambda) \tag{3.12}$$

on $\mathrm{\acute{E}t}/\mathcal{U}_{-1}$ given objectwise by the forgetful functor. To prove this, we need to check that the square

$$\begin{tikzcd} \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{V};\Lambda) \arrow[r, "-\otimes_\Lambda \chi\Lambda"] \arrow[d, "e^*"] & \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{V};\chi\Lambda) \arrow[d, "e^*"] \\ \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{V}';\Lambda) \arrow[r, "-\otimes_\Lambda \chi\Lambda"] & \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{V}';\chi\Lambda) \end{tikzcd}$$

is right adjointable for every map $e : \mathcal{V} \to \mathcal{V}'$ in $\mathrm{\acute{E}t}/\mathcal{U}_{-1}$. This follows from Lemma 3.4.3 which implies that $e^*\chi_{\mathcal{V}}\Lambda \to \chi_{\mathcal{V}'}\Lambda$ is an equivalence. That said, we define (3.11) to be the restriction of (3.12). That the hypotheses of [Lur17, Lemma 5.2.2.37] are satisfied is clear:

- hypothesis (1) of loc. cit. follows from Proposition 3.2.1;
- hypothesis (2) of loc. cit. follows from [Lur17, Corollary 4.2.3.2];
- hypothesis (3) of loc. cit. is clear since the ∞-categories $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(U_n;\Lambda)$ are presentable;
- hypothesis (4) of loc. cit., and more generally the right adjointability of the squares

$$\begin{tikzcd} \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{V};\chi\Lambda) \arrow[r, "e^*"] \arrow[d] & \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{V}';\chi\Lambda) \arrow[d] \\ \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{V};\Lambda) \arrow[r, "e^*"] & \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{V}';\Lambda), \end{tikzcd}$$

  for $e : \mathcal{V}' \to \mathcal{V}$ in $\mathrm{\acute{E}t}/\mathcal{U}_{-1}$, is clear by construction.

This completes the proof. ◻

Lemma 3.5.2. *The natural transformation*

$$\widetilde{\xi}^{\otimes} : \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\chi\Lambda)^{\otimes} \to \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}((-)^{\mathrm{rig}};\Lambda)^{\otimes}$$

is a morphism in $\mathrm{Fun}(\mathrm{FSch}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}))$. *Moreover, in the following two cases, if we restrict this natural transformation to the subcategory* $\mathcal{V} \subset \mathrm{FSch}$*, we get a morphism in* $\mathrm{Fun}(\mathcal{V}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega}))$.

(1) We work in the non-hypercomplete case and, if τ is the étale topology, we assume that Λ is eventually coconnective. In this case, we may take $\mathcal{V}$ to be the wide subcategory of FSch *consisting of quasi-compact morphisms.*

(2) We work in the hypercomplete case. In this case, $\mathcal{V}$ is the subcategory whose objects are those formal schemes $\mathcal{S}$ such that $\mathcal{S}^{\mathrm{rig}}$ is (Λ,τ)-admissible and whose morphisms are the quasi-compact and quasi-separated ones.

Proof. By [Lur17, Theorem 3.4.4.2], $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\chi\Lambda)^{\otimes}$ is a presentable monoidal ∞-category for every $\mathcal{S} \in \mathrm{FSch}$. Moreover, the image of $-\otimes_\Lambda \chi\Lambda : \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda) \to \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\chi\Lambda)$ generates $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\chi\Lambda)$ by colimits. This follows from Proposition 3.1.14 since the right adjoint to $-\otimes_\Lambda \chi\Lambda$ is conservative by [Lur17, Corollary 4.2.3.2]. By [Lur17, Corollary 3.4.4.6], this right adjoint also preserves all colimits, which implies that $-\otimes_\Lambda \chi\Lambda$ preserves compact objects. In particular, we see that $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\chi\Lambda)$ is compactly generated when $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)$ is. Thus, the second part of the statement follows easily from Propositions 2.4.22 and 3.2.3. ◻

Our next goal is to prove the continuity property for $\mathbf{FSH}^{(\mathrm{eff},\wedge)}_{\tau}(-;\chi\Lambda)$.

Theorem 3.5.3. *Let $(\mathcal{S}_\alpha)_\alpha$ be a cofiltered inverse system of quasi-compact and quasi-separated formal schemes with affine transition maps, and let $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$ be the limit of this system. We assume one of the following two alternatives.*

(1) We work in the non-hypercomplete case. When τ is the étale topology, we assume furthermore that Λ is eventually coconnective.

(2) We work in the hypercomplete case, and $\mathcal{S}$ and $\mathcal{S}^{\mathrm{rig}}$ as well as the $\mathcal{S}_\alpha$'s and the $\mathcal{S}^{\mathrm{rig}}_\alpha$'s are (Λ,τ)-admissible. When τ is the étale topology, we assume furthermore that Λ is eventually coconnective or that the numbers $\mathrm{pvcd}_\Lambda(\mathcal{S}_\alpha)$ and $\mathrm{pvcd}_\Lambda(\mathcal{S}^{\mathrm{rig}}_\alpha)$ are bounded independently of α.

Then the obvious functor

$$\operatorname*{colim}_{\alpha} \mathbf{FSH}^{(\mathrm{eff},\wedge)}_{\tau}(\mathcal{S}_\alpha;\chi\Lambda) \to \mathbf{FSH}^{(\mathrm{eff},\wedge)}_{\tau}(\mathcal{S};\chi\Lambda), \tag{3.13}$$

where the colimit is taken in Pr^{L}, is an equivalence.

Remark 3.5.4. Compared to the analogous statements for rigid analytic and formal motives (see Theorem 2.5.1 and Proposition 3.2.4), we have to assume, in the non-hypercomplete case, that Λ is eventually coconnective when τ is the étale topology. This is due to Lemma 3.5.7 below, that we were only able to prove under this extra assumption which insures the compact generation of the ∞-categories of $\chi\Lambda$-modules in formal motives.

We will obtain Theorem 3.5.3 as a consequence of Theorem 2.5.1 and Proposition 3.2.4. To do so, we need some ∞-categorical facts. We start with the following result, which is well-known but for which we couldn't find a reference.

Lemma 3.5.5. *Let $\mathcal{C}^\otimes$ be a monoidal ∞-category admitting colimits which are compatible with the monoidal structure. Then, the forgetful functor $\mathrm{ff} : \mathrm{Mod}(\mathcal{C}) \to \mathcal{C}$ commutes with filtered colimits.*

Proof. By [Lur17, Theorem 4.5.3.1], we have a coCartesian fibration $\mathrm{Mod}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C})$. By [Lur17, Corollary 3.4.4.6(2)], for every $A \in \mathrm{CAlg}(\mathcal{C})$, the ∞-category $\mathrm{Mod}_A(\mathcal{C})$ admits colimits and the forgetful functor $\mathrm{ff}_A : \mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ is colimit-preserving. Also, the base change functor $\mathrm{Mod}_A(\mathcal{C}) \to \mathrm{Mod}_B(\mathcal{C})$, associated to a morphism $A \to B$ in $\mathrm{CAlg}(\mathcal{C})$, is colimit-preserving since it admits a right adjoint. Moreover, by [Lur17, Corollaries 3.2.3.2 & 3.2.3.3], the ∞-category $\mathrm{CAlg}(\mathcal{C})$ admits colimits and the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ preserves the filtered ones. Using [Lur09, Proposition 4.3.1.5(2) & Corollary 4.3.1.11], we deduce that $\mathrm{Mod}(\mathcal{C})$ admits colimits and that they are computed as follows. Let $p : K \to \mathrm{Mod}(\mathcal{C})$ be a diagram and let $q : K \to \mathrm{CAlg}(\mathcal{C})$ be the diagram obtained by composing with the forgetful functor. Let $A_\infty \in \mathrm{CAlg}(\mathcal{C})$ be a colimit of q and let $p' : K \to \mathrm{Mod}_{A_\infty}(\mathcal{C})$ be a diagram endowed with a morphism $p \to p'$ in $\mathrm{Mod}(\mathcal{C})^K$ given by coCartesian edges. (See the beginning of the proof of [Lur09, Corollary 4.3.1.11].) Then, the colimit of p is equivalent to the colimit of p' computed in $\mathrm{Mod}_{A_\infty}(\mathcal{C})$.

Now assume that K is a filtered partially ordered set, and let L be the subset of $K \times K$ consisting of those pairs (i, j) with $i \leq j$. We endow L with the induced order. Consider the commutative square

$$\begin{tikzcd} K \arrow[r, "p"] \arrow[d] & \mathrm{Mod}(\mathcal{C}) \arrow[d] \\ L \arrow[r, "\tilde{q}"] & \mathrm{CAlg}(\mathcal{C}), \end{tikzcd}$$

where the vertical left arrow is the diagonal map given by $i \mapsto (i,i)$ and $\tilde{q}$ is the diagram obtained by composing q with the map $L \to K$ given by $(i,j) \mapsto j$. Let $\tilde{p} : L \to \mathrm{Mod}(\mathcal{C})$ be the relative left Kan extension (in the sense of [Lur09, Definition 4.3.2.2]). Setting $A_i = q(i)$ and $M_i = p(i)$, we have informally $\tilde{p}(i,j) = A_j \otimes_{A_i} M_i$. The diagrams p and $\tilde{p}$ have the same colimits, so it is enough to show that $\mathrm{ff}(\operatorname{colim} \tilde{p}) \simeq \operatorname{colim} \mathrm{ff} \circ \tilde{p}$. Now, a colimit over L can be computed as a double colimit

$$\operatorname*{colim}_{(i,j)\in L} \simeq \operatorname*{colim}_{i\in K} \operatorname*{colim}_{j\in K_{i/}}.$$

Moreover, since the diagram $i \mapsto \operatorname{colim}_{j\in K_{i/}} \tilde{p}(i,-)$ lands in $\mathrm{Mod}_{A_\infty}(\mathcal{C})$, its colimit commutes with ff_{A_∞} as mentioned above. Thus, it is enough to prove the statement for the diagrams $\tilde{p}(i,-) : K_{i/} \to \mathrm{Mod}(\mathcal{C})$. Said differently, we may assume that p takes an edge of K to a coCartesian edge of the coCartesian fibration $\mathrm{Mod}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C})$.

We may assume that K has an initial object $o \in K$. We have a natural transformation between the following two functors $\mathrm{Mod}_{A_o}(\mathcal{C}) \to \mathcal{C}$.

(1) The first one sends $M \in \mathrm{Mod}_{A_o}(\mathcal{C})$ to the colimit in $\mathcal{C}$ of the diagram $i \mapsto \mathrm{ff}_{A_i}(A_i \otimes_{A_o} M)$.
(2) The second one sends $M \in \mathrm{Mod}_{A_o}(\mathcal{C})$ to $\mathrm{ff}_{A_\infty}(A_\infty \otimes_{A_o} M)$.

We want to show that this natural transformation is an equivalence. (Together with the description of colimits in $\mathrm{Mod}(\mathcal{C})$ given at the beginning, this would complete the proof.) To do so, we remark that the two functors above are colimit-preserving. Using [Lur17, Proposition 4.7.3.14], we reduce to show that this natural transformation is an equivalence on A_o-modules of the form $A_o \otimes M$, with $M \in \mathcal{C}$. In this case, we have to show that the morphism

$$\operatorname*{colim}_{i\in K} \mathrm{ff}(A_i \otimes M) \to \mathrm{ff}(A_\infty \otimes M)$$

is an equivalence. This is clear since $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ commutes with filtered colimits. □

Before stating the next ∞-categorical result, we introduce some notation. Let $\mathcal{C}$ be an ∞-category and $\mathcal{E}^\otimes : \mathcal{C} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ a functor. Consider the commutative triangle

$$\begin{tikzcd} \mathcal{M}^\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathrm{Fin}_* \times \mathcal{D} \arrow[dl, "\mathrm{id}\times p"] \\ & \mathrm{Fin}_* \times \mathcal{C} & \end{tikzcd}$$

obtained by applying Lurie's unstraightening construction [Lur09, §3.2] to the functor sending $X \in \mathcal{C}$ to the commutative triangle

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{E}(X))^\otimes \arrow[rr] \arrow[dr] & & \mathrm{Fin}_* \times \mathrm{CAlg}(\mathcal{E}(X)) \arrow[dl] \\ & \mathrm{Fin}_*. & \end{tikzcd}$$

By Lemma 3.4.6 and [Lur09, Proposition 2.4.2.3(3)], the maps p, q and r are all coCartesian fibrations. Assume that we are given a section A of the coCartesian fibration $p : \mathcal{D} \to \mathcal{C}$, and consider $\mathcal{M}^\otimes_A = \mathcal{M}^\otimes \times_{\mathcal{D},A} \mathcal{C}$. The obvious functor $\mathcal{M}^\otimes_A \to \mathrm{Fin}_* \times \mathcal{C}$ is a coCartesian fibration. By Lurie's straightening construction [Lur09, §3.2], it determines a functor

$$\mathrm{Mod}_A(\mathcal{E})^\otimes : \mathcal{C} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).$$

For proving Theorem 3.5.3, we will use the following general result.

Lemma 3.5.6. *Assume that $\mathcal{C}$ is filtered and set $\mathcal{E}_\infty^\otimes = \operatorname{colim}_{\mathcal{C}} \mathcal{E}^\otimes$. (Here and below, the colimit is taken in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.) Let $\widetilde{A} : \mathcal{C} \to \mathrm{CAlg}(\mathcal{E}_\infty)$ be the composition of the section A with the obvious functor $\mathcal{D} \to \mathrm{CAlg}(\mathcal{E}_\infty)$, and set $A_\infty = \operatorname{colim} \widetilde{A}$. Then there is an equivalence*

$$\operatorname*{colim}_{\mathcal{C}} \mathrm{Mod}_A(\mathcal{E})^\otimes \simeq \mathrm{Mod}_{A_\infty}(\mathcal{E}_\infty)^\otimes. \tag{3.14}$$

Proof. By [Lur17, Corollary 3.2.3.2], the forgetful functor $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$ detects filtered colimits. Therefore, it is enough to prove that

$$\operatorname{colim} \mathrm{Mod}_A(\mathcal{E}) \to \mathrm{Mod}_{A_\infty}(\mathcal{E}_\infty)$$

is an equivalence, where the colimit is taken in Pr^{L}. By [Lur17, Corollary 4.5.1.6], the ∞-category $\mathrm{Mod}_{A(c)}(\mathcal{E}(c))$ is equivalent to the ∞-category $\mathrm{LMod}_{A(c)}(\mathcal{E}(c))$ of left-$A(c)$-modules, for every $c \in \mathcal{C}$, and similarly for $\mathrm{Mod}_{A_\infty}(\mathcal{E}_\infty)$. In fact, [Lur17, Corollary 4.5.1.6] shows also that the functor $\mathrm{Mod}_A(\mathcal{E}) : \mathcal{C} \to \mathrm{Pr}^{\mathrm{L}}$ is equivalent to the functor $\mathrm{LMod}_A(\mathcal{E}) : \mathcal{C} \to \mathrm{Pr}^{\mathrm{L}}$ which is constructed similarly as above. More explicitly, one applies Lurie's unstraightening construction [Lur09, §3.2] to the functor sending $c \in \mathcal{C}$ to the functor $\mathrm{LMod}(\mathcal{E}(c)) \to \mathrm{Alg}(\mathcal{E}(c))$ (see [Lur17, Definition 4.2.1.13 & Example 4.2.1.18]) to get a morphism of coCartesian fibrations

$$\begin{tikzcd} \mathcal{M}'^\otimes \arrow[rr, "r'"] \arrow[dr, "q'"'] & & \mathcal{D}' \arrow[dl, "p'"] \\ & \mathcal{C}. & \end{tikzcd}$$

Then, the functor $\mathrm{LMod}_A(\mathcal{E})$ is obtained by applying Lurie's straightening construction [Lur09, §3.2] to the coCartesian fibration $\mathcal{M}'_A = \mathcal{M}' \times_{\mathcal{D}', A} \mathcal{C} \to \mathcal{C}$. That said, we are left to show that

$$\operatorname*{colim}_{\mathcal{C}} \mathrm{LMod}_A(\mathcal{E}) \to \mathrm{LMod}_{A_\infty}(\mathcal{E}_\infty) \tag{3.15}$$

is an equivalence, where the colimit is taken in Pr^{L}. Using the functor $\widehat{\Theta} : \mathrm{Pr}^{\mathrm{Alg}} \to \mathrm{Pr}^{\mathrm{Mod}}$ of [Lur17, Construction 4.8.3.24 & Notation 4.8.5.10] and the forgetful functor $\mathrm{ff} : \mathrm{Pr}^{\mathrm{Mod}} \to \mathrm{Pr}^{\mathrm{L}}$, we may rewrite (3.15) as

$$\operatorname*{colim}_{\mathcal{C}} \mathrm{ff} \circ \widehat{\Theta}(\mathcal{E}, A) \to \mathrm{ff} \circ \widehat{\Theta}(\mathcal{E}_\infty, A_\infty). \tag{3.16}$$

We give below an informal description of the objects we have just introduced and refer the reader to loc. cit. for the precise definitions:

- $\mathrm{Pr}^{\mathrm{Alg}}$ is the ∞-category whose objects are pairs $(\mathcal{X}^\otimes, R)$ consisting of a presentable monoidal ∞-category $\mathcal{X}^\otimes$ and an associative algebra $R \in \mathrm{Alg}(\mathcal{X})$;
- $\mathrm{Pr}^{\mathrm{Mod}} \simeq \mathrm{LMod}(\mathrm{Pr}^{\mathrm{L}})$ is the ∞-category whose objects are pairs $(\mathcal{X}^\otimes, \mathcal{Y})$ consisting of a presentable monoidal ∞-category $\mathcal{X}^\otimes$ and an $\mathcal{X}^\otimes$-module $\mathcal{Y}$ in $\mathrm{Pr}^{\mathrm{L},\otimes}$;
- $\widehat{\Theta}$ sends $(\mathcal{X}^\otimes, R)$ to $(\mathcal{X}^\otimes, \mathrm{Mod}_R(\mathcal{X}))$ and ff sends $(\mathcal{X}^\otimes, \mathcal{Y})$ to $\mathcal{Y}$;
- $(\mathcal{E}, A)$ denotes the functor $\mathcal{C} \to \mathrm{Pr}^{\mathrm{Alg}}$ given informally by $c \mapsto (\mathcal{E}(c), A(c))$.

By Lemma 3.5.5, the functor ff commutes with filtered colimits. Using [Lur17, Theorem 4.8.5.11] and [Lur09, Proposition 4.4.2.9], we deduce that $\widehat{\Theta}$ commutes also with filtered colimits. Since $\operatorname{colim}_{\mathcal{C}} (\mathcal{E}, A) \simeq (\mathcal{E}_\infty, A_\infty)$, this proves that (3.16) is an equivalence. □

Using Proposition 3.2.4, Lemma 3.5.6 and the construction of the functor $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\chi\Lambda)$, we see that Theorem 3.5.3 is a consequence of the following lemma.

Lemma 3.5.7. *With the notation and assumptions of Theorem 3.5.3, we have an equivalence*

$$\operatorname*{colim}_{\alpha} f_\alpha^* \chi_{\mathcal{S}_\alpha} \Lambda \to \chi_{\mathcal{S}} \Lambda$$

in $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)$*, where* $f_\alpha : \mathcal{S} \to \mathcal{S}_\alpha$ *is the obvious map.*

Proof. Under the assumptions of Theorem 3.5.3, Theorem 2.5.1 and Proposition 3.2.4 provide us with equivalences in $\mathrm{Pr}^{\mathrm{L}}_\omega$

$$\operatorname*{colim}_{\alpha} \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}_\alpha^{\mathrm{rig}};\Lambda) \simeq \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda) \tag{3.17}$$

$$\text{and} \quad \operatorname*{colim}_{\alpha} \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}_\alpha;\Lambda) \simeq \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda), \tag{3.18}$$

where the colimits are also taken in $\mathrm{Pr}^{\mathrm{L}}_\omega$. (See Propositions 2.4.22 and 3.2.3.) In particular, the ∞-category $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)$ is compactly generated and it suffices to show that a compact object M in this ∞-category induces an equivalence

$$\mathrm{Map}_{\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)}(M, \operatorname*{colim}_{\alpha} f_\alpha^* \chi_{\mathcal{S}_\alpha}\Lambda) \to \mathrm{Map}_{\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)}(M, \chi_{\mathcal{S}}\Lambda). \tag{3.19}$$

For $\beta \leq \alpha$, we denote by $f_{\beta\alpha} : \mathcal{S}_\beta \to \mathcal{S}_\alpha$ the transition map in the inverse system $(\mathcal{S}_\alpha)_\alpha$. Since M is compact, there exists an index ρ and a compact object $M_\rho \in \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}_\rho;\Lambda)$ such that $M \simeq f_\rho^* M_\rho$. We have canonical equivalences:

$$\begin{aligned}
\mathrm{Map}_{\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)}(M, \operatorname*{colim}_{\alpha} f_\alpha^* \chi_{\mathcal{S}_\alpha}\Lambda) &\overset{(1)}{\simeq} \operatorname*{colim}_{\alpha} \mathrm{Map}_{\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)}(M, f_\alpha^* \chi_{\mathcal{S}_\alpha}\Lambda) \\
&\overset{(2)}{\simeq} \operatorname*{colim}_{\alpha\leq\rho} \operatorname*{colim}_{\beta\leq\alpha} \mathrm{Map}_{\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}_\beta;\Lambda)}(f_{\beta\rho}^* M_\rho, f_{\beta\alpha}^* \chi_{\mathcal{S}_\alpha}\Lambda) \\
&\overset{(3)}{\simeq} \operatorname*{colim}_{\beta\leq\rho} \mathrm{Map}_{\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}_\beta;\Lambda)}(f_{\beta\rho}^* M_\rho, \chi_{\mathcal{S}_\beta}\Lambda) \\
&\overset{(4)}{\simeq} \operatorname*{colim}_{\beta\leq\rho} \mathrm{Map}_{\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}_\beta^{\mathrm{rig}};\Lambda)}(f_{\beta\rho}^{\mathrm{rig},*} \xi_{\mathcal{S}_\rho} M_\rho, \Lambda) \\
&\overset{(5)}{\simeq} \mathrm{Map}_{\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda)}(f_\rho^{\mathrm{rig},*} \xi_{\mathcal{S}_\rho} M_\rho, \Lambda) \\
&\overset{(6)}{\simeq} \mathrm{Map}_{\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)}(M, \chi_{\mathcal{S}}\Lambda)
\end{aligned}$$

where

(1) follows from the assumption that M is compact,
(2) follows from the fact that the colimit in (3.18) is taken in $\mathrm{Pr}^{\mathrm{L}}_\omega$,
(3) follows from the cofinality of the diagonal map $\beta \mapsto (\beta \leq \beta)$,
(4) follows from the adjunction $(\xi_{\mathcal{S}_\beta}, \chi_{\mathcal{S}_\beta})$ and the commutation $\xi_{\mathcal{S}_\beta} f_{\beta\rho}^* \simeq f_{\beta\rho}^{\mathrm{rig},*} \xi_{\mathcal{S}_\rho}$,
(5) follows from the fact that the colimit in (3.17) is taken in $\mathrm{Pr}^{\mathrm{L}}_\omega$,
(6) follows from the commutation $f_\rho^{\mathrm{rig},*} \xi_{\mathcal{S}_\rho} \simeq \xi_{\mathcal{S}} f_\rho^*$ and the adjunction $(\xi_{\mathcal{S}}, \chi_{\mathcal{S}})$.

It is easy to see that the composition of the above equivalences coincide with the map (3.19). □

Remark 3.5.8. Lemma 3.5.7 admits a useful extension as follows. Keep the notation and assumptions of Theorem 3.5.3. Let I be the indexing category of the inverse system $(\mathcal{S}_\alpha)_\alpha$ and let $\alpha \mapsto N_\alpha$ be a section of the coCartesian fibration associated to the functor $I^{\mathrm{op}} \to \mathrm{CAT}_\infty$, $\alpha \mapsto \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}_\alpha^{\mathrm{rig}};\Lambda)$. Let $N \in \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S}^{\mathrm{rig}};\Lambda)$ be the colimit of the $f_\alpha^{\mathrm{rig},*} N_\alpha$'s. Then there is an equivalence

$$\operatorname*{colim}_{\alpha} f_\alpha^* \chi_{\mathcal{S}_\alpha} N_\alpha \xrightarrow{\sim} \chi_{\mathcal{S}} N$$

in $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{S};\Lambda)$. This is shown using exactly the same reasoning as in the proof of Lemma 3.5.7.

We finish this subsection with a computation of the stalks of $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\chi\Lambda)$ for the topology rig-τ on FSch.

Theorem 3.5.9. *Let $\mathcal{S}$ be a formal scheme and let $\mathfrak{s} \to \mathcal{S}$ be an algebraic rigid point of $\mathcal{S}$. Assume one of the following two alternatives.*

1. *We work in the non-hypercomplete case and, if τ is the étale topology, we assume that Λ is eventually coconnective.*
2. *We work in the hypercomplete case, and $\mathcal{S}$ and $\mathcal{S}^{\mathrm{rig}}$ are (Λ,τ)-admissible. When τ is the étale topology, we assume furthermore that Λ is eventually coconnective or that the numbers $\mathrm{pvcd}_\Lambda(\mathcal{S}')$, for admissible blowups $\mathcal{S}' \to \mathcal{S}$, are bounded independently of $\mathcal{S}'$.*

Then there is an equivalence of ∞-categories

$$\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\chi\Lambda)_{\mathfrak{s}} \simeq \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{s};\chi\Lambda)$$

where the left-hand side is the stalk of $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(-;\chi\Lambda)$ at $\mathfrak{s}$, i.e., the colimit, taken in Pr^{L}, of the diagram $(\mathfrak{s} \to \mathcal{U} \to \mathcal{S}) \mapsto \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathcal{U};\chi\Lambda)$ with $\mathcal{U} \in \mathrm{FRig\acute{E}t}/\mathcal{S}$.

Proof. This follows from Theorem 3.5.3. Indeed, the condition that $\mathcal{S}^{\mathrm{rig}}$ is (Λ,τ)-admissible implies that $\mathfrak{s}$ is (Λ,τ)-admissible. Moreover, if the numbers $\mathrm{pvcd}_\Lambda(\mathcal{S}')$ are bounded independently of $\mathcal{S}'$ for admissible blowups $\mathcal{S}' \to \mathcal{S}$, then the same is true for the numbers $\mathrm{pvcd}_\Lambda(\mathcal{U})$ for the saturated rig-étale neighbourhoods $\mathfrak{s} \to \mathcal{U} \to \mathcal{S}$. □

3.6. Proof of the main result, I. Fully faithfulness.

Our goal in this subsection is to prove the first part of Theorem 3.3.3 concerning the fully faithfulness of the functor $\widetilde{\xi}_{\mathcal{S}}$. (The second part of this theorem will be proved in the next subsection.) A key ingredient is a projection formula for the functor $\chi_{\mathcal{S}}$ as in the following statement. This projection formula is also a key ingredient in the proof of the extended proper base change theorem for rigid analytic motives, see Theorem 4.1.4 below.

Theorem 3.6.1. *We work under Assumption 3.3.1. Let $\mathcal{S}$ be a formal scheme and set $S = \mathcal{S}^{\mathrm{rig}}$. Then, for $M \in \mathbf{RigSH}_\tau^{(\wedge)}(S;\Lambda)$ and $N \in \mathbf{FSH}_\tau^{(\wedge)}(\mathcal{S};\Lambda)$, the obvious map*

$$\chi_{\mathcal{S}}(M) \otimes N \to \chi_{\mathcal{S}}(M \otimes \xi_{\mathcal{S}}(N)) \tag{3.20}$$

is an equivalence.

We first prove the following reduction.

Lemma 3.6.2. *To prove Theorem 3.6.1, it is enough to consider the alternatives (i), (ii) and (iv) of Assumptions 3.3.1. Moreover, when working under the alternative (iv), we may assume the following extra conditions:*

1. *τ is the étale topology;*
2. *Λ is the Eilenberg–Mac Lane spectrum associated to the ring $\mathbb{Z}/\ell$, with ℓ a prime number invertible on $\mathcal{S}$;*
3. *M and N are compact objects.*

Proof. We split the proof into two parts.

Part 1. Here we show that the conclusion of Theorem 3.6.1 holds under (iii) if it holds under (iv).

We work under the alternative (iii). The problem is local on $\mathcal{S}$. Thus, we may assume that $\mathcal{S}$ is affine, given as a limit of a cofiltered inverse system $(\mathcal{S}_\alpha)_\alpha$ of affine formal schemes such that the $\mathcal{S}_\alpha$'s and their generic fibers $S_\alpha = \mathcal{S}_\alpha^{\mathrm{rig}}$ are (Λ, τ)-admissible. By Theorem 2.5.1 and Proposition 3.2.4, we have equivalences

$$\operatorname*{colim}_\alpha \mathbf{RigSH}_\tau(S_\alpha; \Lambda) \simeq \mathbf{RigSH}_\tau(S; \Lambda) \quad \text{and} \quad \operatorname*{colim}_\alpha \mathbf{FSH}_\tau(\mathcal{S}_\alpha; \Lambda) \simeq \mathbf{FSH}_\tau(\mathcal{S}; \Lambda)$$

in Pr^{L}, and the colimits are taken in Pr^{L}. Using that the tensor product of Pr^{L} commutes with filtered colimits, we deduce an equivalence

$$\operatorname*{colim}_\alpha \left(\mathbf{RigSH}_\tau(S_\alpha; \Lambda) \otimes \mathbf{FSH}_\tau(\mathcal{S}_\alpha; \Lambda)\right) \simeq \mathbf{RigSH}_\tau(S; \Lambda) \otimes \mathbf{FSH}_\tau(\mathcal{S}; \Lambda).$$

Since the functors $\xi_{\mathcal{S}_\alpha}$ and $\chi_{\mathcal{S}_\alpha}$ belong to Pr^{L} and are in adjunction, and since $\xi_{\mathcal{S}}$ is the colimit of the $\xi_{\mathcal{S}_\alpha}$'s, we deduce that $\chi_{\mathcal{S}}$ is the colimit of the $\chi_{\mathcal{S}_\alpha}$'s. (Here we use Propositions 2.4.22 and 3.2.3 to view $\xi_{\mathcal{S}}$ and the $\xi_{\mathcal{S}_\alpha}$'s as functors in $\mathrm{Pr}^{\mathrm{L}}_\omega$ with colimit-preserving right adjoints.) Considering $\chi_{\mathcal{S}}(-) \otimes (-)$ and $\chi_{\mathcal{S}}(- \otimes \xi_{\mathcal{S}}(-))$ as functors from $\mathbf{RigSH}_\tau(S; \Lambda) \otimes \mathbf{FSH}_\tau(\mathcal{S}; \Lambda)$ to $\mathbf{FSH}_\tau(\mathcal{S}; \Lambda)$, and similarly with "$\mathcal{S}_\alpha$" instead of "$\mathcal{S}$", it follows that the natural transformation $\chi_{\mathcal{S}}(-) \otimes (-) \to \chi_{\mathcal{S}}(- \otimes \xi_{\mathcal{S}}(-))$ is the colimit of the natural transformations $\chi_{\mathcal{S}_\alpha}(-) \otimes (-) \to \chi_{\mathcal{S}_\alpha}(- \otimes \xi_{\mathcal{S}_\alpha}(-))$. This reduces us to treat the case where $\mathcal{S}$ and S are (Λ, τ)-admissible. But in this case, we have

$$\mathbf{RigSH}_\tau(S; \Lambda) \simeq \mathbf{RigSH}^\wedge_\tau(S; \Lambda) \qquad \text{and} \qquad \mathbf{FSH}_\tau(\mathcal{S}; \Lambda) \simeq \mathbf{FSH}^\wedge_\tau(\mathcal{S}; \Lambda)$$

by Propositions 2.4.19 and 3.2.2. Therefore, this case is covered by the alternative (iv).

Part 2. Here we assume that the conclusion of Theorem 3.6.1 holds under (i) and (ii), and we show that we may assume conditions (1), (2) and (3) when proving Theorem 3.6.1 under (iv).

Assume the alternative (iv). If τ is the Nisnevich topology, then there is nothing to prove since Theorem 3.6.1 holds under (i). Thus, we may assume that τ is the étale topology. By Propositions 2.4.22 and 3.2.3, the ∞-categories $\mathbf{RigSH}^\wedge_{\acute{e}t}(S; \Lambda)$ and $\mathbf{FSH}^\wedge_{\acute{e}t}(\mathcal{S}; \Lambda)$ are compactly generated, and the functor $\chi_{\mathcal{S}}$ commutes with colimits (since its left adjoint is compact-preserving). This will be used freely in the discussion below.

Let $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ and $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$ be the rationalisations of M and N, and let M_{tor} and N_{tor} be the fibers of $M \to M_{\mathbb{Q}}$ and $N \to N_{\mathbb{Q}}$. Since Theorem 3.6.1 holds under the alternative (ii), we deduce that the morphism (3.20) becomes an equivalence if we replace M by $M_{\mathbb{Q}}$ or N by $N_{\mathbb{Q}}$. Thus, it remains to show that the morphism (3.20) becomes an equivalence if we replace M and N by M_{tor} and N_{tor}. Now, M_{tor} is a coproduct of ℓ-nilpotent objects, where ℓ varies among the prime numbers which are not invertible in $\pi_0\Lambda$, and similarly for N_{tor}. Moreover, every ℓ-nilpotent object is a colimit of compact ℓ-nilpotent objects. Thus, it is enough to show that the morphism (3.20) is an equivalence when M and N are ℓ-nilpotent compact objects.

By Theorems 2.10.3, 2.10.4 and 3.1.10, we have equivalences of ∞-categories

$$\mathrm{Shv}^\wedge_{\acute{e}t}(\acute{\mathrm{E}}\mathrm{t}/S; \Lambda)_{\ell\text{-nil}} \simeq \mathbf{RigSH}^\wedge_{\acute{e}t}(S; \Lambda)_{\ell\text{-nil}} \qquad \text{and} \qquad \mathrm{Shv}^\wedge_{\acute{e}t}(\acute{\mathrm{E}}\mathrm{t}/\mathcal{S}; \Lambda)_{\ell\text{-nil}} \simeq \mathbf{FSH}^\wedge_{\acute{e}t}(\mathcal{S}; \Lambda)_{\ell\text{-nil}}.$$

We denote by M_0 and N_0 the objects of $\mathrm{Shv}^\wedge_{\acute{e}t}(\acute{\mathrm{E}}\mathrm{t}/S; \Lambda)_{\ell\text{-nil}}$ and $\mathrm{Shv}^\wedge_{\acute{e}t}(\acute{\mathrm{E}}\mathrm{t}/\mathcal{S}; \Lambda)_{\ell\text{-nil}}$ corresponding to M and N by these equivalences. It is enough to show that

$$\chi_{\mathcal{S}}(M_0) \otimes_\Lambda N_0 \to \chi_{\mathcal{S}}(M_0 \otimes_\Lambda \xi_{\mathcal{S}}(N_0)) \tag{3.21}$$

is an equivalence. (Here $\xi_{\mathcal{S}}$ is the inverse image functor associated to the morphism of sites $(\text{Ét}/S, \text{ét}) \to (\text{Ét}/\mathcal{S}, \text{ét})$ given by $(-)^{\text{rig}}$, and $\chi_{\mathcal{S}}$ is its right adjoint.) Since M_0 and N_0 are compact, they are eventually connective. It follows from Lemmas 2.4.5 and 2.4.11 (and the analogue of the latter for schemes) that we have equivalences

$$\begin{aligned} \chi_{\mathcal{S}}(M_0) \otimes_\Lambda N_0 &\simeq \lim_r \chi_{\mathcal{S}}(M_0 \otimes_\Lambda \tau_{\le r}\Lambda) \otimes_\Lambda N_0 \\ \chi_{\mathcal{S}}(M_0 \otimes_\Lambda \xi_{\mathcal{S}}(N_0)) &\simeq \lim_r \chi_{\mathcal{S}}((M_0 \otimes_\Lambda \tau_{\le r}\Lambda) \otimes_\Lambda \xi_{\mathcal{S}}(N_0)). \end{aligned}$$

Thus, it is enough to show that (3.21) becomes an equivalence if we replace M_0 by $M_0 \otimes_\Lambda \tau_{\le r}\Lambda$. The latter, being a compact object of $\text{Shv}^\wedge_{\text{ét}}(\text{Ét}/S; \tau_{\le r}\Lambda)$, is eventually connective and coconnective. Thus, if we momentarily renounce on having M_0 compact, which we do, we may assume that M_0 is eventually connective and coconnective. By an easy induction, we may even assume that M_0 is in the heart of $\text{Shv}^\wedge_{\text{ét}}(\text{Ét}/S; \Lambda)$ and that ℓ acts by 0 on M_0, i.e., M_0 is an ordinary étale sheaf of $\pi_0\Lambda/\ell$-modules.

Furthermore, we may take $N_0 = \Lambda_{\text{ét}}(\mathcal{U})/\ell$, with $\mathcal{U}$ an étale formal $\mathcal{S}$-scheme, since the objects of this form and their desuspensions generate $\text{Shv}^\wedge_{\text{ét}}(\text{Ét}/\mathcal{S}; \Lambda)$ under colimits. In this case, we have

$$\begin{aligned} \chi_{\mathcal{S}}(M_0) \otimes_\Lambda N_0 &\simeq \chi_{\mathcal{S}}(M_0) \otimes_{\mathbb{Z}} \mathbb{Z}_{\text{ét}}(\mathcal{U})/\ell \\ &\simeq \chi_{\mathcal{S}}(M_0) \otimes_{\mathbb{Z}/\ell} (\mathbb{Z}_{\text{ét}}(\mathcal{U})/\ell \oplus \mathbb{Z}_{\text{ét}}(\mathcal{U})/\ell[1]), \end{aligned}$$

$$\begin{aligned} \chi_{\mathcal{S}}(M_0 \otimes_\Lambda \xi_{\mathcal{S}}(N_0)) &\simeq \chi_{\mathcal{S}}(M_0 \otimes_{\mathbb{Z}} \xi_{\mathcal{S}}(\mathbb{Z}_{\text{ét}}(\mathcal{U})/\ell)) \\ &\simeq \chi_{\mathcal{S}}(M_0 \otimes_{\mathbb{Z}/\ell} \xi_{\mathcal{S}}(\mathbb{Z}_{\text{ét}}(\mathcal{U})/\ell \oplus \mathbb{Z}_{\text{ét}}(\mathcal{U})/\ell[1])). \end{aligned}$$

This shows that we may assume that $\Lambda = \mathbb{Z}/\ell$ as claimed. It remains to replace M_0 by a compact étale sheaf of $\mathbb{Z}/\ell$-modules to finish the proof. □

To prove Theorem 3.6.1, we need some preliminaries. We start by introducing a new ∞-category of motives. Let $\mathcal{S}$ be a formal scheme and fix a topology $\tau \in \{\text{nis}, \text{ét}\}$.

Definition 3.6.3. We define the ∞-category $\overline{\mathbf{FSH}}^{(\text{eff},\wedge)}_\tau(\mathcal{S}; \Lambda)$ by repeating Definitions 3.1.1 and 3.1.3 while replacing $\text{FSm}/\mathcal{S}$ with the category $\text{FRigSm}/\mathcal{S}$ of rig-smooth formal $\mathcal{S}$-schemes (see Definition 1.3.13 and Remark 1.4.14).

Remark 3.6.4. There are functors relating $\overline{\mathbf{FSH}}^{(\text{eff},\wedge)}_\tau(\mathcal{S}; \Lambda)$ to other ∞-categories of motives considered before. Below, we set as usual $S = \mathcal{S}^{\text{rig}}$.

(1) The inclusion functor $\iota_{\mathcal{S}} : \text{FSm}/\mathcal{S} \to \text{FRigSm}/\mathcal{S}$ induces an adjunction

$$\iota_{\mathcal{S}}^* : \mathbf{FSH}^{(\text{eff},\wedge)}_\tau(\mathcal{S}; \Lambda) \rightleftarrows \overline{\mathbf{FSH}}^{(\text{eff},\wedge)}_\tau(\mathcal{S}; \Lambda) : \iota_{\mathcal{S},*}.$$

The functor $\iota_{\mathcal{S},*}$ is induced by the restriction functor along $\iota_{\mathcal{S}}$, and the functor $\iota_{\mathcal{S}}^*$ is fully faithful and underlies a monoidal functor.

(2) The functor $(-)^{\text{rig}} : \text{FRigSm}/\mathcal{S} \to \text{RigSm}/S$ induces an adjunction

$$\overline{\xi}_{\mathcal{S}} : \overline{\mathbf{FSH}}^{(\text{eff},\wedge)}_\tau(\mathcal{S}; \Lambda) \rightleftarrows \mathbf{RigSH}^{(\text{eff},\wedge)}_\tau(S; \Lambda) : \overline{\chi}_{\mathcal{S}}.$$

By Remark 2.1.14, $\overline{\xi}_{\mathcal{S}}$ is a localisation functor, and $\overline{\chi}_{\mathcal{S}}$ is fully faithful and identifies the ∞-category $\mathbf{RigSH}^{(\text{eff},\wedge)}_\tau(S; \Lambda)$ with the full sub-∞-category of $\overline{\mathbf{FSH}}^{(\text{eff},\wedge)}_\tau(\mathcal{S}; \Lambda)$ spanned by those objects admitting rig-τ-(hyper)descent.

Clearly, we have natural equivalences $\xi_{\mathcal{S}} \simeq \overline{\xi}_{\mathcal{S}} \circ \iota_{\mathcal{S}}^*$ and $\chi_{\mathcal{S}} \simeq \iota_{\mathcal{S},*} \circ \overline{\chi}_{\mathcal{S}}$.

We record the following lemma for later use.

Lemma 3.6.5. *The functor $\iota_{\mathcal{S},*}$ underlies a monoidal functor*

$$\iota^{\otimes}_{\mathcal{S},*} : \overline{\mathbf{FSH}}^{(\mathrm{eff},\wedge)}_{\tau}(\mathcal{S};\Lambda)^{\otimes} \to \mathbf{FSH}^{(\mathrm{eff},\wedge)}_{\tau}(\mathcal{S};\Lambda)^{\otimes} \tag{3.22}$$

which belongs to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

Proof. The functor

$$\iota_{\mathcal{S},*} : \mathrm{PSh}(\mathrm{FRigSm}/\mathcal{S};\Lambda) \to \mathrm{PSh}(\mathrm{FSm}/\mathcal{S};\Lambda) \tag{3.23}$$

underlies a monoidal functor $\iota^{\otimes}_{\mathcal{S},*}$ and admits a right adjoint. (Recall that the tensor product on presheaves is given objectwise, see Remarks 2.1.5 and 2.1.6.) Moreover, it commutes with the τ-(hyper)sheafification functor. Indeed, restricting to the small sites $(\acute{\mathrm{E}}\mathrm{t}/\mathcal{X},\tau)$, for $\mathcal{X}$ in $\mathrm{FSm}/\mathcal{S}$ (resp. $\mathrm{FRigSm}/\mathcal{S}$), detects τ-(hyper)sheaves and τ-local equivalences in the hypercomplete and non-hypercomplete cases. It follows that the functor (3.23) induces a left adjoint functor

$$\iota_{\mathcal{S},*} : \mathrm{Shv}^{(\wedge)}_{\tau}(\mathrm{FRigSm}/\mathcal{S};\Lambda) \to \mathrm{Shv}^{(\wedge)}_{\tau}(\mathrm{FSm}/\mathcal{S};\Lambda) \tag{3.24}$$

underlying a monoidal functor. Moreover, for $\mathcal{X}$ a smooth formal $\mathcal{S}$-scheme, we have an equivalence $\iota_{\mathcal{S},*}(\Lambda_{\tau}(\mathcal{X})) \simeq \Lambda_{\tau}(\mathcal{X})$. Using [Lur09, Proposition 5.5.4.20], it follows that (3.24) preserves $\mathbb{A}^1$-local equivalences inducing a left adjoint functor

$$\iota_{\mathcal{S},*} : \overline{\mathbf{FSH}}^{\mathrm{eff},(\wedge)}_{\tau}(\mathcal{S};\Lambda) \to \mathbf{FSH}^{\mathrm{eff},(\wedge)}_{\tau}(\mathcal{S};\Lambda) \tag{3.25}$$

underlying a monoidal functor. This functor sends $\mathrm{T}_{\mathcal{S}}$ to $\mathrm{T}_{\mathcal{S}}$, and induces a left adjoint functor

$$\iota_{\mathcal{S},*} : \overline{\mathbf{FSH}}^{(\wedge)}_{\tau}(\mathcal{S};\Lambda) \to \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{S};\Lambda) \tag{3.26}$$

underlying a monoidal functor. From the above discussion, we see that the functors (3.25) and (3.26) are right adjoint to the functors $\iota^{*}_{\mathcal{S}}$ in Remark 3.6.4(1), finishing the proof. □

Remark 3.6.6. There is also an obvious functorial dependence of $\overline{\mathbf{FSH}}^{(\mathrm{eff},\wedge)}_{\tau}(\mathcal{S};\Lambda)$ on the formal scheme $\mathcal{S}$. A morphism of formal schemes $f : \mathcal{T} \to \mathcal{S}$ induces an inverse image functor

$$f^{*} : \overline{\mathbf{FSH}}^{(\mathrm{eff},\wedge)}_{\tau}(\mathcal{S};\Lambda) \to \overline{\mathbf{FSH}}^{(\mathrm{eff},\wedge)}_{\tau}(\mathcal{T};\Lambda)$$

which is a left adjoint and underlies a monoidal functor. Moreover, we have natural equivalences

$$f^{*} \circ \iota^{*}_{\mathcal{S}} \simeq \iota^{*}_{\mathcal{T}} \circ f^{*} \qquad \text{and} \qquad f^{\mathrm{rig},*} \circ \overline{\xi}_{\mathcal{S}} \simeq \overline{\xi}_{\mathcal{T}} \circ f^{*}.$$

When f is rig-smooth, f^{*} admits a left adjoint $f_{\sharp}$ and there is a natural equivalence $\overline{\xi}_{\mathcal{S}} \circ f_{\sharp} \simeq f^{\mathrm{rig}}_{\sharp} \circ \overline{\xi}_{\mathcal{T}}$. If f is smooth, we also have a natural equivalence $\iota^{*}_{\mathcal{S}} \circ f_{\sharp} \simeq f_{\sharp} \circ \iota^{*}_{\mathcal{T}}$.

We now state the main technical result needed for proving Theorem 3.6.1.

Proposition 3.6.7. *Let $\mathcal{S}$ be a formal scheme and set $S = \mathcal{S}^{\mathrm{rig}}$. Let M and N be objects of $\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)$ and $\mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{S};\Lambda)$ respectively. We work under one the alternatives (i), (ii) or (iv) of Assumption 3.3.1 and, when working under (iv), we assume the conditions (1), (2) and (3) of Lemma 3.6.2. Then, the obvious morphism*

$$\overline{\chi}_{\mathcal{S}}(M) \otimes \iota^{*}_{\mathcal{S}}(N) \to \overline{\chi}_{\mathcal{S}}(M \otimes \xi_{\mathcal{S}}(N)) \tag{3.27}$$

is an equivalence in $\overline{\mathbf{FSH}}^{(\wedge)}_{\tau}(\mathcal{S};\Lambda)$.

We first explain how Theorem 3.6.1 follows from Proposition 3.6.7.

Proof of Theorem 3.6.1. By Lemma 3.6.2, we may work under one the alternatives (i), (ii) or (iv) of Assumption 3.3.1, and assume the conditions (1), (2) and (3) of Lemma 3.6.2 when working under (iv). Then, we have a chain of equivalences

$$\begin{aligned}\chi_{\mathcal{S}}(M)\otimes N &\overset{(1)}{\simeq} \iota_{\mathcal{S},*}(\overline{\chi}_{\mathcal{S}}(M))\otimes\iota_{\mathcal{S},*}(\iota^*_{\mathcal{S}}(N))\\ &\overset{(2)}{\simeq} \iota_{\mathcal{S},*}(\overline{\chi}_{\mathcal{S}}(M)\otimes\iota^*_{\mathcal{S}}(N))\\ &\overset{(3)}{\simeq} \iota_{\mathcal{S},*}(\overline{\chi}_{\mathcal{S}}(M\otimes\xi_{\mathcal{S}}(N)))\\ &\overset{(4)}{\simeq} \chi_{\mathcal{S}}(M\otimes\xi_{\mathcal{S}}(N))\end{aligned}$$

where

(1) follows from the equivalence $\chi_{\mathcal{S}}\simeq\iota_{\mathcal{S},*}\circ\overline{\chi}_{\mathcal{S}}$ and the fully faithfulness of $\iota^*_{\mathcal{S}}$,
(2) follows from Lemma 3.6.5,
(3) follows from Proposition 3.6.7,
(4) follows from the equivalence $\chi_{\mathcal{S}}\simeq\iota_{\mathcal{S},*}\circ\overline{\chi}_{\mathcal{S}}$.

It is easy to see that the composition of the above equivalences coincides with the natural morphism $\chi_{\mathcal{S}}(M)\otimes N\to\chi_{\mathcal{S}}(M\otimes\xi_{\mathcal{S}}(N))$. □

Proof of Proposition 3.6.7. The morphism (3.27) is given by the following composition

$$\begin{aligned}\overline{\chi}_{\mathcal{S}}(M)\otimes\iota^*_{\mathcal{S}}(N) &\overset{(1)}{\longrightarrow} \overline{\chi}_{\mathcal{S}}\overline{\xi}_{\mathcal{S}}(\overline{\chi}_{\mathcal{S}}(M)\otimes\iota^*_{\mathcal{S}}(N))\\ &\overset{(2)}{\simeq} \overline{\chi}_{\mathcal{S}}(\overline{\xi}_{\mathcal{S}}\overline{\chi}_{\mathcal{S}}(M)\otimes\overline{\xi}_{\mathcal{S}}\iota^*_{\mathcal{S}}(N))\\ &\overset{(3)}{\simeq} \overline{\chi}_{\mathcal{S}}(M\otimes\xi_{\mathcal{S}}(N))\end{aligned}$$

where the equivalence (2) follows from the fact that $\overline{\xi}_{\mathcal{S}}$ is monoidal, and the equivalence (3) follows from the fact that $\overline{\chi}_{\mathcal{S}}$ is fully faithful and the equivalence $\xi_{\mathcal{S}}\simeq\overline{\xi}_{\mathcal{S}}\circ\iota^*_{\mathcal{S}}$. Thus, to prove the proposition, it remains to show that the morphism (1) is an equivalence. This would follows if the object $E=\overline{\chi}_{\mathcal{S}}(M)\otimes\iota^*_{\mathcal{S}}(N)$ belongs to the image of the functor $\overline{\chi}_{\mathcal{S}}$. Recall that the latter identifies $\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)$ with the full sub-∞-category of $\overline{\mathbf{FSH}}^{(\wedge)}_{\tau}(\mathcal{S};\Lambda)$ spanned by those objects admitting rig-τ-(hyper)descent. Thus, we need to show that E is local with respect to morphisms of the form

$$\operatorname*{colim}_{[n]\in\mathbf{\Delta}}\mathrm{M}(\mathcal{U}_\bullet)\to\mathrm{M}(\mathcal{U}_{-1}),\tag{3.28}$$

and their desuspensions and negative Tate twists, where $\mathcal{U}_\bullet$ is a rig-τ-hypercover which we assume to be truncated in the non-hypercomplete case. (Here $\mathcal{U}_{-1}$ is a rig-smooth formal $\mathcal{S}$-scheme and $\mathcal{U}_n$, for $n\in\mathbb{N}$, are rig-étale over $\mathcal{U}_{-1}$.) Since M and N are general objects of $\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)$ and $\mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{S};\Lambda)$, it is enough to show that E is local with respect to (3.28) without worrying about desuspensions and negative Tate twists. By a standard argument, the case of a rig-τ-hypercover $\mathcal{U}$ follows if we can treat the cases of a rig-τ-hypercover $\mathcal{U}'$ refining $\mathcal{U}$ and its base change to each of the $\mathcal{U}_n$'s. Using the description of rig-τ-covers given in Remark 1.4.14 and Proposition 1.4.19, we may thus assume that $\mathcal{U}_\bullet$ satisfies the following, according to the cases $\tau=\mathrm{nis}$ and $\tau=\text{ét}$.

