Castelnuovo–Mumford Regularity of Annihilators, Ext and Tor Modules

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1 Introduction

In his opening address to the *Workshop on Castelnuovo–Mumford Regularity and Applications* at the Max Planck Institute for Mathematics in the Sciences at Leipzig in June 2007, we learned from Professor Eberhard Zeidler, former Director of that Institute, that physicists have a high esteem for algebraic geometry, because it provides so many invariants. Among these invariants Castelnuovo–Mumford regularity is particularly interesting. For example, mathematical physics are interested in degrees of defining equations of characteristic varieties of D-modules, a subject which is closely related to Castelnuovo–Mumford regularity. So, in the PhD thesis [2] of Michael Bächtold, we find the result that the Hilbert function (with respect to an appropriate filtration) of a D-module W over a standard Weyl algebra A bounds from above the degrees of polynomials which are needed to cut out set theoretically the characteristic variety of W. This is true, because the Hilbert function h_M of a graded module M which is generated over the polynomial ring $R = K[x_1, \ldots, x_r]$ by finitely many elements of degree 0 bounds from above the Castelnuovo–Mumford regularity $\mathrm{reg}(\mathrm{Ann}_R(M))$ of the annihilator $\mathrm{Ann}_R(M)$ of M. This was worked out in the MSc thesis [21] of the third author. Let us also mention here the recently finished PhD thesis [5] of Roberto Boldini, which is devoted to a different aspect of characteristic varieties of D-modules.

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Later it turned out that the ideas used in [21] may be combined with some earlier bounding results of [8] to get a number of a priori bounds for the Castelnuovo–Mumford regularity of Ext- and Tor-modules, for example, bounds which hold over arbitrary Noetherian homogeneous rings with local Artinian base ring and for arbitrary finitely generated graded modules over them. We do not insist that one has to use exclusively the results of [8]. Indeed, instead one also could use, for example, results of Chardin–Fall–Nagel [13] to end up with similar bounds.

Our original question asks whether a certain finite collection of invariants of a finitely generated graded module M over a homogeneous Noetherian ring R bounds the Castelnuovo–Mumford regularity of M. This leads to ask for a priori bounds, hence for bounds which apply in a most general setting. Here, being bounded in terms of certain invariants usually is more interesting than the size of the bound. On the other hand, one also can ask for *specific bounds*, for example, bounds which apply only for a specified class of graded R-modules, but which in turn are smaller (and possibly sharp). Already at its beginning, the investigation of Castelnuovo–Mumford regularity shows an interplay of these two aspects (see, e.g., [3, 4, 6, 10, 12, 16, 20]). In the this chapter, clearly the first aspect plays a dominant role. Nevertheless, in the last section, we shall give a bound on the Castelnuovo–Mumford regularity of certain specified Tor-modules which extends earlier bounding results of Eisenbud–Huneke–Ulrich [15] and Caviglia [11].

In Sect. 2 of this chapter, we present some preliminaries, and we give an extension to graded modules of Mumford's basic bounding result for graded ideals in a polynomial ring [20] in terms of Hilbert polynomials—an extension which to some extend may be viewed as folklore. It says that over a Noetherian homogeneous (e.g., standard graded) ring R with local Artinian base ring R_0, the Castelnuovo–Mumford regularity of a finitely generated graded R-module M is bounded in terms of the length of R_0, the degree vector of a homogeneous system of generators of M, the Hilbert polynomial p_M, and the postulation number $p(M)$, of M (see Proposition 5). In view of our first goal, which is to bound the Castelnuovo–Mumford regularity of the annihilator $\mathrm{Ann}_R(M)$ of a finitely generated graded R-module M in terms of the Hilbert function h_M of M, we clearly have to use this result.

In Sect. 3, we give a few preliminaries on filtered modules over filtered rings, especially on D-modules, and introduce in more detail the original question asked by Bächtold on the degrees of equations cutting out set theoretically the characteristic variety of such modules. As this chapter has an expository touch, we allow ourselves to include here a short introduction to characteristic varieties of modules over appropriately filtered K-algebras, especially over Weyl algebras. Readers familiar with the subject therefore might jump what is said in Reminder 1 to Remark 5. For readers who aim to learn more about the subject, we recommend to consult [18, 19], or [14]. After this expository introduction, we tie the link to the Castelnuovo–Mumford regularity of annihilators of graded modules and prove the requested bounding results on their Castelnuovo–Mumford regularity in terms of Hilbert functions (see Theorem 14 and Corollaries 15 and 16). We shall do this by first proving that the Castelnuovo–Mumford regularity of the annihilator $\mathrm{Ann}_R(M)$

of a finitely generated graded R-module M is bounded in terms of invariants of R, the initial degree and the Castelnuovo–Mumford regularity of M (see Theorem 10 and Corollaries 11–13). Then we apply Proposition 5 to get the requested bound for $\operatorname{reg}(\operatorname{Ann}_R(M))$ in terms of the Hilbert function h_M of M.

In Sect. 4 we give an a priori bound for the Castelnuovo–Mumford regularity of the modules $\operatorname{Ext}^i_R(M, N)$ in terms of i, of invariants of R, the initial degrees, the Castelnuovo–Mumford regularities, and the number of generators of M and N, where M and N are finitely generated graded modules over a Noetherian homogeneous ring R with local Artinian base ring R_0 (see Theorem 4 and Corollary 5). As an application we get a simply shaped bound for the Castelnuovo–Mumford regularity of the deficiency modules $K^i(M)$ in terms of i, invariants of R, the initial degree, the Castelnuovo–Mumford regularity and the number of generators of M (see Corollaries 7 and 8).

In Sect. 5—under the same hypothesis as in Sect. 4—we first give a bound for the Castelnuovo–Mumford regularity of the tensor product $M \otimes_R N$ in terms of the invariants mentioned above (see Proposition 3). We then deduce a corresponding a priori bound for the Castelnuovo–Mumford regularity of the modules $\operatorname{Tor}_i^R(M, N)$ (see Theorem 4, Corollary 5, and Remark 6). Then, we leave the field of a priori bounds and establish—over arbitrary rings R as above—an upper bound on the Castelnuovo–Mumford regularity of the modules $\operatorname{Tor}_k^R(M, N)$, provided that at least one of the two modules M or N has finite projective dimension and that $\operatorname{Tor}_i^R(M, N)$ is if dimension ≤ 1 for all $i > 0$ (see Proposition 8). As an application we prove a bounding result for the Castelnuovo–Mumford regularity of the modules $\operatorname{Tor}_k^R(M, N)$ which holds under the hypotheses that $\operatorname{Tor}_1^R(M, N)$ is of dimension ≤ 1 and the singular locus of the scheme $\operatorname{Proj}(R)$ is finite. This will extend the previously mentioned results of Eisenbud–Huneke–Ulrich and Caviglia (see Theorem 10 and Corollary 11).

2 Some Preliminaries

In this section, we fix a few notations and recall some basic facts which we shall use throughout this chapter. For the reader's convenience we also present and prove a result of folklore type which extends Mumford's basic regularity bound [20]. As a basic reference for this section we use [7].

