PROJECTIVE SURFACES OF MAXIMAL SECTIONAL REGULARITY

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ABSTRACT. We study projective surfaces $X \subset \mathbb{P}^r$ (with $r \geq 5$) of maximal sectional regularity and degree $d > r$, hence surfaces for which the Castelnuovo-Mumford regularity $\operatorname{reg}(C)$ of a general hyperplane section curve $C = X \cap \mathbb{P}^{r-1}$ takes the maximally possible value $d-r+3$. We show that each of these surfaces is either a cone over a curve $C \subset \mathbb{P}^{r-1}$ of maximal regularity or else a birational outer linear projection of a smooth rational surface scroll $\widetilde{X} \subset \mathbb{P}^{d+1}$. We prove that the Castelnuovo-Mumford regularity of these surfaces satisfies the equality $\operatorname{reg}(X) = d - r + 3$ and we compute or estimate various of their cohomological invariants as well as their Betti numbers. We study the the extremal variety $\mathbb{F}(X)$ of these surfaces X, that is the closed union of the extremal secant lines of all smooth hyperplane section curves of X. We show that $\mathbb{F}(X)$ is either a plane or that otherwise $r = 5$ and $\mathbb{F}(X)$ is a rational smooth threefold scroll $S(1,1,1) \subset \mathbb{P}^5$.

1. Introduction

For a non-degenerate irreducible projective variety $X \subset \mathbb{P}^r$ defined over an algebraically closed field K, there are various interesting questions regarding the syzygies of the homogeneous vanishing ideal I_X of X. One of the prominent classical problems in this context is to find (least) upper bounds on the number m for which the following properties hold:

($\mathrm{A_m}$) The hypersurfaces of degree m cut out a complete linear system on X.
($\mathrm{B_m}$) X is cut out in $\mathbb{P}^r$ by hypersurfaces of degree m and I_X is generated by homogeneous polynomials of degree $\leq m$.

This bounding problem was completely solved in 1893 for smooth curves X in complex projective 3-space by Castelnuovo [11]. In 1966, Mumford [26] introduced the concept of Castelnuovo regularity (later called Castelnuovo-Mumford regularity) and reformulated the above problem as a bounding problem for this new invariant in terms of the degree, the codimension, the Hilbert coefficients or other projective invariants of X. Recall that Mumford defined the variety $X \subset \mathbb{P}^r$ to be *m-regular* if its sheaf of vanishing ideals $\mathcal{J}_X \subseteq \mathcal{O}_{\mathbb{P}^r}$ satisfies the following cohomological vanishing condition

$$H^i(\mathbb{P}^r, \mathcal{J}_X(m-i)) = 0 \text{ for all } i \geq 1.$$

Keep in mind, that X is k-regular for all $k \geq m$ if it is m-regular. This latter observation gives justification to define the *Castelnuovo-Mumford regularity* $\operatorname{reg}(X)$ of X as the least integer m such that X is m-regular. It is well known that the m-regularity of X implies

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the properties $(\mathrm{A_{m-1}})$ and $(\mathrm{B_m})$. Moreover, property $(\mathrm{B_m})$ has the geometric consequence that each $(m+1)$-secant line to X is actually contained in X.

A well known conjecture due to Eisenbud and Goto (see [15]) says that

$$\mathrm{reg}(X) \leq d - c + 1,$$

where d is the degree and c is the codimension of $X \subset \mathbb{P}^r$. This conjecture has been proved so far only for irreducible but not necessarily smooth curves $X \subset \mathbb{P}^r$ by Gruson-Lazarsfeld-Peskine [20], and for smooth complex surfaces by Pinkham [29] and Lazarsfeld [25]. Moreover in [20] the curves in $\mathbb{P}^r$, whose regularity takes the maximally possible value $d - r + 2$ are classified: they are either of degree $\leq r + 1$ or else smooth rational curves having a $(d - r + 2)$-secant line.

An attempt to push further this latter classification is to study arbitrary *varieties of extremal regularity*, that is non-degenerate irreducible varieties $X \subset \mathbb{P}^r$ of codimension c and degree d which satisfy the inequality $\mathrm{reg}(X) \geq d-c+1$. This idea is a basic guideline for our paper.

To present our approach in more detail, we suppose that the degree d and the codimension c of our non-degenerate irreducible variety $X \subset \mathbb{P}^r$ satisfy $3 \leq c < r$ and $d > c + 2$. A $(d-c+1)$-secant line $\mathbb{L}$ to X, which is not contained in X is called an *extremal secant line* to X. If X admits such an extremal secant line, it must be of extremal regularity. So, a possible generalization of the classification of curves of extremal (and hence maximal) regularity given in [20], is to classify varieties of extremal regularity by the "size" of their set

$$\Sigma^{\circ}_{d-c+1}(X) := \{\mathbb{L} \in \mathbb{G}(1, \mathbb{P}^r) \mid \#(\mathbb{L} \cap X) = d - c + 1\}$$

of extremal secant lines, where $\mathbb{G}(k, \mathbb{P}^r)$ denotes the Grassmannian of k-spaces $\mathbb{P}^k \subset \mathbb{P}^r$ and $\#Z$ denotes the length of the Noetherian scheme Z. As $\Sigma^{\circ}_{d-c+1}(X)$ is locally closed in $\mathbb{G}(1, \mathbb{P}^r)$, its size is naturally measured by its dimension

$$\mathfrak{d}(X) := \dim(\Sigma^{\circ}_{d-c+1}(X)) = \dim\left(\overline{\Sigma^{\circ}_{d-c+1}(X)}\right).$$

The *special extremal secant lines* to X – hence the extremal secant lines to subspace section curves of X which have maximal regularity – are of particular interest for our investigation. To make this more explicit, we write $\mathbb{PU}(X)$ for the set of all subspaces $\mathbb{E} \in \mathbb{G}(c + 1, \mathbb{P}^r)$ for which $X \cap \mathbb{E} \subset \mathbb{E} = \mathbb{P}^{c+1}$ is a (non-degenerate integral) curve of maximal regularity. Then – as $c + 1 \geq 4$ – for each $\mathbb{E} \in \mathbb{PU}(X)$, the curve $X \cap \mathbb{E}$ has a unique $(d - c + 1)$-secant line $\mathbb{L}_{\mathbb{E},X} \in \Sigma^{\circ}_{d-c+1}(X)$. We now consider the set

$${}^*\Sigma^{\circ}_{d-c+1}(X) := \{\mathbb{L}_{\mathbb{E},X} \mid \mathbb{E} \in \mathbb{PU}(X)\}$$

of special extremal secant lines to X and measure its size by the dimension of its closure, thus by

$${}^*\mathfrak{d}(X) := \dim\left(\overline{{}^*\Sigma^{\circ}_{d-c+1}(X)}\right).$$

Clearly ${}^*\mathfrak{d}(X)$ also measures the size of the set $\mathbb{PU}(X)$ of "good" $c+1$ subspaces $\mathbb{E}$ of $\mathbb{P}^r$, and one expects that X behaves nicely if this latter set is "big" – that is contains a dense open subset of $\mathbb{G}(c+1, \mathbb{P}^r)$. In this case, we say that X is of *maximal sectional regularity*. Our first main result relates the above concepts as follows (see Theorem 2.8).

1.1. **Theorem.** *Assume that* $1 \leq t \leq r-3$ *and let* $X \subset \mathbb{P}^r$ *be a non-degenerate irreducible projective variety of dimension* t *and degree* $> r-t+2$. *Then,* ${}^*\mathfrak{d}(X) = \mathfrak{d}(X) \leq 2t-2$. *with equality at the second place if and only if* X *is of maximal sectional regularity.*

Besides of this brief look at varieties of extremal regularity of arbitrary dimension, we shall concentrate to the case of surfaces of extremal regularity, and within this case, we focus to *surfaces of maximal sectional regularity*, hence to non-degenerate irreducible surfaces $X \subset \mathbb{P}^r$ with $\mathfrak{d}(X) = {}^*\mathfrak{d}(X) = 2$.

We attack the investigation of these surfaces via a detour, which is of interest on its own: namely, a study of *sectionally rational varieties*, hence of non-degenerate irreducible varieties $X \subset \mathbb{P}^r$ of codimension c such that $X \cap \mathbb{E}$ is a rational curve for a general space $\mathbb{E} \in \mathbb{G}(c+1, \mathbb{P}^r)$. It turns out that these varieties are outer linear projections of varieties of minimal degree, provided they are either surfaces with finite non-normal locus or else the base field K has characteristic 0 (see Theorem 3.3). Then we prove a bounding result for the Castelnuovo-Mumford regularity of almost *non-singular projections* of varieties which satisfy the syzygetic property $N_{2,p}$ of Green-Lazarsfeld (see Theorem 3.5). As an application we show that sectionally rational surfaces with finite non-normal locus satisfy the conjectural inequality of Eisenbud-Goto (see Corollary 3.6). As surfaces of maximal sectional regularity are sectionally rational, these results directly apply to them.

For an arbitrary non-degenerate irreducible surface $X \subset \mathbb{P}^r$ of degree d, we may consider the closed union

$$\mathbb{F}^+(X) := \overline{\bigcup_{\mathbb{L} \in \Sigma^\circ_{d-r+3}(X)} \mathbb{L}}$$

of all extremal secant lines to X, which we call the *extended extremal variety* of X. Moreover, if $r \geq 5$, we call the closed union

$$\mathbb{F}(X) := \overline{\bigcup_{\mathbb{L} \in {}^*\Sigma^\circ_{d-r+3}(X)} \mathbb{L}}$$

of all special extremal secant lines to X the *extremal variety* of X. Observe that

$$\mathbb{F}(X) \subseteq \mathbb{F}^+(X).$$

Using these concepts, we can formulate the following survey of the structure theory of surfaces of maximal sectional regularity.

1.2. **Theorem.** *Let* $r \geq 5$ *and let* $X \subset \mathbb{P}^r$ *be a non-degenerate irreducible projective surface of degree* $d \geq r+1$ *which is of maximal sectional regularity and not a cone. Then*

(a) *(See Theorem 4.3 (a)(3)) There is a smooth rational normal surface scroll* $\widetilde{X} \subset \mathbb{P}^{d+1}$ *and a linear subspace* $\Lambda \in \mathbb{G}(d-r, \mathbb{P}^{r+1})$ *which avoids* $\widetilde{X}$ *and such that the linear projection map*

$$\pi'_\Lambda : \mathbb{P}^{d+1} \setminus \Lambda \twoheadrightarrow \mathbb{P}^r$$

induces a finite birational morphism

$$\pi_\Lambda := \pi'_\Lambda \restriction \widetilde{X} \twoheadrightarrow X.$$

(b) *(See Theorem 4.3 (b))* $\operatorname{reg}(X) = d-r+3$, *so that the surface* X *satisfies in particular the Eisenbud-Goto conjecture.*

(c) *(See Theorem 5.14 (a), (b) and Theorem 6.3 (a)) Either*
   (1) $r = 5$, $X \subset \mathbb{F}(X)$ *and* $\mathbb{F}(X)$ *is projectively equivalent to the rational normal threefold scroll* $S(1,1,1) \subset \mathbb{P}^5$, *or else*
   (2) $\mathbb{F}(X) = \mathbb{P}^2$ *and* $X \cap \mathbb{F}(X)$ *is a plane curve of degree* $d - r + 3$.

(d) *(See Proposition 5.2 (a),(b), Theorem 5.14 (e) and Theorem 6.3 (h)) The set* ${}^*\Sigma^\circ_{d-r+3}(X)$ *of all special extremal secant lines to* X *is open in its closure* $\overline{{}^*\Sigma^\circ_{d-r+3}(X)}$ *and hence locally closed in the Grassmannian* $\mathbb{G}(1,\mathbb{P}^r)$. *Moreover the following statements hold*
   (1) *In the case (1) of statement (c) we have* ${}^*\Sigma^\circ_{d-2}(X) = \Sigma^\circ_{d-2}(X)$ *and the image of* $\overline{{}^*\Sigma^\circ_{d-2}(X)}$ *under the Plücker embedding* $\psi : \mathbb{G}(1,\mathbb{P}^5) \to \mathbb{P}^{14}$ *is a Veronese surface in* $\mathbb{P}^5$.
   (2) *In the case (2) of statement (c), the image of* $\overline{{}^*\Sigma^\circ_{d-r+3}(X)}$ *under the Plücker embedding* $\psi : \mathbb{G}(1.\mathbb{P}^r) \to \mathbb{P}^{\binom{r+1}{2}-1}$ *is a plane.*

If the surface $X \subset \mathbb{P}^r$ is obtained as an outer linear projection of a smooth rational normal surface scroll $\widetilde{X} \subset \mathbb{P}^{d+1}$ from a subspace $\Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1} \setminus \widetilde{X}$ we write $X = \widetilde{X}_\Lambda$. Keep in mind that the class group $\mathrm{Cl}(\widetilde{Y})$ of a smooth rational normal n-fold scroll $\widetilde{Y} \subset \mathbb{P}^s$ is generated by a hyperplane section $H = \widetilde{Y} \cap \mathbb{P}^{s-1}$ and a ruling $F = \mathbb{P}^{n-1}$ of $\widetilde{Y}$. Using this terminology, we can classify all surfaces of maximal sectional regularity as follows.

1.3. **Theorem.** *Let* $r \geq 5$ *and let* $X \subset \mathbb{P}^r$ *be a non-degenerate irreducible projective surface of degree* $d \geq r+1$ *which is of maximal sectional regularity.*

(a) *(See Theorem 5.14 (a),(b),(e)) The following two statements are equivalent:*
   (i) $r = 5$ *and* $\mathbb{F}(X)$ *is a rational 3-fold scroll* $S(1,1,1) \subset \mathbb{P}^5$.
   (ii) X *is contained in a scroll* $S(1,1,1) \subset \mathbb{P}^5$ *as a divisor which is linearly equivalent to* $H + (d-3)F$.

(b) *(See Definition and Remark 5.1 (C) and Theorem 6.13 (a)) The following two statements are equivalent:*
   (i) $\mathbb{F}(X) = \mathbb{P}^2$.
   (ii) X *is either*
      (1) *a cone over a curve* $C \subset \mathbb{P}^{r-1} (\subset \mathbb{P}^r)$ *of maximal regularity, or else*
      (2) *is equal to* $\widetilde{X}_\Lambda$, *where* $\Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1}$ *is contained in the linear span* $\langle D \rangle = \mathbb{P}^{d-r+3} \subset \mathbb{P}^{d+1}$ *of a divisor* $D \in |H + (3-r)F|$, *and the induced linear projection map*

$$\pi'_\Lambda \restriction : \langle D \rangle \setminus \Lambda \twoheadrightarrow \mathbb{P}^2$$

      is generically one-to-one along D.

Finally, we study some cohomological and homological invariants of a surface $X \subset \mathbb{P}^r$ of degree d which is of maximal sectional regularity. By Theorem 1.2 (b) we know that such a surface is $(d-r+3)$-regular, but not $(d-r+2)$-regular. In particular the triplet

$$\nu(X) := \big(h^1(\mathbb{P}^r, \mathcal{J}_X(d-r+1)), h^2(\mathbb{P}^r, \mathcal{J}_X(d-r)), h^3(\mathbb{P}^r, \mathcal{J}_X(d-r-1))\big)$$

is non-zero, and moreover the *index of normality*

$$N(X) := \sup\{n \in \mathbb{Z} \mid h^1(\mathbb{P}^r, \mathcal{J}_X(n)) \neq 0\}$$

of X cannot exceed $d-r+1$. To measure, how far the surface X is away from being locally Cohen-Macaulay, we also introduce the invariant

$$\mathrm{e}(X) := \sum_{x\in X, closed} \mathrm{length}\big(H^1_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x})\big),$$

which has the property that

$$h^2(\mathbb{P}^r, \mathcal{J}_X(n)) = \mathrm{e}(X) \text{ for all } n \ll 0.$$

Moreover, for a closed subscheme $Z \subset \mathbb{P}^r$ with homogeneous vanishing ideal $I_Z \subset S$, we use

$$\mathrm{depth}(Z) := \mathrm{depth}(S/I_Z)$$

to denote the *arithmetic depth* of $Z \subset \mathbb{P}^r$.

Besides of these cohomological invariants of X, we also shall investigate the syzygetic behavior of X, hence the *Betti numbers*

$$\beta_{i,j}(X) := \dim\big(\mathrm{Tor}_i^S(K, S/I_X)\big)_{i+j} \quad (\text{ with } K := S/\bigoplus_{n>0} S_n).$$

In the special case in which $r = 5$ and $\mathbb{F}(X) = S(1,1,1)$, we describe X as a divisor on $S(1,1,1)$ and hence may determine some of the previously introduced invariants (see Theorem 5.14).

1.4. **Theorem.** *Let $X \subset \mathbb{P}^5$ be a surface of degree $d > 5$ which is of maximal sectional regularity and such that $\mathbb{F}(X) = S(1,1,1)$. Then*

$$N(X) = d-4, \quad \mathrm{depth}(X) = 1, \quad \mathrm{e}(X) = 0,$$

$$\nu(X) = \Big(\binom{d-3}{2}, 0, 0\Big) \text{ and } \beta_{1,b-3}(X) = \binom{d-1}{2}.$$

In the general situation, that is if $\mathbb{F}(X)$ is a plane, we have the following result.

1.5. **Theorem.** *Let $5 \le r < d$ and let $X \subset \mathbb{P}^r$ be a surface of degree d and maximal sectional regularity such that $\mathbb{F}(X) = \mathbb{P}^2$. Set $Y := X \cup \mathbb{F}(X)$. Then*

- (a) *(See Theorem 6.3 (d)(2),(4) and (e)(2))*
  - (1) $\nu(X) = (h^1(\mathbb{P}^r, \mathcal{J}_X(d-r+1)), 1, 0)$;
  - (2) $\mathrm{e}(X) = h^2(\mathbb{P}^r, \mathcal{J}_X) = h^2(\mathbb{P}^r, \mathcal{J}_Y) + \binom{d-r+2}{2}$.
- (b) *(See Theorem 6.3 (f)) The pair $\tau(X) := \big(\mathrm{depth}(X), \mathrm{depth}(Y)\big)$ satisfies*
  - (1) $\tau(X) = (2,3)$ *if* $r+1 \le d \le 2r-4$;
  - (2) $\tau(X) \in \{(1,1),(2,2),(2.3)\}$ *if* $2r-3 \le d \le 3r-7$;
  - (3) $\tau(X) \in \{(1,1).(2,2)\}$ *if* $3r-6 \le d$.
- (c) *(See Theorem 6.8 (a)) The following four conditions are equivalent*
  - (i) $N(X) \le d-r$;
  - (ii) $\mathrm{reg}(Y) \le d-r+2$;
  - (iii) $\beta_{i,d-r+2}(X) = \binom{r-2}{i-1}$ *for all* $i \ge 1$.
  - (iv) $\beta_{r,d-r+2}(X) = 0$.
- (d) *(See Theorem 6.8 (b)) The following two conditions are equivalent*
  - (i) $\beta_{1,d-r+2}(X) = 1$;

(ii) *The homogeneous vanishing ideal I_Y of Y in S is generated by the homogeneous polynomials of degree $\leq d-r+2$ which are contained in the homogeneous vanishing ideal I_X of X in S, thus $I_Y = \big((I_X)_{\leq d-r+2}\big)$.*

Moreover, these two conditions hold, if the equivalent conditions of statement (c) are satisfied, and they imply that each proper extremal secant line to X is contained in $\mathbb{F}(X)$ and hence that the $(d-r+3)$-secant variety $\mathrm{Sec}_{d-r+3}(X)$ of X is equal to Y.

We shall provide examples of surfaces X of extremal regularity, which show that $\mathfrak{d}(X)$ can take all values in the set $\{-1, 0, 1, 2\}$ (see Construction and Examples 7.1), even for smooth surfaces X which are sectionally rational and occur as divisors on smooth rational normal threefold scrolls – and that there are indeed many such examples with $\mathfrak{d}(X) = -1$, that is without extremal secant lines. This latter fact is noteworthy as Gruson-Lazarsfeld-Peskine's paper [20] lead to the expectation, that there are only "a few exceptional varieties of extremal regularity having no extremal secant line".

We also shall provide examples, which show that in the general case where $X \subset \mathbb{P}^r$ is a surface of maximal sectional regularity whose extremal variety $\mathbb{F}(X)$ is a plane, all pairs $\tau(X) = \big(\mathrm{depth}(X), \mathrm{depth}(Y)\big)$ listed in Theorem 1.4 (b) may indeed occur (see Construction and Examples 7.2, Examples 7.3, 7.4 and 7.5).

Finally let us mention, the following problem, which we believe to have an affirmative answer (see Problem and Remark 7.7).

1.6. **Problem.** *Let $5 \leq r < d$ and let $X \subset \mathbb{P}^r$ be a non-degenerate irreducible surface of degree d which is of maximal sectional regularity. Is it true, that the following three statements are equivalent?*

(i) $N(X) \leq d - r$.
(ii) $\beta_{1,d-r+2}(X) = 1$.
(iii) $\mathbb{F}(X) = \mathbb{P}^2$.

We know, that the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold (see the implication (i) $\Rightarrow$ (iii) in statement (a) and the implication (i) $\Rightarrow$ (ii) in statement (b) of Theorem 6.8).

2. Geometry of Extremal Secant Lines and Varieties of Maximal Sectional Regularity

In this section, we consider the set of all *extremal secant lines*, i.e. of all proper $d-c+1$-secant lines of a non-degenerate irreducible projective variety $X \subset \mathbb{P}^r$ of degree d and codimension $c < r$. Varieties, which admit such extremal secant lines are of *extremal (Castelnuovo-Mumford) regularity*. Among the varieties of extremal regularity, those of *maximal sectional regularity* are of particular interest for our investigation. The main result of this section characterizes varieties of maximal sectional regularity as those which "have the largest possible set of extremal secant lines". We first fix a few notations, which we shall keep for the rest of our paper.

2.1. **Notation and Convention.** (A) By $\mathbb{N}_0$ and $\mathbb{N}$ we respectively denote the set of non-negative and of positive integers. If $(Z, \mathcal{O}_Z)$ is a Noetherian scheme, we respectively denote by $\mathrm{Reg}(Z)$, $\mathrm{CM}(Z)$, $\mathrm{Nor}(Z)$, $S_2(Z)$ the *locus of regular, Cohen-Macaulay, normal* and *of S_2-points* of Z. Moreover we denote the *length* of the scheme Z by $\#Z$, thus

$$\#Z := \mathrm{length}(\mathcal{O}_Z) \quad (\in \mathbb{N}_0 \cup \{\infty\}).$$

The *singular locus* of a morphism $f : Y \to Z$ of Noetherian schemes will be denoted by $\mathrm{Sing}(f)$. So, $\mathrm{Sing}(f)$ is the least closed subset W of Z such that restriction of f gives rise to an isomorphism

$$f \restriction : Y \setminus f^{-1}(W) \xrightarrow{\cong} Z \setminus W.$$

If $\mathrm{Sing}(f)$ is a finite set, we say that f is *almost non-singular*.
(B) Once for all we fix an algebraically closed field K, an integer $r \in \mathbb{N}$ and always write S for the polynomial ring $K[x_0, \ldots, x_r]$. We furnish S with its standard grading and write S_+ for the irrelevant ideal $\bigoplus_{n\in\mathbb{N}} S_n = (x_0, \ldots, x_r)$ of S. If $\mathfrak{a} \subset S$ is a graded ideal of S, we use $\mathfrak{a}^{\mathrm{sat}}$ to denote the *saturation* $\bigcup_{n\in\mathbb{N}}(\mathfrak{a} :_S (S_+)^n)$ of $\mathfrak{a}$.
(C) If $Z \subseteq \mathbb{P}^r := \mathrm{Proj}(S)$ is a closed subscheme, $I_Z \subseteq S$ is used to denote the *homogeneous vanishing ideal* of Z in S and $\mathcal{J}_Z \subseteq \mathcal{O}_{\mathbb{P}^r}$ is used to denote the *sheaf of vanishing ideals* of Z in $\mathcal{O}_{\mathbb{P}^r}$. Keep in mind, that $I_Z \subset S$ is a graded saturated ideal, which equals S if and only if $Z = \emptyset$, that $\mathcal{J}_Z = \widetilde{I_Z}$ is the coherent sheaf of $\mathcal{O}_{\mathbb{P}^r}$-modules induced by I_Z, and that $I_Z = H^0_*(\mathbb{P}^r, \mathcal{J}_Z) = \bigoplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{J}_Z(n))$.
(D) Moreover $X \subset \mathbb{P}^r$ always denotes an irreducible non-degenerate variety with sheaf of vanishing ideals $\mathcal{J} := \mathcal{J}_X \subset \mathcal{O}_{\mathbb{P}^r_K}$, homogeneous vanishing ideal $I := I_X \subseteq S$ and *homogeneous coordinate ring* $A = A_X := S/I$.

As (Castelnuovo-Mumford) regularity is of basic significance for our paper, we recall a few facts and define a few notions related to this invariant.

2.2. **Notation and Reminder.** (A) If $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a finitely generated graded S-module, and $i \in \mathbb{N}_0$, we use

$$H^i(M) = \bigoplus_{n\in\mathbb{Z}} H^i(M)_n$$

to denote the i-th local cohomology module of M with respect to the irrelevant ideal $S_+ = \bigoplus_{n\in\mathbb{N}} S_n$ of S, furnished with its natural grading. Moreover we also set in this situation

$$h^i(M)_n := \dim_K \big(H^i(M)_n\big) \text{ for all } i \in \mathbb{N}_0 \text{ and all } n \in \mathbb{Z}.$$

Keep in mind that $h^i(M)_n$ is always finite and vanishes whenever $i > \dim(M)$ or $n \gg 0$.

If $\mathcal{F} := \widetilde{M}$ is the coherent sheaf of $\mathcal{O}_{\mathbb{P}^r}$-modules induced by M, the *Serre-Grothendieck Correspondence* yields an exact sequence of graded S-modules

$$0 \longrightarrow H^0(M) \longrightarrow M \longrightarrow H^0_*(\mathbb{P}^r, \mathcal{F}) \longrightarrow H^1(M) \longrightarrow 0$$

and isomorphisms of graded S-modules

$$H^i_*(\mathbb{P}^r, \mathcal{F}) \cong H^{i+1}(M) \text{ for all } i \in \mathbb{N},$$

where $H^i_*(\mathbb{P}^r, \mathcal{F})$ denotes the graded S-module $\bigoplus_{n\in\mathbb{Z}} H^i(\mathbb{P}^r, \mathbb{F}(n))$.

In particular, if $\emptyset \neq Z \subsetneq \mathbb{P}^r$ is a closed subscheme with sheaf of vanishing ideals $\mathcal{J}_Z \subset \mathcal{O}_{\mathbb{P}^r}$ and homogeneous vanishing ideal $I_Z = \bigoplus H^0(\mathbb{P}^r, \mathcal{J}_Z(n)) \subset S$, we have

$$H^i(\mathbb{P}^r, \mathcal{O}_Z(n)) \cong H^i(Z, \mathcal{O}_Z(n)) \cong H^{i+1}(S/I_Z)_n \text{ for all } i \in \mathbb{N} \text{ and all } n \in \mathbb{Z}$$

and

$$H^i(\mathbb{P}^r, \mathcal{J}_Z(n)) \cong H^i(S/I_Z)_n \text{ if either } i \neq r \text{ and } n \in \mathbb{Z}, \text{ or else } i = r \text{ and } n \geq -r.$$

(B) In the setting of part (A) and for any $k \in \mathbb{N}_0$ we denote the *(Castelnuovo-Mumford) regularity of M at and above level k* by $\operatorname{reg}^k(M)$, hence

$$\operatorname{reg}^k(M) = \inf\{a \in \mathbb{Z} \mid H^i(M)_{i+n} = 0, \forall i \geq k, \forall n > a\}.$$

The *(Castelnuovo-Mumford) Regularity* $\operatorname{reg}(M)$ of M is defined to be the regularity of M at and above level 0, thus

$$\operatorname{reg}(M) := \operatorname{reg}^0(M) = \inf\{a \in \mathbb{Z} \mid H^i_{S_+}(M)_{i+n} = 0, \forall i \in \mathbb{N}_0, \forall n > a\},$$

The *(Castelnuovo-Mumford) regularity* of a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r}$-modules $\mathcal{F}$ will be denoted by $\operatorname{reg}(\mathcal{F})$, hence

$$\operatorname{reg}(\mathcal{F}) = \inf\{a \in \mathbb{Z} \mid H^i(\mathbb{P}^r, \mathcal{F}(i+n)) = 0, \forall i \in \mathbb{N}, \forall n \geq a\}.$$

If $\mathcal{F} := \widetilde{M}$ denotes the coherent sheaf of $\mathcal{O}_{\mathbb{P}^r}$-modules induced by the finitely generated graded S-module M, it follows from the Serre-Grothendieck Correspondence that

$$\operatorname{reg}(\mathcal{F}) = \operatorname{reg}^2(M).$$

If $\emptyset \neq Z \subsetneq \mathbb{P}^r$ is a closed subscheme with sheaf of vanishing ideals $\mathcal{J}_Z \subseteq \mathcal{O}_{\mathbb{P}^r}$ and homogeneous vanishing ideal $I_Z \subset S$, the *(Castelnuovo-Mumford) regularity* of Z is denoted by $\operatorname{reg}(Z)$. So, according to the observations made in part (A) we have

$$\operatorname{reg}(Z) = \operatorname{reg}(\mathcal{J}_Z) = \operatorname{reg}(I_Z) = \operatorname{reg}(S/I_Z) + 1.$$

We now define the notions of variety of extremal regularity and of maximal sectional regularity, which latter is the basic concept of our paper.

