

BOUNDING PATTERNS FOR THE COHOMOLOGY OF VECTOR BUNDLES

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ABSTRACT. Let $t \in \mathbb{N}$, let K be a field and let \mathcal{V}_K^t denote the class of all algebraic vector bundles over the projective space \mathbb{P}_K^t .

The *cohomology table* of a bundle $\mathcal{E} \in \mathcal{V}_K^t$ is defined as the family of non-negative integers $h_{\mathcal{E}} := (h^i(\mathbb{P}_K^t, \mathcal{E}(n)))_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}}$. A set $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ is said to be a *bounding pattern* for the cohomology of vector bundles over \mathbb{P}_K^t , if for each family $(h^{(i,n)})_{(i,n) \in \mathbb{S}}$ of non-negative integers, the set of cohomology tables

$$\{h_{\mathcal{E}} \mid \mathcal{E} \in \mathcal{V}_K^t : h_{\mathcal{E}}^i(n) \leq h^{(i,n)} \text{ for all } (i,n) \in \mathbb{S}\}$$

is finite. Our main result says that this is the case if and only if \mathbb{S} contains a *quasi-diagonal of width t* , that is a set of the form

$$\{(i, n_i) \mid i = 0, \dots, t\} \text{ with integers } n_0 > n_1 > \dots > n_t.$$

1. INTRODUCTION

This note continues our investigation [BJL2], which was devoted to the question “*What bounds cohomology of a projective scheme X with coefficients in a coherent sheaf \mathcal{F} ?*” The aim of the present paper – which is inspired by the “twin“ Master theses [C] and [K] – is to study this question in the special case where X is a projective space over a field and the coherent sheaf of coefficients \mathcal{F} is locally free and hence an algebraic vector bundle.

Our aim is to show that the bounding patterns for the cohomology of vector bundles over projective spaces are the same as the bounding patterns for the cohomology of coherent sheaves over projective varieties. We namely shall prove that a set $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ bounds cohomology (in the sense of [BJL2]) in the class of all pairs (X, \mathcal{F}) – where X is a projective scheme over an Artinian ring and \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules of dimension $\leq t$ – if and only if it bounds cohomology in the class of all algebraic vector bundles over a projective t -space over a given field K . To make this precise, we first introduce some notation and recall some basic notions.

Notation 1.1. (A) By \mathbb{Z} we denote the set of integers. If $c \in \mathbb{Z}$ we set $\mathbb{Z}_{\leq c} := \{n \in \mathbb{Z} \mid n \leq c\}$ and $\mathbb{Z}_{\geq c} := \{n \in \mathbb{Z} \mid n \geq c\}$. We write $\mathbb{N} := \mathbb{Z}_{\geq 1}$ and $\mathbb{N}_0 := \mathbb{Z}_{\geq 0}$.

(B) Fix $t \in \mathbb{N}_0$ and let \mathcal{S}^t be the class which consists of all pairs (X, \mathcal{F}) in which $X = \text{Proj}(R)$ is a projective scheme induced by a Noetherian homogeneous ring $R = \bigoplus_{n \geq 0} R_n =$

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$R_0[x_1, \dots, x_r]$ ($r \in \mathbb{N}$ and $x_1, \dots, x_r \in R_1$) over an Artinian base ring R_0 and \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules with $\dim(\mathcal{F}) \leq t$.

(C) For each pair $(X, \mathcal{F}) \in \mathcal{S}^t$, each non-negative integer i and each integer n let

$$h_{\mathcal{F}}^i(n) = h^i(X, \mathcal{F}(n)) := \text{length}_{R_0}(H^i(X, \mathcal{F}(n)))$$

denote the (finite) R_0 -length of the i -th cohomology group $H^i(X, \mathcal{F}(n))$ of X with coefficients in the n -th twist $\mathcal{F}(n)$ of \mathcal{F} .

Definition 1.2. (A) For $(X, \mathcal{F}) \in \mathcal{S}^t$ we define the *cohomology table* of (X, \mathcal{F}) as the family of non-negative integers

$$h_{\mathcal{F}} := (h_{\mathcal{F}}^i(n))_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}}.$$

(B) We say that a subset $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ *bounds cohomology in the subclass* $\mathcal{C} \subseteq \mathcal{S}^t$ if for each family $(h^{(i,n)})_{(i,n) \in \mathbb{S}}$ of non-negative integers $h^{(i,n)} \in \mathbb{N}_0$ the set of cohomology tables

$$\{h_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{C} \text{ and } \forall (i, n) \in \mathbb{S} : h_{\mathcal{F}}^i(n) \leq h^{(i,n)}\}$$

is finite. We express this also by saying that the set $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ is a *bounding pattern for the cohomology in the class* \mathcal{C} .