Notation 1. Let $\mathbb{N}_0$ denote the set of nonnegative integers and let $\mathbb{N}$ denote the set of positive integers.

Throughout let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with Artinian local base ring $(R_0, \mathfrak{m}_0)$ and irrelevant ideal $R_+ := \bigoplus_{n \in \mathbb{N}} R_n$. Observe in particular that there are finitely many elements $l_1, l_2, \ldots, l_r \in R_1$ such that $R = R_0[l_1, l_2, \ldots, l_r]$, $R_+ = \langle l_1, l_2, \ldots, l_2 \rangle$ and $\mathfrak{m} := \mathfrak{m}_0 \oplus R_+$ is the unique homogeneous maximal ideal of R.

Next we recall a few basic facts on local cohomology of graded R-modules and Castelnuovo–Mumford regularity.

Reminder 2. If $T = \bigoplus_{n\in\mathbb{Z}} T_n$ is a graded R-module we define the *beginning* (or the *initial degree*) and the *end* of T, respectively, by

$$\mathrm{beg}(T) := \inf\{n \in \mathbb{Z} \mid T_n \neq 0\}, \quad \mathrm{end}(T) := \sup\{n \in \mathbb{Z} \mid T_n \neq 0\}.$$

Moreover, the *generating degree* of the graded R-module T is defined by

$$\mathrm{gendeg}(T) := \inf\{n \in \mathbb{Z} \mid T = \sum_{m\leq n} RT_m\}.$$

We always use the convention that $\inf(\bullet)$ and $\sup(\bullet)$ are formed in $\mathbb{Z} \cup \{-\infty, +\infty\}$ with $\inf(\varnothing) := \infty$ and $\sup(\varnothing) := -\infty$. Obviously, we have

$$T \neq 0 \Rightarrow \mathrm{beg}(T) \leq \mathrm{gendeg}(T) \leq \mathrm{end}(T).$$

If the R-module T is finitely generated, we have $\mathrm{gendeg}(T) \leq \infty$.

For each nonnegative integer $i \in \mathbb{N}_0$ and each graded R-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$ let $H^i_{R_+}(M)$ denote the ith local cohomology module of M with respect to the irrelevant ideal R_+ of R. The R-modules $H^i_{R_+}(M) = \bigoplus_{n\in\mathbb{Z}} H^i_{R_+}(M)_n$ carry a natural grading, the graded R-modules $H^i_{R_+}(M)$ are Artinian, and so their graded parts $H^i_{R_+}(M)_n$ are R_0-modules of finite length in all degrees $n \in \mathbb{Z}$ and vanish for all $n \gg 0$. Moreover, if $r := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1)$ denotes the minimal number of generators of the R_0 module R_1, we have $H^i_{R_+}(M) = 0$ for all $i > r$.

Let M be a finitely generated graded R-module and let $k \in \mathbb{N}_0$. The *(Castelnuovo–Mumford) regularity of M at and above level k* is defined by

$$\mathrm{reg}^k(M) := \sup\{\mathrm{end}(H^i_{R_+}(M)) + i \mid i \geq k\}.$$

Observe that $\mathrm{reg}^k(M) < \infty$. The *(Castelnuovo–Mumford) regularity of M* at all is defined as the Castelnuovo–Mumford regularity of M at and above level 0, thus by

$$\mathrm{reg}(M) := \mathrm{reg}^0(M) = \sup\{\mathrm{end}(H^i_{R_+}(M)) + i \mid i \in \mathbb{N}_0\}.$$

We always have the inequality

$$\mathrm{gendeg}(M) \leq \mathrm{reg}(M).$$

We constantly use without further mention the behavior of regularities in short exact sequences of finitely generated graded R-modules and the fact that regularities are not affected if one considers M as a graded S-module by means of a surjective homomorphism of homogeneous Noetherian rings $\phi : S \twoheadrightarrow R$.

For simplicity, we also introduce the *width* of the finitely generated graded R-module M. e.g. the span between the regularity and the initial degree of M:

$$\mathrm{w}(M) := \max\{0, \mathrm{reg}(M) - \mathrm{beg}(M) + 1\}.$$

Observe that $\mathrm{w}(M) > 0$ if and only if $M \neq 0$, whereas $\mathrm{w}(M) = 0$ means that $M = 0$. When $R = K[x_1, x_2, \ldots, x_r]$ is a polynomial ring over a field K, the width of M is precisely the number of rows in the Betti-diagram of M, so

$$\mathrm{w}(M) = \sup\{\mathrm{end}(\mathrm{Tor}_R^i(R/R_+, M)) - \mathrm{beg}(M) - i \mid i \in \mathbb{N}_0\}.$$

We recall a few basic facts on Hilbert polynomials of graded R-modules.

Reminder 3. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded R-module. We denote the *Hilbert polynomial* of M by p_M so that

$$\mathrm{length}_{R_0}(M_n) = p_M(n) \text{ for all } n \gg 0.$$

We also introduce the *postulation number* of M that is the invariant

$$p(M) := \sup\{n \in \mathbb{Z} \mid \mathrm{length}_{R_0}(M_n) \neq p_M(n)\} \quad \in \mathbb{Z} \cup \{-\infty\}.$$

Hilbert polynomials behave additively in short exact sequences of finitely generated graded R-modules. Moreover, the Hilbert polynomial and the postulation number of a finitely generated graded R-module are not affected if one considers M as a graded S-module by means of a surjective homomorphism $S \twoheadrightarrow R$ of Noetherian homogeneous R_0-algebras.

For each $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ we may consider the nonnegative integer

$$h_M^i(n) := \mathrm{length}_{R_0}(H_{R_+}^i(M)_n),$$

which vanishes for all $n \gg 0$ and for all $i > \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1)$. *Serre's formula* yields (see [7, 17.1.6])

$$p_M(n) = \mathrm{length}_{R_0}(M_n) - \sum_{i \in \mathbb{N}_0} (-1)^i h_M^i(n) \text{ for all } n \in \mathbb{Z}.$$

One obvious consequence of this formula is the estimate

$$\mathrm{reg}(M) \le \max\{\mathrm{reg}^1(M), p(M) + 1\}.$$

Next, we quote the following auxiliary result, which will play a crucial role in our later arguments

Lemma 4. *Assume that R is a Cohen–Macaulay ring of dimension $r > 0$ and multiplicity e. Let $f : W \longrightarrow V$ be a homomorphism of finitely generated graded R-modules. If $V \neq 0$ is generated by μ homogeneous elements and*

$$\alpha := \min\{\mathrm{beg}(V), \mathrm{reg}(V) - \mathrm{reg}(R)\},$$

then we have

$$\mathrm{reg}(\mathrm{Im}(f)) \leq \big[\max\{\mathrm{gendeg}(W), \mathrm{reg}(V) + 1\} + e(\mu + 1) - \alpha\big]^{2^{r-1}} + \alpha.$$

Proof. This is nothing else than Corollary 6.2 of [8]. ◻

Finally, we give the announced extension of Mumford's regularity bound. It says that the regularity of a finitely generated graded R-module M is bounded in terms of the length of the base ring R_0, the Hilbert polynomial p_M, the postulation number $p(M)$ and the degrees $a_1, a_2, \ldots, a_\mu$ of generators of M.