2.3. **Remark and Definition.** (A) Let the irreducible non-degenerate variety $X \subset \mathbb{P}^r$ be of degree d and codimension $c < r$. Then, the conjectural regularity inequality of Eisenbud-Goto [15] says that

$$\operatorname{reg}(X) \leq d - c + 1.$$

We say that X is of *extremal regularity* if $\operatorname{reg}(X) \geq d - c + 1$.
(B) If $C = X \subset \mathbb{P}^r$ is a curve, the conjectural inequality of part (A) holds according to Gruson-Lazarsfeld-Peskine [20], so that $\operatorname{reg}(C) \leq d - r + 2$ in this case. Thus, C is of extremal regularity if and only if $\operatorname{reg}(C)$ takes its maximally possible value $d - r + 2$. We therefore say in this situation, that the curve C is of *maximal regularity*. Curves of maximal regularity have some important properties, which shall be of particular interest for our later investigations. Namely according to [20] and [7][Remark 3.1 (C)] we can say:

(a) If $r \geq 3$ and $d > r + 1$, each curve $C \subset \mathbb{P}^r$ of degree d and of maximal regularity is smooth and rational.
(b) If $r \geq 3$ and $d > r + 1$, each curve $C \subset \mathbb{P}^r$ of degree d and of maximal regularity has a $(d - r + 2)$-secant line $\mathbb{L} = \mathbb{P}^1 \subset \mathbb{P}^r$.
(c) If in addition $r \geq 4$, the $(d - r + 2)$-secant line of statement (b) is uniquely determined by C.

(C) If $k \leq r$ is a non-negative integer, we write

$$\mathbb{G}(k, \mathbb{P}^r) := \{\mathbb{P}^k \mid \mathbb{P}^k \subseteq \mathbb{P}^r\}$$

for the *Grassmannian* of all k-subspaces $\mathbb{P}^k \subseteq \mathbb{P}^r$.

Let $X \subset \mathbb{P}^r$ be as in part (A). We say that X is of *maximal sectional regularity* if $X \cap \mathbb{P}^{c+1} \subset \mathbb{P}^{c+1}$ is a curve of maximal regularity for a general space $\mathbb{P}^{c+1} \in \mathbb{G}(c+1, \mathbb{P}^r)$. So, according to statement (a) of part (B) we can say:

> If $c \geq 2$ and if $X \subset \mathbb{P}^r$ is of maximal sectional regularity, then $X \cap \mathbb{P}^{c+1} \subset \mathbb{P}^{c+1}$ is a non-degenerate smooth and rational curve of degree d and of regularity $d-c+1$ for a general space $\mathbb{P}^{c+1} \in \mathbb{G}(c+1, \mathbb{P}^r)$.

(D) If $Z \subset \mathbb{P}^r$ is a closed subscheme of dimension > 1 and $\mathbb{H} = \mathbb{P}^{r-1} \subset \mathbb{P}^r$ is a general hyperplane, we have $\operatorname{reg}(Z \cap \mathbb{H}) \leq \operatorname{reg}(Z)$. So, by induction on the dimension $\dim(X)$ of X, we obtain:

> A variety $X \subset \mathbb{P}^r$ of maximal sectional regularity is of extremal regularity.

The previously defined concepts are intimately related to the existence of highly secant lines. We therefore recall a few preliminary facts on secant lines and secant varieties.

2.4. **Notation and Reminder.** (A) Let $Z \subset \mathbb{P}^r$ be a closed subscheme, let $m \in \mathbb{N}_0$ and consider the set

$$\Sigma_m(Z) := \{\mathbb{L} \in \mathbb{G}(1, \mathbb{P}^r) \mid \#(Z \cap \mathbb{L}) \geq m\}$$

of all *m-secant lines* to Z. This set is closed in $\mathbb{G}(1, \mathbb{P}^r)$. To see this, let $d \in \mathbb{N}$ be such that the homogeneous vanishing ideal $I_Z \subset S = K[x_0, \ldots, x_r]$ is generated by homogeneous polynomials of degree $\leq d$. Let $\mathbb{L} \in \mathbb{G}(1, \mathbb{P}^r)$ and let $I_{\mathbb{L}} \subset S$ denote the homogeneous vanishing ideal of $\mathbb{L}$. Then, the ideal $I_Z + I_{\mathbb{L}} \subset S$ is generated by homogeneous polynomials of degree $\leq d$ so that its regularity is bounded in terms of d and r only – for example by the inequality $\operatorname{reg}(I_Z + I_{\mathbb{L}}) \leq (2d)^{2^{r-1}}$ (see [12]). Now, fix some integer $t \geq \max\{(2d)^{2^{r-1}}, m\}$. Then, the vanishing ideal $I_{Z \cap \mathbb{L}} = (I_Z + I_{\mathbb{L}})^{\mathrm{sat}} \subset S$ of $Z \cap \mathbb{L}$ coincides with $I_Z + I_{\mathbb{L}}$ in all degrees $\geq t$. With $N := \dim_K(S_t)$ it follows, that $\mathbb{L} \in \Sigma_m(Z)$ if and only if $\dim_K((I_Z)_t + (I_{\mathbb{L}})_t) \leq N - m$. This means, that the set $\Sigma_m(Z)$ is the preimage of the set

$$V := \{\mathbb{P} \in \mathbb{G}(N-t-2, |\mathcal{O}_{\mathbb{P}^r}(t)|) \mid \dim\langle \mathbb{P}, |\mathcal{J}_Z(t)|\rangle < N - m\} \subset \mathbb{G}(N-t-2, |\mathcal{O}_{\mathbb{P}^r}(t)|)$$

under the morphism

$$\Phi : \mathbb{G}(1, \mathbb{P}^r) \to \mathbb{G}(N-t-2, |\mathcal{O}_{\mathbb{P}^r}(t)|), \quad \mathbb{L} \mapsto |\mathcal{J}_{\mathbb{L}}(t)| = |(I_{\mathbb{L}})_t|.$$

According to [21, Chapter 6] the above set V is closed in the Grassmannian $\mathbb{G}(N-t-2, |\mathcal{O}_{\mathbb{P}^r}(t)|)$. Therefore $\Sigma_m(Z)$ is a closed subset of the Grassmannian $\mathbb{G}(1, \mathbb{P}^r)$.

(B) Keep the previous notations and hypotheses, let $T \subset \mathbb{G}(1, \mathbb{P}^r)$ be a closed set and consider the *coincidence variety*

$$Y(T) := \{(x, \mathbb{L}) \in \mathbb{P}^r \times \mathbb{G}(1, \mathbb{P}^r) \mid x \in \mathbb{L} \in T\}$$

which is a closed subset of $\mathbb{P}^r \times \mathbb{G}(1, \mathbb{P}^r)$. Moreover, consider the two morphisms

$$p : Y(T) \to \mathbb{P}^r \text{ and } q : Y(T) \to \mathbb{G}(1, \mathbb{P}^r).$$

induced by the canonical projections. Then, we have

$$\mathcal{S}(T) := \bigcup_{\mathbb{L}\in T} \mathbb{L} = p(q^{-1}(T))$$

and hence $\mathcal{S}(T) \subset \mathbb{P}^r$ is closed.

Applying this to the closed set $T := \Sigma_m(Z) \subset \mathbb{G}(1, \mathbb{P}^r)$, we obtain that the *m-secant variety*

$$\mathrm{Sec}_m(Z) := \mathcal{S}(\Sigma_m(Z)) = \bigcup_{\mathbb{L}\in\Sigma_m(Z)} \mathbb{L} \subset \mathbb{P}^r$$

of Z is closed in $\mathbb{P}^r$.

(C) We also shall use the notations

$$\Sigma_\infty(Z) := \{\mathbb{L} \in \mathbb{G}(1, \mathbb{P}^r) \mid \#(Z \cap \mathbb{L}) = \infty\} = \{\mathbb{L} \in \mathbb{G}(1, \mathbb{P}^r) \mid \mathbb{L} \subseteq Z\}$$

and

$$\mathrm{Sec}_\infty(Z) := \mathcal{S}(\Sigma_\infty(Z)) = \bigcup_{L\in\Sigma_\infty(Z)} \mathbb{L} = \bigcup_{\mathbb{P}^1=\mathbb{L}\subseteq X} \mathbb{L}.$$

Observe that we have the inclusions

$$\Sigma_\infty(Z) \subseteq \Sigma_m(Z) \text{ and } \mathrm{Sec}_\infty(Z) \subseteq \mathrm{Sec}_m(Z) \text{ for all } m \in \mathbb{N}_0$$

with equality at both places if the vanishing ideal $I_Z \subseteq S$ is generated by polynomials of degree $< m$. Hence $\Sigma_\infty(Z) \subseteq \mathbb{G}(1, \mathbb{P}^r)$ and $\mathrm{Sec}_\infty(Z) \subseteq \mathbb{P}^r$ are closed subsets, too. In particular, for each $m \in \mathbb{N}_0$, the set

$$\Sigma^\circ_m(Z) := \Sigma_m(Z) \setminus \Sigma_\infty(Z)$$

of *proper m-secant lines* to Z is locally closed in $\mathbb{G}(1, \mathbb{P}^r)$. Moreover $\Sigma^\circ_m(Z) \neq \emptyset$ implies that I_Z needs homogeneous generators of degree $\geq m$. So, for each $m \in \mathbb{N}_0$ we have the implication:

$$\text{If } \Sigma^\circ_m(Z) \neq \emptyset, \text{ then } \mathrm{reg}(Z) \geq m.$$

In the sequel we are interested in the set $\Sigma^\circ_{d-c+1}(X)$ of proper extremal secant lines to a variety $X \subset \mathbb{P}^r$ of codimension c and degree d. Among these proper extremal secant lines, those which are extremal secant lines of a curve of maximal regularity $X \cap \mathbb{E}$ with $\mathbb{E} \subset \mathbb{G}(c+1, \mathbb{P}^r)$ will be of particular interest later. We therefore introduce the following notions.

2.5. **Definition and Remark.** (A) Let $X \subset \mathbb{P}^r$ be a non-degenerate irreducible projective variety of codimension $c < r$ and degree d, and let

$$\mathbb{PU}(X) := \{\mathbb{E} \in \mathbb{G}(c+1, \mathbb{P}^r) \mid X \cap \mathbb{E} \subset \mathbb{E} \text{ is an integral curve of maximal regularity}\}.$$

Observe, that this set contains a non-empty open subset of $\mathbb{G}(c+1, \mathbb{P}^r)$ if and only if X is of maximal sectional regularity.

(B) Keep the above notations, assume that $c \geq 3$ and $d > c+2$ and let $\mathbb{E} \in \mathbb{PU}(X)$. Then, according to statements (b) and (c) of Remark and Definition 2.3 (B) the curve of maximal regularity $X \cap \mathbb{E} \subset \mathbb{E} = \mathbb{P}^{c+1}$ has a unique $(d-c+1)$-secant line, which we denote by $\mathbb{L}_{\mathbb{E},X}$. So, this secant line is characterized by the property

$$\#\big((X \cap \mathbb{E}) \cap \mathbb{L}_{\mathbb{E},X}\big) = \#(X \cap \mathbb{L}_{\mathbb{E},X}) = d-c+1.$$

Using this notation we define the set

$$^*\Sigma^\circ_{d-c+1}(X) := \{\mathbb{L}_{\mathbb{E},X} \mid \mathbb{E} \in \mathbb{PU}(X)\} \subseteq \Sigma^\circ_{d-c+1}(X)$$

and call the lines, which belong to this set the *special extremal secant lines* to X.
(C) Keep the previous notations and assume that $3 \leq c < r-1$ and $d > c+2$. Let $\mathbb{H} = \mathbb{P}^{r-1} \subset \mathbb{P}^r$ be a general hyperplane so that $X \cap \mathbb{H} \subset \mathbb{H} = \mathbb{P}^{r-1}$ is a non-degenerate reduced and irreducible variety of codimension $c \geq 3$ and of degree $d > c+2$. Let $\mathbb{E} \in \mathbb{PU}(X \cap \mathbb{H})$. Then $X \cap \mathbb{E} = (X \cap \mathbb{H}) \cap \mathbb{E}$ yields that $\mathbb{E} \in \mathbb{PU}(X)$ and $\mathbb{L}_{\mathbb{E},X\cap\mathbb{H}} = \mathbb{L}_{\mathbb{E},X}$. Therefore we get

$$\mathbb{PU}(X \cap \mathbb{H}) \subseteq \mathbb{PU}(X) \cap \mathbb{G}(c+1, \mathbb{H}) \text{ and } {}^*\Sigma^\circ_{d-c+1}(X \cap \mathbb{H}) \subseteq {}^*\Sigma^\circ_{d-c+1}(X) \cap \mathbb{G}(1, \mathbb{H}).$$

Thus we have

$$\overline{{}^*\Sigma^\circ_{d-c+1}(X \cap \mathbb{H})} \subseteq \overline{{}^*\Sigma^\circ_{d-c+1}(X)} \cap \mathbb{G}(1, \mathbb{H}).$$

2.6. **Definition and Remark.** (A) Let $X \subset \mathbb{P}^r$ be an irreducible projective variety of degree d and codimension $c < r$. The proper $(d-c+1)$-secant lines to X, hence the lines $\mathbb{L}$ which belong to the locally closed subset $\Sigma^\circ_{d-c+1}(X)$ (see Notation and Reminder 2.4 (C)) are called *proper extremal secant lines* to X. To measure the size of the set of all proper extremal secant lines to X, we introduce the invariant

$$\mathfrak{d}(X) := \dim\left(\Sigma^\circ_{d-c+1}(X)\right) = \dim\left(\overline{\Sigma^\circ_{d-c+1}(X)}\right)$$

with the usual convention that $\dim(\emptyset) = -1$. According to Notation and Reminder 2.4 (C) we can say

$$\text{If } \mathfrak{d}(X) \geq 0, \text{ then } X \subset \mathbb{P}^r \text{ is a variety of extremal regularity.}$$

(B) Keep the notations of part (A) and assume that $3 \leq c < r$ and $d > c+2$. To measure the size of the set ${}^*\Sigma^\circ_{d-c+1}(X)$ (see Definition and Remark 2.5) of all special extremal secant lines to X, we define the invariant

$$^*\mathfrak{d}(X) := \dim\left(\overline{{}^*\Sigma^\circ_{d-c+1}(X)}\right).$$

Now, we are heading for the main result of this section. We begin with the following auxiliary result.

2.7. **Lemma.** *Let Σ be a closed subset of $\mathbb{G}(1, \mathbb{P}^r)$ and let $\mathbb{H} = \mathbb{P}^{r-1} \subset \mathbb{P}^r$ be a general hyperplane. Then, the following statements hold.*

(a) *If $\dim(\Sigma) \leq 1$, then $\Sigma \cap \mathbb{G}(1, \mathbb{H}) = \emptyset$.*
(b) *If $\dim(\Sigma) \geq 2$, then each irreducible component W of $\Sigma \cap \mathbb{G}(1, \mathbb{H})$ satisfies*

$$\dim(W) = \dim(\Sigma) - 2.$$

Proof. A result of Kleiman (see [23, Corollary 4]) says that for an integral algebraic group scheme G, an integral scheme X with transitive G-action, for two closed integral subschemes $Y, Z \subset X$ and for general $g \in G$, all irreducible components of $g(Y) \cap Z$ have dimension $\dim(Y) + \dim(Z) - \dim(X)$.

If we fix a hyperplane $\mathbb{H}_0 \subset \mathbb{P}^r$ and apply this result with $G = \mathrm{Aut}(\mathbb{P}^r)$, $X = \mathbb{G}(1, \mathbb{P}^r)$, $Y := \mathbb{G}(1, \mathbb{H}_0)$ and $Z = \Sigma$, and keep in mind that

$$\dim(\mathbb{G}(1, \mathbb{P}^r)) - \dim(\mathbb{G}(1, \mathbb{H}_0)) = 2(r-2) - 2(r-1) = 2,$$

we get our claim. ☐

2.8. **Theorem.** *Assume that $1 \leq t \leq r-3$ and let $X \subset \mathbb{P}^r$ be a non-degenerate irreducible projective variety of dimension t and degree $> r-t+2$. Then, the following statements hold.*

(a) ${}^*\mathfrak{d}(X) = \mathfrak{d}(X) \leq 2t-2$.
(b) *X is of maximal sectional regularity if and only if $\mathfrak{d}(X) = 2t-2$.*

Proof. Let $d := \deg(X)$ and let $c := r-t$. As ${}^*\Sigma^\circ_{d-c+1}(X) \subseteq \Sigma^\circ_{d-c+1}(X)$ we have the inequality

$$ {}^*\mathfrak{d}(X) \leq \mathfrak{d}(X). $$

We proceed by by induction on $t = \dim(X)$.

First, let $t = 1$, so that $r \geq 4$ and $X \subset \mathbb{P}^r$ is a curve of degree $d > r+1$. If $\mathfrak{d}(X) \geq 0$, it follows by Definition and Remark 2.6 (A), that X is of maximal regularity. But then, by Definition and Remark 2.3 (B) (c), X has precisely one extremal secant line, so that $\mathfrak{d}(X) = {}^*\mathfrak{d}(X) = 0$. This observation proves statements (a) and (b) in the case $t = 1$.

So, let $t > 1$ and let $\mathbb{H} = \mathbb{P}^{r-1} \subset \mathbb{P}^r$ be a general hyperplane. Then $X \cap \mathbb{H} \subset \mathbb{H} = \mathbb{P}^{r-1}$ is a non-degenerate irreducible projective variety of codimension $c \geq 3$, of dimension $t-1$ and of degree $d > c+2$ – which is of maximal sectional regularity if and only if $X \subset \mathbb{P}^r$ is. So by induction we first have

$$ {}^*\mathfrak{d}(X \cap \mathbb{H}) = \mathfrak{d}(X \cap \mathbb{H}) \leq 2t-4, $$

with equality at the second place if and only if and only if $X \cap \mathbb{H} \subset \mathbb{H} = \mathbb{P}^{r-1}$ is of maximal sectional regularity – hence, if and only if $X \subset \mathbb{P}^r$ is of maximal sectional regularity.

Moreover, we have

$$ \Sigma^\circ_{d-c+1}(X \cap \mathbb{H}) = \Sigma^\circ_{d-c+1}(X) \cap \mathbb{G}(1, \mathbb{H}). $$

On application of Lemma 2.7, with $\Sigma := \overline{\Sigma^\circ_{d-c+1}(X)}$ it follows hat

$$ \mathfrak{d}(X) = \dim\left(\Sigma^\circ_{d-c+1}(X)\right) = \dim\left(\Sigma^\circ_{d-c+1}(X \cap \mathbb{H})\right) + 2 = \mathfrak{d}(X \cap \mathbb{H}) + 2 \leq 2t-2, $$

with equality at the last place if and only if X is of maximal sectional regularity.

It remains to show that ${}^*\mathfrak{d}(X) \geq \mathfrak{d}(X)$. In view of Definition and Remark 2.5 (C) we have

$$ \mathfrak{d}(X \cap \mathbb{H}) = {}^*\mathfrak{d}(X \cap \mathbb{H}) = \dim\left(\overline{{}^*\Sigma^\circ_{d-c+1}(X \cap \mathbb{H})}\right) \leq \dim\left(\overline{{}^*\Sigma^\circ_{d-c+1}(X)} \cap \mathbb{G}(1, \mathbb{H})\right). $$

If we apply Lemma 2.7 with $\Sigma = \overline{{}^*\Sigma^\circ_{d-c+1}(X)}$, we get

$$ {}^*\mathfrak{d}(X) = \dim\left(\overline{{}^*\Sigma^\circ_{d-c+1}(X)}\right) = \dim\left(\overline{{}^*\Sigma^\circ_{d-c+1}(X)} \cap \mathbb{G}(1, \mathbb{H})\right) + 2 $$

and hence

$$ {}^*\mathfrak{d}(X) \geq \mathfrak{d}(X \cap \mathbb{H}) + 2 = \mathfrak{d}(X). $$

☐

3. Sectionally Rational Varieties

As we shall see, surfaces of maximal sectional regularity have at most finitely many singular points and their generic hyperplane section is a smooth rational curve. As a consequence, these surfaces are almost non-singular projections of rational normal surface scrolls. In this section, we approach this fact in a more general setting, by showing first, that (under certain restrictions) projective varieties whose general linear section curve is rational, are birational projections of varieties of minimal degree. We first fix some notations concerning projections.

3.1. **Notation and Convention.** Let $r' \geq r$ and let $X' \subset \mathbb{P}^{r'}$ be an irreducible and non-degenerate variety. If $\Lambda = \mathbb{P}^{r'-r-1} \subset \mathbb{P}^{r'}$ is a linear subspace, such that $X' \cap \Lambda = \emptyset$ and $\pi'_\Lambda : \mathbb{P}^{r'} \setminus \Lambda \twoheadrightarrow \mathbb{P}^r$ is a *linear projection with center* Λ, we write $X'_\Lambda := \pi'_\Lambda(X')$ for the *projected image* $\pi'_\Lambda(X') \subset \mathbb{P}^r$ of X' and $\pi_\Lambda : X' \twoheadrightarrow X'_\Lambda$ for the finite morphism induced by the projection π'_Λ. If $X = X'_\Lambda$, we say that X is *a projection of* X' *(from the center* Λ*)* and call $X' \subset \mathbb{P}^{r'}$ a *projecting variety* of X. If the induced projection morphism $\pi_\Lambda : X' \twoheadrightarrow X$ is in addition almost non-singular, we say that X is an *almost non-singular projection of* X' *(from the center* Λ*)*.

Next, we define the basic concept of this section.

3.2. **Definition.** A non-degenerate irreducible variety $X \subset \mathbb{P}^r$ of codimension $c < r$ is said to be *sectionally rational* if $X \cap \mathbb{P}^{c+1} \subset \mathbb{P}^{c+1}$ is a (possibly singular) rational curve for a general space $\mathbb{P}^{c+1} \in \mathbb{G}(c+1, \mathbb{P}^r)$.

Now, we are ready to prove the announced result on sectionally rational varieties. The reader should be aware of the fact, that in our proof, we use a Bertini Theorem for linear systems of ample divisors found in [17], which is known only in characteristic 0. Nevertheless, in the case of sectionally rational surfaces with finite non-normal locus – the case we are actually heading for – we can avoid the use of the mentioned Bertini Theorem and thus get a characteristic free statement.

3.3. **Theorem.** *Let* $X \subset \mathbb{P}^r$ *be sectionally rational. Set* $t := \dim X$ *and* $d := \deg X$*. Assume that either*

(1) $\operatorname{Char} K = 0$, *or else*
(2) $t = 2$ *and* $X \setminus \operatorname{Nor}(X)$ *is finite.*

Then, X *is a projection of a variety of minimal degree. More precisely, there is a variety* $\widetilde{X} \subset \mathbb{P}^{d+t-1}$ *of minimal degree and a subspace* $\Lambda = \mathbb{P}^{d+t-r-2} \subset \mathbb{P}^{d+t-1}$ *with* $\widetilde{X} \cap \Lambda = \emptyset$ *such that (in the notation introduced in Notation and Convention 3.1)*

(a) $\widetilde{X}_\Lambda = X$*;*
(b) $X \setminus \operatorname{Nor}(X) = \operatorname{Sing}(\pi_\Lambda : \widetilde{X} \twoheadrightarrow X)$.

Proof. Let $\nu : Y \to X$ be the normalization of X, so that Y is a normal projective variety and ν is a finite surjective morphism with $\operatorname{Sing}(\nu) = X \setminus \operatorname{Nor}(X)$. Consider the ample invertible sheaf of $\mathcal{O}_Y$-modules $\mathcal{L} := \nu^*(\mathcal{O}_X(1))$. Let $h_1, \ldots, h_{t-1} \in S_1$ be general linear forms and consider the irreducible varieties $X_i := X \cap \operatorname{Proj}(S/\sum_{j=1}^{t-i} h_j S)$ and their preimages $Y_i := \nu^{-1}(X_i) \quad (i = 1, \ldots t)$. As the linear forms h_j are general, X_i is not contained in $\mathrm{Sing}(\nu)$ and so, the schemes Y_i are irreducible and the induced finite morphisms

$$\nu_i := \nu \restriction: Y_i \twoheadrightarrow X_i, \quad i = 1 \ldots, t$$

are birational. Assume first, that $\mathrm{Char}\, K = 0$. As the closed subscheme $Y_i \subset Y$ is cut out by the $t - i$ general divisors $\nu^*(h_j) \in |\mathcal{L}| \quad (j = 1, \ldots, t-i)$, it is normal by the Bertini Theorem [17, Corollary 3.4.2]. So, the sequence $Y_1 \subset Y_2 \subset \ldots \subset Y_t = Y$ forms a ladder with normal rungs of the polarized variety $(Y, \mathcal{L})$ in the sense of Fujita [18, Definition 3.1, pg.28]. As $\nu_1 : Y_1 \twoheadrightarrow X_1$ is birational, it follows that $Y_1 \cong \mathbb{P}^1$.

Assume now, that $t = 2$ and the non-normal locus $X \setminus \mathrm{Nor}(X) = \mathrm{Sing}(\nu)$ of X is finite. As X is a surface, it follows that the singular locus $X \setminus \mathrm{Reg}(X)$ is finite too, So, as h_1 is general, the curve $X_1 \subset X$ is smooth and disjoint to $\mathrm{Sing}(\nu)$. It follows that $X_1 \cong \mathbb{P}^1$ and $\nu_1 : Y_1 \twoheadrightarrow X_1$ is an isomorphism, so that also in this case $Y_1 \cong \mathbb{P}^1$. Hence, again we have a ladder $Y_1 \subset Y_2 = Y$ of the polarized variety $(Y, \mathcal{L})$ with normal rungs and $Y_1 \cong \mathbb{P}^1$.

Thus, in both cases the sectional genera in the sense of Fujita [18, 2.1, pp.25,28] satisfy $g(Y, \mathcal{L}) = g(Y_1, \mathcal{L} \restriction_{Y_1}) = 0$. Therefore, by [18, Proposition 3.4], the Δ-genus $\Delta(X, \mathcal{L})$ of the polarized variety $(X, \mathcal{L})$ equals zero. So, according to [18, Corollary 4.12] we have

$$\Gamma_*(Y, \mathcal{L}) := \bigoplus_{n \in \mathbb{N}_0} \Gamma(Y, \mathcal{L}^{\otimes^n}) = K[\Gamma(Y, \mathcal{L})]$$

and $\mathcal{L}$ is very ample over Y. According to Fujita's Classification Theorem [18, Theorem 5.15], it now follows, that $|\mathcal{L}|$ induces a closed immersion $i : Y \to \mathbb{P}^s = \mathbb{P}(\Gamma(Y, \mathcal{L}))$ such that $\widetilde{X} := i(Y) \subset \mathbb{P}^s$ is a variety of minimal degree and that, in addition, there is a projection $\pi'_\Lambda : \mathbb{P}^s \setminus \Lambda \twoheadrightarrow \mathbb{P}^r$ from a subspace $\Lambda = \mathbb{P}^{s-r-1} \subset \mathbb{P}^s$ with $\widetilde{X} \cap \Lambda = \emptyset$ such that $\widetilde{X}_\Lambda = X$ and $\nu = \pi_\Lambda \circ i$. As $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$ is birational, we have $\dim \widetilde{X} = t$ and $\deg \widetilde{X} = d$. As $\widetilde{X} \subset \mathbb{P}^s$ is of minimal degree, it follows that $d = s - t + 1$ and hence $s = d + t - 1$. This proves our claim. $\square$

Observe, that according to the previous result, sectionally rational surfaces with finite non-normal locus are almost non-singular projections of surface scrolls or the Veronese surface. One important consequence of this is, that these surfaces satisfy the conjectural inequality of Eisenbud-Goto [15] for the Castelnuovo-Mumford regularity. Again, we shall give a more general approach to this fact and show, that an almost non-singular outer projection X'_Λ from a subspace $\Lambda = \mathbb{P}^{r'-r-1} \subset \mathbb{P}^{r'}$ of a variety $X' \subset \mathbb{P}^{r'}$ which satisfies the Green-Lazarsfeld condition $N_{2,p}$ for some $p > r'-r$, satisfies the Eisenbud-Goto inequality. We begin with some preparations concerning Betti numbers.