(nis) The morphism of formal simplicial schemes $\mathcal{U}_\bullet\to\mathcal{U}_{-1}$ (here $\bullet\geq 0$) factors through an admissible blowup $\widetilde{\mathcal{U}}_{-1}\to\mathcal{U}_{-1}$ and the resulting morphism $\mathcal{U}_\bullet\to\widetilde{\mathcal{U}}_{-1}$ is a Nisnevich hypercover of $\widetilde{\mathcal{U}}_{-1}$ which is truncated in the non-hypercomplete case.

(ét) The morphism of formal simplicial schemes $\mathcal{U}_\bullet \to \mathcal{U}_{-1}$ (here $\bullet \geq 0$) factors through an admissible blowup $\widetilde{\mathcal{U}}_{-1} \to \mathcal{U}_{-1}$ and the resulting morphism $\mathcal{U}_\bullet \to \widetilde{\mathcal{U}}_{-1}$ factors as

$$\mathcal{U}_\bullet \xrightarrow{(2)} \widetilde{\mathcal{U}}_\bullet \xrightarrow{(1)} \widetilde{\mathcal{U}}_{-1}$$

where (1) is a Nisnevich hypercover of $\widetilde{\mathcal{U}}_{-1}$ which is truncated in the non-hypercomplete case and (2) is a relative hypercover for the topology generated by finite rig-étale coverings (in the sense of Definition 1.4.16(3)) which is also truncated in the non-hypercomplete case.

We denote by "rigfét" the topology on formal schemes generated by finite rig-étale coverings. Since E admits Nisnevich (hyper)descent by construction, we see that the result would follow if we can prove the following two properties (where we denote by $\mathrm{M} : \mathrm{FRigSm}/\mathcal{S} \to \overline{\mathbf{FSH}}{}^{(\wedge)}_\tau(\mathcal{S};\Lambda)$ the "associated motive" functor as in Definitions 2.1.15 and 3.1.3):

(A) E is local with respect to morphisms $\mathrm{M}(\mathcal{V}) \to \mathrm{M}(\mathcal{U})$, where $\mathcal{V} \to \mathcal{U}$ is an admissible blowup;

(B) if τ is the étale topology, then E is local with respect to morphisms of the form

$$\operatorname*{colim}_{[n]\in\mathbf{\Delta}} \mathrm{M}(\mathcal{V}_\bullet) \to \mathrm{M}(\mathcal{V}_{-1}), \tag{3.29}$$

where $\mathcal{V}_\bullet$ is a hypercover for the topology rigfét, which we assume to be truncated in the non-hypercomplete case.

We split the rest of the proof into several parts. In the first part, we prove property (A). In the second part, we establish a preliminary fact for proving property (B). In the remaining parts, we prove property (B) assuming one of the alternatives (ii) or (iv) in Assumption 3.3.1.

Part 1. Here we prove property (A). We start by introducing some notations. We denote by $f : \mathcal{U} \to \mathcal{S}$ the structural morphism and by $e : \mathcal{V} \to \mathcal{U}$ the admissible blowup, and we set $g = f \circ e$. Since $\mathrm{M}(\mathcal{U}) = f_\sharp\Lambda$ and $\mathrm{M}(\mathcal{V}) = g_\sharp\Lambda$ (see Remark 3.6.6), it is enough to show that the obvious morphism

$$\mathrm{Map}_{\overline{\mathbf{FSH}}{}^{(\wedge)}_\tau(\mathcal{U};\Lambda)}(\Lambda, f^*E) \to \mathrm{Map}_{\overline{\mathbf{FSH}}{}^{(\wedge)}_\tau(\mathcal{V};\Lambda)}(\Lambda, g^*E)$$

is an equivalence. This map can be identified with

$$\mathrm{Map}_{\mathbf{FSH}^{(\wedge)}_\tau(\mathcal{U};\Lambda)}(\Lambda, \iota_{\mathcal{U},*}f^*E) \to \mathrm{Map}_{\mathbf{FSH}^{(\wedge)}_\tau(\mathcal{V};\Lambda)}(\Lambda, \iota_{\mathcal{V},*}g^*E)$$

which is induced by a morphism $\iota_{\mathcal{U},*}f^*E \to e_*\iota_{\mathcal{V},*}g^*E$ in $\mathbf{FSH}^{(\wedge)}_\tau(\mathcal{U};\Lambda)$, and it is enough to show that the latter is an equivalence. We have a chain of equivalences

$$\begin{array}{rcl} \iota_{\mathcal{U},*}f^*E & = & \iota_{\mathcal{U},*}f^*(\overline{\chi}_{\mathcal{S}}(M) \otimes \iota^*_{\mathcal{S}}(N)) \\ & \overset{(1)}{\simeq} & (\iota_{\mathcal{U},*}f^*\overline{\chi}_{\mathcal{S}}(M)) \otimes (\iota_{\mathcal{U},*}f^*\iota^*_{\mathcal{S}}(N)) \\ & \overset{(2)}{\simeq} & \chi_{\mathcal{U}}(f^{\mathrm{rig},*}(M)) \otimes f^*(N) \end{array}$$

where (1) follows from Lemma 3.6.5 and (2) follows from the natural equivalences

$$f^* \circ \overline{\chi}_{\mathcal{S}} \simeq \overline{\chi}_{\mathcal{U}} \circ f^{\mathrm{rig},*}, \quad \iota_{\mathcal{U},*} \circ \overline{\chi}_{\mathcal{U}} \simeq \chi_{\mathcal{U}}, \quad f^* \circ \iota^*_{\mathcal{S}} \simeq \iota^*_{\mathcal{U}} \circ f^* \quad \text{and} \quad \iota_{\mathcal{U},*} \circ \iota^*_{\mathcal{U}} \simeq \mathrm{id}.$$

The same applies with "$\mathcal{V}$" and "g" instead of "$\mathcal{U}$" and "f". Thus, we are left to show that the morphism

$$\chi_{\mathcal{U}}(f^{\mathrm{rig},*}(M)) \otimes f^*(N) \to e_*(\chi_{\mathcal{V}}(g^{\mathrm{rig},*}(M)) \otimes g^*(N))$$

is an equivalence. Since e_σ is a projective morphism, we may use Theorem 3.1.10 and the projective projection formula for algebraic motives (see [Ayo07a, Théorème 2.3.40] and Proposition 2.2.12(1) in the rigid analytic setting) to rewrite the above morphism as

$$\chi_{\mathcal{U}}(f^{\mathrm{rig},*}(M)) \otimes f^*(N) \to e_*(\chi_{\mathcal{V}}(g^{\mathrm{rig},*}(M))) \otimes f^*(N).$$

The result follows now from the commutation $e_* \circ \chi_{\mathcal{V}} \simeq \chi_{\mathcal{U}} \circ e^{\mathrm{rig}}_*$ and the fact that $e^{\mathrm{rig}} : \mathcal{V}^{\mathrm{rig}} \to \mathcal{U}^{\mathrm{rig}}$ is an isomorphism (which implies that $e^{\mathrm{rig}}_* \circ g^{\mathrm{rig},*} \simeq f^{\mathrm{rig},*}$).

Part 2. Until the end of the proof, τ will be the étale topology. In this part, we formulate a property which implies property (B) for a fixed hypercover $\mathcal{V}_\bullet$; see property (B′) below.

For $n \geq -1$, we denote by $g_n : \mathcal{V}_n \to \mathcal{S}$ and $e_n : \mathcal{V}_n \to \mathcal{V}_{-1}$ the obvious morphisms. As in the first part, we need to prove that

$$\mathrm{Map}_{\mathbf{FSH}^{(\wedge)}_\tau(\mathcal{V}_{-1};\Lambda)}(\Lambda, \iota_{\mathcal{V}_{-1},*} g^*_{-1} E) \to \lim_{[n]\in\mathbf{\Delta}} \mathrm{Map}_{\mathbf{FSH}^{(\wedge)}_\tau(\mathcal{V}_n;\Lambda)}(\Lambda, \iota_{\mathcal{V}_n,*} g^*_n E)$$

is an equivalence. As explained in the first part, we have an equivalence

$$\iota_{\mathcal{V}_n,*} g^*_n E \simeq \chi_{\mathcal{V}_n}(g^{\mathrm{rig},*}_n(M)) \otimes g^*_n(N),$$

and it is enough to prove that

$$\chi_{\mathcal{V}_{-1}}(g^{\mathrm{rig},*}_{-1}(M)) \otimes g^*_{-1}(N) \to \lim_{[n]\in\mathbf{\Delta}} e_{n,*}\left(\chi_{\mathcal{V}_n}(g^{\mathrm{rig},*}_n(M)) \otimes g^*_n(N)\right)$$

is an equivalence in $\mathbf{FSH}^{(\wedge)}_{\text{ét}}(\mathcal{V}_{-1};\Lambda)$. Since $e_{n,\sigma}$ is a finite morphism, we may use Theorem 3.1.10 and the projective projection formula for algebraic motives to rewrite the above morphism as

$$\begin{array}{rcl}
\chi_{\mathcal{V}_{-1}}(g^{\mathrm{rig},*}_{-1}(M)) \otimes g^*_{-1}(N) & \to & \lim\limits_{[n]\in\mathbf{\Delta}} \left(e_{n,*}(\chi_{\mathcal{V}_n}(g^{\mathrm{rig},*}_n(M))) \otimes g^*_{-1}(N)\right) \\
& \simeq & \lim\limits_{[n]\in\mathbf{\Delta}} \left(\chi_{\mathcal{V}_{-1}}(e^{\mathrm{rig}}_{n,*}(g^{\mathrm{rig},*}_n(M))) \otimes g^*_{-1}(N)\right) \\
& \simeq & \lim\limits_{[n]\in\mathbf{\Delta}} \left(\chi_{\mathcal{V}_{-1}}(e^{\mathrm{rig}}_{n,*} e^{\mathrm{rig},*}_n(g^{\mathrm{rig},*}_{-1}(M))) \otimes g^*_{-1}(N)\right).
\end{array}$$

Since $g^{\mathrm{rig},*}_{-1}(M)$ belongs to $\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{V}^{\mathrm{rig}}_{-1};\Lambda)$, it admits (hyper)descent with respect to $\mathcal{V}^{\mathrm{rig}}_\bullet$. Using that $\chi_{\mathcal{V}_{-1}}$ is a right adjoint functor, we deduce that the morphism

$$\chi_{\mathcal{V}_{-1}}(g^{\mathrm{rig},*}_{-1}(M)) \to \lim_{[n]\in\mathbf{\Delta}} \chi_{\mathcal{V}_{-1}}(e^{\mathrm{rig}}_{n,*} e^{\mathrm{rig},*}_n(g^{\mathrm{rig},*}_{-1}(M))).$$

is an equivalence. Thus, we see that property (B) follows from the following property:

(B′) Set $A^\bullet = \chi_{\mathcal{V}_{-1}}(e^{\mathrm{rig}}_{\bullet,*} e^{\mathrm{rig},*}_\bullet(g^{\mathrm{rig},*}_{-1}(M)))$ and $B = g^*_{-1}(N)$. Then, the obvious morphism

$$(\lim_{[n]\in\mathbf{\Delta}} A^n) \otimes B \to \lim_{[n]\in\mathbf{\Delta}} (A^n \otimes B) \tag{3.30}$$

is an equivalence in $\mathbf{FSH}^{(\wedge)}_{\text{ét}}(\mathcal{V}_{-1};\Lambda)$.

Part 3. Here we prove property (B) assuming that $\pi_0\Lambda$ is a $\mathbb{Q}$-algebra.

For a formal scheme $\mathcal{X}$, the site $(\mathrm{FRig\acute{E}t}/\mathcal{X}, \mathrm{rigf\acute{e}t})$ has zero global and local Λ-cohomological dimensions. Indeed, let $\mathcal{F}$ be an ordinary rigfét-sheaf of $\mathbb{Q}$-vector spaces on $\mathrm{FRig\acute{E}t}/\mathcal{X}$. For every finite rig-étale covering $\mathcal{X}'' \to \mathcal{X}'$ in $\mathrm{FRig\acute{E}t}/\mathcal{X}$, there is a normalised transfer map $\mathcal{F}(\mathcal{X}'') \to \mathcal{F}(\mathcal{X}')$ which is a section to the restriction map. (This map can be constructed rigfét-locally on $\mathcal{X}'$, and thus we may assume that $\mathcal{X}''$ is isomorphic to a finite coproduct of copies of $\mathcal{X}'$.) Using these normalised transfer maps, one can show that the Čech cohomology of $\mathcal{X}$ with values in $\mathcal{F}$ vanishes in degrees ≥ 1. More precisely, given a finite rig-étale cover $\mathcal{X}' \to \mathcal{X}$, one can build, using the normalised transfer maps, a contracting homotopy from $\mathcal{F}(\mathcal{X}'_\bullet)$, where $\mathcal{X}'_\bullet$ is the Čech nerve of $\mathcal{X}' \to \mathcal{X}$, to the constant simplicial complex $\mathcal{F}(\mathcal{X})$. We leave the easy details to the reader. By Corollary 2.4.6, it follows that every rigfét-sheaf of Λ-modules on $\mathrm{FRigÉt}/\mathcal{X}$ is automatically a rigfét-(hyper)sheaf. (Indeed, although Λ is not assumed to be eventually coconnective, the condition that $\pi_0\Lambda$ is a $\mathbb{Q}$-algebra implies that there exists a morphism of commutative ring spectra $\mathbb{Q} \to \Lambda$, and thus we may replace Λ by $\mathbb{Q}$ in order to apply Corollary 2.4.6.)

By the above discussion, it is enough to check property (B) when $\mathcal{V}_\bullet$ is the Čech nerve associated to a finite rig-étale covering $e_0 : \mathcal{V}_0 \to \mathcal{V}_{-1}$. Moreover, we may assume that the formal $\mathcal{V}_{-1}$-scheme $\mathcal{V}_0$ admits an action of a finite group G which is simply transitive on the geometric fibers of $e_0^{\mathrm{rig}} : \mathcal{V}_0^{\mathrm{rig}} \to \mathcal{V}_{-1}^{\mathrm{rig}}$. The Čech nerve $\mathcal{V}_\bullet$ can be refined by the following rigfét-hypercover

$$\cdots \mathcal{V}_0 \times G \times G \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} \mathcal{V}_0 \times G \rightrightarrows \mathcal{V}_0 \longrightarrow \mathcal{V}_{-1}. \tag{3.31}$$

Since the latter has the same form when base-changed to each $\mathcal{V}_n$, we are left to prove property (B) with (3.31) instead of $\mathcal{V}_\bullet$. As explained in the second part, it suffices to prove property (B′) for (3.31). In this case, the cosimplicial object $A^\bullet$ defines an action of G on $A^0 \in \mathbf{FSH}_{\text{ét}}^{(\wedge)}(\mathcal{V}_{-1};\Lambda)$, and we may rewrite (3.30) as

$$(A^0)^G \otimes B \to (A^0 \otimes B)^G.$$

That this is an equivalence follows from the fact that taking the "G-invariant subobject" in a $\mathbb{Q}$-linear ∞-category is equivalent to taking the image of the projector $|G|^{-1}\sum_{g\in G} g$.

Part 4. Here we prove property (B) under the alternative (iv) and assuming conditions (1), (2) and (3) of Lemma 3.6.2.

By Theorems 2.10.3, 2.10.4 and 3.1.10, we have equivalences of ∞-categories

$$\mathrm{Shv}_{\text{ét}}^{\wedge}(\text{Ét}/\mathcal{X}^{\mathrm{rig}};\mathbb{Z}/\ell) \simeq \mathbf{RigSH}_{\text{ét}}^{\wedge}(\mathcal{X}^{\mathrm{rig}};\mathbb{Z}/\ell) \qquad \text{and} \qquad \mathrm{Shv}_{\text{ét}}^{\wedge}(\text{Ét}/\mathcal{X};\mathbb{Z}/\ell) \simeq \mathbf{FSH}_{\text{ét}}^{\wedge}(\mathcal{X};\mathbb{Z}/\ell)$$

for every formal $\mathcal{S}$-scheme $\mathcal{X}$. Let $M_0 \in \mathrm{Shv}_{\text{ét}}^{\wedge}(\text{Ét}/\mathcal{X}^{\mathrm{rig}};\mathbb{Z}/\ell)$ and $N_0 \in \mathrm{Shv}_{\text{ét}}^{\wedge}(\text{Ét}/\mathcal{X};\mathbb{Z}/\ell)$ be the étale hypersheaves corresponding to M and N by these equivalences. Set $A_0^\bullet = \chi_{\mathcal{V}_{-1}}(e_{\bullet,*}^{\mathrm{rig}} e_\bullet^{\mathrm{rig},*}(g_{-1}^{\mathrm{rig},*}(M_0)))$ and $B_0 = g_{-1}^*(N_0)$. We need to prove that

$$(\lim_{[n]\in\mathbf{\Delta}} A_0^n) \otimes B_0 \to \lim_{[n]\in\mathbf{\Delta}} (A_0^n \otimes B_0) \tag{3.32}$$

is an equivalence in $\mathrm{Shv}_{\text{ét}}^{\wedge}(\text{Ét}/\mathcal{V}_{-1};\mathbb{Z}/\ell)$. We will do this by proving that (3.32) induces an equivalence at every geometric point $v \to \mathcal{V}_{-1,\sigma}$. Since M_0 and N_0 are compact, $A_0^\bullet$ and $A_0^\bullet \otimes B_0$ are eventually connective and coconnective as cosimplicial objects, i.e., uniformly in the cosimplicial degree. Since the homotopy limit of a cosimplicial object in complexes of $\mathbb{Z}/\ell$-modules can be computed using the total complex of the associated double complex, this implies that

$$(\lim_{[n]\in\mathbf{\Delta}} A_0^n)_v \simeq \lim_{[n]\in\mathbf{\Delta}} (A_0^n)_v \qquad \text{and} \qquad (\lim_{[n]\in\mathbf{\Delta}} (A_0^n \otimes B))_v \simeq \lim_{[n]\in\mathbf{\Delta}} ((A_0^n)_v \otimes B_v).$$

Thus, the fiber of (3.32) at v can be identified with the map

$$(\lim_{[n]\in\mathbf{\Delta}} (A_0^n)_v) \otimes (B_0)_v \to \lim_{[n]\in\mathbf{\Delta}} ((A_0^n)_v \otimes (B_0)_v).$$

That the latter is an equivalence follows from the fact that $(B_0)_v$ is a perfect complex of $\mathbb{Z}/\ell$-modules (which is a consequence of the assumption that N is compact). □

The method used for proving Theorem 3.6.1 can be also used to prove the following result.

Proposition 3.6.8. *We work under Assumption 3.3.1. Let $\mathcal{S}$ be a formal scheme and set $S = \mathcal{S}^{\mathrm{rig}}$. The functor $\chi_{\mathcal{S}} : \mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda) \to \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{S};\Lambda)$ preserves colimits.*

Proof. This is clear under the alternatives (iii) and (iv) which imply that $\xi_{\mathcal{S}}$ belongs to $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ by Propositions 2.4.22 and 3.2.3. Thus, it is enough to consider the alternatives (i) and (ii).

By Lemma 3.6.5, the functor $\iota_{\mathcal{S},*}$ preserves colimits. Since $\chi_{\mathcal{S}} = \iota_{\mathcal{S},*} \circ \overline{\chi}_{\mathcal{S}}$, it is enough to show that the functor $\overline{\chi}_{\mathcal{S}}$ preserves colimits. The latter is fully faithful with essential image the full-subcategory of $\overline{\mathbf{FSH}}^{(\wedge)}_{\tau}(\mathcal{S};\Lambda)$ spanned by those objects admitting rig-τ-(hyper)descent. Thus, it is enough to show that the property of admitting rig-τ-(hyper)descent is preserved under colimits.

Let $E : I \to \overline{\mathbf{FSH}}^{(\wedge)}_{\tau}(\mathcal{S};\Lambda)$ be a diagram with colimit $E(\infty)$ and such that $E(\alpha)$ admits rig-τ-(hyper)descent for every $\alpha \in I$. We need to show that $E(\infty)$ admits rig-τ-(hyper)descent. As in the proof of Proposition 3.6.7, we reduce to show the following two properties:

(A) $E(\infty)$ is local with respect to morphisms $\mathrm{M}(\mathcal{V}) \to \mathrm{M}(\mathcal{U})$, where $\mathcal{V} \to \mathcal{U}$ is an admissible blowup;

(B) if τ is the étale topology, then $E(\infty)$ is local with respect to morphisms of the form

$$\operatorname*{colim}_{[n]\in\mathbf{\Delta}} \mathrm{M}(\mathcal{V}_{\bullet}) \to \mathrm{M}(\mathcal{V}_{-1}),$$

where $\mathcal{V}_{\bullet}$ is a hypercover for the topology rigfét, which we assume to be truncated in the non-hypercomplete case.

We split the rest of the proof into two parts.

Part 1. Here we prove property (A). We start by introducing some notations. We denote by $f : \mathcal{U} \to \mathcal{S}$ the structural morphism and by $e : \mathcal{V} \to \mathcal{U}$ the admissible blowup, and we set $g = f \circ e$. We need to show that the obvious morphism

$$\mathrm{Map}_{\overline{\mathbf{FSH}}^{(\wedge)}_{\tau}(\mathcal{U};\Lambda)}(\Lambda, f^*E(\infty)) \to \mathrm{Map}_{\overline{\mathbf{FSH}}^{(\wedge)}_{\tau}(\mathcal{V};\Lambda)}(\Lambda, g^*E(\infty))$$

is an equivalence. As in the first part of the proof of Proposition 3.6.7, it is enough to show that

$$\iota_{\mathcal{U},*} f^*E(\infty) \to e_*\iota_{\mathcal{V},*} g^*E(\infty)$$

is an equivalence. Since the objects $E(\alpha)$ admit rig-τ-(hyper)descent, for $\alpha \in I$, we deduce that the morphisms

$$\iota_{\mathcal{U},*} f^*E(\alpha) \to e_*\iota_{\mathcal{V},*} g^*E(\alpha)$$

are equivalences. Since the functors f^*, g^*, $\iota_{\mathcal{U},*}$ and $\iota_{\mathcal{V},*}$ preserve colimits (see Lemma 3.6.5), it suffices to show that the functor $e_* : \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{V};\Lambda) \to \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{U};\Lambda)$ preserves colimits. By Theorem 3.1.10, it is equivalent to show that the functor $e_{\sigma,*} : \mathbf{SH}^{(\wedge)}_{\tau}(\mathcal{V}_{\sigma};\Lambda) \to \mathbf{SH}^{(\wedge)}_{\tau}(\mathcal{U}_{\sigma};\Lambda)$ preserves colimits. This follows from the fact that e_{σ} is projective which implies that $e_{\sigma,*} \simeq e_{\sigma,!}$ admits a right adjoint $e^!_{\sigma}$; see [Ayo07a, Théorème 1.7.17].

Part 2. Here we prove property (B). In particular, we work under the alternative (ii) and assume that τ is the étale topology.

For $n \geq -1$, we denote by $g_n : \mathcal{V}_n \to \mathcal{S}$ and $e_n : \mathcal{V}_n \to \mathcal{V}_{-1}$ the obvious morphisms. As in the second part of the proof of Proposition 3.6.7, we need to show that

$$\iota_{\mathcal{V}_{-1},*} g^*_{-1} E(\infty) \to \lim_{[n]\in\mathbf{\Delta}} e_{n,*}\iota_{\mathcal{V}_n,*} g^*_n E(\infty)$$

is an equivalence. Since the objects $E(\alpha)$ admit rigét-(hyper)descent, for $\alpha \in I$, we deduce that the morphisms

$$\iota_{\mathcal{V}_{-1},*} g_{-1}^* E(\alpha) \to \lim_{[n]\in\mathbf{\Delta}} e_{n,*}\iota_{\mathcal{V}_n,*} g_n^* E(\alpha)$$

are equivalences. For $n \geq -1$, the functors g_n^*, $\iota_{\mathcal{V}_n,*}$ and $e_{n,*}$ commute with colimits. (For the second one, we use Lemma 3.6.5 and, for the third one, we use that $e_{n,\sigma}$ is finite which implies that $e_{n,\sigma,*} \simeq e_{n,\sigma,!}$ admits a right adjoint $e_{n,\sigma}^!$; see [Ayo07a, Théorème 1.7.17].) Therefore, it is enough to show that the obvious morphism

$$\operatorname*{colim}_{\alpha\in I} \lim_{[n]\in\mathbf{\Delta}} e_{n,*}\iota_{\mathcal{V}_n,*} g_n^* E(\alpha) \to \lim_{[n]\in\mathbf{\Delta}} \operatorname*{colim}_{\alpha\in I} e_{n,*}\iota_{\mathcal{V}_n,*} g_n^* E(\alpha) \tag{3.33}$$

is an equivalence. Now, as explained in the third part of the proof of Proposition 3.6.7, we may assume from the beginning that $\mathcal{V}_\bullet$ is of the form (3.31). In this case, the morphism (3.33) can be rewritten as follows:

$$\operatorname*{colim}_{\alpha\in I} (e_{0,*}\iota_{\mathcal{V}_0,*} g_0^* E(\alpha))^G \to (\operatorname*{colim}_{\alpha\in I} e_{0,*}\iota_{\mathcal{V}_0,*} g_0^* E(\alpha))^G.$$

That this is an equivalence follows from the fact that taking the "G-invariant subobject" in a $\mathbb{Q}$-linear ∞-category is equivalent to taking the image of the projector $|G|^{-1}\sum_{g\in G} g$. □

With Theorem 3.6.1 and Proposition 3.6.8 at hand, we can prove the first assertion in Theorem 3.3.3.

Proof of Theorem 3.3.3(1). We need to show that the unit map $\mathrm{id} \to \widetilde{\chi}_{\mathcal{S}} \circ \widetilde{\xi}_{\mathcal{S}}$ is an equivalence. Clearly, $\widetilde{\xi}_{\mathcal{S}}$ preserves colimits and the same is true for $\widetilde{\chi}_{\mathcal{S}}$ by Proposition 3.6.8 combined with [Lur17, Corollary 3.4.4.6(2)]. It is thus enough to show that $M \to \widetilde{\chi}_{\mathcal{S}}\widetilde{\xi}_{\mathcal{S}} M$ is an equivalence for M varying in a set of objects generating $\mathbf{FSH}_\tau^{(\wedge)}(\mathcal{S};\chi\Lambda)$ under colimits. Thus, we may assume that M is a free $\chi_{\mathcal{S}}\Lambda$-module, i.e., that $M \simeq \chi_{\mathcal{S}}(\Lambda) \otimes N$ for some $N \in \mathbf{FSH}_\tau^{(\wedge)}(\mathcal{S};\Lambda)$. In this case, the unit map coincides with the obvious map $\chi_{\mathcal{S}}(\Lambda) \otimes N \to \chi_{\mathcal{S}}\xi_{\mathcal{S}}(N)$ which is an equivalence by Theorem 3.6.1. □

3.7. **Proof of the main result, II. Sheafification.**

Our goal in this subsection is to prove the second part of Theorem 3.3.3. Using [Lur09, Corollaries 3.2.2.5 & 3.2.3.2], this is equivalent to proving the following statement.

Theorem 3.7.1. *We work under Assumption 3.3.2. The morphism of* Pr^{L}*-valued presheaves*

$$\widetilde{\xi} : \mathbf{FSH}_{\text{ét}}^{(\wedge)}(-;\chi\Lambda) \to \mathbf{RigSH}_{\text{ét}}^{(\wedge)}((-)^{\mathrm{rig}};\Lambda)$$

exhibits $\mathbf{RigSH}_{\text{ét}}^{(\wedge)}((-)^{\mathrm{rig}};\Lambda)$ *as the rig-étale sheaf associated to* $\mathbf{FSH}_{\text{ét}}^{(\wedge)}(-;\chi\Lambda)$.

Remark 3.7.2. In the hypercomplete case, Theorem 3.7.1, combined with Theorem 2.3.4, shows that the étale sheafification of $\mathbf{FSH}_{\text{ét}}^{\wedge}(-;\chi\Lambda)$ is already an étale hypersheaf.

Remark 3.7.3. Let $\mathcal{S}$ be a formal scheme.

(1) Recall that a sieve $H \subset \mathcal{S}$ is a sub-presheaf of $\mathcal{S}$ considered as a presheaf on FSch. A formal H-scheme is a formal $\mathcal{S}$-scheme such that the structural morphism $\mathcal{T} \to \mathcal{S}$ factors through H. We say that H is generated by a family $(\mathcal{S}_i \to \mathcal{S})_i$ if H is equal to the union of the images of the morphisms $\mathcal{S}_i \to \mathcal{S}$ considered as morphisms of presheaves on FSch. Equivalently, H is the smallest sieve of $\mathcal{S}$ such that the $\mathcal{S}_i$'s are formal H-schemes.

(2) We say that a sieve $H \subset \mathcal{S}$ is a rig-étale sieve if the inclusion $H \subset \mathcal{S}$ becomes an isomorphism after rig-étale sheafification. Equivalently, H contains the sieve generated by a rig-étale cover of $\mathcal{S}$. (Of course, this also makes sense for any other topology.)

We will need the following definition.

Definition 3.7.4. Let $\mathcal{S}$ be a formal scheme.

(1) A formal $\mathcal{S}$-scheme $\mathcal{U}$ is said to be nearly smooth (resp. étale) if, locally on $\mathcal{U}$, it is of finite type and there exists a finite morphism $\mathcal{U}' \to \mathcal{U}$ from a smooth (resp. étale) formal $\mathcal{S}$-scheme $\mathcal{U}'$ inducing an isomorphism $\mathcal{U}'^{\mathrm{rig}} \simeq \mathcal{U}^{\mathrm{rig}}$ on generic fibers.
(2) Let $H \subset \mathcal{S}$ be a sieve. A formal $\mathcal{S}$-scheme $\mathcal{U}$ is said to be H-potentially nearly smooth (resp. étale) if $\mathcal{U} \times_{\mathcal{S}} \mathcal{T}$ is nearly smooth (resp. étale) over $\mathcal{T}$ for every formal H-scheme $\mathcal{T}$. If H is generated by a family $(\mathcal{S}_i \to \mathcal{S})_i$, it is enough to ask that $\mathcal{U} \times_{\mathcal{S}} \mathcal{S}_i$ is nearly smooth (resp. étale) over $\mathcal{S}_i$ for every i.
(3) A formal $\mathcal{S}$-scheme $\mathcal{U}$ is said to be potentially nearly smooth (resp. étale) if it is H-potentially nearly smooth (resp. étale) for some rig-étale sieve $H \subset \mathcal{S}$.

As usual, we say that a morphism of formal schemes $\mathcal{T} \to \mathcal{S}$ is (H-potentially, potentially) nearly smooth if the formal $\mathcal{S}$-scheme $\mathcal{T}$ is so.

Remark 3.7.5. It follows immediately from the definition that the class of nearly smooth (resp. étale) morphisms is stable under base change and composition. Similarly, the class of potentially nearly smooth (resp. étale) morphisms is stable under base change. It follows from Proposition 3.7.7 below that the class of potentially nearly étale morphisms is also stable under composition if we restrict to quasi-compact and quasi-separated formal schemes. However, this is not the case for the class of potentially nearly smooth morphisms.

We gather a few properties concerning the notion of (potentially) nearly étale morphisms in the following proposition.

Proposition 3.7.6.

(1) A nearly étale morphism of formal schemes is rig-étale.
(2) Let $f : \mathcal{T} \to \mathcal{S}$ be a potentially nearly étale morphism of formal schemes. Then, there exists a rig-étale cover $g : \mathcal{T}' \to \mathcal{T}$ such that $f \circ g$ is rig-étale.[8]
(3) A quasi-compact and quasi-separated rig-étale morphism of formal schemes is potentially nearly étale.

Proof. Assertion (1) is clear. Indeed, the notion of rig-étaleness is local for the rig topology (see Definition 1.3.3(2)) and a finite morphism $\mathcal{U}' \to \mathcal{U}$ as in Definition 3.7.4(1) is a rig cover.

We now prove (2). By assumption, there is a rig-étale cover $e : \mathcal{S}' \to \mathcal{S}$ such that $f' : \mathcal{T}' = \mathcal{T} \times_{\mathcal{S}} \mathcal{S}' \to \mathcal{S}'$ is nearly étale. By (1), we know that f' is rig-étale. If follows that $e \circ f' : \mathcal{T}' \to \mathcal{S}$ is also rig-étale. Now, remark that $g : \mathcal{T}' \to \mathcal{T}$, which is a base change of e, is a rig-étale cover. This proves the second assertion.

It remains to prove (3). Let $f : \mathcal{T} \to \mathcal{S}$ be a quasi-compact and quasi-separated rig-étale morphism. Our goal is to show that f is potentially nearly étale. The problem is local on $\mathcal{S}$ for the rig-étale topology and, since f is quasi-compact and quasi-separated, it is local for the Zariski topology on $\mathcal{T}$. Thus, we may assume that $\mathcal{S} = \mathrm{Spf}(A)$, with A an adic ring of principal ideal type,

[8] It is plausible that f itself is rig-étale, but we didn't strive to prove this since we do not need it.

and $\mathcal{T} = \mathrm{Spf}(B)$, with B a rig-étale adic A-algebra such that the zero ideal of B is saturated. We fix a generator $\pi \in A$ of an ideal of definition.

We will show that every algebraic geometric rigid point $\mathfrak{s} : \mathrm{Spf}(V) \to \mathcal{S}$ admits a rig-étale neighbourhood $\mathcal{U}_{\mathfrak{s}}$ such that $\mathcal{T} \times_{\mathcal{S}} \mathcal{U}_{\mathfrak{s}}$ is nearly étale over $\mathcal{U}_{\mathfrak{s}}$. This suffices to conclude.

Fix $\mathfrak{s}$ as above. Consider the rig-étale V-algebra $W = V \widehat{\otimes}_A B/(0)^{\mathrm{sat}}$. Arguing as in the proof of Proposition 1.4.19, we see that $\mathrm{Spf}(W)$ is the completion of a quasi-finite affine flat V-scheme, necessarily of finite presentation by [FK18, Chapter 0, Corollary 9.2.8]. From Zariski's main theorem [Gro66, Chapitre IV, Théorème 8.12.6], we deduce that $\mathrm{Spf}(W)$ is an open formal subscheme of $\mathrm{Spf}(W')$ where W' is a finite flat V-algebra. Moreover since $V[\pi^{-1}]$ is an algebraically closed field it follows that $W'[\pi^{-1}]$ is a finite direct product of copies of $V[\pi^{-1}]$. Replacing $\mathcal{S}$ with a rig-étale neighbourhood of $\mathfrak{s}$ and $\mathcal{T}$ with an open covering, we may assume that W is the completion of a localisation of W', i.e., there exists $u \in W'$ which is invertible in $W'[\pi^{-1}]$ and such that W is the completion of $W'[u^{-1}]$.

Using that $W'[\pi^{-1}]$ is a direct product of copies of $V[\pi^{-1}]$, we may find a morphism of V-algebras

$$V[t]/((t - a_1) \cdots (t - a_r)) \to W',$$

inducing an isomorphism after inverting π, where the a_i's belong to V and such that two distinct a_i's differ additively by an invertible element of $V[\pi^{-1}]$. We may extend this morphism into a presentation

$$V\langle t, s_1, \ldots, s_m\rangle/((t - a_1) \cdots (t - a_r), \pi^N s_1 - P_1, \ldots, \pi^N s_m - P_m)^{\mathrm{sat}} \simeq W' \tag{3.34}$$

where $N \in \mathbb{N}$ is large enough and the P_i's are polynomials in $V[t]$. The left-hand side of the isomorphism (3.34) gives a presentation of the rig-étale V-algebra W' as in Definition 1.3.3. Using Proposition 1.3.8 and Lemma 1.4.26, we may assume that the a_i's and the coefficients of the P_j's belong to the image of the map

$$\operatorname*{colim}_{\mathrm{Spf}(V) \to \mathcal{U} \to \mathcal{S}} \mathcal{O}(\mathcal{U}) \to V, \tag{3.35}$$

where the colimit is over affine rig-étale neighbourhoods of $\mathfrak{s}$ in $\mathcal{S}$. Similarly, we may assume that $u \in W'$ is the image of a polynomial $Q \in A[t, s_1, \ldots, s_m]$ with coefficients in the image of (3.35). Thus, we may find a rig-étale neighbourhood $\mathcal{U}_{\mathfrak{s}} = \mathrm{Spf}(A_{\mathfrak{s}})$ of $\mathfrak{s}$ and lifts $\widetilde{a}_i$'s, $\widetilde{P}_j$'s and $\widetilde{Q}$ to $A_{\mathfrak{s}}$ of the a_i's, P_j's and Q. We then set

$$C'_{\mathfrak{s}} = A_{\mathfrak{s}}\langle t, s_1, \ldots, s_m\rangle/((t - \widetilde{a}_1) \cdots (t - \widetilde{a}_r), \pi^N s_1 - \widetilde{P}_1, \ldots, \pi^N s_m - \widetilde{P}_m)^{\mathrm{sat}}$$

$$\text{and} \qquad C_{\mathfrak{s}} = C'_{\mathfrak{s}}\langle v\rangle/(v \cdot \widetilde{Q} - 1).$$

Refining $\mathcal{U}_{\mathfrak{s}}$, we may assume that two $\widetilde{a}_i$'s differ by an invertible element of $A_{\mathfrak{s}}[\pi^{-1}]$. This insures that $C'_{\mathfrak{s}}$ is a rig-étale $A_{\mathfrak{s}}$-algebra. By construction, we have an isomorphism

$$V \widehat{\otimes}_{A_{\mathfrak{s}}} C_{\mathfrak{s}}/(0)^{\mathrm{sat}} \simeq W \simeq V \widehat{\otimes}_A B/(0)^{\mathrm{sat}}.$$

Using Corollary 1.3.10, we may refine $\mathcal{U}_{\mathfrak{s}}$ and assume that

$$C_{\mathfrak{s}} \simeq A_{\mathfrak{s}} \widehat{\otimes}_A B/(0)^{\mathrm{sat}}.$$

Therefore, to conclude, it is enough to see that $\mathrm{Spf}(C'_{\mathfrak{s}})$ is nearly étale over $\mathrm{Spf}(A_{\mathfrak{s}})$ for $\mathcal{U}_{\mathfrak{s}}$ sufficiently small. After refining $\mathcal{U}_{\mathfrak{s}}$ if necessary, we may assume that the classes of the $\widetilde{P}_i$'s in the ring $A_{\mathfrak{s}}[t]/((t - \widetilde{a}_1) \cdots (t - \widetilde{a}_r))$, divided by π^N, are algebraic over this ring. (Indeed, the P_i's satisfy the analogous property.) In this case, the claim is clear since the normalisation of $C'_{\mathfrak{s}}$ in $C'_{\mathfrak{s}}[\pi^{-1}]$ is then a finite direct product of copies of the normalisation of $A_{\mathfrak{s}}$ in $A_{\mathfrak{s}}[\pi^{-1}]$. $\square$

Proposition 3.7.7. *Let $\mathcal{T} \to \mathcal{S}$ be a quasi-compact and quasi-separated potentially nearly étale morphism of formal schemes. Let $\mathcal{V}$ be a potentially nearly smooth formal $\mathcal{T}$-scheme. Then $\mathcal{V}$ is also potentially nearly smooth as a formal $\mathcal{S}$-scheme.*

Proof. The problem is local on $\mathcal{S}$ for the rig-étale topology. Thus, we may assume that $\mathcal{S}$ and $\mathcal{T}$ are quasi-compact and quasi-separated, and that the morphism $\mathcal{T} \to \mathcal{S}$ is nearly étale. The problem is also local on $\mathcal{T}$. Thus, we may assume that there is a finite morphism $\mathcal{T}_1 \to \mathcal{T}$ from an étale formal $\mathcal{S}$-scheme $\mathcal{T}_1$ inducing an isomorphism on generic fibers. It is clearly enough to show that the formal $\mathcal{S}$-scheme $\mathcal{T}_1 \times_{\mathcal{T}} \mathcal{V}$ is potentially nearly smooth over $\mathcal{S}$. Thus, we may replace $\mathcal{T}$ with $\mathcal{T}_1$ and $\mathcal{V}$ with $\mathcal{T}_1 \times_{\mathcal{T}} \mathcal{V}$, and assume that $\mathcal{T} \to \mathcal{S}$ is étale. Let $\mathcal{T}' \to \mathcal{T}$ be a rig-étale cover such that $\mathcal{V} \times_{\mathcal{T}} \mathcal{T}'$ is nearly smooth over $\mathcal{T}'$. By Lemma 3.7.8 below, there is a rig-étale cover $\mathcal{S}' \to \mathcal{S}$ and and a morphism of formal $\mathcal{T}$-schemes $\mathcal{T} \times_{\mathcal{S}} \mathcal{S}' \to \mathcal{T}'$. We claim that the formal $\mathcal{S}'$-scheme $\mathcal{V} \times_{\mathcal{S}} \mathcal{S}'$ is nearly smooth. Indeed, we have an isomorphism $\mathcal{V} \times_{\mathcal{S}} \mathcal{S}' \simeq \mathcal{V} \times_{\mathcal{T}} (\mathcal{T} \times_{\mathcal{S}} \mathcal{S}')$ and the formal $\mathcal{T} \times_{\mathcal{S}} \mathcal{S}'$-scheme $\mathcal{V} \times_{\mathcal{T}} (\mathcal{T} \times_{\mathcal{S}} \mathcal{S}')$ is nearly smooth since it is a base change of the formal $\mathcal{T}'$-scheme $\mathcal{V} \times_{\mathcal{T}} \mathcal{T}'$. The structural morphism of the formal $\mathcal{S}'$-scheme $\mathcal{V} \times_{\mathcal{S}} \mathcal{S}'$ is thus the composition of two nearly smooth morphisms

$$\mathcal{V} \times_{\mathcal{T}} (\mathcal{T} \times_{\mathcal{S}} \mathcal{S}') \to \mathcal{T} \times_{\mathcal{S}} \mathcal{S}' \to \mathcal{S}'.$$

This finishes the proof since nearly smooth morphisms are preserved under composition. □

Lemma 3.7.8. *Let $\mathcal{T} \to \mathcal{S}$ be a quasi-compact and quasi-separated étale morphism of formal schemes, and let $\mathcal{T}' \to \mathcal{T}$ be a rig-étale cover. Then there exists a rig-étale cover $\mathcal{S}' \to \mathcal{S}$ and a morphism of $\mathcal{T}$-schemes $\mathcal{T} \times_{\mathcal{S}} \mathcal{S}' \to \mathcal{T}'$.*

Proof. This is proven in the same manner as Corollary 1.4.30. Given an algebraic geometric rigid point $\mathfrak{s} \to \mathcal{S}$, we consider $\mathfrak{t} = \mathfrak{s} \times_{\mathcal{S}} \mathcal{T}$. This is a quasi-compact and quasi-separated étale formal $\mathfrak{s}$-scheme. Thus $\mathfrak{t}$ is a disjoint union of quasi-compact open formal subschemes of $\mathfrak{s}$. In particular, the morphism $\mathfrak{t} \to \mathcal{T}$ factors through $\mathcal{T}'$. We then use Corollary 1.4.20 and Lemma 1.4.26 to conclude. □

Definition 3.7.9. Let $\mathcal{S}$ be a formal scheme.

(1) Let $K \subset \mathcal{S}$ be a sieve. A rig-étale sieve $H \subset \mathcal{S}$ is said to be K-potentially nearly étale if it can be generated by a family $(\mathcal{S}_i \to \mathcal{S})_i$ consisting of rig-étale morphisms which are K-potentially nearly étale.
(2) A rig-étale sieve $H \subset \mathcal{S}$ is said to be potentially nearly étale if it is K-potentially nearly étale for some rig-étale sieve $K \subset \mathcal{S}$.

Corollary 3.7.10. *Let $\mathcal{S}$ be a quasi-compact and quasi-separated formal scheme. Let $H \subset \mathcal{S}$ be a rig-étale sieve. Then, we may refine H by a rig-étale sieve which is potentially nearly étale.*

Proof. After refinement, we may assume that H is generated by a rig-étale cover $(\mathcal{S}_i \to \mathcal{S})_{i\in I}$ where I is finite and every $\mathcal{S}_i$ is a quasi-compact and quasi-separated rig-étale formal $\mathcal{S}$-scheme. By Proposition 3.7.6(3), each $\mathcal{S}_i$ is K_i-potentially nearly étale over $\mathcal{S}$ for some rig-étale sieve $K_i \subset \mathcal{S}$. It follows that H is K-potentially nearly étale, with $K = \cap_i K_i$ which is a rig-étale sieve since I is finite. □

Notation 3.7.11.

(1) Given a presheaf of sets H on FSch, we denote by $\mathbf{FSH}^{(\wedge)}_{\text{ét}}(H;\chi\Lambda)$ the object of Pr^{L} obtained by evaluating on H the right Kan extension of $\mathbf{FSH}^{(\wedge)}_{\text{ét}}(-;\chi\Lambda)$ along the Yoneda embedding $\mathrm{FSch}^{\mathrm{op}} \to \mathcal{P}(\mathrm{FSch})^{\mathrm{op}}$. We define similarly $\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(H^{\mathrm{rig}};\Lambda)$.

(2) Let $\mathcal{S}$ be a formal scheme and $H \subset \mathcal{S}$ a rig-étale sieve. We denote by

$$\widetilde{\xi}_H : \mathbf{FSH}^{(\wedge)}_{\text{ét}}(H; \chi\Lambda) \to \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}}; \Lambda) \tag{3.36}$$

the functor obtained by evaluating on H the right Kan extension of $\widetilde{\xi}$ and then composing with the equivalence $\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(H^{\text{rig}}; \Lambda) \simeq \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}}; \Lambda)$ provided by Theorem 2.3.4.

Notation 3.7.12. Let $\mathcal{S}$ be a formal scheme and $H \subset \mathcal{S}$ a rig-étale sieve. We denote by

$$\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}}; \Lambda)_{\langle H \rangle} \subset \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}}; \Lambda)$$

the full sub-∞-category generated under colimits, desuspensions and negative Tate twists by motives of the form $\mathrm{M}(\mathcal{U}^{\text{rig}})$ where $\mathcal{U}$ is a formal $\mathcal{S}$-scheme which is H-potentially nearly smooth.

Proposition 3.7.13. *We work under Assumption 3.3.2. Let $\mathcal{S}$ be a quasi-compact and quasi-separated formal scheme and let $H \subset \mathcal{S}$ be a rig-étale sieve.*

(1) The functor (3.36) *is fully faithful and its essential image contains* $\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}}; \Lambda)_{\langle H \rangle}$.

(2) Assume that the sieve H is K-potentially nearly étale for a rig-étale sieve $K \subset \mathcal{S}$. Then the essential image of (3.36) *is contained in* $\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}}; \Lambda)_{\langle K \rangle}$.

Proof. Up to equivalences, the functor $\widetilde{\xi}_H$ is given by

$$\lim_{\mathcal{T} \to H} \mathbf{FSH}^{(\wedge)}_{\text{ét}}(\mathcal{T}; \chi\Lambda) \to \lim_{\mathcal{T} \to H} \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{T}^{\text{rig}}; \Lambda),$$

where the limit is over the category of formal H-schemes. Since limits in CAT_∞ preserve fully faithful embeddings, Theorem 3.3.3(1), proved in Subsection 3.6, implies that the functor $\widetilde{\xi}_H$ is fully faithful. Moreover, an object $M \in \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}}; \Lambda)$ belongs to the essential image of $\widetilde{\xi}_H$ if and only if, for every $e : \mathcal{T} \to \mathcal{S}$ factoring through H, $e^{\text{rig},*}M$ belongs to the essential image of $\widetilde{\xi}_{\mathcal{T}}$. This shows the first assertion. Indeed, if $\mathcal{U}$ is a formal $\mathcal{S}$-scheme which is H-potentially nearly smooth and $e : \mathcal{T} \to \mathcal{S}$ as before, then $e^{\text{rig},*}\mathrm{M}(\mathcal{U}^{\text{rig}}) \simeq \mathrm{M}((\mathcal{U} \times_{\mathcal{S}} \mathcal{T})^{\text{rig}})$ is a colimit of objects of the form $\mathrm{M}(\mathcal{V}^{\text{rig}}) \simeq \xi_{\mathcal{T}}\mathrm{M}(\mathcal{V})$ where $\mathcal{V}$ is a smooth formal $\mathcal{T}$-scheme admitting a finite morphism to an open formal subscheme of $\mathcal{U} \times_{\mathcal{S}} \mathcal{T}$ which induces an isomorphism on generic fibers. (Recall that such $\mathcal{V}$'s exist locally on the nearly smooth formal $\mathcal{T}$-scheme $\mathcal{U} \times_{\mathcal{S}} \mathcal{T}$.)

To prove the second assertion, we assume that H is generated by a rig-étale cover $(\mathcal{S}_i \to \mathcal{S})_i$ such that the formal $\mathcal{S}$-schemes $\mathcal{S}_i$ are rig-étale and K-potentially nearly étale. We want to show that the essential image of $\widetilde{\xi}_H$ is contained in $\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}}; \Lambda)_{\langle K \rangle}$. Let M be in the essential image of $\widetilde{\xi}_H$. Let $\mathcal{T} = \coprod_i \mathcal{S}_i$ and form the Čech nerve $\mathcal{T}_\bullet$. Denote by $e_n : \mathcal{T}_n \to \mathcal{S}$ the obvious morphism. Then

$$\operatorname*{colim}_{[n] \in \mathbf{\Delta}} e^{\text{rig}}_{n,\sharp} e^{\text{rig},*}_n M \to M$$

is an equivalence. (Indeed, by the projection formula, the simplicial object $e^{\text{rig}}_{\bullet,\sharp} e^{\text{rig},*}_\bullet M$ is equivalent to $\mathrm{M}(\mathcal{T}^{\text{rig}}_\bullet) \otimes M$ and $\mathcal{T}^{\text{rig}}_\bullet \to \mathcal{S}^{\text{rig}}$ is a truncated étale hypercover of $\mathcal{S}^{\text{rig}}$.) Since $e^{\text{rig},*}_n M$ belongs to the essential image of $\widetilde{\xi}_{\mathcal{T}_n}$, it is enough to show that the essential image of $e^{\text{rig}}_{n,\sharp} \circ \widetilde{\xi}_{\mathcal{T}_n}$ is contained in $\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}}; \Lambda)_{\langle K \rangle}$. This would follow if we can prove that for every smooth formal $\mathcal{T}_n$-scheme $\mathcal{V}$ the formal $\mathcal{S}$-scheme $\mathcal{V}$ is K-potentially nearly smooth. This is a direct consequence of the definitions (and also a special case of Proposition 3.7.7). □

Recall that $\mathrm{L}_{\text{rigét}}$ denotes the rig-étale sheafification functor. In particular, $\mathrm{L}_{\text{rigét}}\mathbf{FSH}^{(\wedge)}_{\text{ét}}(-; \chi\Lambda)$ is the rig-étale sheaf associated to the Pr^{L}-valued presheaf $\mathbf{FSH}^{(\wedge)}_{\text{ét}}(-; \chi\Lambda)$.