Proposition 5. *Let $p \in \mathbb{Q}[x]$ be a polynomial, let $\mu \in \mathbb{N}$, and let $\boldsymbol{a} := (a_1, a_2, \ldots, a_\mu) \in \mathbb{Z}^\mu$ with $a_1 \leq a_2, \leq \cdots \leq a_\mu$. Then, there is a function*

$$F_{p,\boldsymbol{a}} : \mathbb{N}^2 \times \mathbb{Z} \longrightarrow \mathbb{Z}$$

such that whenever $\lambda := \mathrm{length}(R_0)$, $r := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1)$ and M, is a finitely generated graded R-module such that $p_M = p$, $p(M) \leq \pi$, and $M = \sum_{i=1}^{\mu} Rm_i$ with $m_i \in M_{a_i}$ for $i = 1, 2, \ldots, \mu$, we have

$$\mathrm{reg}(M) \leq F_{p,\boldsymbol{a}}(\lambda, r, \pi).$$

Proof. Let $r, \mu \in \mathbb{N}$, let $\boldsymbol{a} := (a_1, a_2, \ldots, a_\mu) \in \mathbb{Z}^\mu$ with $a_1 \leq a_2 \leq \cdots \leq a_\mu$, and let $p \in \mathbb{Q}[x]$. According to Theorem 17.2.7 of [7], there is a function

$$G_{r,p,\boldsymbol{a}} : \mathbb{N} \longrightarrow \mathbb{Z}$$

such that whenever $S = R_0[x_1, x_2, \ldots, x_r]$ is a polynomial with Artinian local base ring R_0 ring with $\mathrm{length}(R_0) = \lambda$ and $N \subset \bigoplus_{i=1}^{\mu} S(-a_i) =: U$ is a graded submodule such that the graded S-module $M := U/N$ satisfies $p_M = p$, we have

$$\mathrm{reg}^2(N) \leq G_{r,p,\boldsymbol{a}}(\lambda).$$

In view of the short exact sequence of graded S-modules

$$0 \longrightarrow N \longrightarrow U \longrightarrow M \longrightarrow 0,$$

as $\mathrm{reg}(U) = a_\mu$ and by Reminder 3 we get

$$\mathrm{reg}(M) \leq \max\{a_\mu, G_{r,p,\boldsymbol{a}}(\lambda), p(M) + 1\}.$$

Define

$$F_{p,\boldsymbol{a}}(\lambda, r, \pi) := \max\{a_\mu, G_{r,p,\boldsymbol{a}}(\lambda).\pi + 1\}.$$

If R and M satisfy the requirements of our proposition, then there is a surjective homomorphism of homogeneous Noetherian R_0-algebras $\phi : S \twoheadrightarrow R$ and we may consider M as an S-module by means of ϕ. In particular, we may write $M = U/N$ for some graded submodule $N \subset U$. As $\text{reg}(M)$, p_M and $p(M)$, are not affected if we consider M as an S-module we get the requested inequality. □

3 Characteristic Varieties of D-Modules and the Regularity of Annihilators

As mentioned in the introduction, this chapter grew out of a problem concerning characteristic varieties of D-modules. In this section, we aim to introduce this problem in more detail and present its solution, which bases on a bound for the regularity of the annihilator of a finitely generated graded module over a polynomial ring over a field. We first recall a few elementary facts on Weyl algebras and D-modules. Our suggested reference for this is [14], although we partly use our own terminology. We start in a slightly more general setting, for which we recommend the references [18] and [19].

Reminder 1. Let K be a field and let A be a unital associative K-algebra which carries a *filtration* $A_\bullet = (A_i)_{i\in\mathbb{N}_0}$ so that each A_i is a K-subspace of A such that

$$A_i \subseteq A_{i+1} \text{ for all } i \in \mathbb{N}_0, \quad 1 \in A_0, \quad A = \bigcup_{i\in\mathbb{N}_0} A_i \quad \text{ and}$$

$$A_i A_j \subseteq A_{i+j} \text{ for all } i, j \in \mathbb{N}_0,$$

where by definition $A_i A_j := \sum_{(f,g)\in A_i\times A_j} Kfg$. To simplify notation, we set $A_i = 0$ for all $i < 0$. The *associated graded* ring of A with respect to the filtration $A_\bullet$ is defined as the graded K-algebra

$$\text{Gr}(A) = \text{Gr}_{A_\bullet}(A) = \bigoplus_{i\in\mathbb{N}_0} A_i/A_{i-1},$$

with multiplication induced by $(f + A_{i-1})(g + A_{j-1}) := fg + A_{i+j-1}$ for all $i, j \in \mathbb{N}_0$, all $f \in A_i$ and all $g \in A_j$. The filtration $A_\bullet$ is said to be *commutative* if

$$fg - gf \in A_{i+j-1} \text{ for all } i, j \in \mathbb{N}_0 \text{ and for all } f \in A_i \text{ and all } g \in A_j.$$

In this situation, the associated graded ring $\text{Gr}(A)$ is commutative. The filtration $A_\bullet$ is said to be *very good* if is commutative and

$$A_0 = K, \quad \dim_K(A_1) < \infty, \quad \text{ and } A_i = A_1 A_{i-1} \text{ for all } i \in \mathbb{N}.$$

Clearly in this situation, the associated graded ring is a commutative homogeneous Noetherian K-algebra. If $A_\bullet$ is a very good filtration of A, we say that $(A, A_\bullet)$—or briefly A—is a *very well-filtered* K-algebra.

Let W be left A-module which carries an $A_\bullet$*-filtration* $W_\bullet = (W_i)_{i\in\mathbb{Z}}$ so that each W_i is a K-subspace of W and moreover

$$W_i \subseteq W_{i+1} \text{ for all } i \in \mathbb{Z}, \quad W = \bigcup_{i\in\mathbb{Z}} W_i \quad \text{and}$$

$$A_i W_j \subseteq W_{i+j} \text{ for all } (i, j) \in \mathbb{N}_0 \times \mathbb{Z},$$

where by definition $A_i W_j := \sum_{(f,w)\in A_i\times W_j} Kfw$. The *associated graded module* of W with respect to the filtration $W_\bullet$ is the graded $\mathrm{Gr}(A)$-module

$$\mathrm{Gr}(W) = \mathrm{Gr}_{W_\bullet}(W) := \bigoplus_{j\in\mathbb{Z}} W_j / W_{j-1},$$

with scalar multiplication induced by $(f + A_{i-1})(w + W_{j-1}) := fw + W_{i+j-1}$ for all $(i, j) \in \mathbb{N}_0 \times \mathbb{Z}$, all $f \in A_i$ and all $w \in W_j$.