3.4. **Notation and Reminder.** (A) We have $K = S/S_+$ and denote the i-th *Betti number* in degree $i + j$ of a finitely generated graded S-module $M \neq 0$ by

$$\beta_{i,j}(M) := \dim_K \left(\mathrm{Tor}_i^S(K, M)_{i+j} \right) \text{ for } i \in \mathbb{N} \text{ and } j \in \mathbb{Z}.$$

Keep in mind the well known fact, that the *initial degree*

$$\mathrm{Indeg}(M) := \inf\{n \in \mathbb{Z} \mid M_n \neq 0\},$$

the regularity and the depth of M bound the range of pairs of indices (i, j) with non-zero Betti numbers as follows:

$$\begin{aligned}\operatorname{Indeg}(M) &= \min\{j \in \mathbb{Z} \mid \beta_{0,j}(M) \neq 0\} = \min\{j \in \mathbb{Z} \mid \beta_{i,j}(M) \neq 0 \text{ for some } i \in \mathbb{N}_0\}\\ \operatorname{reg}(M) &= \max\{j \in \mathbb{Z} \mid \beta_{i,j}(M) \neq 0 \text{ for some } i \in \mathbb{N}_0\},\\ \operatorname{depth}(M) &= \max\{i \in \mathbb{N}_0 \mid \beta_{r+1-i,j}(M) \neq 0 \text{ for some } j \in \mathbb{Z}\}.\end{aligned}$$

Moreover, if $i \in \mathbb{N}_0$ and $l \in \mathbb{Z}$ are such that $\beta_{i,j}(M) = 0$ for all $j \leq l$, then $\beta_{k,j}(M) = 0$ for all $k \geq i$ and all $j \leq l$.

Finally, if $I \subsetneq S$ is a homogeneous ideal, we have

$$\beta_{i,j}(I) = \beta_{i+1,j-1}(S/I) \text{ for all } i \in \mathbb{N}_0 \text{ and all } j \in \mathbb{Z}.$$

If $\emptyset \neq Z \subsetneq \mathbb{P}^r$ is a closed subscheme with homogeneous vanishing ideal $I_Z \subset S$, we define the *Betti numbers* of Z by

$$\beta_{i,j} = \beta_{i,j}(Z) := \beta_{i,j}(S/I_Z) = \beta_{i-1,j+1}(I_Z) \text{ for all } i \in \mathbb{N} \text{ and all } j \in \mathbb{Z}.$$

Observe that $\beta_{0,0}(Z) = 1$ and $\beta_{0,j} = 0$ for all $j \neq 0$. Moreover, we have $\operatorname{depth}(S/I_Z) = \operatorname{depth}(Z) > 0$. In addition by Notation and Reminder 2.2 (B), we have $\operatorname{reg}(S/I_Z) = \operatorname{reg}(Z) - 1$. Finally if Z is non-degenerate, we have $\beta_{1,0}(Z) = 0$ and hence $\beta_{i,0}(Z) = 0$ for all $i \geq 1$. So, for a non-degenerate closed subscheme $\emptyset \neq Z \subsetneq \mathbb{P}^r$ we usually only list the Betti numbers

$$\beta_{i,j}(Z) \quad \text{with} \quad 1 \leq i \leq r + 1 - \operatorname{depth}(Z) \quad (\leq r) \quad \text{and} \quad 1 \leq j < \operatorname{reg}(Z).$$

(B) Let $p \in \mathbb{N}$. The graded ideal $I \subset S$ is said to satisfy the *(Green-Lazarsfeld) property* $N_{2,p}$ (see [19]) if the Betti numbers of S/I satisfy the condition

$$\beta_{i,j} := \beta^S_{i,j}(S/I) = \beta^S_{i-1,j+1}(I) = 0 \text{ whenever } i \leq p \text{ and } j \neq 1,$$

– hence if and only if the minimal free resolution of I – up to the homological degree p – has the form

$$\ldots \to S^{\beta_{p,1}}(-p-1) \to \ldots \to S^{\beta_{1,1}}(-2) \to I \to 0.$$

The closed subscheme $Z \subset \mathbb{P}^r$ is said to satisfy the *property* $N_{2,p}$ if its homogeneous vanishing ideal $I_Z \subset S$ satisfies the property $N_{2,p}$.

Now, we may prove the announced regularity bound for almost non-singular projections of $N_{2,p}$-varieties.

3.5. **Theorem.** *Let $r' \geq r$ be integers, let $X' \subset \mathbb{P}^{r'}$ be a non-degenerate projective variety of dimension ≥ 2 which satisfies the property $N_{2,p}$ for some $p \geq \max\{2, r' - r + 1\}$. Let $\Lambda = \mathbb{P}^{r'-r-1}$ be a subspace such that $X' \cap \Lambda = \emptyset$. Let*

$$X := X'_\Lambda \subset \mathbb{P}^r$$

and assume that the induced finite morphism $\pi_\Lambda : X' \twoheadrightarrow X$ is almost non-singular. Then

(a) *The homogeneous vanishing ideal $I_X \subset S$ of X is generated by homogeneous polynomials of degrees $\leq r' - r + 2$.*
(b) $\operatorname{reg}(X) \leq \max\{\operatorname{reg}(X'), r' - r + 2\}$.

Proof. Let $I_{X'} \subset S' := K[x_0, \ldots, x_{r'}]$ be the homogeneous vanishing ideal of $X' \subset \mathbb{P}^{r'} = \mathrm{Proj}(S')$ and let $A' := S'/I_{X'}$ be the homogeneous coordinate ring of X'. We consider A' as a finitely generated graded S-module and set $t := r' - r$. As X' satisfies the condition $N_{2,p}$ with $p \geq \max\{2, t+1\}$, it follows by [1, Theorem 3.6], that the minimal free presentation of A' has the shape

$$S^s(-2) \xrightarrow{v} S \oplus S^t(-1) \xrightarrow{q} A' \to 0$$

for some $s \in \mathbb{N}$. Moreover, the coordinate ring $A = S/I_X$ of X is nothing else than the image $q(S)$ under q of the direct summand $S \subset S \oplus S^t(-1)$. Therefore

$$A'/A \cong \mathrm{Coker}\big(u : S^s(-2) \to S^t(-1)\big),$$

where u is the composition of the map $v : S^s(-2) \to S \oplus S^t(-1)$ with the canonical projection map $w : S \oplus S^t(-1) \twoheadrightarrow S^t(-1)$. Hence, the S-module $(A'/A)(1)$ is generated by t homogeneous elements of degree 0 and related in degree 1. As $\mathrm{Sing}(\pi_\Lambda)$ is finite, we have $\dim(A'/A) \leq 1$. So, it follows by [13, Corollary 2.4] that $\mathrm{reg}\big((A'/A)(1)\big) \leq t - 1$, whence $\mathrm{reg}(A'/A) \leq t$. Now, the short exact sequence $0 \to A \to A' \to A'/A \to 0$ implies that $\mathrm{reg}(A) \leq \max\{\mathrm{reg}(A'), t+1\}$. It follows that $\mathrm{reg}(X) = \mathrm{reg}(A) + 1 \leq \max\{\mathrm{reg}(A') + 1, t+2\} = \max\{\mathrm{reg}(X'), t+2\} = \max\{\mathrm{reg}(X'), r' - r + 2\}$. This proves claim (b).

To prove claim (a), observe that $I_X = \mathrm{Ker}(q) \cap S$ occurs in the short exact sequence of graded S-modules $0 \to I_X \to \mathrm{Im(v)} \xrightarrow{w\restriction} \mathrm{Im(u)} \to 0$, where $w \restriction$ is the restriction of the above projection map w. In particular, we may identify $w \restriction$ with the canonical map $S^s(-2)/\mathrm{Ker}(v) \twoheadrightarrow S^s(-2)/\mathrm{Ker}(u)$. It follows, that $I_X \cong \mathrm{Ker}(u)/\mathrm{Ker}(v)$. In view of the exact sequence $0 \to \mathrm{Ker}(u) \to S^s(-2) \xrightarrow{u} S^t(-1) \to A'/A \to 0$, we now finally get $\mathrm{reg}(\mathrm{Ker}(u)) \leq t + 2 = r' - r + 2$. Therefore $\mathrm{Ker}(u)$ is generated in degrees $\leq r' - r + 2$, and hence so is I_X. This proves statement (a). $\square$

The reader may have noticed, that our proof is inspired by arguments originally found in the papers [20] and [27]. Nevertheless, the use of [13, Corollary 2.4] allows us to argue in a more direct way.

3.6. **Corollary.** *(Compare* [24, Theorem 5.2]*) Assume that the projective variety $X \subset \mathbb{P}^r$ is sectionally rational and that its non-normal locus $X \setminus \mathrm{Nor}(X)$ is finite. Suppose that either* Char $K = 0$ *or X is a surface. Then, the regularity of X satisfies the Eisenbud-Goto inequality:*

$$\mathrm{reg}(X) \leq \deg(X) - \mathrm{codim}_{\mathbb{P}^r}(X) + 1.$$

Proof. We may assume that $X \subsetneq \mathbb{P}^r$. Set $d := \deg(X)$ and $\dim(X) := t$. According to Theorem 3.3 there is a variety $\widetilde{X} \subset \mathbb{P}^{d+t-1}$ of minimal degree and a subspace $\Lambda = \mathbb{P}^{d+t-r-2}$ such that $\widetilde{X} \cap \Lambda = \emptyset$, $X = \widetilde{X}_\Lambda$ and $\mathrm{Sing}(\pi_\Lambda : \widetilde{X} \twoheadrightarrow X) = X \setminus \mathrm{Nor}(X)$. In particular $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$ is almost non-singular. Moreover $\widetilde{X} \subset \mathbb{P}^{d+t-1}$ satisfies the conditions $N_{2,p}$ for all $p \in \mathbb{N}$, so that $\mathrm{reg}(\widetilde{X}) = 2$. Therefore, by Theorem 3.5 we get $\mathrm{reg}(X) \leq (d + t - 1) - r + 2 = d - (r - t) + 1$ and this proves our claim. $\square$

Motivated by our aim to study surfaces of maximal sectional regularity, we now shall focus on sectionally rational surfaces $X \subset \mathbb{P}^r$ with $5 \leq r < \deg(X)$ which have finite non-normal locus. One easy consequence of Theorem 3.3 is, that these surfaces are almost non-singular projections of (possibly singular) normal surface scrolls. Our proof of Theorem 3.3 relies on Fujita's classification of polarized varieties [18] of Δ-genus 0, which on its turn relies on a powerful result of Ekedahl [16]. We therefore take it for justified, to furnish in Theorem 3.8 a second and more direct argument, showing that sectionally rational surfaces with finite non-normal locus are projections of surface scrolls. Instead of Fujita's geometric arguments, we use a purely cohomological idea, which on its turn has the disadvantage to apply only in the special situation we shall be looking at. On the other hand, this specific approach gives slightly more insight and shows at once, that the normalization and the finite Macaulayfication of our surfaces coincide. Again, we begin with some preparations.

3.7. **Notation and Reminder.** (A) (see [9]) Assume now, that $X \subset \mathbb{P}^r = \mathrm{Proj}(S)$ is a non-degenerate irreducible projective surface of degree d, homogeneous vanishing ideal $I \subset S$ and homogeneous coordinate ring $A = S/I$. Let $\mathfrak{a} \subsetneq A_+ = S_+A$ be the graded radical ideal which defines the non-Cohen-Macaulay locus $X \setminus \mathrm{CM}(X)$ of X. Observe that height $\mathfrak{a} \geq 2$, so that the ideal transform

$$B(A) := D_{\mathfrak{a}}(A) = \varinjlim \mathrm{Hom}_A(\mathfrak{a}^n, A) = \bigcup_{n\in\mathbb{N}} (A :_{\mathrm{Quot(A)}} \mathfrak{a}^n) = \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathrm{CM}(X), \mathcal{O}_X(n))$$

of A with respect to $\mathfrak{a}$ is a positively graded finite birational integral extension domain of A. In particular $B(A)_0 = K$. Moreover $B(A)$ has the second Serre-property S_2. As $\mathrm{Proj}(B(A))$ is of dimension 2, it thus is a locally Cohen-Macaulay scheme.
If E is a finite graded integral extension domain of A which satisfies the property S_2, we have $A \subset B(A) \subset E$. So $B(A)$ is the least finite graded integral extension domain which has the property S_2. Therefore, we call $B(A)$ the *S_2-cover of A*. We also can describe $B(A)$ as the endomorphism ring $\mathrm{End}(K(A), K(A))$ of the canonical module $K(A) = K^3(A) = \mathrm{Ext}_S^{r-2}(A, S(-r-1))$ of A.
(B) Let the notations be as in part (A). Then, the inclusion map $A \to B(A)$ gives rise to a finite morphism

$$\pi : \widetilde{X} := \mathrm{Proj}(B(A)) \twoheadrightarrow X, \text{ with } \mathrm{Sing}(\pi) = X \setminus \mathrm{CM}(X).$$

In particular π is almost non-singular and hence birational. Moreover, for any finite morphism $\rho : Y \twoheadrightarrow X$ such that Y is locally Cohen-Macaulay, there is a unique morphism $\sigma : Y \to \widetilde{X}$ such that $\rho = \pi \circ \sigma$. In addition σ is an isomorphism if and only if $\mathrm{Sing}(\rho) = X \setminus \mathrm{CM}(X)$. Therefore, the morphism $\pi : \widetilde{X} \twoheadrightarrow X$ is addressed as the *finite Macaulayfication* of X. Keep in mind, that – unlike to what happens with normalization – there may be proper birational morphisms $\tau : Z \twoheadrightarrow X$ with Z locally Cohen-Macaulay, which do not factor through π (see [3]).
(C) Let $X \subset \mathbb{P}^r$ be as in part (A). We introduce the invariants

$$\mathrm{e}_x(X) := \mathrm{length}(H^1_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x})), (x \in X \text{ closed }) \text{ and } \mathrm{e}(X) := \sum\nolimits_{x\in X,\text{closed}} \mathrm{e}_x(X).$$

Note that the latter counts the *number of non-Cohen-Macaulay points* of X in a weighted way. Keep in mind that

$$\mathrm{e}(X) = h^1(X, \mathcal{O}_X(n)) \text{ for all } n \ll 0.$$

(D) Let $X \subset \mathbb{P}^r$ and $A = S/I$ be as above. We denote the *arithmetic depth* of X by $\mathrm{depth}(X)$, hence $\mathrm{depth}(X) := \mathrm{depth}(S/I)$.

Now, we are ready to formulate and to prove the conclusive result of this section. As previously announced and justified, we shall offer two different proves of statement (c) of the theorem to follow.

3.8. **Theorem.** *Let $d \in \mathbb{N}$ with $5 \leq r < d$. Assume that the non-degenerate irreducible surface $X \subset \mathbb{P}^r_K$ of degree d is sectionally rational and has finite non-normal locus $X \setminus \mathrm{Nor}(X)$. Then*

(a) $\mathrm{reg}(X) \leq d - r + 3$ *and* $\mathrm{Reg}(X) = \mathrm{Nor}(X) = \mathrm{CM}(X)$.

(b) *The cohomology of X satisfies the following conditions*
  (1) *For all $n \leq 0$ it holds $h^2(\mathbb{P}^r, \mathcal{J}_X(n)) = \mathrm{e}(X)$.*
  (2) *For all $n \geq 0$ it holds $h^2(\mathbb{P}^r, \mathcal{J}_X(n+1)) \leq h^2(\mathbb{P}^r, \mathcal{J}_X(n))$.*
  (3) *For all $n \geq -1$ it holds $h^3(\mathbb{P}^r, \mathcal{J}_X(n)) = 0$.*

(c) *The S_2-cover $B = B(A)$ of $A = S/I$ is the homogeneous coordinate ring of a surface scroll $\widetilde{X} \subset \mathbb{P}^{d+1}_K$ of degree d and there is a subspace $\Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1}$ such that $\widetilde{X} \cap \Lambda = \emptyset$ and*
  (1) $X = \widetilde{X}_\Lambda$,
  (2) $\mathrm{Sing}(\pi_\Lambda : \widetilde{X} \twoheadrightarrow X) = X \setminus \mathrm{CM}(X)$.

(d) *There is a unique non-negative integer $a \leq d/2$ such that the scroll $\widetilde{X} \subset \mathbb{P}^{d+1}$ of statement (c) is projectively equivalent to the standard rational surface scroll $S(a, d-a) \subset \mathbb{P}^{d+1}$, and moreover $a > 0$ if and only if X is not a cone.*

(e) *The morphism $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$ of statement (c) provides the normalization as well as the finite Macaulayfication of X.*

(f) *If $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$ is as in statement (c) and $p \in X \setminus \mathrm{Reg}(X)$, we have*
  (1) $1 < \#(\pi_\Lambda)^{-1}(p) \leq \mathrm{mult}_p(X)$, *with equality at the second place X is not a cone with vertex p and if $\mathrm{mult}_{\widetilde{p}}((\pi_\Lambda)^{-1}(p)) \leq 2$ for all $\widetilde{p} \in (\pi_\Lambda)^{-1}(p)$.*
  (2) $\#(\pi_\Lambda)^{-1}(p) \leq \mathrm{e}_p(X) + 1$ *with equality if and only if p is a Buchsbaum point of the surface X.*

(g) *If $\mathbb{M} = \mathbb{P}^s \subset \mathbb{P}^r$ is a linear subspace whose intersection with X is finite, then*

$$\#(\mathrm{Reg}(X) \cap \mathbb{M}) + 2\#\big((X \cap \mathbb{M})_{\mathrm{red}} \setminus \mathrm{Reg}(X)\big) \leq d - r + s + 2.$$

Proof. (b): Let $h \in S_1 \setminus \{0\}$ be a general linear form. As X has at most finitely many singular points, the curve $C_h := \mathrm{Proj}(A/hA)$ is smooth and rational. Therefore $H^0(C_h, \mathcal{O}_{C_h}) = H^0(X, \mathcal{O}_X) = K$ and $H^1(C_h, \mathcal{O}_{C_h}(n)) = 0$ for all $n \geq 0$. Hence, the exact sequences

$$0 \to H^0(X, \mathcal{O}_X(n)) \to H^0(C_h, \mathcal{O}_{C_h}(n)) \to H^1(X, \mathcal{O}_X(n-1)) \to H^1(X, \mathcal{O}_X(n))$$

$$\to H^1(C_h, \mathcal{O}_{C_h}(n)) \to H^2(X, \mathcal{O}_X)(n-1)) \to H^2(X, \mathcal{O}_X(n)) \to 0$$

show that

$$h^1(X, \mathcal{O}_X(-1)) = h^1(X, \mathcal{O}_X), \quad h^1(X, \mathcal{O}_X(n+1)) \leq h^1(X, \mathcal{O}_X(n)) \text{ for all } n \geq 0$$

and

$$h^2(X, \mathcal{O}_X(n)) = 0 \text{ for all } n \geq -1.$$

Moreover, as the non-normal locus $X \backslash \mathrm{Nor}(X)$ of X is finite, it follows from [2, Proposition 5.2] that

$$\mathrm{e}(X) \leq h^1(X, \mathcal{O}_X(n-1)) \leq \max\{h^1(X, \mathcal{O}_X(n)) - r, \mathrm{e}(X)\}, \text{ for all } n \leq 0.$$

Therefore we obtain $\mathrm{e}(X) = h^1(X, \mathcal{O}_X(n))$ for all $n \leq 0$. As $h^i(\mathbb{P}^r, \mathcal{J}_X(n)) = h^{i-1}(X, \mathcal{O}_X(n))$ for all $n \in \mathbb{Z}$ and $i = 2, 3$, we get our claims (1), (2) and (3).

(c):*First Proof*: According to Theorem 3.3 there is a surface $\widetilde{X} \subset \mathbb{P}^{d+1}$ of minimal degree and a subspace $\Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1}$, such that

$$\widetilde{X} \cap \Lambda = \emptyset, \quad X = \widetilde{X}_\Lambda \text{ and } \mathrm{Sing}(\pi_\Lambda : \widetilde{X} \twoheadrightarrow X) = X \setminus \mathrm{Nor}(X).$$

Observe, that $\widetilde{X} \subset \mathbb{P}^{d+1}$ is either a (possible singular) surface scroll or the Veronese surface in $\mathbb{P}^5$. As $d \geq 6$, the latter cannot occur, and so $\widetilde{X} \subset \mathbb{P}^{d+1}$ is a surface scroll. Now, let E be the homogeneous coordinate ring of $\widetilde{X} \subset \mathbb{P}^{d+1}$. As E is a Cohen-Macaulay ring, we have canonical inclusions of graded rings $A \subset B(A) \subset E$ (see Notation and Reminder 3.7 (A)). As X is a surface, we have $X \setminus \mathrm{CM}(X) \subset X \setminus \mathrm{Nor}(X) = \mathrm{Sing}(\pi_\Lambda) = |\,\mathrm{Proj}(A/\mathfrak{a})|$ and hence $E \subset \bigcup_{n \in \mathbb{N}} (A :_{\mathrm{Quot(A)}} \mathfrak{a}^n) = B(A)$. Therefore $E = B(A) = B$ and statement (c) is shown.

Alternative Proof: Our first aim is to show that $\dim_K(B_1) = d + 2$. As $h \in S_1 \setminus \{0\}$ is general, we have $\mathrm{Rad}\,(\mathfrak{a}, h) = A_+$. So, comparing local cohomology furnishes a short exact sequence of graded local cohomology modules (see [10, (8.1.2)])

$$0 \to H^1_{(h)}(H^1_{\mathfrak{a}}(A)) \to H^2_{A_+}(A) \to H^0_{(h)}(H^2_{\mathfrak{a}}(A)) \to 0.$$

Observe that with $D := D_{A_+}(A) = \varinjlim \mathrm{Hom}_A((A_+)^n, A) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$, the kernel of the natural map $B/A \twoheadrightarrow B/D$ is S_+-torsion. As $\mathrm{Rad}(\mathfrak{a}, h) = A_+$, it follows

$$H^1_{(h)}(H^1_{\mathfrak{a}}(A)) = H^1_{A_+}(H^1_{\mathfrak{a}}(A)) = H^1_{A_+}(B/A) = H^1_{A_+}(B/D).$$

By Lemma 2.4 of [9] we have $\dim_K((B/D)_n) = \mathrm{e}(X)$ for all $n \gg 0$. Consequently $\dim_K(D_{A_+}(B/D)_n) = \mathrm{e}(X)$ for all $n \in \mathbb{Z}$. As $(B/D)_0 = 0$ it follows that

$$\dim_K(H^1_{(h)}(H^1_{\mathfrak{a}}(A))_0) + \dim_K(H^1_{A_+}(B/D)_0) = \mathrm{e}(X).$$

By statement (b) we also have $\dim_K(H^2_{A_+}(A)_0) = H^1(X, \mathcal{O}_X) = \mathrm{e}(X)$. So, the above sequence shows that $H^0_{(h)}(H^2_{\mathfrak{a}}(A))_0 = 0$. Therefore the multiplication map $h : H^2_{\mathfrak{a}}(A)_0 \to H^2_{\mathfrak{a}}(A)_1$ is injective. Now, applying the functor $D_{\mathfrak{a}}(\bullet)$ to the exact sequence $0 \to A(-1) \stackrel{h}{\to} A \to A/hA \to 0$ and observing once more that $\mathrm{Rad}\,(\mathfrak{a}, h) = A_+$, we get the exact sequence of K-vector spaces

$$0 \to D_{\mathfrak{a}}(A)_0 \to D_{\mathfrak{a}}(A)_1 \to D_{A_+}(A/hA)_1 \to H^2_{\mathfrak{a}}(A)_0 \stackrel{h}{\to} H^2_{\mathfrak{a}}(A)_1.$$

As $D_{\mathfrak{a}}(A)_0 = B_0 = K$ and as, in addition, the last map in this sequence is injective, we end up with $\dim_K(B_1) = \dim_K(D_{\mathfrak{a}}(A)_1) = \dim_K(D_{A_+}(A/hA)_1) + 1$. As $C_h \subset \mathrm{Proj}(S/hS) = \mathbb{P}^{r-1}$ is a non-degenerate smooth rational curve of degree d, the K-vector space $D_{A_+}(A/hA)_1 \cong H^0(C_h, \mathcal{O}_{C_h}(1))$ has dimension $d+1$, so that indeed $\dim_K(B_1) = d+2$.

Now, consider the non-degenerate closed subscheme $\widetilde{X} := \mathrm{Proj}(K[B_1]) \subset \mathbb{P}^{d+1}$. As $K[B_1]$ is a finite birational integral extension domain of A, the scheme $\widetilde{X} \subset \mathbb{P}^{d+1}$ is a non-degenerate irreducible and reduced surface of degree d. It follows in particular that $\widetilde{X} \subset \mathbb{P}^{d+1}$ is a surface of minimal degree and hence (as $d \geq 6$) a (possibly singular) surface scroll. In particular $K[B_1]$ is a Cohen-Macaulay ring which contains A and is contained in the S_2-cover $B(A)$ of A. Thus $K[B_1] = B(A)$ (see Notation and Reminder 3.7 (A)) and hence $B = B(A)$ is the homogeneous coordinate ring of the surface scroll $\widetilde{X} \subset \mathbb{P}^{d+1}$.

Moreover, the inclusion map $A \to B(A)$ gives rise to a finite morphism $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$, induced by a linear projection $\pi'_\Lambda : \mathbb{P}^{d+1} \setminus \Lambda \twoheadrightarrow \mathbb{P}^r$ from a subspace $\Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1}$ disjoint to $\widetilde{X} \subset \mathbb{P}^{d+1}$, so that indeed $X = \widetilde{X}_\Lambda$. Finally, by Notation and Reminder 3.7 (B), we have $\mathrm{Sing}(\pi_\Lambda : \widetilde{X} \twoheadrightarrow X) = X \setminus \mathrm{CM}(X)$. So, statement (c) is shown.

(d): Clearly, $\widetilde{X}$ is projectively equivalent to a rational surface scroll $S(a, d-a) \subset \mathbb{P}^{d+1}$ for some non-negative integer $a \leq d/2$. The uniqueness of a follows for example by [9, Lemma 5.5]. The remaining claim is clear from the well known fact that $\widetilde{X}_\Lambda$ is a cone if and only if $\widetilde{X}$ is – and hence if and only if $a = 0$.

(a): By statement (c) and Corollary 3.6, we see that $\mathrm{reg}(X) \leq d - r + 3$. As X is a surface, we have $\mathrm{Reg}(X) \subseteq \mathrm{Nor}(X) \subseteq \mathrm{CM}(X)$. It thus remains to show that $\mathrm{CM}(X) \subseteq \mathrm{Reg}(X)$. To do so, let $x \in \mathrm{CM}(X)$ be a closed point. Then, by statement (d) we have $x \notin \mathrm{Sing}(\pi_\Lambda)$. Hence x has a unique preimage $\widetilde{x} \in \widetilde{X}$ and moreover $\mathcal{O}_{\widetilde{X},\widetilde{x}} \cong \mathcal{O}_{X,x}$. Assume that $\widetilde{x} \notin \mathrm{Reg}(\widetilde{X})$. Then $\widetilde{x}$ is a vertex point of $\widetilde{X}$ and hence the tangent space $\mathrm{T}_{\widetilde{x}}(\widetilde{X})$ of $\widetilde{X}$ at $\widetilde{x}$ has dimension $d+1 > r \geq \dim_K(\mathrm{T}_x(X))$. This leads to the contradiction that $x \in \mathrm{Sing}(\pi_\Lambda)$. Therefore $\widetilde{x} \in \mathrm{Reg}(\widetilde{X})$ and hence $x \in \mathrm{Reg}(X)$.

(e): By its construction, $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$ provides the finite Macaulayfication of X (see Notation and Reminder 3.7 (B)). According to statement (a) we have $\mathrm{Sing}(\pi_\Lambda) = X \setminus \mathrm{Nor}(X) = \mathrm{Sing}(\rho)$, where $\rho : Y \twoheadrightarrow X$ is the normalization of X. As Y is a normal surface, it is locally Cohen-Macaulay. Therefore π_Λ provides also the normalization of X (see Notation and Reminder 3.7 (B)).

(f): As p is an isolated point of $X \setminus \mathrm{Reg}(X) = \mathrm{Sing}(\pi_\Lambda)$, it follows that the $\mathcal{O}_{X,p}$-module $((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p/\mathcal{O}_{X,p}$ is of finite length. As $\widetilde{X}$ is locally Cohen-Macaulay, the finitely generated $\mathcal{O}_{X,p}$-module $((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p$ is Cohen-Macaulay. From this, we get

$$\mathrm{mult}_p(X) = \mathrm{mult}_{\mathfrak{m}_{X,p}}\left(((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p\right) \geq \mathrm{length}\left(((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p/\mathfrak{m}_{X,p}((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p\right) = \#(\pi_\Lambda^{-1}(p))$$

and also $((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p/\mathcal{O}_{X,p} \cong H^1_{\mathfrak{m}_{X,p}}(\mathcal{O}_{X,p})$. As $\left((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}}\right)_p$ is a proper finite integral extension domain of $\mathcal{O}_{X,p}$,the Nakayma Lemma yields that $\mathrm{length}\left(((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p/\mathfrak{m}_{X,p}((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p\right) > 1$. Therefore we have

$$1 < \#((\pi_\Lambda)^{-1}(p)) = \mathrm{length}\left(((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p/\mathfrak{m}_{X,p}((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p\right) \leq \mathrm{length}\left(((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p/\mathfrak{m}_{X,p}\right) =$$

$$= \mathrm{length}\left(((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p/\mathcal{O}_{X,p}\right) + 1 = \mathrm{e}_p(X) + 1.$$

This proves the inequalities in claims (1) and (2).