Definition 1.3. (See [BJL2, Definition 4.2].) A subset $\mathbb{T} \subset \mathbb{N}_0 \times \mathbb{Z}$ is called a *quasi-diagonal of width* t , if there is a sequence of integers $(n_i)_{i=0}^t$ with $n_t < n_{t-1} < \dots < n_0$ such that

$$\mathbb{T} = \{(i, n_i) \mid i = 0, \dots, t\}.$$

The main result of [BJL2] will allow us to say (see Theorem 2.3)

A subset $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ bounds cohomology in \mathcal{S}^t if and only if it contains a quasi-diagonal of width t .

Clearly, if $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ bounds cohomology in the class \mathcal{S}^t , it does so in any subclass \mathcal{C} of \mathcal{S}^t . Conversely, one also might ask for “*natural small subclasses*” \mathcal{C} of \mathcal{S}^t with the property that a set $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ which bounds cohomology in the subclass \mathcal{C} does so in the whole class \mathcal{S}^t . The aim of this paper is to show that the algebraic vector bundles over a projective space \mathbb{P}_K^t form such natural small subclasses of \mathcal{S}^t . To make this precise we introduce the following notation.

Notation 1.4. For a field K we write

$$\mathcal{V}_K^t := \{(X, \mathcal{F}) \in \mathcal{S}^t \mid X = \mathbb{P}_K^t \text{ and } \mathcal{F} \text{ is locally free}\}$$

for the subclass of \mathcal{S}^t which consists of all locally free sheaves of finite rank – hence of all algebraic vector bundles – over the projective space \mathbb{P}_K^t .

Now, fix a field K . Then, our main result says (see Theorem 4.1)

A subset $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ bounds cohomology in \mathcal{V}_K^t if and only if it contains a quasi-diagonal of width t .

This obviously shows that the set $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ bounds cohomology in the class \mathcal{S}^t if and only if it does so in the subclass \mathcal{V}_K^t of \mathcal{C} (see Corollary 4.3).

It is natural to ask, whether in this statement one can replace the class \mathcal{V}_K^t by its subclass $\text{ind } \mathcal{V}_K^t$ of indecomposable algebraic vector bundles over \mathbb{P}_K^t . In general this is not true (see Example 5.2). But under some additional conditions on the set \mathbb{S} one can indeed replace the class \mathcal{V}_K^t by its subclass $\text{ind } \mathcal{V}_K^t$ in the above equivalence, as we show in Corollary 5.3.

2. BOUNDING PATTERNS FOR THE COHOMOLOGY OF PROJECTIVE SCHEMES

Throughout this section let $t \in \mathbb{N}_0$. We recall a few basic facts which were proved in [BJL2] and we conclude from these that a subset $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ bounds cohomology in the class \mathcal{S}^t (see Notation 1.1 (B) and Definition 1.2 (B)) if and only if it contains a quasi-diagonal of width t (see Definition 1.3).

Reminder 2.1. (A) (See [BJL1, Definition 5.2].) Let \mathcal{D}^{t+1} denote the class of all pairs (R, M) in which $R = \bigoplus_{n \in \mathbb{N}_0} R_n = R_0[x_1, \dots, x_r]$ ($r \in \mathbb{N}_0$ and $x_1, \dots, x_r \in R_1$) is a Noetherian homogeneous ring with Artinian base ring R_0 and $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is a finitely generated graded R -module with $\dim(M) \leq t + 1$.

(B) (See [BS, 17.1.4] for example.) If $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ is a Noetherian homogeneous ring we write R_+ for the irrelevant ideal $\bigoplus_{n \in \mathbb{N}} R_n$ of R . Moreover by $D_{R_+}^i(\bullet)$ we denote the i -th right derived functor of the R_+ -transform functor $D_{R_+}(\bullet) = \varinjlim_n \text{Hom}_R((R_+)^n, \bullet)$.

Now, let $(R, M) \in \mathcal{D}^{t+1}$. Then, for each $i \in \mathbb{N}_0$ the R -module $D_{R_+}^i(M)$ carries a natural grading and for all $n \in \mathbb{Z}$ the n -th graded component $D_{R_+}^i(M)_n$ of $D_{R_+}^i(M)$ is an R_0 -module of finite length. Moreover $D_{R_+}^i(M) = 0$ for all $i > t$ and $D_{R_+}^i(M)_n = 0$ for all $i > 0$ and all $n \gg 0$. In particular, for each pair $(i, n) \in \mathbb{N}_0 \times \mathbb{Z}$ we may define the non-negative integer $d_M^i(n) := \text{length}_{R_0}(D_{R_+}^i(M)_n)$. This finally allows to define the *cohomology table* of (R, M) as the family of non-negative integers

$$(d_M^i(n))_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}}.$$

(C) We say that a subset $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ *bounds cohomology in the subclass* $\mathcal{C} \subseteq \mathcal{D}^{t+1}$, if for each family $(d^{(i,n)})_{(i,n) \in \mathbb{S}}$ of non-negative integers $d^{(i,n)} \in \mathbb{N}_0$ the set of cohomology tables

$$\{d_M \mid (R, M) \in \mathcal{C} \text{ and } \forall (i, n) \in \mathbb{S} : d_M^i(n) \leq d^{(i,n)}\}$$

is finite. We express this also by saying that the set $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ is a *bounding pattern for the cohomology in the class* \mathcal{C} .