We say that two $A_\bullet$-filtrations $W_\bullet^{(1)}, W_\bullet^{(2)}$ are *equivalent* if there is some $r \in \mathbb{N}_0$ such that

$$W_{i-r}^{(1)} \subseteq W_i^{(2)} \subseteq W_{i+r}^{(1)} \text{ for all } i \in \mathbb{Z}.$$

Note that in this situation for all $i \in \mathbb{N}$ and all $f \in A_i$ we have the implication

$$fW_j^{(1)} \subseteq W_{j+i-1}^{(1)} \text{ for all } j \in \mathbb{Z} \quad \Rightarrow \quad f^{2r+1} W_j^{(2)} \subseteq W_{j+(2r+1)i-1}^{(2)} \text{ for all } j \in \mathbb{Z}.$$

So, if the filtration $A_\bullet$ is commutative, we can say:

If $W_\bullet^{(1)}$ is equivalent to $W_\bullet^{(2)}$, then $\sqrt{\mathrm{Ann}_{\mathrm{Gr}(A)}(\mathrm{Gr}_{W_\bullet^{(1)}}(W))} = \sqrt{\mathrm{Ann}_{\mathrm{Gr}(A)}(\mathrm{Gr}_{W_\bullet^{(2)}}(W))}$.

Remark and Definition 2. Let $V \subseteq W$ be a K-subspace such that $AV = W$. Then $A_\bullet V := (A_j V)_{j\in\mathbb{Z}}$ defines an $A_\bullet$-filtration on W, which we call the $A_\bullet$*-filtration induced by* V. If $V^{(1)}, V^{(2)} \subseteq W$ are two K-subspaces of finite dimension such that $W = AV^{(k)}$ for $k = 1, 2$, the induced filtrations $A_\bullet V^{(1)}$ and $A_\bullet V^{(2)}$ are equivalent so that by Reminder 1 we have $\sqrt{\mathrm{Ann}_{\mathrm{Gr}(A)}(\mathrm{Gr}_{A_\bullet V^{(1)}}(W))} = \sqrt{\mathrm{Ann}_{\mathrm{Gr}(A)}(\mathrm{Gr}_{A_\bullet V^{(2)}}(W))}$.

Assume now, that the filtration $A_\bullet$ of A is commutative and that the left A-module W is finitely generated. Then, there is a finite-dimensional K-subspace $V \subseteq W$ such that $W = AV$. According to our previous observation, the closed subset

$$\mathbb{V}(W) = \mathbb{V}_{A_\bullet}(W) := \mathrm{Spec}\left(\mathrm{Gr}(A)/\left(\mathrm{Ann}_{\mathrm{Gr}(A)}(\mathrm{Gr}_{A_\bullet V}(W))\right)\right) \subseteq \mathrm{Spec}(\mathrm{Gr}(A))$$

does not depend on our choice of V and hence is determined by the filtration $A_\bullet$ and the module W. It is called the *characteristic variety* of the finitely generated left A-module W with respect to the commutative filtration $A_\bullet$ of A.

Remark and Definition 3. Let W be a left A-module equipped with an $A_\bullet$-filtration $W_\bullet$. We say that the $A_\bullet$-filtration $W_\bullet$ is *very good*, if

$$W_j = 0 \text{ for all } j < 0, \quad \dim_K(W_0) < \infty \quad \text{ and } W_j = A_j W_0 \text{ for all } j \in \mathbb{N}.$$

Thus, the very good $A_\bullet$-filtrations of W are precisely the filtrations $A_\bullet V$ induced by a K-subspace $V \subseteq W$ of finite dimension. So, W admits a very good filtration if and only if it is finitely generated, and then all good filtrations are equivalent. If $W_\bullet$ is a good filtration of W, we say that $(W, W_\bullet)$—or briefly W—is *very well-filtered* (with respect to the filtration $A_\bullet$).

Assume that $W = (W, W_\bullet)$ is very well-filtered with respect to $A_\bullet$. Then, the associated graded module $\mathrm{Gr}(W) = \mathrm{Gr}_{W_\bullet}(W)$ of W with respect to $W_\bullet$ is generated by finitely many homogeneous elements of degree 0. In particular one may define the *Hilbert function* $h_W = h_{(W,W_\bullet)}$ of W with respect to $W_\bullet$ as the Hilbert function of the graded $\mathrm{Gr}(A)$-module $\mathrm{Gr}(W) = \mathrm{Gr}_{W_\bullet}(W)$, hence

$$h_W(j) = h_{(W,W_\bullet)}(j) = h_{\mathrm{Gr}(W)}(j) = \dim_K(W_j / W_{j-1}) \text{ for all } j \in \mathbb{Z}.$$

Example 4. Let K be a field of characteristic 0, let $n \in \mathbb{N}$, and let $E_n(K) := \mathrm{End}_K(K[X_1, \ldots, X_n])$ denote the endomorphism ring of the polynomial ring $K[X_1, \ldots, X_n]$. For all $i \in \{1, \ldots, n\}$ we identify X_i with the K-endomorphism on $K[X_1, \ldots, X_n]$ given by multiplication with X_i, and we write D_i for the partial derivative with respect to X_i on $K[X_1, \ldots, X_n]$. Then, the nth *Weyl algebra* over K is defined as the subring

$$A_n(K) := K\langle X_1, \ldots, X_n, D_1, \ldots, D_n\rangle \subseteq E_n(K)$$

of $E_n(K)$ generated by the multiplication endomorphisms X_i and the partial derivatives D_i. The ring $A_n(K)$ is a unital associative central K-algebra and its elements are called *partial differential operators* on $K[X_1, \ldots, X_n]$. The elements

$$X^\nu D^\mu := X_1^{\nu_1} \ldots X_n^{\nu_n} D_1^{\mu_1} \ldots D_n^{\mu_n} \in A_n(K), \text{ with } \nu := (\nu_1, \ldots, \nu_n),$$
$$\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}_0^n$$

are called *elementary partial differential operators*. One has the *Heisenberg relations*

$$[X_i, X_j] = 0, \quad [D_i, D_j] = 0, \quad [D_i, X_j] = \delta_{ij} \text{ for all } i, j \in \{1, \ldots, n\},$$

where $[\bullet, \bullet]$ denotes the commutator operation and δ_{ij} denotes the Kronecker symbol.

It follows from the Heisenberg relations that the elementary differential operators form a K-vector space basis of $A_n(K)$. Therefore, each element $f \in A_n(K)$ may be written as $f = \sum_{\nu,\mu \in \mathbb{N}_0^n} a_{\nu\mu} X^\nu D^\mu$ with uniquely determined coefficients $a_{\nu\mu} \in K$

which vanish for all but finitely many pairs (ν, μ). So, if $f \neq 0$, we may define the *degree* of f by $\deg(f) := \max\{|\nu| + |\mu| \mid (\nu, \mu) \in \mathbb{N}_0^n : a_{\nu\mu} \neq 0\}$. In addition, we set $\deg(0) := -\infty$. Now, one gets a filtration $A_\bullet$ on $A = A_n(K)$ given by $A_i := \{f \in A \mid \deg(f) \leq i\}$, the so-called *degree filtration*—a commutative very good filtration on A. More precisely, if $x_1, \ldots, x_{2n}$ are indeterminates, one has an isomorphism of graded K algebras:

$$K[x_1, \ldots, x_{2n}] \xrightarrow{\cong} \mathrm{Gr}_{A_\bullet}(A), \quad x_i \mapsto X_i + A_0, \quad x_{i+n} \mapsto D_i + A_0$$
$$\text{for all } i \in \{1, \ldots, n\}.$$

We assume from now on, that the Weyl algebra $A = A_n(K)$ is always endowed with its degree filtration.