Proposition 3.7.14. *We work under Assumption 3.3.2. Let $\mathcal{S}$ be a quasi-compact and quasi-separated formal scheme. Then the functor*

$$\mathrm{L}_{\text{rigét}}\mathbf{FSH}^{(\wedge)}_{\text{ét}}(\mathcal{S};\chi\Lambda) \to \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}};\Lambda) \tag{3.37}$$

is fully faithful with essential image the full sub-∞-category generated under colimits, desuspensions and negative Tate twists by motives of the form $\mathrm{M}(\mathcal{U}^{\text{rig}})$ *where* $\mathcal{U}$ *is a formal* $\mathcal{S}$*-scheme which is potentially nearly smooth. In fact, we can restrict to those* $\mathcal{U}$*'s which are smooth over a quasi-compact and quasi-separated rig-étale formal* $\mathcal{S}$*-scheme.*

Proof. We split the proof into three steps.

Step 1. Let $\mathrm{L}^1_{\text{rigét}}$ be the endofunctor on presheaves over FSch described informally as follows. Given a formal scheme $\mathcal{S}$ and a presheaf $\mathcal{F}$ with values in an ∞-category admitting limits and colimits, we have

$$\mathrm{L}^1_{\text{rigét}}(\mathcal{F})(\mathcal{S}) = \operatorname*{colim}_{H\subset\mathcal{S}} \overline{\mathcal{F}}(H)$$

where $\overline{\mathcal{F}}$ is the right Kan extension along the Yoneda embedding and the colimit is over the rig-étale sieves $H \subset \mathcal{S}$. For a precise construction of such an endofunctor, we refer the reader to [Lur09, Construction 6.2.2.9 & Remark 6.2.2.12].[9] (In loc. cit., this is done for presheaves with values in $\mathcal{S}$, but the construction makes sense for more general presheaves.)

Let $\mathcal{S}$ be a quasi-compact and quasi-separated formal scheme. Let $\mathrm{Sv}(\mathcal{S})$ be the set of rig-étale sieves of $\mathcal{S}$ ordered by containment and let $\mathrm{Sv}'(\mathcal{S})$ be the subset of $\mathrm{Sv}(\mathcal{S}) \times \mathrm{Sv}(\mathcal{S})$, endowed with the induced order, consisting of those pairs (H, K) such that H is K-potentially nearly étale. We have two projections $\mathrm{Sv}'(\mathcal{S}) \to \mathrm{Sv}(\mathcal{S})$ which are cofinal by Corollary 3.7.10 and [Lur09, Theorem 4.1.3.1]. By Proposition 3.7.13, every pair $(H, K) \in \mathrm{Sv}'(\mathcal{S})$ gives rise to a sequence of fully faithful embeddings

$$\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}};\Lambda)_{\langle H\rangle} \to \mathbf{FSH}^{(\wedge)}_{\text{ét}}(H;\chi\Lambda) \to \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}};\Lambda)_{\langle K\rangle} \to \mathbf{FSH}^{(\wedge)}_{\text{ét}}(K;\chi\Lambda),$$

in which we identified $\mathbf{FSH}^{(\wedge)}_{\text{ét}}(H;\chi\Lambda)$ with its essential image under $\widetilde{\xi}_H$, and similarly for K instead of H. Passing to the colimit over $\mathrm{Sv}'(\mathcal{S})$ and using the cofinality of the two projections $\mathrm{Sv}'(\mathcal{S}) \to \mathrm{Sv}(\mathcal{S})$, we obtain an equivalence in Pr^{L}:

$$\mathrm{L}^1_{\text{rigét}}\mathbf{FSH}^{(\wedge)}_{\text{ét}}(\mathcal{S};\chi\Lambda) \simeq \operatorname*{colim}_{H\subset\mathcal{S}} \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}};\Lambda)_{\langle H\rangle}.$$

Since the sub-∞-categories $\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}};\Lambda)_{\langle H\rangle}$ are generated under colimits by a set of compact generators of $\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}};\Lambda)$, it follows immediately that the induced functor

$$\widetilde{\xi}^1_{\mathcal{S}} : \mathrm{L}^1_{\text{rigét}}\mathbf{FSH}^{(\wedge)}_{\text{ét}}(\mathcal{S};\chi\Lambda) \to \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(\mathcal{S}^{\text{rig}};\Lambda)$$

is fully faithful with essential image the full sub-∞-category generated under colimits, desuspensions and negative Tate twists by motives of the form $\mathrm{M}(\mathcal{U}^{\text{rig}})$ where $\mathcal{U}$ is a formal $\mathcal{S}$-scheme which is potentially nearly smooth.

[9] This can be found in the electronic version of [Lur09] on the author's webpage, but not in the published version.

Step 2. Here, we prove that $\mathrm{L}^1_{\mathrm{rig\acute{e}t}}\mathbf{FSH}^{(\wedge)}_{\acute{e}t}(-;\chi\Lambda)$, restricted to $\mathrm{FSch}^{\mathrm{qcqs}}$, is already a rig-étale sheaf. This will prove the statement except for the last sentence.

We argue as in the proof of Proposition 3.7.13. Let $H \subset \mathcal{S}$ be a rig-étale sieve generated by a finite family $(\mathcal{S}_i \to \mathcal{S})_i$ such that the $\mathcal{S}_i$'s are quasi-compact and rig-étale over $\mathcal{S}$. We consider the functor

$$\widetilde{\xi}^1_H : \mathrm{L}^1_{\mathrm{rig\acute{e}t}}\mathbf{FSH}^{(\wedge)}_{\acute{e}t}(H;\chi\Lambda) \to \mathbf{RigSH}^{(\wedge)}_{\acute{e}t}(\mathcal{S}^{\mathrm{rig}};\Lambda)$$

defined as in Notation 3.7.11(2). This is a fully faithful functor with essential image the sub-∞-category spanned by those $M \in \mathbf{RigSH}^{(\wedge)}_{\acute{e}t}(\mathcal{S}^{\mathrm{rig}};\Lambda)$ such that $e^{\mathrm{rig},*}M$ belongs to the essential image of $\widetilde{\xi}^1_{\mathcal{T}}$ for every $e : \mathcal{T} \to \mathcal{S}$ factoring through H. Our goal is to show that $\widetilde{\xi}^1_{\mathcal{S}}$ and $\widetilde{\xi}^1_H$ have the same essential image.

Let $\mathcal{T} = \coprod_i \mathcal{S}_i$ and form the Čech nerve $\mathcal{T}_\bullet$ associated to $\mathcal{T} \to \mathcal{S}$. Let $e_n : \mathcal{T}_n \to \mathcal{S}$ be the obvious morphism. Let M be in the essential image of $\widetilde{\xi}^1_H$. We have an equivalence

$$\operatorname*{colim}_{[n]\in\mathbf{\Delta}} e^{\mathrm{rig}}_{n,\sharp}e^{\mathrm{rig},*}_n M \to M.$$

Therefore, it is enough to show that $e^{\mathrm{rig}}_{n,\sharp}e^{\mathrm{rig},*}_n M$ belongs to the essential image of $\widetilde{\xi}^1_{\mathcal{S}}$. Using the description of the essential image of $\widetilde{\xi}^1_H$ given above, it suffices to show that $e^{\mathrm{rig}}_{n,\sharp}$ takes the essential image of $\widetilde{\xi}^1_{\mathcal{T}_n}$ to the essential image of $\widetilde{\xi}^1_{\mathcal{S}}$. This follows from the description of the essential images of $\widetilde{\xi}^1_{\mathcal{S}}$ and $\widetilde{\xi}^1_{\mathcal{T}_n}$ given above, and the fact that a potentially nearly smooth formal $\mathcal{T}_n$-scheme is also potentially nearly smooth as a formal $\mathcal{S}$-scheme which follows from Propositions 3.7.6(3) and 3.7.7.

Step 3. It remains to show the last assertion in the statement, concerning the generators under colimits of the essential image of (3.37). Let $\mathcal{C}$ be the sub-∞-category of $\mathbf{RigSH}^{(\wedge)}_{\acute{e}t}(\mathcal{S}^{\mathrm{rig}};\Lambda)$ generated under colimits, desuspension and negative Tate twists by $\mathrm{M}(\mathcal{V}^{\mathrm{rig}})$, with $\mathcal{V}$ smooth over a rig-étale formal $\mathcal{S}$-scheme. We want to show that $\mathcal{C}$ coincides with the essential image of (3.37). By the previous steps, it is enough to show that $\mathrm{M}(\mathcal{U}^{\mathrm{rig}}) \in \mathcal{C}$ for every potentially nearly smooth formal $\mathcal{S}$-scheme $\mathcal{U}$. Let $\mathcal{T} \to \mathcal{S}$ be a rig-étale cover such that $\mathcal{U} \times_{\mathcal{S}} \mathcal{T}$ is nearly smooth over $\mathcal{T}$. Let $\mathcal{T}_\bullet$ be the Čech nerve associated to $\mathcal{T} \to \mathcal{S}$. Since

$$\mathrm{M}(\mathcal{U}^{\mathrm{rig}}) \simeq \operatorname*{colim}_{[n]\in\mathbf{\Delta}} \mathrm{M}((\mathcal{U} \times_{\mathcal{S}} \mathcal{T}_n)^{\mathrm{rig}}),$$

it is enough to show that $\mathrm{M}((\mathcal{U} \times_{\mathcal{S}} \mathcal{T}_n)^{\mathrm{rig}}) \in \mathcal{C}$ for every $n \in \mathbb{N}$. The problem is local on $\mathcal{U} \times_{\mathcal{S}} \mathcal{T}_n$. Since the latter is nearly smooth, we are reduced to show that $\mathrm{M}(\mathcal{V}^{\mathrm{rig}}) \in \mathcal{C}$ if $\mathcal{V}$ is a formal $\mathcal{T}_n$-scheme admitting a finite morphism $\mathcal{V}' \to \mathcal{V}$ inducing an isomorphism $\mathcal{V}'^{\mathrm{rig}} \simeq \mathcal{V}^{\mathrm{rig}}$ and such that $\mathcal{V}'$ is smooth over $\mathcal{T}_n$. This is clear since $\mathrm{M}(\mathcal{V}'^{\mathrm{rig}}) \in \mathcal{C}$ by construction. □

Corollary 3.7.15. *Let $(\mathcal{S}_\alpha)_\alpha$ be a cofiltered inverse system of quasi-compact and quasi-separated formal schemes with affine transition morphisms, and let $\mathcal{S} = \lim_\alpha \mathcal{S}_\alpha$. Assume one of the following conditions.*

(1) We work under the alternative (iii) of Assumption 3.3.1.

(2) We work under the alternative (iv) of Assumption 3.3.1. We assume furthermore that Λ is eventually coconnective or that the numbers $\mathrm{pvcd}_\Lambda(\mathcal{S}^{\mathrm{rig}}_\alpha)$ are bounded independently of α.

Then, we have an equivalence in Pr^{L}:

$$\operatorname*{colim}_{\alpha} \mathrm{L}_{\mathrm{rig\acute{e}t}}\mathbf{FSH}_{\acute{e}t}(\mathcal{S}_\alpha;\chi\Lambda) \simeq \mathrm{L}_{\mathrm{rig\acute{e}t}}\mathbf{FSH}_{\acute{e}t}(\mathcal{S};\chi\Lambda).$$

Proof. This follows from Theorem 2.5.1, Proposition 3.7.14 and the following assertion. Given a rig-étale formal $\mathcal{S}$-scheme $\mathcal{T}$ and a smooth formal $\mathcal{T}$-scheme $\mathcal{V}$, we can find, locally for the rig topology on $\mathcal{T}$ and $\mathcal{V}$, an index α_0, a rig-étale formal $\mathcal{S}_{\alpha_0}$-scheme $\mathcal{T}_{\alpha_0}$, a smooth formal $\mathcal{T}_{\alpha_0}$-scheme $\mathcal{V}_{\alpha_0}$, and isomorphisms of formal $\mathcal{S}$-schemes

$$\mathcal{T}/(0)^{\mathrm{sat}} \simeq \lim_{\alpha\leq\alpha_0} \mathcal{T}_\alpha/(0)^{\mathrm{sat}} \qquad \text{and} \qquad \mathcal{V}/(0)^{\mathrm{sat}} \simeq \lim_{\alpha\leq\alpha_0} \mathcal{V}_\alpha/(0)^{\mathrm{sat}}.$$

(As usual, for $\alpha \leq \alpha_0$, we set $\mathcal{T}_\alpha = \mathcal{T}_{\alpha_0} \times_{\mathcal{S}_{\alpha_0}} \mathcal{S}_\alpha$ and similarly for $\mathcal{V}_\alpha$.) To prove this assertion, we may assume that the $\mathcal{S}_\alpha = \mathrm{Spf}(A_\alpha)$'s are affine, that $\mathcal{T} = \mathrm{Spf}(B)$ with B adic rig-étale over $A = \mathrm{colim}_\alpha A_\alpha$ and admitting a presentation as in Definition 1.3.3, and $\mathcal{V} = \mathrm{Spf}(C)$ with C an adic B-algebra étale over $B\langle t_1, \ldots, t_m\rangle$. Then, the result follows easily from Corollary 1.3.10. □

Remark 3.7.16. Recall that our goal in this subsection is to prove Theorem 3.7.1. This is equivalent to the statement that the morphism of rig-étale Pr^{L}-valued sheaves

$$\mathrm{L}_{\mathrm{rig\acute{e}t}}\widetilde{\xi} : \mathrm{L}_{\mathrm{rig\acute{e}t}}\mathbf{FSH}^{(\wedge)}_{\acute{e}t}(-;\chi\Lambda) \to \mathbf{RigSH}^{(\wedge)}_{\acute{e}t}((-)^{\mathrm{rig}};\Lambda) \tag{3.38}$$

is an equivalence under Assumption 3.3.2. Clearly, it is enough to do so after restricting (3.38) to affine formal schemes. Every affine formal scheme is the limit of a cofiltered inverse system of $(\Lambda, \text{ét})$-admissible affine formal schemes with $(\Lambda, \text{ét})$-admissible generic fiber. Thus, when working under the alternative (iii) of Assumption 3.3.1, Theorem 2.5.1 and Corollary 3.7.15 allow us to restrict (3.38) further to the subcategory of $(\Lambda, \text{ét})$-admissible affine formal schemes with $(\Lambda, \text{ét})$-admissible generic fiber. By Propositions 2.4.19 and 3.2.2, we are then automatically working under the alternative (iv) of Assumption 3.3.1. Said differently, to prove Theorem 3.7.1 we may work from this point onwards under the alternative (iv) of Assumption 3.3.1. In particular, since we only consider formal schemes with finite dimensional generic fibers, (3.38) is a morphism of rig-Nisnevich hypersheaves. (See the proof of Lemma 2.4.18.) As a consequence, it is enough to show that (3.38) induces equivalences on the stalks for the rig-Nisnevich topology. Using Theorem 2.8.6 and the analogous statement for $\mathrm{L}_{\mathrm{rig\acute{e}t}}\mathbf{FSH}^{(\wedge)}_{\acute{e}t}(-;\chi\Lambda)$ which follows in the same way from Corollary 3.7.15, we are left to show the following statement.

Proposition 3.7.17. *Let s be a rigid point and set $\mathfrak{s} = \mathrm{Spf}(\kappa^+(s))$. Assume the following conditions:*

(1) every prime number is invertible either in $\kappa^+(s)$ or in $\pi_0\Lambda$;
(2) when working in the non-hypercomplete case, Λ is eventually coconnective.

Then, $\mathbf{RigSH}^{(\wedge)}_{\acute{e}t}(s;\Lambda)$ is generated under colimits, desuspension and negative Tate twists by motives of the form $\mathrm{M}(\mathcal{U}^{\mathrm{rig}})$ with $\mathcal{U}$ smooth over a rig-étale formal $\mathfrak{s}$-scheme (or, equivalently, by the motives $M(U)$ with U smooth with good reduction over an étale rigid analytic s-space).

Proof. This is a generalisation of [Ayo15, Theorem 2.5.34], and we will adapt the proof of loc. cit. to our situation. Let $\mathcal{C}(s)$ be the sub-∞-category of $\mathbf{RigSH}^{(\wedge)}_{\acute{e}t}(s;\Lambda)$ generated under colimits, desuspension and negative Tate twists by motives of the form $\mathrm{M}(\mathcal{U}^{\mathrm{rig}})$, with $\mathcal{U}$ smooth over a rig-étale formal $\mathfrak{s}$-scheme. Note that $\mathcal{C}(s)$ is equally generated by motives of the form $\mathrm{M}(U)$, with U smooth with good reduction over an étale rigid analytic s-space. Our goal is to show that $\mathcal{C}(s)$ is equal to $\mathbf{RigSH}^{(\wedge)}_{\acute{e}t}(s;\Lambda)$. We divide the proof into several steps.

Step 1. Here we show that it is enough to prove the proposition under the following assumptions:

- $\pi_0\Lambda$ is a $\mathbb{Q}$-algebra;
- $\kappa(s)$ is algebraically closed and $\kappa^+(s)$ has finite height.

In particular, s is $(\Lambda, \text{ét})$-admissible and we will be working in the hypercomplete case.

Indeed, we can find a cofiltered inverse system of rigid points $(s_\alpha)_\alpha$ with $s \sim \lim_\alpha s_\alpha$ such that the valuation rings $\kappa^+(s_\alpha)$ have finite ranks and the fields $\kappa(s_\alpha)$ have finite virtual Λ-cohomological dimensions. We set $\mathfrak{s}_\alpha = \mathrm{Spf}(\kappa^+(s_\alpha))$ so that $\mathfrak{s} = \lim_\alpha \mathfrak{s}_\alpha$. Our goal is to prove that $\mathcal{C}(s) = \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(s;\Lambda)$ and, by Lemma 2.1.20, it is enough to show that $\mathrm{M}(\mathcal{V}^{\mathrm{rig}}) \in \mathcal{C}(s)$ for $\mathcal{V}$ a rig-smooth formal $\mathfrak{s}$-scheme. Moreover, we may assume that $\mathcal{V} = \mathrm{Spf}(A)$ where A is an adic $\kappa^+(s)$-algebra which is rig-étale over $\kappa^+(s)\langle t_1, \cdots, t_m\rangle$. Thus, using Corollary 1.3.10, there is an index α and a rig-smooth formal $\mathfrak{s}_\alpha$-scheme $\mathcal{V}_\alpha$ such that $\mathcal{V}^{\mathrm{rig}} = \mathcal{V}^{\mathrm{rig}}_\alpha \times_{s_\alpha} s$. Since $\mathcal{C}(s)$ contains the image of $\mathcal{C}(s_\alpha)$ by the inverse image functor along $s \to s_\alpha$, we see that it is enough to show that $\mathrm{M}(\mathcal{V}^{\mathrm{rig}}_\alpha) \in \mathcal{C}(s_\alpha)$. Thus, we may replace s by s_α and assume that s is $(\Lambda, \text{ét})$-admissible. In particular, by Proposition 2.4.19, the non-hypercomplete case is then covered by the hypercomplete case. Also, the ∞-category $\mathbf{RigSH}^{\wedge}_{\text{ét}}(s;\Lambda)$ is compactly generated by Proposition 2.4.22.

Next, we explain how to reduce to the case where $\pi_0\Lambda$ is a $\mathbb{Q}$-algebra. Let $M \in \mathbf{RigSH}^{\wedge}_{\text{ét}}(s;\Lambda)$ and consider the cofiber sequence $M \to M_{\mathbb{Q}} \to M_{\mathrm{tor}}$ where $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ is the rationalisation of M. The motive M_{tor} is a direct coproduct of ℓ-nilpotent motives M_ℓ for ℓ non invertible in $\pi_0\Lambda$. By Theorem 2.10.3, we have an equivalence of ∞-categories

$$\mathrm{Shv}^{\wedge}_{\text{ét}}(\text{Ét}/s;\Lambda)_{\ell\text{-nil}} \simeq \mathbf{RigSH}^{\wedge}_{\text{ét}}(s;\Lambda)_{\ell\text{-nil}}.$$

This implies that M_ℓ belongs to the sub-∞-category of $\mathbf{RigSH}^{\wedge}_{\text{ét}}(s;\Lambda)$ generated under colimits by motives of the form $\mathrm{M}(U)$, where U is an étale rigid analytic s-space. This show that M_{tor} belongs to $\mathcal{C}(s)$, and we are left to show that $M_{\mathbb{Q}}$ belongs to $\mathcal{C}(s)$. To do so, we may replace Λ with $\Lambda_{\mathbb{Q}}$ and assume that $\pi_0\Lambda$ is a $\mathbb{Q}$-algebra.

It remains to explain how to reduce to the case where $\kappa(s)$ is algebraically closed. Let $\kappa^+(\overline{s})$ be the adic completion of a valuation ring extending $\kappa^+(s)$ inside a separable closure of $\kappa(s)$, and let $\kappa(\overline{s})$ be the fraction field of $\kappa^+(\overline{s})$. This defines a geometric algebraic point $\overline{s}$ over s as in Construction 1.4.27(2). We have $\overline{s} \sim \lim_\alpha \overline{s}_\alpha$ where $(\overline{s}_\alpha)_\alpha$ is the cofiltered inverse system of rigid points such that $\kappa(\overline{s}_\alpha)/\kappa(s)$ is a finite separable extension contained in $\kappa(\overline{s})$. Using Theorem 2.5.1 and arguing as above, we have an equivalence in $\mathrm{Pr}^{\mathrm{L}}_\omega$:

$$\mathcal{C}(\overline{s}) \simeq \operatorname*{colim}_{\alpha} \mathcal{C}(\overline{s}_\alpha). \tag{3.39}$$

Denote by $e : \overline{s} \to s$, $e_\alpha : \overline{s} \to \overline{s}_\alpha$ and $r_\alpha : \overline{s}_\alpha \to s$ the obvious morphisms. Consider a compact motive $M \in \mathbf{RigSH}^{\wedge}_{\text{ét}}(s;\Lambda)$ and assume that we know that $e^*M \in \mathcal{C}(\overline{s})$. Since e^*M is compact, the equivalence (3.39) implies that there exists α_0 and a compact object $N \in \mathcal{C}(\overline{s}_{\alpha_0})$ such that $e^*M = e^*_{\alpha_0}N$. In particular, the two compact objects $r^*_{\alpha_0}M$ and N of $\mathbf{RigSH}^{\wedge}_{\text{ét}}(\overline{s}_{\alpha_0};\Lambda)$ become equivalent when pulled back to $\overline{s}$. By Theorem 2.5.1, they actually become equivalent when pulled back to $\overline{s}_\alpha$, for $\alpha \leq \alpha_0$ sufficiently small. This shows that $r^*_\alpha M$ belongs to $\mathcal{C}(\overline{s}_\alpha)$. We now conclude as in the second step of the proof of Proposition 3.7.14: using the Čech nerve associated to $\overline{s}_\alpha \to s$, we reduce to show that, for $n \geq 1$,

$$M \otimes \mathrm{M}(\overbrace{\overline{s}_\alpha \times_s \cdots \times_s \overline{s}_\alpha}^{n\,\text{times}}) \simeq (r_{\alpha,\sharp}r^*_\alpha M) \otimes \mathrm{M}(\overbrace{\overline{s}_\alpha \times_s \cdots \times_s \overline{s}_\alpha}^{n-1\,\text{times}}),$$

belongs to $\mathcal{C}(s)$ which is clear.

Step 2. In the remainder of the proof, we work under the two assumptions introduced in the first step. We set $K = \kappa(s)$, $V = \kappa^+(s)$ and we fix $\pi \in V$ a generator of an ideal of definition. We set $\eta = \mathrm{Spec}(K)$ and use a subscript "η" to denote the fiber at η of a V-scheme. By Lemma 2.1.20, the

∞-category $\mathbf{RigSH}^{\wedge}_{\text{ét}}(s;\Lambda)$ is generated under colimits by the motives $\mathrm{M}(Y)$, for $Y \in \mathrm{RigSm}^{\mathrm{qc}}/s$, and their desuspensions and negative Tate twists. We will show that $\mathrm{M}(Y) \in \mathcal{C}(s)$ by induction on the relative dimension d of $|Y|$ over $|s|$. The case of relative dimension zero is clear because Y is then étale over s. In general, the problem is local on Y. Thus, by Proposition 1.3.15, we may assume that Y is the π-adic completion $\widehat{P}$ of a V-scheme P of finite presentation and generically smooth. Replacing P with the Zariski closure of P_η, we may also assume that P is flat over V.

Step 3. (This is analogous to the second step in the proof of [Ayo15, Théorème 2.5.34].) In this step, we will prove the following preliminary assertion. Let $E \subset P$ be a closed subscheme, generically of codimension ≥ 1, and let $Z = \widehat{E}^{\mathrm{rig}}$ considered as a closed rigid analytic subspace of Y. Then the relative motive $\mathrm{M}(Y/Y \smallsetminus Z)$, defined as the cofiber of $\mathrm{M}(Y \smallsetminus Z) \to \mathrm{M}(Y)$, belongs to $\mathcal{C}(s)$. The proof of this uses the induction on the relative dimension of $|Y|$ over $|s|$, and we will argue by a second induction on the dimension of E_η. The base case for the second induction is when E_η is empty: the relative motive is then zero and the claim is obvious. Let $E' \subset E$ be the closure of the singularity locus of E_η and $Z' = \widehat{E'}^{\mathrm{rig}}$. Since $\kappa(s)$ is algebraically closed and hence perfect, E'_η has codimension ≥ 1 in E_η. By the second induction, we may assume that $\mathrm{M}(Y/Y \smallsetminus Z')$ belongs to $\mathcal{C}(s)$. We are thus left to show that $\mathrm{M}(Y \smallsetminus Z'/Y \smallsetminus Z)$ belongs to $\mathcal{C}(s)$. The rigid analytic space $Y \smallsetminus Z'$ is not necessarily quasi-compact, but we may write it as a filtered union of quasi-compact opens $Y_\alpha = (\widehat{P_\alpha})^{\mathrm{rig}}$ where P_α are open subschemes of admissible blowups of P, not meeting the closure of E'_η. Thus, we are left to show that $\mathrm{M}(Y_\alpha/Y_\alpha \smallsetminus Z_\alpha)$ belongs to $\mathcal{C}(s)$ with Z_α the generic fiber of the formal completion of $E_\alpha = E \times_P P_\alpha$. Replacing Y with Y_α and E with E_α, we are thus reduced to showing that $\mathrm{M}(Y/Y \smallsetminus Z)$ belongs $\mathcal{C}(s)$ under the assumption that E_η is smooth.

As usual, we may also assume that P is affine, and that E is flat over V. Now, assume we are given a finite type morphism $e : \widetilde{P} \to P$ and a closed subscheme $\widetilde{E} \subset \widetilde{P}$ with the following properties:

- e_η is étale, $\widetilde{E} \subset e^{-1}(E)$ and $\widetilde{E}_\eta = e_\eta^{-1}(E_\eta)$;
- the induced morphism $\widetilde{E} \to E$ is proper and an isomorphism $\widetilde{E}_\eta \simeq E_\eta$ on generic fibers.

Then, letting $\widetilde{Y}$ and $\widetilde{Z}$ be the generic fibers of the π-adic completions of $\widetilde{P}$ and $\widetilde{E}$, we have, by étale excision, an isomorphism $\mathrm{M}(\widetilde{Y}/\widetilde{Y} \smallsetminus \widetilde{Z}) \simeq \mathrm{M}(Y/Y \smallsetminus Z)$. Using this principle twice, we may assume that P is isomorphic to $E \times \mathbb{A}^c$, for some $c \geq 1$, and that $E \subset P$ is the zero section. In this case, the relative motive $\mathrm{M}(Y/Y \smallsetminus Z)$ is isomorphic to $\mathrm{M}(Z)(c)[2c]$, and we may conclude using the induction on the relative dimension of Y.

Step 4. (This is analogous to the third step in the proof of [Ayo15, Théorème 2.5.34].) In this step, we show that we may assume P to be "poly-stable". By means of [Ber99, Lemma 9.2], applied to some compactification of P, we may find a proper surjective morphism $e : Q \to P$ with the following properties:

- there is a finite group G acting on the P-scheme Q, a dense open subscheme $L \subset P_\eta$ with inverse image $M = e^{-1}(L)$ dense in Q_η, and such that $M \to M/G$ is a finite étale Galois cover with group G and $M/G \to L$ is a universal homeomorphism;
- the projection $Q \to \mathrm{Spec}(V)$ factors as a composition of

$$Q = Q_d \xrightarrow{f_d} Q_{d-1} \to \ldots \to Q_1 \xrightarrow{f_1} Q_0 = \mathrm{Spec}(V)$$

and, for every $1 \leq i \leq d$, the morphism f_i decomposes, étale locally on the source and the target, as

$$\mathrm{Spec}(B) \xrightarrow{\text{étale}} \mathrm{Spec}(A[u,v]/(uv-a)) \to \mathrm{Spec}(A) \tag{3.40}$$

with A a flat V-algebra of finite type, u and v two indeterminates, and $a \in A$ invertible in $A[\pi^{-1}]$.

In particular, we see that the f_i's have relative dimension 1 and that the $(f_i)_\eta$'s are smooth.

Let $E \subset P$ be the closure of $P_\eta \smallsetminus L$ in P and $F \subset Q$ the closure of $Q_\eta \smallsetminus M$ in Q. By the second step, it is enough to prove that $\mathrm{M}(\widehat{P}^{\mathrm{rig}} \smallsetminus \widehat{E}^{\mathrm{rig}})$ belongs to $\mathcal{C}(s)$. By Lemma 3.7.18 below, $\mathrm{M}(\widehat{P}^{\mathrm{rig}} \smallsetminus \widehat{E}^{\mathrm{rig}})$ is a direct summand of $\mathrm{M}(\widehat{Q}^{\mathrm{rig}} \smallsetminus \widehat{F}^{\mathrm{rig}})$ and it is enough to see that the latter is in $\mathcal{C}(s)$. Using the second step again, we see that it is enough to show that $\mathrm{M}(\widehat{Q}^{\mathrm{rig}})$ belongs to $\mathcal{C}(s)$. Thus, replacing P with Q and Y with $\widehat{Q}^{\mathrm{rig}}$, we may assume that the projection $P \to \mathrm{Spec}(V)$ can be factored as a composition

$$P = P_d \xrightarrow{f_d} P_{d-1} \to \ldots \to P_1 \xrightarrow{f_1} P_0 = \mathrm{Spec}(V) \tag{3.41}$$

with f_i given, étale locally on the source and the target, by (3.40).

Step 5. We now conclude the proof. We argue by induction on the number of integers $i \in \{1, \ldots, d\}$ such that f_i is not smooth. If all the f_i's are smooth, then the formal scheme $\widehat{P}$ is smooth over $\mathrm{Spf}(V)$ and $\mathrm{M}(Y) \in \mathcal{C}(s)$ by construction. Now suppose that at least one of the f_i's is not smooth. Arguing as in [Ayo15, page 332],[10] we may assume that $f_d : P_d \to P_{d-1}$ is not smooth. The problem is local for the étale topology on Y. (More precisely, if $Y_\bullet \to Y$ is a truncated étale hypercover then it is enough to prove that $\mathrm{M}(Y_n) \in \mathcal{C}(s)$ for $n \geq 0$.) Therefore, we may assume that a factorization as in (3.40) exists globally for f_d, i.e., that f_d is a composition of

$$P = P_d \xrightarrow{\text{étale}} P_{d-1}[u,v]/(uv-a) \to P_{d-1}$$

for some $a \in \mathcal{O}(P_{d-1})$ which is invertible in $\mathcal{O}((P_{d-1})_\eta)$. Arguing by étale excision as in [Ayo15, page 333], we conclude that it suffices to treat the case where $P = P_{d-1}[u,v]/(uv-a)$.

We set $R = P_{d-1}$. By the induction on the relative dimension of $|Y| \to |s|$, we know that $\mathrm{M}(\widehat{R}^{\mathrm{rig}})$ belongs to $\mathcal{C}(s)$. Consider the blowup $e : W \to R[u]$ of the ideal (a, u). Since a is invertible on R_η, e_η is an isomorphism and $\widehat{W}^{\mathrm{rig}} \simeq \widehat{R}^{\mathrm{rig}} \times \mathbb{B}^1$. Moreover, W admits a Zariski cover given by $P = R[u,v]/(uv-a)$ and $P' = R[u,w]/(aw-u) \simeq R[w]$ intersecting at $P'' = R[u,v,v^{-1}]/(uv-a) \simeq R[v,v^{-1}]$. Thus, we have a cofiber sequence

$$\mathrm{M}(\widehat{R}^{\mathrm{rig}} \times \mathbb{U}^1) \to \mathrm{M}(\widehat{P}^{\mathrm{rig}}) \oplus \mathrm{M}(\widehat{R}^{\mathrm{rig}} \times \mathbb{B}^1) \to \mathrm{M}(\widehat{R}^{\mathrm{rig}} \times \mathbb{B}^1)$$

showing that $\mathrm{M}(Y)$ is isomorphic to $\mathrm{M}(\widehat{R}^{\mathrm{rig}}) \oplus \mathrm{M}(\widehat{R}^{\mathrm{rig}})(1)[1]$. This finishes the proof. □

Lemma 3.7.18. *Let S be a rigid analytic space, $f : Y \to X$ a morphism of smooth rigid analytic S-spaces and G a finite group acting on the rigid analytic X-space Y. Assume that $Y \to Y/G$ is a finite étale cover and that $Y/G \to X$ is a universal homeomorphism. Assume also that the order of G is invertible in $\pi_0\Lambda$ and that every prime number is invertible either in $\mathcal{O}(X)$ or in $\pi_0\Lambda$. Then, in the ∞-category $\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(S;\Lambda)$, the morphism $\mathrm{M}(Y) \to \mathrm{M}(X)$ induced by f exhibits $\mathrm{M}(X)$ as the image of the projector $|G|^{-1}\sum_{g\in G} g$ acting on $\mathrm{M}(Y)$.*

[10]We remind the reader that the page references to [Ayo15] correspond to the published version.

Proof. Let $\pi_X : X \to S$ and $\pi_Y : Y \to S$ be the structural morphisms. Since $\mathrm{M}(X) = \pi_{X,\sharp}\pi_X^*\Lambda$, there is an equivalence of copresheaves

$$\mathrm{Map}_{\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(S\,;\Lambda)}(\mathrm{M}(X), -) \simeq \mathrm{Map}_{\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(S\,;\Lambda)}(\Lambda, \pi_{X,*}\pi_X^*(-)),$$

and similarly for Y instead of X. Thus, by Yoneda's lemma, it is enough to show that, for every $M \in \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(S\,;\Lambda)$, the obvious morphism $\pi_{X,*}\pi_X^*M \to \pi_{Y,*}\pi_Y^*M$ exhibits $\pi_{X,*}\pi_X^*M$ as the image of the projector $|G|^{-1}\sum_{g\in G} g$ acting on $\pi_{Y,*}\pi_Y^*M$. Set $X' = Y/G$ and let $\pi_{X'} : X' \to S$ be the structural morphism. By étale descent, the image of the projector $|G|^{-1}\sum_{g\in G} g$ acting on $\pi_{Y,*}\pi_Y^*M$ is equivalent to $\pi_{X',*}\pi_{X'}^*M$. Thus, we need to show that the natural transformation $\pi_{X,*}\pi_X^* \to \pi_{X',*}\pi_{X'}^*$ is an equivalence. This follows from the fact that the unit morphism $\mathrm{id} \to e_*e^*$ is an equivalence, which is a consequence of Theorem 2.9.7. □

Now that we have completed the proof of Theorem 3.7.1, we record the following generalisation of Proposition 3.7.17.

Corollary 3.7.19. *Let S be a rigid analytic space. Assume the following conditions:*

(1) every prime number is invertible either in every $\kappa^+(s)$ for $s \in |S|$ or in $\pi_0\Lambda$;
(2) when working in the non-hypercomplete case, Λ is eventually coconnective.

Then $\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(S\,;\Lambda)$ is generated under colimits, desuspension and negative Tate twists by the motives $\mathrm{M}(U)$ with U smooth with good reduction over an étale rigid analytic S-space.

Proof. The problem is local on S. Thus, we may assume that $S = \mathrm{Spf}(A)^{\mathrm{rig}}$ with A an adic ring. We may write A as the colimit in the category of adic rings of a filtered direct system $(A_\alpha)_\alpha$ such that the $\mathcal{S}_\alpha = \mathrm{Spf}(A_\alpha)$ and the $S_\alpha = \mathrm{Spf}(A_\alpha)^{\mathrm{rig}}$ are $(\Lambda, \text{ét})$-admissible. Arguing as in the first step of the proof of Proposition 3.7.17, we see that it is enough to prove the corollary for each S_α. Said differently, we may assume that $\mathcal{S} = \mathrm{Spf}(A)$ and S are $(\Lambda, \text{ét})$-admissible. By Theorem 3.7.1, we have an equivalence

$$\mathrm{L}_{\text{rigét}}\mathbf{FSH}^{(\wedge)}_{\text{ét}}(\mathcal{S};\chi\Lambda) \simeq \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(S\,;\Lambda).$$

We may now conclude using Proposition 3.7.14. □

Corollary 3.7.20. *Let S be a rigid analytic space and assume the conditions (1) and (2) of Corollary 3.7.19. For every $U \in \text{Ét}^{\mathrm{qcqs}}/S$, denote by $f_U : U \to S$ the structural morphism and choose a formal model $\mathcal{U}$ of U. Then, the functors*

$$\chi_{\mathcal{U}} \circ f_U^* : \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(S\,;\Lambda) \to \mathbf{FSH}^{(\wedge)}_{\text{ét}}(\mathcal{U};\Lambda),$$

for $U \in \text{Ét}^{\mathrm{qcqs}}/S$, form a conservative family. In fact, the same is true if we restrict to those U's admitting affine formal models of principal ideal type.

Proof. This follows immediately from Proposition 3.1.14 and Corollary 3.7.19. □

We end the subsection with the following statement.

Theorem 3.7.21. *We assume that τ is the étale topology and work under one of the alternatives (ii), (iii) and (iv) of Assumption 3.3.1. Let s be a geometric rigid point and set $\mathfrak{s} = \mathrm{Spf}(\kappa^+(s))$. Then*

$$\widetilde{\xi}_{\mathfrak{s}} : \mathbf{FSH}^{(\wedge)}_{\text{ét}}(\mathfrak{s};\chi\Lambda) \to \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(s;\Lambda)$$

is an equivalence of ∞-categories.

Proof. When working under (iii) or (iv), this is a direct consequence of Theorem 3.3.3(2) and the fact that every rig-étale cover of $\mathfrak{s}$ splits. In the generality considered in the statement, we argue as follows. The functor $\widetilde{\xi}_{\mathfrak{s}}$ is fully faithful by Theorem 3.3.3(1). Since this functor preserves colimits, it remains to see that its image generates $\mathbf{RigSH}^{(\wedge)}_{\text{ét}}(s;\Lambda)$ under colimits. This follows from Proposition 3.7.17 and the fact that an étale rigid analytic s-space is a coproduct of open subspaces. □

3.8. **Complement.**

Theorem 3.3.3 is especially useful if we have a handle on the commutative algebras $\chi_{\mathcal{S}}\Lambda$, for $\mathcal{S} \in \mathrm{FSch}$. Our goal in this subsection is to obtain a purely algebro-geometric description of these commutative algebras, i.e., one that does not involve rigid analytic geometry. In order to do so, we need to assume that τ is the étale topology; the case of the Nisnevich topology seems to require techniques of resolution of singularities which are stronger than what is available.

Given a formal scheme $\mathcal{S}$, we will implicitly identify the ∞-categories $\mathbf{SH}^{(\mathrm{eff},\wedge)}_{\tau}(\mathcal{S}_\sigma;\Lambda)$ and $\mathbf{FSH}^{(\mathrm{eff},\wedge)}_{\tau}(\mathcal{S};\Lambda)$ by means of Theorem 3.1.10. In particular, $\chi_{\mathcal{S}}\Lambda$ will be considered as a commutative algebra in $\mathbf{SH}^{(\mathrm{eff},\wedge)}_{\tau}(\mathcal{S}_\sigma;\Lambda)$. Our goal is to prove Theorem 3.8.1 below. The proof will occupy most of the subsection, and it is inspired by the proof of [Ayo15, Théorème 1.3.38].

Theorem 3.8.1. *Let B be a scheme, $B_\sigma \subset B$ a closed subscheme locally of finite presentation up to nilimmersion, and $B_\eta \subset B$ its open complement. Consider the functor*

$$\chi_B : \mathbf{SH}^{(\wedge)}_{\text{ét}}(B_\eta;\Lambda) \to \mathbf{SH}^{(\wedge)}_{\text{ét}}(B_\sigma;\Lambda)$$

given by $\chi_B = i^* \circ j_*$, where $i : B_\sigma \to B$ and $j : B_\eta \to B$ are the obvious immersions. Assume that every prime number is invertible either in $\pi_0\Lambda$ or in $\mathcal{O}(B)$. Assume one of the following alternatives.

(1) We work in the non-hypercomplete case and Λ is eventually coconnective;
(2) We work in the hypercomplete case and B is $(\Lambda, \text{ét})$*-admissible.*

Let $\widehat{B}$ be the formal completion of B at B_σ. (Note that $B_\sigma = \widehat{B}_\sigma$ up to nilimmersion.) Then, there is an equivalence $\chi_B\Lambda \simeq \chi_{\widehat{B}}\Lambda$ of commutative algebras in $\mathbf{SH}^{(\wedge)}_{\text{ét}}(B_\sigma;\Lambda)$.

Remark 3.8.2. One has a good handle on the motive $\chi_B\Lambda$ in many situations. For example, if B is regular and B_σ is a principal regular divisor in B, then $\chi_B\Lambda \simeq \Lambda \oplus \Lambda(-1)[-1]$. This follows from absolute purity; see Corollary 3.8.32 below. More generally, absolute purity can be used to give a precise description of $\chi_B\Lambda$ when B_σ is a normal crossing divisor of a regular scheme B. In general, assuming that B is quasi-excellent, one can access $\chi_B\Lambda$ using techniques of resolution of singularities to reduce to the case where B is regular and B_σ is a normal crossing divisor. In fact, these techniques will also be used in the proof of Theorem 3.8.1.

Remark 3.8.3. Let k be a field of characteristic zero having finite virtual Λ-cohomological dimension. In the non-hypercomplete case, assume that Λ is eventually coconnective. Let K be the discretely valued field $k((\pi))$ and $R \subset K$ its valuation ring. For $n \in \mathbb{N}^\times$, we denote by $K_n = K[\pi^{1/n}]$ the finite extension of K obtained by adjoining an n-th root of unity, and $R_n \subset K_n$ its valuation ring. Also, we let K_∞ be the completion of $\bigcup_{n\in\mathbb{N}^\times} K_n$ and $R_\infty \subset K_\infty$ its valuation ring. Using Theorem 3.8.1 (and Remark 3.8.2), we obtain canonical equivalences of commutative algebras

$$\chi_{R_n}\Lambda \simeq q_{n,*}\Lambda$$

where $q_n : T_n \to \mathrm{Spec}(k)$ is the structural projection of the 1-dimensional torus $T_n \simeq \mathbb{G}_{\mathrm{m}}$ given by $\mathrm{Spec}(k[\pi^{1/n}, \pi^{-1/n}])$. It follows formally that we have an equivalence of ∞-categories

$$\mathbf{SH}^{(\wedge)}_{\text{ét}}(k, \chi_{R_n}\Lambda) \simeq \mathbf{uSH}^{(\wedge)}_{\text{ét}}(T_n; \Lambda),$$

where $\mathbf{uSH}^{(\wedge)}_{\text{ét}}(T_n; \Lambda)$ is the full sub-∞-category of $\mathbf{SH}^{(\wedge)}_{\text{ét}}(T_n; \Lambda)$ generated under colimits by the image of the functor q_n^*. Letting n go to ∞, we obtain an equivalence of ∞-categories

$$\mathbf{SH}^{(\wedge)}_{\text{ét}}(k, \chi_{R_\infty}\Lambda) \simeq \mathbf{uSH}^{(\wedge)}_{\text{ét}}(T_\infty; \Lambda), \tag{3.42}$$

where T_∞ is the pro-torus given by the spectrum of $\bigcup_{n\in\mathbb{N}^\times} k[\pi^{1/n}, \pi^{-1/n}]$ and $\mathbf{uSH}^{(\wedge)}_{\text{ét}}(T_\infty; \Lambda)$ is defined similarly as for the T_n's. Now, assume furthermore that k is algebraically closed. Then the valued field K_∞ is also algebraically closed. Combining the equivalence (3.42) with Theorem 3.7.21, we obtain an equivalence of ∞-categories

$$\mathbf{uSH}^{(\wedge)}_{\text{ét}}(T_\infty; \Lambda) \simeq \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(K_\infty; \Lambda). \tag{3.43}$$

Moreover, one can check that this equivalence is given by the composition of

$$\mathbf{uSH}^{(\wedge)}_{\text{ét}}(T_\infty; \Lambda) \subset \mathbf{SH}^{(\wedge)}_{\text{ét}}(T_\infty; \Lambda) \to \mathbf{SH}^{(\wedge)}_{\text{ét}}(K_\infty; \Lambda) \xrightarrow{\mathrm{An}^*} \mathbf{RigSH}^{(\wedge)}_{\text{ét}}(K_\infty; \Lambda).$$

In fact, by Galois descent, one can show that the equivalence (3.43) is also true without assuming that k is algebraically closed. We obtain in this way a weak version of [Ayo15, Scholie 1.3.26(1)] (for the étale topology and after replacing K with K_∞). See also [Ayo15, Théorème 2.5.75] for a similar statement for motives with transfers.

Our first task is to construct a morphism of commutative algebras $\chi_B\Lambda \to \chi_{\widehat{B}}\Lambda$ which we will eventually prove to be an equivalence. In order to do so, we need a digression on the notion of rigid analytic schemes, generalising [Ayo15, Définition 1.4.1].

Definition 3.8.4. A rigid analytic scheme S is a triple $(S_\eta, \widehat{S}, \iota_S)$ consisting of a rigid analytic space S_η, called the generic fiber of S, a formal scheme $\widehat{S}$, called the completion of S, and an open immersion $\iota_S : \widehat{S}^{\mathrm{rig}} \to S_\eta$. (We think of S as obtained from S_η and $\widehat{S}$ by gluing along $\widehat{S}^{\mathrm{rig}}$.) Given a rigid analytic scheme S, we set $S_\sigma = \widehat{S}_\sigma$ and call it the special fiber of S. A morphism of rigid analytic schemes $f : T \to S$ is a pair of morphisms $(f_\eta, \widehat{f})$, where $f_\eta : T_\eta \to S_\eta$ is a morphism of rigid analytic spaces and $\widehat{f} : \widehat{T} \to \widehat{S}$ is a morphism of formal schemes, and such that $\iota_S \circ \widehat{f_\eta} = f_\eta \circ \iota_T$. The morphism f is said to be étale (resp. smooth) if both f_η and $\widehat{f}$ are étale (resp. smooth).

Notation 3.8.5. We denote by RigSch the category of rigid analytic schemes. Given a rigid analytic scheme S, we denote by RigSch/S the overcategory of rigid analytic S-schemes and $\text{Ét}/S$ (resp. RigSm/S) its full subcategory consisting of étale (resp. smooth) objects.

Remark 3.8.6.

(1) We have a fully faithful embedding RigSpc $\to$ RigSch sending a rigid analytic space S to the triple $(S, \emptyset, \emptyset \to S)$. We will identify RigSpc with its essential image in RigSch.
(2) We have a fully faithful embedding FSch $\to$ RigSch sending a formal scheme $\mathcal{S}$ to the triple $(\mathcal{S}^{\mathrm{rig}}, \mathcal{S}, \mathrm{id}_{\mathcal{S}^{\mathrm{rig}}})$. We will identify FSpc with its essential image in RigSch.

Remark 3.8.7. A morphism j of rigid analytic schemes is said to be a closed (resp. an open) immersion if both j_η and $\widehat{j}$ are closed (resp. open) immersions. Given a closed immersion $Z \to S$

of rigid analytic schemes, the complement $S \smallsetminus Z$ is defined to be the rigid analytic scheme given by the triple $(S_\eta \smallsetminus Z_\eta, \widehat{S} \smallsetminus \widehat{Z}, \iota_{S \smallsetminus Z})$ where $\iota_{S\smallsetminus Z}$ is obtained by restriction and corestriction from ι_S. We have an obvious open immersion $S \smallsetminus Z \to S$.

We warn the reader about the following notation clash: given a closed immersion of formal schemes $\mathcal{Z} \to \mathcal{S}$, then "$\mathcal{S} \smallsetminus \mathcal{Z}$" can mean two different things. It can mean the open formal subscheme of $\mathcal{S}$ supported on the open subset $|\mathcal{S}| \smallsetminus |\mathcal{Z}|$ of $|\mathcal{S}|$. It can also mean the rigid analytic scheme obtained as the complement of $\mathcal{Z}$ in $\mathcal{S}$ considered as rigid analytic schemes. Each time there is a risk of confusion, we will specify if the complementation is taken in the category of formal schemes or the category of rigid analytic schemes.

Next, we generalise Construction 1.1.15.

Construction 3.8.8. Let B be a scheme, $B_\sigma \subset B$ a closed subscheme locally of finite presentation up to nilimmersion, and $B_\eta \subset B$ its open complement. There exists an analytification functor

$$(-)^{\mathrm{an}} : \mathrm{Sch}^{\mathrm{lft}}/B \to \mathrm{RigSch}/\widehat{B} \tag{3.44}$$

which is uniquely determined by the following two properties.

(1) It is compatible with gluing along open immersions.
(2) For a separated finite type B-scheme X with an open immersion $X \to \overline{X}$ into a proper B-scheme, and complement $Y = \overline{X} \smallsetminus X$, we have

$$X^{\mathrm{an}} = \widehat{\overline{X}} \smallsetminus (\widehat{Y}) \tag{3.45}$$

where, for a B-scheme W, $\widehat{W}$ is the formal completion of W at $W_\sigma = W \times_B B_\sigma$.

We stress that in (3.45) the complement is taken in the category of rigid analytic schemes.

Remark 3.8.9. Keep the notation of Construction 3.8.8. The functor (3.44) commutes with finite limits, and preserves étale and smooth morphisms, closed immersions and complementary open immersions, as well as proper morphisms. For $X \in \mathrm{Sch}^{\mathrm{lft}}/B$, we have a canonical isomorphism $(X^{\mathrm{an}})_\eta \simeq (X_\eta)^{\mathrm{an}}$ so there is no ambiguity in writing "X^{an}_η". The formal completions of X and X^{an} are canonically isomorphic, i.e., $\widehat{X^{\mathrm{an}}} \simeq \widehat{X}$, and we have isomorphisms $(X^{\mathrm{an}})_\sigma \simeq X_\sigma \simeq (X_\sigma)^{\mathrm{an}}$ up to nilimmersions.

Definition 3.8.10. Let $(f_i : S_i \to S)_i$ be a family of étale morphisms of rigid analytic schemes. We say that this family is an étale (resp. Nisnevich) cover if both families $(f_{i,\eta} : S_{i,\eta} \to S_\eta)_i$ and $(\widehat{f_i} : \widehat{S}_i \to \widehat{S})_i$ are étale (resp. Nisnevich) covers. The topology generated by étale (resp. Nisnevich) covers is called the étale (resp. Nisnevich) topology and is denoted by "ét" (resp. "nis").

Notation 3.8.11. Let X be a rigid analytic scheme. We denote by $\mathbb{B}^n_X$ the relative n-dimensional ball given by the triple $(\mathbb{B}^n_{X_\eta}, \mathbb{A}^n_{\widehat{X}}, \mathrm{id}_{\mathbb{B}^n} \times \iota_X)$. Similarly, we denote by $\mathbb{U}^1_X \subset \mathbb{B}^1_X$ the relative unit circle given by the triple $(\mathbb{U}^1_{X_\eta}, \mathbb{A}^1_{\widehat{X}} \smallsetminus 0_{\widehat{X}}, \mathrm{id}_{\mathbb{U}^1} \times \iota_X)$.