If p is not a vertex point of X, we have $(\pi_\Lambda)^{-1}(p) \subset \operatorname{Reg}(\widetilde{X})$ and the equality in claim (1) follows easily (see [5, Lemma 3.2]). Observe that p is a Buchsbaum point of X if and only if $\mathfrak{m}_{X,p}H^1_{\mathfrak{m}_{X,p}}(\mathcal{O}_{X,p}) = 0$, hence if and only if $\mathfrak{m}_{X,p}((\pi_\Lambda)_*\mathcal{O}_{\widetilde{X}})_p = \mathfrak{m}_{X,p}$. This proves the equivalence of claim (2).

(g): Let $\Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1}$ be as in (c) and let $\pi'_\Lambda : \mathbb{P}^{d+1} \setminus \Lambda \twoheadrightarrow \mathbb{P}^r$ be a projection centered at Λ such that π_Λ coincides with the restriction $\pi'_\Lambda \restriction_{\widetilde{X}}$ of π'_Λ to $\widetilde{X}$. By our hypotheses

$$\#\big(\pi_\Lambda^{-1}(X \cap \mathbb{M})\big) = \#\big(\widetilde{X} \cap \overline{(\pi'_\Lambda)^{-1}(\mathbb{M})}\big) < \infty.$$

As $\widetilde{X} \subset \mathbb{P}^{d+1}$ is a rational normal scroll and $(\pi'_\Lambda)^{-1}(\mathbb{M}) = \mathbb{P}^{s+d-r+1} \subset \mathbb{P}^{d+1}$ we have the inequality $\#\big(\widetilde{X} \cap \overline{(\pi'_\Lambda)^{-1}(\mathbb{M})}\big) \leq d - r + s + 2$, so that

$$\#\big((\pi_\Lambda)^{-1}(X \cap \mathbb{M})\big) \leq d - r + s + 2.$$

As $\operatorname{Sing}(\pi_\Lambda) = X \setminus \operatorname{Reg}(X)$ we have an isomorphism $(\pi_\Lambda)^{-1}(\operatorname{Reg}(X) \cap \mathbb{M}) \cong \operatorname{Reg}(X) \cap \mathbb{M}$, so that

$$\#\big((\pi_\Lambda)^{-1}(\operatorname{Reg}(X) \cap \mathbb{M})\big) = \#\big(\operatorname{Reg}(X) \cap \mathbb{M}\big).$$

By claim (1) of statement (f) we have $\#((\pi_\Lambda)^{-1}(p) \geq 2$ for all $p \in (X \cap \mathbb{M}) \setminus \operatorname{Reg(X)}$, so that

$$2\#\big((X \cap \mathbb{M})_{\text{red}} \setminus \operatorname{Reg}(X)\big) \leq \#\big((\pi_\Lambda)^{-1}(X \cap \mathbb{M}) \setminus \operatorname{Reg}(X)\big).$$

As the fiber $(\pi_\Lambda)^{-1}(X \cap \mathbb{M})$ is the disjoint union of its two subschemes $(\pi_\Lambda)^{-1}(\operatorname{Reg}(X) \cap \mathbb{M})$ and $(\pi_\Lambda)^{-1}((X \cap \mathbb{M}) \setminus \operatorname{Reg}(X))$, our claim follows. $\square$

4. Surfaces of Maximal Sectional Regularity

In section 2 we already have introduced varieties of maximal sectional regularity. In this section, we will focus to the special case of surfaces of maximal sectional regularity. We first recall a few basic facts around the notion of surface of maximal sectional regularity. Then, we give a characterization of these surfaces in terms of what we call their sectional regularity. Finally, we prove a structure theorem which gives a number of fundamental properties of such surfaces and which mainly follows from Theorem 3.8.

4.1. **Notation and Reminder.** (A) Let $X \subset \mathbb{P}^r = \operatorname{Proj}(S)$ is a non-degenerate irreducible projective surface of degree d, with homogeneous vanishing ideal $I \subset S$ and homogeneous coordinate ring $A = S/I$. For each linear form $h \in S_1 \setminus \{0\}$ we write

$$\mathbb{H}_h := \operatorname{Proj}(S/hS)$$

for the hyperplane in $\mathbb{P}^r$ defined by h and

$$C_h := \operatorname{Proj}(A/hA) = X \cap \mathbb{H}_h$$

for the corresponding hyperplane section of X.

(B) (see [8]) We keep the above hypotheses and notations. The *sectional regularity* $\operatorname{sreg}(X)$ of the surface X is defined as the least regularity of a hyperplane section of X:

$$\operatorname{sreg}(X) := \min\{\operatorname{reg}(C_h) \mid h \in S_1 \setminus \{0\}\}.$$

For each $h \in S_1 \setminus \{0\}$ we have $\operatorname{reg}(C_h) \leq \operatorname{reg}(X)$ so that in particular

$$\operatorname{sreg}(X) \leq \operatorname{reg}(X).$$

(C) Let the notations be as in part (A). Let $i \in \mathbb{N}$, $n \in \mathbb{Z}$ and $h \in S_1 \setminus \{0\}$. Then the exact sequence of K-vector spaces

$$H^i(\mathbb{P}^r, \mathcal{J}_X(n-1)) \xrightarrow{h} H^i(\mathbb{P}^r, \mathcal{J}_X(n)) \to H^i(\mathbb{H}_h, \mathcal{J}_{C_h}(n)) \to$$
$$H^{i+1}(\mathbb{P}^r, \mathcal{J}_X(n-1)) \xrightarrow{h} H^{i+1}(\mathbb{P}^r, \mathcal{J}_X(n))$$

shows, that the set

$$\mathbb{W}^i_n(X) := \{h \in S_1 \setminus \{0\} \mid H^i(\mathbb{H}_h, \mathcal{J}_{C_h}(n)) = 0\}$$

is open in S_1 for all $n \in \mathbb{Z}$ and is equal to $S_1 \setminus \{0\}$ if $n \geq \operatorname{reg}(X)$. So, for each $s \in \mathbb{Z}$ the set $\{h \in S_1 \setminus \{0\} \mid \operatorname{reg}(C_h) \leq s\} = \bigcap_{i=1}^{r+1} \bigcap_{n \geq s} \mathbb{W}^i_{n-i}(X)$ is open in S_1. Applying this with $s := \operatorname{sreg} X$, we see that the set

$$\mathbb{W}(X) := \{h \in S_1 \setminus \{0\} \mid \operatorname{reg}(C_h) = \operatorname{sreg}(X)\}$$

is open and dense in S_1 and keeping in mind the first Bertini Theorem, we obtain that the set

$$\mathbb{U}(X) := \{h \in \mathbb{W}(X) \mid C_h \text{ is integral }\}$$

is open and dense in S_1. Observe that in the notations of Definition and Remark 2.5 (A) we have

$$\mathbb{P}\mathbb{U}(X) = \{\mathbb{H}_h \mid h \in \mathbb{U}(X)\}.$$

(D) Keep the previous notations and hypotheses. Then, for each $h \in \mathbb{U}(X)$, the regularity bound for curves due to Gruson-Lazarsfeld-Peskine [20] yields that $\operatorname{reg}(C_h) \leq d - r + 3$, so that

$$\operatorname{sreg}(X) \leq d - r + 3.$$

In [8] we did define surfaces of maximal sectional regularity as those, whose sectional regularity takes the maximal possible value. That this definition coincides with our definition given in Remark and Definition 2.3 (C) is the subject of the following result, in which we use the notations introduced in Definition and Remarks 2.5 and 2.6.

4.2. **Proposition.** *Let $5 \leq r < d$ and let $X \subset \mathbb{P}^r$ be a non-degenerate irreducible projective surface of degree d.*

(a) *The following statements are equivalent*
   (i) *The surface X is of maximal sectional regularity;*
   (ii) $\operatorname{sreg}(X) = d - r + 3$;
   (iii) ${}^*\mathfrak{d}(X) = 2$;
   (iv) $\mathfrak{d}(X) = 2$.

(b) *If the equivalent conditions (i) – (iv) of statement (a) are satisfied, then we have*
   (1) *For each $h \in \mathbb{U}(X)$, the curve $C_h \subset \mathbb{H}_h$ is of maximal regularity $d-r+3$ and hence smooth, rational and with a unique extremal secant line $\mathbb{L}_h := \mathbb{L}_{\mathbb{H}_h,X}$.*
   (2) ${}^*\Sigma^\circ_{d-r+3}(X) = \{\mathbb{L}_h \mid h \in \mathbb{U}(X)\}$.

Proof. (a): (i) $\Leftrightarrow$ (ii): According to Notation and Reminder 4.1 (C), the set

$$\mathbb{U}(X) = \{h \in S_1 \setminus \{0\} \mid C_h \subset \mathbb{H}_h \text{ is an integral curve with} \operatorname{reg}(C_h) = \operatorname{sreg}(X)\}$$

is dense and open in S_1. According to Remark and Definition 2.3 (C) the surface $X \subset \mathbb{P}^r$ is of maximal sectional regularity if and only if there is a dense open subset $\mathcal{U}$ of $S_1 \setminus \{0\}$

such that $C_h \subset \mathbb{H}_h$ is an integral curve with $\operatorname{reg}(C_h) = d-r+3$. This gives the requested equivalence.

The equivalences (i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) hold by Theorem 2.8.

(b): Assume that the equivalent conditions (i) – (iv) of statement (a) hold. Claims (1) and (2) follow from the fact that $\mathbb{PU}(X) = \{\mathbb{H}_h \mid h \in \mathbb{U}(X)\}$ (see Notation and Reminder 4.1 (C)) and Definition and Remark 2.5 (C). □

Now we are able to formulate and to prove our first main result on the structure of surfaces of maximal sectional regularity.

4.3. **Theorem.** *Let $5 \leq r < d$ and assume that the non-degenerate irreducible surface $X \subset \mathbb{P}^r$ of degree d is of maximal sectional regularity. Then*

- (a) *The surface X is sectionally rational and $X \setminus \operatorname{Nor}(X)$ is finite. In particular*
  - (1) $\operatorname{reg}(X) = d-r+3$ *and* $\operatorname{Reg}(X) = \operatorname{Nor}(X) = \operatorname{CM}(X)$.
  - (2) *For all $n \leq 0$ it holds $h^2(\mathbb{P}^r, \mathcal{J}_X(n)) = \mathrm{e}(X)$,*
    for all $n \geq 0$ it holds $h^2(\mathbb{P}^r, \mathcal{J}_X(n+1)) \leq h^2(\mathbb{P}^r, \mathcal{J}_X(n))$, and
    for all $n \geq -1$ it holds $h^3(\mathbb{P}^r, \mathcal{J}_X(n)) = 0$.
  - (3) *There is a unique non-negative integer $a \leq \frac{d}{2}$ and a subspace $\Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1}$ disjoint to the rational surface scroll $\widetilde{X} := S(a, d-a) \subset \mathbb{P}^{d+1}$ such that $X = \widetilde{X}_\Lambda$ and $\operatorname{Sing}(\pi_\Lambda : \widetilde{X} \twoheadrightarrow X) = X \setminus \operatorname{Reg}(X)$. Moreover the morphism $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$ provides the normalization and the finite Macaulayfication of X and the number a is positive if and only if $X \subset \mathbb{P}^r$ is a not cone.*
  - (4) *If $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$ is as in claim (3) and $p \in X \setminus \operatorname{Reg}(X)$, we have*
    - (i) $1 < \#(\pi_\Lambda)^{-1}(p) \leq \operatorname{mult}_p(X)$, *with equality at the second place if X is not a cone with vertex p and if $\operatorname{mult}_{\widetilde{p}}((\pi_\Lambda)^{-1}(p)) \leq 2$ for all $\widetilde{p} \in (\pi_\Lambda)^{-1}(p)$.*
    - (ii) $\#(\pi_\Lambda)^{-1}(p) \leq \mathrm{e}_p(X) + 1$ *with equality if and only if p is a Buchsbaum point of the surface X.*
  - (5) *If $\mathbb{M} = \mathbb{P}^s \subset \mathbb{P}^r$ is a linear subspace whose intersection with X is finite, then*
    $$\#(\operatorname{Reg}(X) \cap \mathbb{M}) + \#\big((X \cap \mathbb{M})_{\mathrm{red}} \setminus \operatorname{Reg}(X)\big) \leq d-r+s+2.$$
- (b) $\operatorname{reg}(X) = d-r+3$, $\mathbb{W}(X) = S_1 \setminus \{0\}$ *and* $\operatorname{depth}(X) \leq 2$.
- (c) *If $h \in \mathbb{U}(X)$, then*
  $$W_h := \operatorname{Join}(X, \mathbb{L}_h) \subset \mathbb{P}^r$$
  is a rational 4-fold scroll of type $S(0,0,b_h, r-b_h-3)$ with $0 \leq b_h \leq \frac{r-3}{2}$ and $\mathbb{L}_h = S(0,0)$.
- (d) $h^0(\mathbb{P}^r, \mathcal{J}_X(2)) \geq \binom{r-3}{2}$.

Proof. (a): According to Proposition 4.2 (b) (1), the hyperplane section curve $X \cap \mathbb{H}_h = C_h$ is smooth and rational for all h in the dense open subset $\mathbb{U}(X)$ of S_1. This shows at once, that X is sectionally rational and has finite non-normal locus. So, we get statement (a) as well as its claims (1)–(5) on application of Notation and Reminder 4.1 (B) and Theorem 3.8.

(b): By Theorem 3.8 (a) and Notation and Reminder 4.1 (B) we have $\operatorname{reg}(X) = d-r+3 = \operatorname{sreg} X$. It follows that $\operatorname{reg}(C_h) \leq \operatorname{reg}(X) = d-r+3$ for all $h \in S_1 \setminus \{0\}$ and hence $\mathbb{W}(X) = S_1 \setminus \{0\}$ (see Notation and Reminder 4.1 (C)). To prove the inequality $\operatorname{depth}(X) \leq 2$, assume that $\operatorname{depth}(X) > 1$ and let $h \in \mathbb{U}(X)$. Then, the ring $S/(I,h)$ is the homogeneous coordinate ring of the curve $C_h \subset \mathbb{H}_h$, which is of maximal regularity and hence of arithmetic depth 1. It follows that $\operatorname{depth}(X) = \operatorname{depth}(S/I) = 2$.

(c): Let $h \in \mathbb{U}(X)$ and observe that $X \cap \mathbb{L}_h \subset \operatorname{Reg}(X)$. Let $W_h := \mathrm{Join(X, \mathbb{L}_h)} \subset \mathbb{P}^r$. Then $W_h \subset \mathbb{P}^r$ is a non-degenerate irreducible projective variety of dimension 4. Let $\mathbb{I} := \mathbb{P}^{r-2} \subset \mathbb{P}^r$ be a general $(r-2)$-plane and let $\varrho_h : X \setminus \mathbb{L}_h \to W_h \cap \mathbb{I} \subset \mathbb{I}$ be the finite dominant morphism obtained by projecting from $\mathbb{L}_h$. A general $(r-2)$-plane $\mathbb{E} = \mathbb{P}^{r-2} \subset \mathbb{P}^r$, which contains $\mathbb{L}_h$, satisfies $\#(X \cap \mathbb{E}) = d$. Therefore it follows that $W_h \cap \mathbb{I} \subset \mathbb{I}$ is of degree $d-(d-r+3) = r-3 = \operatorname{codim}_{\mathbb{P}^r}(W_h)+1$. So, $W_h \subset \mathbb{P}^r$ is of minimal degree and moreover $\mathbb{L}_h \subset \operatorname{Sing}(W_h)$. Hence, W_h is either projectively equivalent to a scroll $S(0,0,b_h,r-b_h-3) \subset \mathbb{P}^r$ for some non-negative integer $b_h \leq \frac{r-3}{2}$ and $\mathbb{L}_h = S(0,0)$, or else $r = 7$ and W_h is a cone with vertex $\mathbb{L}_h$ over a Veronese surface contained in some subspace $\mathbb{P}^5 \subset \mathbb{P}^7$. As X is the projected image of a surface scroll $\widetilde{X} \subset \mathbb{P}^{r+1}$, it contains a one-dimensional family of lines disjoint to $\mathbb{L}_h$, so that W_h contains a one-dimensional family of 3-spaces containing $\mathbb{L}_h$. This excludes the second case.

(d): This follows from statement (c), as $h^0(\mathbb{P}^r, \mathcal{J}_X(2)) \geq h^0(\mathbb{P}^r, \mathcal{J}_{W_h}(2)) = \binom{r-3}{2}$. □

5. Extremal Varieties

In this section we first define the notion of extremal variety $\mathbb{F}(X)$ of a surface $X \subset \mathbb{P}^r$ of maximal sectional regularity as the closed union of all special extremal secant lines to X hence of all lines in the set ${}^*\Sigma^\circ_{d-r+3}(X) = \{\mathbb{L}_h \mid h \in \mathbb{U}(X))$ (see Propostion 4.2 (b)). As a main result of this section, we will show that $\mathbb{F}(X)$ is either a plane or a smooth rational 3-fold scroll and that the latter case only occurs if $r = 5$.

5.1. **Definition and Remark.** (A) Let $5 \leq r < d$ and let $X \subset \mathbb{P}^r$ be a surface of degree d which is of maximal sectional regularity. We write

$$\mathbb{F}(X) := \overline{\bigcup_{h \in \mathbb{U}(X)} \mathbb{L}_h} = \overline{\bigcup_{\mathbb{L} \in {}^*\Sigma^\circ_{d-r+3}(X)} \mathbb{L}} \subset \mathbb{P}^r$$

for the closure of the union of all special extremal secant lines to X and call $\mathbb{F}(X)$ the *extremal variety* of X.

(B) As $\mathbb{U}(X)$ is an infinite set of $(d-r+3)$-secant lines of X, we clearly must have

$$\dim(X \cap \mathbb{F}(X)) \geq 1 \text{ and } \dim(\mathbb{F}(X)) \geq 2.$$

(C) Assume first, that our surface $X \subset \mathbb{P}^r$ is a cone with vertex p over the irreducible non-degenerate curve $C = X \cap \mathbb{H} \subset \mathbb{H}$ of degree d, where $\mathbb{H} = \mathbb{P}^{r-1} \subset \mathbb{P}^r$ is a hyperplane which avoids p. Now, for all $h \in \mathbb{U}(X)$ we must have $p \notin \mathbb{H}_h$, so that C and C_h are isomorphic via the projection $\pi_p : \mathbb{H} \xrightarrow{\cong} \mathbb{H}_h$ from p. This shows, that $C \subset \mathbb{H}$ is a curve of maximal regularity and that its unique extremal secant line $\mathbb{L}$ is mapped onto $\mathbb{L}_h$ by π_p. Therefore, if X is a cone, we see:

(1) $\mathbb{F}(X) = \mathbb{P}^2 \subset \mathbb{P}^r$ with $p \in \mathbb{F}(X)$.
(2) $\{\mathbb{L}_h \mid h \in \mathbb{U}(X)\} = \{\mathbb{L} \in \Sigma_{d-r+3}(X) \mid p \notin \mathbb{L}\}$.

(D) Assume now, that $X \subset \mathbb{P}^r$ is an arbitrary non-degenerate irreducible surface of degree d. Then, instead of the previously introduced extremal variety $\mathbb{F}(X)$ of X one also can introduce the *extended extremal variety* of X as the closed union

$$\mathbb{F}^+(X) := \overline{\bigcup_{\mathbb{L} \in \Sigma^\circ_{d-r+3}(X)} \mathbb{L}}$$

of all proper extremal secant lines to X. If $X \subset \mathbb{P}^r$ is of maximal sectional regularity, we clearly have

$$\mathbb{F}(X) \subseteq \mathbb{F}^+(X),$$

and it might indeed be of interest to know, under which circumstances we have equality in this context.

In fact, we may extend claim (2) of Definition and Remark 5.1 (C) to arbitrary surfaces of maximal sectional regularity as follows.

5.2. **Proposition.** *Let the notations and hypotheses be as in Definition and Remark 5.1.*

(a) *The following equalities hold*

$$\{\mathbb{L}_h \mid h \in \mathbb{U}(X)\} = \{\mathbb{L} \in \Sigma_{d-r+3}(X) \mid X \cap \mathbb{L} \textit{ is a finite subset of } \operatorname{Reg}(X)\}$$
$$= \{\mathbb{L} \in \Sigma_3(X) \mid X \cap \mathbb{L} \textit{ is a finite subset of } \operatorname{Reg}(X)\}.$$

(b) *The set* $\{\mathbb{L}_h \mid h \in \mathbb{U}(X)\} = {}^*\Sigma^\circ_{d-r+3}(X)$ *is locally closed in* $\mathbb{G}(1, \mathbb{P}^r)$.

(c) *If* $p \in X \setminus \operatorname{Reg}(X)$, *then* $\dim(X \cap \langle p, \mathbb{L}_h \rangle) > 0$ *for all* $h \in \mathbb{U}(X)$.

Proof. (a): The inclusion "$\subseteq$" between the first and the second set follows from the fact, that $C_h := X \cap \mathbb{H}_h$ is smooth for each $h \in \mathbb{U}(X)$ and hence can only contain smooth points of X. The inclusion between the second and the third set in our statement is immediate.

So, let $\mathbb{L} \in \Sigma_3(X)$ such that $X \cap \mathbb{L}$ is finite and contained in $\operatorname{Reg}(X)$, and assume that $\mathbb{L}_h \neq \mathbb{L}$ for all $h \in \mathbb{U}(X)$. We aim for a contradiction.

By Theorem 4.3 (a)(3) we may write $X = \widetilde{X}_\Lambda$ and $\operatorname{Sing}(\pi_\Lambda : \widetilde{X} \twoheadrightarrow X) = X \setminus \operatorname{Reg}(X)$, where $\widetilde{X} \subset \mathbb{P}^{d+1}$ is a surface scroll, $\Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1}$ is a subspace disjoint to $\widetilde{X}$ and π_Λ is induced by a linear projection $\pi'_\Lambda : \mathbb{P}^{d+1} \setminus \Lambda \twoheadrightarrow \mathbb{P}^r$. Let $\widetilde{\mathbb{L}} := \overline{(\pi'_\Lambda)^{-1}(\mathbb{L})} = \mathbb{P}^{d-r+2} \subset \mathbb{P}^{d+1}$. Then, the set $\widetilde{X} \cap \widetilde{\mathbb{L}} = (\pi_\Lambda)^{-1}(X \cap \mathbb{L})$ is finite.

Let $\widetilde{\mathbb{H}} = \mathbb{P}^d \subset \mathbb{P}^{d+1}$ be a general hyperplane which contains the space $\widetilde{\mathbb{L}}$. If $\widetilde{X}$ is not a cone, we may conclude by [5, Remark 2.3 (B)], that the intersection $\widetilde{X} \cap \widetilde{\mathbb{H}} \subset \widetilde{\mathbb{H}}$ is a rational normal curve. If $\widetilde{X}$ is a cone, the fact that $\mathbb{L}$ avoids the singular locus of X implies that $\widetilde{\mathbb{L}}$ does not contain the vertex of $\widetilde{X}$ and we end up again with the conclusion that $\widetilde{X} \cap \widetilde{\mathbb{H}} \subset \widetilde{\mathbb{H}}$ is a rational normal curve.

By Theorem 4.3 (b), there is some $h \in \mathbb{W}(X)$ such that $\mathbb{H}_h = \pi'_\Lambda(\widetilde{\mathbb{H}} \setminus \Lambda) = \mathbb{P}^{r-1} \subset \mathbb{P}^r$. As $\widetilde{\mathbb{H}}$ is general and $X \cap \mathbb{L}$ avoids the finite set $X \setminus \operatorname{Reg}(X) = \operatorname{Sing}(\pi_\Lambda)$, the intersection $C_h = X \cap \mathbb{H}_h$ avoids the set $\operatorname{Sing}(\pi_\Lambda)$. Therefore the induced map $\pi_\Lambda \restriction: \widetilde{X} \cap \widetilde{\mathbb{H}} \to C_h$ is an isomorphism and hence $C_h \subset \mathbb{H}_h$ is a smooth rational curve of degree d, so that $h \in \mathbb{U}(X)$. By our assumption we have $\mathbb{L} \neq \mathbb{L}_h$, hence $\mathbb{V} := \langle \mathbb{L}_h, \mathbb{L} \rangle = \mathbb{P}^s \subset \mathbb{H}_h$ with $s \in \{2, 3\}$. As $C_h \subset \mathbb{H}_h = \mathbb{P}^{r-1}$ is an irreducible non-degenerate curve and $s \leq 3 < r - 1$, the intersection $X \cap \mathbb{V}$ is finite.

As $\mathbb{L} \neq \mathbb{L}_h$ we have $\#(X \cap (\mathbb{L} \cup \mathbb{L}_h)) \geq \#(X \cap \mathbb{L}) + \#(X \cap \mathbb{L}_h) - \varepsilon$ with $\varepsilon = 1$ if $\mathbb{L}$ and $\mathbb{L}_h$ meet in a point of X, and $\varepsilon = 0$ otherwise. In the first case, we have $s = 2$, so that always $3 - \varepsilon \geq s$. Therefore, we obtain

$$\infty > \#(X\cap\mathbb{V}) \geq \#(X\cap(\mathbb{L}\cup\mathbb{L}_h)) \geq \#(X\cap\mathbb{L})+\#(X\cap\mathbb{L}_h)-\varepsilon \geq 3+d-r+3-\varepsilon \geq d-r+s+3.$$

As $\mathbb{L} \cup \mathbb{L}_h \subset \operatorname{Reg}(X)$ this contradicts Theorem 4.3 (a)(5).

(b): This is clear, as $\Sigma_{d-r+3}(X) \subset \mathbb{G}(1, \mathbb{P}^r)$ is closed (see Notation and Reminder 2.4 (B)) and $X \setminus \operatorname{Reg}(X)$ is finite (see Theorem 4.3 (a)).

(c): Let $h \in \mathbb{U}(X)$, and observe that by statement (a) we have $\mathbb{M} := \langle p, \mathbb{L}_h \rangle = \mathbb{P}^2$. Assume that $\#(X \cap \mathbb{M}) < \infty$. By statement (a) we have $X \cap \mathbb{L}_h \subset \operatorname{Reg}(X) \cap \mathbb{M}$. By our choice of p we have $p \in (X \cap \mathbb{M}) \setminus \operatorname{Reg}(X)$. Therefore we get

$$d - r + 5 \leq \#(X \cap \mathbb{L}_h) + 2 \leq \#\big(\operatorname{Reg}(X) \cap \mathbb{M}\big) + \#\big((X \cap \mathbb{M})_{\text{red}} \setminus \operatorname{Reg}(X)\big).$$

But this contradicts Theorem 4.3 (a)(5) and hence proves our claim. □

We now give a first criterion for the planarity of the extremal variety $\mathbb{F}(X)$ of our surface $X \subset \mathbb{P}^r$ of maximal sectional regularity. We begin with the following auxiliary result.

5.3. **Lemma.** *Let $5 \leq r < d$, let $X \subset \mathbb{P}^r$ be a surface of degree d, and let $\mathbb{F} = \mathbb{P}^2 \subset \mathbb{P}^r$ be a plane, such that $\dim(X \cap \mathbb{F}) = 1$. Then, the following statements hold*

(a) *If $\deg_{\mathbb{F}}(X\cap\mathbb{F}) \geq d-r+3$, then X is of maximal sectional regularity and $\mathbb{F}(X) = \mathbb{F}$.*
(b) *If X is of maximal sectional regularity and $\deg_{\mathbb{F}}(X \cap \mathbb{F}) \geq 3$, then $\mathbb{F}(X) = \mathbb{F}$.*
(c) *If X is of maximal sectional regularity and $\mathbb{L}_h \subset \mathbb{F}$ for general $h \in \mathbb{U}(X)$, then $\mathbb{F}(X) = \mathbb{F}$.*

Proof. (a): Set $t := \deg_{\mathbb{F}}(X \cap \mathbb{F})$ and let $\mathbb{H} = \mathbb{P}^{r-1} \subset \mathbb{P}^r$ be a general hyperplane. Then, the line $\mathbb{L} := \mathbb{F}\cap\mathbb{H}$ is t-secant to the integral curve $C := X\cap\mathbb{H} \subset \mathbb{H}$ of degree d. Therefore $t \leq \operatorname{reg}(C) \leq d - r + 3$, whence $t = d - r + 3$. Thus $X \cap \mathbb{H} = C \subset \mathbb{H} = \mathbb{P}^{r-1}$ is a curve of maximal regularity $d - r + 3$. As $\mathbb{H}$ is general, we may assume that $\mathbb{H} = \mathbb{H}_h$ for some $h \in \mathbb{U}(X)$ and therefore $\operatorname{sreg}(X) = d - r + 3$.