Remark 5. The finitely generated left A-modules are called *D-modules* over A. For each D-module W over A, the characteristic variety of W is a closed subset of an affine $2n$-space over K:

$$\mathbb{V}(W) \subset \mathrm{Spec}(\mathrm{Gr}(A)) = \mathrm{Spec}(K[x_1, \ldots, x_{2n}]) = \mathbb{A}_K^{2n}.$$

We endow W with a very good filtration $W_\bullet$ so that its associated graded module $\mathrm{Gr}(W) = \mathrm{Gr}_\bullet(W)$ is generated by finitely many homogeneous elements of degree 0. The very well-filtered D-module $W = (W, W_\bullet)$ has a Hilbert function $h_W = h_{(W, W_\bullet)}$.

We now formulate in its original form the problem concerning the degrees of homogeneous polynomials which set theoretically cut out the characteristic variety of a D-module, posed to us by Bächtold.

Problem 6. Let $W = (W, W_\bullet)$ be a very well-filtered D-module. Do n and the Hilbert function $h_W = h_{(W, W_\bullet)}$ bound from above the degree of homogeneous polynomials in $K[x_1, \ldots, x_{2n}]$ which are needed to cut out the set $\mathbb{V}(W)$ from $\mathbb{A}_K^{2n}$?

Remark 7. By the definition of characteristic variety, the bound we are asking for in Problem 6 is on its turn bounded from above by $\mathrm{gendeg}\big(\mathrm{Ann}_{\mathrm{Gr}(A)}(\mathrm{Gr}(W))\big)$. So, it suffices to bound from above this latter invariant in terms of the Hilbert function $h_{(W, W_\bullet)} = h_{\mathrm{Gr}(W)}$. This is what we are heading for, and this is also what finally was stated in Lemma 7.41 of [2]. This Lemma was used there, to prove a certain uniformity result, which says that, over a C^∞-manifold M, the "global characteristic generically agrees with the point-wise characteristic" (see Theorem 7.39 of [2]).

Remark 8. According to Remark 7, the problem posed in Problem 6 is solved if, for a polynomial ring $R = K[x_1, \ldots, x_r]$ over a field K and a graded R-module M which is generated by finitely many homogeneous elements of degree 0, the generating degree $\mathrm{gendeg}(\mathrm{Ann}_R(M))$ of the annihilator of M is bounded in terms of r and the Hilbert function h_M of M. As $\mathrm{gendeg}(\mathrm{Ann}_R(M)) \leq \mathrm{reg}(\mathrm{Ann}_R(M))$, it suffices indeed to show that the regularity of the annihilator of M is bounded in terms of r and the Hilbert function of M.

Remark 9. Let the notation and hypotheses as in Remark 8 and assume that M is generated by μ homogeneous elements of degree 0. We aim to find an upper bound on $\mathrm{reg}(\mathrm{Ann}_R(M))$ which depends only on r and h_M. In fact, the Hilbert function is a somehow enigmatic object, as it is not clear (e.g., from the computational point of view) what it means "to know" a function $h : \mathbb{Z} \to \mathbb{N}_0$. Such arithmetic functions may encode an uncountable variety of information, and thus are not accessible for finitistic considerations. We therefore prefer to replace the function h_M by finitistic invariants (which are known in a "philosophical sense" if h_M is). We thus aim to bound $\mathrm{reg}(\mathrm{Ann}_R(M))$ in terms of r, $\mu = h_M(0)$, the Hilbert polynomial p_M of M, and the postulation number $p(M)$ of M.

We shall do this in a more general context. Hence, from now on, let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be as in Notation 1, that is, a Noetherian homogeneous ring with Artinian local base ring $(R_0, \mathfrak{m}_0)$ and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R-module.

We begin with the following bounding result for the regularity $\mathrm{reg}(\mathrm{Ann}_R(M))$ of the annihilator $\mathrm{Ann}_R(M)$ of the graded R-module M.

Theorem 10. *Let $r := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0R_1) > 0$, set $\lambda := \mathrm{length}(R_0)$, $\rho := \mathrm{reg}(R)$. If $M \neq 0$ is generated by μ homogeneous elements,*

$$\beta := \mathrm{reg}(M) + \mathrm{gendeg}(M) - 2\mathrm{beg}(M) \quad \textit{and}\ \alpha := \mathrm{beg}(M) - \mathrm{gendeg}(M),$$

then we have

$$\mathrm{reg}(\mathrm{Ann}_R(M)) \le \max\{\rho, \left[\beta + \lambda(\mu^2+1)+1\right]^{2^{r-1}} + \alpha + 1\}.$$

Proof. By our definition of the number r there is a surjective homomorphism of homogeneous R_0-algebras $\phi : S = R_0[x_1, x_2, \ldots, x_r] \twoheadrightarrow R$, where S is a standard graded polynomial ring over R_0. Clearly, the invariants $\mathrm{gendeg}(M)$, $\mathrm{beg}(M)$, and μ are not affected, if we consider M as a graded S-module by means of ϕ. In addition, the invariants $\mathrm{reg}(M)$ and $\mathrm{reg}(\mathrm{Ann}_R(M))$ are not affected if we consider M and $\mathfrak{a} := (\mathrm{Ann}_R(M))$ as graded S-modules by means of ϕ.

We now set $\mathfrak{b} := \mathrm{Ann}_S(M) = \phi^{-1}(\mathfrak{a})$ so that we have an isomorphism of graded S-modules $R/\mathfrak{a} \cong S/\mathfrak{b}$ and hence a short exact sequence of graded S-modules

$$0 \longrightarrow \mathfrak{a} \longrightarrow R \longrightarrow S/\mathfrak{b} \longrightarrow 0.$$

Consequently we have $\mathrm{reg}(\mathfrak{a}) \le \max\{\mathrm{reg}(R), \mathrm{reg}(S/\mathfrak{b}) + 1\}$. So, it suffices to show that

$$\mathrm{reg}(S/\mathfrak{b}) \le \left[\beta + \lambda(\mu^2+1)+1\right]^{2^{r-1}} + \alpha.$$

Observe that we have an exact sequence of R-modules

$$0 \longrightarrow \mathfrak{b} \longrightarrow S \xrightarrow{\epsilon} \mathrm{Hom}_S(M, M), \quad (x \mapsto \epsilon(x) := x\mathrm{Id}_M)$$

and an epimorphism of graded S-modules

$$\pi : \bigoplus_{i=1}^{\mu} S(-a_i) \twoheadrightarrow M, \quad \mathrm{beg}(M) = a_1 \le a_2 \le \cdots \le a_\mu = \mathrm{gendeg}(M).$$

In particular we obtain an induced monomorphism of graded S-modules

$$0 \longrightarrow \mathrm{Hom}_S(M, M) \xrightarrow{g:=\mathrm{Hom}_S(\pi, M)} \mathrm{Hom}_S\left(\bigoplus_{i=1}^{\mu} S(-a_i), M\right) = \bigoplus_{i=1}^{\mu} M(a_i) =: V.$$