Definition 3.8.12. Given a rigid analytic scheme S, we define the monoidal ∞-category of rigid analytic motives $\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(S;\Lambda)^\otimes$ from the smooth étale site $(\mathrm{RigSm}/S, \tau)$ using the interval $\mathbb{B}^1_S$ and the motive of $\mathbb{U}^1_S$ pointed by the unit section, just as in Definitions 2.1.11 and 2.1.15.

Remark 3.8.13. Many of the results that we have established for ∞-categories of motives over rigid analytic spaces hold true for ∞-categories of motives over rigid analytic schemes, and often the proof we gave can be read in the context of rigid analytic schemes. This is the case for instance for Proposition 2.2.1. Moreover, Proposition 2.2.3 holds true for rigid analytic schemes, except that the proof of the localisation property requires some extra arguments. These extra arguments can be found in the proof of [Ayo15, Proposition 1.4.21]. Proposition 2.2.7 also extends: with the notation of Construction 3.8.8, the contravariant functor

$$X \mapsto \mathbf{RigSH}_\tau^{(\wedge)}(X^{\mathrm{an}}; \Lambda), \quad f \mapsto f^{\mathrm{an},*}$$

from $\mathrm{Sch}^{\mathrm{lft}}/B$ to Pr^{L} is a stable homotopical functor in the sense that it satisfies the ∞-categorical versions of the properties (1)–(6) listed in [Ayo07a, §1.4.1].

Keep the notation as in Construction 3.8.8. Given a B-scheme X which is locally of finite type, the analytification functor (3.44) induces a premorphism of sites

$$\mathrm{An}_X : (\mathrm{RigSm}/X^{\mathrm{an}}, \tau) \to (\mathrm{Sm}/X, \tau). \tag{3.46}$$

By the functoriality of the construction of the ∞-categories of motives, (3.46) induces a functor

$$\mathrm{An}_X^* : \mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(X; \Lambda) \to \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(X^{\mathrm{an}}; \Lambda). \tag{3.47}$$

(This generalises the functor (2.13).) Given a morphism $f : Y \to X$ in $\mathrm{Sch}^{\mathrm{lft}}/B$, there is an equivalence $f^{\mathrm{an},*} \circ \mathrm{An}_X^* \simeq \mathrm{An}_Y^* \circ f^*$. In fact, the generalisation of Proposition 2.2.13 holds true: we have a morphism of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$-valued presheaves

$$\mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(-; \Lambda)^\otimes \to \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}((-)^{\mathrm{an}}; \Lambda)^\otimes \tag{3.48}$$

on $\mathrm{Sch}^{\mathrm{lft}}/B$. Also, note that if Z is a B_σ-scheme which is locally of finite type, then An_Z^* is an equivalence of ∞-categories.

Notation 3.8.14. Let B be a scheme, $B_\sigma \subset B$ a closed subscheme locally of finite presentation, and $B_\eta \subset B$ its open complement.

(1) Given a B-scheme X, we set $X_\sigma = X \times_B B_\sigma$ and $X_\eta = X \times_B B_\eta$, and we define the functor

$$\chi_X : \mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(X_\eta; \Lambda) \to \mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(X_\sigma; \Lambda) \tag{3.49}$$

as in the statement of Theorem 3.8.1. More precisely, we denote by $i : X_\sigma \to X$ and $j : X_\eta \to X$ the obvious inclusions, and set $\chi_X = i^* \circ j_*$.

(2) Given a rigid analytic $\widehat{B}$-scheme X, we define the functor

$$\chi_X : \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(X_\eta; \Lambda) \to \mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(X_\sigma; \Lambda) \tag{3.50}$$

similarly. More precisely, we denote by $i : X_\sigma \to X$ and $j : X_\eta \to X$ the obvious inclusions, and set $\chi_X = i^* \circ j_*$.

Remark 3.8.15. In the T-stable case, the collection of functors $\{\chi_X\}_X$, for $X \in \mathrm{Sch}/B$, is part of a specialisation system in the sense of [Ayo07b, Définition 3.1.1]. In fact, this specialisation system is considered in [Ayo07b, Exemple 3.1.4] where it is called the canonical specialisation system. Similarly, the collection of functors $\{\chi_{X^{\mathrm{an}}} \circ \mathrm{An}_{X_\eta}^*\}_X$, for $X \in \mathrm{Sch}/B$, is part of a specialisation system; see [Ayo15, Proposition 1.4.41]. There are natural transformations

$$\rho_X : \chi_X \to \chi_{X^{\mathrm{an}}} \circ \mathrm{An}_{X_\eta}^*, \tag{3.51}$$

given by the composition of

$$\chi_X = i^* \circ j_* \simeq \mathrm{An}_{X_\sigma}^* \circ i^* \circ j_* \simeq i^{\mathrm{an},*} \circ \mathrm{An}_X^* \circ j_* \to i^{\mathrm{an},*} \circ j_*^{\mathrm{an}} \circ \mathrm{An}_{X_\eta}^* \simeq \chi_{X^{\mathrm{an}}} \circ \mathrm{An}_{X_\eta}^*,$$

which are part of a morphism of specialisation systems; see [Ayo15, Lemme 1.4.42].

Remark 3.8.16. The natural transformation ρ_X is independent of B in the following way. Let $B' \in \mathrm{Sch}^{\mathrm{lft}}/B$ and $X \in \mathrm{Sch}^{\mathrm{lft}}/B'$. Then we have two natural transformations "$\chi_X \to \chi_{X^{\mathrm{an}}} \circ \mathrm{An}^*_{X_\eta}$", one associated with X considered as a B-scheme and one associated with X considered as a B'-scheme. We claim that these two natural transformations are equivalent. To explain how, we write momentarily $\chi_{(X/B)^{\mathrm{an}}}$, $\mathrm{An}^*_{X_\eta/B}$, etc., to stress the dependency on the scheme B. There is a canonical isomorphism

$$(X/B')^{\mathrm{an}} \simeq (X/B)^{\mathrm{an}} \times_{(B'/B)^{\mathrm{an}}} \widehat{B'},$$

and hence an open immersion of rigid analytic $\widehat{B}$-schemes $\iota : (X/B')^{\mathrm{an}} \to (X/B)^{\mathrm{an}}$ inducing an isomorphism on special fibers. Moreover, we have natural equivalences

$$\chi_{(X/B)^{\mathrm{an}}} \simeq \chi_{(X/B')^{\mathrm{an}}} \circ \iota^*_\eta \qquad \text{and} \qquad \mathrm{An}^*_{X_\eta/B'} \simeq \iota^*_\eta \circ \mathrm{An}^*_{X_\eta/B}.$$

Modulo these equivalences, the two natural transformations "$\chi_X \to \chi_{X^{\mathrm{an}}} \circ \mathrm{An}^*_{X_\eta}$" give the same natural transformation $\chi_X \to \chi_{(X/B')^{\mathrm{an}}} \circ \iota^*_\eta \circ \mathrm{An}^*_{X_\eta/B}$.

Lemma 3.8.17. *Let X be a rigid analytic $\widehat{B}$-scheme. The functor* (3.50) *is equivalent to the composition of*

$$\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(X_\eta;\Lambda) \xrightarrow{\iota^*_X} \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(\widehat{X}^{\mathrm{rig}};\Lambda) \xrightarrow{\chi_{\widehat{X}}} \mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(X_\sigma;\Lambda),$$

where $\chi_{\widehat{X}}$ is the functor introduced in Notation 3.1.12.

Proof. For the sake of clarity, we will momentarily write "$\chi'_{\widehat{X}}$" instead of "$\chi_{\widehat{X}}$" for the functor introduced in Notation 3.1.12 and use "$\chi_{\widehat{X}}$" to denote the functor introduced in Notation 3.8.14(2) with $\widehat{X}$ considered as a rigid analytic $\widehat{B}$-scheme via the fully faithful embedding $\mathrm{FSch} \to \mathrm{RigSch}$.

We have an equivalence $\chi_X \simeq \chi_{\widehat{X}} \circ \iota^*_X$ which follows from the fact that $(\iota_X)_\sigma$ is the identification $\widehat{X}_\sigma \simeq X_\sigma$. Thus, to prove the lemma, it is enough to show that the two functors

$$\chi_{\widehat{X}},\ \ \chi'_{\widehat{X}} : \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(\widehat{X}_\eta;\Lambda) \to \mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(X_\sigma;\Lambda)$$

are equivalent. (Note that $\widehat{X}_\eta = \widehat{X}^{\mathrm{rig}}$; here we use "$\widehat{X}_\eta$" because we want to think about $\widehat{X}$ as a rigid analytic scheme via the fully faithful embedding of Remark 3.8.6(2).) In order to do that, we remark that the base change functor $\mathrm{RigSm}/\widehat{X} \to \mathrm{Sm}/X_\sigma$ factors as follows

$$\mathrm{RigSm}/\widehat{X} \xrightarrow{\widehat{(-)}} \mathrm{FSm}/\widehat{X} \xrightarrow{(-)_\sigma} \mathrm{Sm}/X_\sigma.$$

We deduce immediately from the construction of the ∞-categories of motives that the inverse image functor $i^* : \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(\widehat{X};\Lambda) \to \mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(X_\sigma;\Lambda)$ is the composition of

$$\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(\widehat{X};\Lambda) \xrightarrow{\widehat{(-)}^*} \mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\widehat{X};\Lambda) \xrightarrow{\sigma^*} \mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(X_\sigma;\Lambda)$$

where σ^* is the equivalence of Theorem 3.1.10 and $\widehat{(-)}^*$ is the functor that takes the motive of a rigid analytic $\widehat{X}$-scheme to the motive of its formal completion. The formal completion functor $\widehat{(-)}$ is right adjoint to the obvious inclusion $\mathrm{inc} : \mathrm{FSm}/\widehat{X} \to \mathrm{RigSch}/\widehat{X}$. It follows that $\widehat{(-)}^*$ is right adjoint to the functor

$$\mathrm{inc}^* : \mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\widehat{X};\Lambda) \to \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(\widehat{X};\Lambda).$$

This means that we have an equivalence $\widehat{(-)}^* \simeq \mathrm{inc}_*$. In conclusion, we see that $\chi_{\widehat{X}}$ is equivalent to the composition of

$$\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(\widehat{X}_\eta;\Lambda) \xrightarrow{j_*} \mathbf{RigSH}^{(\mathrm{eff},\wedge)}_\tau(\widehat{X};\Lambda) \xrightarrow{\mathrm{inc}_*} \mathbf{FSH}^{(\mathrm{eff},\wedge)}_\tau(\widehat{X};\Lambda) \xrightarrow{\sigma^*} \mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(X_\sigma;\Lambda).$$

Since $j^* \circ \mathrm{inc}^*$ is clearly equivalent to the functor $\xi_{\widehat{X}}$ from Notation 3.1.12, the result follows. $\square$

Corollary 3.8.18. *The functor $\chi_{\widehat{B}}$ obtained by taking $X = \widehat{B}$ in Notation 3.8.14(2) coincides with the functor $\chi_{\widehat{B}}$ obtained by taking $\mathcal{S} = \widehat{B}$ in Notation 3.1.12.*

From Corollary 3.8.18, we see that Theorem 3.8.1 follows from the following statement.

Theorem 3.8.19. *Let B be a scheme, $B_\sigma \subset B$ a closed subscheme locally of finite presentation up to nilimmersion, and $B_\eta \subset B$ its open complement. Assume that every prime number is invertible either in $\pi_0\Lambda$ or in $\mathcal{O}(B)$. Assume one of the following alternatives.*

1. *We work in the non-hypercomplete case and Λ is eventually coconnective;*
2. *We work in the hypercomplete case and B is $(\Lambda, \text{ét})$-admissible.*

Then, for every $X \in \mathrm{Sch}^{\mathrm{lft}}/B$, the natural transformation $\rho_X : \chi_X \to \chi_{X^{\mathrm{an}}} \circ \mathrm{An}^*_{X_\eta}$, between functors from $\mathbf{SH}^{(\wedge)}_{\text{ét}}(X_\eta; \Lambda)$ to $\mathbf{SH}^{(\wedge)}_{\text{ét}}(X_\sigma; \Lambda)$, is an equivalence.

We start by proving a reduction.

Lemma 3.8.20. *To prove Theorem 3.8.19, we may assume that Λ is eventually coconnective and that B is essentially of finite type over* $\mathrm{Spec}(\mathbb{Z})$*. In particular, there is no need to distinguish the non-hypercomplete and the hypercomplete cases.*

Proof. We first explain how to reduce to the case where Λ is eventually coconnective. For this, we only need to consider the alternative (2). It follows from Propositions 2.4.22 and 3.2.3 that ρ_X is a natural transformation between colimit-preserving functors between compactly generated categories. Thus, it is enough to prove that $\chi_X M \to \chi_{X^{\mathrm{an}}}\mathrm{An}^*_{X_\eta} M$ is an equivalence for $M \in \mathbf{SH}^{\wedge}_{\text{ét}}(X_\eta; \Lambda)$ compact. Arguing as in the second part of the proof of Lemma 3.6.2, we reduce to the following two cases:

- $\pi_0\Lambda$ is a $\mathbb{Q}$-algebra;
- M is ℓ-nilpotent for a prime ℓ invertible on B.

In the first case, we may replace Λ by $\mathbb{Q}$ and assume that Λ is eventually coconnective as claimed. In the second case, let $M_0 \in \mathrm{Shv}^{\wedge}_{\text{ét}}(\text{Ét}/X_\eta; \Lambda)_\ell$ be the object corresponding to M by the equivalence

$$\mathrm{Shv}^{\wedge}_{\text{ét}}(\text{Ét}/X_\eta; \Lambda)_{\ell\text{-nil}} \simeq \mathbf{SH}^{\wedge}_{\text{ét}}(X_\eta; \Lambda)_{\ell\text{-nil}}$$

provided by Theorem 2.10.4. Using also Theorem 2.10.3, we reduce to show that $\chi_X M_0 \to \chi_{X^{\mathrm{an}}}\mathrm{An}^*_{X_\eta} M_0$ is an equivalence. (Here the functors χ_X, $\chi_{X^{\mathrm{an}}}$ and $\mathrm{An}^*_{X_\eta}$ are defined on étale hypersheaves of Λ-modules by the same formulas as their motivic versions.) Using Lemma 2.4.5, one obtains equivalences

$$\chi_X M_0 \simeq \lim_r \chi_X(M_0 \otimes_\Lambda \tau_{\le r}\Lambda) \qquad \text{and} \qquad \chi_{X^{\mathrm{an}}}\mathrm{An}^*_{X_\eta} M_0 \simeq \lim_r \chi_{X^{\mathrm{an}}}\mathrm{An}^*_{X_\eta}(M_0 \otimes_\Lambda \tau_{\le r}\Lambda).$$

(Indeed, as M_0 is compact, the inverse system $(M_0 \otimes_\Lambda \tau_{\le r}\Lambda)_r$ consists of eventually coconnective étale sheaves and is eventually constant on homotopy sheaves.) This shows that we may replace M and Λ by $M \otimes_\Lambda \tau_{\le r}\Lambda$ and $\tau_{\le r}\Lambda$, and assume that Λ is eventually coconnective as claimed.

We now assume that Λ is eventually coconnective and explain how to reduce to the case where B is essentially of finite type over $\mathrm{Spec}(\mathbb{Z})$. By Propositions 2.4.19 and 3.2.2, the alternative (2) is covered by the alternative (1). By Remark 3.8.16, we only need to consider the case $X = B$. The problem is local on B, so we may assume that B is affine given as a limit of a cofiltered inverse system $(B_\alpha)_\alpha$ of affine schemes which are essentially of finite type over $\mathbb{Z}$. We may also assume that there are closed subschemes $B_{\alpha,\sigma} \subset B_\alpha$ such that, for every $\beta \le \alpha$, $B_{\beta,\sigma}$ is the inverse image of

$B_{\alpha,\sigma}$, and B_σ is the limit of the inverse system $(B_{\alpha,\sigma})_\alpha$. Set $B_{\alpha,\eta} = B_\alpha \smallsetminus B_{\alpha,\sigma}$ so that B_η is the limit of the inverse system $(B_{\alpha,\eta})_\alpha$. Let $i_\alpha : B_{\alpha,\sigma} \to B_\alpha$ and $j_\alpha : B_{\alpha,\eta} \to B_\alpha$ be the obvious immersions, and let $f_\alpha : B \to B_\alpha$ and $f_{\beta\alpha} : B_\beta \to B_\alpha$ be the obvious morphisms.

We need to show that $\chi_B M \to \chi_{\widehat{B}} \operatorname{An}^*_{X_\eta} M$ is an equivalence for all $M \in \mathbf{SH}_{\text{ét}}(B_\eta; \Lambda)$. Since the three functors χ_B, $\chi_{\widehat{B}}$ and $\operatorname{An}^*_{X_\eta}$ commute with colimits, we may assume that M is compact. By Proposition 2.5.11, we have an equivalence

$$\mathbf{SH}_{\text{ét}}(B; \Lambda) \simeq \operatorname*{colim}_{\alpha} \mathbf{SH}_{\text{ét}}(B_\alpha; \Lambda)$$

in Pr^{L}, and similarly for B_σ and B_η. Since $M \in \mathbf{SH}_{\text{ét}}(B_\eta; \Lambda)$ is assumed compact, we may find an index α_0, a compact object $M_{\alpha_0} \in \mathbf{SH}_{\text{ét}}(B_{\alpha_0,\eta}; \Lambda)$ and an equivalence $f^*_{\alpha_0,\eta} M_{\alpha_0} \simeq M$. We set $M_\alpha = f^*_{\alpha\alpha_0,\eta} M$. With this, we have an equivalence

$$j_* M \simeq \operatorname*{colim}_{\alpha \le \alpha_0} f^*_\alpha j_{\alpha,*} M_\alpha.$$

(It is not totally obvious how to construct such an equivalence. One needs to argue as in the proof of Lemma 3.5.7; see also Remark 3.5.8.) Applying i^*, we deduce an equivalence

$$\chi_B M \simeq \operatorname*{colim}_{\alpha \le \alpha_0} f^*_\alpha \chi_{B_\alpha} M_\alpha. \tag{3.52}$$

Similarly, by Remark 3.5.8 and using Corollary 3.8.18, we have an equivalence

$$\chi_{\widehat{B}} \operatorname{An}^*_{B_\eta} M \simeq \operatorname*{colim}_{\alpha \le \alpha_0} f^{\mathrm{an},*}_\alpha \chi_{\widehat{B}_\alpha} \operatorname{An}^*_{B_{\alpha,\eta}} M_\alpha. \tag{3.53}$$

Therefore, it is enough to show that $\chi_{B_\alpha} M_\alpha \to \chi_{\widehat{B}_\alpha} \operatorname{An}^*_{B_{\alpha,\eta}} M_\alpha$ is an equivalence. In particular, we may assume that B is quasi-excellent and $(\Lambda, \text{ét})$-admissible. In this case, since Λ is eventually coconnective, we are automatically working in the hypercomplete case by Propositions 2.4.19 and 3.2.2. This finishes the proof. □

Our next task is to prove the following weak version of Theorem 3.8.19 (which we are able to justify even when τ is the Nisnevich topology).

Proposition 3.8.21. *Let B be a quasi-excellent (Λ, τ)-admissible scheme, $B_\sigma \subset B$ a closed subscheme, and $B_\eta \subset B$ its open complement. If τ is the étale topology, assume that every prime number is invertible either in $\pi_0\Lambda$ or in $\mathcal{O}(B)$. Then, there is a natural transformation $\chi_{\widehat{B}} \circ \operatorname{An}^*_{B_\eta} \to \chi_B$, between functors from $\mathbf{SH}^\wedge_\tau(B_\eta; \Lambda)$ to $\mathbf{SH}^\wedge_\tau(B_\sigma; \Lambda)$, which is a section to the natural transformation ρ_B, i.e., such that the composition of*

$$\chi_{\widehat{B}} \circ \operatorname{An}^*_{B_\eta} \to \chi_B \xrightarrow{\rho_B} \chi_{\widehat{B}} \circ \operatorname{An}^*_{B_\eta}$$

is the identity.

To prove Proposition 3.8.21 we need a digression. (Compare with [Ayo15, page 112].[11])

Construction 3.8.22. Let $\mathcal{S}$ be a formal scheme. We denote by $\mathrm{FRigSm}_{\mathrm{af}}/\mathcal{S}$ the full subcategory of $\mathrm{FSch}/\mathcal{S}$ spanned by rig-smooth formal $\mathcal{S}$-schemes which are affine. Consider the functor

$$\mathfrak{D}_{\mathcal{S}} : \mathrm{FRigSm}_{\mathrm{af}}/\mathcal{S} \to \mathrm{Sch} \tag{3.54}$$

sending an affine formal scheme $\mathrm{Spf}(A)$ over $\mathcal{S}$ to the scheme $\mathrm{Spec}(A)$. Consider also the two related functors $\mathfrak{D}_{\mathcal{S},\sigma}$ and $\mathfrak{D}_{\mathcal{S},\eta}$ between the same categories, sending an affine formal scheme

[11]We remind the reader that the page references to [Ayo15] correspond to the published version.

$\mathrm{Spf}(A)$ over $\mathcal{S}$ to the schemes $\mathrm{Spf}(A)_\sigma$ and $\mathrm{Spec}(A) \smallsetminus \mathrm{Spf}(A)_\sigma$ respectively. We consider $\mathfrak{D}_\mathcal{S}$, $\mathfrak{D}_{\mathcal{S},\sigma}$ and $\mathfrak{D}_{\mathcal{S},\eta}$ as diagrams of schemes and define the smooth τ-sites

$$(\mathrm{Sm}/\mathfrak{D}_\mathcal{S}, \tau), \quad (\mathrm{Sm}/\mathfrak{D}_{\mathcal{S},\sigma}, \tau) \quad \text{and} \quad (\mathrm{Sm}/\mathfrak{D}_{\mathcal{S},\eta}, \tau)$$

as in [Ayo07b, §4.5.1]. To fix the notation, let us recall that an object of $\mathrm{Sm}/\mathfrak{D}_\mathcal{S}$ is a pair $(\mathcal{U}, V)$ consisting of an object $\mathcal{U} \in \mathrm{FRigSm}_{\mathrm{af}}/\mathcal{S}$ and a smooth $\mathcal{O}(\mathcal{U})$-scheme V. The topology τ on $\mathrm{Sm}/\mathfrak{D}_\mathcal{S}$ is generated by families of the form $((\mathrm{id}_\mathcal{U}, e_i) : (\mathcal{U}, V_i) \to (\mathcal{U}, V))_i$ where the family $(e_i)_i$ is a cover for the topology τ.

The ∞-category $\mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_\mathcal{S};\Lambda)$ is constructed from the site $(\mathrm{Sm}/\mathfrak{D}_\mathcal{S}, \tau)$, using the interval $\mathbb{A}^1$ and the motive of $\mathbb{A}^1 \smallsetminus 0$ pointed by the unit section, as in Definitions 2.1.11 and 2.1.15 (or Definition 3.1.1 and 3.1.3), and similarly for $\mathfrak{D}_{\mathcal{S},\sigma}$ and $\mathfrak{D}_{\mathcal{S},\eta}$. (For a construction using the language of model categories, see [Ayo07b, §4.5.2].) We note here that $\mathbb{A}^1$ (resp. $\mathbb{A}^1 \smallsetminus 0$) is considered as a presheaf of sets on $\mathrm{Sm}/\mathfrak{D}_\mathcal{S}$, sending $(\mathcal{U}, V)$ to $\mathcal{O}(V)$ (resp. $\mathcal{O}^\times(V)$). This presheaf is not representable unless $\mathcal{S}$ is affine, but the Cartesian product with this presheaf preserves representable presheaves. (For instance, we have $\mathbb{A}^1 \times (\mathcal{U}, V) = (\mathcal{U}, \mathbb{A}^1_V)$.) We have morphisms of diagrams of schemes $\mathrm{i} : \mathfrak{D}_{\mathcal{S},\sigma} \to \mathfrak{D}_\mathcal{S}$ and $\mathrm{j} : \mathfrak{D}_{\mathcal{S},\eta} \to \mathfrak{D}_\mathcal{S}$, and we define the functor

$$\chi_{\mathfrak{D}_\mathcal{S}} : \mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_{\mathcal{S},\eta};\Lambda) \to \mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_{\mathcal{S},\sigma};\Lambda) \tag{3.55}$$

to be the composite $\mathrm{i}^* \circ \mathrm{j}_*$.

Similarly, consider the functor

$$\mathfrak{D}_\mathcal{S}^{\mathrm{an}} : \mathrm{FRigSm}_{\mathrm{af}}/\mathcal{S} \to \mathrm{RigSch} \tag{3.56}$$

sending an affine formal scheme $\mathrm{Spf}(A)$ over $\mathcal{S}$ to $\mathrm{Spf}(A)$ considered as a rigid analytic scheme. Consider also the related functor $\mathfrak{D}_{\mathcal{S},\eta}^{\mathrm{an}}$ between the same categories, sending an affine formal scheme $\mathrm{Spf}(A)$ over $\mathcal{S}$ to the rigid analytic space $\mathrm{Spf}(A)^{\mathrm{rig}}$. We consider $\mathfrak{D}_\mathcal{S}^{\mathrm{an}}$ and $\mathfrak{D}_{\mathcal{S},\eta}^{\mathrm{an}}$ as diagrams of rigid analytic schemes and define the smooth τ-sites $(\mathrm{RigSm}/\mathfrak{D}_\mathcal{S}^{\mathrm{an}}, \tau)$ and $(\mathrm{RigSm}/\mathfrak{D}_{\mathcal{S},\eta}^{\mathrm{an}}, \tau)$ as in [Ayo07b, §4.5.1]. The ∞-category $\mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_\mathcal{S}^{\mathrm{an}};\Lambda)$ is constructed from the site $(\mathrm{RigSm}/\mathfrak{D}_\mathcal{S}^{\mathrm{an}}, \tau)$, using the interval $\mathbb{B}^1$ and the motive of $\mathbb{U}^1$ pointed by the unit section, as in Definitions 2.1.11 and 2.1.15, and similarly for $\mathfrak{D}_{\mathcal{S},\eta}^{\mathrm{an}}$. We have morphisms of diagrams of rigid analytic schemes $\mathrm{i}^{\mathrm{an}} : \mathfrak{D}_{\mathcal{S},\sigma} \to \mathfrak{D}_\mathcal{S}^{\mathrm{an}}$ and $\mathrm{j}^{\mathrm{an}} : \mathfrak{D}_{\mathcal{S},\eta}^{\mathrm{an}} \to \mathfrak{D}_\mathcal{S}^{\mathrm{an}}$, and we define the functor

$$\chi_{\mathfrak{D}_\mathcal{S}^{\mathrm{an}}} : \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_{\mathcal{S},\eta}^{\mathrm{an}};\Lambda) \to \mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_{\mathcal{S},\sigma};\Lambda) \tag{3.57}$$

to be the composite $\mathrm{i}^{\mathrm{an},*} \circ \mathrm{j}^{\mathrm{an}}_*$. The analytification functor induces functors

$$\mathrm{An}^*_{\mathfrak{D}_\mathcal{S}} : \mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_\mathcal{S};\Lambda) \to \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_\mathcal{S}^{\mathrm{an}};\Lambda) \qquad \text{and}$$
$$\mathrm{An}^*_{\mathfrak{D}_{\mathcal{S},\eta}} : \mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_{\mathcal{S},\eta};\Lambda) \to \mathbf{RigSH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_{\mathcal{S},\eta}^{\mathrm{an}};\Lambda).$$

We may then define a natural transformation

$$\rho_{\mathfrak{D}_\mathcal{S}} : \chi_{\mathfrak{D}_\mathcal{S}} \to \chi_{\mathfrak{D}_\mathcal{S}^{\mathrm{an}}} \circ \mathrm{An}^*_{\mathfrak{D}_\eta} \tag{3.58}$$

as in Remark 3.8.15.

Remark 3.8.23. The functor (3.56) factors through the subcategory $\mathrm{FSch} \subset \mathrm{RigSch}$ and defines a diagram of formal schemes that we denote by $\mathfrak{D}_\mathcal{S}^{\mathrm{for}}$. As in Construction 3.8.22, we can define an ∞-category $\mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_\mathcal{S}^{\mathrm{for}};\Lambda)$ of formal motives over $\mathfrak{D}_\mathcal{S}^{\mathrm{for}}$ using the smooth site $(\mathrm{FSm}/\mathfrak{D}_\mathcal{S}^{\mathrm{for}}, \tau)$. Moreover, we have an equivalence of ∞-categories

$$\sigma^* : \mathbf{FSH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_\mathcal{S}^{\mathrm{for}};\Lambda) \xrightarrow{\sim} \mathbf{SH}_\tau^{(\mathrm{eff},\wedge)}(\mathfrak{D}_{\mathcal{S},\sigma};\Lambda)$$

as in Theorem 3.1.10.

Lemma 3.8.24. *The functor $\chi_{\mathfrak{D}^{\mathrm{an}}_{\mathcal{S}}}$ coincides with the composition of*

$$\mathbf{RigSH}^{(\mathrm{eff},\wedge)}_{\tau}(\mathfrak{D}^{\mathrm{an}}_{\mathcal{S},\eta};\Lambda) \xrightarrow{\chi_{\mathfrak{D}^{\mathrm{for}}_{\mathcal{S}}}} \mathbf{FSH}^{(\mathrm{eff},\wedge)}_{\tau}(\mathfrak{D}^{\mathrm{for}}_{\mathcal{S}};\Lambda) \xrightarrow{\sigma^*} \mathbf{SH}^{(\mathrm{eff},\wedge)}_{\tau}(\mathfrak{D}_{\mathcal{S},\sigma};\Lambda)$$

where $\chi_{\mathfrak{D}^{\mathrm{for}}_{\mathcal{S}}}$ is the restriction along the functor $(-)^{\mathrm{rig}} : \mathrm{FSm}/\mathfrak{D}^{\mathrm{for}}_{\mathcal{S}} \to \mathrm{RigSm}/\mathfrak{D}^{\mathrm{an}}_{\mathcal{S},\eta}$ sending a pair $(\mathcal{U},\mathcal{V})$ to $(\mathcal{U},\mathcal{V}^{\mathrm{rig}})$.

Proof. This is diagrammatic version of Lemma 3.8.17 which is proven in the same way. □

Remark 3.8.25. There are five diagonal functors emanating from $\mathrm{FRigSm}_{\mathrm{af}}/\mathcal{S}$ and taking values in the categories $\mathrm{Sm}/\mathfrak{D}_{\mathcal{S}}$, $\mathrm{Sm}/\mathfrak{D}_{\mathcal{S},\sigma}$, $\mathrm{Sm}/\mathfrak{D}_{\mathcal{S},\eta}$, $\mathrm{RigSm}/\mathfrak{D}^{\mathrm{an}}_{\mathcal{S}}$ and $\mathrm{RigSm}/\mathfrak{D}^{\mathrm{an}}_{\mathcal{S},\eta}$. These functors will be denoted respectively by diag, diag_{σ}, diag_{η}, $\mathrm{diag}^{\mathrm{an}}$ and $\mathrm{diag}^{\mathrm{an}}_{\eta}$. They send an affine formal scheme $\mathcal{U} = \mathrm{Spf}(A)$ over $\mathcal{S}$ to the pairs $(\mathcal{U},\mathrm{Spec}(A))$, $(\mathcal{U},\mathcal{U}_{\sigma})$, $(\mathcal{U},\mathrm{Spec}(A)\smallsetminus\mathcal{U}_{\sigma})$, $(\mathcal{U},\mathcal{U})$ and $(\mathcal{U},\mathcal{U}^{\mathrm{rig}})$ respectively. We now concentrate on the case of diag, but what we are going to say can be adapted to the remaining four diagonal functors. The functor diag induces an adjunction

$$\mathrm{diag}^* : \mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af}}/\mathcal{S};\Lambda) \rightleftarrows \mathrm{PSh}(\mathrm{Sm}/\mathfrak{D}_{\mathcal{S}};\Lambda) : \mathrm{diag}_*$$

where diag_* is the restriction functor. As in Remark 2.1.19, we denote by $T_{\mathcal{S}}$ (instead of $\mathrm{T}_{\mathcal{S}}$) the cofiber of the split inclusion of $\Lambda(\mathcal{S}) \to \Lambda(\mathbb{A}^1_{\mathcal{S}}\smallsetminus 0_{\mathcal{S}})$ (without τ-(hyper)sheafification), and similarly for $T_{\mathfrak{D}_{\mathcal{S}}}$. (Here $\mathcal{S}$ and $\mathbb{A}^1_{\mathcal{S}}\smallsetminus 0_{\mathcal{S}}$ are considered as presheaves of sets on $\mathrm{FRigSm}_{\mathrm{af}}/\mathcal{S}$ which are not necessarily representable.) Noting that $\mathrm{diag}_*(T_{\mathfrak{D}_{\mathcal{S}}}) \simeq T_{\mathcal{S}}$, we may extend the above adjunction to T-spectra:

$$\mathrm{diag}^* : \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af}}/\mathcal{S};\Lambda)) \rightleftarrows \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{Sm}/\mathfrak{D}_{\mathcal{S}};\Lambda)) : \mathrm{diag}_*.$$

Here, by abuse of notation, we write $\mathrm{Spt}_T(\mathrm{PSh}(-;\Lambda))$ for the ∞-category associated to the simplicial category $\mathrm{Spt}_T(\mathrm{PSh}_{\Delta}(-;\Lambda))$ endowed with its levelwise global model structure; compare with Remark 2.1.19. We have the following equivalences

$$\mathrm{diag}_{\sigma,*} \simeq \mathrm{diag}_* \circ \mathrm{i}_* \simeq \mathrm{diag}^{\mathrm{an}}_* \circ \mathrm{i}^{\mathrm{an}}_*, \quad \mathrm{diag}_{\eta,*} \simeq \mathrm{diag}_* \circ \mathrm{j}_* \quad \text{and} \quad \mathrm{diag}^{\mathrm{an}}_{\eta,*} \simeq \mathrm{diag}^{\mathrm{an}}_* \circ \mathrm{j}^{\mathrm{an}}_*.$$

Moreover, there are natural equivalences $\mathrm{An}^*_{\mathfrak{D}_{\mathcal{S}}} \circ \mathrm{diag}^* \simeq \mathrm{diag}^{\mathrm{an},*}$ and $\mathrm{An}^*_{\mathfrak{D}_{\mathcal{S},\eta}} \circ \mathrm{diag}^*_{\eta} \simeq \mathrm{diag}^{\mathrm{an},*}_{\eta}$ inducing natural transformations

$$\mathrm{diag}_* \to \mathrm{diag}^{\mathrm{an}}_* \circ \mathrm{An}^*_{\mathfrak{D}_{\mathcal{S}}} \qquad \text{and} \qquad \mathrm{diag}_{\eta,*} \to \mathrm{diag}^{\mathrm{an}}_{\eta,*} \circ \mathrm{An}^*_{\mathfrak{D}_{\mathcal{S},\eta}}. \tag{3.59}$$

Lemma 3.8.26. *Below, we consider $\mathrm{diag}_{\eta,*}$ and $\mathrm{diag}^{\mathrm{an}}_{\eta,*}$ as ordinary functors on ordinary categories of presheaves of sets. Given a rigid analytic space W over $\mathcal{S}^{\mathrm{rig}}$, we denote also by W the presheaf of sets on $\mathrm{FRigSm}_{\mathrm{af}}/\mathcal{S}$ given by $W(\mathcal{X}) = \mathrm{Hom}_{\mathcal{S}^{\mathrm{rig}}}(\mathcal{X}^{\mathrm{rig}}, W)$.*

(1) Let $(\mathcal{U},V)$ be an object of $\mathrm{Sm}/\mathfrak{D}_{\mathcal{S},\eta}$ which we identify with the presheaf of sets it represents. Denote by V^{an} the analytification of V with respect to the adic ring $\mathcal{O}(\mathcal{U})$. Then, there is a morphism of presheaves of sets

$$\mathrm{diag}_{\eta,*}(\mathcal{U},V) \to V^{\mathrm{an}} \tag{3.60}$$

which induces an isomorphism after sheafification for the rig topology.

(2) Let $(\mathcal{U},V)$ be an object of $\mathrm{RigSm}/\mathfrak{D}^{\mathrm{an}}_{\mathcal{S},\eta}$ which we identify with the presheaf of sets it represents. Then, there is a morphism of presheaves of sets

$$\mathrm{diag}^{\mathrm{an}}_{\eta,*}(\mathcal{U},V) \to V \tag{3.61}$$

which induces an isomorphism after sheafification for the rig topology.

Proof. We only prove the first part, which is slightly more interesting. Set $A = \mathcal{O}(\mathfrak{U})$ and let $\mathcal{T} = \mathrm{Spf}(B)$ be a rig-smooth affine formal $\mathcal{S}$-scheme. A section of $\mathrm{diag}_{\eta,*}(\mathfrak{U}, V)$ on $\mathcal{T}$ is a pair (f, g) consisting of a morphism of formal $\mathcal{S}$-schemes $f : \mathcal{T} \to \mathfrak{U}$ and a morphism of schemes $g : \mathrm{Spec}(B) \smallsetminus \mathcal{T}_\sigma \to V$ over $\mathrm{Spec}(A) \smallsetminus \mathfrak{U}_\sigma$. This gives rise to a section of the $(\mathrm{Spec}(B) \smallsetminus \mathcal{T}_\sigma)$-scheme $V \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ and, by analytification over $\mathcal{T}$, to a morphism $\mathcal{T}^{\mathrm{rig}} \to V^{\mathrm{an}} \times_{\mathcal{S}^{\mathrm{rig}}} \mathcal{T}^{\mathrm{rig}}$. This defines the morphism of presheaves (3.60). It remains to see that this morphism induces an equivalence on stalks for the rig topology. To do so, we evaluate (3.60) on a rig point $\mathfrak{t} = \mathrm{Spf}(R)$ over $\mathcal{S}$, with R an adic valuation ring with fraction field K. We may replace $\mathcal{S}$ with $\mathfrak{t}$ and assume that V is a smooth K-scheme. The question being local, we may assume that V is compactifiable over R and fix an open immersion $V \to \overline{V}$ into a proper R-scheme $\overline{V}$. In this case, the evaluation of (3.60) on $\mathfrak{t}$ is the obvious map between

(1) the set of K-points $x : \mathrm{Spec}(K) \to V$;
(2) the set of R-points $\mathfrak{x} : \mathrm{Spf}(R) \to \widehat{\overline{V}}$ such that there exists an admissible blowup $\overline{V}' \to \overline{V}$ with the property that the lift $\mathfrak{x}' : \mathrm{Spf}(R) \to \widehat{\overline{V}'}$ of $\mathfrak{x}$ factors through the complement of the special fiber of the Zariski closure of $\overline{V}'_\eta \smallsetminus V$ in $\overline{V}'$. (See Construction 1.1.15.)

To give a morphism of formal R-schemes $\mathfrak{x} : \mathrm{Spf}(R) \to \widehat{\overline{V}}$ is equivalent to giving a morphism of R-schemes $\widetilde{x} : \mathrm{Spec}(R) \to \overline{V}$, and the condition in (2) corresponds to the condition that $\widetilde{x}$ sends $\mathrm{Spec}(K)$ to V. Hence, the set described in (2) can be identified with

(2′) the set of R-points $\widetilde{x} : \mathrm{Spec}(R) \to \overline{V}$ sending $\mathrm{Spec}(K)$ to V.

That the obvious map between (1) and (2′) is a bijection is clear. (Note that the existence of this map follows from the valuative criterion of properness but, once the existence of this map is granted, it is clearly a bijection.) □

Recall that the weak equivalences of the stable $(\mathbb{B}^1, \tau)$-local model structure are called the stable $(\mathbb{B}^1, \tau)$-local equivalences; see Remark 2.1.19. Similarly, we have the notions of stable $(\mathbb{A}^1, \tau)$-local equivalences and stable $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalences. For later use, we record the following result.

Lemma 3.8.27.
(1) The functor

$$\mathrm{diag}_{\eta,*} : \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{Sm}/\mathfrak{D}_{\mathcal{S},\eta}; \Lambda)) \to \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af}}/\mathcal{S}; \Lambda))$$

takes a stable $(\mathbb{A}^1, \tau)$-local equivalence to a stable $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalence.
(2) The functor

$$\mathrm{diag}^{\mathrm{an}}_{\eta,*} : \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{RigSm}/\mathfrak{D}^{\mathrm{an}}_{\mathcal{S},\eta}; \Lambda)) \to \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af}}/\mathcal{S}; \Lambda))$$

takes a stable $(\mathbb{B}^1, \tau)$-local equivalence to a stable $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalence.

Proof. We only treat the first part; the second part is proven in the same way. The functor $\mathrm{diag}_{\eta,*}$ commutes with colimits. Thus, by [Lur09, Proposition 5.5.4.20], it is enough to show that $\mathrm{diag}_{\eta,*}$ transforms the following types of morphisms

(1) $\mathrm{colim}_{[n]\in\Delta} \Lambda(\mathfrak{U}, V_n) \to \Lambda(\mathfrak{U}, V_{-1})$, where $V_\bullet$ is a τ-hypercover,
(2) $\Lambda(\mathfrak{U}, V) \to \Lambda(\mathfrak{U}, \mathbb{A}^1_V)$,
(3) a morphism of T-spectra $F \to F'$ such that $F_n \to F'_n$ is an equivalence for n large enough,

into $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalences, for (1) and (2), and into stable $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalences, for (3). The case of (3) is obvious, so we only need to discuss morphisms of type (1) and (2).

In (1) and (2) above, $\mathcal{U}$ is an affine formal scheme which is rig-smooth over $\mathcal{S}$. We set $U = \operatorname{Spec}(\mathcal{O}(\mathcal{U}))$, $U_\sigma = \mathcal{U}_\sigma$ and $U_\eta = U \smallsetminus U_\sigma$. Then V and the V_n's, for $n \geq -1$, are smooth U_η-schemes. By Lemma 3.8.26(1), $\operatorname{diag}_{\eta,*}$ takes morphisms of type (1) and (2) to morphisms which are rig-locally equivalent to

(1′) $\operatorname{colim}_{[n]\in\Delta} \Lambda(V_n^{\mathrm{an}}) \to \Lambda(V_{-1}^{\mathrm{an}})$,
(2′) $\Lambda(V^{\mathrm{an}}) \to \Lambda((\mathbb{A}^1_V)^{\mathrm{an}})$,

where we use the notation introduced in aforementioned lemma. By Remark 2.1.14, it is enough to show that (1′) and (2′) are $(\mathbb{B}^1, \tau)$-equivalences in $\mathrm{PSh}(\mathrm{RigSm}/\mathcal{S}^{\mathrm{rig}}; \Lambda)$ which is obvious. □

We now state the main technical result needed for proving Proposition 3.8.21. (Compare with [Ayo15, Théorème 1.3.37].)

Proposition 3.8.28. *Let B be a quasi-excellent (Λ, τ)-admissible scheme, $B_\sigma \subset B$ a closed subscheme locally of finite presentation, and $B_\eta \subset B$ its open complement. If τ is the étale topology, assume that every prime number is invertible either in $\pi_0\Lambda$ or in $\mathcal{O}(B)$.*

(1) Consider the commutative diagram of diagrams of schemes

$$\begin{tikzcd} \mathfrak{D}_{\widehat{B},\eta} \arrow[r, "\mathfrak{j}"] \arrow[d, "\mathfrak{u}_\eta"] & \mathfrak{D}_{\widehat{B}} \arrow[d, "\mathfrak{u}"] & \mathfrak{D}_{\widehat{B},\sigma} \arrow[l, "\mathfrak{i}"'] \arrow[d, "\mathfrak{u}_\sigma"] \\ B_\eta \arrow[r, "j"] & B & B_\sigma. \arrow[l, "i"'] \end{tikzcd}$$

Then, the composite functor

$$\operatorname{diag}_{\sigma,*} \circ \mathfrak{i}^* \circ \mathfrak{j}_* \circ \mathfrak{u}_\eta^* : \mathbf{SH}^\wedge_\tau(B_\eta; \Lambda) \to \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af}}/\widehat{B}; \Lambda)) \tag{3.62}$$

takes values in $\mathbf{RigSH}^\wedge_\tau(\widehat{B}^{\mathrm{rig}}; \Lambda)$ considered as the full sub-∞-category of the target of (3.62) *spanned by those objects which are stably $(\mathbb{A}^1, \text{rig-}\tau)$-local.*

(2) Consider the commutative diagram of diagrams of rigid analytic schemes

$$\begin{tikzcd} \mathfrak{D}^{\mathrm{an}}_{\widehat{B},\eta} \arrow[r, "\mathfrak{j}^{\mathrm{an}}"] \arrow[d, "\mathfrak{u}^{\mathrm{an}}_\eta"] & \mathfrak{D}^{\mathrm{an}}_{\widehat{B}} \arrow[d, "\mathfrak{u}^{\mathrm{an}}"] & \mathfrak{D}_{\widehat{B},\sigma} \arrow[l, "\mathfrak{i}^{\mathrm{an}}"'] \arrow[d, "\mathfrak{u}_\sigma"] \\ \widehat{B}^{\mathrm{rig}} \arrow[r, "j^{\mathrm{an}}"] & \widehat{B} & B_\sigma. \arrow[l, "i^{\mathrm{an}}"'] \end{tikzcd}$$

Then, the composite functor

$$\operatorname{diag}^{\mathrm{an}}_{\sigma,*} \circ \mathfrak{i}^{\mathrm{an},*} \circ \mathfrak{j}^{\mathrm{an}}_* \circ \mathfrak{u}^{\mathrm{an},*}_\eta : \mathbf{RigSH}^\wedge_\tau(\widehat{B}^{\mathrm{rig}}; \Lambda) \to \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af}}/\widehat{B}; \Lambda)) \tag{3.63}$$

takes values in $\mathbf{RigSH}^\wedge_\tau(\widehat{B}^{\mathrm{rig}}; \Lambda)$ considered as the full sub-∞-category of the target of (3.63) *spanned by those objects which are stably $(\mathbb{A}^1, \text{rig-}\tau)$-local. Moreover, the induced endofunctor of $\mathbf{RigSH}^\wedge_\tau(\widehat{B}^{\mathrm{rig}}; \Lambda)$ is equivalent to the identity functor.*

Proof. We start with part (2) which is easier. Let $\operatorname{diag}^{\mathrm{for}} : \mathrm{FRigSm}_{\mathrm{af}}/\widehat{B} \to \mathrm{FSm}/\mathfrak{D}^{\mathrm{for}}_{\widehat{B}}$ be the diagonal functor sending an affine formal scheme $\mathcal{U}$ to the pair $(\mathcal{U}, \mathcal{U})$, and let $\operatorname{diag}^{\mathrm{for}}_*$ be constructed as in Remark 3.8.25. We have an equivalence $\operatorname{diag}^{\mathrm{for}}_* \circ \sigma_* \simeq \operatorname{diag}_{\sigma,*}$, where σ_* is restriction along the functor $(-)_\sigma : \mathrm{FSm}/\mathfrak{D}^{\mathrm{for}}_{\widehat{B}} \to \mathrm{Sm}/\mathfrak{D}_{\widehat{B},\sigma}$. By Lemma 3.8.24 and Theorem 3.1.10, the composite functor (3.63) is equivalent to the composite functor

$$\mathrm{diag}^{\mathrm{for}}_* \circ \chi_{\mathfrak{D}^{\mathrm{for}}_{\widehat{B}}} \circ \mathfrak{u}^{\mathrm{an},*}_\eta : \mathbf{RigSH}^\wedge_\tau(\widehat{B}^{\mathrm{rig}}; \Lambda) \to \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af}}/\widehat{B}; \Lambda)). \tag{3.64}$$

Now, $\chi_{\mathfrak{D}^{\mathrm{for}}_{\widehat{B}}}$ is restriction along the functor $(-)^{\mathrm{rig}} : \mathrm{FSm}/\mathfrak{D}^{\mathrm{for}}_{\widehat{B}} \to \mathrm{RigSm}/\mathfrak{D}^{\mathrm{an}}_{\widehat{B}^{\mathrm{rig}}}$ and $\mathfrak{u}^*_\eta$ is restriction along along the functor $\mathrm{RigSm}/\mathfrak{D}^{\mathrm{an}}_{\widehat{B}^{\mathrm{rig}}} \to \mathrm{RigSm}/\widehat{B}^{\mathrm{rig}}$ sending a pair $(\mathcal{U}, \mathcal{V})$ to $\mathcal{V}^{\mathrm{rig}}$. It follows that the composite functor (3.64) is restriction along the functor $(-)^{\mathrm{rig}} : \mathrm{FRigSm}_{\mathrm{af}}/\widehat{B} \to \mathrm{RigSm}/\widehat{B}^{\mathrm{rig}}$. The claim now follows from Remark 2.1.14.

We now concentrate on part (1). We fix an object $M \in \mathbf{SH}^\wedge_\tau(B_\eta; \Lambda)$. Our goal is to show that $\mathrm{diag}_{\sigma,*}\mathrm{i}^*\mathrm{j}_*\mathfrak{u}^*_\eta M$ belongs to the full sub-∞-category

$$\mathbf{RigSH}^\wedge_\tau(\widehat{B}^{\mathrm{rig}}; \Lambda) \subset \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af}}/\widehat{B}; \Lambda)). \tag{3.65}$$

The proof of this is similar to the proof of Proposition 3.6.7 and, instead of repeating large portions of that proof we will refer to it when possible. It follows from Propositions 2.4.22 and 3.2.3 that the sub-∞-category (3.65) is closed under colimits and that the functors $\mathrm{diag}_{\sigma,*}$, i^*, j_* and $\mathfrak{u}^*_\eta$ are colimit-preserving. Thus, we may assume that M is compact. We split the proof into several steps.

Step 1. Arguing as in the second part of the proof of Lemma 3.6.2, we may assume one of the following alternatives:

(1) τ is the Nisnevich topology;
(2) $\pi_0\Lambda$ is a $\mathbb{Q}$-algebra;
(3) τ is the étale topology and M is ℓ-nilpotent for a prime ℓ invertible on B.

Moreover, we claim that under the alternative (3), we may assume that Λ is eventually coconnective. To prove this, let $M_0 \in \mathrm{Shv}^\wedge_{\text{ét}}(\text{Ét}/B_\eta; \Lambda)_{\ell\text{-nil}}$ be the object corresponding to M by the equivalence

$$\mathrm{Shv}^\wedge_{\text{ét}}(\text{Ét}/B_\eta; \Lambda)_{\ell\text{-nil}} \simeq \mathbf{SH}^\wedge_{\text{ét}}(B_\eta; \Lambda)_{\ell\text{-nil}}$$

provided by Theorem 2.10.4. Then, as a T-spectrum, M is given at level m by $\iota^*_{B_\eta} M_0(m)[m]$, where $\iota^*_{B_\eta}$ is as in Notation 2.10.7. (See [Ayo14a, Corollary 4.9] in the case where Λ is an Eilenberg–Mac Lane spectrum; the general case can be treated similarly.) Similarly, as a T-spectrum, $\mathrm{i}^*\mathrm{j}_*\mathfrak{u}^*_\eta M$ is given at level m by $\iota^*_{\mathfrak{D}_{\widehat{B},\sigma}}\mathrm{i}^*\mathrm{j}_*\mathfrak{u}^*_\eta M_0(m)[m]$. Using this and Lemma 2.4.5, one deduces an equivalence

$$\mathrm{diag}_{\sigma,*}\mathrm{i}^*\mathrm{j}_*\mathfrak{u}^*_\eta M \simeq \lim_r \mathrm{diag}_{\sigma,*}\mathrm{i}^*\mathrm{j}_*\mathfrak{u}^*_\eta (M \otimes_\Lambda \tau_{\le r}\Lambda).$$

Since the sub-∞-category (3.65) is stable under limits, we deduce that it is enough to prove the result for $M \otimes_\Lambda \tau_{\le r}\Lambda$. This proves our claim.