(D) According to [BJL2, Corollary 4.10] a set $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ bounds cohomology in the class \mathcal{D}^{t+1} if and only if \mathbb{S} contains a quasi-diagonal of width t (see 1.3).

Remark 2.2. (A) (See [H, Chapter II] for example.) Keep in mind that there is a surjection

$$\Phi : \mathcal{D}^{t+1} \twoheadrightarrow \mathcal{S}^t, \quad (R, M) \mapsto (\text{Proj}(R), \widetilde{M}),$$

where \widetilde{M} is the sheaf of $\mathcal{O}_{\text{Proj}(R)}$ -modules induced by the graded R -module M .

(B) (See [BS, Chapter 20] for example.) For each pair $(R, M) \in \mathcal{D}^{t+1}$ and each pair $(i, n) \in \mathbb{N}_0 \times \mathbb{Z}$, the *Serre-Grothendieck Correspondence* yields an isomorphism of R_0 -modules $H^i(\text{Proj}(R), \widetilde{M}(n)) \cong D_{R_+}^i(M)_n$, so that $h_{\widetilde{M}} = d_M$. Hence, in particular we see that the set $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ bounds cohomology in the class $\mathcal{C} \subseteq \mathcal{D}^{t+1}$ if and only if it bounds cohomology in the class $\Phi(\mathcal{C}) \subseteq \mathcal{S}^t$.

Theorem 2.3. *Let $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$. Then \mathbb{S} bounds cohomology in the class \mathcal{S}^t if and only if it contains a quasi-diagonal of width t .*

Proof. This is clear by Reminder 2.1 (D) and Remark 2.2 (B). \square

Remark 2.4. (A) If $\mathbb{S} \subseteq \{0, \dots, t\}$ contains a quasi-diagonal of width t , then according to Theorem 2.3 it bounds cohomology in any subclass $\mathcal{C} \subseteq \mathcal{S}^t$.

(B) Let K be a field. If the set \mathbb{S} contains a quasi-diagonal of width t , then it bounds cohomology in the class \mathcal{V}_K^t of algebraic vector bundles over \mathbb{P}_K^t .

(C) Observe, that all pairs $(X, \mathcal{F}) \in \mathcal{S}^0$ have constant cohomology table $h_{\mathcal{F}}$. So a set $\mathbb{S} \subseteq \{0\} \times \mathbb{Z}$ bounds cohomology in the class \mathcal{S}^0 if and only if $\mathbb{S} \neq \emptyset$.

3. COMBINATORIAL PATTERNS AND QUASI-DIAGONALS

Notation 3.1. (A) Throughout this section, we fix a positive integer $t \in \mathbb{N}$ and a set $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$.

(B) We also consider the *diagonal projection*

$$\varrho : \{0, \dots, t\} \times \mathbb{Z} \longrightarrow \mathbb{Z}; \quad (i, n) \mapsto \varrho(i, n) := i + n.$$

(C) If $\mathbb{U} \subseteq \mathbb{Z}$ is a set of integers, we form $\inf \mathbb{U}$ and $\sup \mathbb{U}$ within the set $\mathbb{Z} \cup \{-\infty, \infty\}$, with the usual convention that $\inf \emptyset := \infty$ and $\sup \emptyset := -\infty$.

Definition 3.2. (A) Let $i \in \{0, \dots, t\}$. We define the *beginning* and the *end* of the set \mathbb{S} at level i respectively by:

$$\begin{aligned} \text{beg}^i(\mathbb{S}) &:= \inf\{n \in \mathbb{Z} \mid (i, n) \in \mathbb{S}\}; \\ \text{end}^i(\mathbb{S}) &:= \sup\{n \in \mathbb{Z} \mid (i, n) \in \mathbb{S}\}. \end{aligned}$$

(B) The *height* and the *depth* of \mathbb{S} are defined respectively by:

$$\begin{aligned} \text{height}(\mathbb{S}) &:= \sup\{i \in \mathbb{Z} \mid \exists n \in \mathbb{Z} : (i, n) \in \mathbb{S}\}; \\ \text{depth}(\mathbb{S}) &:= \inf\{i \in \mathbb{Z} \mid \exists n \in \mathbb{Z} : (i, n) \in \mathbb{S}\}. \end{aligned}$$