(b): Let $\mathbb{P}^1 = \mathbb{L} \subset \mathbb{F}$ be a general line. Then $X \cap \mathbb{L} \subset \operatorname{Reg}(X)$ and $\#(X \cap \mathbb{L}) = \#\big((X \cap \mathbb{F}) \cap \mathbb{L}\big) = \deg_{\mathbb{F}}(X \cap \mathbb{F}) \geq 3$. So, by Proposition 5.2 (a) we have $\deg_{\mathbb{F}}(X \cap \mathbb{F}) = \#(X \cap \mathbb{L}) \geq d - r + 3$. Now, we may conclude by statement (a).

(c): Clearly, for general $h \in \mathbb{U}(X)$, we have

$$d - r + 3 = \#(C_h \cap \mathbb{L}_h) = \#(X \cap \mathbb{L}_h) \leq \#\big((X \cap \mathbb{F}) \cap \mathbb{H}_h\big) = \deg(X \cap \mathbb{F}).$$

So, our claim follows by statement (b). □

Now, we can prove the announced criterion for the planarity of the extremal variety.

5.4. **Proposition.** *Let $5 \leq r < d$ and let $X \subset \mathbb{P}^r$ be surface of degree d and of maximal sectional regularity. Then, the following conditions are equivalent.*

(i) *$\mathbb{F}(X)$ is a plane.*
(ii) $\dim(\mathbb{F}(X)) = 2$
(iii) *For all $h_1, h_2 \in \mathbb{U}(X)$ the lines $\mathbb{L}_{h_1}$ and $\mathbb{L}_{h_2}$ meet.*

(iv) *There is a non-empty open set $U \subset \mathbb{U}(X)$ such that the lines $\mathbb{L}_{h_1}$ and $\mathbb{L}_{h_2}$ meet for all $h_1, h_2 \in U$.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i): Assume that $\mathbb{F}(X)$ is of dimension 2 and write $\mathbb{F}(X) = D_1 \cup \cdots \cup D_t \cup E$, where $t \in \mathbb{N}$, $D_1, \ldots, D_t \subset \mathbb{P}^r$ are the different 2-dimensional irreducible components of $\mathbb{F}(X)$ and $E \subset \mathbb{P}^r$ is closed, reduced and of dimension ≤ 1. For general $h \in \mathbb{U}(X)$, there is an index $i_h \in \{1, \ldots, t\}$ such that $\mathbb{L}_h \subset D_{i_h}$. So, without loss of generality we may assume that there is a non-empty open set $U \subset \mathbb{U}(X)$ such that $\mathbb{L}_h \subset D_1$ for all $h \in U$. By Bertini we thus find a non-empty open set $W \subset U$ such that $D_1 \cap \mathbb{H}_h \subset \mathbb{P}^r$ is an integral closed subscheme of dimension 1 and of degree $\deg(D_1)$ for all $h \in W$. As $\mathbb{L}_h \subset D_1 \cap \mathbb{H}_h$ for all $h \in U$, it follows, that $D_1 \cap \mathbb{H}_h = \mathbb{L}_h$ for all $h \in W$. This shows, that $D_1 \subset \mathbb{P}^r$ is a plane which contains $\mathbb{L}_h$ for all $h \in W$, and hence proves our claim by Lemma 5.3 (c).

The implications (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious, so that it remains to show the implication (iv) $\Rightarrow$ (i). We find $h_1, h_2 \in U$ such that $\mathbb{L}_{h_1}$ and $\mathbb{L}_{h_2}$ are distinct, and hence span a plane $\mathbb{P}^2 = \mathbb{F} \subset \mathbb{P}^r$. Clearly, there is an non-empty open set $W \subset U$ such that $\mathbb{L}_h$ avoids the common point of $\mathbb{L}_{h_1}$ and $\mathbb{L}_{h_2}$ for all $h \in W$. So, for all $h \in W$ we must have $\mathbb{L}_h \subset \mathbb{F}$. By Lemma 5.3 (c) it follows that $\mathbb{F}(X) = \mathbb{F}$. □

Our next aim is to give a link between the invariant b_h of Theorem 4.3 (c) and the nature of the extremal variety $\mathbb{F}(X)$ of X. We begin with a few preparations.

5.5. **Notation and Remark.** (A) Let $5 \leq r \leq d$ and let $C \subset \mathbb{P}^{r-1}$ be a curve of degree d, of maximal regularity and with extremal secant line $\mathbb{L}_C$. Then, the variety $\mathrm{Join}(\mathbb{L}_C, C) \subset \mathbb{P}^{r-1}$ is known to be a threefold scroll of type $S(0,0,r-3)$ with vertex $\mathbb{L}_C = S(0,0) \subset S(0,0,r-3) = \mathrm{Join}(\mathbb{L}_C, C)$.

(B) Keep the above notations and let $X = \widetilde{X}_\Lambda \subset \mathbb{P}^r$ be a non-conic surface of maximal sectional regularity, where $\widetilde{X} = S(a, d-a) \subset \mathbb{P}^{d+1}$, and where the subspace $\Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1}$ and also the induced projection morphism $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$ are defined as in Theorem 4.3 (a). Observe in particular that $a > 0$, so that the scroll $\widetilde{X}$ is smooth. We fix a canonical projection $\varphi : \widetilde{X} \twoheadrightarrow \mathbb{P}^1$. For each closed point $x \in \mathbb{P}^1$, let $\mathbb{L}(x)$ denote the ruling line $\varphi^{-1}(x)$ of $\widetilde{X}$ and let $\mathbb{L}_\Lambda(x) := \pi_\Lambda(\mathbb{L}(x)) = \mathbb{P}^1 \subset \mathbb{P}^r$. Then it obviously holds

$$X = \bigcup_{x \in \mathbb{P}^1} \mathbb{L}_\Lambda(x).$$

(C) Keep the previous notations and hypotheses and let $p \in X$ be a closed point. As $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$ is finite and almost non-singular with $\mathrm{Sing}(\pi_\Lambda) = X \setminus \mathrm{Reg}(X)$, we have

$$1 \leq \#\{x \in \mathbb{P}^1 \mid p \in \mathbb{L}_\Lambda(x)\} < \infty, \text{ with equality at the first place if } p \in \mathrm{Reg}(X).$$

Now, let $C \subset X$ be a closed integral subscheme of dimension 1 whose linear span $\langle C \rangle \subseteq \mathbb{P}^r$ satisfies the condition $2 \leq \dim\langle C \rangle \leq r-1$. As $\mathrm{Sing}(\pi_\Lambda) = X \setminus \mathrm{Reg}(X)$ is a finite set, the closed subscheme $\widetilde{C} := \overline{(\pi_\Lambda)^{-1}(C \cap \mathrm{Reg}(X))} \subset \widetilde{X}$ is integral and of dimension 1 with $2 \leq \dim\langle \widetilde{C} \rangle \leq d$, hence a degenerate prime divisor on X, and thus a curve section of $\widetilde{X}$. Therefore

$$\#(\widetilde{C} \cap \mathbb{L}(x)) = 1 \text{ for all } x \in \mathbb{P}^1.$$

From this it follows by the previous observation that

$$\#(C \cap \mathbb{L}_\Lambda(x)) = 1, \text{ for a general closed point } x \in \mathbb{P}^1.$$

5.6. **Lemma.** *Let the notations and hypotheses be as in Notation and Reminder 5.5 (A). Let $C \subset Y$, where $Y = S(0,0,r-3) \subset \mathbb{P}^{r-1}$ is a threefold scroll. Then the vertex $\mathbb{L} = S(0,0) \subset S(0,0,r-3)$ of Y equals $\mathbb{L}_C$ and $Y = \mathrm{Join}(\mathbb{L}_C, C)$*

Proof. We assume that $\mathbb{L} \neq \mathbb{L}_C$ and aim for a contradiction. As $\#(C \cap \mathbb{L}_C) > 2$ we have $\mathbb{L}_C \subset Y$ and hence $\langle \mathbb{L}, \mathbb{L}_C \rangle \subset Y$, so that $\mathbb{L}$ and $\mathbb{L}_C$ are coplanar. Now, the linear projection map $\pi_{\mathbb{L}} : Y \setminus \mathbb{L} \to S(r-3) \subset \mathbb{P}^{r-3}$ induces a dominant morphism $C \setminus (C \cap \mathbb{L}) \to S(r-3)$. As C is smooth, this morphism may be extended to a surjective morphism $\phi : C \twoheadrightarrow S(r-3)$. This implies that

$$\deg(\phi) = \frac{d - \#(C \cap \mathbb{L})}{\deg_{\mathbb{P}^{r-3}}(S(r-3))} \leq \frac{d}{r-3}.$$

As $\mathbb{L}$ and $\mathbb{L}_C$ are coplanar, there is a point $z \in S(r-3)$ such that $\phi(C \cap \mathbb{L}_C) = \{z\}$. As $S(r-3)$ is smooth, this implies that

$$\deg(\phi) = \#\phi^{-1}(z) \geq \#(C \cap \mathbb{L}_C) = d - r + 3.$$

The two previous inequalities imply that $\frac{d}{r-3} \geq d - r + 3$, which is impossible if $d \geq r$. This contradiction shows that $\mathbb{L} = \mathbb{L}_C$ and hence proves our claim. $\square$

5.7. **Lemma.** *Let $5 \leq r < d$ and let $X \subset \mathbb{P}^r$ be a surface of degree d and of maximal sectional regularity. Then, for each $h \in \mathbb{U}(X)$, the linear projection $\pi'_{\mathbb{L}_h} : \mathbb{P}^r \setminus \mathbb{L}_h \twoheadrightarrow \mathbb{P}^{r-2}$ from $\mathbb{L}_h = \mathbb{P}^1 \subset \mathbb{P}^r$ induces a birational morphism*

$$\pi_{\mathbb{L}_h} : X \setminus (X \cap \mathbb{L}_h) \to Z_h := \overline{\pi_{\mathbb{L}_h}\big(X \setminus (X \cap \mathbb{L}_h)\big)} \subset \mathbb{P}^{r-2},$$

and $Z_h \subset \mathbb{P}^{r-2}$ is a rational surface scroll of type $S(b_h, r-3-b_h)$, where $b_h \leq \frac{r-3}{2}$ is as in Theorem 4.3 (c). Moreover, if X is non-conic and with $\mathcal{S}_h := \{x \in \mathbb{P}^1 \mid \mathbb{L}_\Lambda(x) \cap \mathbb{L}_h \neq \emptyset\}$, the following statements hold:

- (a) *$1 \leq \#\mathcal{S}_h \leq d - r + 3$, and $\pi_{\mathbb{L}_h}(\mathbb{L}_\Lambda(x))$ is a line if $x \in \mathbb{P}^1 \setminus \mathcal{S}_h$ and a point if $x \in \mathcal{S}_h$.*
- (b) *If $z \in Z_h$ is a general point, there is a unique point $x_z \in \mathbb{P}^1$ with $z \in \pi_{\mathbb{L}_h}(\mathbb{L}_\Lambda(x_z))$.*
- (c) *The set $T_h := \mathrm{Sing}(\pi_{\mathbb{L}_h}) \cap \pi_{\mathbb{L}_h}(X \setminus (X \cap \mathbb{L}_h))$ is finite.*
- (d) *If $b_h > 0$ and $C \subset X$ is an integral closed subscheme of dimension 1 with $4 \leq \dim\langle C \rangle \leq r-1$, then $C' := \overline{\pi_{\mathbb{L}_h}(C \setminus (C \cap \mathbb{L}_h)}$ is a curve section of Z_h.*

Proof. Observe that $\pi_{\mathbb{L}_h}$ coincides with the restriction $\pi'_{\mathbb{L}_h} \restriction_{X \setminus (X \cap \mathbb{L}_h)}$ of the linear projection map $\pi'_{\mathbb{L}_h} : \mathbb{P}^r \setminus \mathbb{L}_h \twoheadrightarrow \mathbb{P}^{r-2}$ to $X \setminus (X \cap \mathbb{L}_h)$. Now, in the notations of Theorem 4.3 (c), we have $\mathbb{L}_h = S(0,0) \subset S(0,0,b_h,r-3-b_h) = W_h = \mathrm{Join}(\mathbb{L}_h, X)$, so that

$$Z_h = \overline{\pi'_{\mathbb{L}_h}(W_h \setminus \mathbb{L}_h)} = S(b_h, r-b_h-3) \subset \mathbb{P}^{r-2}.$$

As X is a union of lines, the same is true for the (irreducible non-degenerate) closed subset $Z_h \subset \mathbb{P}^{r-2}$. In particular we must have $\dim Z_h \geq 2$. Now, let $p \in Z_h$ be a general point. Then $\pi_{\mathbb{L}_h}^{-1}(p)$ is a finite non-empty set and $\mathbb{M} := \overline{(\pi'_{\mathbb{L}_h})^{-1}(p)} = \mathbb{P}^2 \subset \mathbb{P}^r$ is a plane which contains $\mathbb{L}_h$. As $X \cap \mathbb{L}_h \subset \mathrm{Reg}(X)$ (see Proposition 5.2 (a), as $X \setminus \mathrm{Reg}(X)$ is finite and because $p \in Z_h$ is general, we may assume that $X \cap \mathbb{M} \subset \mathrm{Reg}(X)$. By Theorem 4.3 (a)(5) we thus obtain

$$\#(\pi_{\mathbb{L}_h}^{-1}(p)) + \#(X \cap \mathbb{L}_h) = \#(\pi_{\mathbb{L}_h}^{-1}(p) \cup \big(X \cap \mathbb{L}_h)\big) = \#(X \cap \mathbb{M}) \leq d - r + 4.$$

As $\#(X \cap \mathbb{L}_h) = d - r + 3$, it follows that $\#(\pi^{-1}(p)) \leq 1$. So $\pi_{\mathbb{L}_h} : X \to Z_h$ is indeed birational.

Assume from now on, that X is non-conic.

(a): This follows by Notation and Remark 5.5 (C) and the fact that $X \cap \mathbb{L}_h \subset \mathrm{Reg}(X)$ (see Proposition 5.2 (a)).

(b): Let $q \in \mathbb{L}$ be general. Then $\pi_{\mathbb{L}_h}^{-1}(q)$ consists of a single point $p \in X$, which indeed belongs to $\mathrm{Reg}(X)$. Now, we may again conclude by Notation and Remark 5.5 (C).

(c): Let $t \in U_h = T_h \cap \mathrm{Reg}(Z_h) = \mathrm{Sing}(\pi_{\mathbb{L}_h}) \cap \pi_{\mathbb{L}_h}(X \setminus (X \cap \mathbb{L}_h)) \cap \mathrm{Reg}(Z_h)$. As t is a regular point of Z_h and the morphism $\pi_{\mathbb{L}_h} : X \to Z_h$ is birational, it follows that $\pi_{\mathbb{L}_h}^{-1}(t) \subset X$ has pure dimension 1. As X is a surface, U_h must be finite. As Z_h has at most one singularity, this proves our claim.

(d): Observe, that $C' \subset Z_h$ is an integral closed subscheme of dimension 1 but not a line. So, it suffices to show, that $\#(C' \cap \mathbb{L}) \leq 1$ for a general ruling line $\mathbb{L}$ of a fixed ruling family of Z_h . Let $z \in Z_h$ be general. We consider the line $\mathbb{L} := \pi_{\mathbb{L}_h}(\mathbb{L}_\Lambda(x_z))$ of statement (b). Assume first, that $Z_h \neq S(1,1)$, so that Z_h admits only one family of ruling lines. As z is general in Z_h, the line $\mathbb{L}$ is the unique ruling line of Z_h passing through z. By statement (c) we may assume that $\mathbb{L}$ avoids the set $\mathrm{Sing}(\pi_{\mathbb{L}_h}) \subset Z_h$, so that $\pi_{\mathbb{L}_h}^{-1}(\mathbb{L}) = \mathbb{L}_\Lambda(x_z)$. Hence by Notation and Remark 5.5 (C) we get $\#\pi_{\mathbb{L}_h}^{-1}(C' \cap \mathbb{L}) = \#(C \cap \mathbb{L}_\Lambda(x_z)) \leq 1$.

If $Z_h = S(1,1)$, one of the two ruling families of Z_h contains the line $\mathbb{L} := \pi_{\mathbb{L}_h}(\mathbb{L}_\Lambda(x_z))$ for general $z \in Z_h$. Now we may conclude as above. □

5.8. **Proposition.** *Let $5 \leq r < d$ and let $X \subset \mathbb{P}^r$ be a surface of degree d and of maximal sectional regularity. Let the notations be as in Theorem 4.3 (c).*

- (a) *If $b_h = 0$ for some $h \in \mathbb{U}(X)$, then it holds*
  - (1) *$b_{h'} = 0$ and $\mathbb{F}(X) = S(0,0,0) \subset S(0,0,0,r-3) = W_{h'}$ for all $h' \in \mathbb{U}(X)$;*
  - (2) *$\mathbb{F}(X) = \mathbb{P}^2$.*
- (b) *If $b_h > 0$ for some $h \in \mathbb{U}(X)$, then it holds*
  - (1) *$r = 5$ and, in addition, for all $h' \in \mathbb{U}(X)$ we have $b_{h'} = 1$ and $\mathbb{L}_{h'}$ is either equal or disjoint to $\mathbb{L}_h$;*
  - (2) *$X \subset \mathbb{F}(X)$ and $\dim(\mathbb{F}(X)) = 3$.*

Proof. Fix some $h \in \mathbb{U}(X)$ and consider the non-empty open subset of $\mathbb{U}(X)$ defined by

$$U := \{h' \in \mathbb{U}(X) \mid \mathbb{L}_{h'} \nsubseteq \mathbb{H}_h\} = \{h' \in \mathbb{U}(X) \mid \mathbb{L}_{h'} \neq \mathbb{L}_h\}.$$

As in Theorem 4.3 (c), let $W_h := \mathrm{Join}(\mathbb{L}_h, X) = S(0,0,b_h, r-3-b_h) \subset \mathbb{P}^r$. Then, for any $h' \in U$ we have $\#(W_h \cap \mathbb{L}_{h'}) \geq \#(X \cap \mathbb{L}_{h'}) \geq d - r + 3 > 2$, thus $\mathbb{L}_{h'} \subset W_h$ and hence

$$C_{h'} \cup \mathbb{L}_{h'} \subset \mathbb{H}_{h'} \cap W_h.$$

Keep in mind, that

$$V_{h,h'} := \mathbb{H}_{h'} \cap W_h \subset \mathbb{H}_{h'} = \mathbb{P}^{r-1}$$

is a threefold scroll of type $S(0, b_h, r-3-b_h)$. Observe that the vertex q of $V_{h,h'}$ is the intersection of $\mathbb{L}_h$ with $\mathbb{H}_{h'}$. Observe also that the restriction of the linear projection $\pi'_{\mathbb{L}_H} : \mathbb{P}^r \setminus \mathbb{L}_h \twoheadrightarrow \mathbb{P}^{r-2}$ to $\mathbb{H}_{h'} \setminus \{q\}$ yields the linear projection π'_q centered at q, thus:

$$\mathbb{L}_h \cap \mathbb{H}_{h'} = \{q\} \quad \text{and} \quad \pi'_{\mathbb{L}_h} \restriction_{\mathbb{H}_{h'} \setminus \{q\}} = \pi'_q : \mathbb{H}_{h'} \setminus \{q\} \twoheadrightarrow \mathbb{P}^{r-2}.$$

Suppose first, that $b_h = 0$ and let $\mathbb{L}$ be the line $S(0,0) \subset V_{h,h'} = S(0,0,r-3)$. By Lemma 5.6 it follows that $\mathbb{L} = \mathbb{L}_{h'}$ and hence that $\mathbb{L}_{h'} \subset S(0,0,0) \subset S(0,0,0,r-3-b_h) = W_h$. Now, Lemma 5.3 (c) implies that $\mathbb{F}(X) = S(0,0,0) = \mathbb{P}^2$.

Suppose now, that $b_h > 0$ and let $h' \in U$. According to Lemma 5.7 (d), the curve $C' := \overline{\pi_{\mathbb{L}_h}(C_{h'} \setminus (C_{h'} \cap \mathbb{L}_h))}$ is a curve section of $Z_h = S(b_h, r-3-b_h)$, with $\langle C' \rangle = \pi'_{\mathbb{L}_h}(\mathbb{H}_{h'}) = \mathbb{P}^{r-2}$. Moreover by Lemma 5.7 (c), the set $C' \cap \operatorname{Sing}(\pi_{\mathbb{L}_h})$ is finite, so that the induced morphism $\pi_{\mathbb{L}_h} \restriction : C_{h'} \setminus (C_{h'} \cap \mathbb{L}_h) \to C'$ is birational. As C' is smooth and rational, this latter morphism extends to a unique isomorphism

$$\varphi : C_{h'} \xrightarrow{\cong} C'.$$

We aim to show first, that the two distinct lines $\mathbb{L}_h$ and $\mathbb{L}_{h'}$ are disjoint. Assume that $\mathbb{L}_h$ and $\mathbb{L}_{h'}$ are not disjoint. Then they must meet in the vertex q of the scroll $V_{h,h'} = S(0, b_h, r-3-b_h)$. Hence, the point $p = \pi'_{\mathbb{L}_h}(\mathbb{L}_{h'} \setminus \{q\}) \in Z_h$ satisfies $p \in C'$ and $\#\varphi^{-1}(p) = \#(\mathbb{L}_{h'} \cap C_{h'}) = d - r + 3 > 1$, a contradiction. This proves the stated disjointness of the extremal secant lines $\mathbb{L}_{h'}$ and $\mathbb{L}_h$.

By Proposition 5.4 it now follows, that $\mathbb{F}(X)$ is not a plane if $b_h > 0$. Applying the previous arguments to all $h' \in \mathbb{U}(X)$ instead of h, we see that either $b_{h'} = 0$ for all $h' \in \mathbb{U}(X)$ or $b_{h'} > 0$ for all $h' \in \mathbb{U}(X)$. This observation completes in particular the proof of statement (a).

It remains to complete the proof of statement (b). We first aim to show, that

$$\dim(\mathbb{F}(X)) \leq 1 + \dim(X \cap \mathbb{F}(X)).$$

To do so, we consider the coincidence set

$$Y := \{(x, \mathbb{L}_h) \mid h \in \mathbb{U}(X) \text{ and } x \in \mathbb{L}_h\} \subset \mathbb{P}^r \times \mathbb{G}(1, \mathbb{P}^r)$$

– which is locally closed in $\mathbb{G}(1, \mathbb{P}^r)$ by Proposition 5.2 (b) – and its locally closed subset

$$T := Y \cap \big(X \times \mathbb{G}(1, \mathbb{P}^r)\big) = \{(x, \mathbb{L}) \in Y \mid x \in X\} \subset \mathbb{P}^r \times \mathbb{G}(1, \mathbb{P}^r).$$

Now, the projection morphism $\varrho : \mathbb{P}^r \times \mathbb{G}(1, \mathbb{P}^r) \twoheadrightarrow \mathbb{P}^r$ maps T to $X \cap \mathbb{F}(U)$ and hence induces a morphism $\varrho \restriction : T \to X \cap \mathbb{F}(X)$. As any two distinct lines $\mathbb{L}_h$ and $\mathbb{L}_{h'}$ with $h, h' \in \mathbb{U}(X)$ are disjoint, the map $\varrho \restriction$ is injective, so that

$$\dim(T) \leq \dim(X \cap \mathbb{F}(X)).$$

Moreover we have $\varrho(Y) = \bigcup_{h \in \mathbb{U}(X)} \mathbb{L}_h$, so that $\mathbb{F}(X) = \overline{\varrho(Y)}$ and hence

$$\dim(\mathbb{F}(X)) \leq \dim(Y).$$

The projection morphism $\sigma : \mathbb{P}^r \times \mathbb{G}(1, \mathbb{P}^r) \twoheadrightarrow \mathbb{G}(1, \mathbb{P}^r)$ satisfies $\sigma(Y) = \sigma(T) = \{\mathbb{L}_h \mid h \in \mathbb{U}(X)\}$. Moreover for each $h \in \mathbb{U}(X)$ we have $\sigma^{-1}(\mathbb{L}_h) \cap Y = \{(x, \mathbb{L}_h) \mid x \in \mathbb{L}_h\} \cong \mathbb{P}^1$. This shows, that

$$\dim(Y) \leq \dim\left(\overline{\sigma(T)}\right) + 1 \leq \dim(T) + 1,$$

so that indeed $\dim(\mathbb{F}(X)) \leq \dim(Y) \leq \dim(T) + 1 \leq \dim(X \cap \mathbb{F}(X)) + 1$.

As $\mathbb{F}(X) \neq \mathbb{P}^2$ we have $\dim(\mathbb{F}(X)) \geq 3$ (see Proposition 5.4 and Definition and Remark 5.1 (B)). It follows that $X \subset \mathbb{F}(X)$ and $\dim(\mathbb{F}(X)) = 3$. This proves claim (2) of statement (b).

It remains to complete the prove of claim (1) of statement (b). Observe that the line $\mathbb{M}_{h'} := \pi'_{\mathbb{L}_h}(\mathbb{L}_{h'}) \subset \mathbb{P}^{r-2}$ satisfies $\varphi(C_{h'} \cap \mathbb{L}_{h'}) = \pi'_{\mathbb{L}_h}(C_{h'} \cap \mathbb{L}_{h'}) \subset Z_h \cap \mathbb{M}_{h'}$. As φ is an isomorphism, we have $\#(C' \cap \mathbb{M}_{h'}) = \#\varphi(C_{h'} \cap \mathbb{L}_{h'}) = \#(C_{h'} \cap \mathbb{L}_{h'}) = d - r + 3 > 2$, and hence $\mathbb{M}_{h'} \subset Z_h$.

As $\mathbb{L}_{h'} \cap \mathbb{L}_h = \emptyset$, the line $\mathbb{L}_{h'} \subset V_{h,h'}$ avoids the vertex q of $V_{h.h'}$ and hence is not contained in any of the ruling planes of $V_{h,h'}$. As the ruling lines of the surface scroll Z_h are precisely the images of the ruling planes of $V_{h,h'}$ under the linear projection map $\pi'_q : \mathbb{H}_{h'} \setminus \{q\} \twoheadrightarrow \mathbb{P}^{r-2}$ centered at q, it follows that $\mathbb{M}_{h'} = \pi'_{\mathbb{L}_h}(\mathbb{L}_{h'}) = \pi'_q(\mathbb{L}_{h'}) \subset Z_h$ is not a ruling line of Z_h. So $\mathbb{M}_{h'}$ must be a line section of Z_h, and hence $b_h = 1$.

Our next aim is to show that $r = 5$. Assume to the contrary, that $r \geq 6$. Then $\mathbb{L}_{h'} \subset V_{h,h'} = S(0, 1, r-4) \subset S(0, 0, 1, r-4) = W_h$ shows that $\mathbb{L}_{h'} \subset S(0, 0, 1) = \mathbb{P}^3$ for all $h' \in \mathbb{U}(X)$, so that $\mathbb{F}(X) \subset \mathbb{P}^3$. But now, by claim (b)(2) we get that $X \subset \mathbb{P}^3$, and this contradiction shows, that indeed $r = 5$.

Applying this to arbitrary $h' \in \mathbb{U}(X)$ instead of h, we get in particular that $b_{h'} = 1$ for all $h \in \mathbb{U}(X)$, and claim (1) of statement (b) is shown completely. □

We may summarize the previous result as follows:

5.9. **Theorem.** *Let $5 \leq r < d$ and let $X \subset \mathbb{P}^r$ be a surface of degree d which is of maximal sectional regularity. Then, in the notations of Theorem 4.3 (c) we have*

(a) $\mathbb{F}(X)$ *is a plane if and only if $b_h = 0$ for all $h \in \mathbb{U}(X)$, and this is always the case if $r \geq 6$.*
(b) $\mathbb{F}(X)$ *is not a plane if and only if $r = 5$ and $b_h = 1$ for all $h \in \mathbb{U}(X)$.*

Proof. This is immediate by Proposition 5.8. □

We now aim to describe the extremal variety $\mathbb{F}(X)$ of a surface $X \subset \mathbb{P}^5$ of maximal sectional regularity in case it is not a plane. We begin with some preparations.

5.10. **Notation and Reminder.** (A) Let $s > 2$ and let $Y \subset \mathbb{P}^s$ be a smooth rational normal surface scroll with projection morphism $\varphi : Y \twoheadrightarrow \mathbb{P}^1$. Let $F := \varphi^{-1}(x) = \mathbb{P}^1 \in \mathrm{Div}(Y)$ denote a fibre and $H := Y \cap \mathbb{P}^{s-1} \in \mathrm{Div}(Y)$ a general hyperplane section, and let $\overline{F}, \overline{H} \in \mathrm{Cl}(Y)$ be the corresponding divisor classes. As $H \subset Y$ is a section of φ we have $\mathrm{Cl}(Y) = \mathbb{Z}\overline{H} \oplus \mathbb{Z}\overline{F}$, so that any divisor $D \in \mathrm{Div}(Y)$ is linearly equivalent to a divisor of the form $aH + bF$, with uniquely determined integers $a, b \in \mathbb{Z}$ (see [22, Proposition V.2.3]).
(B) Let the notations be as in part (A) and keep in mind that $H \cdot H = \deg(Y)$, $H \cdot F = 1$ and $F \cdot F = 0$. Let $a, b \in \mathbb{Z}$ and let $D \in |aH + bF|$. If follows that $\deg(D) = D \cdot H = a\deg(Y) + b$. As D is effective, it is linearly equivalent to a divisor of the form $\sum_{i=1}^t n_i C_i + cF$ with pairwise distinct prime divisors $C_1, \ldots, C_t$ which are not fibers, with $n_1, \ldots, n_t > 0$, $c \geq 0$ and with $t \geq 0$. If follows, that

$$a = (aH + bF) \cdot F = D \cdot F = (\sum_{i=1}^t n_i C_i + cF) \cdot F = \sum_{i=1}^t n_i C_i \cdot F \geq \sum_{i=1}^t n_i,$$

with equality at the last place if and only if $C_1, \dots, C_t \subset Y$ are sections.