In conclusion, when τ is the étale topology, we may assume that Λ is eventually coconnective. (Indeed, if $\pi_0\Lambda$ is a $\mathbb{Q}$-algebra, there is a morphism $\mathbb{Q} \to \Lambda$ and we may replace Λ by $\mathbb{Q}$.)

Step 2. From now on, we set $E = \mathrm{diag}^*_\sigma \mathrm{i}^*\mathrm{j}_*\mathfrak{u}^*_\eta M$ and, for $m \in \mathbb{N}$, we denote by E_m the m-th level of the T-spectrum E. In this step, we show that E admits levelwise hyperdescent for the rig-Nisnevich topology. Arguing as in the beginning of the proof of Proposition 3.6.7, we need to show that E_m has descent for every rig-Nisnevich hypercover $\mathcal{U}_\bullet$ in $\mathrm{FRigSm}_{\mathrm{af}}/\widehat{B}$ admitting a morphism of augmented simplicial formal schemes $\widetilde{\mathcal{U}}_\bullet \to \mathcal{U}_\bullet$ such that:

- $\widetilde{\mathcal{U}}_\bullet$ is a Nisnevich hypercover;
- $\widetilde{\mathcal{U}}_{-1} \to \mathcal{U}_{-1}$ is an admissible blowup;

- $\widetilde{\mathcal{U}}_n \to \mathcal{U}_n$ is an isomorphism for $n \geq 0$.

In particular, we see that $\widetilde{\mathcal{U}}_n$ is affine except possibly when $n = -1$. For $n \geq -1$, we set $U_n = \mathrm{Spec}(\mathcal{O}(\mathcal{U}_n))$ and, for $n \geq 0$, we set $\widetilde{U}_n = U_n$. Since $\widetilde{\mathcal{U}}_{-1} \to \mathcal{U}_{-1}$ is an admissible blowup, it is the formal completion of a unique blowup $e : \widetilde{U}_{-1} \to U_{-1}$ with center supported on $\mathcal{U}_{-1,\sigma} \subset U_{-1}$. For $n \geq -1$, we set $U_{n,\sigma} = \mathcal{U}_{n,\sigma}$, $\widetilde{U}_{n,\sigma} = \widetilde{\mathcal{U}}_{n,\sigma}$, $U_{n,\eta} = U_n \smallsetminus U_{n,\sigma}$ and $\widetilde{U}_{n,\eta} = \widetilde{U}_n \smallsetminus U_{n,\sigma}$. We denote by $u_n : U_n \to B$ and $\widetilde{u}_n : \widetilde{U}_n \to B$ the obvious morphisms.

Since M can be shifted and twisted, it suffices to prove that the map

$$\mathrm{Map}(\Lambda(\mathcal{U}_{-1}), E_0) \to \lim_{[n]\in\mathbf{\Delta}} \mathrm{Map}(\Lambda(\mathcal{U}_n), E_0)$$

is an equivalence, where the mapping spaces are taken in $\mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af}}/\widehat{B};\Lambda)$. Looking at the definition of E_0, we see that this map is equivalent to

$$\begin{array}{rcl} \mathrm{Map}_{\mathbf{SH}^{\wedge}_{\tau}(U_{-1,\sigma};\Lambda)}(\Lambda, \chi_{U_{-1}} u^*_{-1,\eta} M) & \to & \lim\limits_{[n]\in\mathbf{\Delta}} \mathrm{Map}_{\mathbf{SH}^{\wedge}_{\tau}(U_{n,\sigma};\Lambda)}(\Lambda, \chi_{U_n} u^*_{n,\eta} M) \\ & = & \lim\limits_{[n]\in\mathbf{\Delta}} \mathrm{Map}_{\mathbf{SH}^{\wedge}_{\tau}(\widetilde{U}_{n,\sigma};\Lambda)}(\Lambda, \chi_{\widetilde{U}_n} \widetilde{u}^*_{n,\eta} M). \end{array} \tag{3.66}$$

For $n \geq 0$, we let $v_n : \widetilde{U}_n \to \widetilde{U}_{-1}$ be the obvious morphism. Since B is quasi-excellent, the v_n's are regular morphisms. By Lemma 3.8.29 below, the morphism

$$\chi_{\widetilde{U}_n} \widetilde{u}^*_{n,\eta} M \to v^*_{n,\sigma} \chi_{\widetilde{U}_{-1}} \widetilde{u}^*_{-1,\eta} M$$

is an equivalence. Therefore, the left-hand side in (3.66) is equivalent to

$$\lim_{[n]\in\mathbf{\Delta}} \mathrm{Map}_{\mathbf{SH}^{\wedge}_{\tau}(\widetilde{U}_{n,\sigma};\Lambda)}(\Lambda, v^*_{n,\sigma} \chi_{\widetilde{U}_{-1}} \widetilde{u}^*_{-1,\eta} M).$$

Since $\widetilde{U}_{\bullet,\sigma}$ is a Nisnevich hypercover, the latter is equivalent to $\mathrm{Map}_{\mathbf{SH}^{\wedge}_{\tau}(\widetilde{U}_{-1,\sigma};\Lambda)}(\Lambda, \chi_{\widetilde{U}_{-1}} u^*_{-1,\eta} M)$. Thus, we are left to show that the morphism

$$\chi_{U_{-1}} u^*_{-1,\eta} M \to e_{\sigma,*} \chi_{\widetilde{U}_{-1}} \widetilde{u}^*_{-1,\eta} M$$

is an equivalence. This follows from the projective base change theorem and the fact that e_η is an isomorphism.

Step 3. In this step and the next one, we assume that τ is the étale topology and we prove that E admits levelwise hyperdescent for the rig-étale topology. By the second step, we already know that E admits levelwise hyperdescent for the rig-Nisnevich topology. Thus, arguing as in the beginning of the proof of Proposition 3.6.7, it remains to show that E has levelwise descent for the topology rigfét.

In this step, we deal with the case where $\pi_0\Lambda$ is a $\mathbb{Q}$-algebra. As explained in the third part of the proof of Proposition 3.6.7, we only need to show that E has levelwise descent for a rigfét-hypercover of the form

$$\cdots \mathcal{V}_0 \times G \times G \Rrightarrow \mathcal{V}_0 \times G \rightrightarrows \mathcal{V}_0 \longrightarrow \mathcal{V}_{-1}. \tag{3.67}$$

where $\mathcal{V}_{-1}$ is an affine rig-smooth formal $\widehat{B}$-scheme and $\mathcal{V}_0 \to \mathcal{V}_{-1}$ is a finite rig-étale covering admitting an action of a finite group G which is simply transitive on the geometric fibers of $\mathcal{V}_0^{\mathrm{rig}} \to \mathcal{V}_{-1}^{\mathrm{rig}}$. For $n \in \{-1, 0\}$, we set $V_n = \mathrm{Spec}(\mathcal{O}(\mathcal{V}_n))$, $V_{n,\sigma} = \mathcal{V}_{n,\sigma}$ and $V_{n,\eta} = V_n \smallsetminus V_{n,\sigma}$. We also denote by $v_{-1} : V_{-1} \to B$, $v_0 : V_0 \to B$ and $e : V_0 \to V_{-1}$ the obvious morphisms. For later use, we note that $e_\eta : V_{0,\eta} \to V_{-1,\eta}$ is a finite étale cover admitting an action of G which is simply transitive on geometric fibers.

Since M can be shifted and twisted, it suffices to prove that the map

$$\mathrm{Map}(\Lambda(\mathcal{V}_{-1}), E_0) \to \mathrm{Map}(\Lambda(\mathcal{V}_0), E_0)^G$$

is an equivalence, where the mapping spaces are taken in $\mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af}}/\widehat{B}; \Lambda)$. Looking at the definition of E_0, we see that this map is equivalent to

$$\mathrm{Map}_{\mathbf{SH}^{\wedge}_{\mathrm{\acute{e}t}}(V_{-1,\sigma};\Lambda)}(\Lambda, \chi_{V_{-1}} v^*_{-1,\eta} M) \to \mathrm{Map}_{\mathbf{SH}^{\wedge}_{\mathrm{\acute{e}t}}(V_{0,\sigma};\Lambda)}(\Lambda, \chi_{V_0} v^*_{0,\eta} M)^G.$$

Thus, it is enough to show that

$$\chi_{V_{-1}} v^*_{-1,\eta} M \to (e_{\sigma,*}\chi_{V_0} v^*_{0,\eta} M)^G \simeq (\chi_{V_{-1}} e_{\eta,*} v^*_{-1,\eta} M)^G$$

is an equivalence. (The equivalence above follows from the proper base change theorem and the fact that e is finite.) Taking the "G-invariant subobject" in a $\mathbb{Q}$-linear ∞-category is equivalent to taking the image of the projector $|G|^{-1}\sum_{g\in G} g$, and hence it commutes with the functor $\chi_{V_{-1}}$. Thus, it is enough to show that $v^*_{-1,\eta} M_0 \to (e_{\eta,*} e^*_\eta v^*_{-1,\eta} M_0)^G$ is an equivalence, which follows from étale descent in $\mathbf{SH}^{\wedge}_{\mathrm{\acute{e}t}}(V_{-1,\eta}; \Lambda)$.

Step 4. Here we complete the proof that E admits levelwise hyperdescent for the rig-étale topology. By the first and the third steps, we may assume that M is ℓ-nilpotent and that Λ is eventually coconnective. Let $M_0 \in \mathrm{Shv}^{\wedge}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/B_\eta; \Lambda)_{\ell\text{-nil}}$ be the object corresponding to M by the equivalence

$$\mathrm{Shv}^{\wedge}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/B_\eta; \Lambda)_{\ell\text{-nil}} \simeq \mathbf{SH}^{\wedge}_{\mathrm{\acute{e}t}}(B_\eta; \Lambda)_{\ell\text{-nil}}$$

provided by Theorem 2.10.4. As in the third step, it suffices to show descent for the rigfét-hypercover (3.67) and it is enough to prove that

$$\chi_{V_{-1}} v^*_{-1,\eta} M \to (\chi_{V_{-1}} e_{\eta,*} v^*_{-1,\eta} M)^G$$

is an equivalence. Using Theorem 2.10.4, we may as well prove that

$$\chi_{V_{-1}} v^*_{-1,\eta} M_0 \to (\chi_{V_{-1}} e_{\eta,*} v^*_{-1,\eta} M_0)^G$$

is an equivalence. Since Λ is eventually coconnective and M_0 is compact, we deduce that the étale sheaf M_0 is also eventually coconnective. Taking the "G-invariant subobject" commutes with direct images and, if we restrict to eventually coconnective étale sheaves, it also commute with inverse images. (The latter assertion can be proven using an explicit model for the G-invariant functor; see the fourth part of the proof of Proposition 3.6.7 for a similar argument.) Thus, as in the previous step, it is enough to show that $v^*_{-1,\eta} M_0 \to (e_{\eta,*} e^*_\eta v^*_{-1,\eta} M_0)^G$ is an equivalence, which follows from étale descent in $\mathrm{Shv}^{\wedge}_{\mathrm{\acute{e}t}}(\mathrm{\acute{E}t}/V_{-1,\eta}; \Lambda)$.

Step 5. In this last step, we check that E is levelwise $\mathbb{A}^1$-invariant and an Ω-spectrum. Since M can be shifted, it is enough to show that the maps

$$\begin{aligned} &\mathrm{Map}(\Lambda(\mathcal{U}), E_m) \to \mathrm{Map}(\Lambda(\mathbb{A}^1_{\mathcal{U}}), E_m), \\ &\mathrm{Map}(\Lambda(\mathcal{U}), E_m) \to \mathrm{fib}\{\mathrm{Map}(\Lambda(\mathbb{A}^1_{\mathcal{U}} \smallsetminus 0_{\mathcal{U}}), E_{m+1}) \xrightarrow{1^*} \mathrm{Map}(\Lambda(\mathcal{U}), E_{m+1})\} \end{aligned} \tag{3.68}$$

are equivalences for every $\mathcal{U} \in \mathrm{FRigSm}_{\mathrm{af}}/\widehat{B}$.

Set $U = \mathrm{Spec}(\mathcal{O}(\mathcal{U}))$, $U_\sigma = \mathcal{U}_\sigma$ and $U_\eta = U \smallsetminus U_\sigma$. Let $\mathcal{V}$ be an affine smooth formal $\mathcal{U}$-scheme, and set $V = \mathrm{Spec}(\mathcal{O}(\mathcal{V}))$, $V_\sigma = \mathcal{V}_\sigma$ and $V_\eta = V \smallsetminus V_\sigma$. Denote by $u : U \to B$ and $g : V \to U$ the obvious morphisms. Then we have equivalences

$$\begin{aligned}
\mathrm{Map}(\Lambda(\mathcal{U}), E_m) &\simeq \mathrm{Map}_{\mathbf{SH}^\wedge_\tau(U_\sigma;\Lambda)}(\Lambda(-m), \chi_U u_\eta^* M),\\
\\
\mathrm{Map}(\Lambda(\mathcal{V}), E_m) &\simeq \mathrm{Map}_{\mathbf{SH}^\wedge_\tau(V_\sigma;\Lambda)}(\Lambda(-m), \chi_V g_\eta^* u_\eta^* M),\\
&\overset{(1)}{\simeq} \mathrm{Map}_{\mathbf{SH}^\wedge_\tau(V_\sigma;\Lambda)}(\Lambda(-m), g_\sigma^* \chi_U u_\eta^* M),\\
&\overset{(2)}{\simeq} \mathrm{Map}_{\mathbf{SH}^\wedge_\tau(U_\sigma;\Lambda)}(\Lambda(-m), g_{\sigma,*} g_\sigma^* \chi_U u_\eta^* M).
\end{aligned}$$

The equivalence (1) follows from Lemma 3.8.29 below and the fact that g is regular. The equivalence (2) follows by adjunction. Letting $p : \mathbb{A}^1_{U_\sigma} \to U_\sigma$ and $q : \mathbb{A}^1_{U_\sigma} \smallsetminus 0_{U_\sigma} \to U_\sigma$ be the obvious projections, we deduce that the maps (3.68) are equivalent to the following ones:

$$\mathrm{Map}_{\mathbf{SH}^\wedge_\tau(U_\sigma;\Lambda)}(\Lambda(-m), \chi_U u_\eta^* M) \to \mathrm{Map}_{\mathbf{SH}^\wedge_\tau(U_\sigma;\Lambda)}(\Lambda(-m), p_* p^* \chi_U u_\eta^* M),$$
$$\mathrm{Map}_{\mathbf{SH}^\wedge_\tau(U_\sigma;\Lambda)}(\Lambda(-m), \chi_U u_\eta^* M) \to \mathrm{Map}_{\mathbf{SH}^\wedge_\tau(U_\sigma;\Lambda)}(\Lambda(-m-1), \mathrm{fib}\{1^* : q_* q^* \chi_U u_\eta^* M \to \chi_U u_\eta^* M\})$$

which are clearly equivalences as needed. □

The following lemma was used in the proof of Proposition 3.8.28. We prove it in a greater generality than needed because of its potential usefulness.

Lemma 3.8.29 (Regular base change)**.** *Consider a Cartesian square of schemes*

$$\begin{tikzcd}
Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\
X' \arrow[r, "g"] & X
\end{tikzcd}$$

with X locally noetherian, g regular, and f quasi-compact and quasi-separated. Assume one of the following alternatives:

1. *(1) we work in the non-hypercomplete case and, when τ is the étale topology, we assume furthermore that Λ is eventually coconnective;*
2. *(2) we work in the hypercomplete case, and the schemes X, X', Y and Y' are (Λ,τ)-admissible.*

Then, the natural transformation $g^* \circ f_* \to f'_* \circ g'^*$, between functors from $\mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(Y;\Lambda)$ to $\mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(X';\Lambda)$, is an equivalence.

Proof. This is a generalisation of [Ayo15, Corollary 1.A.4] and, as in loc. cit., its proof consists in reducing to the smooth base change theorem using Popescu's theorem on regular algebras and Proposition 2.5.11. However, here we need an extra argument to reduce to the case where Λ is eventually coconnective so that Proposition 2.5.11 applies. The problem being local on X, X' and Y, we may assume that X, X', Y and Y' are affine. (This uses the hypothesis that f is quasi-compact and quasi-separated.) By Proposition 3.2.3, the ∞-category $\mathbf{SH}^{(\mathrm{eff},\wedge)}_\tau(X;\Lambda)$ is compactly generated, and similarly for X', Y and Y'. By the same proposition, the functors f_* and f'_* are colimit-preserving, and thus belong to Pr^{L}. (The same is obviously true for g^* and g'^*.)

We first prove the lemma under the alternative (1). By [Pop86, Theorem 1.8], the X-scheme X' is a limit of a cofiltered inverse system $(X'_\alpha)_\alpha$ of smooth affine X-schemes. For each α, consider a

Cartesian square

$$\begin{tikzcd} Y'_\alpha \arrow[r, "g'_\alpha"] \arrow[d, "f'_\alpha"] & Y \arrow[d, "f"] \\ X'_\alpha \arrow[r, "g_\alpha"] & X. \end{tikzcd}$$

By the smooth base change theorem, we have commutative squares in Pr^{L}

$$\begin{tikzcd} \mathbf{SH}^{(\mathrm{eff})}_\tau(Y'_\alpha;\Lambda) \arrow[d, "f'_{\alpha,*}"] & \mathbf{SH}^{(\mathrm{eff})}_\tau(Y;\Lambda) \arrow[l, "g'^*_\alpha"'] \arrow[d, "f_*"] \\ \mathbf{SH}^{(\mathrm{eff})}_\tau(X'_\alpha;\Lambda) & \mathbf{SH}^{(\mathrm{eff})}_\tau(X;\Lambda) \arrow[l, "g^*_\alpha"'] \end{tikzcd}$$

Taking the colimit in Pr^{L} of these squares yields a commutative square expressing that $g^* \circ f_*$ is equivalent to $f'_* \circ g'^*$ as needed. (This is actually not obvious; one needs to argue as in the proof of Theorem 2.7.1. We leave the details to the reader.)

Next, we prove the lemma under the alternative (2). Using Proposition 3.2.2, we may conclude using the lemma under the alternative (1) if τ is the Nisnevich topology or if Λ is eventually coconnective and, more generally, if Λ is an algebra over an eventually coconnective commutative ring spectrum. In particular, we may assume that τ is the étale topology, and the result holds if $\pi_0\Lambda$ is a $\mathbb{Q}$-algebra. Arguing as in the second part of the proof of Lemma 3.6.2, it remains to prove that $g^*f_*M \to f'_*g'^*M$ is an equivalence when $M \in \mathbf{SH}^{(\mathrm{eff}),\wedge}_{\acute{e}t}(Y;\Lambda)_{\ell\text{-nil}}$, for some prime ℓ invertible on X. Moreover, we may assume that M is compact. By Theorem 2.10.4, it is enough to show that $g^*f_*M_0 \to f'_*g'^*M_0$ is an equivalence for $M_0 \in \mathrm{Shv}^\wedge_{\acute{e}t}(\acute{E}t/Y;\Lambda)_{\ell\text{-nil}}$. Using Lemma 2.4.5, one deduces equivalences

$$g^*f_*M_0 \simeq \lim_r g^*f_*(M_0 \otimes_\Lambda \tau_{\leq r}\Lambda) \qquad \text{and} \qquad f'_*g'^*M_0 \simeq \lim_r f'_*g'^*(M_0 \otimes_\Lambda \tau_{\leq r}\Lambda).$$

Thus, we may replace M and Λ with $M \otimes_\Lambda \tau_{\leq r}\Lambda$ and $\tau_{\leq r}\Lambda$. We are then automatically working under the alternative (1), and the result follows. □

Proof of Proposition 3.8.21. We have a commutative square of natural transformations

$$\begin{tikzcd} \mathrm{diag}_{\eta,*} \circ \mathrm{u}^*_\eta \arrow[r, "\alpha"] \arrow[d, "\beta'"] & \mathrm{diag}_{\sigma,*} \circ \mathrm{i}^* \circ \mathrm{j}_* \circ \mathrm{u}^*_\eta \arrow[d, "\beta"] \\ \mathrm{diag}^{\mathrm{an}}_{\eta,*} \circ \mathrm{u}^{\mathrm{an},*}_\eta \circ \mathrm{An}^*_{B_\eta} \arrow[r, "\alpha'"] & \mathrm{diag}_{\sigma,*} \circ \mathrm{i}^{\mathrm{an},*} \circ \mathrm{j}^{\mathrm{an}}_* \circ \mathrm{u}^{\mathrm{an},*}_\eta \circ \mathrm{An}^*_{B_\eta}. \end{tikzcd} \tag{3.69}$$

The natural transformation α is obtained from $\mathrm{j}_* \to \mathrm{i}_* \circ \mathrm{i}^* \circ \mathrm{j}_*$ by applying diag_*, and similarly for the natural transformation α'. The natural transformation β is deduced from (3.58) (with $\mathcal{S} = \widehat{B}$). Finally, the natural transformation β' is deduced from the second natural transformation in (3.59) (with $\mathcal{S} = \widehat{B}$) and the equivalence $\mathrm{u}^{\mathrm{an},*}_\eta \circ \mathrm{An}^*_{B_\eta} \simeq \mathrm{An}^*_{\mathfrak{D}_{\widehat{B},\eta}} \circ \mathrm{u}^*_\eta$.

We claim that the natural transformation $\beta \circ \alpha$ is given by stable $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalences. We will prove this by showing that α' is an equivalence and that β' is given by stable $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalences. We then use this to finish the proof of the proposition. We split the remainder of the proof into three steps accordingly.

Step 1. Here we prove that α' is an equivalence. In fact, even the natural transformation

$$\mathrm{diag}^{\mathrm{an}}_{\eta,*} \to \mathrm{diag}_{\sigma,*} \circ \mathrm{i}^{\mathrm{an},*} \circ \mathrm{j}^{\mathrm{an}}_{*} = \mathrm{diag}_{\sigma,*} \circ \chi_{\mathfrak{D}^{\mathrm{an}}_{\widehat{B}}}$$

is an equivalence. Indeed, by Lemma 3.8.24 and Theorem 3.1.10, we have an equivalence

$$\mathrm{diag}_{\sigma,*} \circ \chi_{\mathfrak{D}^{\mathrm{an}}_{\widehat{B}}} \simeq \mathrm{diag}^{\mathrm{for}}_{*} \circ \chi_{\mathfrak{D}^{\mathrm{for}}_{\widehat{B}}}.$$

(See the beginning of the proof of Proposition 3.8.28.) Thus, we need to show that the natural transformation

$$\mathrm{diag}^{\mathrm{an}}_{\eta,*} \to \mathrm{diag}^{\mathrm{for}}_{*} \circ \chi_{\mathfrak{D}^{\mathrm{for}}_{\widehat{B}}}$$

is an equivalence. This follows from the equality $\mathrm{diag}^{\mathrm{an}}_{\eta} = (-)^{\mathrm{rig}} \circ \mathrm{diag}^{\mathrm{for}}$ and the fact that $\chi_{\mathfrak{D}^{\mathrm{for}}_{\widehat{B}}}$ is restriction along the functor $(-)^{\mathrm{rig}} : \mathrm{FSm}/\mathfrak{D}^{\mathrm{for}}_{\widehat{B}} \to \mathrm{FSm}/\mathfrak{D}^{\mathrm{an}}_{\widehat{B},\eta}$.

Step 2. Here we prove that β' is given by stable $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalences. Since all the functors composing the source and the target of β' are colimit-preserving and since stable $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalences are preserved by colimits, it is enough to show that

$$\beta'_M : \mathrm{diag}_{\eta,*}\mathrm{u}^*_{\eta}M \to \mathrm{diag}^{\mathrm{an}}_{\eta,*}\mathrm{u}^{\mathrm{an},*}_{\eta}\mathrm{An}^*_{B_\eta}M$$

is a stable $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalence when M is of the form $\mathrm{L}_{\mathbb{A}^1,\tau,\mathrm{st}}\mathrm{Sus}^m_T\Lambda(X)$ for $n \in \mathbb{N}$ and $X \in \mathrm{Sm}/B_\eta$. (Here, $\mathrm{L}_{\mathbb{A}^1,\tau,\mathrm{st}}$ is the stable $(\mathbb{A}^1, \tau)$-localisation functor and Sus^m_T is the left adjoint sending a T-spectrum to its m-th level.) We have an equivalence

$$\mathrm{u}^*_{\eta}M \simeq \mathrm{L}_{\mathbb{A}^1,\tau,\mathrm{st}}\mathrm{u}^*_{\eta}\mathrm{Sus}^m_T\Lambda(X)$$

where, on the right-hand side, $\mathrm{u}^*_{\eta} : \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{Sm}/B_\eta; \Lambda)) \to \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{Sm}/\mathfrak{D}_{\widehat{B},\eta}; \Lambda))$ is the inverse image functor on T-spectra of presheaves of Λ-modules. Using Lemma 3.8.27(1), we deduce a stable $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalence

$$\mathrm{diag}_{\eta,*}\mathrm{u}^*_{\eta}\mathrm{Sus}^m_T\Lambda(X) \to \mathrm{diag}_{\eta,*}\mathrm{u}^*_{\eta}M.$$

Similarly, we have $\mathrm{An}^*_{B_\eta}M \simeq \mathrm{L}_{\mathbb{B}^1,\tau,\mathrm{st}}\mathrm{Sus}^m_T\Lambda(X^{\mathrm{an}})$. Arguing as before and using Lemma 3.8.27(2), we deduce a stable $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalence

$$\mathrm{diag}^{\mathrm{an}}_{\eta,*}\mathrm{u}^{\mathrm{an},*}_{\eta}\mathrm{Sus}^m_T\Lambda(X^{\mathrm{an}}) \to \mathrm{diag}^{\mathrm{an}}_{\eta,*}\mathrm{u}^{\mathrm{an},*}_{\eta}\mathrm{An}^*_{B_\eta}M.$$

The result follows now by remarking that the obvious morphism

$$\mathrm{diag}_{\eta,*}\mathrm{u}^*_{\eta}\mathrm{Sus}^m_T\Lambda(X) \to \mathrm{diag}^{\mathrm{an}}_{\eta,*}\mathrm{u}^{\mathrm{an},*}_{\eta}\mathrm{Sus}^m_T\Lambda(X^{\mathrm{an}})$$

is an isomorphism.

Step 3. We are now ready to finish the proof of the proposition. By Proposition 3.8.28(1), the functor $\mathrm{diag}_{\sigma,*} \circ \mathrm{i}^* \circ \mathrm{j}_* \circ \mathrm{u}^*_{\eta}$ takes values in the ∞-subcategory spanned by stably $(\mathbb{A}^1, \text{rig-}\tau)$-local objects. Therefore, α factors through the functor $\mathrm{L}_{\mathbb{A}^1,\text{rig-}\tau,\mathrm{st}} \circ \mathrm{diag}_{\eta,*} \circ \mathrm{u}^*_{\eta}$ and the composition of

$$\mathrm{L}_{\mathbb{A}^1,\text{rig-}\tau,\mathrm{st}} \circ \mathrm{diag}_{\eta,*} \circ \mathrm{u}^*_{\eta} \xrightarrow{\widetilde{\alpha}} \mathrm{diag}_{\sigma,*} \circ \mathrm{i}^* \circ \mathrm{j}_* \circ \mathrm{u}^*_{\eta} \xrightarrow{\beta} \mathrm{diag}_{\sigma,*} \circ \mathrm{i}^{\mathrm{an},*} \circ \mathrm{j}^{\mathrm{an}}_* \circ \mathrm{u}^{\mathrm{an},*}_{\eta} \circ \mathrm{An}^*_{B_\eta}$$

is given by stable $(\mathbb{A}^1, \text{rig-}\tau)$-local equivalences (by the first and second steps). Since the source and the target of this composition take values in the ∞-subcategory spanned by stably $(\mathbb{A}^1, \text{rig-}\tau)$-local objects (by Proposition 3.8.28(2) for the target), this composition is in fact a natural equivalence. Thus, we have shown that β admits a section. Applying the restriction functor

$$\mathrm{r}_* : \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{FRigSm}_{\mathrm{af}}/\widehat{B}; \Lambda)) \to \mathrm{Spt}_T(\mathrm{PSh}(\mathrm{FSm}_{\mathrm{af}}/\widehat{B}; \Lambda))$$

to β, we deduce a natural transformation

$$\mathfrak{r}_*(\beta) : \mathfrak{r}_* \circ \mathrm{diag}_{\sigma,*} \circ \mathfrak{i}^* \circ \mathfrak{j}_* \circ \mathfrak{u}_\eta^* \to \mathfrak{r}_* \circ \mathrm{diag}_{\sigma,*} \circ \mathfrak{i}^{\mathrm{an},*} \circ \mathfrak{j}_*^{\mathrm{an}} \circ \mathfrak{u}_\eta^{\mathrm{an},*} \circ \mathrm{An}_{B_\eta}^*$$

admitting a section. We claim that this natural transformation is equivalent to $\rho_B : \chi_B \to \chi_{\widehat{B}} \circ \mathrm{An}_{B_\eta}^*$. We only explain how to identify $\mathfrak{r}_* \circ \mathrm{diag}_{\sigma,*} \circ \mathfrak{i}^* \circ \mathfrak{j}_* \circ \mathfrak{u}_\eta^*$ with χ_B; the identification of $\mathfrak{r}_* \circ \mathrm{diag}_{\sigma,*} \circ \mathfrak{i}^{\mathrm{an},*} \circ \mathfrak{j}_*^{\mathrm{an}} \circ \mathfrak{u}_\eta^{\mathrm{an},*}$ with $\chi_{\widehat{B}}$ is similar and easier.

Denote by $\mathfrak{D}_{\widehat{B}}^{\mathrm{sm}}$ the diagram of schemes obtained by restricting the functor $\mathfrak{D}_{\widehat{B}}$ to the subcategory $\mathrm{FSm}_{\mathrm{af}}/\widehat{B} \subset \mathrm{FRigSm}_{\mathrm{af}}/\widehat{B}$. Define $\mathfrak{D}_{\widehat{B},\sigma}^{\mathrm{sm}}$ and $\mathfrak{D}_{\widehat{B},\eta}^{\mathrm{sm}}$ similarly and denote by

$$\mathfrak{i}^{\mathrm{sm}} : \mathfrak{D}_{\widehat{B},\sigma}^{\mathrm{sm}} \to \mathfrak{D}_{\widehat{B}}^{\mathrm{sm}} \qquad \text{and} \qquad \mathfrak{j}^{\mathrm{sm}} : \mathfrak{D}_{\widehat{B},\eta}^{\mathrm{sm}} \to \mathfrak{D}_{\widehat{B}}^{\mathrm{sm}}$$

the obvious inclusions. We also consider the diagonal functor $\mathrm{diag}_\sigma^{\mathrm{sm}} : \mathrm{FSm}/\widehat{B} \to \mathrm{Sm}/\mathfrak{D}_{\widehat{B},\sigma}^{\mathrm{sm}}$ sending a formal scheme $\mathcal{U}$ to the pair $(\mathcal{U}, \mathcal{U}_\sigma)$. With these notations, we have an equivalence

$$\mathfrak{r}_* \circ \mathrm{diag}_{\sigma,*} \circ \mathfrak{i}^* \circ \mathfrak{j}_* \circ \mathfrak{u}_\eta^* \simeq \mathrm{diag}_{\sigma,*}^{\mathrm{sm}} \circ \mathfrak{i}^{\mathrm{sm},*} \circ \mathfrak{j}_*^{\mathrm{sm}} \circ \mathfrak{u}_\eta^{\mathrm{sm},*}.$$

Now, remark that the diagram of schemes $\mathfrak{D}_{\widehat{B}}^{\mathrm{sm}}$ takes values in regular B-schemes. By Lemma 3.8.29, we deduce an equivalence

$$\mathfrak{u}_\sigma^{\mathrm{sm},*} \circ \chi_B = \mathfrak{u}_\sigma^{\mathrm{sm},*} \circ i^* \circ j_* \simeq \mathfrak{i}^{\mathrm{sm},*} \circ \mathfrak{j}_*^{\mathrm{sm}} \circ \mathfrak{u}_\eta^{\mathrm{sm},*}.$$

We conclude by remarking that $\mathrm{diag}_{\sigma,*}^{\mathrm{sm}} \circ \mathfrak{u}_\sigma^{\mathrm{sm},*}$ is equivalent to the identity functor. □

We are now almost ready to finish the proof of Theorem 3.8.19, but we still need two results which are of independent interest. The following is a version of [Ayo07a, Proposition 2.2.27(2)] with integral coefficients.

Proposition 3.8.30. *Let B be a $(\Lambda, \text{ét})$-admissible scheme, $B_\sigma \subset B$ a closed subscheme, and $B_\eta \subset B$ its open complement. Assume one of the following alternatives:*

- *B is quasi-compact and quasi-excellent of characteristic zero;*
- *B is of finite type over a quasi-compact and quasi-excellent scheme of dimension ≤ 1.*

Assume that every prime number is invertible either in $\pi_0\Lambda$ or in $\mathcal{O}(B)$. Then, the ∞-category $\mathbf{SH}_{\text{ét}}^{\wedge}(B_\eta; \Lambda)$ is compactly generated, up to desuspension and Tate twists, by motives of the form $f_{\eta,*}\Lambda$, where $f : X \to B$ is a proper morphism with X regular and such that X_σ is a normal crossing divisor.

Proof. By [Tem08, Theorem 1.1] and [dJ97, Theorem 5.13], given a finite type B-scheme X with X_η integral and dense in X, we may find a proper morphism $e : X' \to X$ such that:

(1) X' is regular and X'_σ is a strict normal crossing divisor of X';
(2) X'_η is integral and dense in X', and $X' \to X$ is dominant and generically finite;
(3) there exists a finite group G acting on the X-scheme X' and a dense open $U \subset X_\eta$ with inverse image $U' \subset X'_\eta$, such that the morphism $U' \to U$ factors as a finite étale Galois cover $U' \to U'/G$ with group G and a universal homeomorphism $U'/G \to U$.

Now, let $\mathcal{T}$ (resp. $\mathcal{T}'$) be the smallest full sub-∞-category of $\mathbf{SH}_{\text{ét}}^{\wedge}(B_\eta; \Lambda)$ closed under colimits, desuspension and Tate twists, and containing the motives of the form $f_{\eta,*}\Lambda$, where $f : X \to B$ is a proper morphism (resp. a proper morphism with X regular and X_σ a normal crossing divisor). By [Ayo07a, Lemme 2.2.23], we have $\mathcal{T} = \mathbf{SH}_{\text{ét}}^{\wedge}(B_\eta; \Lambda)$, and it is enough to show that $\mathcal{T} \subset \mathcal{T}'$. Said differently, we need to show that $f_{\eta,*}\Lambda \in \mathcal{T}'$ for any proper morphism $f : X \to B$. We argue by induction on the dimension of X_η.

Given a dense open immersion $j : U \to X_\eta$, we have an equivalence

$$(f_{\eta,*}\Lambda \in \mathcal{T}') \Leftrightarrow (f_{\eta,*}j_!\Lambda \in \mathcal{T}') \tag{3.70}$$

by the induction hypothesis and the localisation property. Thus, given a proper morphism $e_1 : X_1 \to X$ such that $e_1^{-1}(U)$ is dense in $X_{1,\eta}$ and $e_1^{-1}(U) \simeq U$, we may replace X with X_1. Applying this to the normalisation of X, we reduce to the case where X is integral and X_η dense in X.

Now, let $e : X' \to X$, G, U and U' be as in (1)–(3) above. Set $f' = f \circ e$, and denote by $j : U \to X_\eta$ and $j' : U' \to X'_\eta$ the obvious inclusions. Then $f'_{\eta,*}\Lambda \in \mathcal{T}'$ by definition and $f'_{\eta,*}j'_!\Lambda \in \mathcal{T}'$ by the equivalence (3.70), for X' instead of X, which is also valid under the induction hypothesis since X'_η has the same dimension as X_η. Moreover, by the equivalence (3.70), we only need to show that $f_{\eta,*}j_!\Lambda \in \mathcal{T}'$. Since $\mathcal{T}'$ is closed under colimits, it is enough to show that

$$f_{\eta,*}j_!\Lambda \simeq \operatorname*{colim}_G f'_{\eta,*}j'_!\Lambda$$

where $f'_{\eta,*}j'_!\Lambda$ is endowed with the G-action induced from the action of G on X'. Let $u : U' \to U$ and $v : U'/G \to U$ be the obvious morphisms. Since e is proper, we have $f'_{\eta,*}j'_!\Lambda \simeq f_{\eta,*}j_!u_*\Lambda$. Since $f_{\eta,*}$ and $j_!$ commute with colimits, we have

$$\operatorname*{colim}_G f'_{\eta,*}j'_!\Lambda \simeq f_{\eta,*}j_!(\operatorname*{colim}_G u_*\Lambda).$$

Thus, we are left to show that $\Lambda \to \operatorname{colim}_G u_*\Lambda$ is an equivalence. By étale descent, we have $v_*\Lambda \simeq \operatorname{colim}_G u_*\Lambda$ and by Theorem 2.9.7 we have $\Lambda \simeq v_*\Lambda$. This finishes the proof. □

The following is a generalisation of [Ayo14a, Théorème 7.4].

Proposition 3.8.31. *Let S be a regular $(\Lambda, \text{ét})$-admissible scheme and assume that every prime number is invertible either in $\pi_0\Lambda$ or in $\mathcal{O}(S)$. Let*

$$\begin{tikzcd} T' \arrow[r, "s'"] \arrow[d, "t'"] & T \arrow[d, "t"] \\ S' \arrow[r, "s"] & S \end{tikzcd}$$

be a transversal square of closed immersions in the sense of [Ayo14a, Définition 7.2]. *Then, the morphism $s'^*t^!\Lambda \to t'^!s^*\Lambda$ is an equivalence in $\mathbf{SH}^\wedge_{\text{ét}}(T';\Lambda)$.*

Proof. More generally, given a Λ-module $M \in \mathrm{Mod}_\Lambda$, we will prove that $s'^*t^!M \to t'^!s^*M$ is an equivalence. Since the functors s^*, $t^!$, s'^* and t'^* are colimit-preserving, we may assume that M is compact. When Λ is the Eilenberg–Mac Lane spectrum associated to an ordinary ring, this is [Ayo14a, Théorème 7.4]. It follows that the proposition is known if $\pi_0\Lambda$ is a $\mathbb{Q}$-algebra or, said differently, if we replace M by $M_\mathbb{Q} = M \otimes \mathbb{Q}$. Thus, we are left to treat the case where M is ℓ-nilpotent for a prime ℓ invertible on S. We may apply Theorem 2.10.4 and work with the ∞-categories of étale sheaves $\mathrm{Shv}^\wedge_{\text{ét}}(\text{Ét}/(-);\Lambda)_{\ell\text{-nil}}$ instead of $\mathbf{SH}^\wedge_{\text{ét}}(-;\Lambda)$. We have equivalences

$$t^!M \simeq \lim_r t^!(M \otimes_\Lambda \tau_{\leq r}\Lambda) \qquad \text{and} \qquad t'^!M \simeq \lim_r t'^!(M \otimes_\Lambda \tau_{\leq r}\Lambda).$$

Since S is $(\Lambda, \text{ét})$-admissible and M is compact, Lemma 2.4.5 implies that the inverse system $(t^!(M \otimes_\Lambda \tau_{\leq r}\Lambda))_r$ in $\mathrm{Shv}^\wedge_{\text{ét}}(\text{Ét}/T;\Lambda)$ is eventually constant on homotopy sheaves. It follows that

$$s'^*t^!M \simeq \lim_r s'^*t^!(M \otimes_\Lambda \tau_{\leq r}\Lambda).$$

Thus, it is enough to prove that the maps

$$s'^* t^! (M \otimes_\Lambda \tau_{\leq r}\Lambda) \to t'^! s^* (M \otimes_\Lambda \tau_{\leq r}\Lambda)$$

are equivalences. Said differently, we may assume that Λ is eventually coconnective. By an easy induction, we reduce to the case where Λ is the Eilenberg–Mac Lane spectrum associated to $\mathbb{Z}/\ell$. (See the proof of Lemma 3.6.2.) In this case, the result is proven in [Ayo14a, Proposition 7.8] as a consequence of Gabber's absolute purity [ILO14, Exposé XVI, Théorème 3.1.1]. ◻

Corollary 3.8.32. *Let B be a $(\Lambda, \text{ét})$-admissible scheme, $B_\sigma \subset B$ a closed subscheme, and $B_\eta \subset B$ its open complement. Below, we use Notation 3.8.14.*

(1) Assume that B is regular and that B_σ is a regular subscheme of codimension c defined as the vanishing locus of a global regular sequence $a_1, \ldots, a_c \in \mathcal{O}(B)$. Then, we have equivalences

$$i^! \Lambda \simeq \Lambda(-c)[-2c] \qquad \textit{and} \qquad \chi_B \Lambda \simeq \Lambda \oplus \Lambda(-c)[-2c+1]$$

in $\mathbf{SH}^{\wedge}_{\text{ét}}(B_\sigma; \Lambda)$.

(2) Assume that B is regular and that B_σ is a strict normal crossing divisor. Let $D \subset B_\sigma$ be an irreducible component and D° the intersection of D with the regular locus of $(B_\sigma)_{\text{red}}$. Let $u : D^\circ \to D$ and $v : D \to B_\sigma$ be the obvious inclusions. The morphism

$$v^* \chi_B \Lambda \to u_* u^* v^* \chi_B \Lambda$$

is an equivalence in $\mathbf{SH}^{\wedge}_{\text{ét}}(D; \Lambda)$.

Proof. For the first assertion, we consider the commutative diagram with Cartesian squares

$$\begin{tikzcd}
B_\eta \arrow[r, "j"] \arrow[d, "a_\eta"] & B \arrow[d, "a"] & B_\sigma \arrow[l, "i"'] \arrow[d, "a_\sigma"] \\
\mathbb{A}^c_B \smallsetminus 0_B \arrow[r, "j_0"] & \mathbb{A}^c_B & B, \arrow[l, "i_0"']
\end{tikzcd}$$

where a is the section of $\mathbb{A}^c_B \to B$ induced by the c-tuple $(a_1, \ldots, a_c)$ and i_0 is the zero section. By Proposition 3.8.31, we have equivalences $i^! \Lambda \simeq a_\sigma^* i_0^! \Lambda$ and $\chi_B \Lambda \simeq a_\sigma^* \chi_{\mathbb{A}^c_B} \Lambda$, which enable us to conclude.

We now pass to the second assertion. Since the problem is local over B, we may assume that $(B_\sigma)_{\text{red}}$ is defined by an equation of the form $a_1 \cdots a_c = 0$, where $a_1, \ldots, a_c$ is a regular sequence. Consider the commutative diagram with Cartesian squares

$$\begin{tikzcd}
B_\eta \arrow[r, "j"] \arrow[d, "a_\eta"] & B \arrow[d, "a"] & B_\sigma \arrow[l, "i"'] \arrow[d, "a_\sigma"] \\
U \arrow[r, "j'"] & \mathbb{A}^c_B & E, \arrow[l, "i'"']
\end{tikzcd}$$

where E is defined by the equation $t_1 \cdots t_c = 0$, with $(t_1, \ldots, t_c)$ a system of coordinates on $\mathbb{A}^c$, and $U = \mathbb{A}^c_B \smallsetminus E$. For $I \subset \{1, \ldots, c\}$ nonempty, we let $D_I \subset B_\sigma$ and $H_I \subset E$ be the closed subschemes defined by the equations $\prod_{i\in I} a_i = 0$ and $\prod_{i\in I} t_i = 0$ respectively. We have transversal squares

$$\begin{tikzcd} D_I \arrow[r, "i_I"] \arrow[d, "a_I"] & B \arrow[d, "a"] \\ H_I \arrow[r, "i'_I"] & \mathbb{A}^c_B. \end{tikzcd}$$

By Proposition 3.8.31, we deduce equivalences $a_I^* i'^!_I \Lambda \simeq i^!_I \Lambda$. Since $i^! \Lambda$ and $i'^! \Lambda$ can be built from the $i^!_I \Lambda$'s and the $i'^!_I \Lambda$'s using the same recipe, we deduce that the obvious map $a_\sigma^* i'^! \Lambda \to i^! \Lambda$ is an equivalence. It follows that

$$a_\sigma^* \chi_{\mathbb{A}^c_B} \Lambda \to \chi_B \Lambda$$

is also an equivalence. We may assume that $D = D_1$. We set $H = H_1$ and define H° as in the statement. We also let $v' : H \to E$ and $u' : H^\circ \to H$ be the obvious inclusions. By [Ayo07b, Théorème 3.3.11], the obvious map

$$v'^* \chi_{\mathbb{A}^c_B} \Lambda \to u'_* u'^* v'^* \chi_{\mathbb{A}^c_B} \Lambda$$

is an equivalence. We have a commutative diagram

$$\begin{tikzcd} a_1^* v'^* \chi_{\mathbb{A}^c_B} \Lambda \arrow[r, "\sim"] \arrow[d, "\sim"] & v^* a_\sigma^* \chi_{\mathbb{A}^c_B} \Lambda \arrow[r, "\sim"] & v^* \chi_B \Lambda \arrow[d] \\ a_1^* u'_* u'^* v'^* \chi_{\mathbb{A}^c_B} \Lambda \arrow[r] & u_* u^* v^* a_\sigma^* \chi_{\mathbb{A}^c_B} \Lambda \arrow[r, "\sim"] & u_* u^* v^* \chi_B \Lambda. \end{tikzcd}$$

So, we are left to show that the morphism $a_1^* u'_* u'^* v'^* \chi_{\mathbb{A}^c_B} \Lambda \to u_* u^* v^* a_\sigma^* \chi_{\mathbb{A}^c_B} \Lambda$ is an equivalence. For this, we remark that $u'^* v'^* \chi_{\mathbb{A}^c_B} \Lambda \simeq \Lambda \oplus \Lambda(-1)[-1]$ in $\mathbf{SH}^\wedge_{\text{ét}}(H^\circ; \Lambda)$, and that this morphism is equivalent to

$$a_1^* u'_* (\Lambda \oplus \Lambda(-1)[-1]) \to u_* (\Lambda \oplus \Lambda(-1)[-1]).$$

Thus, it remains to show that $b^* z'^! \Lambda \to z^! \Lambda$ is an equivalence, with $z : D \smallsetminus D^\circ \to D$, $z' : H \smallsetminus H^\circ \to H$ and $b : D \smallsetminus D^\circ \to H \smallsetminus H^\circ$ the obvious morphisms. This is proven in the same way we proved above that $a_\sigma^* i'^! \Lambda \to i^! \Lambda$ was an equivalence. □

We are finally ready to conclude.

Proof of Theorem 3.8.19. By Lemma 3.8.20, we may assume that B is essentially of finite type over $\mathrm{Spec}(\mathbb{Z})$ and work in the hypercomplete case. Since the source and target of ρ_X consist of colimit-preserving functors, it is enough to prove that $\chi_X M \to \chi_{X^{\mathrm{an}}} \mathrm{An}^*_{X_\eta} M$ is an equivalence when M belongs to set of compact generators of $\mathbf{SH}^\wedge_{\text{ét}}(X_\eta; \Lambda)$. By Proposition 3.8.30, we may assume that $M = f_{\eta,*}\Lambda$ where $f : Y \to X$ is a proper morphism such that Y is regular and Y_σ is a normal crossing divisor. By the proper base change theorem, we have equivalences

$$\chi_X f_{\eta,*} \Lambda \simeq f_{\sigma,*} \chi_Y \Lambda \qquad \text{and} \qquad \chi_{X^{\mathrm{an}}} \mathrm{An}^*_{X_\eta} f_{\eta,*} \Lambda \simeq f_{\sigma,*} \chi_{Y^{\mathrm{an}}} \mathrm{An}^*_{Y_\eta} \Lambda \simeq f_{\sigma,*} \chi_{Y^{\mathrm{an}}} \Lambda.$$

Thus, replacing X with Y, we may assume that X is regular and X_σ a strict normal crossing divisor and, in this case, we only need to show that $\chi_X \Lambda \to \chi_{X^{\mathrm{an}}} \Lambda$ is an equivalence. By Proposition 3.8.21, this morphism admits a section, and thus $\chi_{X^{\mathrm{an}}} \Lambda$ is the image of a projector p of $\chi_X \Lambda$. We need to prove that p is the identity, and it is enough to show this after restriction to each irreducible component of X_σ. Using Corollary 3.8.32(2), it is enough to do so after restricting to the regular locus of X_σ. Said differently, we may assume that X_σ is a regular divisor.

From now on, we assume that X is regular and that X_σ is a regular divisor defined by the zero locus of $a \in \mathcal{O}(X)$. We denote by p the projector of χ_X provided by Proposition 3.8.21. Our goal is to show that p acts on $\chi_X\Lambda \simeq \Lambda \oplus \Lambda(-1)[-1]$ by the identity, and it is enough to show that p is an equivalence. First, note that we have a commutative square

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d, equal] & \chi_X\Lambda \arrow[d, "p"] \\ \Lambda \arrow[r] & \chi_X\Lambda \end{tikzcd}$$

since p is an algebra endomorphism of $\chi_X\Lambda$. (Indeed, the section constructed in Proposition 3.8.21 respects the natural right-lax monoidal structures.) Thus, with respect to the decomposition $\chi_X\Lambda \simeq \Lambda \oplus \Lambda(-1)[-1]$, p is given by a triangular matrix

$$p = \begin{pmatrix} 1 & r \\ 0 & q \end{pmatrix}.$$

We will show that q is the identity of $\Lambda(-1)[-1]$. To do so, we consider the morphism $\Lambda \to \Lambda(1)[1]$ in $\mathbf{SH}^{\wedge}_{\text{ét}}(X_\eta;\Lambda)$ corresponding to $a \in \mathcal{O}^\times(X_\eta)$, i.e., induced by the section $a : X_\eta \to \mathbb{A}^1_{X_\eta} \smallsetminus 0_{X_\eta}$. Applying χ_X and then $p : \chi_X \to \chi_X$ yields a commutative square

$$\begin{tikzcd} \Lambda \oplus \Lambda(-1)[-1] \arrow[r, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & r \\ 0 & q \end{smallmatrix}\right)"] & \Lambda(1)[1] \oplus \Lambda \arrow[d, "\left(\begin{smallmatrix} 1 & r \\ 0 & q \end{smallmatrix}\right)"] \\ \Lambda \oplus \Lambda(-1)[-1] \arrow[r, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"] & \Lambda(1)[1] \oplus \Lambda. \end{tikzcd}$$

This forces q to be the identity, as needed. $\square$

4. The six-functor formalism for rigid analytic motives

In this section, we develop the six-functor formalism for rigid analytic motives, getting rid of the quasi-projectivity assumption imposed in [Ayo15, §1.4]. The key step in doing so is to prove an extended proper base change theorem for rigid analytic motives; see Theorem 4.1.4 below. An important particularity in the rigid analytic setting is the existence of canonical compactifications (aka., Huber compactifications). We will not make use of these compactifications in defining the exceptional direct image functors, but see Theorem 4.3.20 below.

4.1. **Extended proper base change theorem.**

Our goal in this subsection is to prove a general extended proper base change theorem for rigid analytic motives; see Theorem 4.1.4 below. This will be achieved by reducing to the usual proper base change theorem for algebraic motives. A compatibility property for the functors $\chi_{\mathcal{S}}$, for $\mathcal{S} \in \mathrm{FSch}$, and the operations $f_\sharp$, for f smooth, plays a key role in this reduction; it is given in Theorem 4.1.3 below which we deduce quite easily from Theorem 3.6.1 (which was a key step in proving Theorem 3.3.3). We start by a well-known generalisation of some facts contained in [Ayo07a, Scholie 1.4.1].

Proposition 4.1.1. *Consider a Cartesian square in* FSch

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "g'"] \arrow[d, "f'"] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X}' \arrow[r, "g"] & \mathcal{X} \end{tikzcd}$$

with f proper.