The *width* of \mathbb{S} is defined by

$$w(\mathbb{S}) := \text{height}(\mathbb{S}) - \text{depth}(\mathbb{S}).$$

Reminder 3.3. (A) According to [BH, Definition 1.2], the set \mathbb{S} is called a *combinatorial pattern*, if:

- (a) $\text{depth}(\mathbb{S}) = 0$;
- (b) $(i, n) \in \mathbb{S} \Rightarrow \exists j \leq i : (j, n + i - j + 1) \in \mathbb{S}$;
- (c) $(i, n) \in \mathbb{S} \Rightarrow \exists k \geq i : (k, n + i - k - 1) \in \mathbb{S}$;
- (d) $0 < i \leq t \Rightarrow \text{end}^i(\mathbb{S}) < \infty$.

(B) The set \mathbb{S} is called a *minimal combinatorial pattern*, if it is a combinatorial pattern and if there is no combinatorial pattern $\mathbb{S}' \subsetneq \mathbb{S}$ strictly contained in \mathbb{S} .

(C) Let \mathbb{S} be a combinatorial pattern. Then, according to [BH, 2.1 and 2.2] the following two statements hold:

- (a) The induced projection map $\varrho : \mathbb{S} \rightarrow \mathbb{Z}$ (see Notation 3.1 (B)) is surjective.
- (b) The map $\varrho : \mathbb{S} \rightarrow \mathbb{Z}$ is bijective if and only if the combinatorial pattern \mathbb{S} is minimal.

(D) Let

$$\mathbb{F}^t := \{\underline{a} := (a_j)_{j=1}^p \mid p \in \mathbb{N}_0 \text{ and } 1 \leq a_1 \leq a_2 \leq \dots \leq a_p \leq t - 1\}$$

be the set of all monotonically increasing sequences in $\{1, \dots, t - 1\}$. According to the properties of cohomological patterns (a)–(d) of part (A) and statement (b) of part (C) there is a bijection between the set $\mathbb{Z} \times \mathbb{F}^t$ and the set of all minimal combinatorial patterns of width t . Indeed, for each $b \in \mathbb{Z}$ and each $\underline{a} = (a_1, \dots, a_p) \in \mathbb{F}^t$ ($p \in \mathbb{N}_0$), the set

$$\mathbb{M}_{b, \underline{a}}^t := (\{0\} \times \mathbb{Z}_{\geq b}) \cup \{(a_j, b - j - a_j) \mid 1 \leq j \leq p\} \cup (\{t\} \times \mathbb{Z}_{\leq b - t - p - 1})$$

is a minimal combinatorial pattern of width t . Conversely, each minimal combinatorial pattern of width t can be written as $\mathbb{M}_{b, \underline{a}}^t$ with uniquely determined $b \in \mathbb{Z}$ and $\underline{a} \in \mathbb{F}^t$.

Our next result has been shown in [K].

Theorem 3.4. *Assume that \mathbb{S} contains no quasi-diagonal of width t , that $\text{beg}^t(\mathbb{S}) \neq -\infty$ and $\text{end}^0(\mathbb{S}) \neq \infty$. Then, there is a minimal combinatorial pattern \mathbb{M} of width t such that $\mathbb{M} \cap \mathbb{S} = \emptyset$.*

Proof. Assume that $\text{end}^0(\mathbb{S}) = -\infty$, so that $(0, n) \notin \mathbb{S}$ for all $n \in \mathbb{Z}$. Chose an integer $c \leq \text{beg}^t(\mathbb{S})$. Then, the minimal combinatorial pattern (see Reminder 3.3)

$$\mathbb{M}_{c+t, \emptyset}^t = (\{0\} \times \mathbb{Z}_{\geq c+t}) \cup (\{t\} \times \mathbb{Z}_{\leq c-1})$$

is disjoint to \mathbb{S} .

Assume now that $\text{end}^0(\mathbb{S}) \neq -\infty$, so that $\text{end}^0(\mathbb{S}) \in \mathbb{Z}$. Set

$$b := \text{end}^0(\mathbb{S}) + 1 \text{ and } c := \text{beg}^t(\mathbb{S}).$$

First, assume that $t = 1$. As \mathbb{S} contains no quasi-diagonal of width t , we then have $c \geq b - 1$ so that $\mathbb{M}_{b, \emptyset}^1 = (\{0\} \times \mathbb{Z}_{\geq b}) \cup (\{1\} \times \mathbb{Z}_{\leq b-2})$ is disjoint to \mathbb{S} .

