Blowing-Up the Plane and its Visualization – with a Glance to Singular Algebraic Surfaces:

An Excursion to a Visualizable Side Branch of an Algebraic-Geometric Trunc

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Bl(x,y)

What is Blowing-Up ?

Blowing up is a fundamental construction of Algebraic Geometry.

We illustrate this construction in the case of

Blowing-Up a Disk.

BLOWING-UP A DISK

We consider a disk *D* in the plane *E*. We furnish *E* with a standard system of coordinates, consisting of an *x*-axis and a *y*-axis. So each point *p* in the plane *E* is identified by two numbers *x* and *y*, the *coordinates of* the point *p.*

We fix two polynomials (with real coefficients):

$$f = f(x,y) \quad \text{and} \quad g = g(x,y).$$

Then, the *blowing-up Bl(f,g) von D with respect to f and g* is defined.

WHAT HAPPENS WHILE BLOWING-UP ?

Each point $p = (x,y)$ *of* the disk *D,* which satisfies the two equations

$$f(x,y) = 0 \quad \text{and} \quad g(x,y) = 0$$

is replaced by a *projective line,* which on its turn can be realized as a *circle.*

So, the point *p* is "blown-up" to a circle.

How this circle is inserted into the disk *D* in place of the point *p,* depends on the polynomials *f* and *g.* In order to insert the circle properly into the disk *D,* the disk must be *deformed* appropriately.

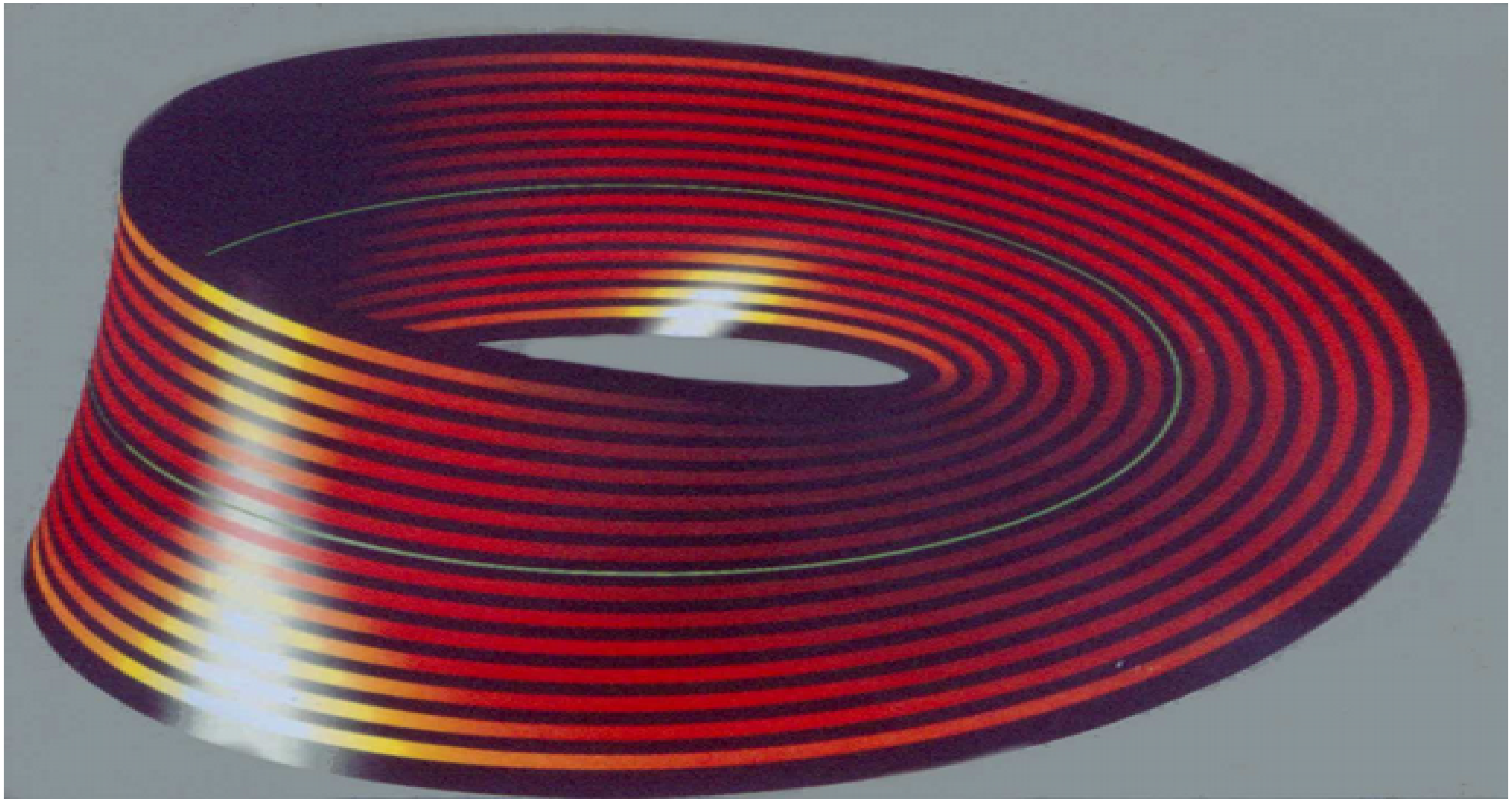
VISUALIZATION OF BLOWING-UP

The blowing-up of a disk with respect to two polynomials can be embedded in 3-dimensional space by means of a standard procedure (*stereographic projection*). Thus:

The blowing-up *Bl(f,g)* of the disk *D* can be realized as a surface in space. The shape of this surface *Bl(f,g)* depends only on the two polynomials *f* and *g*. Through various choices of *f* and *g* one obtains a rich collection of surfaces in 3-space.

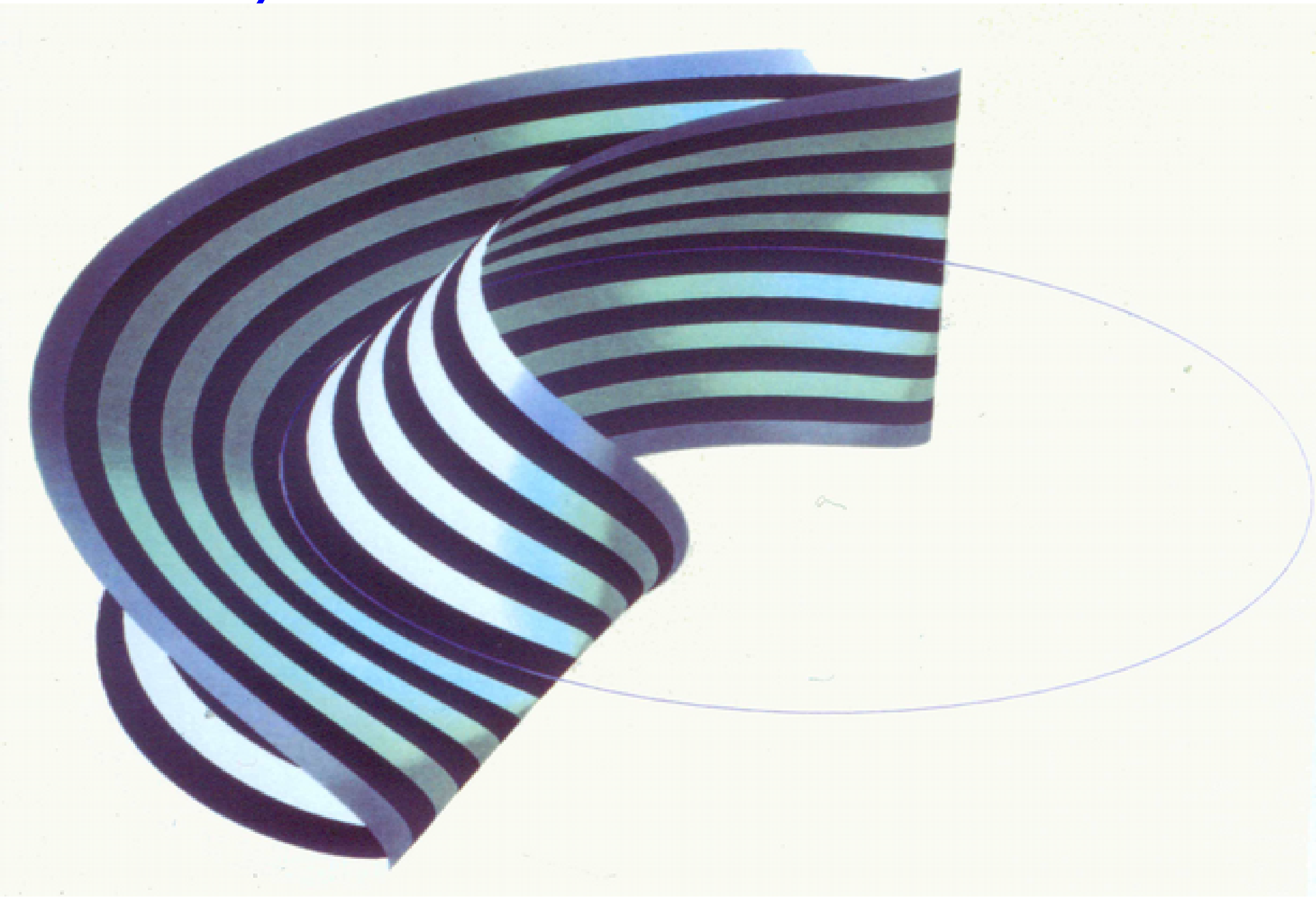
A FIRST EXAMPLE

We choose $f = x$ and $g = y$, e.g. we consider the blowing-up $Bl(x,y)$. It shows up as a *Moebius strip.*