5.11. **Lemma.** *Let H and F be respectively a general hyperplane section and a ruling line of the rational normal surface scroll $Y := S(1,2) \subset \mathbb{P}^4$. Let $C \subset \mathbb{P}^4$ be a non-degenerate integral curve of degree $d > 5$ which is of maximal regularity. If $C \subset S(1,2)$, then C is linearly equivalent to the divisor $H + (d-3)F$ and the line $S(1) \subset S(1,2)$ is the extremal secant line to C.*

Proof. Let C be linearly equivalent to $aH + bF$. By Notation and Reminder 5.10 (B) we have $a \geq 1$. As the surface Y is arithmetically Cohen-Macaulay, we have $H^i(\mathbb{P}^4, \mathcal{J}_Y(1)) = 0$ for $i = 1, 2$ and so, the short exact sequence

$$0 \longrightarrow \mathcal{J}_Y \longrightarrow \mathcal{J}_C \longrightarrow \mathcal{O}_Y(-C) \longrightarrow 0$$

implies an isomorphism $H^1(\mathbb{P}^4, \mathcal{J}_C(1)) \cong H^1\big(\mathbb{P}^4, \mathcal{O}_Y((1-a)H - bF)\big)$. If $a > 1$, we have $H^1\big(\mathbb{P}^4, \mathcal{O}_Y((1-a)H - bF)\big) = 0$, and we get the contradiction that the curve of maximal regularity $C \subset \mathbb{P}^4$ is linearly normal (see Proposition 2.7 a) of [7]). Therefore it holds $a = 1$. As $d = \deg(C) = \deg(Y) + b = 3 + b$ (see Notation and Reminder 5.10 (A)), we obtain $b = d - 3$ and C is linearly equivalent to $H + (d-3)F$.

Moreover the line section $\mathbb{L} = S(1)$ of $Y = S(1,2)$ satisfies the condition

$$\#(C \cap \mathbb{L}) = C \cdot \mathbb{L} = (H + (d-3)F) \cdot \mathbb{L} = d - 2,$$

and hence $\mathbb{L}$ is indeed the extremal secant line to C. □

5.12. **Notation and Reminder.** Let $n \in \mathbb{Z}$. We say, that the closed subscheme $Z \subset \mathbb{P}^r$ is *n-normal* if $H^1(\mathbb{P}^r, \mathcal{J}_Z(n)) = 0$ and we introduce the *index of normality* of Z as

$$N(Z) := \sup\{n \in \mathbb{Z} \mid Z \text{ is not } n\text{-normal}\} = \operatorname{end}(\bigoplus_{n\in\mathbb{Z}} H^1(\mathbb{P}^r, \mathcal{J}_Z(n)) = \operatorname{end}(H^1(S/I_Z)).$$

Keep in mind that $N(Z) \leq \operatorname{reg}(Z) - 2$ and that $N(Z) = -\infty$ if $\operatorname{depth} Z > 1$. In particular, if $5 \leq r < d$ and $X \subset \mathbb{P}^r$ is a surface of maximal sectional regularity and degree d, we have

$$N(X) \leq d - r + 1.$$

5.13. **Lemma.** *Let $5 \leq r < d$, let $Y \subset \mathbb{P}^r$ be an irreducible surface of degree d which is contained in the smooth threefold scroll $Z := S(1, a, r-a-3)$ with $1 \leq a \leq \frac{r-2}{2}$ as a divisor linearly equivalent to $H + (d-r+2)F$. Then, the following statements hold:*

(a) *$N(Y) = d - r + 1$ and*

$$h^1(\mathbb{P}^r, \mathcal{J}_Y(d-r+1)) = \begin{cases} 1 & \text{if } a \geq 2, \\ d-r & \text{if } a = 1 \text{ and } r \geq 6, \text{ and} \\ \binom{d-3}{2} & \text{if } a = 1 \text{ and } r = 5\ . \end{cases}$$

(b) *The minimal number of generators in degree $d-r+3$ of the homogeneous vanishing ideal $I_Y \subset S$ of Y is given by*

$$\beta_{1,d-r+2}(Y) = \begin{cases} 1 & \text{if } a \geq 2, \\ d-r+3 & \text{if } a = 1 \text{ and } r \geq 6, \text{ and} \\ \binom{d-1}{2} & \text{if } a = 1 \text{ and } r = 5. \end{cases}$$

Proof. (a): We set $b := r - a - 3$ and consider the short exact sequence

$$0 \to \mathcal{J}_Z \to \mathcal{J}_Y \to \mathcal{O}_Z(-Y) \to 0.$$

Then, since Z is arithmetically Cohen-Macaulay, we have

$$H^1(\mathbb{P}^r, \mathcal{J}_Y(n)) \cong H^1(Z, \mathcal{O}_Z(-Y + nH)) \text{ for all } n \in \mathbb{Z}.$$

Setting $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ we may write

$$\begin{aligned} H^1(Z, \mathcal{O}_Z(-Y + nH)) &= H^1(Z, \mathcal{O}_Z((n-1)H - (d-r+2)F)) \\ &= H^1(\mathbb{P}^1, \operatorname{sym}^{n-1}\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(-d+r-2)) \\ &= \bigoplus_{0 \le i,j, i+j \le n-1} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i + ja + (n-i-j)b - d + r - 2)). \end{aligned}$$

Altogether, we obtain that $h^1(\mathbb{P}^r, \mathcal{J}_Y(n) = 0$ for all $n > d - r + 1$ and

$$h^1(\mathbb{P}^r, \mathcal{J}_Y(d-r+1)) = \begin{cases} 1 & \text{if } a \ge 2, \\ d-r & \text{if } a = 1 \text{ and } b \ge 2, \text{ and} \\ \binom{d-3}{2} & \text{if } a = b = 1. \end{cases}$$

This proves statement (a).

(b): For this statement, see [28]. □

Now, we are ready to prove the following structure result on non-planar extremal varieties.

5.14. **Theorem.** *Let $5 \le r < d$ and let $X \subset \mathbb{P}^r$ be a surface of degree d and of maximal sectional regularity such that $\mathbb{F}(X)$ is not a plane. Then we have:*

(a) *$r = 5$ and $\mathbb{F}(X) = S(1,1,1)$.*

(b) *X is contained in $\mathbb{F}(X)$, smooth and linearly equivalent to the divisor $H + (d-3)F$, where H is the hyperplane divisor and F is ruling plane of $\mathbb{F}(X) = S(1,1,1)$.*

(c) *$N(X) = d - 4$, $\mathrm{e}(X) = 0$ and moreover*
   (1) *$h^1(\mathbb{P}^r, \mathcal{J}_X(d-4)) = \binom{d-3}{2}$;*
   (2) *$h^2(\mathbb{P}^r, \mathcal{J}_X(n)) = 0$ for all $n \in \mathbb{Z}$;*
   (3) *$h^3(\mathbb{P}^r, \mathcal{J}_X(n)) = 0$ for all $n \ge 0$.*

(d) *$\beta_{1,d-3}(X) = \binom{d-1}{2}$.*

(e) *${}^*\Sigma^\circ_{d-2}(X) = \Sigma^\circ_{d-2}(X) = \Sigma_3(X) \setminus \Sigma_\infty(X)$ and $\mathbb{F}^+(X) = \mathbb{F}(X)$. Moreover $\overline{{}^*\Sigma^\circ_{d-2}(X)} = \Sigma_3(X)$ and the image of this set under the Plücker embedding $\psi : \mathbb{G}(1, \mathbb{P}^5) \to \mathbb{P}^{14}$ is a Veronese surface in a subspace $\mathbb{P}^5 \subset \mathbb{P}^{14}$.*

Proof. (a): By Proposition 5.8 we already know that $r = 5$, $X \subset \mathbb{F}(X)$ and $\dim(\mathbb{F}(X)) = 3$. Now, let $h, h' \in \mathbb{U}(X)$ such that $\mathbb{L}_{h'} \ne \mathbb{L}_h$. Then, according to Proposition 5.8 (b), the two 4-scrolls W_h and $W_{h'}$ in $\mathbb{P}^5$ are both of type $S(0,0,1,1)$ and $\langle \mathbb{L}_h, \mathbb{L}_{h'} \rangle \subset \mathbb{P}^5$ is a 3-space.

As $X \subset W_h$ and $\#(X \cap \mathbb{L}_{h''}) = d - 2 > 2$ for all $h'' \in \mathbb{U}(X)$, we have $\mathbb{F}(X) \subset W_h$. Moreover, as $\mathbb{L}_h$ is the vertex of $W_h = S(0,0,1,1)$ and $\mathbb{L}_{h'} \subset W_h$ it holds $\langle \mathbb{L}_h, \mathbb{L}_{h'} \rangle \subset W_h$. Hence, by symmetry we get

$$X \subset \mathbb{F}(X) \subset W_h \cap W_{h'} \text{ and } \mathbb{P}^3 = \langle \mathbb{L}_h, \mathbb{L}_{h'} \rangle \subset W_h \cap W_{h'}.$$

As W_h and $W_{h'}$ are two distinct integral hyperquadrics in $\mathbb{P}^5$, the intersection $W_h \cap W_{h'}$ is arithmetically Cohen-Macaulay, satisfies $\dim(W_h \cap W_{h'}) = 3$ and $\deg(W_h \cap W_{h'}) = 4$. In particular it follows that $W_h \cap W_{h'} = \langle \mathbb{L}_h, \mathbb{L}_{h'} \rangle \cup D$, where $D \subset \mathbb{P}^5$ is an non-degenerate integral closed subscheme of dimension 3 and degree 3. As $X \subset \mathbb{P}^5$ is non-degenerate and contained in $W_h \cap W_{h'}$, we have $X \subset D$. As $\#(X \cap \mathbb{L}_{h''}) = d - r + 3 > 2$, it follows that $\mathbb{L}_{h''} \subset D$ for all $h'' \in \mathbb{U}(X)$, and hence that $\mathbb{F}(X) = D$.

Observe that D is a scroll of type $S(1,1,1)$ or $S(0,1,2)$ or $S(0,0,3)$. We aim to exclude the latter two cases. Assume first, that $D = S(0,0,3)$. As $X \subset D$ is a Weil divisor, it follows, that X is arithmetically Cohen-Macaulay, which contradicts Theorem 4.3 (b). Assume now, that $D = S(0,1,2)$ and let $h'' \in \mathbb{U}(X)$ be general. Then, we have

$$C_{h''} \subset D \cap \mathbb{H}_{h''} = S(1,2) \subset H_{h''} = \mathbb{P}^4,$$

and according to Lemma 5.11, the line section $S(1)$ of $S(1,2)$ coincides with the extremal secant line $\mathbb{L}_{h''}$ of $C_{h''}$, so that $\mathbb{L}_{h''}$ is contained in the plane $\mathbb{P}^2 = S(0,1) \subset S(0,1,2) = D$ for general $h'' \in \mathbb{U}(X)$, and this contradicts Proposition 5.8 (b)(1). Therefore indeed $\mathbb{F}(X) = D = S(1,1,1)$, and statement (a) is shown completely.

(b): For all $h \in \mathbb{U}(X)$ we have

$$C_h = X \cap \mathbb{H}_h = X \cap (D \cap \mathbb{H}_h) \subset D \cap \mathbb{H}_h = S(1,2) \subset \mathbb{H}_h = \mathbb{P}^4$$

and Lemma 5.11 yields that the divisor X is linearly equivalent to $H + (d-3)F$. It remains to show, that X is smooth.

As X is a divisor of the smooth variety $D = S(1,1,1)$ it is a Cohen-Macaulay variety. So, by Theorem 4.3 (a)(1) it follows indeed that X is smooth.

(c): The equality $N(X) = d - 4$ and claim (1) follow immediately from statements (a) and (b) by Lemma 5.13 (a), whereas the vanishing of $\mathrm{e}(X)$ follows from the fact that X is smooth. Claims (2) and (3) follow from the fact that $\mathrm{e}(X) = 0$ by Theorem 4.3(a)(2).

(d): This follows from statements (a) and (b) by Lemma 5.13 (b).

(e): By statement (b) the surface X is smooth. So, the equalities ${}^*\Sigma^\circ_{d-2}(X) = \Sigma^\circ_{d-2}(X) = \Sigma_3(X) \setminus \Sigma_\infty(X)$ follow immediately by Proposition 5.2 (a),(b). The equality $\overline{{}^*\Sigma^\circ_{d-2}(X)} = \Sigma_3(X)$ now follows easily as $X \subset \mathbb{F}(X)$ (see statement (b)).

To prove the remaining claim, we identify $\mathbb{F}(X) = S(1,1,1)$ with the image of the Segre embedding $\sigma : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$. Let

$$\Theta := \{\mathbb{P}^1 \times \{q\} \mid q \in \mathbb{P}^2\} \subset \mathbb{G}(1, \mathbb{P}^5)$$

denote the closed subset of all fibers under the canonical projection $\mathbb{P}^1 \times \mathbb{P}^2 \twoheadrightarrow \mathbb{P}^2$, and let

$$\Omega := \bigcup_{p \in \mathbb{P}^1} \mathbb{G}(1, \{p\} \times \mathbb{P}^2) \subset \mathbb{G}(1, \mathbb{P}^5)$$

denote the closed subset of all lines contained in some fiber of the canonical projection $\mathbb{P}^1 \times \mathbb{P}^2 \twoheadrightarrow \mathbb{P}^1$, hence in a ruling plane of $S(1,1,1)$. By Proposition 5.8 (b)(1) we have

$$\#\big({}^*\Sigma^\circ_{d-2}(X) \cap \mathbb{G}(1, \{p\} \times \mathbb{P}^2)\big) \leq 1 \text{ for all } p \in \mathbb{P}^1,$$

and hence $\dim({}^*\Sigma^\circ_{d-2}(X) \cap \Omega) \leq 1$. Therefore $U := {}^*\Sigma^\circ_{d-2}(X) \setminus \Omega$ is a locally closed dense subset of $\overline{{}^*\Sigma^\circ_{d-2}(X)}$, consists of line sections of $S(1,1,1)$, and hence is contained in Θ. It follows that $\overline{{}^*\Sigma^\circ_{d-2}(X)} \subseteq \Theta$, so that we get the inclusion

$$\psi(\overline{{}^*\Sigma^\circ_{d-2}(X)}) \subseteq \psi(\Theta).$$

By standard arguments on Plücker embeddings one sees that $\psi(\Theta)$ is the Veronese surface in some subspace $\mathbb{P}^5 \subset \mathbb{P}^{14}$. As the left hand side of the previous inclusion is a surface, we get our claim. □

6. Planar Extremal Varieties

In this section, we give a few results which concern the "general" case in which the extremal variety is a plane. The case of surfaces of maximal sectional regularity which are cones is understood by what is said in Definition and Remark 5.1 (C). Therefore, we restrict ourselves to consider the case of non-conic surfaces of maximal sectional regularity. Observe, that according to Theorem 5.9, our results apply to all non-conic surfaces of maximal sectional regularity in $\mathbb{P}^r$ with $r \geq 6$. We begin with the following auxiliary results.

6.1. **Lemma.** *Let $s > 1$, let $C \subset \mathbb{P}^s$ be a closed subscheme of dimension 1 and degree d and let $\mathbb{H} = \mathbb{P}^{s-1} \subset \mathbb{P}^s$ be a hyperplane. Then*

$$\#(C \cap \mathbb{H}) \geq d \textit{ with equality if and only if } \mathrm{Ass}_C(\mathcal{O}_C) \cap \mathbb{H} = \emptyset.$$

Proof. Let $R = K \oplus R_1 \oplus R_2 \oplus \ldots = K[R_1]$ be the homogeneous coordinate ring of C and let $f \in R_1$ be such that $C \cap \mathbb{H} = \mathrm{Proj}(R/fR)$. Let $H_R(t) = dt + c$ be the Hilbert polynomial of R. Then, the two exact sequences

$$0 \to fR \to R \to R/fR \to 0 \text{ and } 0 \to (0 :_R f)(-1) \to R(-1) \to fR \to 0$$

yield that the Hilbert polynomial of R/fR is given by

$$H_{R/fR}(t) = d + H_{(0:_R f)}(t-1).$$

Observe that the polynomial $H_{(0:_R f)}(t-1)$ vanishes if and only if $(0 :_R f)_t = 0$ for all $t \gg 0$, hence if and only if

$$f \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R) \setminus \{R_+\}} \mathfrak{p}.$$

But this latter condition is equivalent to the requirement that $\mathrm{Ass}(C) \cap \mathbb{H} = \emptyset$. □

6.2. **Lemma.** *Let $5 \leq r$ and let $C \subset \mathbb{P}^{r-1}$ be a curve of degree $d \geq 3r-6$ which is of maximal regularity and with extremal secant line $\mathbb{L} \subset \mathbb{P}^{r-1}$. Then* $\mathrm{depth}(C \cup \mathbb{L}) = 1$.

Proof. Since C is contained in a rational 3-fold scroll $\mathbb{S} := S(0,0,r-3)$, we have

$$\dim_K(I_C)_2 \geq \dim_K(I_{\mathbb{S}})_2 = \binom{r-1}{2} - 2r + 5.$$

Assume now, that $\mathrm{depth}(C \cup \mathbb{L}) \neq 1$, so that $C \cup \mathbb{L}$ is arithmetically Cohen Macaulay. Then, by Proposition 3.6 of [7] it follows that $\dim_K(C)_2 = \binom{r}{2} - d - 1$, whence $\binom{r-1}{2} - 2r + 5 \leq \binom{r}{2} - d - 1$, thus the contradiction that $d \leq 3r - 7$. □

Now, we can formulate and prove the first main result of this section. We use the notations introduced in Notation and Reminder 2.2 and 3.7 (C), (D).

6.3. **Theorem.** *Let $5 \leq r < d$ and let $X \subset \mathbb{P}^r$ be a surface of degree d and of maximal sectional regularity which is not a cone. Assume that $\mathbb{F} = \mathbb{F}(X)$ is a plane, set $C := X \cap \mathbb{F}$, $Y := X \cup \mathbb{F}$ and let I and L respectively denote the homogeneous vanishing ideal of X and of $\mathbb{F}$ in S. Then the following statements hold*

(a) *Each line $\mathbb{L} \subset \mathbb{F}$ which is not contained in X, satisfies $\#(C \cap \mathbb{L}) = \#(X \cap \mathbb{L}) = d - r + 3$. In particular, $C \subset \mathbb{F}$ is a curve of degree $d - r + 3$ and has no closed associated points.*

(b) $X \setminus \mathrm{Reg}(X) \subseteq C \setminus \mathrm{Reg}(C)$.

(c) *$I_{d-r+3} \setminus I \cap L \neq \emptyset$ and for each $f \in I_{d-r+3} \setminus I \cap L$ it holds $I = (I \cap L, f)$.*

(d) (1) *$h^1(\mathbb{P}^r, \mathcal{J}_X(n)) = h^1(\mathbb{P}^r, \mathcal{J}_Y(n))$ for all $n \in \mathbb{Z}$.*
(2) *$h^2(\mathbb{P}^r, \mathcal{J}_X(d-r)) = 1$ and $h^2(\mathbb{P}^r, \mathcal{J}_X(n)) = 0$ for all $n > d - r$.*
(3) *$h^2(\mathbb{P}^r, \mathcal{J}_Y(n)) = 0$ for all $n \geq d - r$.*
(4) *$h^3(\mathbb{P}^r, \mathcal{J}_X(n-1)) = h^3(\mathbb{P}^r, \mathcal{J}_Y(n)) = 0$ for all $n \geq 0$.*

(e) (1) *$h^2(\mathbb{P}^r, \mathcal{J}_X(n)) = h^2(\mathbb{P}^r, \mathcal{J}_Y(n)) + \max\{0, \binom{-n+d-r+2}{2}\}$ for all $n \geq 0$.*
(2) *$\mathrm{e}(X) = h^2(\mathbb{P}^r, \mathcal{J}_Y(0)) + \binom{d-r+2}{2} = h^2(\mathbb{P}^r, \mathcal{J}_X(n))$ for all $n \leq 0$.*
(3) *$h^2(\mathbb{P}^r, \mathcal{J}_Y(n+1)) \geq h^2(\mathbb{P}^r, \mathcal{J}_Y(n))$ for all $n \leq 0$, with equality for $n = 0$.*
(4) *$h^2(\mathbb{P}^r, \mathcal{J}_Y(n+1)) \leq \max\{0, h^2(\mathbb{P}^r, \mathcal{J}_Y(n)) - 1\}$ for all $n > 0$.*
(5) *If $h^2(\mathbb{P}^r, \mathcal{J}_Y(0) = 0$, then $h^2(\mathbb{P}^r, \mathcal{J}_Y(n)) = 0$ for all $n \in \mathbb{Z}$.*

(f) *For the pair $\tau(X) := \big(\mathrm{depth}(X), \mathrm{depth}(Y)\big)$ we have*
(1) $\tau(X) = (2, 3)$ *if $r + 1 \leq d \leq 2r - 4$;*
(2) $\tau(X) \in \{(1, 1), (2, 2), (2.3)\}$ *if $2r - 3 \leq d \leq 3r - 7$;*
(3) $\tau(X) \in \{(1, 1).(2, 2)\}$ *if $3r - 6 \leq d$.*

(g) $\mathrm{reg}(Y) \leq d - r + 3$.

(h) *The image of $\overline{{}^*\Sigma^\circ_{d-r+3}(X)}$ under the Plücker embedding $\psi : \mathbb{G}(1, \mathbb{P}^r) \to \mathbb{P}^{\binom{r+1}{2}-1}$ is a plane.*

Proof. In the proof, we prefer to use local cohomology instead of sheaf cohomology. So keep in mind that

$$H^i(\mathbb{P}^r, \mathcal{J}_X(n)) = H^i(S/I)_n \text{ and } H^i(\mathbb{P}^r, \mathcal{J}_Y(n)) = H^i(S/I \cap L)_n \text{ for } i = 1, 2, 3 \text{ and } n \in \mathbb{Z}.$$

(a): First let $h \in \mathbb{U}(X)$. Then $\mathbb{L}_h \subset \mathbb{F}$ and $\#(C \cap \mathbb{L}_h) = \#(X \cap \mathbb{L}_h) = \#(C_h \cap \mathbb{L}_h) = d - r + 3$. This shows, that $C \subset \mathbb{F}$ is a closed subscheme of dimension 1 and degree $d - r + 3$. Now, let $\mathbb{L} \subset \mathbb{F}$ be an arbitrary line which is not contained in X. As $C \subset \mathbb{F}$ is of dimension 1 and of degree $d - r + 3$ we have $\#(X \cap \mathbb{L}) = \#(C \cap \mathbb{L}) \geq d - r + 3$. As $\mathrm{reg}(X) = d - r + 3$ (see Theorem 4.3 (b)) we also have $\#(X \cap \mathbb{L}) \leq d - r + 3$, so that indeed $\#(X \cap \mathbb{L}) = d - r + 3$. Now, it follows by Lemma 6.1 that C has no closed associated point.

(b): Let $p \in X \setminus \mathrm{Reg}(X)$. We first show, that $p \in C$. Assume that this is not the case so that $p \notin \mathbb{F}$. Now, as seen previously, a general hyperplane $\mathbb{H} = \mathbb{P}^{r-1} \subset \mathbb{P}^r$ which runs through p has the property that $(X \cap \mathbb{H})_{\mathrm{red}} \subset \mathbb{H}$ is a non-degenerate irreducible curve. Set $\mathbb{L} := \mathbb{H} \cap \mathbb{F}$. Then $\#(X \cap \langle p, \mathbb{L}\rangle) < \infty$. In particular the line $\mathbb{L} \subset \mathbb{F}$ is not contained in X, so that $\#(X \cap \mathbb{L}) = d - r + 3$ by statement (a). As $\mathbb{H}$ is general, the line $\mathbb{L}$ also avoids the finite set $X \setminus \mathrm{Reg}(X)$. It follows by Proposition 5.2 (a) that there is some $h \in \mathbb{U}(X)$ such that $\mathbb{L} = \mathbb{L}_h$, and statement (c) of that same proposition yields the contradiction that $\#(X \cap \langle p, \mathbb{L}\rangle) = \infty$. Therefore we have indeed $p \in C$.

Now, let $\mathbb{L} \subset \mathbb{F}$ be a general line which runs through p. As $\mathbb{L}$ avoids all singular points of X different from p it follows by Theorem 4.3 (a)(5) that $\#(X \cap \mathbb{L} \setminus p) + 2 \leq d - r + 3$. So, by statement (a) we get

$$d - r + 3 - \mathrm{mult}_p(C \cap \mathbb{L}) = \#(C \cap \mathbb{L} \setminus p) \leq \#(X \cap \mathbb{L} \setminus p) \leq d - r + 1$$

and hence $\mathrm{mult}_p(C \cap \mathbb{L}) \geq 2$. This shows, that p is a singular point of C.

(c): According to statement (a), there is a homogeneous polynomial $g \in S_{d-r+3} \setminus L$ such that the homogeneous vanishing ideal $(I + L)^{\mathrm{sat}} \subset S$ of C in S can be written as (L, g). In particular we have $I_{\leq d-r+2} \subset L$. As $\mathrm{reg}(X) = d - r + 3$, the ideal $I \subset S$ is generated by homogeneous polynomials of degree $\leq d - r + 3$. As $g \notin L$ it follows that I_{d-r+3} is not contained in L and hence that $I + L = (L, f)$ for all $f \in I_{d-r+3} \setminus I \cap L$. Therefore $I = I \cap (I + L) = I \cap (L, f) = (I \cap L, f)$ for all such f.

(d): According to statement (c) we have an exact sequence

$$0 \to (S/L)(-d + r - 3) \to S/I \cap L \to S/I \to 0.$$

Applying cohomology and observing that $H^i((S/L)(-d+r-3))_n$ vanishes if either $i \neq 3$ or else if $i = 3$ and $n \geq d - r + 1$, and keeping in mind Theorem 4.3 (a)(2), we get claims (1) and (4) and also that

$$H^2(S/I)_n = H^2(S/I \cap L)_n = 0 \text{ for all } n > d - r.$$

It remains to show that $H^2(S/I)_{d-r} \cong K$ and $H^2(S/I \cap L)_{d-r} = 0$. To this end, let $h \in \mathbb{U}(X)$ and consider the induced exact sequence

$$H^1(S/I) \longrightarrow H^1(S/(I,h)^{\mathrm{sat}}) \longrightarrow H^2(S/I)(-1) \xrightarrow{h} H^2(S/I) \longrightarrow H^2(S/(I,h)^{\mathrm{sat}}).$$

As $S/(I,h)^{\mathrm{sat}}$ is the homogeneous coordinate ring of the curve of maximal regularity $C_h \subset \mathrm{Proj}(S/hS) = \mathbb{P}^{r-1}$ we get from Proposition 2.7 of [7], that $H^1(S/(I,h)^{\mathrm{sat}})_{d-r+1} \cong K$ and $H^2(S/(I,h)^{\mathrm{sat}})_n = 0$ for all $n \geq 0$.

As $H^2(S/I)_{d-r+1} = 0$, it follows, that $h^2(S/I)_{d-r} = \dim(H^2(S/I)_{d-r}) \leq 1$. As $H^2(S/L) = 0$ the first exact sequence used above induces exact sequences

$$0 \to H^2(S/I \cap L)_n \longrightarrow H^2(S/I))_n \longrightarrow H^3(S/L)_{n-d+r-3} \longrightarrow H^3(S/I \cap L)_n$$

for all $n \in \mathbb{Z}$. In view of claim (4) we thus get exact sequences

$$0 \to H^2(S/I \cap L)_n \longrightarrow H^2(S/I))_n \longrightarrow H^3(S/L)_{n-d+r-3} \longrightarrow 0 \text{ for all } n \geq 0.$$

We apply this with $n = d - r$. As $H^3(S/L)_{-3} \cong H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \cong K$ it follows that $\dim(H^2(S/I)_{d-r}) = \dim(H^2(S/I \cap L)_{d-r}) + 1$. In view of our previous observation, this shows that

$$H^2(S/I)_{d-r} \cong K \text{ and } H^2(S/I \cap L)_{d-r} = 0.$$

(e): If we apply the last exact sequence used in the proof of statement (d) for all $n \geq 0$ and bear in mind that $h^3(S/L)_{n-d+r-3} = h^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n - d + r - 3)) = \binom{-n-d-r+2}{2}$ we get claim (1).