So we get a composition map

$$S \xrightarrow{f:=g\circ\epsilon} V, \quad \text{with} \quad \mathrm{Im}(f) = \mathrm{Im}(\epsilon) \cong S/\mathfrak{b}.$$

Now, observe that S is a Cohen–Macaulay ring of dimension r with $\mathrm{gendeg}(S) = \mathrm{reg}(S) = 0$ and with multiplicity λ. Moreover the S-module V is generated by μ^2 homogeneous elements. Furthermore, we have

$$\mathrm{beg}(V) = \mathrm{beg}(M) - a_\mu = \mathrm{beg}(M) - \mathrm{gendeg}(M) = \alpha$$

and

$$\mathrm{reg}(V) = \mathrm{reg}(M) - a_1 = \mathrm{reg}(M) - \mathrm{beg}(M) \ge 0 = \mathrm{gendeg}(S).$$

In particular we have $\min\{\mathrm{beg}(V), \mathrm{reg}(V) - \mathrm{reg}(S)\} = \mathrm{beg}(V) = \alpha$ and $\mathrm{reg}(V) \ge 0$. So, if we apply Lemma 4 to the above homomorphism $f : S \longrightarrow V$ and observe that $\mathrm{Im}(f) \cong S/\mathfrak{b}$, we obtain indeed

$$\begin{aligned}\mathrm{reg}(S/\mathfrak{b}) &\le \left[\mathrm{reg}(M) - \mathrm{beg}(M) + 1 + \lambda(\mu^2+1) - \mathrm{beg}(M) + \mathrm{gendeg}(M)\right]^{2^{r-1}} + \alpha \\ &= \left[\beta + \lambda(\mu^2 + 1) + 1\right]^{2^{r-1}} + \alpha. \qquad \square\end{aligned}$$

As an immediate consequence we now get the an upper bound for the regularity of the annihilator of M in terms of the two invariants $\rho := \mathrm{reg}(R), \lambda := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1)$ of the ring R and the three invariants $\mathrm{reg}(M), \mathrm{beg}(M), \mu := \dim_{R_0/\mathfrak{m}_0}(M/(\mathfrak{m}_0 R + R_1 R)M)$ of the module M of Theorem 10.

Corollary 11. *Let R, M, r, λ, ρ, and μ be as in Theorem 10. Then it holds*

$$\mathrm{reg}(\mathrm{Ann}_R(M)) \le \max\{\rho, \left[2\big(\mathrm{reg}(M) - \mathrm{beg}(M)\big) + \lambda(\mu^2 + 1) + 1\right]^{2^{r-1}} + 1\}.$$

Proof. This is clear by Theorem 10 as $\mathrm{gendeg}(M) \leq \mathrm{reg}(M)$ and $\alpha = \mathrm{beg}(M) - \mathrm{gendeg}(M) \leq 0$. □

This bound becomes particularly simple if R is a polynomial ring.

Corollary 12. *Let $r \in \mathbb{N}$ and assume that $R = R_0[x_1, x_2, \ldots, x_r]$ is a standard graded polynomial ring. If $M \neq 0$ is generated by μ homogeneous elements, then we have*

$$\mathrm{reg}(\mathrm{Ann}_R(M)) \leq \left[2\big(\mathrm{reg}(M) - \mathrm{beg}(M)\big) + \mathrm{length}(R_0)(\mu^2+1) + 1\right]^{2^{r-1}} + 1.$$

Proof. This follows immediately from Corollary 11 as $\mathrm{reg}(R) = 0$. □

The following special case covers the situation of primary interest.

Corollary 13. *Let $R = K[x_1, x_2, \ldots, x_r]$ be a polynomial ring over the field K. If $M \neq 0$ is a graded R-module which is generated by μ homogeneous elements of degree 0, then we have*

$$\mathrm{reg}(\mathrm{Ann}_R(M)) \leq \left[2\mathrm{reg}(M) + \mu^2 + 2\right]^{2^{r-1}} + 1.$$

To answer affirmatively our original question on characteristic varieties of D-modules, we can use the following result, in which the function $F_{p,\boldsymbol{a}}$ is as in Proposition 5.

Theorem 14. *Let r, λ, and ρ be as in Theorem 10, let $\mu \in \mathbb{N}$, and let $\boldsymbol{a} := (a_1, a_2, \ldots, a_\mu) \in \mathbb{Z}^\mu$ with $a_1 \leq a_2 \leq \cdots \leq a_\mu$. If $M = \sum_{i=1}^{\mu} Rm_i$ is a finitely generated graded R-module with $m_i \in M_{a_i}$ for $i = 1, 2, \ldots .\mu$, then we have*

$$\mathrm{reg}(\mathrm{Ann}_R(M)) \leq \max\{\rho, \left[F_{p_M,\boldsymbol{a}}(\lambda, r, p(M)) + a_\mu - 2a_1 + \lambda(\mu^2+1) + 1\right]^{2^{r-1}} + a_1 - a_\mu + 1\}.$$

In particular, we have

$$\mathrm{reg}(\mathrm{Ann}_R(M)) \leq \max\{\rho, \left[2\big(F_{p_M,\boldsymbol{a}}(\lambda, r, p(M)) - a_1\big) + \lambda(\mu^2+1) + 1\right]^{2^{r-1}} + 1\}.$$

Proof. This is immediate by Theorem 10, respectively, Corollary 11 and Proposition 5 as $\mathrm{beg}(M) = a_1$ and $\mathrm{gendeg}(M) = a_\mu$. □

Now, we have reached the goal set out in Remark 9 by the special case of the previous bound in which $\mathbf{0} \in \mathbb{Z}^\mu$.

Corollary 15. *Let K be a field and R be a Noetherian homogeneous K-algebra set $r := \dim_K(R_1)$ and $\rho := \mathrm{reg}(R)$. If $M \neq 0$ is a graded R-module which is generated by μ homogeneous elements of degree 0, then we have*

$$\mathrm{reg}(\mathrm{Ann}_R(M)) \leq \max\{\rho, \left[2F_{p_M,\mathbf{0}}(1, r, p(M)) + \mu^2 + 2\right]^{2^{r-1}} + 1\}.$$

Finally, we also recover the bound we suggested to look for in Remark 9.

Corollary 16. *Let $R = K[x_1, x_2, \ldots, x_r]$ be a polynomial ring over the field K and let $M \neq 0$ be a graded R-module which is generated by μ homogeneous elements of degree 0. Then the regularity of the annihilator of M is bounded in terms of the number r of indeterminates, the Hilbert polynomial p_M of M, the postulation number $p(M)$ of M, and the number μ of generators of M. More precisely, we have*

$$\operatorname{reg}(\operatorname{Ann}_R(M)) \leq \left[2F_{p_M,\mathbf{0}}(1, r, p(M)) + \mu^2 + 2)\right]^{2^{r-1}} + 1.$$

4 A Regularity Bound for Ext-Modules

The aim of this section is to give an upper bound on the Castelnuovo–Mumford regularity of the modules $\operatorname{Ext}^i_R(M, N)$ in terms of the number r of linear forms, which are needed to generate R as an R_0-algebra, and the regularities and initial degrees of the modules M and N. We begin with the case $i = 0$ and give a bound on the regularity of the graded R-module $\operatorname{Hom}_R(M, N)$.