A LESS KNOWN EXAMPLE

We take $f = x^2$ and $g = y^2$. We obtain a (curved) *double Whitney umbrella.*

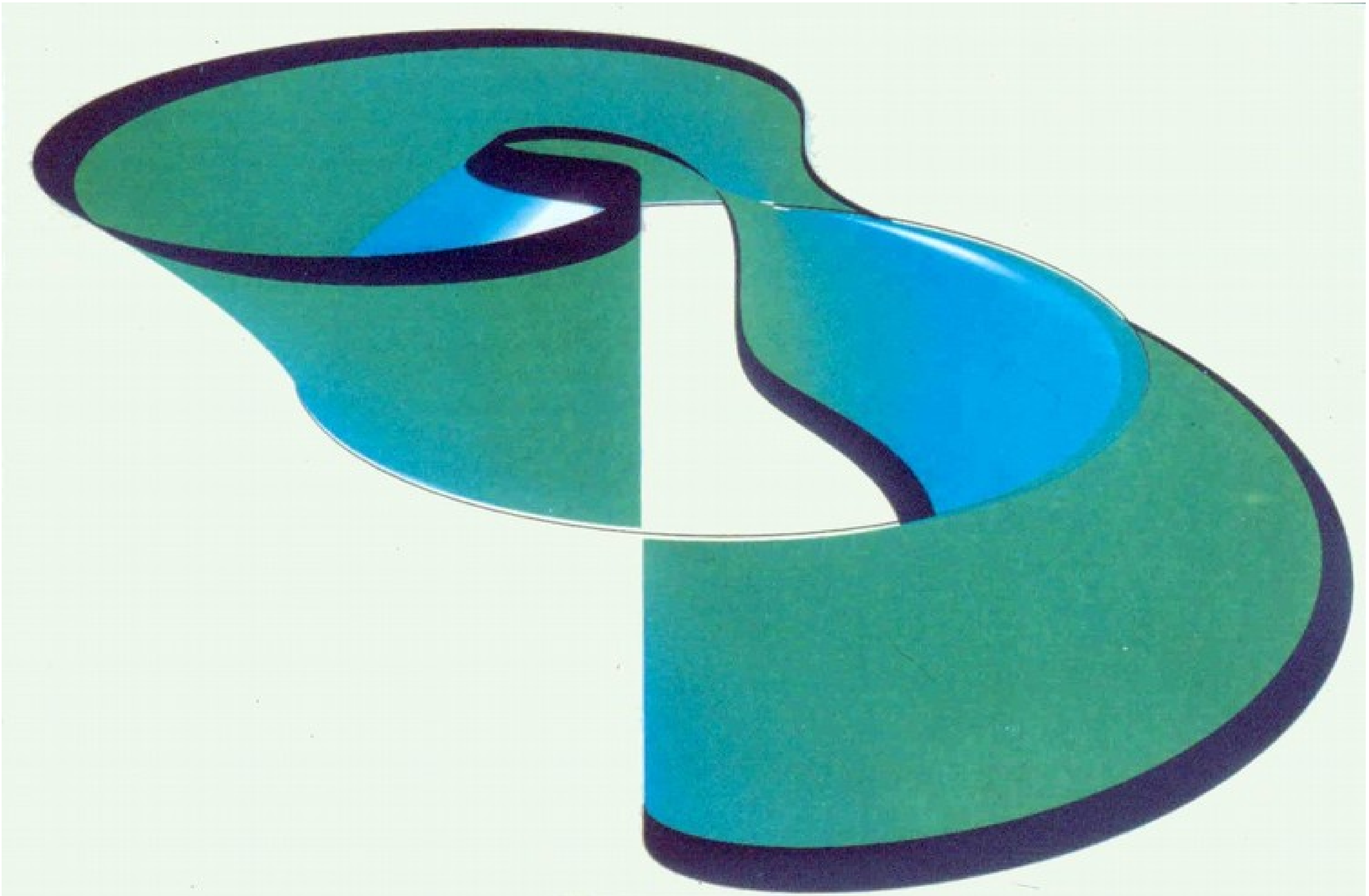


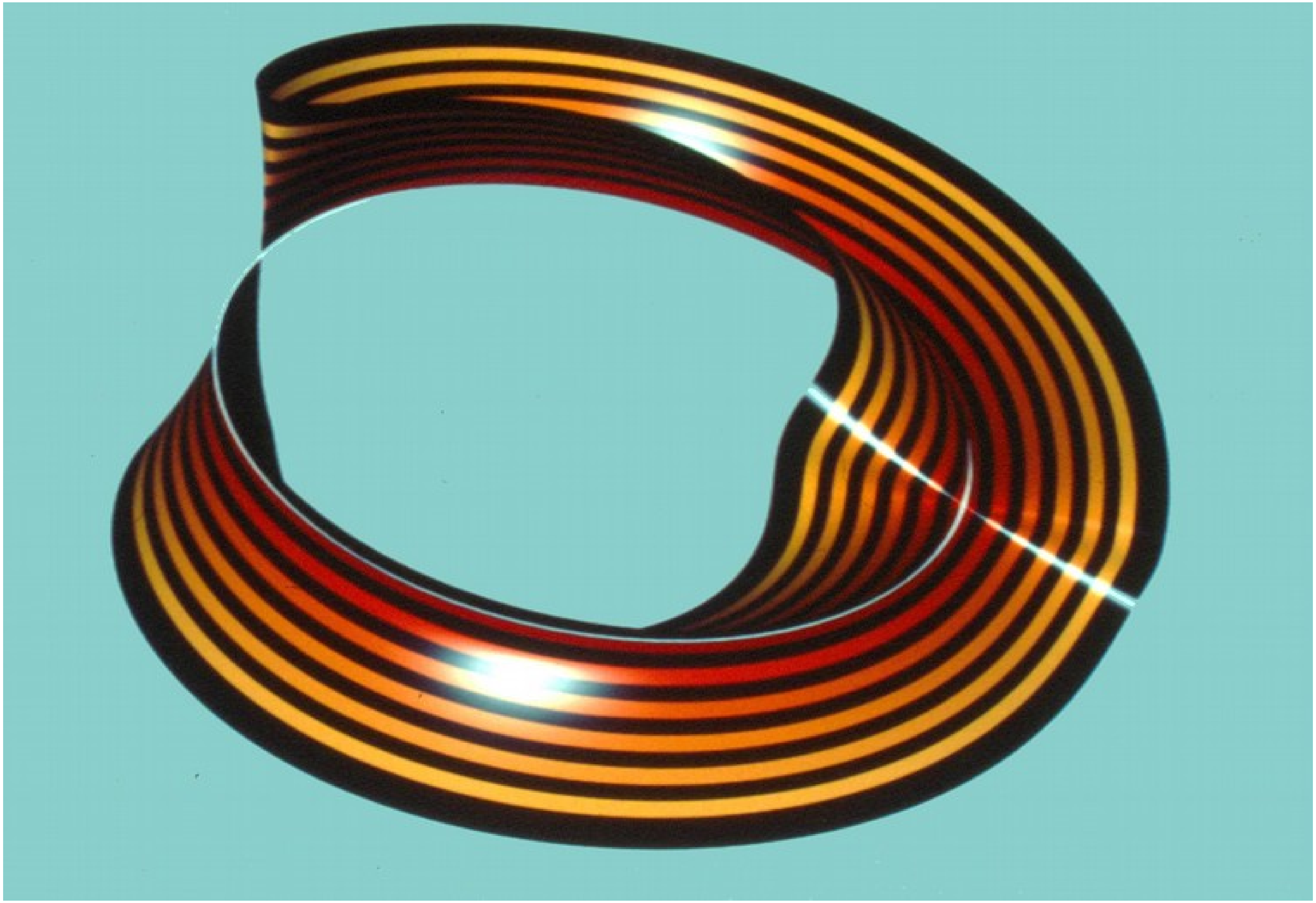
FURTHER EXAMPLES

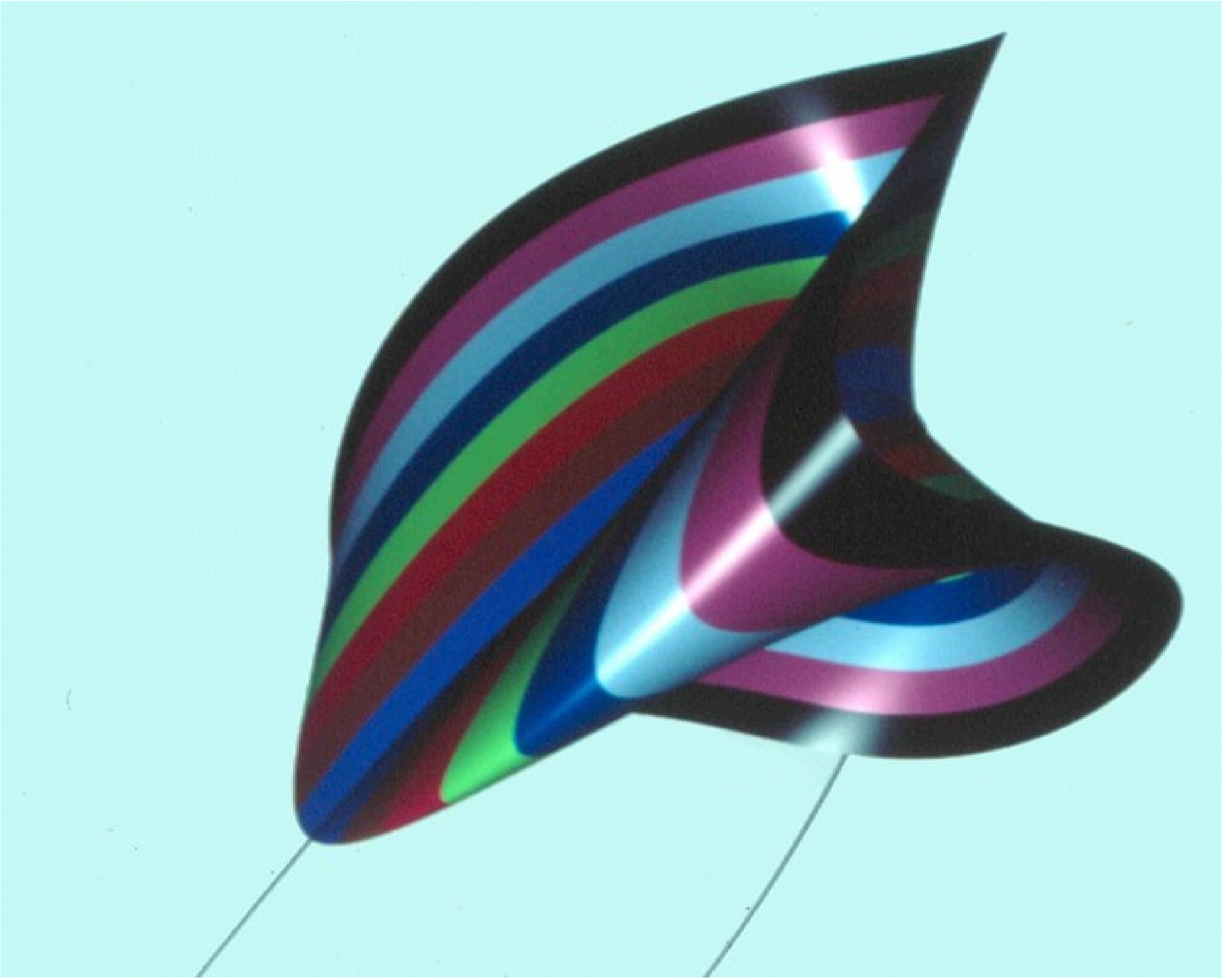