So, assume from now on that $t > 1$. Our aim is to construct a sequence of integers $(a_1, a_2, \dots, a_p) = \underline{a} \in \mathbb{F}^t$ with $p \in \mathbb{N}_0$ such that $\mathbb{M}_{b, \underline{a}}^t \cap \mathbb{S} = \emptyset$. To achieve this, we construct a sequence of integers $0 =: a_0 < a_1 \leq a_2 \leq \dots \leq a_i \leq a_{i+1} \leq \dots \leq a_p < t$ with the following properties:

- (1) $(a_i, b - i - a_i) \notin \mathbb{S}$ for all $i \in \{0, \dots, p\}$.

- (2) If $0 \leq i < p$ and $a_i < a_{i+1}$, then $(j, b - i - j - 1) \in \mathbb{S}$ for all j with $a_i \leq j < a_{i+1}$.
(3) $p \geq b - c - t$.

By our choice of b and a_0 , condition (1) is satisfied with 0 instead of p . If $b - c - t \leq 0$ we can set $p = 0$ and the requested sequence is constructed.

Thus, assume from now on that $b - c - t > 0$, so that $c < b - t$. As $(0, b - 1), (t, c) \in \mathbb{S}$ and \mathbb{S} contains no quasi-diagonal of width t , we have

$$1 \leq a_1 := \inf\{k \in \mathbb{N}_0 \mid (k, b - k - 1) \notin \mathbb{S}\} \leq t - 1.$$

For this choice of a_1 , condition (2) is satisfied with 1 instead of p .

Assume now, that the integers $a_0 < a_1 \leq \dots \leq a_s < t$ ($s \in \mathbb{N}$) are already constructed such that conditions (1) and (2) are satisfied with s instead of p .

If $s \geq b - c - t$, we set $p := s$ and condition (3) is satisfied. So the requested sequence is constructed in this case. Therefore, it remains to define a_{s+1} if $s < b - c - t$. To do so, we distinguish two cases.

If $(a_s, b - s - a_s - 1) \notin \mathbb{S}$, we set $a_{s+1} := a_s$. Then, clearly, the sequence a_0, a_1, \dots, a_{s+1} satisfies the requirements (1) and (2) with $s + 1$ instead of p and we have extended our sequence in the requested way.

So, finally assume that $(a_s, b - s - a_s - 1) \in \mathbb{S}$. We find integers $0 = i_0 < i_1 < \dots < i_r < p$ ($r \in \mathbb{N}_0$) such that

$$\{i_0, \dots, i_r\} = \{i \mid 0 \leq i < s \text{ and } a_i < a_{i+1}\}.$$

Set

$$\begin{aligned} \mathbb{T}_1 &:= \bigcup_{k=0}^r \{(j, b - i_k - j - 1) \mid a_{i_k} \leq j < a_{i_{k+1}}\} \text{ and} \\ \mathbb{T}_2 &:= \{(j, b - s - j - 1) \mid a_s \leq j < t\} \cup \{(t, c)\}. \end{aligned}$$

As $s < b - c - t$ it follows that $\mathbb{T}_1 \cup \mathbb{T}_2$ is a quasi-diagonal of width t . Keep in mind that by condition (1) we have $\mathbb{T}_1 \subseteq \mathbb{S}$. As \mathbb{S} does not contain a quasi-diagonal of width t , the set \mathbb{T}_2 cannot be contained in \mathbb{S} . As $(a_s, b - s - a_s - 1), (t, c) \in \mathbb{T}_2 \cap \mathbb{S}$ it follows that $(k, b - k - 1) \notin \mathbb{S}$ for some k with $a_s < k < t$. Choose k minimal with this property and set $a_{s+1} := k$. Then the sequence a_0, \dots, a_s, a_{s+1} satisfies the properties (1) and (2) with $s + 1$ instead of p and again we have extended our sequence in the requested way.

So altogether, the requested sequence is constructed. Setting $\underline{a} = (a_1, \dots, a_p)$ we easily see by conditions (1) and (3) that indeed $\mathbb{M}_{b, \underline{a}}^t \cap \mathbb{S} = \emptyset$. \square

4. BOUNDING PATTERNS FOR THE COHOMOLOGY OF VECTOR BUNDLES

Throughout this section, let $t \in \mathbb{N}_0$. We now are ready to prove the announced main result.