Lemma 1. *Let $r := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1) > 0$, $\lambda := \operatorname{length}(R_0)$ and let*

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W$$

be an exact sequence of finitely generated graded R-modules. If $W \neq 0$ is generated by μ homogeneous elements, then we have

$$\operatorname{reg}(U) \leq \max\{\operatorname{reg}(V), \left[\max\{\operatorname{gendeg}(V), \operatorname{reg}(W) + 1\} + \lambda(\mu + 1) - \operatorname{beg}(W)\right]^{2^{r-1}} + \operatorname{beg}(W) + 1\}$$

Proof. The short exact sequence of graded R-modules

$$0 \longrightarrow U \xrightarrow{f} V \longrightarrow \operatorname{Im}(g) \longrightarrow 0$$

gives $\operatorname{reg}(U) \leq \max\{\operatorname{reg}(V), \operatorname{reg}(\operatorname{Im}(g)) + 1\}$. Hence, it suffices to show that

$$\operatorname{reg}(\operatorname{Im}(g)) \leq \left[\max\{\operatorname{gendeg}(V), \operatorname{reg}(W)+1\}+\lambda(\mu+1)-\operatorname{beg}(W)\right]^{2^{r-1}} + \operatorname{beg}(W).$$

According to our definition of the number r there is a surjective homomorphism of homogeneous R_0-algebras $\phi : S = R_0[x_1, x_2, \ldots, x_r] \twoheadrightarrow R$, in which S is a standard graded polynomial ring over R_0. None of the numerical invariants occurring in the requested inequality are affected if we consider U, V and W as graded S-modules by means of ϕ. Thus, we may replace R by S and hence assume that $R = R_0[x_1, x_2, \ldots, x_r]$ is a polynomial ring. In particular R then is

Cohen–Macaulay of dimension r of multiplicity λ and satisfies $\operatorname{reg}(R) = 0$. Now, we get the requested inequality if we apply Lemma 4 to the homomorphism $V \xrightarrow{g} W$. □

Lemma 2. *Let* $r := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1) > 0$, $\lambda := \operatorname{length}(R_0)$*; let* M *and* N *be two non-zero finitely generated graded* R*-modules and suppose that there is an exact sequence of graded* R*-modules*

$$\bigoplus_{j=1}^{\sigma} R(-b_j) \longrightarrow \bigoplus_{i=1}^{\mu} R(-a_i) \longrightarrow M \longrightarrow 0$$

with integers $b_1 \leq b_2 \leq \cdots \leq b_\sigma$ *and* $a_1 \leq a_2 \leq \cdots \leq a_\mu$*. If* $N \neq 0$ *is generated by* ν *homogeneous elements,*

$$\beta := \max\{\operatorname{gendeg}(N) - a_1, \operatorname{reg}(N) - b_1 + 1\}, \quad \text{and } \gamma := \operatorname{beg}(N) - b_\sigma,$$

then we have

$$\operatorname{reg}(\operatorname{Hom}_R(M,N)) \leq \max\{\operatorname{reg}(N) - a_1, \left[\beta + \lambda(\sigma\nu + 1) - \gamma\right]^{2^{r-1}} + \gamma + 1\}.$$

Proof. Apply Lemma 1 to the induced exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M,N) \longrightarrow \bigoplus_{i=1}^{\mu} N(a_i) \longrightarrow \bigoplus_{j=1}^{\sigma} N(b_j)$$

and observe that

$$\operatorname{gendeg}\left(\bigoplus_{i=1}^{\mu} N(a_i)\right) = \operatorname{gendeg}(N) - a_1, \quad \operatorname{reg}\left(\bigoplus_{i+1}^{\mu} N(a_i)\right) = \operatorname{reg}(N) - a_1,$$

$$\operatorname{beg}\left(\bigoplus_{j=1}^{\sigma} N(b_j)\right) = \operatorname{beg}(N) - b_\sigma, \quad \operatorname{reg}\left(\bigoplus_{j=1}^{\sigma} N(b_j)\right) = \operatorname{reg}(N) - b_1$$

and that $\bigoplus_{j=1}^{\sigma} N(b_j)$ is generated by $\sigma\nu$ homogeneous elements. □

Proposition 3. *Let* $r := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1) > 0$, $\lambda = \operatorname{length}(R_0)$*. If* M *and* N *are two nonzero graded* R*-modules which are generated, respectively, by* μ *and* ν *homogeneous elements, then we have*

$$\operatorname{reg}(\operatorname{Hom}_R(M,N)) \leq \left[\mathrm{w}(M) + \mathrm{w}(N) - 1 + \left(\tbinom{\mathrm{w}(M)+r}{r-1}\lambda\mu\nu + 1\right)\lambda\right]^{2^{r-1}} + \operatorname{beg}(N) - \operatorname{beg}(M).$$

Proof. Again, there is a surjective homomorphism $\phi : S = R_0[x_1, x_2, \ldots, x_r] \twoheadrightarrow R$ of homogeneous R_0-algebras. Observe in particular that the graded S-modules $\mathrm{Hom}_S(M, N)$ and $\mathrm{Hom}_R(M, N)$ are isomorphic. So, the numerical invariants occurring in our statement are not affected if we consider M and N as graded S-modules by means of ϕ. Hence, we may once more assume that $R = R_0[x_1, x_2, \ldots, x_r]$ is a polynomial ring. Now, let

$$\bigoplus_{j=1}^{\sigma} R(-b_j) \longrightarrow \bigoplus_{i=1}^{\mu} R(-a_i) \longrightarrow M \longrightarrow 0$$

with $b_1 \leq b_2 \leq \cdots \leq b_\sigma$ and $a_1 \leq a_2 \leq \cdots \leq a_\mu$ be a minimal free presentation of M. Then, as

$$\mathrm{gendeg}(N) \leq \mathrm{reg}(N), \text{ and } \mathrm{reg}(M) + 1 \geq b_\sigma \geq b_1 \geq a_1 + 1 = \mathrm{beg}(M) + 1,$$

we get the following inequalities:

$$\begin{aligned} \beta &:= \max\{\mathrm{gendeg}(N) - a_1, \mathrm{reg}(N) - b_1 + 1\} \leq \mathrm{reg}(N) - \mathrm{beg}(M), \\ \gamma &:= \mathrm{beg}(N) - b_\sigma \leq \mathrm{beg}(N) - \mathrm{beg}(M) - 1, \\ -\gamma &\leq \mathrm{reg}(M) - \mathrm{beg}(N) + 1. \end{aligned}$$

Moreover, by the minimality of our presentation, we have

$$\sigma \leq \mathrm{length}_{R_0}\left(\Big(\bigoplus_{i=1}^{\mu} R(-a_i) \Big)_{\leq \mathrm{reg}(M)+1} \right) \leq \lambda\mu \binom{\mathrm{w}(M)+r}{r-1}.$$

Thus, we may conclude by Lemma 2. □

Now, we are ready to prove the main result of this section.