Choosing more complicated polynomials *f* and *g,* one clearly obtains more complicated surfaces *Bl(f,g).*

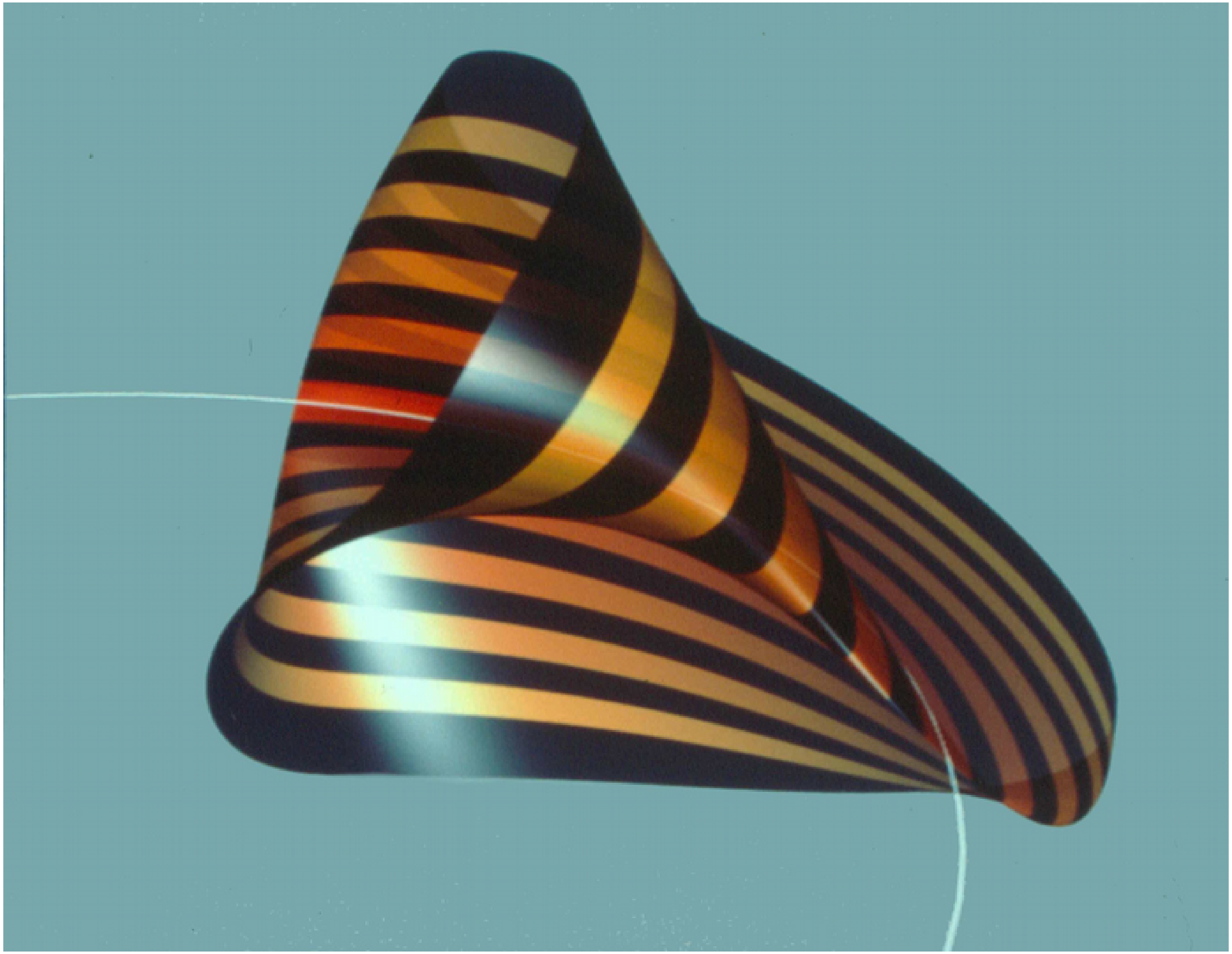
But quickly, one usually reaches the limit of the available computing power and the quality of the pictures suffers...