Claim (2) follows immediately from claim (1), as $h^2(S/I)_0 = h^1(X, \mathcal{O}_X) = \mathrm{e}(X)$ (see Theorem 4.3 (a)(2)).

Now, let $h \in \mathbb{U}(X)$ be general. Then $C_h = X \cap \mathbb{H}_h \subset \mathbb{H}_h = \mathbb{P}^{r-1}$ is a curve of degree d and maximal regularity, with extremal secant line $\mathbb{L}_h := \mathbb{F} \cap \mathbb{H}_h$. Moreover the ring $S/(I \cap L, h)^{\mathrm{sat}}$ is the homogeneous coordinate ring of $C_h \cup \mathbb{L}_h$ in S. So, by Proposition 2.7 c),d) of [7] we have $H^1(S/(I \cap L, h)^{\mathrm{sat}})_{n+1} = 0$ for all $n \leq 0$ and – in addition – that $H^2(S/(I \cap L, h)^{\mathrm{sat}})_{n+1} = 0$ for all $n \geq 0$. Now, the induced exact sequence

$$H^1(S/(I \cap L, h)^{\mathrm{sat}}) \longrightarrow H^2(S/I \cap L)(-1) \xrightarrow{h} H^2(S/I \cap L) \longrightarrow H^2(S/(I \cap L, h)^{\mathrm{sat}})$$

proves claim (3) and shows that the map $H^2(S/I \cap L)(-1) \xrightarrow{h} H^2(S/I \cap L)$ is an epimorphism in all positive degrees.

Finally, by Remark 3.2 B) of [7], the graded S-module $H^1(S/(I \cap L, h)^{\mathrm{sat}})$ is generated by homogeneous elements of degree 2. Consequently the same holds for the kernel of the map $H^2(S/I \cap L)(-1) \xrightarrow{h} H^2(S/I \cap L)$ in the above sequence. As this map is an epimorphism in all positive degrees, we get claim (4). Claim (5) is an immediate consequence of claim (4).

(f): We keep the above notations. It follows easily by Theorem 4.3 (c) and the claims proved in statement (d), that

$$\tau(X) \in \{(1,1), (2,2), (2,3)\}.$$

This proves in particular claim (2).

Now, let $d \leq 2r - 4 = 2(r-1) - 2$. Then, by Proposition 3.5 of [7] it follows that the 2-dimensional ring $S/(I \cap L, h)^{\mathrm{sat}}$ is Cohen-Macaulay, so that $H^1(S/(I \cap L, h)^{\mathrm{sat}}) = 0$. Thus, the above exact sequence shows that the map $H^2(S/I \cap L)(-1) \xrightarrow{h} H^2(S/I \cap L)$ is injective. The short exact sequence

$$H^1(S/I \cap L)(-1) \xrightarrow{h} H^1(S/I \cap L) \longrightarrow H^1(S/(I \cap L, h)^{\mathrm{sat}})$$

also shows, that the map $H^1(S/I \cap L)(-1) \xrightarrow{h} H^1(S/I \cap L)$ is surjective. It follows, that $H^1(S/I \cap L) = H^2(S/I \cap L) = 0$, so that $\mathrm{depth}(S/I \cap L) = 3$. In view of the above observation it follows that $\tau(X) = (2,3)$, and this proves claim (1).

Now, let $d \geq 3r - 6$. It follows by Lemma 6.2 that $\mathrm{depth}(S/(I \cap L, h)^{\mathrm{sat}}) = 1$. Consequently, we must have $\mathrm{depth}(S/I \cap L) \leq 2$ and our previous observation gives $\tau(X) \in \{(1,1), (2,2)\}$. This proves claim (3).

(g): This is immediate by claims (1),(3) and (4) of statement (d).

(h): By statement (a) we have $\overline{{}^*\Sigma^{\circ}_{d-r+3}(X)} = \mathbb{G}(1, \mathbb{F}) = \mathbb{G}(1, \mathbb{P}^2)$. Standard arguments show that $\psi\big(\mathbb{G}(1, \mathbb{P}^2)\big)$ is a plane in $\mathbb{P}^{\binom{r+1}{2}-1}$. This proves our claim. □

6.4. **Corollary.** *Let the notations and hypotheses be as be as in Theorem 6.3. Then it holds*

(a) $\mathrm{e}(X) = 0$ *or else* $\mathrm{e}(X) \geq \binom{d-r+2}{2}$.

(b) $\mathrm{e}(X) \geq \binom{d-r+2}{2}$ *if and only if* $\mathbb{F}(X) = \mathbb{P}^2$.

Proof. If $\mathbb{F}(X)$ is a plane, it follows by Theorem 6.3 (e)(2) that $\mathrm{e}(X) \geq \binom{d-r+2}{2}$. If $\mathbb{F}(X)$ is not a plane it follows by Theorem 5.14 that X is smooth and hence satisfies $\mathrm{e}(X) = 0$. This proves our claim. □

Our next main result is devoted to the relations between the index of normality $N(X)$, the Betti numbers $\beta_{i,j}(X)$ and the nature of the union $X \cup \mathbb{F}(X)$, where X is a surface of maximal sectional regularity with planar extremal variety $\mathbb{F}(X)$. We begin with two auxiliary results.

6.5. **Lemma.** *Let $5 \leq r < d$, let $X \subset \mathbb{P}^r$ be a surface of degree d and of maximal sectional regularity which is not a cone and assume that $\mathbb{F}(X)$ is a plane. Let $Y := X \cup \mathbb{F}(X)$ and set $m := \operatorname{reg}(Y)$. Then the following statements hold:*

(a) *For all $i \geq 1$ we have*

$$\beta_{i,j}(X) = \begin{cases} \beta_{i,j}(Y) & \text{for } 1 \leq j \leq m-1, \\ \beta_{i,j}(Y) = 0 & \text{for } m \leq j \leq d-r+1, \\ \beta_{i,d-r+2}(Y) + \binom{r-2}{i-1} & \text{for } j = d-r+2. \end{cases}$$

(b) *$m \leq d-r+2$ if and only if $\beta_{i,j}(X) = \binom{r-2}{i-1}$ for all $i \geq 1$.*

Proof. Let I and L respectively denote the homogeneous vanishing ideals of X and $\mathbb{F}(X)$ in S, so that $\beta_{i,j}(X) = \beta_{i,j}(S/I)$ and $\beta_{i,j}(Y) = \beta_{i,j}(S/I \cap L)$ for all $i, j \in \mathbb{N}$.
(a): By statement (c) of Theorem 6.3 we have an exact sequence

$$0 \to (S/L)(-d+r-3) \to S/I \cap L \to S/I \to 0,$$

which induces exact sequences

$$\operatorname{Tor}_i^S(K, S/L)_{i+j-d+r-3} \to \operatorname{Tor}_i^S(K, S/I \cap L)_{i+j} \to \operatorname{Tor}_i^S(K, S/I)_{i+j}$$
$$\to \operatorname{Tor}_{i-1}^S(K, S/L)_{(i-1)+j-d+r-2} \to \operatorname{Tor}_{i-1}^S(K, S/I \cap L)_{(i-1)+j+1}$$

After an appropriate change of coordinates in S we may assume that $L = \langle x_3, \ldots, x_r \rangle$, and this shows that for all $k \in \mathbb{N}_0$ we have

$$\dim_K \big(\operatorname{Tor}_k^S(K, S/L)_{k+l} \big) = \beta_{k,l}(S/L) = \begin{cases} 0 & \text{if } l \neq 0 \\ \binom{r-2}{k} & \text{if } l = 0 \end{cases}$$

Therefore, the above exact sequences make us end up with isomorphisms

$$\operatorname{Tor}_i^S(K, S/I \cap L)_{i+j} \cong \operatorname{Tor}_i^S(K, S/I)_{i+j} \text{ for all } i \geq 1 \text{ and all } j \in \{1, 2, \ldots, d-r+1\}.$$

As $\operatorname{reg}(S/I \cap L) = \operatorname{reg}(Y) - 1 = m - 1$, we have $\beta_{i,j}(S/I \cap L) = 0$ for all $i \geq 1$ and all $j \geq m$. So by the above isomorphisms we get the requested values of $\beta_{i,j}(S/I)$ for all $i \geq 1$ and all $j \in \{1, \ldots, d-r+1\}$.

As $\operatorname{reg}(S/I \cap L) = \operatorname{reg}(Y) - 1 \leq d-r+2$ (see Theorem 6.3 (g)), the last module in the above exact sequences vanishes for $j = d-r+2$. So, our previous observation on the Betti numbers $\beta_{k,l}(S/L)$ yields a short exact sequence

$$0 \to \operatorname{Tor}_i^S(K, S/I \cap L)_{i+d-r+2} \to \operatorname{Tor}_i^S(K, S/I)_{i+d-r+2} \to K^{\binom{r-2}{i-1}} \to 0 \text{ for all } i \geq 1,$$

which shows that $\beta_{i,d-r+2}(S/I) = \beta_{i,d-r+2}(S/I \cap L) + \binom{r-2}{i-1}$, and this proves our claim.
(b): As already said above, we have $\operatorname{reg}(S/I \cap L) = \operatorname{reg}(Y) - 1 \leq d-r+2$, whence $\beta_{i,j}(Y) = 0$ for all $i \geq 1$ and all $j \geq d-r+3$. From this we conclude by statement (a). ∎

6.6. **Notation and Reminder.** Let $T = \bigoplus_{n\in\mathbb{Z}} T_n$ be a graded S-module. Then, we denote the *socle* of T by $\mathrm{Soc}(T)$, thus:

$$\mathrm{Soc}(T) := (0 :_T S_+) \cong \mathrm{Hom}_S(K, T) = \mathrm{Hom}_S(S/S_+, T).$$

Keep in mind that the socle of a graded Artinian S-module T is a K-vector space of finite dimension which vanishes if and only if T does.

6.7. **Lemma.** *Let the notations and hypotheses be as in Lemma 6.5 and let $I \subset S$ denote the homogeneous vanishing ideal of X. Then we have the following statements*

(a) $\mathrm{Soc}(H^1(S/I))(-r-1) \cong \mathrm{Tor}^S_r(K, S/I)$.
(b) *If* $\mathrm{depth}(X) = 1$, *then* $H^1\big(\mathbb{P}^r, \mathcal{J}_X(N(X))\big) \cong \mathrm{Tor}^S_r(K, S/I)_{N(X)+r+1}$.
(c) $N(X) \leq d - r$ *if and only if* $\beta_{r,d-r+2}(X) = 0$.

Proof. (a): If $\mathrm{depth}(X) > 1$, both of the occurring modules vanish and our claim is obvious. So, we assume that $\mathrm{depth}(X) = 1$ and consider the total ring of sections $D := D_{S_+}(S/I) = \bigoplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{O}_X(n))$ of X, as well as the short exact sequence

$$0 \longrightarrow S/I \longrightarrow D \longrightarrow H^1(S/I) \longrightarrow 0.$$

We apply the Koszul functor $K(\underline{x}; \bullet)$ with respect to the sequence $\underline{x} := x_0, x_1, \ldots, x_r$ to this sequence and end up in homology with an exact sequence

$$H_{r+1}(\underline{x}; D) \to H_{r+1}(\underline{x}; H^1(S/I)) \to H_r(\underline{x}; S/I) \to H_r(\underline{x}; D).$$

As $\mathrm{depth}(D) > 1$ the first and the last module in this sequence vanish, so that

$$H_{r+1}(\underline{x}; H^1(S/I)) \cong H_r(\underline{x}; S/I).$$

As the Koszul complex $K(\underline{x}, S)$ provides a free resolution of $K = S/S_+$ and $K(\underline{x}; S/I) \cong K(\underline{x}; S) \otimes_S S/I$ we have $H_r(\underline{x}; S/I) \cong \mathrm{Tor}^S_r(K, S/I)$. As the sequence $\underline{x}$ has length $r+1$, we have $H_{r+1}(\underline{x}; H^1(S/I)) \cong \mathrm{Soc}(H^1(S/I))(-r-1)$. Altogether, we now obtain statement (a).

(b): As $N(X) = \mathrm{end}(H^1(S/I))$, we have

$$H^1\big(\mathbb{P}^r, \mathcal{J}_X(N(X))\big) \cong H^1(S/I)_{N(X)} = \mathrm{Soc}(H^1(S/I))_{N(X)}.$$

Now, our claim follows immediately by statement (a).

(c): If $\mathrm{depth}(X) > 1$ we have $N(X) = -\infty$ and $\beta_{r,d-r+2}(X) = 0$, so that our claim is true. We thus may assume that $\mathrm{depth}(X) = 1$. As $\mathrm{reg}(X) = d-r+3$ we have $N(X) \leq d-r+1$ and $\mathrm{Tor}^S_r(K, S/I)_{r+l} = 0$ for all $l \geq d-r+3$. Now, we may conclude by statement (b). □

Now, we are ready to give the announced main result.

6.8. **Theorem.** *Let $5 \leq r < d$ and assume that the surface $X \subset \mathbb{P}$ is non-conic, has degree d and is of maximal sectional regularity.*

(a) *The following statements are equivalent:*
 (i) $N(X) \leq d - r$.
 (ii) $\mathbb{F}(X) = \mathbb{P}^2$ *and* $\mathrm{reg}(X \cup \mathbb{F}(X)) \leq d - r + 2$.
 (iii) $\beta_{i,d-r+2}(X) = \binom{r-2}{i-1}$ *for all* $i \geq 1$.
 (iv) $\mathbb{F}(X) = \mathbb{P}^2$ *and* $\beta_{r,d-r+2}(X) = 0$.
(b) *The following statements are equivalent:*

(i) $\beta_{1,d-r+2}(X) = 1$.
(ii) $\mathbb{F}(X) = \mathbb{P}^2$ *and* $I \cap L = (I_{\leq d-r+2})$, *where* I *and* L *are the homogeneous vanishing ideals of* X *respectively of* $\mathbb{F}(X)$ *in* S.

Moreover, if the equivalent statements (b) (i) and (ii) hold, then we have (in the notations introduced in Notation and Reminder 2.4 and Definition and Remark 5.1):
(1) *If* $\mathbb{L} \in \Sigma^{\circ}_{d-r+3}(X)$, *then* $\mathbb{L} \subset \mathbb{F}(X)$.
(2) $\mathrm{Sec}_{d-r+3}(X) = X \cup \mathbb{F}(X)$.
(3) $\mathbb{F}^{+}(X) = \mathbb{F}(X)$.

(c) *The equivalent conditions (i) – (iv) of statement (a) imply the equivalent conditions (i) and (ii) (and hence also the claims (1), (2) and (3)) of statement (b).*

Proof. (a): (i) ⇒ (ii): Let $N(X) \leq d-r$. It follows by Theorem 5.14 (c), that $\mathbb{F}(X) = \mathbb{P}^2$. Let I and $L \subset S$ respectively denote the homogeneous vanishing ideals of X and $\mathbb{F}(X)$. According to Theorem 6.3 (d)(1) we have $\mathrm{end}(H^1(S/I \cap L) = \mathrm{end}(H^1(S/I) = N(X) \leq d-r$. So, it follows by Theorem 6.3 (d)(3),(4) that $\mathrm{reg}(S/I \cap L) \leq d-r+1$, whence $\mathrm{reg}(X) \cup \mathbb{F}(X) = \mathrm{reg}(I \cap L) \leq d-r+2$.

(ii) ⇒ (i): As $\mathrm{end}(H^1(S/I)) = N(X)$, this is an easy consequence of Theorem 6.3 (d)(1),(3) and (4).

(ii) ⇒ (iii): This follows by Lemma 6.5.

(iii) ⇒ (ii): Assume that statement (iii) holds. Then we have in particular that $\beta_{1,d-r+2}(X) = 1$. By Theorem 5.14 (d) it follows that $\mathbb{F}(X) = \mathbb{P}^2$. Now, we may again conclude by Lemma 6.5.

(iii) ⇔ (iv): This is clear by Lemma 6.7.

(b): (i) ⇒ (ii): Assume that $\beta_{1,d-r+2}(X) = 1$. Then Theorem 5.14 (d) implies that $\mathbb{F}(X) = \mathbb{P}^2$. If follows by Theorem 6.3 (c) that $(I_{\leq d-r+2}) = I \cap L$.

(ii)⇒(i): This follows immediately by Theorem 6.3 (c).

Assume now, that the equivalent statements (i) and (ii) hold, let $\mathbb{L} \in \Sigma_{d-r+3}(X)$ be not contained in X and let $M \subset S$ be the homogeneous vanishing ideal of $\mathbb{L}$. Then, $(I_{d-r+2}) \subset M$. As $I \cap L = (I_{\leq d-r+2})$ it follows that $I \cap L \subset M$. As $\mathbb{L}$ is not contained in X, the ideal I is not contained in M. It follows that $L \subset M$, and hence that $\mathbb{L} \subset \mathbb{F}(X)$. This proves claim (1). Claim (2) is immediate by claim (1), as X is a union of lines and each line $\mathbb{L} \subset \mathbb{P}^r$ with $\#(X \cap \mathbb{L}) > d-r+3$ is contained in X. Now claim (3) is obvious, too.

(c): This follows from the fact that statement (a)(iii) implies that $\beta_{1,d-r+2}(X) = 1$. □

We have seen above, that surfaces X of maximal sectional regularity and sub-maximal index of normality $N(X) < d-r+1$ (see Notation and Reminder 5.12) have a planar extremal variety and show an interesting behavior of Betti numbers. We therefore can expect, that in the extremal case $N(X) = -\infty$ – hence in the case $\mathrm{depth}(X) = 2$ – we get further detailed information on the Betti numbers if X is of “small degree“. Our next main result is devoted to this case.

6.9. **Theorem.** *Let* $5 \leq r < d$, *assume that the surface* $X \subset \mathbb{P}^r$ *of degree* d *is of maximal sectional regularity and satisfies* $\mathrm{depth}(X) = 2$. *Let* $I \subset S$ *be the homogeneous vanishing ideal of* X *and let* $J := (I_{\leq d-r+2}) \subset S$ *be the ideal generated by all polynomials of degree* $\leq d-r+2$ *in* I.

(a) $\mathbb{F}(X) = \mathbb{P}^2$ *and* $J = I \cap L$, *where* $L \subset S$ *is the homogeneous vanishing ideal of* $\mathbb{F}(X)$ *in* S.

(b) *If the ring* S/J *is Cohen-Macaulay – hence if* $\tau(X) = (2,3)$ *– we have* $\operatorname{reg}(S/J) = \operatorname{reg}(X \cup \mathbb{F}(X)) - 1 = 2$ *and*

$$h^2(\mathbb{P}^r, \mathcal{J}_X(n)) = \begin{cases} \mathrm{e}(X) = \binom{d-r+2}{2} \text{ for all } n \leq 0, \\ \binom{d-r-n+2}{2} \text{ for } 0 < n \leq d-r, \\ 0 \text{ for } n > d-r. \end{cases}$$

(c) *Let* $d \leq 2r-5$. *Then, setting*

$$a_i := (d-r+1)\binom{r-1}{i} + \binom{r-2}{i-1}, \quad c_i := (d-1)\binom{r-2}{i} - \binom{r-2}{i+1},$$

for all $i \in \{1, \ldots, r\}$, *we have*

$$\operatorname{Tor}_i^S(K, S/I) = K^{u_i}(-i-1) \oplus K^{v_i}(-i-2) \oplus K^{\binom{r-2}{i-1}}(-i-d-r-2)$$

with

$$\begin{aligned} u_1 &= \binom{r}{2} - d - 1, \\ u_i &= c_i - a_i \qquad && \text{for } 2 \leq i \leq 2r-d-3, \\ u_i &\leq c_i && \text{for } 2r-d-2 \leq i \leq r-1. \end{aligned}$$

and

$$\begin{aligned} v_i &= 0 && \text{for } 1 \leq i \leq 2r-d-4 \text{ and } i = r, \\ v_i &= u_{i+1} + a_{i+1} - c_{i+1} \qquad && \text{for } 2r-d-3 \leq i \leq r-3, \\ v_{r-2} &= d-r. \end{aligned}$$

Proof. (a): This is clear by Theorem 6.8.

(b): Let S/J be Cohen Macaulay. Then, according to statement (a) $S/I \cap L$ is Cohen Macaulay. Now, let $h \in \mathbb{U}(X)$. As the ring $S/I \cap L$ is Cohen Macaulay and h is a non-zero divisor with respect to $S/I \cap L$, it follows that the homogeneous coordinate ring $S/\langle I \cap L, h\rangle$ of $(X \cup \mathbb{F}(X)) \cap \mathbb{H}_h = C_h \cup \mathbb{L}_h$ is Cohen-Macaulay. So, by [5, Theorem 3.3] we have $\operatorname{reg}(S/\langle J, h\rangle) = \operatorname{reg}(S/\langle I \cap L, h\rangle) = 2$ and the fact that h is a non-zero divisor with respect to $S/I \cap L = S/J$ implies that $\operatorname{reg}(S/J) = 2$. By our hypothesis we also have $H^2_*(\mathbb{P}^r, \mathcal{J}_{X \cup \mathbb{F}(X)}) = H^2(S/I \cap L) = 0$. So, by Theorem 6.3 (e) (1) and Theorem 6.9 (b)(2) the values of $h^2(\mathbb{P}^r, \mathcal{J}_X(n))$ are as stated above.

(c): Let $d \leq 2r-5$ and let $h \in \mathbb{U}(X)$. As $\operatorname{depth}(X) > 1$, the ring $S/\langle I, h\rangle$ is the homogeneous coordinate ring of the curve C_h in S. As h is a non-zero divisor with respect to S/I, we have $\beta^S_{i,j}(S/I) = \beta_{i,j}(S/\langle I, h\rangle$ for all $i, j \geq 1$. Now our claim follows immediately by the approximation of the Betti numbers $\beta_{i,j}(S/\langle I, h\rangle) = \beta_{i,j}(C_h)$ of the homogeneous the curve of maximal regularity $C_h \subset \mathbb{H}_h$ given in [8, Theorem 1.2]. $\square$

We now briefly revisit the special case of surfaces $X \subset \mathbb{P}^r$ of degree $r+1$.

6.10. **Remark.** (s. [4], [9]) (A) Assume that $r \geq 5$ and let our surface $X \subset \mathbb{P}^r$ be of degree $r+1$. Then, we can distinguish 9 cases, which show up by their numerical invariants as presented in the following table. Here $\sigma(X)$ denotes the *sectional genus* of X, that is the arithmetic genus of the generic hyperplane section curve $C_h \quad (h \in \mathbb{U}(X))$ or equivalently, the sectional genus of the polarized surface $(X, \mathcal{O}_X(1))$ in the sense of Fujita [18].

Case	sreg(X)	depth(X)	$\sigma(X)$	e(X)	$h^1_A(1)$	$h^1_A(2)$
1	2	3	2	0	0	0
2	3	2	1	0	0	0
3	3	2	1	1	0	0
4	3	1	1	0	1	0
5	3	2	0	2	0	0
6	3	1	0	1	1	≤ 1
7	3	1	0	0	2	≤ 2
8	4	2	0	3	0	0
9	4	1	0	0	2	3

The case 9 occurs only if $r = 5$. In [4] and [9] we listed indeed two more cases 10 and 11, of which we did not know at that time, whether they might occur at all. For these two cases we had $\text{sreg}(X) = 4 = d - r + 3$ and $\text{e}(X) \in \{1, 2\}$. As these surfaces would be of maximal sectional regularity, this would contradict Corollary 6.4. So, surfaces which fall under the cases 10 and 11 cannot occur at all. In the case 9 we have $\text{e}(X) = 0$, and hence by Corollary 6.4 and Theorem 5.14 we must have $r = 5$ and $\mathbb{F}(X) = S(1, 1, 1)$ in this case.

(B) In view of Theorem 4.3, the surfaces of types 8 and 9 are of particular interest, as they are the ones of maximal sectional regularity within all the 9 listed types. Observe, that among all surfaces X of degree $r+1$ in $\mathbb{P}^r$, those of type 8 are precisely the ones X which are of maximal sectional regularity and of arithmetic depth ≥ 2. If $r \geq 6$, the surfaces of type 8 are precisely the ones which are of maximal sectional regularity.

(C) Observe, that in the cases 5 – 9 we have $\sigma(X) = 0$. This means, that the surfaces which fall under these 5 types are all sectionally rational and have finite non-normal locus. So, by Theorem 3.3, these surfaces are almost non-singular projections of a rational normal surface scroll $\widetilde{X} = S(a, r+1-a)$ with $0 \leq a \leq \frac{r+1}{2}$, even if they are cones (see [8, Corollary 5.11] for the non-conic case). So, according to Theorem 3.8 the surfaces X of types 5 – 9 all satisfy the Eisenbud-Goto inequality $\text{reg}(X) \leq 4$, with equality in the cases 8 and 9 (see Theorem 4.3 (b)). In the cases 1 – 5, the values of $h^i(\mathbb{P}^r, \mathcal{J}_X(n)) =: h^i(S/I)_n \quad (i = 1, 2, n \in \mathbb{Z})$ (see [9, Reminder 2.2 (C),(D)]) show, that $\text{reg}(X) = 3$. In the case 6 we may have $\text{reg}(X) = 3$ whereas in the case 7, we know even that $\text{reg}(X)$ my take both values 3 and 4 (see [9, Example 3.5, Examples 3.4 (A),(B),(C)]). This shows in particular, that there are sectionally rational surfaces $X \subset \mathbb{P}^r$ of degree $r+1$ with finite non-normal locus and $\text{sreg}(X) < \text{reg}(X)$.

We now make explicit the Betti numbers of the surfaces $X \subset \mathbb{P}^r$ of degree $r+1$ which are of maximal sectional regularity and of arithmetic depth 1, thus of the surfaces which fall under the type 8 of Remark 6.10.

6.11. **Corollary.** *Assume that the surface $X \subset \mathbb{P}^r_K$ is of degree $r+1$. Then*

(a) *The following conditions are equivalent:*
  (i) *The surface X is of type 8.*
  (ii) $\mathrm{e}(X) = 3$.
  (iii) $\mathrm{sreg}(X) = 4$ *and* $\mathrm{depth}(X) = 2$.
  (iv) $\mathrm{sreg}(X) = 4$ *and X does not fall under the case 9 of Remark 6.10.*

(b) *If the above equivalent conditions hold, and in the notations of Theorem 6.9 for all $i \in \{1, \ldots, r\}$ we have*

$$\mathrm{Tor}_i^S(K, S/I) = K^{u_i}(-i-1) \oplus K^{v_i}(-i-2) \oplus K^{\binom{r-2}{i-1}}(-i-3)$$

with

$$u_1 = \binom{r-1}{2} - 3,$$
$$u_i = (r-1)\binom{r-2}{i} - \binom{r-2}{i+1} - 3\binom{r-2}{i-1} \text{ for } 2 \le i \le r-4,$$
$$u_{r-3} \in \{0, r-2\} \text{ and } \quad u_i = 0 \text{ for } i \ge r-2.$$

and

$$v_i = 0 \text{ for } 1 \le i \le r-5 \text{and } i \ge r-1,$$
$$v_{r-4} = u_{r-3} + (r-1)\binom{r-2}{2} - 3\binom{r-2}{3} - r + 2,$$
$$v_{r-3} = 2r-4,$$
$$v_{r-2} = 3.$$

Proof. Statement (a) follows easily on use of the table in Remark 6.10 (A).
(b): As $\mathrm{depth}(X) = 2$, the surface X has the same Betti numbers as its hyperplane section curve $C_h \subset \mathbb{H}_h = \mathbb{P}^{r-1}$ ($h \in \mathbb{U}(X)$). As X is of maximal sectional regularity, we may apply [6, Theorem 6.12] to the curve C_h in order to obtain information on the Betti numbers of X. In particular we get that $u_{r-3} \in \{0, r-3\}$. If $r = 5$ our claim follows easily from the mentioned theorem of [6]. If $r \ge 6$ we may conclude directly by Theorem 6.9 (d). □

The last main result of this section characterizes non-conic surfaces of maximal sectional regularity in terms of projections of smooth rational surface scrolls which are generically injective along appropriate effective divisors on these scrolls. We first prove an auxiliary result.