(1) The commutative square

$$\begin{tikzcd} \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{X};\Lambda) \arrow[r, "f^*"] \arrow[d, "g^*"] & \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{Y};\Lambda) \arrow[d, "g'^*"] \\ \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{X}';\Lambda) \arrow[r, "f'^*"] & \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{Y}';\Lambda) \end{tikzcd}$$

is right adjointable, i.e., the natural transformation $g^* \circ f_* \to f'_* \circ g'^*$ is an equivalence.

(2) If g is smooth, the commutative square

$$\begin{tikzcd} \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{X}';\Lambda) \arrow[r, "f'^*"] \arrow[d, "g_\sharp"] & \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{Y}';\Lambda) \arrow[d, "g'_\sharp"] \\ \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{X};\Lambda) \arrow[r, "f^*"] & \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{Y};\Lambda) \end{tikzcd}$$

is right adjointable, i.e., the natural transformation $g_\sharp \circ f'_* \to f_* \circ g'_\sharp$ is an equivalence.

Proof. By Theorem 3.1.10, we reduce to show the statement for a Cartesian square in Sch

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

with f proper. When f is projective, this is covered by [Ayo07a, Scholie 1.4.1]; see also [Ayo14a, Proposition 3.5]. The passage from the projective to the proper case is a well-known procedure, that we revisit here because we don't know a reference in the generality we are considering. (Under noetherianness assumptions, an argument can be found in the proof of [CD19, Proposition 2.3.11(2)].)

The question is local on X, so we may assume that X is quasi-compact and quasi-separated. Using a covering of Y by finitely many affine open subschemes, assertion (1) (resp. assertion (2)) follows if we can prove that the natural transformation

$$g^* \circ f_* \circ v_\sharp \to f'_* \circ g'^* \circ v_\sharp \qquad (\text{resp. } g_\sharp \circ f'_* \circ v'_\sharp \to f_* \circ g'_\sharp \circ v'_\sharp)$$

is an equivalence for every open immersion $v : V \to Y$ with base change $v' : V' \to Y'$. Letting $g'' : V' \to V$ be the base change of g', this natural transformation can be rewritten as follows:

$$g^* \circ (f_* \circ v_\sharp) \to (f'_* \circ v'_\sharp) \circ g''^* \qquad (\text{resp. } g_\sharp \circ (f'_* \circ v'_\sharp) \to (f_* \circ v_\sharp) \circ g''_\sharp).$$

By the refined version of Chow's lemma given in [Con07, Corollary 2.6], we may find a blowup $e : Z \to Y$, with centre disjoint from V, such that $h = f \circ e$ is a projective morphism. Let $w : V \to Z$ be the open immersion such that $v = e \circ w$. Set $Z' = Z \times_Y Y'$ and let $e' : Z' \to Y'$, $h' : Z' \to X'$

and $w' : V' \to Z'$ be the base change of e, h and w along g. Using [Ayo07a, Scholie 1.4.1], we have natural equivalences $v_\sharp \simeq e_* \circ w_\sharp$ and $v'_\sharp \simeq e'_* \circ w'_\sharp$. Thus, we may rewrite the above natural transformation as follows:

$$g^* \circ (h_* \circ w_\sharp) \to (h'_* \circ w'_\sharp) \circ g''^* \qquad (\text{resp. } g_\sharp \circ (h'_* \circ w'_\sharp) \to (h_* \circ w_\sharp) \circ g''_\sharp).$$

Thus, we may replace f and f' by h and h', thereby reducing the general case to the case of a projective morphism. □

Lemma 4.1.2. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a proper morphism of formal schemes. Then, the functor*

$$f_* : \mathbf{FSH}^{(\wedge)}_\tau(\mathcal{Y};\Lambda) \to \mathbf{FSH}^{(\wedge)}_\tau(\mathcal{X};\Lambda)$$

is colimit-preserving and thus admits a right adjoint.

Proof. By Theorem 3.1.10, we reduce to show the statement for a proper morphism of schemes $f : Y \to X$. When f is projective, this follows from [Ayo07a, Théorème 1.7.17]. In general, we may assume that X is quasi-compact and quasi-separated, and reduce to show that $f_* \circ v_\sharp$ is colimit-preserving for every open immersion $v : V \to Y$ with V affine. Then, we use the refined version of Chow's lemma given in [Con07, Corollary 2.6], to find a blowup $Y' \to Y$ with centre disjoint from V and such that $Y' \to X$ is projective. We conclude using the equivalence $f_* \circ v_\sharp \simeq f'_* \circ v'_\sharp$ where $f' : Y' \to X$ and $v' : V \to Y'$ are the obvious morphisms. □

Our main task in this subsection is to prove a variant of Proposition 4.1.1 for rigid analytic motives. (A version of Proposition 4.1.1(a) holds true in the rigid analytic setting even without assuming that f is proper but under some mild technical assumptions; see Theorem 2.7.1. We will explain below how to remove these technical assumptions when f is assumed to be proper.) A key ingredient is provided by the following theorem.

Theorem 4.1.3. *We work under Assumption 3.3.1. Let $f : \mathcal{T} \to \mathcal{S}$ be a smooth morphism of formal schemes. The commutative square*

$$\begin{tikzcd}
\mathbf{FSH}^{(\wedge)}_\tau(\mathcal{T};\Lambda) \arrow[r, "\xi_\mathcal{T}"] \arrow[d, "f_\sharp"] & \mathbf{RigSH}^{(\wedge)}_\tau(\mathcal{T}^{\mathrm{rig}};\Lambda) \arrow[d, "f^{\mathrm{rig}}_\sharp"] \\
\mathbf{FSH}^{(\wedge)}_\tau(\mathcal{S};\Lambda) \arrow[r, "\xi_\mathcal{S}"] & \mathbf{RigSH}^{(\wedge)}_\tau(\mathcal{S}^{\mathrm{rig}};\Lambda)
\end{tikzcd}$$

is right adjointable, i.e., the induced natural transformation $f_\sharp \circ \chi_\mathcal{T} \to \chi_\mathcal{S} \circ f^{\mathrm{rig}}_\sharp$ is an equivalence.

Proof. We split the proof into two steps. In the first one, we consider the case where f is an open immersion and, in the second one, we treat the general case.

Step 1. Here we treat the case of an open immersion $j : \mathcal{U} \to \mathcal{S}$. For $M \in \mathbf{RigSH}^{(\wedge)}_\tau(\mathcal{S}^{\mathrm{rig}};\Lambda)$, we have a commutative diagram

$$\begin{tikzcd}
\chi_\mathcal{S}(M) \otimes j_\sharp\Lambda \arrow[r, "(1)"] \arrow[d, "\sim"] & \chi_\mathcal{S}(M \otimes \xi_\mathcal{S} j_\sharp\Lambda) \arrow[r, "\sim"] & \chi_\mathcal{S}(M \otimes j^{\mathrm{rig}}_\sharp\Lambda) \arrow[d, "\sim"] \\
j_\sharp j^* \chi_\mathcal{S} M \arrow[r, "\sim"] & j_\sharp \chi_\mathcal{U}\, j^{\mathrm{rig},*}M \arrow[r, "(2)"] & \chi_\mathcal{S} j^{\mathrm{rig}}_\sharp j^{\mathrm{rig},*}M,
\end{tikzcd}$$

where all the arrows, except the labeled ones, are equivalences for obvious reasons. By Theorem 3.6.1, the morphism (1) is also an equivalence, and hence the same is true for the morphism (2). Thus, the natural transformation $j_\sharp \circ \chi_{\mathcal{U}} \to \chi_{\mathcal{S}} \circ j_\sharp^{\mathrm{rig}}$ becomes an equivalence when applied to the functor $j^{\mathrm{rig},*}$. Since the latter is essentially surjective, the result follows.

Step 2. Here we treat the general case. Clearly, the problem is local on $\mathcal{S}$. We claim that it is also local on $\mathcal{T}$. Indeed, let $(u_i : \mathcal{T}_i \to \mathcal{T})_i$ be an open covering of $\mathcal{T}$. The ∞-category $\mathbf{RigSH}_\tau^{(\wedge)}(\mathcal{T}^{\mathrm{rig}}; \Lambda)$ is generated under colimits by the images of the functors $u_{i,\sharp}^{\mathrm{rig}}$. Clearly, the functors $f_\sharp$ and $f_\sharp^{\mathrm{rig}}$ are colimit-preserving. By Proposition 3.6.8, the same is true for $\chi_{\mathcal{T}}$ and $\chi_{\mathcal{S}}$. Thus, it is enough to prove that the natural transformations $f_\sharp \circ \chi_{\mathcal{T}} \circ u_{i,\sharp}^{\mathrm{rig}} \to \chi_{\mathcal{S}} \circ f_\sharp^{\mathrm{rig}} \circ u_{i,\sharp}^{\mathrm{rig}}$ are equivalences. Using the first step, this natural transformation is equivalent to $(f \circ u_i)_\sharp \circ \chi_{\mathcal{T}_i} \to \chi_{\mathcal{S}} \circ (f \circ u_i)_\sharp^{\mathrm{rig}}$ which brings us to prove the theorem for the morphisms $f \circ u_i$. This proves our claim.

The problem being local on $\mathcal{T}$ and $\mathcal{S}$, we may assume that there is a closed immersion $i : \mathcal{T} \to \mathbb{A}^n_{\mathcal{S}}$. We may also assume that there is an étale neighbourhood of $\mathcal{T}$ in $\mathbb{A}^n_{\mathcal{S}}$ which is isomorphic to an étale neighbourhood of the zero section $\mathcal{T} \to \mathbb{A}^m_{\mathcal{T}}$ (where m is the codimension of the immersion i). Thus, letting $p : \mathbb{A}^n_{\mathcal{S}} \to \mathcal{S}$ be the obvious projection, we have natural equivalences

$$p_\sharp \circ i_* \simeq f_\sharp(m)[2m] \qquad \text{and} \qquad p_\sharp^{\mathrm{rig}} \circ i_*^{\mathrm{rig}} \simeq f_\sharp^{\mathrm{rig}}(m)[2m].$$

Moreover, the following diagram is commutative

$$\begin{tikzcd}
p_\sharp \circ i_* \circ \chi_{\mathcal{T}} \arrow[r, "\sim"] \arrow[d, "\sim"] & p_\sharp \circ \chi_{\mathbb{A}^n_{\mathcal{S}}} \circ i_*^{\mathrm{rig}} \arrow[r] & \chi_{\mathcal{S}} \circ p_\sharp^{\mathrm{rig}} \circ i_*^{\mathrm{rig}} \arrow[d, "\sim"] \\
f_\sharp \circ \chi_{\mathcal{T}}(m)[2m] \arrow[rr] & & \chi_{\mathcal{S}} \circ f_\sharp^{\mathrm{rig}}(m)[2m].
\end{tikzcd}$$

This shows that it suffices to treat the case of the projection $p : \mathbb{A}^n_{\mathcal{S}} \to \mathcal{S}$.

Let $j : \mathbb{A}^n_{\mathcal{S}} \to \mathbb{P}^n_{\mathcal{S}}$ be an open immersion into the relative projective space of dimension n and let $q : \mathbb{P}^n_{\mathcal{S}} \to \mathcal{S}$ be the obvious projection. The morphism $p_\sharp \circ \chi_{\mathbb{A}^n_{\mathcal{S}}} \to \chi_{\mathcal{S}} \circ p_\sharp^{\mathrm{rig}}$ is equivalent to the composition of

$$q_\sharp \circ j_\sharp \circ \chi_{\mathbb{A}^n_{\mathcal{S}}} \to q_\sharp \circ \chi_{\mathbb{P}^n_{\mathcal{S}}} \circ j_\sharp^{\mathrm{rig}} \to \chi_{\mathcal{S}} \circ q_\sharp^{\mathrm{rig}} \circ j_\sharp^{\mathrm{rig}}$$

and the first morphism is an equivalence by the first step. Thus, we are left to treat the case of $q : \mathbb{P}^n_{\mathcal{S}} \to \mathcal{S}$. By [Ayo07a, Théorème 1.7.17] and Corollary 2.2.9, we have equivalences

$$q_\sharp \simeq q_* \circ \mathrm{Th}(\Omega_q) \qquad \text{and} \qquad q_\sharp^{\mathrm{rig}} \simeq q_*^{\mathrm{rig}} \circ \mathrm{Th}(\Omega_{q^{\mathrm{rig}}}),$$

and the following square

$$\begin{tikzcd}
q_\sharp \circ \chi_{\mathbb{P}^n_{\mathcal{S}}} \arrow[r, "\sim"] \arrow[d] & q_* \circ \mathrm{Th}(\Omega_q) \circ \chi_{\mathbb{P}^n_{\mathcal{S}}} \arrow[d, "\sim"] \\
\chi_{\mathcal{S}} \circ q_\sharp^{\mathrm{rig}} \arrow[r, "\sim"] & \chi_{\mathcal{S}} \circ q_*^{\mathrm{rig}} \circ \mathrm{Th}(\Omega_{q^{\mathrm{rig}}})
\end{tikzcd}$$

is commutative. This finishes the proof. □

Here is the main result of this subsection.

Theorem 4.1.4 (Extended proper base change)**.** *Consider a Cartesian square in* RigSpc

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

with f proper.

(1) The commutative square

$$\begin{tikzcd} \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda) \arrow[r, "f^*"] \arrow[d, "g^*"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda) \arrow[d, "g'^*"] \\ \mathbf{RigSH}^{(\wedge)}_{\tau}(X';\Lambda) \arrow[r, "f'^*"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(Y';\Lambda) \end{tikzcd}$$

is right adjointable, i.e., the natural transformation $g^* \circ f_* \to f'_* \circ g'^*$ is an equivalence.

(2) If g is smooth, the commutative square

$$\begin{tikzcd} \mathbf{RigSH}^{(\wedge)}_{\tau}(X';\Lambda) \arrow[r, "f'^*"] \arrow[d, "g_\sharp"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(Y';\Lambda) \arrow[d, "g'_\sharp"] \\ \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda) \arrow[r, "f^*"] & \mathbf{FSH}^{(\wedge)}_{\tau}(Y;\Lambda) \end{tikzcd}$$

is right adjointable, i.e., the natural transformation $g_\sharp \circ f'_* \to f_* \circ g'_\sharp$ is an equivalence.

Proof. The question is local on X and X'. Thus, we may assume that X and X' are quasi-compact and quasi-separated. We split the proof into three steps. The first two steps concern part (2): in the first step we show that it is enough to treat the case where g has good reduction, and in the second step we prove part (2) while working in the non-hypercomplete case and assuming that τ is the Nisnevich topology. Finally, in the third step, we use what we learned in the second step to prove the theorem in complete generality.

Step 1. Here, we assume that part (2) is known when g has good reduction and we explain how to deduce it in general. The problem being local on X', we may assume that our Cartesian square is the composition of two Cartesian squares

$$\begin{tikzcd} Y' \arrow[r, "e'"] \arrow[d, "f'"] & Y_1 \arrow[r, "h'"] \arrow[d, "f_1"] & Y \arrow[d, "f"] \\ X' \arrow[r, "e"] & X_1 \arrow[r, "h"] & X \end{tikzcd}$$

where e is étale and h is smooth with good reduction. (For instance, we may assume that h is the projection of a relative ball.) By assumption, part (2) is known for the right square, so it remains to prove it for the left square. Said differently, we may assume that g is étale. Using Lemma 4.1.5 below, we reduce further to the case where g is finite étale. In this case, there is a natural equivalence $g_\sharp \simeq g_*$ constructed as follows. Consider the Cartesian square

$$\begin{tikzcd} X' \times_X X' \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X' \arrow[d, "g"] \\ X' \arrow[r, "g"] & X, \end{tikzcd}$$

and the diagonal embedding $\Delta : X' \to X' \times_X X'$ which is a clopen immersion. Since g is locally projective, we may use Proposition 2.2.12(2) which implies that the natural transformation

$$g_\sharp \circ \mathrm{pr}_{1,*} \to g_* \circ \mathrm{pr}_{2,\sharp}$$

is an equivalence. Applying this equivalence to the functor $\Delta_\sharp \simeq \Delta_*$, we get the equivalence $g_\sharp \simeq g_*$. Similarly, we have an equivalence $g'_\sharp \simeq g'_*$. Moreover, modulo these equivalences, the natural transformation $g_\sharp \circ f'_* \to f_* \circ g'_\sharp$ coincides with the obvious equivalence $g_* \circ f'_* \simeq f_* \circ g'_*$. This proves the claimed reduction.

Step 2. We now prove part (2) of the statement under Assumption 3.3.1 so that we can use Theorem 4.1.3. (More precisely, we will assume that all the formal models used below satisfy this assumption.) In the third step we explain how to get rid of this assumption.

The problem being local on X and X', we may also assume that f is the generic fiber of a proper morphism $\widetilde{f} : \mathcal{Y} \to \mathcal{X}$ in FSch and that g is the generic fiber of a smooth morphism $\widetilde{g} : \mathcal{X}' \to \mathcal{X}$ of formal schemes (since g can be assumed to have good reduction, by the first step). We form a Cartesian square

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "\widetilde{g}'"] \arrow[d, "\widetilde{f}'"] & \mathcal{Y} \arrow[d, "\widetilde{f}"] \\ \mathcal{X}' \arrow[r, "\widetilde{g}"] & \mathcal{X}. \end{tikzcd}$$

For every quasi-compact and quasi-separated smooth rigid analytic X-space L, with structural morphism $p_L : L \to X$, choose a formal model $\mathcal{L}$ which is a finite type formal $\mathcal{X}$-scheme. By Proposition 3.1.15, when L varies, the functors

$$\chi_{\mathcal{L}} \circ p_L^* : \mathbf{RigSH}_\tau^{(\wedge)}(X; \Lambda) \to \mathbf{FSH}_\tau^{(\wedge)}(\mathcal{L}; \Lambda)$$

form a conservative family. Therefore, it is enough to show that the natural transformation

$$\chi_{\mathcal{L}} \circ p_L^* \circ g_\sharp \circ f'_* \to \chi_{\mathcal{L}} \circ p_L^* \circ f_* \circ g'_\sharp$$

is an equivalence for each $p_L : L \to X$ and $\mathcal{L}$ as above. Letting f_L, f'_L, g_L and g'_L be the base change of the morphisms f, f', g and g' along $p_L : L \to X$, and using Proposition 2.2.1, we reduce to show that the natural transformation

$$\chi_{\mathcal{L}} \circ g_{L,\sharp} \circ f'_{L,*} \to \chi_{\mathcal{L}} \circ f_{L,*} \circ g'_{L,\sharp}$$

is an equivalence. Thus, replacing X with L and $\mathcal{X}$ with $\mathcal{L}$, we may concentrate on the natural transformation

$$\chi_{\mathcal{X}} \circ g_\sharp \circ f'_* \to \chi_{\mathcal{X}} \circ f_* \circ g'_\sharp.$$

Using Theorem 4.1.3, we can rewrite this natural transformation as follows:

$$\widetilde{g}_\sharp \circ \widetilde{f}'_* \circ \chi_{\mathcal{Y}'} \to \widetilde{f}_* \circ \widetilde{g}'_\sharp \circ \chi_{\mathcal{Y}'}.$$

We now conclude using Proposition 4.1.1(2).

Step 3. In this step, we will prove the theorem in complete generality. By Theorem 2.7.1 and the second step, the theorem is known for the ∞-categories $\mathbf{RigSH}_{\mathrm{nis}}(-;\Lambda)$, i.e., when τ is the Nisnevich topology and we work in the non-hypercomplete case. This will be our starting point. (Of course, by the second step, the theorem is known more generally, e.g., when τ is the Nisnevich topology and we work in the hypercomplete case, but this will not be used below.)

For a rigid analytic space S, the functor $\mathrm{L}_S : \mathbf{RigSH}_{\mathrm{nis}}(S;\Lambda) \to \mathbf{RigSH}_{\tau}^{(\wedge)}(S;\Lambda)$ is a localisation functor with respect to the set $\mathcal{H}_S$ consisting of maps of the form $\operatorname{colim}_{[n]\in\mathbf{\Delta}} \mathrm{M}(T_n) \to \mathrm{M}(T_{-1})$, and their desuspensions and negative Tate twists, where $T_\bullet$ is a τ-hypercover which is assumed to be truncated in the non-hypercomplete case. We claim that the functor

$$f_* : \mathbf{RigSH}_{\mathrm{nis}}(Y;\Lambda) \to \mathbf{RigSH}_{\mathrm{nis}}(X;\Lambda)$$

takes $\mathcal{H}_Y$-equivalences to $\mathcal{H}_X$-equivalences, and that the same is true for f'_*. Assuming this claim, one has equivalences $\mathrm{L}_X \circ f_* \simeq f_* \circ \mathrm{L}_Y$, and similarly for f'_*. Since the functors L_Y and $\mathrm{L}_{Y'}$ are essentially surjective on objects, it suffices to prove that the natural transformations

$$g^* \circ f_* \circ \mathrm{L}_Y \to f'_* \circ g'^* \circ \mathrm{L}_Y \quad \text{and} \quad g_\sharp \circ f'_* \circ \mathrm{L}_{Y'} \to f_* \circ g'_\sharp \circ \mathrm{L}_{Y'}$$

are equivalences. Thus, using our claim and the obvious analogous commutations for g^*, $g_\sharp$, g'^* and $g'_\sharp$, the above natural transformations are equivalent to

$$\mathrm{L}_{X'} \circ g^* \circ f_* \to \mathrm{L}_{X'} \circ f'_* \circ g'^* \quad \text{and} \quad \mathrm{L}_X \circ g_\sharp \circ f'_* \to \mathrm{L}_X \circ f_* \circ g'_\sharp,$$

and the result follows.

It remains to prove our claim, and it is enough to consider the case of f (which is a general proper morphism). Using a covering of Y by finitely many affine open subspaces, we see that it suffices to show that $f_* \circ v_\sharp$ takes $\mathcal{H}_V$-equivalences to $\mathcal{H}_X$-equivalences for every open immersion $v : V \to Y$ such that V admits a locally closed immersion into a relative projective space $P \simeq \mathbb{P}^n_X$ over X. (For what we mean by a locally closed immersion, see Definition 1.1.14. For the existence of a cover by open subspaces with the required property, see the proof of Proposition 4.2.2(2) below.) Let $U \subset P$ be an open subspace containing V as a closed subset. Set $Q = Y \times_X P$, $W = V \times_X U$, $W_1 = V \times_X P$ and $W_2 = Y \times_X U$. Thus, Q is a proper rigid analytic X-space, and W, W_1 and W_2 are open subspaces of Q containing Y, via the diagonal embedding $Y \to Q$, as a closed subset. We have a commutative diagram of immersions with Cartesian squares

$$\begin{tikzcd}
V \arrow[r, equal] \arrow[d, equal] \arrow[dr, "t"] & V \arrow[dr, "t_2"] & \\
V \arrow[dr, "t_1"'] & W \arrow[r, "e_2"] \arrow[d, "e_1"] \arrow[dr, "w"] & W_2 \arrow[d, "w_2"] \\
& W_1 \arrow[r, "w_1"] & Q.
\end{tikzcd}$$

Using Proposition 2.2.3(4), we obtain equivalences $e_{1,\sharp} \circ t_* \simeq t_{1,*}$ and $e_{2,\sharp} \circ t_* \simeq t_{2,*}$. Applying this to $w_{1,\sharp}$ and $w_{2,\sharp}$, we obtain equivalences

$$w_{1,\sharp} \circ t_{1,*} \simeq w_\sharp \circ t_* \simeq w_{2,\sharp} \circ t_{2,*}. \tag{4.1}$$

Now, consider the commutative diagram with a Cartesian square

$$\begin{tikzcd} V \arrow[r, "t_1"] \arrow[dr, equal] & W_1 \arrow[r, "w_1"] \arrow[d, "q'"] & Q \arrow[d, "q"] \\ & V \arrow[r, "v"] & Y. \end{tikzcd}$$

By the second step, we deduce equivalences of functors from $\mathbf{RigSH}_{\mathrm{nis}}(V;\Lambda)$ to $\mathbf{RigSH}_{\mathrm{nis}}(Y;\Lambda)$:

$$v_\sharp \simeq v_\sharp \circ q'_* \circ t_{1,*} \simeq q_* \circ w_{1,\sharp} \circ t_{1,*} \simeq q_* \circ w_\sharp \circ t_*.$$

Thus, it will be enough to show that the functor $f_* \circ q_* \circ w_\sharp \circ t_*$ takes $\mathcal{H}_V$-equivalences to $\mathcal{H}_X$-equivalences. Next, consider the commutative diagram with Cartesian squares

$$\begin{tikzcd} V \arrow[r, "t_2"] \arrow[d, equal] & W_2 \arrow[r, "w_2"] \arrow[d, "h'"] & Q \arrow[d, "h"] \\ V \arrow[r, "s"] & U \arrow[r, "u"] & P. \end{tikzcd}$$

By the second step, we we deduce equivalences of functors from $\mathbf{RigSH}_{\mathrm{nis}}(V;\Lambda)$ to $\mathbf{RigSH}_{\mathrm{nis}}(Y;\Lambda)$:

$$u_\sharp \circ s_* \simeq u_\sharp \circ h'_* \circ t_{2,*} \simeq h_* \circ w_{2,\sharp} \circ t_{2,*} \simeq h_* \circ w_\sharp \circ t_*.$$

Since $p \circ h = f \circ q$ with $p : P \to X$ the structural projection of the relative projective space P, we are left to show that $p_* \circ u_\sharp \circ s_*$ takes $\mathcal{H}_V$-equivalences to $\mathcal{H}_X$-equivalences. This is actually true for each of the functors p_*, $u_\sharp$ and s_*. For the first one, we use the equivalence $p_* \simeq p_\sharp \circ \mathrm{Th}^{-1}(\Omega_p)$ provided by Corollary 2.2.9. For the second one, this is clear, and for the third one, this follows from Lemma 2.2.4. □

The following lemma was used in the first step of the proof of Theorem 4.1.4.

Lemma 4.1.5. *Let $f : T \to S$ be an étale morphism of rigid analytic spaces. Then, locally on S and T, we may find a commutative triangle*

$$\begin{tikzcd} T \arrow[r, "j"] \arrow[dr, "f"'] & T' \arrow[d, "f'"] \\ & S \end{tikzcd}$$

where j is an open immersion and f' is a finite étale morphism.

Proof. This is a well-known fact. In the generality we are considering here, it can be proven by adapting the argument used in proving Proposition 3.7.6(3). More precisely, it is enough to show that a rig-étale morphism of formal schemes $f : \mathcal{T} \to \mathcal{S}$ is locally, for the rig topology on $\mathcal{S}$ and $\mathcal{T}$, the composition of an open immersion and a finite rig-étale morphism. We argue locally around a rigid point $\mathfrak{s} : \mathrm{Spf}(V) \to \mathcal{S}$ corresponding to $s \in |\mathcal{S}^{\mathrm{rig}}|$. As in the proof of Proposition 3.7.6(3), we may assume that the formal scheme $\mathfrak{s} \times_{\mathcal{S}} \mathcal{T}/(0)^{\mathrm{sat}}$ is the formal spectrum of the π-adic completion of an algebra of the form

$$V\langle t, s_1, \ldots, s_m\rangle/(R, \pi^N s_1 - P_1, \ldots, \pi^N s_m - P_m)^{\mathrm{sat}}[Q^{-1}] \tag{4.2}$$

where $R \in V[t]$ is a monic polynomial which is separable over $V[\pi^{-1}]$, and $Q \in V[t, s_1, \ldots, s_m]$. (The polynomial R is the analogue of the polynomial $(t - a_1) \cdots (t - a_r)$ in (3.34). Here, since $V[\pi^{-1}]$ is not algebraically closed, our polynomial R will not split in general.) The remainder of the argument is identical to the one used in the proof of Proposition 3.7.6(3). □

The following is a corollary of the proof of Theorem 4.1.4.

Corollary 4.1.6. *Let $f : Y \to X$ be a proper morphism of rigid analytic spaces. Then, the functor*

$$f_* : \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda) \to \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)$$

is colimit-preserving and thus admits a right adjoint.

Proof. This is true for the functor

$$f_* : \mathbf{RigSH}_{\mathrm{nis}}(Y;\Lambda) \to \mathbf{RigSH}_{\mathrm{nis}}(X;\Lambda)$$

by Proposition 2.4.22(1). The result in general follows from the fact that this functor takes $\mathcal{H}_Y$-equivalences to $\mathcal{H}_X$-equivalences as shown in the third step of the proof of Theorem 4.1.4. □

We end this subsection by establishing the projection formula for direct images along proper morphisms.

Proposition 4.1.7.

(1) *Let $f : \mathcal{Y} \to \mathcal{X}$ be a proper morphism of formal schemes. For $M \in \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{X};\Lambda)$ and $N \in \mathbf{FSH}^{(\wedge)}_{\tau}(\mathcal{Y};\Lambda)$, the morphism*

$$M \otimes f_*N \to f_*(f^*M \otimes N)$$

is an equivalence.

(2) *Let $f : Y \to X$ be a proper morphism of rigid analytic spaces. For $M \in \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)$ and $N \in \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)$, the morphism*

$$M \otimes f_*N \to f_*(f^*M \otimes N)$$

is an equivalence.

Proof. We only prove the second part. The proof of the first part is similar: in the argument below, use Proposition 4.1.1 and Lemma 4.1.2 instead of Theorem 4.1.4, and Corollary 4.1.6.

The functor f_* is colimit-preserving by Corollary 4.1.6. Hence, it is enough to prove the result when M varies in a set of generators under colimits for the ∞-category $\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)$. Thus, we may assume that $M = g_\sharp\Lambda$ where $g : X' \to X$ is a smooth morphism. We form the Cartesian square

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X. \end{tikzcd}$$

By Proposition 2.2.1(2), we have natural equivalences

$$M \otimes (-) \simeq g_\sharp \circ g^*(-) \qquad \text{and} \qquad (f^*M) \otimes (-) \simeq g'_\sharp \circ g'^*(-).$$

Modulo these equivalences, the morphism of the statement is the composition of

$$g_\sharp g^* f_* N \to g_\sharp f'_* g'^* N \to f_* g'_\sharp g'^* N.$$

The result follows now from Theorem 4.1.4. □

Recall that an object in a monoidal ∞-category $\mathcal{C}^\otimes$ is strongly dualisable if it is so as an object of the homotopy category of $\mathcal{C}$ endowed with the induced monoidal structure. The following is a well-known consequence of the projection formula for proper direct images.

Corollary 4.1.8.

(1) Let $f : \mathcal{Y} \to \mathcal{X}$ be a smooth and proper morphism of formal schemes. Then $f_\sharp\Lambda$ is strongly dualisable in the monoidal ∞-category $\mathbf{FSH}_\tau^{(\wedge)}(\mathcal{X};\Lambda)^\otimes$ and its dual is $f_*\Lambda$.
(2) Let $f : Y \to X$ be a smooth and proper morphism of rigid analytic spaces. Then $f_\sharp\Lambda$ is strongly dualisable in the monoidal ∞-category $\mathbf{RigSH}_\tau^{(\wedge)}(X;\Lambda)^\otimes$ and its dual is $f_*\Lambda$.

Proof. We only treat the case of rigid analytic motives. We need to show that there is an equivalence between the endofunctors $\underline{\mathrm{Hom}}(f_\sharp\Lambda, -)$ and $(f_*\Lambda) \otimes -$. We have natural equivalences

$$\underline{\mathrm{Hom}}(f_\sharp\Lambda, -) \overset{(1)}{\simeq} f_* f^*(-) \simeq f_*(\Lambda \otimes f^* M) \overset{(2)}{\simeq} f_*(\Lambda) \otimes M$$

where (1) is deduced by adjunction from the smooth projection formula $f_\sharp\Lambda \otimes - \simeq f_\sharp \circ f^*(-)$ (see Proposition 2.2.1(2)) and (2) is deduced from Proposition 4.1.7(2). □

4.2. **Weak compactifications.**

In this subsection, we discuss the notion of a weak compactification of a rigid analytic S-space. For us, it will be enough to know that weak compactifications exist locally. We will also briefly discuss Huber's compactifications.

Definition 4.2.1. Let $f : Y \to X$ be a morphism of rigid analytic spaces. A weak compactification of f is a commutative triangle

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[dr, "f"'] & W \arrow[d, "h"] \\ & X \end{tikzcd} \tag{4.3}$$

of rigid analytic spaces, where i is a locally closed immersion and h a proper morphism. (See Definition 1.1.14.) By abuse of language, we say that h is a weak compactification of f or that W is a weak compactification of Y. We define the category of weak compactifications of f to be the full subcategory of $(\mathrm{RigSpc}/X)_{f/}$ spanned by the weak compactifications of f. We say that f is weakly compactifiable if it admits a weak compactification. (Clearly, for f to be weakly compactifiable, it is necessary that f is separated and locally of finite type.)

Proposition 4.2.2. *Let $f : Y \to X$ be a morphism of rigid analytic spaces.*
(1) The category of weak compactifications of f has fiber products and equalizers. In particular, when $f : Y \to X$ is weakly compactifiable, this category is cofiltered.
(2) Assume that f is locally of finite type. Then, locally on Y, f is weakly compactifiable.

Proof. The first part follows from standard properties of proper morphisms and locally closed immersions. For the second part, since the question is local on Y, we may assume that f factors through an open subspace $U \subset X$ and that $Y \to U$ is the generic fiber of a finite type morphism $\mathcal{Y} \to \mathcal{U}$ between affine formal schemes. In this case, we may factor f as the composition of

$$Y \xrightarrow{s} \mathbb{B}^N_U \xrightarrow{u} \mathbb{P}^N_X \xrightarrow{p} X$$

where s is a closed immersion, u the obvious open immersion and p the obvious projection. □

We will need a short digression concerning the notion of relative interior.

Definition 4.2.3. Let $f : X \to W$ be a morphism between rigid analytic spaces. Let $V \subset W$ be an open subspace. We say that X maps into the interior of V relatively to W and write $f(X) \Subset_W V$ if the closure of $f(|X|)$ in $|W|$ is contained in $|V|$.

Remark 4.2.4. Often we use Definition 4.2.3 when f is a locally closed immersion. In this case, we write simply "$X \Subset_W V$" instead of "$f(X) \Subset_W V$".

Below, we use freely the fact that the underlying topological space of a rigid analytic space is valuative in the sense of [FK18, Chapter 0, Definition 2.3.1].

Lemma 4.2.5. *Let $f : X \to W$ be a morphism between quasi-compact and quasi-separated rigid analytic spaces. A point of $|W|$ belongs to $\overline{f(|X|)}$ if and only if its maximal generisation belongs to $f(|X|)$. Moreover, we have the equalities:*

$$\overline{f(|X|)} = \bigcap_{f(X) \Subset_W V} |V| = \bigcap_{f(X) \Subset_W V} \overline{|V|}. \tag{4.4}$$

Proof. The first assertion follows from [FK18, Chapter 0, Theorem 2.2.26] and the fact that $f(|X|)$ is stable under generisation. It follows that $\overline{f(|X|)}$ is also stable under generisation, which implies that $\overline{f(|X|)}$ is the intersection of its open neighbourhoods. (Indeed, if $w \in |W|$ does not belong to $\overline{f(|X|)}$, we have $\overline{\{w\}} \cap \overline{f(|X|)} = \emptyset$.) This gives the first equality in (4.4). The second equality follows from [FK18, Chapter 0, Proposition 2.3.7]. □

Lemma 4.2.6. *Let $f : X \to W$ be a morphism between quasi-compact and quasi-separated rigid analytic spaces. Let $V \subset W$ be an open subspace such that $f(X) \Subset_W V$. There exists an open subspace $V' \subset W$ such that $f(X) \Subset_W V'$ and $V' \Subset_W V$.*

Proof. By Lemma 4.2.5, we have

$$\overline{f(|X|)} = \bigcap_{f(X) \Subset_W V'} \overline{|V'|} \subset |V|.$$

By [FK18, Chapter 0, Corollary 2.2.12], there exists a quasi-compact open subspace $V' \subset W$ with $f(X) \Subset_W V'$ such that $\overline{|V'|} \subset |V|$ as needed. □

We now discuss Huber's compactifications. We will freely use results and notations from Subsection 1.2. We start with a definition.

Definition 4.2.7.

(1) A Tate ring A is said to be universally uniform if every finitely generated Tate A-algebra is uniform. (Recall that a finitely generated Tate A-algebra is a quotient of $A\langle t\rangle = A_0\langle t\rangle[\pi^{-1}]$ where $t = (t_1, \ldots, t_n)$ is a system of coordinates, $A_0 \subset A$ a ring of definition and $\pi \in A$ a topologically nilpotent unit contained in A_0.) In particular, a universally uniform Tate ring is also stably uniform in the sense of [BV18, pages 30–31]. A Tate affinoid ring R is said to be universally uniform if $R^{\pm}$ is universally uniform.

(2) A universally uniform adic space is a uniform adic space (as in Definition 1.2.6) which is locally isomorphic to $\mathrm{Spa}(A)$, where A is a universally uniform Tate affinoid ring.

Notation 4.2.8.

(1) Let S be a universally uniform adic space. We denote by Adic/S the category of uniform adic S-spaces. We denote by $\mathrm{Adic}^{\mathrm{lft}}/S$ (resp. $\mathrm{Adic}^{\mathrm{sft}}/S$) the full subcategory of Adic/S spanned by those adic S-spaces which are locally of finite type (resp. which are separated of finite type).

(2) Let S be a rigid analytic space. We denote by $\mathrm{RigSpc}^{\mathrm{lft}}/S$ (resp. $\mathrm{RigSpc}^{\mathrm{sft}}/S$) the full subcategory of RigSpc/S spanned by those rigid analytic S-spaces which are locally of finite type (resp. which are separated of finite type).

(3) Let S be a universally uniform adic space. By Corollary 1.2.7, S determines a rigid analytic space which we denote also by S, and we have equivalences of categories $\mathrm{Adic}^{\mathrm{lft}}/S \simeq \mathrm{RigSpc}^{\mathrm{lft}}/S$ and $\mathrm{Adic}^{\mathrm{sft}}/S \simeq \mathrm{RigSpc}^{\mathrm{sft}}/S$.

Notation 4.2.9. Let A be a Tate affinoid ring and B a Tate affinoid A-algebra. We define a new Tate affinoid A-algebra $B_{\mathrm{c}} = (B_{\mathrm{c}}^{\pm}, B_{\mathrm{c}}^{+})$ by setting $B_{\mathrm{c}}^{\pm} = B^{\pm}$ and letting B_{c}^{+} to be the integral closure of the subring $A^{+} + B^{\circ\circ} \subset B$.

The following theorem is due to Huber.

Theorem 4.2.10. *Let S be a quasi-compact and quasi-separated universally uniform adic space. There is a functor* $\mathrm{Adic}^{\mathrm{sft}}/S \to \mathrm{Fun}(\Delta^1, \mathrm{Adic}/S)$ *sending a separated adic S-space of finite type X to an open immersion $j_X : X \to X^{\mathrm{c}}$ over S satisfying the following properties.*

(1) Every point of $|X^{\mathrm{c}}|$ is a specialisation of a point of $|X|$. Moreover, for every $x \in |X|$ and every valuation ring $V \subset \kappa^+(x)$ containing $\kappa^+(s')$ for a specialisation $s' \in |S|$ of the image of x in $|S|$, there exists a unique point $x' \in |X^{\mathrm{c}}|$ which is a specialisation of x and such that $\kappa^+(x') = V$.

(2) The morphism $\mathcal{O}_{X^{\mathrm{c}}} \to j_{X,*}\mathcal{O}_X$ is an isomorphism.

(3) (Compatibility with base change) *If $S' \to S$ is an open immersion, then the morphism*

$$j_X \times_S S' : X \times_S S' \to X^{\mathrm{c}} \times_S S'$$

coincides with $j_{X'} : X' \to X'^{\mathrm{c}}$ where X' is the adic S'-space $X \times_S S'$.

(4) If $S = \mathrm{Spa}(A)$ and $X = \mathrm{Spa}(B)$, then $X^{\mathrm{c}} = \mathrm{Spa}(B_{\mathrm{c}})$.

Proof. This is essentially contained in [Hub96, Theorem 5.1.5]. In loc. cit., it is assumed that adic spaces satisfy one of the conditions in [Hub96, (1.1.1)], but this is only needed to insure universal sheafyness. Here, we use instead universal uniformness and [BV18, Theorem 7]. □

In the next proposition, we denote a uniform adic space and the associated rigid analytic space by the same symbol. (This is an abuse of notation justified by Corollary 1.2.7.)

Proposition 4.2.11. *Let S be a quasi-compact and quasi-separated universally uniform adic space, and let X be a separated adic S-space of finite type. Let $i : X \to W$ be a weak compactification of X over S. Then, X^{c} is naturally a weak limit of the rigid analytic pro-space $(V)_{X \Subset_W V}$ in the sense of Definition 2.8.10.*

Proof. By the universal property of Huber's compactifications (see [Hub96, Theorem 5.1.5]), the locally closed immersion i extends to a morphism $i' : X^{\mathrm{c}} \to W$. Since $i'(|X^{\mathrm{c}}|)$ is contained in the closure of $|X|$ in $|W|$, there is a natural map

$$X^{\mathrm{c}} \to (V)_{X \Subset_W V}, \tag{4.5}$$

and we need to prove that it exhibits X^{c} as a weak limit of $(V)_{X \Subset_W V}$.

We first check that the map

$$|X^{\mathrm{c}}| \to \lim_{X \Subset_W V} |V| \tag{4.6}$$

is a bijection. Injectivity is clear since each locally closed immersion $X \to V$, with $X \Subset_W V$, induces an injection $|X^{\mathrm{c}}| \to |V^{\mathrm{c}}|$ and the map $|X^{\mathrm{c}}| \to |V|$ factors this injection. For surjectivity, we use Lemma 4.2.5 which implies that every point $v \in \lim_{X \Subset_W V} |V|$ is a specialisation of a point $x \in |X|$. Thus, we have $\kappa(v) = \kappa(x)$ and $\kappa^+(s) \subset \kappa^+(v) \subset \kappa^+(x)$. By Theorem 4.2.10(1), the valuation ring $\kappa^+(v) \subset \kappa(x)$ determines a point of $|X^c|$ which is necessarily sent to v by (4.5) since $W \to S$ is separated.

It remains to see that for every point x of $|X^c|$ with image v in $\lim_{X \Subset_W V} |V|$ the map $\kappa(v) \to \kappa(x)$ has dense image. In fact, we have $\kappa(v) \simeq \kappa(x)$. To prove this, we may assume that x belongs to $|X|$, since the residue field of x is equal to the residue field of its maximal generisation and similarly for v. The claimed result is then clear since $X \to (V)_{X \Subset_W V}$ is given by locally closed immersions. □

4.3. **The exceptional functors, I. Construction.**

In this subsection, we define the exceptional functors $f_!$ and $f^!$ associated with a morphism f of rigid analytic spaces which is locally of finite type, and establish some of their basic properties. We start with the easy case of a locally closed immersion.

Lemma 4.3.1. *Let $i : Z \to X$ be a locally closed immersion of rigid analytic spaces. Let $U \subset X$ be an open neighbourhood of Z in which Z is closed. Denote by $s : Z \to U$ and $j : U \to X$ the obvious immersions. Then, the composite functor*

$$j_\sharp \circ s_* : \mathbf{RigSH}_\tau^{(\wedge)}(Z; \Lambda) \to \mathbf{RigSH}_\tau^{(\wedge)}(X; \Lambda)$$

is independent of the choice of U and we denote it by $i_!$.

Proof. Let $U' \subset U$ be an open neighbourhood of Z. Let $s' : Z \to U'$ and $u : U' \to U$ be the obvious immersions. We need to show that $u_\sharp \circ s'_* \simeq s_*$. To do so, we use the Cartesian square

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, "s'"] & Z \arrow[d, "s"] \\ U' \arrow[r, "u"] & U \end{tikzcd}$$

and Proposition 2.2.3(4). □

Lemma 4.3.2. *Let $s : Y \to X$ and $t : Z \to Y$ be locally closed immersions of rigid analytic spaces. There is an equivalence $(s \circ t)_! \simeq s_! \circ t_!$ of functors from $\mathbf{RigSH}_\tau^{(\wedge)}(Z; \Lambda)$ to $\mathbf{RigSH}_\tau^{(\wedge)}(X; \Lambda)$.*

Proof. Indeed, let $U \subset X$ be an open neighbourhood of Y in which Y is closed, and let $V \subset U$ be an open neighbourhood of Z in which Z is closed. Set $W = Y \cap V$. Consider the commutative diagram with a Cartesian square

$$\begin{tikzcd} Z \arrow[r, "e"] \arrow[rr, bend left, "t"] \arrow[dr] & W \arrow[r, "w"] \arrow[d, "d"] & Y \arrow[d, "c"] \arrow[dr, "s"] & \\ & V \arrow[r, "v"] & U \arrow[r, "u"] & X. \end{tikzcd}$$

Using Proposition 2.2.3(4), we have natural equivalences

$$u_\sharp \circ c_* \circ w_\sharp \circ e_* \simeq u_\sharp \circ v_\sharp \circ d_* \circ e_* \simeq (u \circ v)_\sharp \circ (d \circ e)_*$$

as needed. □

Proposition 4.3.3. *Consider a Cartesian square of rigid analytic spaces*

$$\begin{tikzcd} T \arrow[r, "i'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

where i is a locally closed immersion.

(1) There is a natural equivalence $f^* \circ i_! \simeq i'_! \circ f'^*$ *between functors from* $\mathbf{RigSH}_\tau^{(\wedge)}(Z;\Lambda)$ *to* $\mathbf{RigSH}_\tau^{(\wedge)}(Y;\Lambda)$.

(2) Assume that f is a proper morphism. There is a natural equivalence $f_* \circ i'_! \simeq i_! \circ f'_*$ *between functors from* $\mathbf{RigSH}_\tau^{(\wedge)}(T;\Lambda)$ *to* $\mathbf{RigSH}_\tau^{(\wedge)}(X;\Lambda)$.

Proof. Part (1) follows from Proposition 2.2.3(4) and part (2) follows from Theorem 4.1.4. □

We next discuss the case of weakly compactifiable morphisms.

Definition 4.3.4. Let $f : Y \to X$ be a weakly compactifiable morphism of rigid analytic spaces. Choose a weak compactification

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[dr, "f"'] & W \arrow[d, "h"] \\ & X \end{tikzcd}$$

of f and define the functor

$$f_! : \mathbf{RigSH}_\tau^{(\wedge)}(Y;\Lambda) \to \mathbf{RigSH}_\tau^{(\wedge)}(X;\Lambda)$$

by setting $f_! = h_* \circ i_!$. It follows from Corollary 4.1.6 that the functor $f_!$ is colimit-preserving; we denote by $f^!$ its right adjoint. The functors $f_!$ and $f^!$ are called the exceptional direct and inverse image functors.

Lemma 4.3.5. *Keep the notations as in Definition 4.3.4. The functor* $f_!$ *is independent of the choice of the weak compactification of* f.

Proof. Let $i' : Y \to W'$ be a second weak compactification of f and denote by $h' : W' \to X$ the structural projection. Without loss of generality, we may assume that W' is finer than W. Let $U \subset W$ be an open neighbourhood of Y in which Y is closed, and let $U' \subset W'$ be the inverse image of U. We then have a commutative diagram with a Cartesian square

$$\begin{tikzcd} & U' \arrow[r, "j'"] \arrow[d, "g'"] & W' \arrow[d, "g"] \arrow[dr, "h'"] & \\ Y \arrow[ur, "s'"] \arrow[r, "s"'] & U \arrow[r, "j"] & W \arrow[r, "h"'] & X. \end{tikzcd}$$

We need to compare $h_* \circ j_\sharp \circ s_*$ with $h'_* \circ j'_\sharp \circ s'_*$. We have natural transformations

$$h_* \circ j_\sharp \circ s_* \simeq h_* \circ j_\sharp \circ g'_* \circ s'_* \to h_* \circ g_* \circ j'_\sharp \circ s'_* \simeq h'_* \circ j'_\sharp \circ s'_*$$

where the middle one is an equivalence by Theorem 4.1.4. □

Example 4.3.6. Using Lemma 4.3.5 and a well-chosen weak compactification, we obtain the following particular cases.

(1) If $j : U \to X$ is an open immersion, then $j_! \simeq j_\sharp$ and $j^! \simeq j^*$.
(2) If $f : Y \to X$ is proper, then $f_! \simeq f_*$.

Remark 4.3.7. At this point we have constructed, for each weakly compactifiable morphism $f : Y \to X$ of rigid analytic spaces, a functor $f_! : \mathbf{RigSH}^{(\wedge)}_\tau(Y;\Lambda) \to \mathbf{RigSH}^{(\wedge)}_\tau(X;\Lambda)$. Due to the choice of a weak compactification involved in the construction, it is not clear why $f \mapsto f_!$ would be functorial in any sense. The main goal of the remainder of this subsection is to prove that in fact it is, as long as we restrict to morphisms between weakly compactifiable rigid analytic spaces over a fixed base. (Note that morphisms between such spaces are automatically weakly compactifiable, so that our construction applies.)

Notation 4.3.8. Let S be a rigid analytic space.

(1) We denote by $\mathrm{RigSpc}^{\mathrm{wc}}/S \subset \mathrm{RigSpc}/S$ the full subcategory of weakly compactifiable rigid analytic S-spaces. Recall that, by definition, $\mathrm{RigSpc}^{\mathrm{wc}}/S$ is contained in $\mathrm{RigSpc}^{\mathrm{lft}}/S$ (see Notation 4.2.8(2)) and that every object in $\mathrm{RigSpc}^{\mathrm{lft}}/S$ is locally isomorphic to an object of $\mathrm{RigSpc}^{\mathrm{wc}}/S$ by Proposition 4.2.2.
(2) We denote by $\mathrm{RigSpc}^{\mathrm{prop}}/S \subset \mathrm{RigSpc}/S$ the full subcategory of proper rigid analytic S-spaces.
(3) We denote by WComp/S the category whose objects are pairs (X, W) where X is a rigid analytic S-space and W a weak compactification of X. There are functors

$$\mathfrak{d}_S : \mathrm{WComp}/S \to \mathrm{RigSpc}^{\mathrm{wc}}/S \quad \text{and} \quad \mathfrak{w}_S : \mathrm{WComp}/S \to \mathrm{RigSpc}^{\mathrm{prop}}/S$$

sending a pair (X, W) to X and W respectively.

Proposition 4.3.9. *Let S be a rigid analytic space. There is a functor*

$$\mathbf{RigSH}^{(\wedge)}_\tau(-;\Lambda)_! : \mathrm{RigSpc}^{\mathrm{wc}}/S \to \mathrm{Pr}^{\mathrm{L}} \tag{4.7}$$

sending an object X to the ∞-category $\mathbf{RigSH}^{(\wedge)}_\tau(X;\Lambda)$ and a morphism f to the functor $f_!$.

We fix a rigid analytic space S. The functor (4.7) will be constructed below and the fact that it extends the functors in Definition 4.3.4 is proven in Lemma 4.3.14. We start by constructing a similar functor defined on WComp/S.

Notation 4.3.10. Given an object (X, W) in WComp/S, we denote by $\mathbf{RigSH}^{(\wedge)}_\tau((X, W);\Lambda)_!$ the full sub-∞-category of $\mathbf{RigSH}^{(\wedge)}_\tau(W;\Lambda)$ spanned by the essential image of the fully faithful embedding

$$i_! : \mathbf{RigSH}^{(\wedge)}_\tau(X;\Lambda) \to \mathbf{RigSH}^{(\wedge)}_\tau(W;\Lambda), \tag{4.8}$$

where $i : X \to W$ is the given locally closed immersion.

Proposition 4.3.11. *Let $(f, h) : (X', W') \to (X, W)$ be a morphism in* WComp/S.