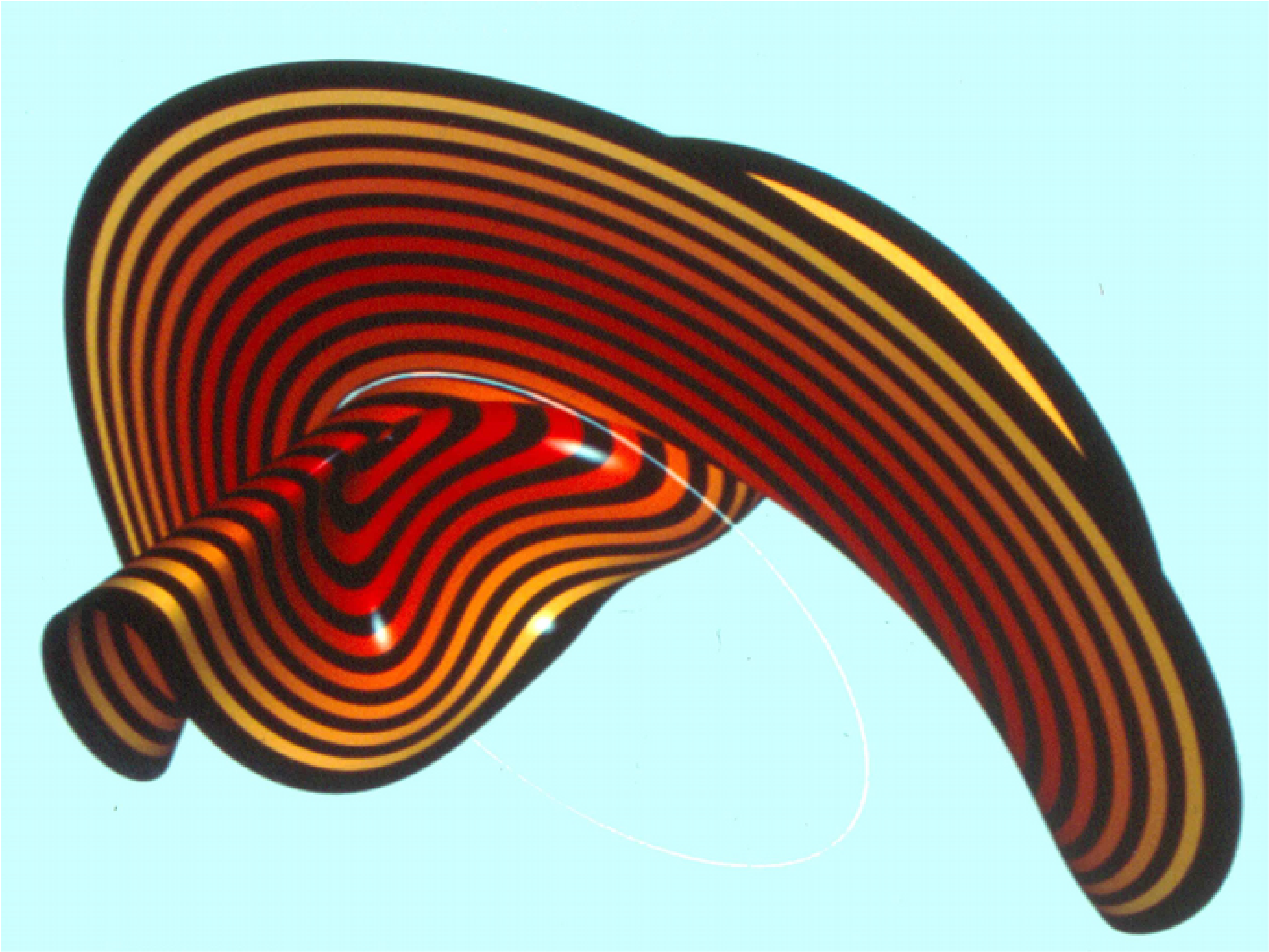
We now shall present a collection of examples, realized as early as 1991 (!) by my student M. Brandenberg in collaboration with Dr. M. Hafner of the Institute of Informatics at the University of Zuerich. (The high quality of the pictures later found much attention).

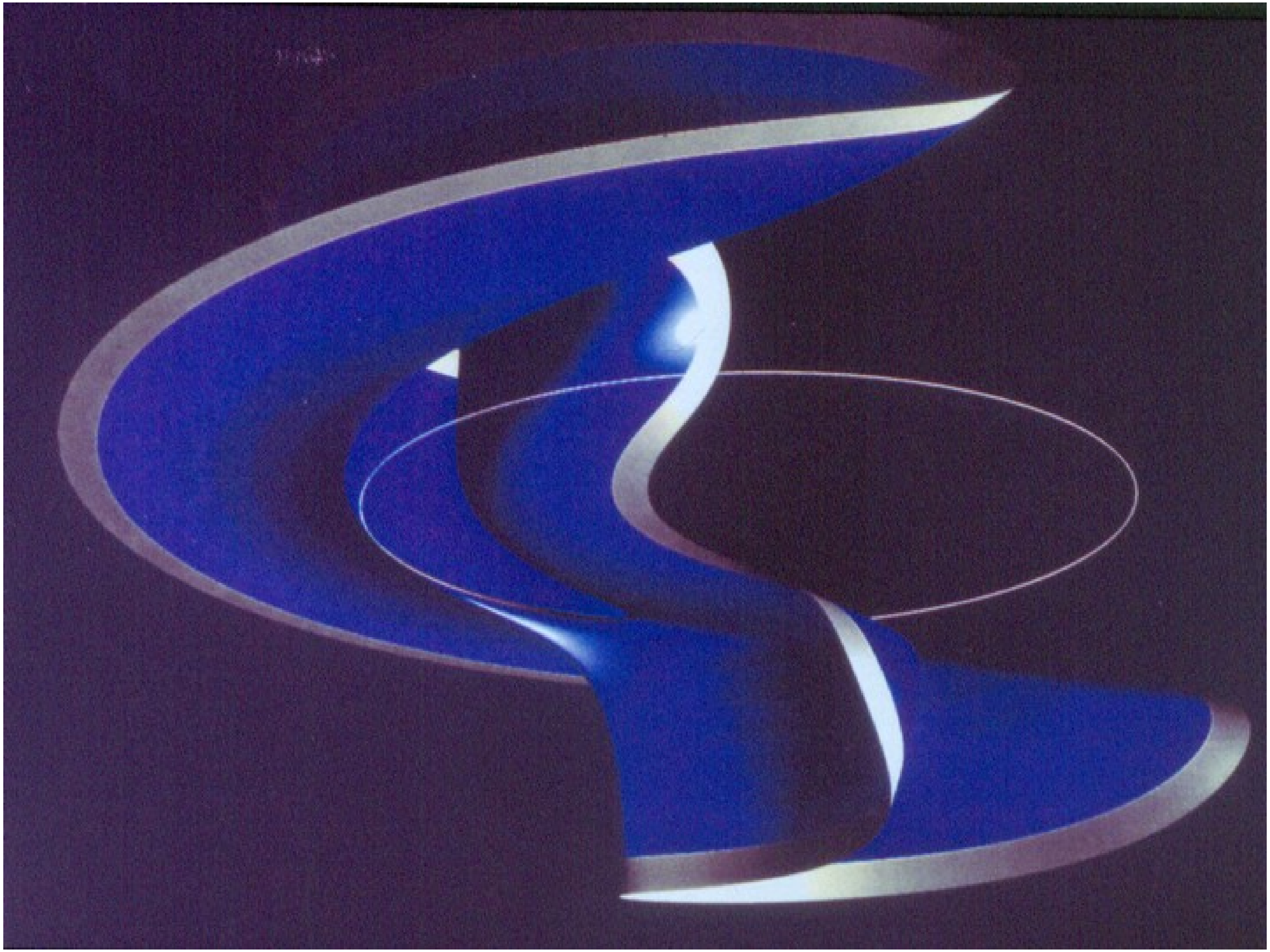












COMMENTS ON THE PICTURES

The disk first is furnished with a specified colouring. During the process of deformation, each point on the disk (which need not be deleted) keeps its colour.

In each of the shown examples only one single point of the disk is deleted and blown-up to a projective line, namely the point *(0,0)*.

The circle which appears on the blowing-up is the blown-up point *(0,0)*. It corresponds to the *exceptional divisor* of the blowing-up *Bl(f,g).* In this situation *Bl(f,g)* is called a *single-point blowing-up* of the disk *D.*

BLOWING-UP MORE THAN ONE POINT

Obviously one is also interested in blowing-up if more than one point of *D* is deleted and blown-up to a projective line. Then, one speaks of *multiple-point blowing-up* of the disk *D*. .