Theorem 4.1. *Assume that \mathbb{S} contains no quasi-diagonal of width t and let K be a field. Then, \mathbb{S} does not bound cohomology in the class \mathcal{V}_K^t of algebraic vector bundles over \mathbb{P}_K^t .*

Proof. The case $t = 0$ is obvious by Remark 2.4 (C). So, let $t > 0$. Assume first, that $\text{beg}^t(\mathbb{S}) > -\infty$ and $\text{end}^0(\mathbb{S}) < \infty$. Then, by Theorem 3.4 there is a minimal combinatorial pattern \mathbb{M} of width t such that $\mathbb{M} \cap \mathbb{S} = \emptyset$. According to [BH, Proposition 4.5] there is a non-zero locally free indecomposable sheaf $\mathcal{E} \in \mathcal{V}_K^t$ such that

$$\mathcal{P}(\mathcal{E}) := \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid h_{\mathcal{E}}^i(n) \neq 0\} = \mathbb{M}.$$

As $\mathcal{E}^{\oplus r} \in \mathcal{V}_K^t$ and $h_{\mathcal{E}^{\oplus r}}^i(n) = rh_{\mathcal{E}}^i(n)$ for all $r \in \mathbb{N}$ and all $(i, n) \in \mathbb{N}_0 \times \mathbb{Z}$ it follows that the set of cohomology tables $\{h_{\mathcal{E}^{\oplus r}} \mid r \in \mathbb{N}\}$ is an infinite subset of

$$\{h_{\mathcal{F}} \mid \mathcal{F} \in \mathcal{V}_K^t : h_{\mathcal{F}}^i(n) = 0 \text{ for all } (i, n) \in \mathbb{S}\}.$$

Therefore \mathbb{S} does not bound cohomology in the class \mathcal{V}_K^t in this case.

Assume now, that $\text{beg}^t(\mathbb{S}) = -\infty$ or $\text{end}^0(\mathbb{S}) = \infty$. For each $r \in \mathbb{N}_0$, set

$$\mathbb{S}_{[r]} := \mathbb{S} \cap (\{0, \dots, t\} \times \{-r, -r+1, \dots, r-1, r\}).$$

Observe that

$$\mathbb{S} = \bigcup_{r \in \mathbb{N}_0} \mathbb{S}_{[r]}.$$

Clearly, for each $r \in \mathbb{N}_0$ the set $\mathbb{S}_{[r]}$ contains no quasi-diagonal of width t and moreover satisfies $\text{beg}^t(\mathbb{S}_{[r]}) \geq -r$ and $\text{end}^0(\mathbb{S}_{[r]}) \leq r$. So, according to Theorem 3.4 for each $r \in \mathbb{N}_0$ there is a minimal combinatorial pattern $\mathbb{M}_{[r]}$ of width t such that

$$\mathbb{M}_{[r]} \cap \mathbb{S}_{[r]} = \emptyset.$$

According to [BH, Proposition 4.5], for each $r \in \mathbb{N}_0$ we find a locally free sheaf $\mathcal{E}_{[r]} \in \mathcal{V}_K^t$ such that

$$\mathcal{P}(\mathcal{E}_{[r]}) := \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid h_{\mathcal{E}_{[r]}}^i(n) \neq 0\} = \mathbb{M}_{[r]}.$$

Now, fix a pair $(i, n) \in \mathbb{S}$. Then, there is an integer $s(i, n) \in \mathbb{N}_0$ such that $(i, n) \in \mathbb{S}_{[r]}$ for all $r \geq s(i, n)$. Consequently $h_{\mathcal{E}_{[r]}}^i(n) = 0$ for all $r \geq s(i, n)$. Therefore

$$h^{(i, n)} := \sup\{h_{\mathcal{E}_{[r]}}^i(n) \mid r \in \mathbb{N}_0\} \in \mathbb{N}_0$$

and hence

$$h_{\mathcal{E}_{[r]}}^i(n) \leq h^{(i, n)}, \text{ for all } (i, n) \in \mathbb{S} \text{ and all } r \in \mathbb{N}_0.$$

Our next aim is to show that the set of minimal combinatorial patterns

$$\mathbb{T} := \{\mathbb{M}_{[r]} \mid r \in \mathbb{N}_0\}$$

is not finite. Observe first, that for each $r \in \mathbb{N}_0$ there is some $n(r) \in \mathbb{N}_0$ such that $(t, -n), (0, n) \in \mathbb{M}_{[r]}$ for all $n \geq n(r)$. Assume now that \mathbb{T} is finite, so that there are finitely many integers r_1, \dots, r_k with $\mathbb{T} = \{\mathbb{M}_{[r_1]}, \dots, \mathbb{M}_{[r_k]}\}$. As $\mathbb{S} = \bigcup_{r \in \mathbb{N}_0} \mathbb{S}_{[r]}$ and by our choice of the patterns $\mathbb{M}_{[r]}$, we then obtain

$$\mathbb{S} \cap \bigcap_{j=1}^k \mathbb{M}_{[r_k]} = \emptyset.$$

It follows that $(t, -n), (0, n) \notin \mathbb{S}$ for all $n \geq m := \max\{n(r_j) \mid j = 1, \dots, k\}$. But this implies that $\text{beg}^0(\mathbb{S}) \geq -m$ and $\text{end}^0(\mathbb{S}) \leq m$, a contradiction!