(1) The functor

$$h_* : \mathbf{RigSH}^{(\wedge)}_\tau(W';\Lambda) \to \mathbf{RigSH}^{(\wedge)}_\tau(W;\Lambda)$$

takes $\mathbf{RigSH}^{(\wedge)}_\tau((X', W');\Lambda)_!$ into $\mathbf{RigSH}^{(\wedge)}_\tau((X, W);\Lambda)_!$, and induces a functor

$$(f, h)_! : \mathbf{RigSH}^{(\wedge)}_\tau((X', W');\Lambda)_! \to \mathbf{RigSH}^{(\wedge)}_\tau((X, W);\Lambda)_!. \tag{4.9}$$

(2) There is a commutative square

$$\begin{array}{ccc} \mathbf{RigSH}^{(\wedge)}_{\tau}(X';\Lambda) & \longrightarrow & \mathbf{RigSH}^{(\wedge)}_{\tau}((X',W');\Lambda)_! \\ \Big\downarrow{\scriptstyle f_!} & & \Big\downarrow{\scriptstyle (f,h)_!} \\ \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda) & \longrightarrow & \mathbf{RigSH}^{(\wedge)}_{\tau}((X,W);\Lambda)_! \end{array}$$

where the horizontal arrows are equivalences.

(3) If f is an isomorphism, then $(f,h)_!$ is an equivalence of ∞-categories.

Proof. Consider the commutative diagram with a Cartesian square

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[rr, bend left, "i'"] \arrow[dr, "f"'] & V \arrow[r, "v"] \arrow[d, "h'"] & W' \arrow[d, "h"] \\ & X \arrow[r, "i"] & W. \end{tikzcd}$$

By Lemma 4.3.2, we have $i'_! \simeq v_! \circ u_!$. Thus, the essential image of $i'_!$ is contained in the essential image of $v_!$. On the other hand, by Proposition 4.3.3(2), we have $h_* \circ v_! \simeq i_! \circ h'_*$. Thus, h_* takes the essential image of $v_!$ into the essential image of $i_!$, which proves the first statement.

Next, we verify the second statement. Note that V is a weak compactification of X' over X. Thus, by Lemma 4.3.5, we have $f_! \simeq h'_* \circ u_!$. Using Proposition 4.3.3(2) again, we obtain natural equivalences

$$i_! \circ f_! \simeq i_! \circ h'_* \circ u_! \simeq h_* \circ v_! \circ u_! \simeq h_* \circ i'_!.$$

This gives the commutative square in the second statement. Finally, the third statement follows from the second one using Lemma 4.3.5. $\square$

Notation 4.3.12. We will denote by

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^* : \mathrm{RigSpc}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}} \tag{4.10}$$

the functor from Proposition 2.1.21 (in the T-stable case and after forgetting the monoidal structure) and by

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)_* : \mathrm{RigSpc} \to \mathrm{Pr}^{\mathrm{R}} \tag{4.11}$$

the functor deduced from (4.10) using the equivalence $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}$. By Corollary 4.1.6, the restriction of (4.11) to $\mathrm{RigSpc}^{\mathrm{prop}}/S$ yields a Pr^{L}-valued functor. In particular, we have a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(\mathfrak{w}_S(-);\Lambda)_* : \mathrm{WComp}/S \to \mathrm{Pr}^{\mathrm{L}}. \tag{4.12}$$

To go further, we need the following well-known general lemma.

Lemma 4.3.13. *Let B be a simplicial set and let $\mathcal{C} : B \to \mathrm{CAT}_\infty$ be a diagram of ∞-categories. Assume that, for each vertex $x \in B_0$, we are given a full sub-∞-category $\mathcal{C}'(x) \subset \mathcal{C}(x)$. Assume also that, for every edge $e \in B_1$, the functor $\mathcal{C}(e_0) \to \mathcal{C}(e_1)$ takes $\mathcal{C}'(e_0)$ into $\mathcal{C}'(e_1)$. Then, there exists a diagram $\mathcal{C}' : B \to \mathrm{CAT}_\infty$ and a natural transformation $\mathcal{C}' \to \mathcal{C}$ such that for every edge $e \in B_1$, $\mathcal{C}'(e)$ is equivalent to the functor induced from $\mathcal{C}(e)$ on the sub-∞-categories $\mathcal{C}'(e_0)$ and $\mathcal{C}'(e_1)$.*

Proof. By Lurie's unstraightening [Lur09, §3.2], one reduces to prove an analogous statement for coCartesian fibrations which is easy and left to the reader. $\square$

By Proposition 4.3.11(1), we may apply Lemma 4.3.13 to the functor (4.12) and the full sub-∞-categories introduced in Notation 4.3.10. This yields a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}((-,-);\Lambda)_! : \mathrm{WComp}/S \to \mathrm{Pr}^{\mathrm{L}}. \tag{4.13}$$

(The fact that this functor lands in Pr^{L}, and not just in CAT_{∞}, follows from Corollary 4.1.6 together with Proposition 4.3.11(2).) By left Kan extension along the functor $\mathfrak{d}_S$, we obtain from (4.13) a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)_! : \mathrm{RigSpc}^{\mathrm{wc}}/S \to \mathrm{Pr}^{\mathrm{L}}. \tag{4.14}$$

The following lemma shows that this left Kan extension behaves as we want it to.

Lemma 4.3.14. *The obvious natural transformation*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}((-,-);\Lambda)_! \to \mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)_! \circ \mathfrak{d}_S \tag{4.15}$$

is an equivalence. In particular, the functor (4.14) *sends a morphism* $f : Y \to X$ *in* $\mathrm{RigSpc}^{\mathrm{wc}}/S$ *to the functor* $f_!$ *of Definition 4.3.4.*

Proof. Given an object $X \in \mathrm{RigSpc}^{\mathrm{wc}}/S$, there is an equivalence in Pr^{L}:

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)_! \simeq \operatorname*{colim}_{(Y,W)\in(\mathrm{WComp}/S)_{/X}} \mathbf{RigSH}^{(\wedge)}_{\tau}((Y,W);\Lambda)_! \tag{4.16}$$

where the category $(\mathrm{WComp}/S)_{/X}$ consists of pairs (Y,W) with Y a rigid analytic X-space and W a compactification of Y over S. Fix a weak compactification P of X over S. There is an obvious forgetful functor

$$\alpha : (\mathrm{WComp}/S)_{/(X,P)} \to (\mathrm{WComp}/S)_{/X}$$

admitting a right adjoint β given by $(Y,W) \mapsto (Y, W \times_S P)$. Moreover, it follows from Proposition 4.3.11(3), that the counit of the adjunction $\alpha \circ \beta \to \mathrm{id}$ induces an equivalence between the functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}((-,-);\Lambda)_! : (\mathrm{WComp}/S)_{/X} \to \mathrm{Pr}^{\mathrm{L}}$$

and its composition with the endofunctor $\alpha \circ \beta$ of $(\mathrm{WComp}/S)_{/X}$. Since β is right adjoint to α, composition with β is equivalent to left Kan extension along α. This implies that the colimit in (4.16) is equivalent to

$$\operatorname*{colim}_{(Y,W)\in(\mathrm{WComp}/S)_{/(X,P)}} \mathbf{RigSH}^{(\wedge)}_{\tau}((Y,W);\Lambda)_! \simeq \mathbf{RigSH}^{(\wedge)}_{\tau}((X,P);\Lambda)_!$$

since (X,P) is the final object of $(\mathrm{WComp}/S)_{/(X,P)}$. This proves the lemma. □

Corollary 4.3.15. *Let X be a weakly compactifiable rigid analytic S-space, and let* Op/X *be the category of open subspaces of X. Then, the functors*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)_! : \mathrm{Op}/X \to \mathrm{Pr}^{\mathrm{L}} \quad \textit{and} \quad \mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^* : (\mathrm{Op}/X)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}} \tag{4.17}$$

are exchanged by the equivalence $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}$.

Proof. Let P be a weak compactification of X. Then, for every open subspace $U \subset X$, P is also a weak compactification of U. Thus, we have a functor $\mathrm{Op}/X \to \mathrm{WComp}/S$ given by $U \mapsto (U,P)$. Therefore, by Lemma 4.3.14, the first functor in (4.17) is equivalent to the functor given by $U \mapsto \mathbf{RigSH}^{(\wedge)}_{\tau}((U,P);\Lambda)_!$. It is immediate from the construction of (4.13) that this functor is equivalent to the one sending an open immersion $u : U \to X$ to the essential image of the fully faithful embedding $u_\sharp$. This proves the corollary. □

Remark 4.3.16. Using the equivalence $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}$, the functor (4.7) gives rise to a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{!} : (\mathrm{RigSpc}^{\mathrm{wc}}/S)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}} \tag{4.18}$$

sending a morphism f to the functor $f^{!}$.

Proposition 4.3.17. *The functor* (4.18) *is a* Pr^{R}*-valued sheaf for the analytic topology.*

Proof. It is enough to show that, for every $X \in \mathrm{RigSpc}^{\mathrm{wc}}/S$, the restriction of (4.18) to Op/X is a sheaf for the analytic topology. This follows from Corollary 4.3.15 and Theorem 2.3.4. (Indeed, the inclusion functors $\mathrm{Pr}^{\mathrm{L}} \to \mathrm{CAT}_{\infty}$ and $\mathrm{Pr}^{\mathrm{R}} \to \mathrm{CAT}_{\infty}$ are limit-preserving by [Lur09, Proposition 5.5.3.13 & Theorem 5.5.3.18].) □

Corollary 4.3.18. *There is a unique extension of* (4.7) *into a functor*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)_{!} : \mathrm{RigSpc}^{\mathrm{lft}}/S \to \mathrm{Pr}^{\mathrm{L}} \tag{4.19}$$

such that the following condition is satisfied. The functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{!} : (\mathrm{RigSpc}^{\mathrm{lft}}/S)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}, \tag{4.20}$$

obtained from (4.19) *using the equivalence* $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}$*, is a* Pr^{R}*-valued sheaf for the analytic topology.*

Proof. This follows from Proposition 4.3.17 using Lemma 2.1.4. Indeed, a Pr^{R}-valued τ-sheaf on a site $(\mathcal{C},\tau)$ is equivalent to a limit-preserving functor on $\mathrm{Shv}_{\tau}(\mathcal{C})^{\mathrm{op}}$; see Definition 2.3.1. □

Remark 4.3.19. At this point, it is unclear that the ∞-category $\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)_{!}$ is equivalent to the ∞-category $\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)$ for a general object $X \in \mathrm{RigSpc}^{\mathrm{lft}}/S$. This will be proven in Subsection 4.4; see Corollary 4.4.23 below. When X is weakly compactifiable, this is already stated in Proposition 4.3.9.

We end this subsection with the following result relating our approach to the one in [Hub96, §5.2].

Theorem 4.3.20. *Let X and Y be quasi-compact and quasi-separated uniform adic spaces, and let $f : Y \to X$ be a weakly compactifiable morphism of rigid analytic spaces. Let $f^{\mathrm{c}} : Y^{\mathrm{c}} \to X$ be the projection of Huber's compactification of Y over X, and $j : Y \to Y^{\mathrm{c}}$ the obvious inclusion. Assume one of the following two alternatives.*

(1) We work in the non-hypercomplete case, and X is locally of finite Krull dimension. When τ is the étale topology, we assume furthermore that Λ is eventually coconnective.
(2) We work in the hypercomplete case, and X is (Λ,τ)-admissible (see Definition 2.4.14).

Then, the functor $f_{!}$ of Definition 4.3.4 coincides with the composite functor $f^{\mathrm{c}}_{*} \circ j_{\sharp}$.

Proof. Fix a weak compactification W of Y over X, and let $h : W \to X$ and $i : Y \to W$ be the given morphisms. The morphism i extends to a morphism $i' : Y^{\mathrm{c}} \to W$. We have $f^{\mathrm{c}}_{*} \simeq h_{*} \circ i'_{*}$. Thus, we only need to show that there is an equivalence $i'_{*} \circ j_{\sharp} \simeq i_{!}$. The Cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{j} & Y^{\mathrm{c}} \\ \| & & \downarrow{\scriptstyle i'} \\ Y & \xrightarrow{i} & W \end{array}$$

and Proposition 4.3.3(1) give an equivalence $i'^* \circ i_! \simeq j_! = j_\sharp$. Thus, it is enough to show that the morphism

$$i_! \to i'_* \circ i'^* \circ i_!$$

is an equivalence. By Proposition 4.2.11, Y^c is the weak limit of the rigid analytic pro-space $(V)_{Y \Subset_W V}$. It follows from Theorem 2.8.15 that there is an equivalence of ∞-categories

$$\operatorname*{colim}_{Y \Subset_W V} \mathbf{RigSH}^{(\wedge)}_\tau(V; \Lambda) \to \mathbf{RigSH}^{(\wedge)}_\tau(Y^c; \Lambda).$$

Arguing as in the proof of Lemma 3.5.7 (see also Remark 3.5.8), we deduce an equivalence

$$\operatorname*{colim}_{Y \Subset_W V} r_{V,*} \circ r_V^* \simeq i'_* \circ i'^*$$

where, for an open subspace $U \subset W$, $r_U : U \to W$ denotes the obvious inclusion. Therefore, it is enough to prove that

$$i_! \to r_{V,*} \circ r_V^* \circ i_!$$

is an equivalence for every $V \subset W$ such that $Y \Subset_W V$. Letting Q be the open subspace of W with underlying topological space $|Q| = |W| \smallsetminus \overline{|Y|}$, we have $W = V \cup Q$. So it suffices to prove that

$$r_V^* \circ i_! \to r_V^* \circ r_{V,*} \circ r_V^* \circ i_! \quad \text{and} \qquad r_Q^* \circ i_! \to r_Q^* \circ r_{V,*} \circ r_V^* \circ i_!$$

are equivalences. For the first one, we use that $r_V^* \circ r_{V,*} \simeq \mathrm{id}$. For the second one, we use Proposition 4.3.3(1) and the fact that $Y \times_W Q = \emptyset$, which imply that the source and the target of the natural transformation are the zero functor. □

Remark 4.3.21. Theorem 4.3.20 can be extended to separable morphisms of finite type which are not assumed to be weakly compactifiable. Indeed, one can construct a variant of the functor (4.7) using Huber's compactifications (instead of weak compactifications) and show that this new functor coincides with (4.7) on $\mathrm{RigSpc}^{\mathrm{wc}}/S$ and gives rise to a sheaf for the analytic topology via the equivalence $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}$. We will not pursue this further in this paper, and leave it to the interested reader.

4.4. **The exceptional functors, II. Exchange.**

The goal of this subsection is to prove Theorem 4.4.2 below and derive a few consequences. This theorem can be seen as a strengthening of Corollary 4.3.18 and gives a way to encapsulate the coherence properties of the exchange equivalences between the ordinary inverse (resp. direct) image functors and the exceptional direct (resp. inverse) image functors. It should be mentioned that Theorem 4.4.2 is not the best possible statement one could hope for. For a better statement, we refer to Theorem 4.4.31 below whose proof relies unfortunately on unproven claims in [GR17] concerning $(\infty, 2)$-categories. However, Theorem 4.4.2 is probably good enough in practice.

Notation 4.4.1. Given a simplicial set B and a diagram $\mathcal{C} : B \to \mathrm{CAT}_\infty$, we denote by $\int_B \mathcal{C} \to B$ a coCartesian fibration classified by $\mathcal{C}$. When B is an ordinary category and $\mathcal{C}$ takes values in the sub-∞-category of CAT_∞ spanned by ordinary categories, we take for $\int_B \mathcal{C}$ the ordinary category given by the Grothendieck construction. In particular, objects of $\int_B \mathcal{C}$ are represented by pairs (b, c) where $b \in B$ and $c \in \mathcal{C}(b)$.

Theorem 4.4.2. *There are functors*

$$\begin{aligned}\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{*}_{!} &: \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{lft}} \to \mathrm{Pr}^{\mathrm{L}} \\ \mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{!}_{*} &: \left(\int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{lft}}\right)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}\end{aligned} \tag{4.21}$$

which are exchanged by the equivalence $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}$ *and which admit the following informal description.*

- *These functors send an object* (S, X)*, with* S *a rigid analytic space and* X *an object of* $\mathrm{RigSpc}^{\mathrm{lft}}/S$*, to the* ∞*-category* $\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)$.
- *These functors send an arrow* $(g, f) : (S, Y) \to (T, X)$*, consisting of morphisms* $g : T \to S$ *and* $f : T \times_S Y \to X$*, to the functors* $f_! \circ g'^*$ *and* $g'_* \circ f^!$ *respectively, with* $g' : T \times_S Y \to Y$ *the base change of* g.

Moreover, the functors in (4.21) *satisfy the following properties.*

(1) The ordinary functors

$$\begin{aligned}\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{*} &: \mathrm{RigSpc}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}} \\ \mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)_{*} &: \mathrm{RigSpc}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}\end{aligned} \tag{4.22}$$

(as in Notation 4.3.12) are obtained from the functors in (4.21) *by composition with the diagonal functor* $\mathrm{RigSpc}^{\mathrm{op}} \to \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{lft}}$*, given by* $S \mapsto (S, S)$.

(2) For a rigid analytic space S*, the functors*

$$\begin{aligned}\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)_{!} &: \mathrm{RigSpc}^{\mathrm{lft}}/S \to \mathrm{Pr}^{\mathrm{L}} \\ \mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{!} &: \mathrm{RigSpc}^{\mathrm{lft}}/S \to \mathrm{Pr}^{\mathrm{R}}\end{aligned} \tag{4.23}$$

(as in Corollary 4.3.18) are obtained from the functors in (4.21) *by restriction to* $\mathrm{RigSpc}^{\mathrm{lft}}/S$.

To construct the functors in (4.21), we start with the functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{*,*} : \int_{\mathrm{RigSpc}^{\mathrm{op}}} (\mathrm{RigSpc}^{\mathrm{prop}})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}} \tag{4.24}$$

admitting the following informal description.

- It sends a pair (S, X), with S a rigid analytic space and X an object of $\mathrm{RigSpc}^{\mathrm{prop}}/S$, to the ∞-category $\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)$.
- It sends an arrow $(g, f) : (S, X) \to (T, Y)$, consisting of morphisms $g : T \to S$ and $f : Y \to T \times_S X$, to the functor $f^* \circ g'^*$ with $g' : T \times_S X \to X$ the base change of g.

Said differently, (4.24) is the composition of

$$\int_{\mathrm{RigSpc}^{\mathrm{op}}} (\mathrm{RigSpc}^{\mathrm{prop}})^{\mathrm{op}} \to \mathrm{RigSpc}^{\mathrm{op}} \xrightarrow{\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{*}} \mathrm{Pr}^{\mathrm{L}}$$

where the first functor is given by $(S, X) \mapsto X$. We will apply to the functor (4.24) the following general construction.

Construction 4.4.3. Let B be a simplicial set, $p : \mathcal{E} \to B$ a coCartesian fibration and $\mathfrak{D} : \mathcal{E} \to \mathrm{CAT}_{\infty}$ a functor. We assume the following condition.

($\star$) For every commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

in $\mathcal{E}$, such that g and g' are p-coCartesian, and $p(f)$ and $p(f')$ are identity morphisms, the associated square

$$\begin{tikzcd} \mathfrak{D}(X) \arrow[r] \arrow[d] & \mathfrak{D}(Y) \arrow[d] \\ \mathfrak{D}(X') \arrow[r] & \mathfrak{D}(Y') \end{tikzcd}$$

is right adjointable.

Let $p' : \mathcal{E}' \to B$ be a coCartesian fibration which is opposite to $\mathcal{E}$, i.e., if p is classified by a diagram $\mathcal{C} : B \to \mathrm{CAT}_\infty$, then p' is classified by the diagram $\mathcal{C}^{\mathrm{op}} : B \to \mathrm{CAT}_\infty$ obtained by composing $\mathcal{C}$ with the autoequivalence $(-)^{\mathrm{op}}$ of CAT_∞. In particular, for $b \in B$, the fiber $\mathcal{E}'_b$ of p' at b is equivalent to the opposite of the fiber $\mathcal{E}_b$ of p at b. Similarly, given a p-coCartesian edge $A \to B$ in $\mathcal{E}$, there is an associated p'-coCartesian edge $A' \to B'$ in $\mathcal{E}'$ such that A' and B' are the images of A and B by the equivalences between the fibers of p and the opposite of the fibers of p'.

Then, there exists a diagram $\mathfrak{D}' : \mathcal{E}' \to \mathrm{CAT}_\infty$ which admits the following informal description.

(1) For $b \in B$, the functor $\mathfrak{D}'|_{\mathcal{E}'_b} : \mathcal{E}'_b \to \mathrm{CAT}_\infty$ lands in $\mathrm{CAT}^{\mathrm{R}}_\infty$ and it is deduced from the functor $\mathfrak{D}|_{\mathcal{E}_b} : \mathcal{E}_b \to \mathrm{CAT}^{\mathrm{L}}_\infty$ using the equivalences $\mathcal{E}'_b \simeq (\mathcal{E}_b)^{\mathrm{op}}$ and $\mathrm{CAT}^{\mathrm{R}}_\infty \simeq (\mathrm{CAT}^{\mathrm{L}}_\infty)^{\mathrm{op}}$.
(2) Given a p-coCartesian edge $A \to B$ in $\mathcal{E}$ with corresponding p'-coCartesian edge $A' \to B'$, the associated functor $\mathfrak{D}(A) \to \mathfrak{D}(B)$ is equivalent to the functor $\mathfrak{D}'(A') \to \mathfrak{D}'(B')$.

The diagram $\mathfrak{D}'$ is constructed as follows. Consider the coCartesian fibration $q : \mathcal{F} \to \mathcal{E}$ classified by $\mathfrak{D}$. By [Lur09, Proposition 2.4.2.3(3)], $p \circ q : \mathcal{F} \to B$ is a coCartesian fibration and q sends a $p \circ q$-coCartesian edge to a p-coCartesian edge. Applying straightening to $p \circ q$ and p, we obtain a morphism $\phi : \mathfrak{N} \to \mathcal{C}$ in $\mathrm{Fun}(B, \mathrm{CAT}_\infty)$ between the diagrams $\mathfrak{N} : B \to \mathrm{CAT}_\infty$ and $\mathcal{C} : B \to \mathrm{CAT}_\infty$ classifying $p \circ q$ and q respectively. Note that for $b \in B$, the functor $\phi(b) : \mathfrak{N}(b) \to \mathcal{C}(b)$ is equivalent to the functor $q_b : \mathcal{F}_b \to \mathcal{E}_b$ induced on the fibers of $p \circ q$ and p. Hence, $\phi(b)$ is a coCartesian fibration. Condition ($\star$) is equivalent to the following one.

($\star'$) For every $b \in B$, the coCartesian fibration $\phi(b) : \mathfrak{N}(b) \to \mathcal{C}(b)$ is also a Cartesian fibration and, for every edge $b_0 \to b_1$ in B, the associated commutative square

$$\begin{tikzcd} \mathfrak{N}(b_0) \arrow[r] \arrow[d, "\phi(b_0)"] & \mathfrak{N}(b_1) \arrow[d, "\phi(b_1)"] \\ \mathcal{C}(b_0) \arrow[r] & \mathcal{C}(b_1) \end{tikzcd}$$

is such that the functor $\mathfrak{N}(b_0) \to \mathfrak{N}(b_1)$ takes a $\phi(b_0)$-Cartesian edge to a $\phi(b_1)$-Cartesian edge.

Passing to the opposite ∞-categories, condition ($\star'$) says that the natural transformation $\phi^{\mathrm{op}} : \mathfrak{N}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ sends a vertex $b \in B$ to a coCartesian fibration and an edge of B to a functor preserving coCartesian edges. Applying unstraightening to ϕ^{op}, we obtain a commutative triangle

$$\begin{tikzcd} \mathcal{F}' \arrow[rr, "q'"] \arrow[dr, "p' \circ q'"'] & & \mathcal{E}' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

where p' and $p' \circ q'$ are the coCartesian fibrations classified by $\mathcal{C}^{\mathrm{op}}$ and $\mathfrak{N}^{\mathrm{op}}$. We may assume that q' is a fibration for the coCartesian model structure on $(\mathrm{Set}^+_\Delta)_{/B}$ (see [Lur09, Proposition 3.1.3.7]) which insures that q' is an inner fibration (by using [Lur09, Remark 3.1.3.4]). In this case, q' is also a coCartesian fibration. To prove this, we argue as in the proof of Lemma 3.4.6. More precisely, by [Lur09, Proposition 2.4.2.11], we know that q' is a locally coCartesian fibration and, by [Lur09, Proposition 2.4.2.8], it remains to check that locally q'-coCartesian edges can be composed. This follows from the characterisation of locally q'-coCartesian edges given in [Lur09, Proposition 2.4.2.11] and condition $(\star')$. That said, the announced diagram $\mathfrak{D}' : \mathcal{E}' \to \mathrm{CAT}_\infty$ is the one obtained from q' by straightening and composing with the autoequivalence $(-)^{\mathrm{op}}$ of CAT_∞.

Remark 4.4.4. Continuing with the notation and assumptions of Construction 4.4.3, let $s : B \to \mathcal{E}$ be a coCartesian section. This corresponds, by straightening, to a natural transformation from the constant diagram $\{*\} : B \to \mathrm{CAT}_\infty$ to $\mathcal{C}$. Passing to opposite functors and unstraightening, we obtain another coCartesian section $s' : B \to \mathcal{E}'$. It follows from the construction that the two composites

$$B \xrightarrow{s} \mathcal{E} \xrightarrow{\mathfrak{D}} \mathrm{CAT}_\infty \qquad \text{and} \qquad B \xrightarrow{s'} \mathcal{E}' \xrightarrow{\mathfrak{D}'} \mathrm{CAT}_\infty$$

are the same.

Lemma 4.4.5. *The condition $(\star)$ in Construction 4.4.3 is satisfied for p the coCartesian fibration $\int_{\mathrm{RigSpc}^{\mathrm{op}}}(\mathrm{RigSpc}^{\mathrm{prop}})^{\mathrm{op}} \to \mathrm{RigSpc}^{\mathrm{op}}$, given by $(S, X) \mapsto S$, and $\mathfrak{D}$ the functor* (4.24) *composed with the inclusion* $\mathrm{Pr}^{\mathrm{L}} \to \mathrm{CAT}_\infty$.

Proof. A commutative square as in condition $(\star)$ corresponds to a square of the form

$$\begin{tikzcd} (S, X) \arrow[r, "(\mathrm{id}_S, f)"] \arrow[d, "(g, \mathrm{id}_{X'})"'] & (S, Y) \arrow[d, "(g, \mathrm{id}_{Y'})"] \\ (T, X') \arrow[r, "(\mathrm{id}_T, f')"] & (T, Y'), \end{tikzcd}$$

where $g : T \to S$ is a morphism of rigid analytic spaces, $f : Y \to X$ a morphism in $\mathrm{RigSpc}^{\mathrm{prop}}/S$, and $f' : Y' \to X'$ the base change of f along g. Letting $g' : X' \to X$ and $g'' : Y' \to Y$ be the base changes of g, the functor (4.24) takes the above square to the commutative square of ∞-categories

$$\begin{tikzcd} \mathbf{RigSH}^{(\wedge)}_\tau(X; \Lambda) \arrow[r, "f^*"] \arrow[d, "g'^*"] & \mathbf{RigSH}^{(\wedge)}_\tau(Y; \Lambda) \arrow[d, "g''^*"] \\ \mathbf{RigSH}^{(\wedge)}_\tau(X'; \Lambda) \arrow[r, "f'^*"] & \mathbf{RigSH}^{(\wedge)}_\tau(Y'; \Lambda). \end{tikzcd}$$

The morphism f, being a morphism of proper rigid analytic S-spaces, is proper. Thus, the right adjointability of the above square follows from Theorem 4.1.4(1). □

By Lemma 4.4.5, we may use Construction 4.4.3 to obtain a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_* : \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{prop}} \to \mathrm{Pr}^{\mathrm{L}}. \tag{4.25}$$

More precisely, the composition of (4.25) with $\mathrm{Pr}^{\mathrm{L}} \to \mathrm{CAT}_\infty$ is the functor $\mathfrak{D}'$ when we take for $\mathfrak{D}$ the composition of (4.24) with $\mathrm{Pr}^{\mathrm{L}} \to \mathrm{CAT}_\infty$; that the resulting functor $\mathfrak{D}'$ lands in Pr^{L} follows from Corollary 4.1.6. The functor (4.25) admits the following informal description.

- It sends a pair (S, X), with S a rigid analytic space and X an object of $\mathrm{RigSpc}^{\mathrm{prop}}/S$, to the ∞-category $\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)$.
- It sends an arrow $(g, f) : (S, Y) \to (T, X)$, consisting of morphisms $g : T \to S$ and $f : T \times_S Y \to X$, to the functor $f_* \circ g'^*$ with $g' : T \times_S Y \to Y$ the base change of g.

Integrating the functors $\mathfrak{w}_S$ from Notation 4.3.8, we obtain a functor

$$\mathfrak{w} : \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{WComp} \to \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{prop}}. \tag{4.26}$$

Composing with (4.25), we obtain a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(\mathfrak{w}(-);\Lambda)^*_* : \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{WComp} \to \mathrm{Pr}^{\mathrm{L}}. \tag{4.27}$$

Notation 4.4.6. Given $(S, (X, W)) \in \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{WComp}$, we denote by $\mathbf{RigSH}^{(\wedge)}_{\tau}((X, W);\Lambda)^*_!$ the full sub-∞-category of $\mathbf{RigSH}^{(\wedge)}_{\tau}(W;\Lambda)^*_*$ introduced in Notation 4.3.10, i.e., the essential image of the fully faithful embedding (4.8).

The next statement is a strengthening of Proposition 4.3.11(1).

Proposition 4.4.7. *Given an arrow* $(g, (f, h)) : (S, (Y, Q)) \to (T, (X, P))$ *in* $\int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{WComp}$*, the associated functor*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(Q;\Lambda)^*_* \to \mathbf{RigSH}^{(\wedge)}_{\tau}(P;\Lambda)^*_* \tag{4.28}$$

takes $\mathbf{RigSH}^{(\wedge)}_{\tau}((Y, Q);\Lambda)^*_!$ *into* $\mathbf{RigSH}^{(\wedge)}_{\tau}((X, P);\Lambda)^*_!$ *and induces a functor*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}((Y, Q);\Lambda)^*_! \to \mathbf{RigSH}^{(\wedge)}_{\tau}((X, P);\Lambda)^*_!. \tag{4.29}$$

Proof. Using Proposition 4.3.11(1), we only need to treat the case of a morphism of the form

$$(g, \mathrm{id}, \mathrm{id}) : (S, (Y, Q)) \to (T, T \times_S Y, T \times_S Q).$$

In this case, we need to show that the functor

$$g'^* : \mathbf{RigSH}^{(\wedge)}_{\tau}(Q;\Lambda) \to \mathbf{RigSH}^{(\wedge)}_{\tau}(T \times_S Q;\Lambda),$$

with $g' : T \times_S Q \to Q$ the base change of g, sends the essential image of $i_!$, with $i : Y \to Q$ the given immersion, to the essential image of $i'_!$, with $i' : T \times_S Y \to T \times_S Q$ the base change of i. This follows immediately from Proposition 4.3.3(1). □

Combining Proposition 4.4.7 with Lemma 4.3.13, we deduce a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}((-,-);\Lambda)^*_! : \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{WComp} \to \mathrm{Pr}^{\mathrm{L}}, \tag{4.30}$$

and this functor restricts to (4.13) on WComp/S for every rigid analytic space S. Integrating the functors $\mathfrak{d}_S$ from Notation 4.3.8, we obtain a functor

$$\mathfrak{d} : \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{WComp} \to \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{wc}} \tag{4.31}$$

given by $(S,(X,W)) \mapsto (S,X)$. By left Kan extension along the functor (4.31), we obtain from (4.30) a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_! : \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{wc}} \to \mathrm{Pr}^{\mathrm{L}}. \tag{4.32}$$

We gather a few properties satisfied by this functor in the following lemma.

Lemma 4.4.8.

(1) The obvious natural transformation

$$\mathbf{RigSH}^{(\wedge)}_{\tau}((-,-);\Lambda)^*_! \to \mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_! \circ \mathfrak{d}$$

is an equivalence.

(2) Composing (4.32) *with the diagonal functor*

$$\mathrm{RigSpc}^{\mathrm{op}} \to \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{wc}}$$

yields the ordinary functor $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^* : \mathrm{RigSpc}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$.

(3) For a rigid analytic space S, the restriction of (4.32) *to* $\mathrm{RigSpc}^{\mathrm{wc}}/S$ *is equivalent to the functor* $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)_! : \mathrm{RigSpc}^{\mathrm{wc}}/S \to \mathrm{Pr}^{\mathrm{L}}$ *of Proposition 4.3.9.*

Proof. The third assertion follows from [Lur09, Proposition 4.3.3.10]. Using this and Lemma 4.3.14, we deduce the first assertion. For the second assertion we argue as follows. By the first assertion, it suffices to describe the composition of (4.30) with the diagonal functor

$$\mathrm{RigSpc}^{\mathrm{op}} \to \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{WComp}$$

given by $S \mapsto (S,(S,S))$. In this composition, we may replace (4.30) by (4.27) without changing the result. In other words, our functor is the composition of

$$\mathrm{RigSpc}^{\mathrm{op}} \xrightarrow{\Delta} \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{prop}} \xrightarrow{\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_*} \mathrm{Pr}^{\mathrm{L}}$$

where Δ is the diagonal functor given by $S \mapsto (S,S)$. Since Δ is a coCartesian section, the result follows from Remark 4.4.4. □

For later use, we also record the following fact.

Lemma 4.4.9. *Let S be a rigid analytic space, and let $X \in \mathrm{RigSpc}^{\mathrm{wc}}/S$. Then, the composition of* (4.32) *with the functor*

$$(\mathrm{RigSpc}/S)^{\mathrm{op}} \to \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{wc}},$$

given by $T \mapsto (T, T\times_S X)$, is equivalent to the functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-\times_S X;\Lambda)^* : (\mathrm{RigSpc}/S)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}.$$

Proof. We first reduce to the case where the rigid analytic S-space X is proper. To do so, we fix a weak compactification W of X, and consider the functors

$$\Delta_X : (\mathrm{RigSpc}/S)^{\mathrm{op}} \to \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{wc}} \quad \text{and} \quad \Delta_W : (\mathrm{RigSpc}/S)^{\mathrm{op}} \to \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{wc}}$$

given by $T \mapsto (T, T \times_S X)$ and $T \mapsto (T, T \times_S W)$ respectively. The given immersion $i : X \to W$ induces a natural transformation $\mathfrak{i} : \Delta_X \to \Delta_W$. Applying (4.32), we obtain a natural transformation

$$\mathfrak{i}_! : \mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_! \circ \Delta_X \to \mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_! \circ \Delta_W.$$

On $T \in \mathrm{RigSpc}/S$, the natural transformation $\mathfrak{i}_!$ is given by the fully faithful embedding $(T \times_S i)_!$. It follows that $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_! \circ \Delta_X$ can be obtained from $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_! \circ \Delta_W$ by applying Lemma 4.3.13 to the essential images of the functors $(T \times_S i)_!$, for $T \in \mathrm{RigSpc}/S$. Using Proposition 4.3.3(1), we see that it is enough to prove that $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_! \circ \Delta_W$ is given by $\mathbf{RigSH}^{(\wedge)}_{\tau}(- \times_S W;\Lambda)^*$. Said differently, we may assume that X is proper over S.

We now prove the lemma assuming that X is proper over S. (The argument is the same as the one used for the proof of Lemma 4.4.8(2).) By Lemma 4.4.8(1), it is enough to prove the same conclusion for the composition of (4.30) with the functor

$$\Delta'_X : (\mathrm{RigSpc}/S)^{\mathrm{op}} \to \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{WComp},$$

given by $T \mapsto (T, (T \times_S X, T \times_S X))$. In this composition, we may replace (4.30) by (4.27) without changing the result. Since Δ'_X is a coCartesian section, the result follows from Remark 4.4.4. □

By Lemmas 4.4.8 and 4.4.9, the functor (4.32) admits the following informal description.

- It sends an object (S, X), with S a rigid analytic space and X an object of $\mathrm{RigSpc}^{\mathrm{wc}}/S$, to the ∞-category $\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)$.
- It sends an arrow $(S, Y) \to (T, X)$, consisting of morphisms $g : T \to S$ and $f : T \times_S Y \to X$, to the functor $f_! \circ g'^*$ with $g' : T \times_S Y \to Y$ the base change of g.

Finally, we define the functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_! : \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{lft}} \to \mathrm{Pr}^{\mathrm{L}} \tag{4.33}$$

to be the left Kan extension of (4.32) along the fully faithful inclusion

$$\iota : \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{wc}} \to \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{lft}}. \tag{4.34}$$

Note that the functor (4.33) is an extension of (4.32) in the usual sense, i.e., the restriction of (4.33) along ι is indeed the functor (4.32).

Proposition 4.4.10. *For a rigid analytic space S, the restriction of* (4.33) *to* $\mathrm{RigSpc}^{\mathrm{lft}}/S$ *is equivalent to the functor*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)_! : \mathrm{RigSpc}^{\mathrm{lft}}/S \to \mathrm{Pr}^{\mathrm{L}}$$

of Corollary 4.3.18.

Proof. By [Lur09, Proposition 4.3.3.10], it is enough to show that the functor $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)_!$ in Corollary 4.3.18 is a left Kan extension of the same-named functor in Proposition 4.3.9. Using the equivalence $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}$, it is equivalent to show that the functor $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^!$ in Corollary 4.3.18 is the right Kan extension of the same-named functor in Remark 4.3.16. Since the former was defined as the unique Pr^{R}-valued sheaf for the analytic topology extending the latter, the result follows from Lemma 4.4.11 below. □

Lemma 4.4.11. *Let $(\mathcal{C}', \tau')$ be a site with $\mathcal{C}'$ an ordinary category admitting finite limits. Let $\mathcal{C} \subset \mathcal{C}'$ be a full subcategory closed under finite limits and let τ be the induced topology on $\mathcal{C}$. Assume that the morphism of sites $(\mathcal{C}', \tau') \to (\mathcal{C}, \tau)$ induces an equivalence between the associated ordinary topoi. (Equivalently, every object of $\mathcal{C}'$ admits a cover by objects in $\mathcal{C}$.) Let $\mathcal{D}$ be an ∞-category admitting limits and let $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be a $\mathcal{D}$-valued τ-sheaf on $\mathcal{C}$. Then, the right Kan extension $F' : \mathcal{C}'^{\mathrm{op}} \to \mathcal{D}$ of F along the inclusion $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}'^{\mathrm{op}}$ is a τ'-sheaf. More precisely, F' is the image of F by the equivalence of ∞-categories $\mathrm{Shv}_\tau(\mathcal{C}; \mathcal{D}) \xrightarrow{\sim} \mathrm{Shv}_{\tau'}(\mathcal{C}'; \mathcal{D})$.*

Proof. By Lemma 2.1.4, we have an equivalence of ∞-topoi $\mathrm{Shv}_{\tau'}(\mathcal{C}') \simeq \mathrm{Shv}_\tau(\mathcal{C})$. Since $\mathrm{Shv}_\tau(\mathcal{C}; \mathcal{D})$ can be identified with the ∞-category of limit-preserving functors from $\mathrm{Shv}_\tau(\mathcal{C})$ to $\mathcal{D}$, and similarly for $\mathcal{C}'$, we deduce an equivalence of ∞-categories $\mathrm{Shv}_{\tau'}(\mathcal{C}'; \mathcal{D}) \simeq \mathrm{Shv}_\tau(\mathcal{C}; \mathcal{D})$. This equivalence is given by the restriction functor. Since the restriction of F' to $\mathcal{C}$ is equivalent to F, we only need to prove that F' is a τ'-sheaf. For $d \in \mathcal{D}$, denote by $\mathrm{y}(d) : \mathcal{D} \to \mathcal{S}$ the copresheaf corepresented by d. The functors $\mathrm{y}(d)$, for $d \in \mathcal{D}$, form a conservative family of limit-preserving functors. Thus, it is enough to show that $\mathrm{y}(d)(F')$ is a τ'-sheaf for every $d \in \mathcal{D}$. Since $\mathrm{y}(d)(F')$ is the right Kan extension of $\mathrm{y}(d)(F)$, we are reduced to prove the lemma with $\mathcal{D}$ the ∞-category of spaces $\mathcal{S}$.

Recall that we need to show that F' is a sheaf. Since $\mathcal{D} = \mathcal{S}$, we have at our disposal the sheafification functors, and these commute with restriction along the inclusion $\mathcal{C} \to \mathcal{C}'$. Let F'' be the τ'-sheaf associated to F'. Since $F'|_{\mathcal{C}} \simeq F$ is already a τ-sheaf, it follows that $F' \to F''$ induces an equivalence after restriction to $\mathcal{C}$. By the universal property of the right Kan extension, there must be a map $F'' \to F'$ such that $F' \to F'' \to F'$ is homotopic to the identity of F'. Thus, F' is a retract of the τ'-sheaf F''. This proves that F' is also a τ'-sheaf (and that $F' \simeq F''$). □

Remark 4.4.12. The category

$$\mathfrak{Q} = \left(\int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{lft}} \right)^{\mathrm{op}}$$

admits a natural topology, called the analytic topology and denoted by "an". It is induced by a pretopology $\mathrm{Cov}_{\mathrm{an}}$ in the sense of [SGAIV1, Exposé II, Définition 1.3], which is given as follows. For $(S, X) \in \mathfrak{Q}$, a family $((S_i, X_i) \to (S, X))_i$ belongs to $\mathrm{Cov}_{\mathrm{an}}(S, X)$ if $(S_i \to S)_i$ is an open cover of S and the morphisms $S_i \times_S X \to X_i$ are isomorphisms.

Proposition 4.4.13. *The functor* (4.33) *is a sheaf for the analytic topology on $\mathfrak{Q}$.*

Proof. Fix an object (S_{-1}, X) in $\mathfrak{Q}$ and let $S_\bullet$ be a truncated hypercover of S_{-1} in the analytic topology. We assume that the S_n's are coproducts of open subspaces of S_{-1}. For $n \in \mathbb{N}$, we set $X_n = S_n \times_{S_{-1}} X$ and similarly for every rigid analytic S_{-1}-space. We need to show that

$$\mathbf{RigSH}^{(\wedge)}_\tau((S_{-1}, X); \Lambda)^*_! \to \lim_{[n] \in \mathbf{\Delta}} \mathbf{RigSH}^{(\wedge)}_\tau((S_n, X_n); \Lambda)^*_! \tag{4.35}$$

is an equivalence. By Lemma 4.4.11, the functor

$$\mathbf{RigSH}^{(\wedge)}_\tau(S_n \times_{S_{-1}} -; \Lambda)_! : \mathrm{Op}/X \to \mathrm{Pr}^{\mathrm{L}}$$

is the left Kan extension of its restriction to the subcategory $\mathrm{Op}^{\mathrm{wc}}/X \subset \mathrm{Op}/X$ spanned by those open subspaces of X which are weakly compactifiable over S_{-1}. Using Proposition 4.4.10, we deduce that

$$\mathbf{RigSH}_{\tau}^{(\wedge)}((S_n, X_n); \Lambda)_!^* \simeq \operatorname*{colim}_{U \in \mathrm{Op}^{\mathrm{wc}}/X} \mathbf{RigSH}_{\tau}^{(\wedge)}((S_n, U_n); \Lambda)_!^*$$

where the colimit is taken in Pr^{L}. Thus, we are reduced to showing that

$$\operatorname*{colim}_{U \in \mathrm{Op}^{\mathrm{wc}}/X} \mathbf{RigSH}_{\tau}^{(\wedge)}((S_{-1}, U); \Lambda)_!^* \to \lim_{[n] \in \mathbf{\Delta}} \operatorname*{colim}_{U \in \mathrm{Op}^{\mathrm{wc}}/X} \mathbf{RigSH}_{\tau}^{(\wedge)}((S_n, U_n); \Lambda)_!^* \tag{4.36}$$

is an equivalence. We want to apply [Lur17, Proposition 4.7.4.19] for commuting the limit with the colimit in the right-hand side of (4.36). For this, we need to show that for every $[n'] \to [n]$ in $\mathbf{\Delta}$ and every inclusion $U \to U'$ in $\mathrm{Op}^{\mathrm{wc}}/X$, the associated square

$$\begin{tikzcd} \mathbf{RigSH}_{\tau}^{(\wedge)}((S_n, U_n); \Lambda)_!^* \arrow[r] \arrow[d] & \mathbf{RigSH}_{\tau}^{(\wedge)}((S_n, U'_n); \Lambda)_!^* \arrow[d] \\ \mathbf{RigSH}_{\tau}^{(\wedge)}((S_{n'}, U_{n'}); \Lambda)_!^* \arrow[r] & \mathbf{RigSH}_{\tau}^{(\wedge)}((S_{n'}, U'_{n'}); \Lambda)_!^* \end{tikzcd}$$

is right adjointable. Let $g : S_{n'} \to S_n$ be the morphism induced by $[n'] \to [n]$, and let $g' : U_{n'} \to U_n$ and $g'' : U'_{n'} \to U'_n$ be the morphisms obtained by base change. Let $u : U \to U'$ be the obvious inclusion, and let $u_n : U_n \to U'_n$ and $u_{n'} : U_{n'} \to U'_{n'}$ be the morphisms obtained by base change. Then, using Lemma 4.4.8, and looking back at the construction of (4.30), we see that the above square is equivalent to

$$\begin{tikzcd} \mathbf{RigSH}_{\tau}^{(\wedge)}(U_n; \Lambda) \arrow[r, "u_{n,\sharp}"] \arrow[d, "g'^*"] & \mathbf{RigSH}_{\tau}^{(\wedge)}(U'_n; \Lambda) \arrow[d, "g''^*"] \\ \mathbf{RigSH}_{\tau}^{(\wedge)}(U_{n'}; \Lambda) \arrow[r, "u_{n',\sharp}"] & \mathbf{RigSH}_{\tau}^{(\wedge)}(U'_{n'}; \Lambda) \end{tikzcd}$$

which is clearly right adjointable. Thus, [Lur17, Proposition 4.7.4.19] applies, and we are left to showing that

$$\mathbf{RigSH}_{\tau}^{(\wedge)}((S_{-1}, U); \Lambda)_!^* \to \lim_{[n] \in \mathbf{\Delta}} \mathbf{RigSH}_{\tau}^{(\wedge)}((S_n, U_n); \Lambda)_!^* \tag{4.37}$$

is an equivalence for every $U \in \mathrm{Op}^{\mathrm{wc}}/X$. Said differently, we may assume that X is weakly compactifiable. In this case, we may use Lemma 4.4.9 to rewrite (4.35) as follows:

$$\mathbf{RigSH}_{\tau}^{(\wedge)}(X; \Lambda)^* \to \lim_{[n] \in \mathbf{\Delta}} \mathbf{RigSH}_{\tau}^{(\wedge)}(X_n; \Lambda)^* \tag{4.38}$$

which is indeed an equivalence by Theorem 2.3.4. □

At this stage, Theorem 4.4.2 is proven, except for the assertion that the functors in (4.21) take an object (S, X) to $\mathbf{RigSH}_{\tau}^{(\wedge)}(X; \Lambda)$. We do know this when X weakly compactifiable over S. In order to establish this in general, we will need a few more results about the functors in (4.23). We first introduce a notation which is useful in discussing these results.

Notation 4.4.14. The functors in (4.23) depend on S. To highlight this dependency, we use "$!_S$" in subscript and superscript instead of "!". More explicitly, we denote by $\mathbf{RigSH}_{\tau}^{(\wedge)}(-; \Lambda)_{!_S}$ and $\mathbf{RigSH}_{\tau}^{(\wedge)}(-; \Lambda)^{!_S}$ these functors. Also, given a morphism $f : Y \to X$ in $\mathrm{RigSpc}^{\mathrm{lft}}/S$, we sometimes denote by $f_{!_S}$ and $f^{!_S}$ the images of f by these functors.

Lemma 4.4.15. *Let S be a rigid analytic space and $f : Y \to X$ a morphism in* $\mathrm{RigSpc}^{\mathrm{lft}}/S$*. Let $g : S' \to S$ be a morphism of rigid analytic spaces, and consider the Cartesian square*

$$\begin{tikzcd} Y' \arrow[r, "g''"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g'"] & X \end{tikzcd}$$

where f' is the base change of f by g. Consider the commutative square

$$\begin{tikzcd} \mathbf{RigSH}^{(\wedge)}_{\tau}(X';\Lambda)^{!_{S'}} \arrow[r, "g'_*"] \arrow[d, "f'^{!_{S'}}"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_S} \arrow[d, "f^{!_S}"] \\ \mathbf{RigSH}^{(\wedge)}_{\tau}(Y';\Lambda)^{!_{S'}} \arrow[r, "g''_*"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{!_S}, \end{tikzcd}$$

where g'_* is obtained by applying the second functor in (4.21) *to the arrow* $(g, \mathrm{id}_{X'}) : (S', X') \to (S, X)$ *and similarly for g''_*. This square is left adjointable if f or g is an open immersion.*

Proof. We may consider the commutative square in the statement as a morphism $(f^{!_{S'}}, f^{!_S})$ in $\mathrm{Fun}(\Delta^1, \mathrm{CAT}_\infty)$ between the functors g'_* and g''_*, and our goal is to show that this morphism belongs to the sub-∞-category $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{CAT}_\infty)$ introduced in [Lur17, Definition 4.7.4.16]. By [Lur17, Corollary 4.7.4.18], it would be enough to show that the morphism $(f^{!_{S'}}, f^{!_S})$ is the limit of an inverse system of morphisms in $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{CAT}_\infty)$. By Proposition 4.4.10, the morphism $(f^{!_{S'}}, f^{!_S})$ is the limit of morphisms in $\mathrm{Fun}(\Delta^1, \mathrm{CAT}_\infty)$ given by the following commutative squares

$$\begin{tikzcd} \mathbf{RigSH}^{(\wedge)}_{\tau}(S' \times_S U;\Lambda)^{!_{S'}} \arrow[r] \arrow[d] & \mathbf{RigSH}^{(\wedge)}_{\tau}(U;\Lambda)^{!_S} \arrow[d] \\ \mathbf{RigSH}^{(\wedge)}_{\tau}(S' \times_S V;\Lambda)^{!_{S'}} \arrow[r] & \mathbf{RigSH}^{(\wedge)}_{\tau}(V;\Lambda)^{!_S}, \end{tikzcd}$$

where $U \subset X$ and $V \subset Y \times_X U$ are open subspaces which are weakly compactifiable over S. Moreover, the transition maps in this inverse system are given by commutative squares of the same type. Therefore, it is enough the show that these squares are left adjointable, and thus we may assume that X and Y are weakly compactifiable over S. In this case, we may use the explicit construction in Definition 4.3.4 and Theorem 4.1.4(2) to conclude. □

Lemma 4.4.16. *Let S be a rigid analytic space and $j : S' \to S$ an open immersion. Let $Y \in$* $\mathrm{RigSpc}^{\mathrm{lft}}/S$ *such that the structure morphism $Y \to S$ factors through S'. Then, there exists an equivalence of ∞-categories*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{!_S} \simeq \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{!_{S'}} \tag{4.39}$$

such that the following condition is satisfied. For every morphism $f : Y \to X$ in $\mathrm{RigSpc}^{\mathrm{lft}}/S$*, the functor $f^{!_S}$ is equivalent, modulo* (4.39)*, to the composition of*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_S} \xrightarrow{j'^*} \mathbf{RigSH}^{(\wedge)}_{\tau}(X';\Lambda)^{!_{S'}} \xrightarrow{f'^{!_{S'}}} \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{!_{S'}},$$

where $X' = S' \times_S X$, and $j' : X' \to X$ and $f' : Y \to X'$ are the obvious morphisms.