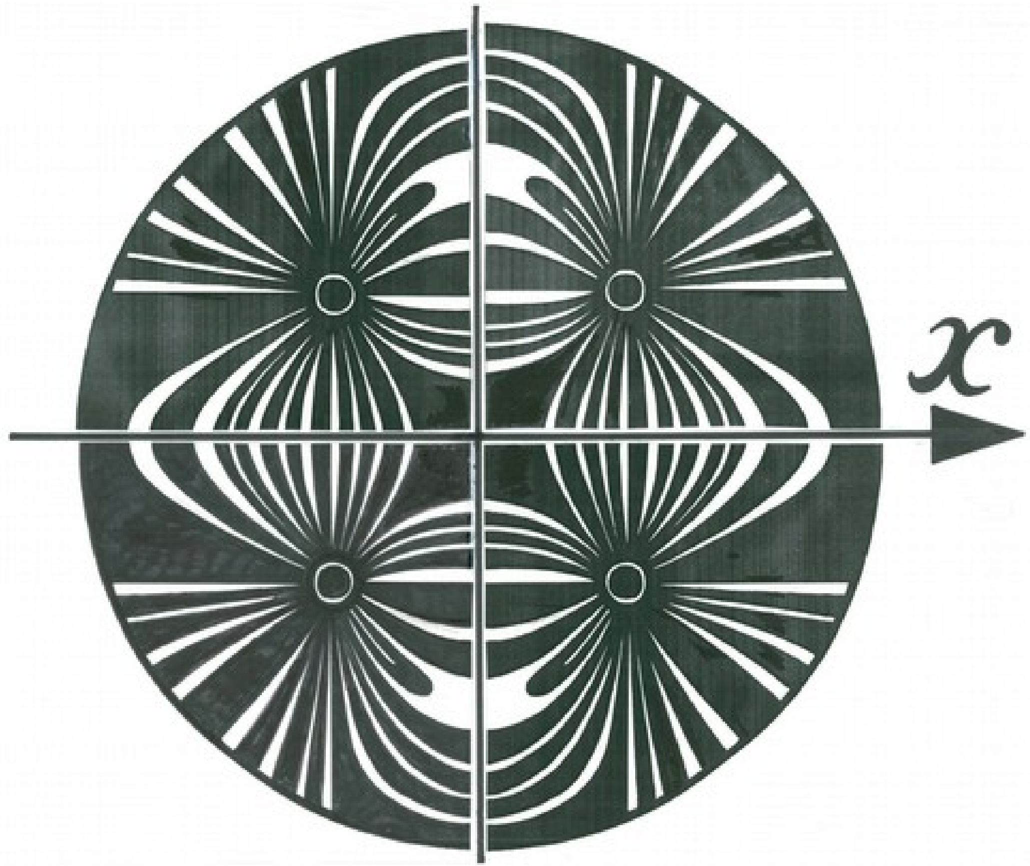
But already with two blown-up points one often reaches the limit of capacity of the computer and the quality of the pictures decays.

Nevertheless one can dare to sketch the shape of some multiple-point blowing-up of *D.*

WE BLOW-UP IN 4 POINTS

We blow up the coloured disk in the four marked points, by choosing:

$f = x^2 - 1$
$g = y^2 -1$



... AND OBTAIN

WHY BLOWING-UP ?

Blowing-up corresponds to a *proper birational morphism* of *Algebraic Geometry.* This aspect of blowing-up is of basic significance for the *(birational) classification of algebraic varieties.*

Another fundamental property of blowing-up is its *resolving effect.* This effect can be illustrated by simple examples.

ON THE RESOLVING EFFECT

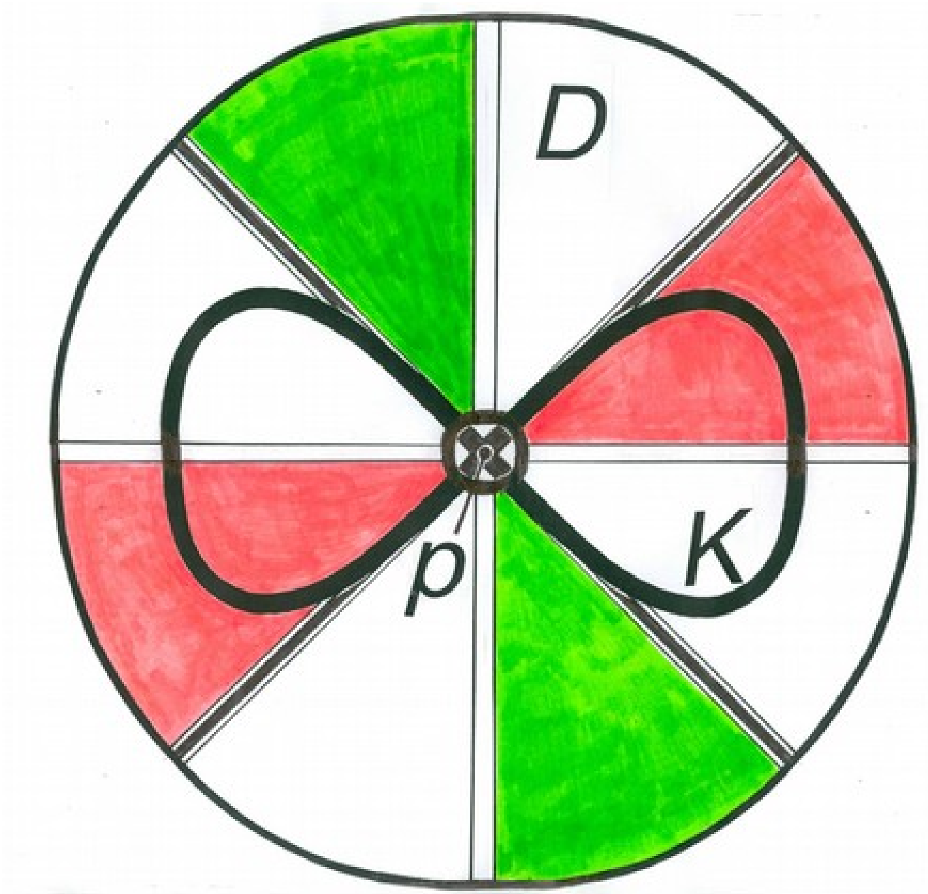
In the disk *D* we consider a curve *K with a singularity* in the centre *(0,0)* of *D.* More precisely, the curve *K* should have a *node* e.g. a *simple double-point,* hence a self intersection with two distinct tangent lines.

We blow up the disk *D* with respect to *x and y and pull back the curve K on the blowing-up Bl(x,y).* We denote the resulting curve on *Bl(x,y)* by *K'.* The curve *K'* appears as a curve on the Moebius strip *Bl(x,y)*.

We illustrate this as follows.

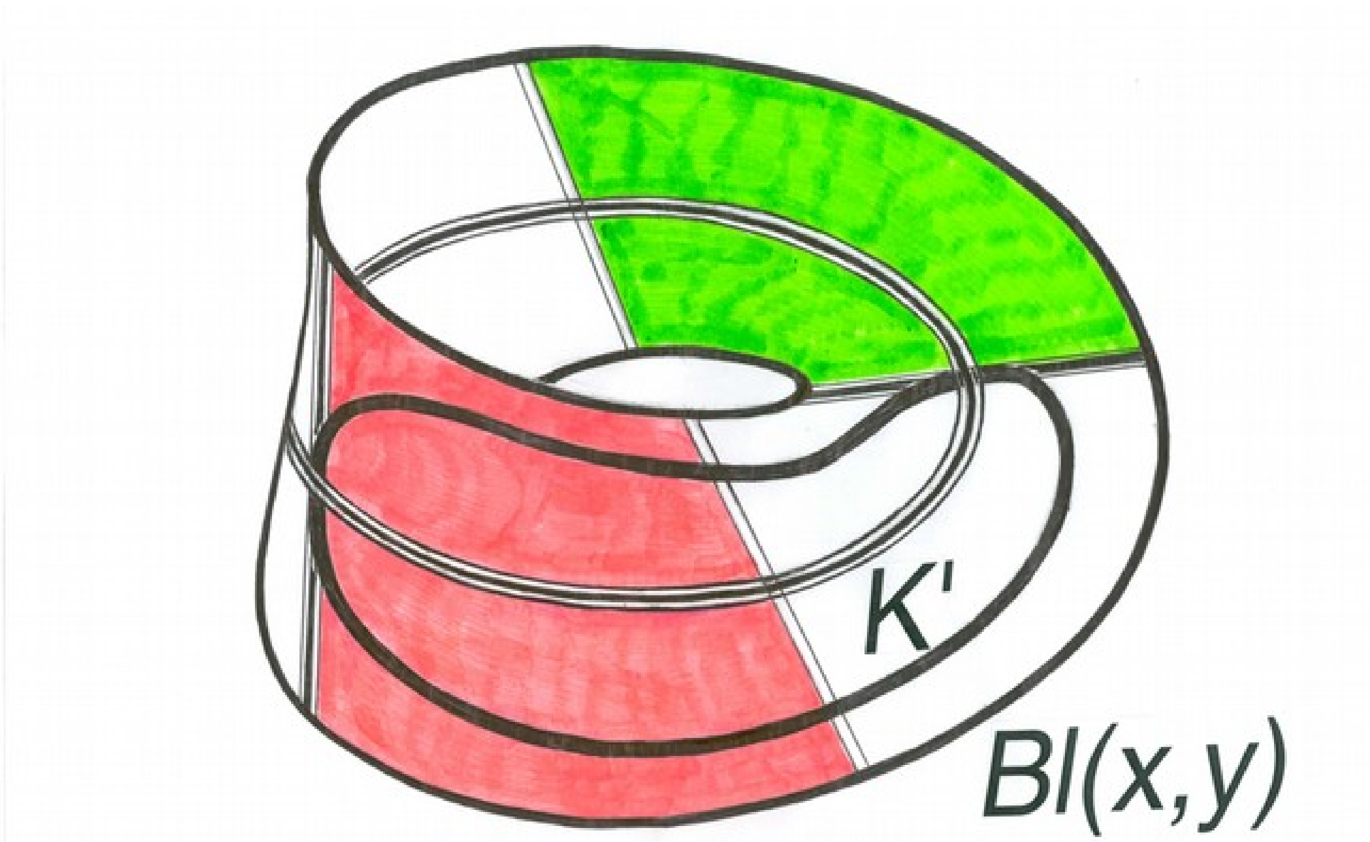
A CURVE SINGULARITY...