6.12. **Lemma.** *Let $d > 2, s > 1$, let $\widetilde{X} \subset \mathbb{P}^{d+1}$ be a smooth rational normal surface scroll and let $\mathbb{K} = \mathbb{P}^s \subset \mathbb{P}^{d+1}$ be such that $\widetilde{X} \cap \mathbb{K} \subset \mathbb{K}$ is a subscheme of dimension 1 and degree $\ge s$. Then*

$$\deg(\widetilde{X} \cap \mathbb{K}) = s \text{ and } \widetilde{X} \cap \mathbb{K} \in \mathrm{Div}(\widetilde{X}).$$

Proof. Assume, that $C := \widetilde{X} \cap \mathbb{K} \notin \mathrm{Div}(\widetilde{X})$. As $\widetilde{X}$ is smooth, this means that C is not a Cartier divisor so that the local vanishing ideal $\mathcal{J}_{C,\widetilde{x}} \subset \mathcal{O}_{\widetilde{X},\widetilde{x}}$ of C is not principal for some closed point $\widetilde{x} \in C$. As $\mathcal{O}_{\widetilde{X},\widetilde{x}}$ is a local factorial domain of dimension 2 and $\mathrm{height}(\mathcal{J}_{C,\widetilde{x}}) = 1$, it follows that

$$\mathfrak{m}_{\widetilde{X},\widetilde{x}} \in \mathrm{Ass}_{\mathcal{O}_{\widetilde{X},\widetilde{x}}}\big(\mathcal{O}_{\widetilde{X},\widetilde{x}}/\mathcal{J}_{C,\widetilde{x}}\big)$$

and hence that $\widetilde{x} \in \mathrm{Ass}(\mathcal{O}_{\widetilde{X}}/\mathcal{J}_C) = \mathrm{Ass}_C(\mathcal{O}_C)$. Now, there is a space $\mathbb{H} = \mathbb{P}^{s-1} \subset \mathbb{K} = \mathbb{P}^s$ such that $\widetilde{x} \in \mathbb{H}$ and $\dim(\widetilde{X} \cap \mathbb{H}) = 0$. According to Lemma 6.1 we now get

$$\infty > \#(\widetilde{X} \cap \mathbb{H}) = \#(C \cap \mathbb{H}) > s.$$

As $\widetilde{X} \subset \mathbb{P}^{d+1}$ is a smooth rational normal scroll and $\mathbb{H} \subset \mathbb{P}^{d+1}$ is an $(s-1)$-space, this is a contradiction, and hence $C \in \mathrm{Div}(\widetilde{X})$.

Assume now, that $\deg(C) > s$. Then for a general $(s-1)$-space $\mathbb{H} \subset \mathbb{K}$ we have again the previous inequalities and hence a contradiction. Therefore $\deg(\widetilde{X} \cap \mathbb{K}) = \deg(C) = s$. □

6.13. **Theorem.** *Let $5 \leq r < d$ and assume that the surface $X \subset \mathbb{P}$ is non-conic, has degree d and is of maximal sectional regularity.*

(a) *The following statements are equivalent:*
   (i) $\mathbb{F}(X) = \mathbb{P}^2$.
   (ii) *$X = \widetilde{X}_\Lambda$, where $\widetilde{X} \subset \mathbb{P}^{d+1}$ is a smooth rational normal surface scroll and $\Lambda = \mathbb{P}^{d-r}$ is disjoint to $\widetilde{X}$ and contained in the linear span $\langle D \rangle = \mathbb{P}^{d-r+3} \subset \mathbb{P}^{d+1}$ of a divisor $D \in |H + (3-r)F|$ (where H and F are defined as in Notation and Reminder 5.10) such that the restriction*

$$\pi'_\Lambda \restriction: \langle D \rangle \setminus \Lambda \twoheadrightarrow \mathbb{P}^2$$

   of the projection map $\pi'_\Lambda : \mathbb{P}^{d+1} \setminus \Lambda \twoheadrightarrow \mathbb{P}^r$ is generically injective along D.

(b) *If the equivalent conditions (i) and (ii) of statement (a) hold, we have*
   (1) $\langle D \rangle = \mathbb{E} := \overline{(\pi'_\Lambda)^{-1}\big(\mathbb{F}(X)\big)}$;
   (2) *$D = (\pi_\Lambda)^{-1}\big(X \cap \mathbb{F}(X)\big) = \widetilde{X} \cap \mathbb{E} = C + \sum_{j=1}^{s} m_j \mathbb{L}_j$, where $C \subset \widetilde{X}$ is a curve section, $\mathbb{L}_1, \ldots, \mathbb{L}_s$ are pairwise different ruling lines of $\widetilde{X}$ and $m_1, \ldots, m_s$ are positive integers with $\sum_{j=1}^{s} m_j = d - r - \deg(C) + 3$.*

Proof. : (a): (i) ⇒ (ii): According to Theorem 6.9 (a) (3) we may write $X = \widetilde{X}_\Lambda$, where $S(a, d-a) = \widetilde{X} \subset \mathbb{P}^{d+1}$ is a smooth rational surface scroll and $\Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1}$ is disjoint to X and the induced projection morphism $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$ is almost non-singular.

Now, let $\mathbb{E} := \overline{(\pi'_\Lambda)^{-1}(\mathbb{F}(X))} = \mathbb{P}^{d-r+3} \subset \mathbb{P}^{d+1}$. Then $D := \widetilde{X} \cap \mathbb{E} = (\pi_\Lambda)^{-1}(X \cap \mathbb{F}(X)) \subset \mathbb{E}$ is a subscheme of dimension 1 and degree $\geq d - r + 3$. By Lemma 6.12 it follows that $D \in \mathrm{Div}(\widetilde{X})$ and $D \subset \mathbb{E}$ is of degree $d - r + 3$.

Now, we write

$$D = \sum_{i=1}^{t} n_i C_i + \sum_{j=1}^{s} m_j \mathbb{L}_j$$

with $s, t \in \mathbb{N}_0$, $n_1, \ldots, n_t; m_1, \ldots, m_s \in \mathbb{N}$, with pairwise distinct prime divisors $C_i \in \mathrm{Div}(\widetilde{X})$ and pairwise distinct ruling lines $\mathbb{L}_1, \ldots, \mathbb{L}_s$ of $\widetilde{X}$ such that

$$\sum_{i=1}^{t} n_i \deg(C_i) + \sum_{j=1}^{s} m_j = \deg(D) = d - r + 3.$$

Now, $t = 0$ would imply that $\sum_{j=1}^{s} m_j = d - r + 3$, and hence that $d - r + 3 < \dim(\langle \sum_{j=1}^{s} m_i \mathbb{L}_j \rangle) = \dim(\langle D \rangle) \leq \dim(\mathbb{E}) = d - r + 3$. This contradiction shows that $t > 0$.

As $D = \widetilde{X} \cap \mathbb{E}$ and each of the curves C_i intersects all ruling lines $\mathbb{L}$ of $\widetilde{X}$, we must have $t = 1$. So, writing $C := C_1$ and $n := n_1$ we obtain $D = nC + \sum_{j=1}^{s} m_j \mathbb{L}_j$ with $n > 0$. As $\langle C \rangle \subseteq \langle D \rangle$ is of dimension $\leq d$, the curve $C \subset \widetilde{X}$ is a section of $\widetilde{X}$.

Moreover, we must have $n = 1$. Otherwise we would get $\mathrm{mult}_C(\widetilde{X} \cap \mathbb{E}) = n > 1$, so that the tangent plane $\mathrm{T}_p(\widetilde{X})$ of $\widetilde{X}$ at a general point $p \in C$ would be a 2-plane contained in $\mathbb{E}$. But this would imply the contradiction, that a general ruling line $\mathbb{L}$ of $\widetilde{X}$ is contained in $\mathbb{E} = \mathbb{P}^{d-r+3}$. Therefore, indeed $n = 1$, and hence

$$D = C + \sum_{j=1}^{s} m_j \mathbb{L}_j \text{ with } \sum_{j=1}^{s} m_j = d - r + 3 - c, \text{ where } c := \deg(C).$$

It follows, that D is linearly equivalent to $C + (d - r + 3 + c)F$ – with $C \cdot F = 1$ – and hence that $D \in |H + (3 - r)F|$.

In particular $\langle D \rangle \subset \mathbb{E}$ is of dimension $d - r + 3$, so that $\langle D \rangle = \mathbb{E} = \mathbb{P}^{d-r+3}$. As $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$ is almost non-singular, the restriction $\pi'_\Lambda \restriction : \mathbb{E} = \langle D \rangle \twoheadrightarrow \mathbb{F}(X) = \mathbb{P}^2$ of $\pi'_\Lambda : \mathbb{P}^{d+1} \twoheadrightarrow \mathbb{P}^r$ is generically injective along D.

(ii) $\Rightarrow$ (i): Let $X = \widetilde{X}_\Lambda$, where

$$\widetilde{X} \subset \mathbb{P}^{d+1}, \quad \Lambda = \mathbb{P}^{d-r} \subset \langle D \rangle = \mathbb{P}^{d-r+3} \text{ and } D \in |H + (3 - r)F|$$

are as in statement (ii). Then $D \subset \widetilde{X} \cap \mathbb{E}$ is of dimension 1 and of degree $d - r + 3$, and $\mathbb{F} := \pi'_\Lambda(\mathbb{E} \setminus \Lambda)$ is a 2-plane in $\mathbb{P}^r$. As $\pi'_\Lambda \restriction \langle D \rangle \twoheadrightarrow \mathbb{F}$ is generically injective, the projection map $\pi_\Lambda : \widetilde{X} \twoheadrightarrow X$ is birational and hence $\pi_\Lambda(D) \subset X \cap \mathbb{F}$ is of dimension 1 and of degree $d - r + 3$. Therefore $X \cap \mathbb{F}$ is of dimension 1 and of degree $d - r + 3$. By Lemma 5.3 (ii) it follows that $\mathbb{F}(X) = \mathbb{F} = \mathbb{P}^2$.

(b): The proof of statement (a) shows in particular that condition (i) of that statement implies the two claims (1) and (2) of statement (b). □

7. Examples and Problems

The central aim of this paper is to investigate non-degenerate irreducible surfaces $X \subset \mathbb{P}^r$ of degree d with maximal sectional regularities, hence surfaces for which the dimension $\mathfrak{d}(X)$ of the set $\Sigma^\circ_{d-r+3}(X)$ of proper extremal secant lines takes its maximally possible value. Our first aim in this present section is to provide examples of smooth surfaces of extremal regularity for which $\mathfrak{d}(X) \in \{-1, 0, 1\}$. In particular we shall see that there are many such surfaces with $\mathfrak{d}(X) = -1$, that is without extremal secant lines at all. The fact, that there are a lot of surfaces of extremal regularity without extremal secant lines does not correspond to the general expectation which arose by the work of Gruson-Lazarsfeld-Peskine [20].

7.1. **Construction and Examples.** (A) Let $a, b, d \in \mathbb{N}$ with $a \leq b$, let $r := a + b + 3$, assume that $d > r$ and consider the smooth threefold rational normal scroll of degree $a + b + 1 = r - 2$

$$Z := S(1, a, b) \subset \mathbb{P}^r.$$

Let $H, F \in \mathrm{Div}(Z)$ respectively be a hyperplane section and a ruling plane of Z, so that each divisor on Z is linearly equivalent to $mH + nF$ for some integers m, n. Let $X \subset \mathbb{P}^r$ be an non-degenerate irreducible surface of degree d which is contained in Z as a divisor linearly equivalent to $H + (d - r + 2)F$. Then one can easily see that

$$h^0(X, \mathcal{O}_X(1)) = h^0(Z, \mathcal{O}_Z(1)) + d - r + 1 = d + 2$$

This means that the linearly normal embedding $X \subset \mathbb{P}^{d+1}$ of X by means of $\mathcal{O}_X(1)$ is of minimal degree and X is the image of a surface $\widetilde{X} \subset \mathbb{P}^{d+1}$ of minimal degree under an isomorphic linear projection $\widetilde{X} \stackrel{\cong}{\to} X$, hence

$$X = \widetilde{X}_\Lambda, \text{ with } \Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1} \text{ and } \Lambda \cap \mathrm{Sec}(\widetilde{X}) = \emptyset.$$

Keep in mind, that $\widetilde{X}$ is either a smooth rational normal surface scroll, a cone over a rational normal curve or the Veronese surface in $\mathbb{P}^5$. As $d + 1 > 5$ and as a cone does not admit a proper isomorphic linear projection, $\widetilde{X} \subset \mathbb{P}^{d+1}$ is a smooth rational normal surface scroll. In particular X must be smooth and sectionally rational. In addition, X is not $(d-r+1)$-normal (see Lemma 5.13) and hence $\mathrm{reg}(X) \geq d-r+3$. By Corollary 3.6 it follows that $\mathrm{reg}(X) = d - r + 3$. In particular, X is a surface of extremal regularity.

Moreover, each line section $\mathbb{L}$ of Z intersects the divisor $H + (d - r + 2)F$ in $d - r + 2$ ruling planes and the hyperplane section H. As $X \in |H + (d - r + 2)F|$ it follows that $\mathbb{L}$ is either contained in X or else a proper $(d - r + 3)$-secant line. Conversely, each proper $(d-r+3)$-secant line to $X \in |H+(d-r+2)F|$ must intersect $d-r+2$ ruling planes and the hyperplane section H, and hence must be contained in Z as a line section. Therefore we have

$$\Sigma^\circ_{d-r+3}(X) = \{\mathbb{L} \in \mathbb{G}(1, \mathbb{P}^r) \mid \mathbb{L} \text{ is a line section of } Z \text{ and } \mathbb{L} \nsubseteq X\}.$$

(B) Suppose first that $a \geq 2$ and that the unique line section $S(1) = \mathbb{L}$ of Z is contained in X. Then, according to the observation made in part (A), we have

$$\Sigma^\circ_{d-r+3}(X) = \emptyset, \text{ and hence } \mathfrak{d}(X) = {}^*\mathfrak{d}(X) = -1.$$

So, in this case $X \subset \mathbb{P}^r$ is a smooth surface of extremal regularity having no proper extremal secant line at all.

(C) Suppose now that yet $a \geq 2$, but that the unique line section $S(1) = \mathbb{L}$ of Z is not contained in X. Then, according to part (A) we have

$$\Sigma^\circ_{d-r+3}(X) = \{\mathbb{L}\}, \text{ and hence } \mathfrak{d}(X) = {}^*\mathfrak{d}(X) = 0.$$

In particular, in this case we and also have

$${}^*\Sigma^\circ_{d-r+3}(X) = \{\mathbb{L}\}.$$

(D) Suppose next, that $a = 1$ and $b \geq 2$. Then, by part (A)

$$\Sigma^{\circ}_{d-r+3}(X) = \{\mathbb{L} \in \mathbb{G}(1.\mathbb{P}^r) \mid \mathbb{L} \subset S(1,1) \text{ and } \mathbb{L} \nsubseteq X\}.,$$

So, in this case the set of proper extremal secant lines $\Sigma^{\circ}_{d-r+3}(X)$ to X is obtained from the set $\{\mathbb{L} \in \mathbb{G}(1.\mathbb{P}^r) \mid \mathbb{L} \subset S(1,1)\}$ of all line sections of Z by removing finitely many lines. Therefore

$$\mathfrak{d}(X) = {}^*\mathfrak{d}(X) = 1,$$

and for the closed union of all proper extremal secant to X lines it holds

$$\mathbb{F}^+(X) = \overline{\bigcup_{\mathbb{L} \in \Sigma^{\circ}_{d-r+3}(X)} \mathbb{L}} = S(1,1) \subset Z.$$

(E) Suppose finally that $a = b = 1$ and hence $r = 5$, so that $Z = S(1,1,1) \subset \mathbb{P}^5$. By statement (A) we now have

$$\Sigma^{\circ}_{d-r+3}(X) = \{\mathbb{L} \in \mathbb{G}(1.\mathbb{P}^r) \mid \mathbb{L} \subset S(1,1,1) = Z \text{ and } \mathbb{L} \nsubseteq X\}.$$

Now, the set $\Sigma^{\circ}_{d-r+3}(X)$ of all proper extremal secant lines to X is obtained by removing from the two-dimensional set $\{\mathbb{L} \in \mathbb{G}(1.\mathbb{P}^r) \mid \mathbb{L} \subset S(1,1,1) = Z\}$ of all lines contained in Z the at most one-dimensional family of all lines $\mathbb{L}$ contained in X. Therefore

$$\mathfrak{d}(X) = {}^*\mathfrak{d}(X) = 2,$$

and the extended extremal variety of X – thus the closed union of all proper extremal secant lines to X – coincides with the extremal variety of X, whence

$$\mathbb{F}(X) = \mathbb{F}^+(X) = \overline{\bigcup_{\mathbb{L} \in \Sigma^{\circ}_{d-r+3}(X)} \mathbb{L}} = S(1,1,1) = Z.$$

Observe, that now the surface X is of maximal sectional regularity and falls under the exceptional case in which the extremal variety $\mathbb{F}(X)$ of X is not a plane.

Next, we aim to present examples which concern surfaces X of maximal sectional regularity which fall under the general case, in which the extremal variety $\mathbb{F}(X)$ of X is a plane. So, let $5 \leq r < d$ and let $X \subset \mathbb{P}^r$ be a non-degenerate irreducible surface of degree d, which is of maximal sectional regularity. We suppose that $\mathbb{F}(X) = \mathbb{P}^2$ and set $Y := X \cup \mathbb{F}(X)$. Then, Theorem 6.3 (f) can be tabulated as follows where $\tau(X)$ denotes the pair $(\operatorname{depth}(X), \operatorname{depth}(Y))$ of the arithmetic depths of X and Y:

d	$r+1 \leq d \leq 2r-4$	$2r-3 \leq d \leq 3r-7$	$d \geq 3r-6$
$\tau(X)$	$(2,3)$	$(1,1), (2,2), (2,3)$	$(1,1), (2,2)$

The aim of this section is to provide examples of surfaces $X \subset \mathbb{P}^r$ of maximal sectional regularity, having all possible $\tau(X)$ listed in the above table. Throughout this section we assume that the characteristic of K is zero.

7.2. **Construction and Examples.** (A) We throughout assume that the characteristic of the base field K is zero. Let a, b be integers such that $3 \leq a \leq b$ and consider the standard smooth rational normal surface scroll

$$\widetilde{X} := S(a,b) \subset \mathbb{P}^{a+b+1}.$$

We shall construct our examples by varying a and b and by projecting $S(a,b)$ from appropriate linear subspaces of $\mathbb{P}^{a+b+1}$. The occurring Betti diagrams have been computed by means of the Computer Algebra System Singular [14].

(B) Let $a \leq b$ and let $X = \widetilde{X}_\Lambda \subset \mathbb{P}^{b+3}$ be the linear projection of $\widetilde{X} = S(a,b)$ from a general $(a-3)$-dimensional subspace $\Lambda = \mathbb{P}^{a-2}$ of $\langle S(a) \rangle = \mathbb{P}^a$. Observe that $X \subset \mathbb{P}^{b+3}$ is a non-degenerate irreducible surface of degree $a+b$. Moreover, the linear projection $\pi'_\Lambda : \mathbb{P}^{a+b+1} \backslash \Lambda \twoheadrightarrow \mathbb{P}^{b+3}$ in question maps the subspace $\mathbb{P}^a \subset \mathbb{P}^{a+b+1}$ onto a plane $\mathbb{P}^2 = \mathbb{F} \subset \mathbb{P}^{b+3}$, so that the the rational normal curve $S(a) = S(a,b) \cap \mathbb{P}^a \subset \mathbb{P}^a$ is mapped birationally onto the plane curve $C_a \cap \mathbb{F} \subset \mathbb{F} := \mathbb{P}^2$ of degree a. As $\deg(C_a) = a = a+b-(b+3)+3$, it follows by Lemma 5.3 (a) that $X \subset \mathbb{P}^{b+3}$ is of maximal sectional regularity and $\mathbb{F}(X) = \mathbb{F} = \mathbb{P}^2$. By Theorem 3.8 (a) we have $\operatorname{reg}(X) = a$. Finally, as $(b+3)+1 \leq a+b \leq 2(b+3)-4$ it follows from Theorem 6.3 (f) that

$$\tau(X) = (2,3).$$

(C) Assume that $b \geq 3$ and let $X = \widetilde{X}_\Lambda \subset \mathbb{P}^{a+3} = \mathbb{P}^r$ be the non-degenerate irreducible surface of degree $d := a+b$ which is obtained by a linear projection of $\widetilde{X} = S(a,b)$ from a general $(b-3)$-dimensional subspace Λ of $\langle S(b) \rangle = \mathbb{P}^b \subset \mathbb{P}^{a+b+1}$. The underlying projection $\pi'_\Lambda : \mathbb{P}^{a+b+1} \backslash \Lambda \twoheadrightarrow \mathbb{P}^r$ maps the space $\mathbb{P}^b$ onto a plane $\mathbb{F} = \mathbb{P}^2 \subset \mathbb{P}^{a+3}$ and induces again a birational map from the rational normal curve $S(b) = \widetilde{X} \cap \mathbb{P}^b$ onto the plane curve $C_b = X \cap \mathbb{F} \subset \mathbb{F} = \mathbb{P}^2$ of degree b. It follows as in part (B) that $X \subset \mathbb{P}^r$ is a surface of maximal sectional regularity, that $\mathbb{F}(X) = \mathbb{F} = \mathbb{P}^2$ and that $\operatorname{reg}(X) = b = d-r+3$.
Now, by Theorem 6.3 (f) we obtain:

$$\text{If } b \leq a+2, \text{ then } \tau(X) = (2,3).$$

(D) To provide examples for further pairs $\tau(X)$, we assume that $a+3 \leq b$ and we vary the center $\Lambda = \mathbb{P}^{b-3} \subset \mathbb{P}^b = \langle S(b) \rangle$ of our projection such the curve $C_b \subset \mathbb{F} = \mathbb{P}^2$ has the appropriate shape. To do so, we first consider the canonical isomorphisms

$$\kappa : \mathbb{P}^1 \stackrel{\cong}{\to} S(b), \quad (s:t) \mapsto (0 : \ldots : 0 : s^b : s^{b-1}t : \ldots : st^{b-1} : t^b) \in \mathbb{P}^{a+b+1}.$$

Then we choose a homogeneous polynomial $f \in K[s,t]$ of degree b which is not divisible by s and by t. Then we chose our center of projection $\Lambda = \mathbb{P}_{b-3} \subset \mathbb{P}^b = \langle S(b) \rangle$ such that the composition

$$\pi \circ \kappa : \mathbb{P}^1 \to C_b \subset \mathbb{F} = \mathbb{P}^2$$

of the previously defined map $\kappa : \mathbb{P}^1 \to S(b)$ with the induced finite birational projection morphism $\pi = \pi_\Lambda \restriction : S(b) \twoheadrightarrow C_b$ is given by

$$\pi \circ \kappa = [s^b, f, t^b] : \mathbb{P}^1 \longrightarrow \mathbb{P}^2, \quad \big((s:t) \mapsto (s^b : f(s,t) : t^b)\big).$$

We denote the corresponding projected image $\widetilde{X}_\Lambda \subset \mathbb{P}^r$ of $\widetilde{X} = S(a,b)$ by X_f and keep in mind that according to part (C) the surface $X_f \subset \mathbb{P}^{a+3} = \mathbb{P}^r$ is of degree $d = a+b$ and of maximal sectional regularity. Clearly, we may identify $\mathbb{F} \subset \mathbb{P}^{a+3} = \mathbb{P}^r$ with the subspace whose non-vanishing homogeneous coordinates sit at the last three places. If we do so, we may write

$$X_f := \{(us^a : us^{a-1}t : \ldots : ust^{a-1} : ut^a : vs^b : vf(s,t) : vt^b) \mid (s,t),(u,v) \in K^2 \setminus \{(0,0)\}\}.$$

After an appropriate choice of f, this latter presentation is accessible to syzygetic computations.

7.3. **Example.** Let $(a,b) = (3,5)$ and $f := s^4t + s^3t^2 + s^2t^3 + st^4$. Then $X_f \subset \mathbb{P}^6$ is of degree $d = 8 (= 2r - 4)$ and the graded Betti numbers $\beta_{i,j} = \beta_{i,j}(X)$ of X are as presented in the following table.

i	1	2	3	4	5	6
$\beta_{i,1}$	6	8	3	0	0	0
$\beta_{i,2}$	4	12	12	4	0	0
$\beta_{i,3}$	0	0	0	0	0	0
$\beta_{i,4}$	1	4	6	4	1	0

By Lemma 6.5 (a) it follows from this graded Betti diagram of X, that

$$\tau(X) = (2,3).$$

7.4. **Example.** Let $(a,b) = (3,8)$ and consider $X_{f_i} \subset \mathbb{P}^6$ $(i = 1,2,3)$ for the following choices of f_i:

(1) $f_1 = s^7t + s^6t^2 + s^5t^3 + s^4t^4 + s^3t^5 + s^2t^6 + st^7$,
(2) $f_2 = s^7t + s^6t^2 + s^5t^3 + s^4t^4 + s^3t^5 + s^2t^6$, and
(3) $f_3 = s^7t + s^6t^2 + s^5t^3 + s^4t^4$.

Then $X_{f_i} \subset \mathbb{P}^6$ is of degree $d = 11$ $(= 2r - 1 = 3r - 7)$ for all $i = 1,2,3$. The graded Betti diagrams of X_{f_1}, X_{f_2} and X_{f_3} are given respectively in the three tables below.

i	1	2	3	4	5	6
$\beta_{i,1}$	6	8	3	0	0	0
$\beta_{i,2}$	0	0	0	0	0	0
$\beta_{i,3}$	4	12	12	4	0	0
$\beta_{i,4}$	0	0	0	0	0	0
$\beta_{i,5}$	1	4	6	4	1	0
$\beta_{i,6}$	0	0	0	0	0	0
$\beta_{i,7}$	1	4	6	4	1	0

i	1	2	3	4	5	6
$\beta_{i,1}$	5	5	0	0	0	0
$\beta_{i,2}$	1	0	1	0	0	0
$\beta_{i,3}$	1	9	11	4	0	0
$\beta_{i,4}$	4	18	32	28	12	2
$\beta_{i,5}$	0	0	0	0	0	0
$\beta_{i,6}$	0	0	0	0	0	0
$\beta_{i,7}$	1	4	6	4	1	0

i	1	2	3	4	5
$\beta_{i,1}$	3	2	0	0	0
$\beta_{i,2}$	10	27	24	7	0
$\beta_{i,3}$	0	0	0	0	0
$\beta_{i,4}$	0	0	0	0	0
$\beta_{i,5}$	0	0	0	0	0
$\beta_{i,6}$	0	0	0	0	0
$\beta_{i,7}$	1	4	6	4	1

By Lemma 6.5 (a) we can see from these tables that

$$\tau(X_{f_1}) = (2,2), \quad \tau(X_{f_2}) = (1,1) \text{ and } \tau(X_{f_3}) = (2,3).$$

7.5. **Example.** Let $(a,b) = (3,9)$ and consider $X_f \subset \mathbb{P}^6$, $(i = 1,2)$ for the two choices

(1) $f_1 = s^8t + s^7t^2 + s^6t^3 + s^5t^4 + s^4t^5 + s^3t^6 + s^2t^7 + st^8$ and
(2) $f_2 = s^8t + s^7t^2 + s^6t^3 + s^5t^4 + s^4t^5 + s^3t^6 + s^2t^7$.

Then $X_{f_i} \subset \mathbb{P}^6$ is of degree $d = 12 \quad (= 2r = 3r - 6)$ for $i = 1,2$. The graded Betti diagrams of X_{f_1} and X_{f_2} are given respectively in the tables below, respectively.

i	1	2	3	4	5	6
$\beta_{i,1}$	6	8	3	0	0	0
$\beta_{i,2}$	0	0	0	0	0	0
$\beta_{i,3}$	2	4	0	0	0	0
$\beta_{i,4}$	1	4	10	6	1	0
$\beta_{i,5}$	0	0	0	0	0	0
$\beta_{i,6}$	1	4	6	4	1	0
$\beta_{i,7}$	0	0	0	0	0	0
$\beta_{i,8}$	1	4	6	4	1	0

i	1	2	3	4	5	6
$\beta_{i,1}$	5	5	0	0	0	0
$\beta_{i,2}$	0	0	1	0	0	0
$\beta_{i,3}$	5	15	15	5	0	0
$\beta_{i,4}$	0	0	0	0	0	0
$\beta_{i,5}$	5	23	42	38	17	3
$\beta_{i,6}$	0	0	0	0	0	0
$\beta_{i,7}$	0	0	0	0	0	0
$\beta_{i,8}$	1	4	6	4	1	0

By Lemma 6.5 (a) we can verify that

$$\tau(X_{f_1}) = (2,2) \text{ and } \tau(X_{f_2}) = (1,1).$$

7.6. **Remark.** The previously given examples of Betti tables have been computed over a base field of characteristic 0. It turns out, that in some of these examples the Betti table varies with the characteristic of the base field. The SINGULAR files with our computations are available upon request to the authors.

7.7. **Problem and Remark.** (A) Let $5 \leq r < d$ and let $X \subset \mathbb{P}^r$ be a non-degenerate surface of degree d which is of maximal sectional regularity. We consider the three conditions

(i) $N(X) \leq d - r$.
(ii) $\beta_{1,d-r+2}(X) = 1$.
(iii) $\mathbb{F}(X) = \mathbb{P}^2$.

(B) By the implication (i) $\Rightarrow$ (iii) given in statement (a) of Theorem 6.8 we have the implication (i) $\Rightarrow$ (ii) among the above three conditions. By the implication (i) $\Rightarrow$ (ii) given in statement (b) of Theorem 6.8 we have the implication (ii) $\Rightarrow$ (iii) among the above three conditions.

We expect, that the converse of both implications holds but could not prove this. So we aim to pose the problem

(P) *Are the three conditions* (i), (ii) *and* (iii) *of part* (A) *equivalent ?*

Observe, that in view of Theorem 5.14 (e) an affirmative answer to this would also answer affirmatively the question, whether the extended extremal variety and the extremal variety of X coincide, hence the question whether

(Q) $\mathbb{F}^+(X) = \mathbb{F}(X)$ *?*

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