Proof. The image of the arrow $(j, \mathrm{id}) : (S, Y) \to (S', Y)$ by the first functor in (4.21) is a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{!_S} \to \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{!_{S'}} \tag{4.40}$$

such that, for every $f : Y \to X$ as in the statement, the square

$$\begin{tikzcd}
\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_S} \arrow[r, "f^{!_S}"] \arrow[d, "j'^*"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{!_S} \arrow[d, "(4.40)"] \\
\mathbf{RigSH}^{(\wedge)}_{\tau}(X';\Lambda)^{!_{S'}} \arrow[r, "f'^{!_{S'}}"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{!_{S'}}
\end{tikzcd}$$

is commutative by Lemma 4.4.15. Thus, to finish the proof, it is enough to show that (4.40) is an equivalence of ∞-categories. By Proposition 4.4.10, the question is local on Y. (Indeed, we may as well prove that the right adjoint of (4.40) is an equivalence of ∞-categories.) Thus, we may assume that Y is weakly compactifiable over S. In this case, we may use the explicit construction in Definition 4.3.4 to conclude. □

Lemma 4.4.17. *Let S be a rigid analytic space and let $j : U \to X$ an open immersion in* $\mathrm{RigSpc}^{\mathrm{lft}}/S$*. Then the functor*

$$j^{!_S} : \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_S} \to \mathbf{RigSH}^{(\wedge)}_{\tau}(U;\Lambda)^{!_S}$$

belongs to Pr^{L} *and hence admits a right adjoint, which we denote by $j_{?_S}$.*

Proof. Indeed, by Proposition 4.4.10, $j^{!_S}$ is a limit in CAT_∞ of functors of the form

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(V;\Lambda)^{!_S} \to \mathbf{RigSH}^{(\wedge)}_{\tau}(U \cap V;\Lambda)^{!_S}$$

for open subspaces $V \subset X$ which are compactifiable over S. By [Lur09, Proposition 5.5.3.13], it is thus enough to prove that $j^{!_S}$ is in Pr^{L} when j is an open immersion between weakly compactifiable rigid analytic S-spaces. In this case, we know that $j^{!_S}$ is equivalent to j^*, and the result follows. □

Lemma 4.4.18. *Let S be a rigid analytic space, and consider a Cartesian square in* $\mathrm{RigSpc}^{\mathrm{lft}}/S$

$$\begin{tikzcd}
V \arrow[r, "v"] \arrow[d, "g"] & Y \arrow[d, "f"] \\
U \arrow[r, "u"] & X,
\end{tikzcd}$$

with u an open immersion (resp. a closed immersion). Then, the commutative square

$$\begin{tikzcd}
\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_S} \arrow[r, "u^{!_S}"] \arrow[d, "f^{!_S}"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(U;\Lambda)^{!_S} \arrow[d, "g^{!_S}"] \\
\mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{!_S} \arrow[r, "v^{!_S}"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(V;\Lambda)^{!_S},
\end{tikzcd}$$

is right adjointable (resp. left adjointable).

Proof. We only consider the case of open immersions; the case of closed immersions is similar. Using Proposition 4.4.10, [Lur17, Corollary 4.7.4.18] and arguing as in the proof of Lemma 4.4.15, we reduce to show the lemma when X and Y are weakly compactifiable over S. In this case, the commutative square of the statement coincides with the one deduced by adjunction from

$$\begin{tikzcd}
\mathbf{RigSH}^{(\wedge)}_{\tau}(V;\Lambda) \arrow[r, "v_\sharp"] \arrow[d, "g_!"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda) \arrow[d, "f_!"] \\
\mathbf{RigSH}^{(\wedge)}_{\tau}(U;\Lambda) \arrow[r, "u_\sharp"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda).
\end{tikzcd}$$

The right adjointability of this square is clear: it follows from the construction of the exceptional direct image functors given in Definition 4.3.4 and Proposition 2.2.1(3). $\square$

Construction 4.4.19. Let S be a rigid analytic space and let $i : Z \to X$ be a locally closed immersion in $\mathrm{RigSpc}^{\mathrm{lft}}/S$. We define a functor

$$i_{?_S} : \mathbf{RigSH}_\tau^{(\wedge)}(Z;\Lambda)_{!_S} \to \mathbf{RigSH}_\tau^{(\wedge)}(X;\Lambda)_{!_S}$$

as follows. Choose an open subspace $U \subset X$ containing Z as a closed subspace, and let $s : Z \to U$ and $j : U \to X$ be the obvious immersions. Define $i_{?_S}$ to be the composite functor $j_{?_S} \circ s_{!_S}$.

Lemma 4.4.20. *Keep the notations of Construction 4.4.19. The functor $i_{?_S}$ is independent of the choice of the open neighbourhood U.*

Proof. Let $U' \subset U$ be an open neighbourhood of Z contained in U. Let $s' : Z \to U'$ and $u : U' \to U$ be the obvious immersions. We need to show that $u_{?_S} \circ s'_{!_S} \simeq s_{!_S}$. We have a Cartesian square

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, "s'"] & Z \arrow[d, "s"] \\ U' \arrow[r, "u"] & U \end{tikzcd}$$

which induces an equivalence $s'^{!_S} \simeq s^{!_S} \circ u_{?_S}$ by Lemma 4.4.18. From this equivalence, we deduce a natural transformation $s_{!_S} \to u_{?_S} \circ s'_{!_S}$. This natural transformation is an equivalence. Indeed, it is enough to check this after applying $u^{!_S}$ and $v^{!_S}$, with $v : U \smallsetminus Z \to U$ the obvious inclusion, and this is easily seen to be true using Lemma 4.4.18 again. $\square$

Lemma 4.4.21. *Let S be a rigid analytic space and $i : Z \to X$ a locally closed immersion in* $\mathrm{RigSpc}^{\mathrm{lft}}/S$*. Let $g : S' \to S$ be a morphism of rigid analytic spaces, and consider the Cartesian square*

$$\begin{tikzcd} Z' \arrow[r, "g''"] \arrow[d, "i'"] & Z \arrow[d, "i"] \\ X' \arrow[r, "g'"] & X \end{tikzcd}$$

where i' is the base change of i by g. Then, there is a commutative square of ∞-categories

$$\begin{tikzcd} \mathbf{RigSH}_\tau^{(\wedge)}(Z';\Lambda)^{!_{S'}} \arrow[r, "g''_*"] \arrow[d, "i'_{?_{S'}}"] & \mathbf{RigSH}_\tau^{(\wedge)}(Z;\Lambda)^{!_S} \arrow[d, "i_{?_S}"] \\ \mathbf{RigSH}_\tau^{(\wedge)}(X';\Lambda)^{!_{S'}} \arrow[r, "g'_*"] & \mathbf{RigSH}_\tau^{(\wedge)}(X;\Lambda)^{!_S} \end{tikzcd}$$

(In the above square, g'_* is obtained by applying the second functor in (4.21) *to the arrow* $(g, \mathrm{id}_{X'}) : (S', X') \to (S, X)$ *and similarly for g''_*.)*

Proof. When i is an open immersion, this follows from Lemma 4.4.15. Thus, we may assume that i is a closed immersion, and we need to prove the analogous statement for the functors $i_{!_S}$ and $i'_{!_{S'}}$. Arguing as in the proof of Lemma 4.4.15, we reduce to the case where X is weakly compactifiable. In this case, the functors $i_{!_S}$ and $i'_{!_{S'}}$ coincide with i_* and i'_*, and the result follows. $\square$

Theorem 4.4.22. *Let S be a rigid analytic space and let $T \in \mathrm{RigSpc}^{\mathrm{lft}}/S$. There is a commutative triangle*

$$\begin{tikzcd}
(\mathrm{RigSpc}^{\mathrm{lft}}/T)^{\mathrm{op}} \arrow[rr] \arrow[dr, "\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{!_T}"'] & & (\mathrm{RigSpc}^{\mathrm{lft}}/S)^{\mathrm{op}} \arrow[dl, "\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{!_S}"] \\
& \mathrm{Pr}^{\mathrm{R}}, &
\end{tikzcd}$$

where the horizontal arrow is the forgetful functor. For $X \in \mathrm{RigSpc}^{\mathrm{lft}}/T$, the induced equivalence of ∞-categories

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_T} \xrightarrow{\sim} \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_S} \tag{4.41}$$

is obtained as follows. Consider the commutative diagram with a Cartesian square

$$\begin{tikzcd}
X \arrow[r, "\delta_X"] \arrow[dr] & T \times_S X \arrow[r, "\mathrm{pr}_X"] \arrow[d] & X \arrow[d] \\
& T \arrow[r, "g"] & S.
\end{tikzcd}$$

Then, the equivalence (4.41) *is the composition of*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_T} \xrightarrow{(\delta_X)_{?_T}} \mathbf{RigSH}^{(\wedge)}_{\tau}(T \times_S X;\Lambda)^{!_T} \xrightarrow{(\mathrm{pr}_X)_*} \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_S}. \tag{4.42}$$

(Here, we denote by $(\mathrm{pr}_X)_*$ the image by the functor $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{!}_{*}$ of the arrow $(g, \mathrm{id}_{T\times_S X}) : (S, X) \to (T, T \times_S X)$.)

Proof. By Proposition 4.4.10 and Lemma 4.4.18, the composite functors (4.42) are part of a morphism of Pr^{R}-valued sheaves on $\mathrm{RigSpc}^{\mathrm{lft}}/T$

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{!_T} \to \mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{!_S}|_{\mathrm{RigSpc}^{\mathrm{lft}}/T}.$$

Thus, it is enough to prove that the composite functor (4.42) is an equivalence under the following assumptions:

- X is weakly compactifiable over S;
- $X \to T$ factors by an open subspace $T' \subset T$ which is weakly compactifiable over S.

The morphism $\delta_X : X \to T \times_S X$ is the composition of the open immersion $j : T' \times_S X \to T \times_S X$ and the morphism $\delta'_X : X \to T' \times_S X$. We deduce that the composition of (4.42) is equivalent to the composition of

$$\begin{tikzcd}
\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_T} \arrow[r, "(\delta'_X)_{?_T}"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(T' \times_S X;\Lambda)^{!_T} \arrow[d, "j_{?_T}"] & \\
& \mathbf{RigSH}^{(\wedge)}_{\tau}(T \times_S X;\Lambda)^{!_T} \arrow[r, "(\mathrm{pr}_X)_*"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_S}
\end{tikzcd}$$

By Lemma 4.4.16, the functor $j^{!_T}$ is equivalent to the composition of

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(T \times_S X;\Lambda)^{!_T} \xrightarrow{j^*} \mathbf{RigSH}^{(\wedge)}_{\tau}(T' \times_S X;\Lambda)^{!_{T'}} \simeq \mathbf{RigSH}^{(\wedge)}_{\tau}(T' \times_S X;\Lambda)^{!_T}.$$

It follows that the functor $j_{?_T}$ is equivalent to the composition of

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(T' \times_S X;\Lambda)^{!_T} \simeq \mathbf{RigSH}^{(\wedge)}_{\tau}(T' \times_S X;\Lambda)^{!_{T'}} \xrightarrow{j_*} \mathbf{RigSH}^{(\wedge)}_{\tau}(T \times_S X;\Lambda)^{!_T}.$$

Thus, modulo the equivalence $\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_T} \simeq \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_{T'}}$, the composition of (4.42) is equivalent to the composition of

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_{T'}} \xrightarrow{(\delta'_X)_{?_{T'}}} \mathbf{RigSH}^{(\wedge)}_{\tau}(T' \times_S X;\Lambda)^{!_{T'}} \xrightarrow{(\mathrm{pr}'_X)_*} \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_S}$$

where $\mathrm{pr}'_X = \mathrm{pr}_X \circ j$. Therefore, it is enough to prove the theorem with T replaced by T'. Said differently, we may assume that X and T are weakly compactifiable over S. In this case, the diagram (4.42) can be identified with

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{*} \xrightarrow{(\delta_X)_*} \mathbf{RigSH}^{(\wedge)}_{\tau}(T \times_S X;\Lambda)^{*} \xrightarrow{(\mathrm{pr}_X)_*} \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{*}$$

whose composition is clearly an equivalence. □

Corollary 4.4.23. *For every rigid analytic space S and every $X \in \mathrm{RigSpc}^{\mathrm{lft}}/S$, there is an equivalence of ∞-categories*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{*} \xrightarrow{\sim} \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_S}. \tag{4.43}$$

Moreover, these equivalences satisfy the following properties.

(1) Given a Cartesian square of rigid analytic spaces

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

with f locally of finite type, there is a commutative square of ∞-categories

$$\begin{tikzcd} \mathbf{RigSH}^{(\wedge)}_{\tau}(X';\Lambda)^{*} \arrow[r, "g'_*"] \arrow[d, "\sim"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{*} \arrow[d, "\sim"] \\ \mathbf{RigSH}^{(\wedge)}_{\tau}(X';\Lambda)^{!_{S'}} \arrow[r, "g'_*"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_S}. \end{tikzcd}$$

(2) Given a rigid analytic space S and an open immersion $j : X' \to X$ in $\mathrm{RigSpc}^{\mathrm{lft}}/S$, we have a commutative square

$$\begin{tikzcd} \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{*} \arrow[r, "j^*"] \arrow[d, "\sim"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(X';\Lambda)^{*} \arrow[d, "\sim"] \\ \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{!_S} \arrow[r, "j^{!_S}"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(X';\Lambda)^{!_S}. \end{tikzcd}$$

Proof. The equivalence (4.43) is the equivalence (4.41) when $X = T$. Property (1) follows easily from the construction of the equivalence (4.43) and Lemma 4.4.21. Property (2) follows from Theorem 4.4.22 combined with Corollary 4.3.15. □

Theorem 4.4.22 shows that the exceptional functors are independent of the base, i.e., the functors $f_{!_S}$ and $f^{!_S}$ are independent of S up to equivalence. Note also that, by Proposition 4.4.10 and Corollary 4.3.18, these functors extend the ones of Definition 4.3.4. This justifies the following definition.

Definition 4.4.24. Let $f : Y \to X$ be a morphism of rigid analytic spaces which is locally of finite type. The functors in adjunction

$$f_! : \mathbf{RigSH}^{(\wedge)}_{\tau}(Y; \Lambda) \rightleftarrows \mathbf{RigSH}^{(\wedge)}_{\tau}(X; \Lambda) : f^!$$

are defined to be the images of the arrow $(\mathrm{id}_X, f) : (X, Y) \to (X, X)$ by the functors in (4.21) modulo the equivalence $\mathbf{RigSH}^{(\wedge)}_{\tau}(Y; \Lambda) \simeq \mathbf{RigSH}^{(\wedge)}_{\tau}(Y; \Lambda)^{!_X}$ given by Corollary 4.4.23. The functors $f_!$ and $f^!$ are called the exceptional direct and inverse image functors.

Remark 4.4.25. Given two morphisms $f : Y \to X$ and $g : Z \to Y$ which are locally of finite type, we have equivalences $f_! \circ g_! \simeq (f \circ g)_!$ and $g^! \circ f^! \simeq (f \circ g)^!$. (This follows from the construction and the equivalences $f_{!_X} \circ g_{!_X} \simeq (f \circ g)_{!_X}$ and $g^{!_X} \circ f^{!_X} \simeq (f \circ g)^{!_X}$.) Therefore, one expects to have functors, from the wide subcategory of RigSpc spanned by locally of finite type morphisms, to Pr^{L} and $(\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}}$, sending a morphism f to the functors $f_!$ and $f^!$. Our method does not give readily such a functor, but techniques from [GR17, Part III] might do. (See Theorem 4.4.31 and Remark 4.4.32 below.)

Proposition 4.4.26. *Consider a Cartesian square of rigid analytic spaces*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

with f locally of finite type. Then, there is a commutative square of ∞-categories

$$\begin{tikzcd} \mathbf{RigSH}^{(\wedge)}_{\tau}(Y; \Lambda) \arrow[r, "g'^*"] \arrow[d, "f_!"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(Y'; \Lambda) \arrow[d, "f'_!"] \\ \mathbf{RigSH}^{(\wedge)}_{\tau}(X; \Lambda) \arrow[r, "g^*"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(X'; \Lambda). \end{tikzcd}$$

Proof. Applying the first functor in (4.21) to the commutative square

$$\begin{tikzcd} (X, Y) \arrow[r] \arrow[d] & (X', Y') \arrow[d] \\ (X, X) \arrow[r] & (X', X'), \end{tikzcd}$$

we get a commutative square of ∞-categories

$$\begin{tikzcd} \mathbf{RigSH}^{(\wedge)}_{\tau}(Y; \Lambda)^{!_X} \arrow[r, "g'^*"] \arrow[d, "f_{!_X}"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(Y'; \Lambda)^{!_{X'}} \arrow[d, "f'_{!_{X'}}"] \\ \mathbf{RigSH}^{(\wedge)}_{\tau}(X; \Lambda)^{!_X} \arrow[r, "g^*"] & \mathbf{RigSH}^{(\wedge)}_{\tau}(X'; \Lambda)^{!_{X'}}. \end{tikzcd}$$

The result follows then from Corollary 4.4.23. $\square$

Proposition 4.4.27. *The composition of the first functor in* (4.21) *with the obvious inclusion*

$$\int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{prop}} \to \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{lft}}$$

is equivalent to the functor (4.25)*. In particular, if $f : Y \to X$ is a proper morphism of rigid analytic spaces, there is an equivalence $f_! \simeq f_*$.*

Proof. This is a direct consequence of the construction. □

Corollary 4.4.28. *Let $f : Y \to X$ be a morphism of rigid analytic spaces. Assume that f admits a factorization $f = p \circ j$ where j is an open immersion and p is a proper morphism. Then, there is an equivalence $f_! \simeq p_* \circ j_\sharp$.*

Proof. This follows from Corollary 4.4.23(2), Remark 4.4.25 and Proposition 4.4.27. □

Theorem 4.4.29 (Ambidexterity)**.** *Let $f : Y \to X$ be a smooth morphism between rigid analytic spaces. There are equivalences $f_! \simeq f_\sharp \circ \mathrm{Th}^{-1}(\Omega_f)$ and $f^! \simeq \mathrm{Th}(\Omega_f) \circ f^*$.*

Proof. We first construct a natural transformation $\alpha_f : f_\sharp \to f_! \circ \mathrm{Th}(\Omega_f)$. Consider the commutative diagram with a Cartesian square

$$\begin{tikzcd} Y \arrow[dr, "\Delta_f"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] & & \\ & Y \times_X Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "f"] \\ & Y \arrow[r, "f"] & X. \end{tikzcd}$$

By Proposition 4.4.26, we have an equivalence $p_{1,!} \circ p_2^* \simeq f^* \circ f_!$. Using the adjunctions $(f_\sharp, f^*)$ and $(p_{2,\sharp}, p_2^*)$, we deduce a natural transformation $f_\sharp \circ p_{1,!} \to f_! \circ p_{2,\sharp}$. Applying the latter to $\Delta_{f,!}$ and using the equivalences $p_{1,!} \circ \Delta_{f,!} \simeq \mathrm{id}$ and $p_{2,\sharp} \circ \Delta_{f,!} \simeq \mathrm{Th}(\Omega_f)$, we get α_f.

We next show that α_f is an equivalence. It is easy to see that α_f is compatible with composition, i.e., that the analogue of [Ayo07a, Proposition 1.7.3] is satisfied. Moreover, if j is an open immersion, α_j is the equivalence $j_\sharp \simeq j_!$. Thus, to show that α_f is invertible, we may argue locally on Y for the analytic topology. Thus, we may assume that Y is weakly compactifiable over X. Choose a weak compactification $i : Y \to W$ and let $g : W \to X$ be the structural morphism. To prove that α_f is invertible, it is enough to show that the natural transformation $f_\sharp \circ p_{1,!} \to f_! \circ p_{2,\sharp}$ is invertible. Unwinding the definitions, we see that it is enough to prove that the natural transformation $f_\sharp \circ \overline{q}_* \to \overline{f}_* \circ q_\sharp$ associated to the Cartesian square

$$\begin{tikzcd} Y \times_X W \arrow[r, "f'"] \arrow[d, "g'"] & W \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

is an equivalence. This is indeed true by Theorem 4.1.4(2). □

There is another way to encapsulate much of the six-functor formalism using $(\infty, 2)$-categories of correspondences (aka., spans). This gives an alternative approach to the constructions of this subsection which is more elegant and more powerful. The technology needed to carry out this approach is developed in [GR17, Part III] but relies, unfortunately, on yet unproven hypotheses in the theory of $(\infty, 2)$-categories; see [GR17, Chapter 10, §0.4]. It is for this reason that we decided to develop a more self-contained approach. However, for the reader who is willing to accept the unproven hypotheses in loc. cit., we briefly explain how this is supposed to work. For a similar discussion in the context of equivariant motives, see [Hoy17, §6.2].

Remark 4.4.30. Given an ∞-category $\mathcal{C}$ with finite limits, there is an associated $(\infty,2)$-category $\mathrm{Corr}(\mathcal{C})$ having the same objects as $\mathcal{C}$, and where 1-morphisms between X and Y are given by spans

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & Y, \end{tikzcd}$$

i.e., maps $(f,g) : Z \to X \times Y$. Given a second span $(f',g') : Z' \to X \times Y$, a 2-morphism $(f',g') \Rightarrow (f,g)$ is a morphism $h : Z' \to Z$ such that $g' = gh$ and $f' = fh$. If P_1, P_2 and P_3 are properties of morphisms in $\mathcal{C}$, we denote by $\mathrm{Corr}(\mathcal{C})^{P_3}_{P_1,P_2}$ the subcategory obtained by imposing P_1, P_2 and P_3 on the morphisms f, g and h above. For details, on the $(\infty,2)$-category $\mathrm{Corr}(\mathcal{C})$, we refer the reader to [GR17, Chapter 7, §1.2]. Below, we will be interested in the $(\infty,2)$-category $\mathrm{Corr}(\mathrm{RigSpc})^{\mathrm{proper}}_{\mathrm{all,wc}}$, where 2-morphisms are given by proper maps, and right legs of spans are requested to be weakly compactifiable while no condition is imposed on left legs.

Theorem 4.4.31. *There is a* 2*-functor*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda) : (\mathrm{Corr}(\mathrm{RigSpc})^{\mathrm{proper}}_{\mathrm{all,wc}})^{2\text{-op}} \to \mathrm{Pr}^{\mathrm{L}} \tag{4.44}$$

sending a span of the form $X \xleftarrow{f} Y \xrightarrow{\mathrm{id}} Y$ *to* f^* *and a span of the form* $Y \xleftarrow{\mathrm{id}} Y \xrightarrow{f} X$ *to* $f_!$. *(Above,* Pr^{L} *is considered as an* $(\infty,2)$*-category in the natural way, i.e., where* 2*-morphisms are given by natural transformations.)*

Proof. We denote by "prop" (resp. "iso", "open", "closed", "imm") the class of proper morphisms (resp. isomorphisms, open immersions, closed immersions, locally closed immersions) in RigSpc. By [GR17, Chapter 7, Theorem 3.2.2] and Theorem 4.1.4(1), there exists a unique 2-functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda) : (\mathrm{Corr}(\mathrm{RigSpc})^{\mathrm{prop}}_{\mathrm{all,prop}})^{2\text{-op}} \to \mathrm{CAT}_{\infty} \tag{4.45}$$

extending the functor $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^* : \mathrm{RigSpc}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$. Also, by the same theorem of loc. cit., there exists a unique 2-functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda) : (\mathrm{Corr}(\mathrm{RigSpc})^{\mathrm{iso}}_{\mathrm{all,open}})^{2\text{-op}} \to \mathrm{CAT}_{\infty} \tag{4.46}$$

extending the same functor. In particular, these two extensions coincide on $(\mathrm{RigSpc}^{\mathrm{qcqs}})^{\mathrm{op}}$. By [GR17, Chapter 7, Theorem 5.2.4] and Proposition 2.2.3, we may glue uniquely (4.46) with the restriction of (4.45) to $(\mathrm{Corr}(\mathrm{RigSpc})^{\mathrm{iso}}_{\mathrm{all,closed}})^{2\text{-op}}$ and get a 2-functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda) : (\mathrm{Corr}(\mathrm{RigSpc})^{\mathrm{iso}}_{\mathrm{all,imm}})^{2\text{-op}} \to \mathrm{CAT}_{\infty} \tag{4.47}$$

By a second application of [GR17, Chapter 7, Theorem 5.2.4] and using Proposition 4.3.3, we can glue uniquely (4.45) and (4.47) to get the 2-functor (4.44) in the statement. $\square$

Remark 4.4.32. We denote by "lft" the class of morphisms which are locally of finite type. It is conceivable that the 2-functor (4.44) can be extended to a 2-functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda) : (\mathrm{Corr}(\mathrm{RigSpc})^{\mathrm{proper}}_{\mathrm{all,lft}})^{2\text{-op}} \to \mathrm{Pr}^{\mathrm{L}} \tag{4.48}$$

sending a span of the form $X \xleftarrow{f} Y \xrightarrow{\mathrm{id}} Y$ to f^* and a span of the form $Y \xleftarrow{\mathrm{id}} Y \xrightarrow{f} X$ to the functor $f_!$ of Definition 4.4.24. We do not pursue this here.

4.5. **Projection formula.**

In this subsection, we explain how to incorporate the projection formula for the exceptional direct image functors into the functor $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_!$ of Theorem 4.4.2.

Theorem 4.5.1. *The functor $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_!$ from Theorem 4.4.2 admits a structure of a module over the composite functor*

$$\int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{lft}} \to \mathrm{RigSpc}^{\mathrm{op}} \xrightarrow{\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}), \tag{4.49}$$

considered as a commutative algebra in the ∞-category of functors from $\int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{lft}}$ to Pr^{L}. (The first functor in (4.49) *is the one given by* $(S, X) \mapsto S$*.) Said differently, there is a functor*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\,\Lambda)^{\otimes}_! : \int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{lft}} \to \mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}) \tag{4.50}$$

which is a lifting of the functor $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_!$ and which is part of a commutative square

$$\begin{tikzcd}
\int_{\mathrm{RigSpc}^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{lft}} \arrow[r, "\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}_!"] \arrow[d] & \mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}) \arrow[d] \\
\mathrm{RigSpc}^{\mathrm{op}} \arrow[r, "\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}"] & \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).
\end{tikzcd}$$

Proof. We only sketch the argument, leaving some details to the reader. The proof consists in revisiting the construction of the functor $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^*_!$ of Theorem 4.4.2, exhibiting step by step a natural module structure over a suitable variant of the algebra (4.49). We start by remarking that the functor (4.24) lifts to a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes,\otimes} : \int_{\mathrm{RigSpc}^{\mathrm{op}}} (\mathrm{RigSpc}^{\mathrm{prop}})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

admitting a natural transformation from the composite functor

$$\int_{\mathrm{RigSpc}^{\mathrm{op}}} (\mathrm{RigSpc}^{\mathrm{prop}})^{\mathrm{op}} \to \mathrm{RigSpc}^{\mathrm{op}} \xrightarrow{\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).$$

(The first functor in the composition above is given by $(S, X) \mapsto S$.) Retaining merely the induced module structure on (4.24), we obtain a commutative square

$$\begin{tikzcd}
\int_{\mathrm{RigSpc}^{\mathrm{op}}} (\mathrm{RigSpc}^{\mathrm{prop}})^{\mathrm{op}} \arrow[r, "\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes,*}"] \arrow[d] & \mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}) \arrow[d] \\
\mathrm{RigSpc}^{\mathrm{op}} \arrow[r, "\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}"] & \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).
\end{tikzcd}$$

With K as in Construction 3.4.4, we set $\mathrm{K}_1 = \langle 1\rangle \times_{\mathrm{Fin}_*, e_0} \mathrm{K}$. We may view the upper horizontal arrow in the previous square as a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\,\Lambda)^{\otimes,*} : \left(\int_{\mathrm{RigSpc}^{\mathrm{op}}} (\mathrm{RigSpc}^{\mathrm{prop}})^{\mathrm{op}}\right) \times \mathrm{K}_1 \to \mathrm{Pr}^{\mathrm{L}}. \tag{4.51}$$

Informally, this functor takes a pair of objects $((S, X), r : \langle 1\rangle \to \langle m\rangle)$ to the tensor product in $\mathrm{Pr}^{\mathrm{L},\otimes}$ of copies of $\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)$, one for each $i \in \{1, \dots, m\}$ different from $r(1)$, and a copy of $\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)$, only when $r(1) \in \{1,\dots,m\}$. Moreover, an arrow of the form $((\mathrm{id}_S,\mathrm{id}_X), s : \langle m\rangle \to \langle n\rangle)$ is sent to a functor induced by the tensor product on $\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)$, and the tensor product of an object of $\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)$ with an object of $\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)$, i.e., the functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)\otimes\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)\to\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda),$$

given by $(M,N)\mapsto f^*(M)\otimes N$ where $f : X \to S$ is the structural morphism. Using this description, it follows from Theorem 4.1.4(1) and Proposition 4.1.7 that the condition $(\star)$ in Construction 4.4.3 is satisfied for the functor (4.51). (What plays the role of the simplicial set "S" in that construction is the category $\mathrm{RigSpc}^{\mathrm{op}}\times\mathrm{K}_1$.) Applying Construction 4.4.3, we obtain a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\,\Lambda)^{\otimes}_{*} : \left(\int_{\mathrm{RigSpc}^{\mathrm{op}}}(\mathrm{RigSpc}^{\mathrm{prop}})\right)\times\mathrm{K}_1\to\mathrm{Pr}^{\mathrm{L}}. \tag{4.52}$$

This functor is easily seen to correspond to a $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}})$-valued functor $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\,\Lambda)^{\otimes}_{*}$ which is a lift of (4.25) and which is part of a commutative square

$$\begin{tikzcd}
\int_{\mathrm{RigSpc}^{\mathrm{op}}}(\mathrm{RigSpc}^{\mathrm{prop}}) \arrow[r, "\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}_{*}"] \arrow[d] & \mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}) \arrow[d] \\
\mathrm{RigSpc}^{\mathrm{op}} \arrow[r, "\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}"] & \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).
\end{tikzcd}$$

Given $(S,(X,W))\in\int_{\mathrm{RigSpc}^{\mathrm{op}}}\mathrm{WComp}$, the sub-$\infty$-category

$$\mathbf{RigSH}^{(\wedge)}((X,W);\Lambda)^{*}_{!}\subset\mathbf{RigSH}^{(\wedge)}_{\tau}(W;\Lambda)^{*}_{*}$$

(see Notations 4.3.10 and 4.4.6) is stable by tensoring with any object of $\mathbf{RigSH}^{(\wedge)}_{\tau}(W;\Lambda)^{*}_{*}$ and, in particular, by the inverse image of any object of $\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)^{*}$. (This is an immediate consequence of Proposition 2.2.1(2).) Applying Lemma 4.3.13 to the restriction of the functor (4.52) to the category $\int_{\mathrm{RigSpc}^{\mathrm{op}}}\mathrm{WComp}$, we obtain a functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}_{!} : \int_{\mathrm{RigSpc}^{\mathrm{op}}}\mathrm{WComp}\to\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}) \tag{4.53}$$

which is a lift of (4.30) and which is part of a commutative square as above. The remainder of the construction follows closely the construction of the functor $\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{*}_{!}$ of Theorem 4.4.2. Namely, we take a left Kan extension of (4.53) along the functor (4.31), and then a second left Kan extension along the fully faithful embedding (4.34). That the resulting functor

$$\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}_{!} : \int_{\mathrm{RigSpc}^{\mathrm{op}}}\mathrm{RigSpc}^{\mathrm{lft}}\to\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}) \tag{4.54}$$

is a lift of (4.32) follows from [Lur09, Proposition 4.3.3.10] and [Lur17, Corollary 3.4.4.6(2)]. □

Proposition 4.5.2. *Let S be a rigid analytic space and $X\in\mathrm{RigSpc}^{\mathrm{lft}}/S$. There exists an equivalence of $\mathbf{RigSH}^{(\wedge)}_{\tau}(S;\Lambda)^{\otimes}$-modules*

$$\mathbf{RigSH}^{(\wedge)}_{\tau}((S,X);\Lambda)^{\otimes}_{!}\simeq\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{\otimes} \tag{4.55}$$

which is a lift of the equivalence of ∞-categories provided by Corollary 4.4.23.

Proof. We want to show that the inverse of the equivalence (4.43) can be naturally lifted to a morphism of $\mathbf{RigSH}^{(\wedge)}_{\tau}(S\,;\Lambda)^{\otimes}$-modules. This equivalence is given by the composition of

$$\mathbf{RigSH}^{(\wedge)}_{\tau}((S,X);\Lambda)^{*}_{!} \xrightarrow{(\mathrm{pr}_2)^*} \mathbf{RigSH}^{(\wedge)}_{\tau}((X,X\times_S X);\Lambda)^{*}_{!} \xrightarrow{(\delta_X)^?} \mathbf{RigSH}^{(\wedge)}_{\tau}((X,X);\Lambda)^{*}_{!}$$

where:

- $\mathrm{pr}_2 : X\times_S X \to X$ is the projection to the second factor and $\delta_X : X \to X\times_S X$ is the diagonal embedding;
- $(\delta_X)^?$ is the left adjoint of the functor $(\delta_X)_?$ as in Construction 4.4.19.

The existence of $(\delta_X)^?$ follows from Proposition 4.4.27 which insures that the functor $i_{!_X}$, for i a closed immersion of rigid analytic X-spaces, admits a left adjoint. The functor $(\mathrm{pr}_2)^*$ admits a natural lift to a morphism of $\mathbf{RigSH}^{(\wedge)}_{\tau}(S\,;\Lambda)^{\otimes}$-modules. So, we are left to prove the same for $(\delta_X)^?$. More generally, it is enough to prove the following assertions (with T a rigid analytic space).

(1) If $j : V \to Y$ is an open immersion in $\mathrm{RigSpc}^{\mathrm{lft}}/T$, the functor

$$j^! : \mathbf{RigSH}^{(\wedge)}_{\tau}((T,Y);\Lambda)^{*}_{!} \to \mathbf{RigSH}^{(\wedge)}_{\tau}((T,V);\Lambda)^{*}_{!}$$

lifts to a morphism of $\mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{\otimes}$-modules.

(2) If $i : Z \to Y$ is a closed immersion in $\mathrm{RigSpc}^{\mathrm{lft}}/T$, the functor

$$i^? : \mathbf{RigSH}^{(\wedge)}_{\tau}((T,Y);\Lambda)^{*}_{!} \to \mathbf{RigSH}^{(\wedge)}_{\tau}((T,Z);\Lambda)^{*}_{!}$$

lifts to a morphism of $\mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{\otimes}$-modules.

For the first assertion, starting with the morphism of $\mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{\otimes}$-modules $j_!$, we need to show that the morphism

$$j^!(A)\otimes B \to j^!(A\otimes B) \tag{4.56}$$

is an equivalence for $A \in \mathbf{RigSH}^{(\wedge)}_{\tau}((T,Y);\Lambda)^{*}_{!}$ and $B \in \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)$. This can be checked locally on Y, and thus we may assume that Y is weakly compactifiable over T. In this case, the morphism (4.56) can be identified with the equivalence $j^*(A)\otimes j^*(B) \simeq j^*(A\otimes B)$. Similarly, for the second assertion, starting with the morphism of $\mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)^{\otimes}$-modules $i_!$, we need to show that the morphism

$$i^?(A\otimes B) \to i^?(A)\otimes B \tag{4.57}$$

is an equivalence for $A \in \mathbf{RigSH}^{(\wedge)}_{\tau}((T,Y);\Lambda)^{*}_{!}$ and $B \in \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)$. This can checked locally on Y, and thus we may assume that Y is weakly compactifiable. In this case, the morphism (4.57) can be identified with the equivalence $i^*(A\otimes B) \simeq i^*(A)\otimes i^*(B)$. □

Corollary 4.5.3 (Projection formula)**.** *Let $f : Y \to X$ be a morphism of rigid analytic spaces which is locally of finite type. Then, the functor*

$$f_! : \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda) \to \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda),$$

as in Definition 4.4.24, admits a lift to a morphism of $\mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)^{\otimes}$-modules. In particular, there is an equivalence

$$M\otimes f_!N \simeq f_!(f^*M\otimes N)$$

for every $M \in \mathbf{RigSH}^{(\wedge)}_{\tau}(X;\Lambda)$ and $N \in \mathbf{RigSH}^{(\wedge)}_{\tau}(Y;\Lambda)$.

Proof. This is an immediate consequence of Theorem 4.5.1 and Proposition 4.5.2. □

Corollary 4.5.4. *Let $f : Y \to X$ be a morphism of rigid analytic spaces which is locally of finite type. Then there are equivalences*

$$f^!\underline{\mathrm{Hom}}(M, M') \simeq \underline{\mathrm{Hom}}(f^*M, f^!M') \quad \text{and} \quad \underline{\mathrm{Hom}}(f_!N, M) \simeq f_*\underline{\mathrm{Hom}}(N, f^!M)$$

for $M, M' \in \mathbf{RigSH}^{(\wedge)}_\tau(X; \Lambda)$ and $N \in \mathbf{RigSH}^{(\wedge)}_\tau(Y; \Lambda)$.

Proof. These are obtained by adjunction from the equivalences

$$(M \otimes -) \circ f_! \simeq f_! \circ (f^*M \otimes -) \qquad \text{and} \qquad (- \otimes f_!N) \simeq f_! \circ (- \otimes N) \circ f^*$$

which are provided by Corollary 4.5.3. □

4.6. **Compatibility with the analytification functor.**

In this last subsection, we prove the compatibility of the exceptional functors with the analytification functor (2.13). We first start with the algebraic analogue of Theorem 4.4.2. (Below, for a scheme S, we denote by $\mathrm{Sch}^{\mathrm{lft}}/S$ the category of locally of finite type S-schemes.)

Theorem 4.6.1. *There are functors*

$$\begin{aligned} \mathbf{SH}^{(\wedge)}_\tau(-; \Lambda)^*_! &: \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{lft}} \to \mathrm{Pr}^{\mathrm{L}} \\ \mathbf{SH}^{(\wedge)}_\tau(-; \Lambda)^!_* &: \left(\int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{lft}}\right)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}} \end{aligned} \tag{4.58}$$

which are exchanged by the equivalence $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}$ and which admit the following informal description.

- *These functors send an object (S, X), with S a scheme and X an object of $\mathrm{Sch}^{\mathrm{lft}}/S$, to the ∞-category $\mathbf{SH}^{(\wedge)}_\tau(X; \Lambda)$.*
- *These functors send an arrow $(g, f) : (S, Y) \to (T, X)$, consisting of morphisms $g : T \to S$ and $f : T \times_S Y \to X$, to the functors $f_! \circ g'^*$ and $g'_* \circ f^!$ respectively, with $g' : T \times_S Y \to Y$ the base change of g.*

Moreover, the functors in (4.58) *satisfy the following properties.*

(1) The ordinary functors

$$\begin{aligned} \mathbf{SH}^{(\wedge)}_\tau(-; \Lambda)^* &: \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}} \\ \mathbf{SH}^{(\wedge)}_\tau(-; \Lambda)_* &: \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}} \end{aligned} \tag{4.59}$$

are obtained from the functors in (4.58) *by composition with the functor $\mathrm{Sch}^{\mathrm{op}} \to \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{lft}}$, given by $S \mapsto (S, S)$.*

(2) For a scheme S, consider the functors

$$\begin{aligned} \mathbf{SH}^{(\wedge)}_\tau(-; \Lambda)_! &: \mathrm{Sch}^{\mathrm{lft}}/S \to \mathrm{Pr}^{\mathrm{L}} \\ \mathbf{SH}^{(\wedge)}_\tau(-; \Lambda)^! &: \mathrm{Sch}^{\mathrm{lft}}/S \to \mathrm{Pr}^{\mathrm{R}} \end{aligned} \tag{4.60}$$

obtained from the functors in (4.58) *by restriction to $\mathrm{Sch}^{\mathrm{lft}}/S$. For a morphism $f : Y \to X$ in $\mathrm{Sch}^{\mathrm{lft}}/S$, denote by $f_!$ and $f^!$ the images of f by these functors respectively. If f is proper there is an equivalence $f_! \simeq f_*$ and if f is smooth there is an equivalence $f^! \simeq \mathrm{Th}(\Omega_f) \circ f^*$.*

(3) The functor $\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^*_!$ can be lifted to a functor

$$\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}_! : \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{lft}} \to \mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}})$$

which is part of a commutative square

$$\begin{tikzcd} \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{lft}} \arrow[r, "\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}_!"] \arrow[d] & \mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}) \arrow[d] \\ \mathrm{Sch}^{\mathrm{op}} \arrow[r, "\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}"] & \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}). \end{tikzcd}$$

Proof. This is the algebraic analogue of the combination of Theorems 4.4.2 and 4.5.1. The proof in the algebraic setting is totally similar to the proof in the rigid analytic setting. However, we spend some lines discussing the construction of the functors in (4.58) in order to introduce some notation which will be useful for the proof of Theorem 4.6.3 below.

Given a scheme S, we denote by $\mathrm{Sch}^{\mathrm{prop}}/S$ the category of proper S-schemes. We also denote by $\mathrm{Sch}^{\mathrm{cp}}/S$ the category of compactifiable S-schemes, i.e., those S-schemes admitting an open immersion into a proper S-scheme. We have an inclusion $\mathrm{Sch}^{\mathrm{cp}}/S \subset \mathrm{Sch}^{\mathrm{sft}}/S$ which is an equality when S is quasi-compact and quasi-separated by Nagata's compactification theorem (see [Con07, Theorem 4.1]). We denote by Comp/S the category whose objects are pairs $(X, \overline{X})$ where X is an S-scheme and $\overline{X}$ is a compactification of X over S. We have a functor $\mathfrak{d}_S : \mathrm{Comp}/S \to \mathrm{Sch}^{\mathrm{cp}}/S$, given by $(X, \overline{X}) \mapsto X$.

The construction of the functors in (4.58) starts with the functor

$$\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^{*,*} : \int_{\mathrm{Sch}^{\mathrm{op}}} (\mathrm{Sch}^{\mathrm{prop}})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}} \tag{4.61}$$

obtained from $\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^*$ by composition with the functor $\int_{\mathrm{Sch}^{\mathrm{op}}}(\mathrm{Sch}^{\mathrm{prop}})^{\mathrm{op}} \to \mathrm{Sch}^{\mathrm{op}}$, given by $(S, X) \mapsto X$. The condition $(\star)$ in Construction 4.4.3 is satisfied for (4.61) by the proper base change theorem (see Proposition 4.1.1(1)). Using this construction, we obtain a functor

$$\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^*_* : \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{prop}} \to \mathrm{Pr}^{\mathrm{L}} \tag{4.62}$$

sending an arrow $(g, f) : (S, Y) \to (T, X)$, consisting of morphisms $g : T \to S$ and $f : T \times_S Y \to X$, to the composite functor $f_* \circ g'^* : \mathbf{SH}^{(\wedge)}_{\tau}(Y;\Lambda) \to \mathbf{SH}^{(\wedge)}_{\tau}(X;\Lambda)$, with $g' : T \times_S Y \to Y$ the base change of g. Let S be a scheme. For $(X, \overline{X})$ in Comp/S, we denote by $\mathbf{SH}^{(\wedge)}_{\tau}((X, \overline{X});\Lambda)^*_!$ the essential image of the fully faithful embedding

$$v_\sharp : \mathbf{SH}^{(\wedge)}_{\tau}(X;\Lambda) \to \mathbf{SH}^{(\wedge)}_{\tau}(\overline{X};\Lambda)$$

where $v : X \to \overline{X}$ is the given open immersion. By Proposition 4.1.1(2), the analogue of Proposition 4.4.7 holds true for the functor (4.62). Thus, we may apply Lemma 4.3.13 to obtain a functor

$$\mathbf{SH}^{(\wedge)}_{\tau}((-,-);\Lambda)^*_! : \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Comp} \to \mathrm{Pr}^{\mathrm{L}}. \tag{4.63}$$

By left Kan extension along the functor $\mathfrak{d} : \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Comp} \to \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{cp}}$, we deduce from (4.63) the functor

$$\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^*_! : \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{cp}} \to \mathrm{Pr}^{\mathrm{L}}. \tag{4.64}$$

The analogue of Lemma 4.4.8 is also valid here. Finally, the first functor in (4.58) is obtained by left Kan extension along $\int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{cp}} \to \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{lft}}$ from (4.64). □

Remark 4.6.2. Theorem 4.6.1 holds true with the same proof for any stable homotopical functor in the sense of [Ayo07a, Définition 1.4.1]. More precisely, given a functor $\mathsf{H}^* : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$, $f \mapsto f^*$ satisfying the ∞-categorical versions of the properties (1)–(6) listed in [Ayo07a, §1.4.1], there are functors

$$\begin{aligned} \mathsf{H}(-)^*_! &: \int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{lft}} \to \mathrm{Pr}^{\mathrm{L}} \\ \mathsf{H}(-)^!_* &: \left(\int_{\mathrm{Sch}^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{lft}}\right)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}} \end{aligned} \tag{4.65}$$

satisfying the properties (1) and (2) of Theorem 4.6.1. Moreover, if H admits a lift to a functor $\mathsf{H}^{\otimes} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ such that the projection formula holds, then property (3) of Theorem 4.6.1 is also satisfied.

Theorem 4.6.3. *Let A be an adic ring. Set $S = \mathrm{Spf}(A)^{\mathrm{rig}}$ and $U = \mathrm{Spec}(A) \smallsetminus \mathrm{Spec}(A/I)$ where $I \subset A$ is an ideal of definition. There is a commutative cube of ∞-categories*

$$\begin{tikzcd}
\int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{lft}} \arrow[rr, "(-)^{\mathrm{an}}"] \arrow[dr, "\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}_!"] \arrow[dd] & & \int_{(\mathrm{RigSpc}^{\mathrm{lft}}/S)^{\mathrm{op}}} \mathrm{RigSpc}^{\mathrm{lft}} \arrow[dr, "\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}_!"] \arrow[dd] & \\
& \mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}) \arrow[rr, equal] \arrow[dd] & & \mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}) \arrow[dd] \\
(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}} \arrow[rr, "(-)^{\mathrm{an}}"] \arrow[dr, "\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}"] & & (\mathrm{RigSpc}^{\mathrm{lft}}/S)^{\mathrm{op}} \arrow[dr, "\mathbf{RigSH}^{(\wedge)}_{\tau}(-;\Lambda)^{\otimes}"] & \\
& \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \arrow[rr, equal] & & \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).
\end{tikzcd}$$

In particular, there is a natural transformation

$$\mathrm{An}^* : \mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^*_! \to \mathbf{RigSH}^{(\wedge)}_{\tau}((-)^{\mathrm{an}};\Lambda)^*_! \tag{4.66}$$

between functors from $\int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{lft}}$ to Pr^{L} which extends the morphism of Pr^{L}-valued presheaves An^* underlying (2.14) *in Proposition 2.2.13.*

Proof. For simplicity, we only construct the natural transformation (4.66). It will be clear from the construction how to lift this natural transformation into a commutative square which is part of a commutative cube as in the statement.

We use the notation introduced in the proof of Theorem 4.6.1. By construction, the functor

$$\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^*_! : \int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{lft}} \to \mathrm{Pr}^{\mathrm{L}}$$

is a left Kan extension along the functor

$$\mathfrak{d}' : \int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}} \mathrm{Comp} \to \int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}} \mathrm{Sch}^{\mathrm{lft}},$$

given by $(S,(X,\overline{X}))\mapsto(S,X)$, of the functor

$$\mathbf{SH}^{(\wedge)}_{\tau}((-,-);\Lambda)^*_! : \int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}}\mathrm{Comp}\to\mathrm{Pr}^{\mathrm{L}}$$

obtained from (4.63) by restriction. (Here, we are combining the two left Kan extensions from the proof of Theorem 4.6.1.) By the universal property of left Kan extensions, it is thus enough to construct a natural transformation

$$\mathrm{An}^* : \mathbf{SH}^{(\wedge)}_{\tau}((-,-);\Lambda)^*_! \to \mathbf{RigSH}^{(\wedge)}_{\tau}((-)^{\mathrm{an}};\Lambda)^*_!\circ\mathfrak{d}'$$

between functors from $\int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}}\mathrm{Comp}$ to Pr^{L}. Now, consider the functors

$$\mathfrak{w} : \int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}}\mathrm{Comp}\to\int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}}\mathrm{Sch}^{\mathrm{prop}}\quad\text{and}\quad\mathfrak{w}' : \int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}}\mathrm{Comp}\to\int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}}\mathrm{Sch}^{\mathrm{lft}}$$

given by $(S,(X,\overline{X}))\mapsto(S,\overline{X})$. The obvious natural transformation $\nu:\mathfrak{d}'\to\mathfrak{w}'$ induces a natural transformation

$$\nu^{\mathrm{an}}_! : \mathbf{RigSH}^{(\wedge)}_{\tau}((-)^{\mathrm{an}};\Lambda)^*_!\circ\mathfrak{d}'\to\mathbf{RigSH}^{(\wedge)}_{\tau}((-)^{\mathrm{an}};\Lambda)^*_!\circ\mathfrak{w}'$$

which is objectwise a fully faithful embedding. Thus, we may obtain $\mathbf{RigSH}^{(\wedge)}_{\tau}((-)^{\mathrm{an}};\Lambda)^*_!\circ\mathfrak{d}'$ from $\mathbf{RigSH}^{(\wedge)}_{\tau}((-)^{\mathrm{an}};\Lambda)^*_!\circ\mathfrak{w}'$ by applying Lemma 4.3.13 to the essential images of the fully faithful embeddings

$$\nu^{\mathrm{an}}_! : \mathbf{RigSH}^{(\wedge)}_{\tau}(X^{\mathrm{an}};\Lambda)\to\mathbf{RigSH}^{(\wedge)}_{\tau}(\overline{X}^{\mathrm{an}};\Lambda)$$

for the objects $(S,(X,\overline{X}))$. Since $\mathbf{SH}^{(\wedge)}_{\tau}((-,-);\Lambda)^*_!$ is constructed from $\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^*_*\circ\mathfrak{w}$ in the same way, we are left to construct a natural transformation

$$\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^*_*\circ\mathfrak{w}\to\mathbf{RigSH}^{(\wedge)}_{\tau}((-)^{\mathrm{an}};\Lambda)^*_!\circ\mathfrak{w}'.$$

The functor $\mathfrak{w}'$ factors through $\int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}}\mathrm{Sch}^{\mathrm{prop}}$. Thus, by Proposition 4.4.27, it is enough to construct a natural transformation

$$\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^*_*\to\mathbf{RigSH}^{(\wedge)}_{\tau}((-)^{\mathrm{an}};\Lambda)^*_*$$

between functors from $\int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}}\mathrm{Sch}^{\mathrm{prop}}$ to Pr^{L}. Equivalently, we need to construct a functor

$$\int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}\times\Delta^1}\mathrm{Sch}^{\mathrm{prop}}\to\mathrm{Pr}^{\mathrm{L}},$$

which restricts to $\mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)^*_*$ over $\{0\}\subset\Delta^1$ and to $\mathbf{RigSH}^{(\wedge)}_{\tau}((-)^{\mathrm{an}};\Lambda)^*_*$ over $\{1\}\subset\Delta^1$. For this, we apply Construction 4.4.3 to the composite functor

$$\int_{(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}\times\Delta^1}(\mathrm{Sch}^{\mathrm{prop}})^{\mathrm{op}}\to\Delta^1\times(\mathrm{Sch}^{\mathrm{lft}}/U)^{\mathrm{op}}\to\mathrm{Pr}^{\mathrm{L}}$$

where the first functor is given by $((S,\epsilon),X)\mapsto(\epsilon,X)$ and the second one classifies the natural transformation $\mathrm{An}^* : \mathbf{SH}^{(\wedge)}_{\tau}(-;\Lambda)\to\mathbf{RigSH}^{(\wedge)}_{\tau}((-)^{\mathrm{an}};\Lambda)$ underlying (2.14) in Proposition 2.2.13. That condition $(\star)$ in Construction 4.4.3 is satisfied, follows from Propositions 2.2.14 and 4.1.1(1), and Theorem 4.1.4(1). □

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Index

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