The point *p* = *(0,0)* is a *singularity* (or a *singular point*) of the curve *K*. The behaviour of *K* in *p is* different from the behaviour of *K* in its other points (the *regular points)*: *K* has in *p* a self-intersection, more precisely a simple double-point.



... IS BLOWN AWAY

The blown up curve *K '* has *no singularity* !



RESOLUTION OF SINGULARITIES

What we have seen in the previous example holds indeed in much more generality. A basic theorem of Algebraic Geometry says (Hironaka, 1964):

Each complex algebraic variety can be desingularized by an appropriate blowing-up (e.g. has a blowing-up which is a variety without singularities).

It is not known yet, whether this result holds in general for algebraic varieties over arbitrary fields.

BLOWING-UP AND ALGEBRA

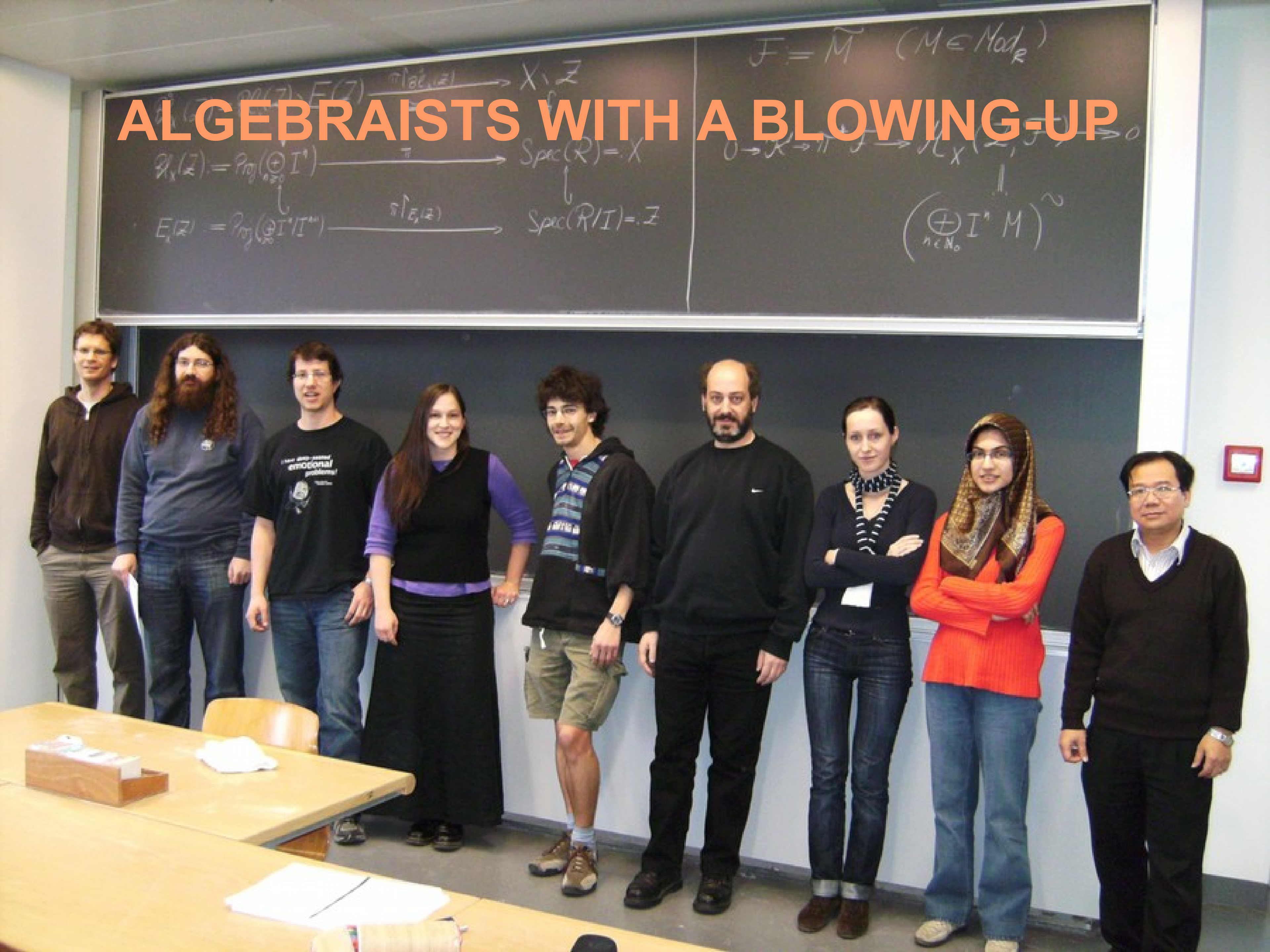
Similarly as the blowing-up of a disc is defined by two polynomials, any blowing-up can indeed be defined by an algebraic object: a so-called *Rees-algebra.*

Instead of studying a blowing-up one can as well study its underlying Rees-algebra.

Therefore, the theory of Rees-algebras is an important field of *Commutative Algebra...*

... and hence one should not be astonished to meet algebraists occupied with blowing-up...