So, the set \mathbb{T} is infinite, as requested. As a consequence, $\{h_{\mathcal{E}_{[r]}} \mid r \in \mathbb{N}_0\}$ is an infinite subset of

$$\{h_{\mathcal{E}} \mid \mathcal{E} \in \mathcal{V}_K^t : h_{\mathcal{E}}^i(n) \leq h^{(i,n)} \text{ for all } (i,n) \in \mathbb{S}\}.$$

This shows that \mathbb{S} does not bound cohomology in \mathcal{V}_K^t . \square

Corollary 4.2. *Let K be a field. Then the set $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ bounds cohomology in the class \mathcal{V}_K^t of algebraic vector bundles over \mathbb{P}_K^t if and only if \mathbb{S} contains a quasi-diagonal of width t .*

Proof. This is clear by Theorem 4.1 and by Remark 2.4 (B). \square

Corollary 4.3. *Let K be a field and let $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$. Then, the following statements are equivalent:*

- (i) \mathbb{S} contains a quasi-diagonal of width t .
- (ii) \mathbb{S} bounds cohomology in the class \mathcal{S}^t .
- (iii) \mathbb{S} bounds cohomology in the class \mathcal{V}_K^t .

Proof. This is clear by Theorem 2.3, Corollary 4.2 and Theorem 4.1. \square

5. INDECOMPOSABLE VECTOR BUNDLES

We keep the hypotheses and notations of the previous sections. It is natural to ask, whether one can replace the class \mathcal{V}_K^t in Corollary 4.2 by the smaller class of all indecomposable vector bundles over \mathbb{P}_K^t . The present section is devoted to this question. We first introduce the following notation.

Notation 5.1. Let $t \in \mathbb{N}_0$ and let K be a field. We write

$$\text{ind } \mathcal{V}_K^t := \{(\mathbb{P}_K^t, \mathcal{E}) \in \mathcal{V}_K^t \mid \mathcal{E} \text{ is indecomposable}\}$$

for the class of all indecomposable algebraic vector bundles over the projective space \mathbb{P}_K^t .

The following example shows that we can not expect in general that the class \mathcal{V}_K^t may be replaced by its subclass $\text{ind } \mathcal{V}_K^t$ in Corollary 4.2.

Example 5.2. Let $t = 1$ and consider the set $\mathbb{S} := \{(0, -1), (1, -1)\} \subset \{0, 1\} \times \mathbb{Z}$. Clearly, \mathbb{S} does not contain a quasi-diagonal. According to the algebraic form of Grothendieck's Splitting Theorem for vector bundles over the projective line (see [G, Théorème 2.1] or [BS', 20.5.9]), (up to isomorphism) the line bundles $\mathcal{O}_{\mathbb{P}_K^1}(n)$ with $n \in \mathbb{Z}$ are precisely the indecomposable algebraic vector bundles over \mathbb{P}_K^1 . So, (up to isomorphism) for each choice of $h^{(0,-1)}, h^{(1,-1)} \in \mathbb{N}_0$ the $h^{(0,-1)} + h^{(1,-1)} + 1$ bundles

$$\mathcal{O}_{\mathbb{P}_K^1}(n) \text{ with } -h^{(1,-1)} \leq n \leq h^{(0,-1)}$$

are precisely the indecomposable algebraic vector bundles \mathcal{E} over the projective line \mathbb{P}_K^1 which satisfy $h_{\mathcal{E}}^i(n) \leq h^{(i,n)}$ for all $(i,n) \in \mathbb{S}$. Hence

$$\#\{h_{\mathcal{E}} \mid \mathcal{E} \in \text{ind } \mathcal{V}_K^1 : \forall (i,n) \in \mathbb{S} : h_{\mathcal{E}}^i(n) \leq h^{(i,n)}\} \leq h^{(0,-1)} + h^{(1,-1)} + 1.$$

This clearly shows that the set \mathbb{S} does indeed bound cohomology in the class $\text{ind } \mathcal{V}_K^1$, although it does not contain a quasi-diagonal.

On the other hand, we have the following result, in which we use the notation

$$\text{Supp}(h) := \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid h^{(i, n)} \neq 0\}$$

for a family $h := (h^{(i, n)})_{(i, n) \in \mathbb{N}_0 \times \mathbb{Z}}$ of non-negative integers $h^{(i, n)}$.