ALGEBRAISTS WITH A BLOWING-UP



VISUALIZATION OF BLOWING-UP AS AN OBJECT OF RESEARCH

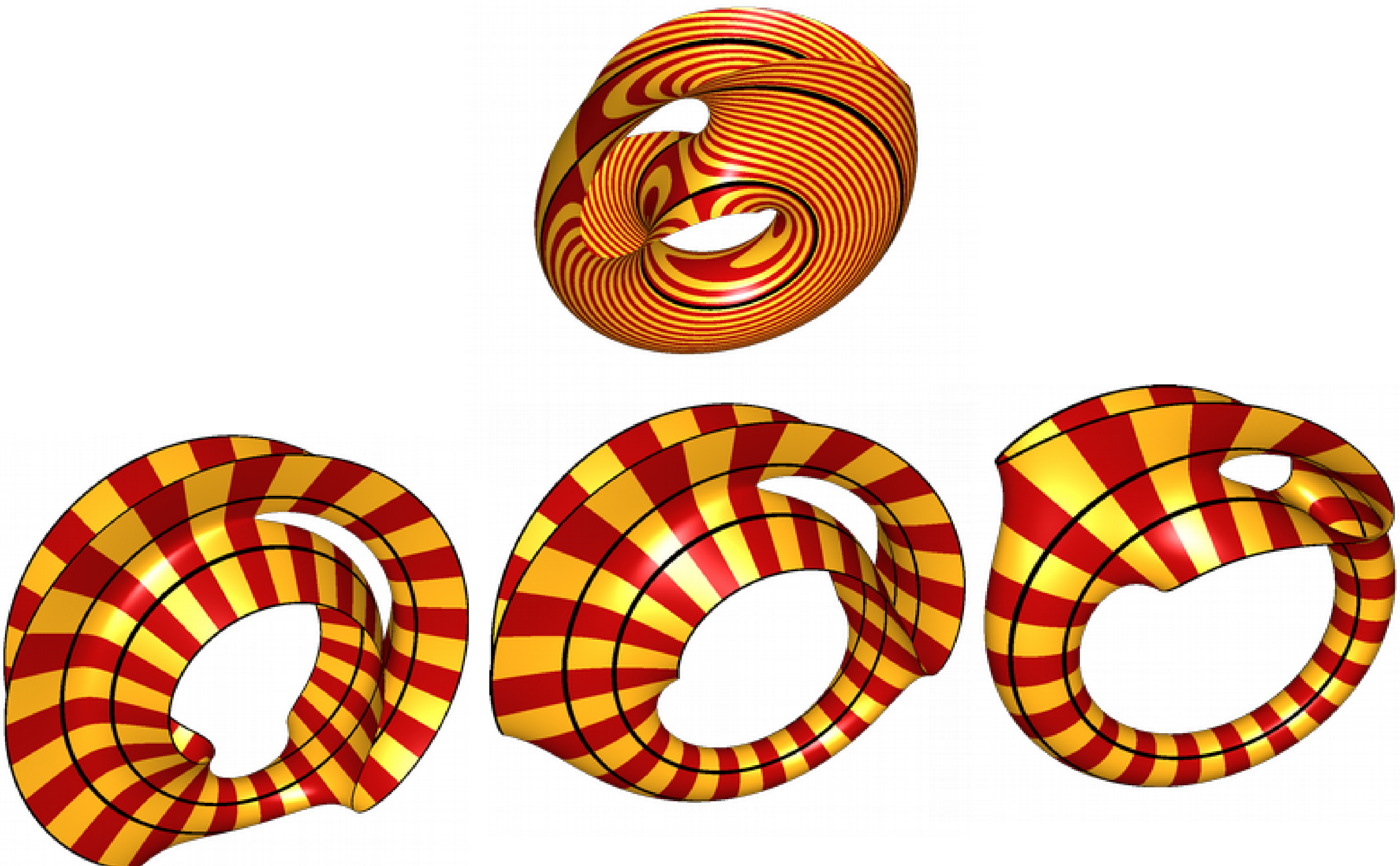
The project *Visualization of Blowing-Up* grew as a side branch of our investigations on *local cohomology of Rees-rings.*

This side branch differs very much from the rather abstract trunc it grew out. It resulted in three Master theses written at the University of Zürich, two Master Theses and a PhD Thesis at the University of Halle (Germany) and a few independent publications.

AN ACTUAL RESEARCH PROJECT

Jointly with the research group of Professor P. Schenzel at the Martin-Luther-Universität Halle (Saale/Germany) and based on a method developed by C. Stussak in his PhD Thesis (written in Halle) one can generate computer images of blowing-up of high resolution (HR) and in short computational time. This allows to produce high resolution images of multiple-point blowing-up. Another application is the generation of high resolution animated images, which allow to visualize *deformations of blowing-up of the disk within an isotopy class… Thus "to produce Movies" !*

FOUR-POINT BLOWING-UP – HIGH RESOLUTION AND ANIMATED



THEORETICAL IMPACTS: Isomorphy ...

There is a natural notion of Isomorphy between two embedded blowing-ups *Bl(f,g)* and *Bl(h,k)* of our disk with respect to to pairs of polynomials. A blowing-up *Bl(f,g)* is called regular, if the Jacobian determinant *J(f,g)* of the pair *(f,g)* does not vanish at any common zero of this pair.

THEOREM 1 (* - Schenzel 2020): Two regular blowups *Bl(f,g)* and *Bl(h,k)* are isomorphic, if and only if:
(a) The two pairs *(f,g)* and *(h,k)* have the same common zeros in our disk, and
(b) the two Jacobian determinants *J(f,g)* and *J(h,k)* have the same sign at each of these common zeros.

REMARK 1: This gives a simple algebraic criterion to decide whether two regular blowing-ups satisfy the geometric condition of being isomorphic.

… and Isotopy

THEOREM 2 (* - Schenzel, 2020): Two arbitrary isomorphic blowing-ups *Bl(f,g)* and *Bl(h,k)* are isotopic within their common isimorphism class.

REMARK 2: In popular terms this means: If two isomorphic blowing-ups *Bl(f,g)* and *Bl(h,k)* are given, there is a movie, starting with the first one and ending with the second one, such that under way one always has isomorphic blowing-ups.

REMARK 3: For our animated visualizations we constantly made use of Theorem 1 and Theorem 2.

ON THE TECHNIQUE OF VISUALIZATION

On use of an implicite presentation of the embedded blowing-up *Bl(f,g),* thanks to the ingenious approach of Ch. Stussak from the research group in Halle – and on use of recent high power graphic carts, interactive animated images (in real time mode) of embedded blowing-ups could be realized. At present, the underlying program is not yet implemented on a commercial computer graphic system.

A basic tool are high power computer graphic algorithms, which also allow to visualize *algebraic surfaces with many singularities.* We now take up this latter theme.

SINGULAR ALGEBRAIC SURFACES

An *algebraic surface* (in 3-space) is the set **of all points** ***(x,y,z)*** **in 3-space, which satisfy a given algebraic equation** ***f(x,y,z) = 0.***

The number of singularities of a ***normal*** **algebraic surface has an upper bound in terms of the degree of the polynomial** ***f*****. (Normal means here, that there are only fintely many singularities at all on the surface).**

The presentation of normal surfaces with many singularities is a test for the standard of a *visualization program.*

THREE EXAMPLES

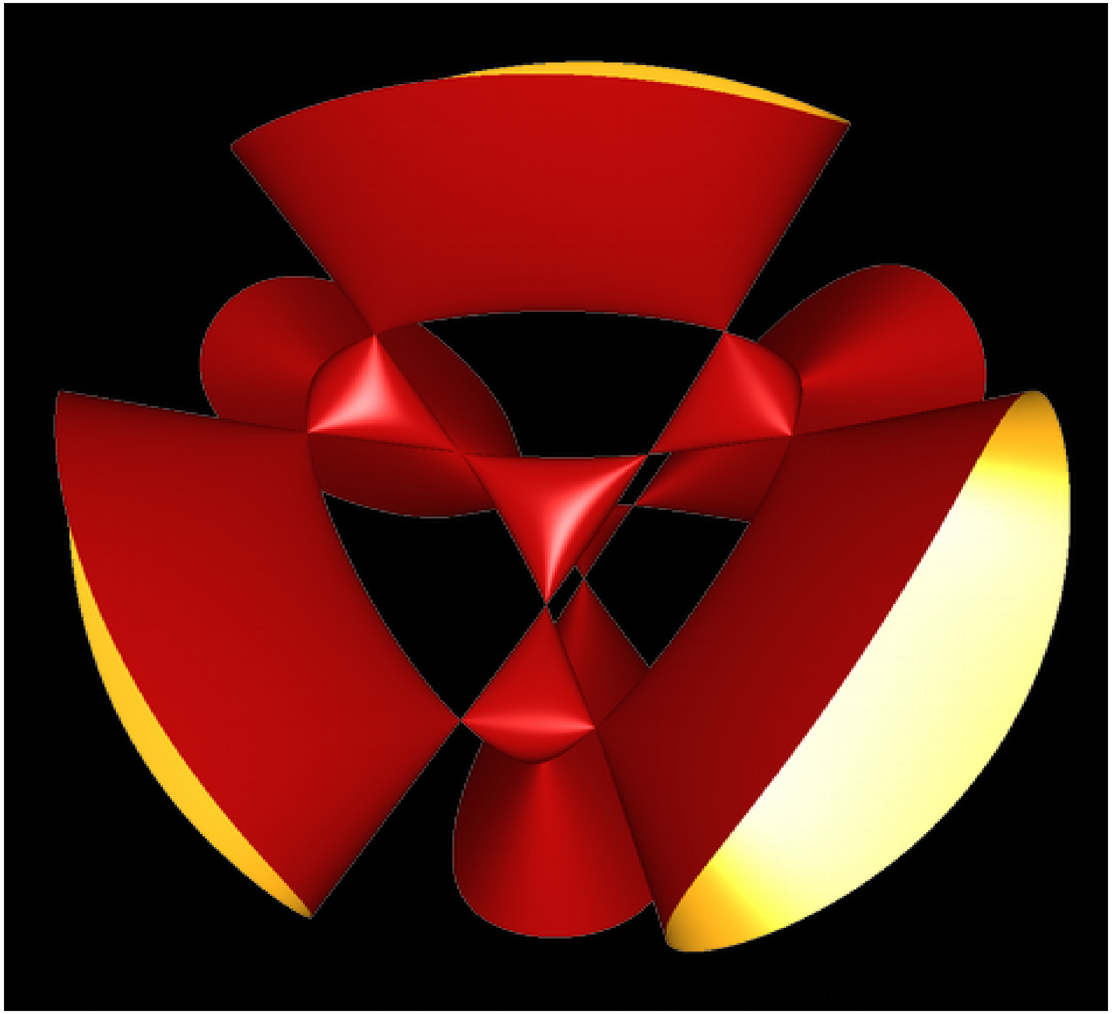
We present three examples of *maximally singular normal algebraic surfaces,* e.g. surfaces whose (finite) number of singularities is maximal for the given degree:

The Kummer quartic: a surface of degree 4 with 16 singularities.

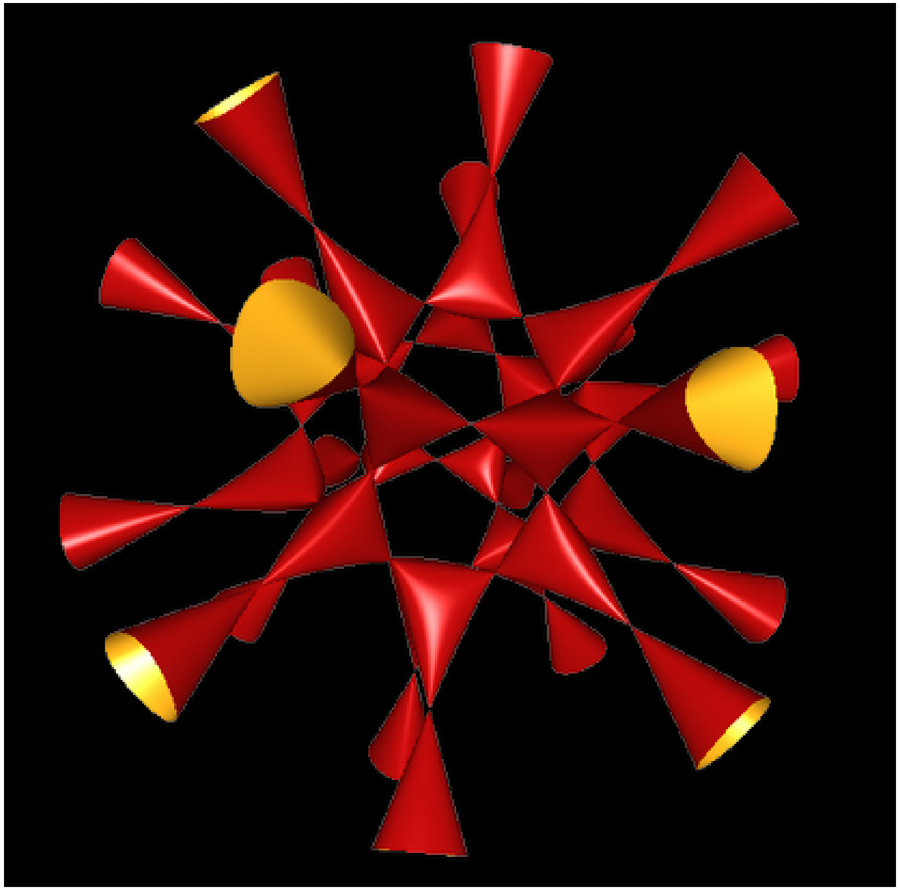
The Barth sextic: a surface of degree 6 with 65 singularities.

The Barth dezic: a surface of degree 10 with 145 singularities.

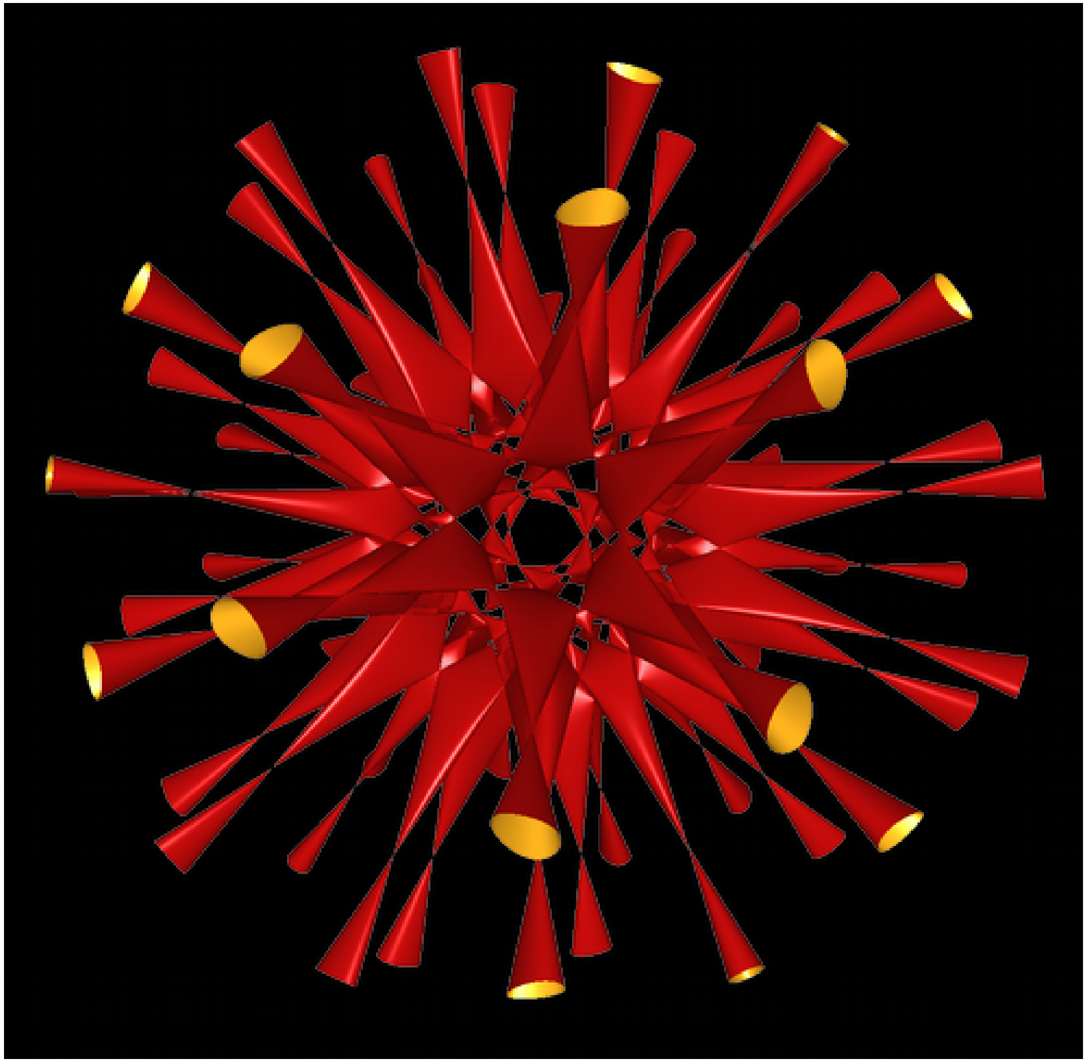
THE KUMMER QUARTIC



THE BARTH SEXTIC



THE BARTH DEZIC



COMMENT ON THE EXAMPLES

The previously shown algebraic surfaces were presented by means of the computer program *SURF*. This program is related to the program *SINGULAR*, a high power computer algebra system developed especially for computations in Algebraic Geometry. The program SINGULAR (like the programs MACAULAY or COCOA, for example) "can compute" algebraic varieties in higher dimensions. With such varieties one leaves (as usually in Algebraic Geometry) the framework of objects which are accessible to visualization and thus has "to replace the eye by algebraic tools". Blowing-up actually quickly gives examples of this, which bring us "beyond visuability"...

WHEN VISUALZATION COMES TO ITS END

Now, let S be any of the three previously shown examples of “surface with many singularities” and let *B* be a ball in three-space which contains all singularities of S. Then, there are indeed three three-variate polynomials $f = f(x,y,z)$, $g = g(x,y,z)$, $h = h(x,y,z)$ with real coefficients, such that the blowing-up $Bl(f,g,h)$ of our ball with respect to these polynomials “resolves the singularities of S”. So, we get a new surface S’ without singularities contained in $Bl(f,g,h)$. But, the “ambient space” $Bl(f,g,h)$ of this surface S’ is of dimension 5. So S’ cannot be visualized anymore. To study this surface we now have “to replace the eye by algebraic tools”, according to what we have already said.


S¹ → Bl(f,g,h)
B = X
J_Z = (f,g,h)
X = D; J_Z = (f,g)
Y
X
Z
π
Bl(x,y)