Corollary 5.3. *Let K be a field and let $\mathbb{S} \subseteq \{0, \dots, t\} \times \mathbb{Z}$ be such that one of the following two conditions is satisfied*

- (1) $\text{beg}^t(\mathbb{S}) = -\infty$ or $\text{end}^0(\mathbb{S}) = \infty$.
- (2) *There is a finite subset $\mathbb{T} \subset \mathbb{S}$ such that $\mathbb{T} \cap \text{Supp}(h_{\mathcal{E}}) \neq \emptyset$ for all non-zero bundles $\mathcal{E} \in \text{ind } \mathcal{V}_K^t$.*

Then the set \mathbb{S} bounds cohomology in the class $\text{ind } \mathcal{V}_K^t$ of indecomposable algebraic vector bundles over \mathbb{P}_K^t if and only if it contains a quasi-diagonal of width t .

Proof. Suppose that \mathbb{S} does not contain a quasi-diagonal of width t . We have to show, that \mathbb{S} does not bound cohomology in the class $\text{ind } \mathcal{V}_K^t$, provided that one of the two conditions (1) or (2) holds.

Assume first, that condition (1) is satisfied. According to [BH, Proposition 4.5] the locally free sheaf $\mathcal{E}_{[r]} \in \mathcal{V}_K^t$ used in the proof of Theorem 4.1 can be chosen to be indecomposable for all $r \in \mathbb{N}_0$. In doing so, we enforce the set $\{h_{\mathcal{E}_{[r]}} \mid r \in \mathbb{N}_0\}$ to be an infinite subset of the set of cohomology tables

$$\{h_{\mathcal{E}} \mid \mathcal{E} \in \text{ind } \mathcal{V}_K^t : h_{\mathcal{E}}^i(n) \leq h^{(i, n)} \text{ for all } (i, n) \in \mathbb{S}\},$$

where, again $h^{(i, n)} := \sup\{h_{\mathcal{E}_{[r]}}^i(n) \mid r \in \mathbb{N}_0\}$ ($\in \mathbb{N}_0$) – as in the proof of Theorem 4.1. This shows, that \mathbb{S} does not bound cohomology in the subclass $\text{ind } \mathcal{V}_K^t$ and hence proves our claim if condition (1) is satisfied.

Assume now that condition (2) holds. As \mathbb{S} does not contain a quasi-diagonal of width t it does not bound cohomology in the class \mathcal{V}_K^t , by Corollary 4.2. Consequently there is a family $(h^{(i, n)})_{(i, n) \in \mathbb{S}}$ such that the set of cohomology tables

$$\mathbb{U} := \{h_{\mathcal{E}} \mid \mathcal{E} \in \mathcal{V}_K^t \text{ and } h_{\mathcal{E}}^i(n) \leq h^{(i, n)} \text{ for all } (i, n) \in \mathbb{S}\}$$

is not finite. Contrary to our claim, we assume now, that \mathbb{S} bounds cohomology in the class $\text{ind } \mathcal{V}_K^t$ of indecomposable algebraic vector bundles over \mathbb{P}_K^t . Then, the set

$$\text{ind } \mathbb{U} := \{h_{\mathcal{E}} \mid \mathcal{E} \in \text{ind } \mathcal{V}_K^t \text{ and } h_{\mathcal{E}}^i(n) \leq h^{(i, n)} \text{ for all } (i, n) \in \mathbb{S}\}$$

is finite. Moreover, by condition (2), each algebraic vector bundle $\mathcal{E} \in \mathcal{V}_K^t$ splits in at most

$$u := \sum_{(i, n) \in \mathbb{T}} h^{(i, n)} \quad (\in \mathbb{N}_0)$$

indecomposable summands. So, for each cohomology table $h \in \mathbb{U}$ we may write

$$h = h_1 + \dots + h_s, \text{ with } 0 \leq s \leq u \text{ and } h_1, \dots, h_s \in \text{ind } \mathbb{U}.$$

As $\text{ind } \mathbb{U}$ is finite, this yields the contradiction that \mathbb{U} is finite. So, \mathbb{S} does not bound cohomology in the class $\text{ind } \mathcal{V}_K^t$ and this proves our claim. \square

Remark 5.4. (A) Observe, that in Example 5.2 the set \mathbb{S} does satisfy none of the two conditions (1) or (2) of Corollary 5.3. Observe also, that in this example we did use the classification of indecomposable algebraic vector bundles over \mathbb{P}_K^1 . A slight modification of the argument performed in Example 5.2 shows indeed, that a set $\mathbb{S} \subseteq \{0, 1\} \times \mathbb{Z}$ bounds cohomology in the class $\text{ind } \mathcal{V}_K^1$ if and only if $w(\mathbb{S}) = 1$ (see Definition 3.2 (B)).

(B) We do not know, whether there is a purely combinatorial necessary and sufficient condition for the set \mathbb{S} to bound cohomology in the class $\text{ind } \mathcal{V}_K^t$ of indecomposable algebraic vector bundles over \mathbb{P}_K^t if $t > 1$.

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