Theorem 4. *Let* $r := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1) > 0$*, let* $\lambda := \mathrm{length}(R_0)$*,* $\rho = \mathrm{reg}(R)$*, and let* M *and* N *be two nonzero graded* R*-modules which are generated, respectively, by* μ *and* ν *homogeneous elements. Then, for each* $i \in \mathbb{N}_0$*, we have*

$$\begin{aligned} &\mathrm{reg}(\mathrm{Ext}^i_R(M, N)) \leq \\ &\Big[\mathrm{w}(M) + \mathrm{w}(N) + i\rho - 1 + \Big(\lambda^{i+1}\mu\nu\binom{\mathrm{w}(M)+r+i\rho}{r-1} \prod_{j=1}^{i} \binom{\mathrm{w}(M)+j\rho+r}{r-1} + 1\Big)\lambda\Big]^{2^{r-1}} \\ &+ \mathrm{beg}(N) - \mathrm{beg}(M) - i. \end{aligned}$$

Proof. The case $i = 0$ is clear by Proposition 3. To treat the cases with $i > 0$ we choose a short exact sequence of graded R-modules

$$0 \longrightarrow M' \longrightarrow \bigoplus_{k=1}^{\mu} R(-a_k) \xrightarrow{\pi} M \longrightarrow 0, \quad \operatorname{beg}(M) = a_1 \le a_2 \le \cdots \le a_\mu$$
$$= \operatorname{gendeg}(M),$$

in which the epimorphism π is minimal. If $M' = 0$, the module M is free, and hence our claim is obvious. So, let $M' \neq 0$ and consider the induced exact sequence of graded R-modules

$$0 \longrightarrow \operatorname{Hom}_R(M,N) \longrightarrow \bigoplus_{k=1}^{\mu} N(a_k) \xrightarrow{f} \operatorname{Hom}_R(M',N) \longrightarrow \operatorname{Ext}^1_R(M,N) \longrightarrow 0$$

and the induced isomorphisms of graded R-modules

$$\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^{i-1}_R(M',N) \text{ for all } i > 1.$$

We first aim to prove our statement in the case $i = 1$. From the above four term exact sequence, we get the estimates

$$\operatorname{reg}(\operatorname{Ext}^1_R(M,N)) \le \max\{\operatorname{reg}(\operatorname{Im}(f)) - 1, \operatorname{reg}(\operatorname{Hom}_R(M',N))\}$$

and

$$\operatorname{reg}(\operatorname{Im}(f)) \le \max\{\operatorname{reg}(\operatorname{Hom}_R(M,N)) - 1, \operatorname{reg}(\bigoplus_{k=1}^{\mu} N(a_k))\}.$$

Our next aim is to make explicit the second estimate. According to Proposition 3, we have

$$\operatorname{reg}(\operatorname{Hom}_R(M,N)) - 1$$
$$\le \left[\operatorname{w}(M) + \operatorname{w}(N) - 1 + \left(\binom{\operatorname{w}(M)+r}{r-1}\lambda\mu\nu + 1\right)\right]^{2^{r-1}} + + \operatorname{beg}(N) - \operatorname{beg}(M) - 1.$$

Moreover, the term $\operatorname{reg}\left(\bigoplus_{k=1}^{\mu} N(a_k)\right) = \operatorname{reg}(N) - a_1 = \operatorname{reg}(N) - \operatorname{beg}(M) = \operatorname{w}(N) + \operatorname{beg}(N) - \operatorname{beg}(M) \le \operatorname{w}(N) + \operatorname{w}(M) + \operatorname{beg}(N) - \operatorname{beg}(M) - 1$ cannot exceed the right-hand side of the above inequality, so that we get the following explicit estimate:

$$\operatorname{reg}(\operatorname{Im}(f)) \le \left[\operatorname{w}(M) + \operatorname{w}(N) - 1 + \left(\binom{\operatorname{w}(M)+r}{r-1}\lambda\mu\nu + 1\right)\lambda\right]^{2^{r-1}} + \operatorname{beg}(N) - \operatorname{beg}(M) - 1.$$

Our next aim is to bound the invariant $\operatorname{reg}(\operatorname{Hom}_R(M',N))$. By our initial minimal short exact sequence, we have $\operatorname{reg}(M') \le \operatorname{reg}(M) + \rho + 1$ and $\operatorname{beg}(M') \ge \operatorname{beg}(M) + 1$, so that we obtain

$$\operatorname{w}(M') \le \operatorname{w}(M) + \rho, \quad -\operatorname{beg}(M') \le -\operatorname{beg}(M) - 1.$$

Let μ' denote the minimal number of homogeneous generators of M'. As $\mathrm{gendeg}(M') \leq \mathrm{reg}(M') \leq \mathrm{reg}(M) + \rho + 1$, we have

$$\mu' \leq \mathrm{length}\big((\bigoplus_{k=1}^{\mu} R(-a_k))_{\leq \mathrm{reg}(M)+\rho+1}\big) \leq \binom{\mathrm{w}(M)+\rho+r}{r-1}\lambda\mu.$$

Using these estimates and applying Proposition 3, we obtain

$$\begin{aligned}
&\mathrm{reg}(\mathrm{Hom}_R(M', N)) \\
&\leq \Big[\mathrm{w}(M) + \rho + \mathrm{w}(N) + \rho - 1 + \Big(\binom{\mathrm{w}(M)+\rho+r}{r-1}^2 \lambda^2 \mu\nu + 1\Big)\lambda\Big]^{2^{r-1}} \\
&\quad + \mathrm{beg}(N) - \mathrm{beg}(M) - 1.
\end{aligned}$$

Observe, that this term exceeds our previous upper bound for $\mathrm{reg}(\mathrm{Im}(f))$. So on use of our very first inequality, we end up with

$$\begin{aligned}
&\mathrm{reg}(\mathrm{Ext}^1_R(M, N)) \\
&\leq \Big[\mathrm{w}(M) + \mathrm{w}(N) + \rho - 1 + \Big(\binom{\mathrm{w}(M)+\rho+r}{r-1}^2 \lambda^2 \mu\nu + 1\Big)\lambda\Big]^{2^{r-1}} \\
&\quad + \mathrm{beg}(N) - \mathrm{beg}(M) - 1.
\end{aligned}$$

This proves our claim if $i = 1$.

For $i > 1$ we now may proceed by induction on use of the previously observed isomorphisms of Ext-modules and keeping in mind the above inequalities $\mathrm{w}(M') \leq \mathrm{w}(M) + \rho$, $-\mathrm{beg}(M') \leq -\mathrm{beg}(M) - 1$, and $\mu' \leq \binom{\mathrm{w}(M)+\rho+r}{r-1}\lambda\mu$. $\square$

In case R is a polynomial ring, this bound becomes simpler in appearance.

Corollary 5. *Assume that M and N are two non-zero graded modules generated by μ respectively, ν homogeneous elements over the polynomial ring $R = R_0[x_1, x_2, \ldots, x_r]$ with $\lambda := \mathrm{length}(R_0)$. Then, for each $i \in \mathbb{N}_0$, we have*

$$\mathrm{reg}(\mathrm{Ext}^i_R(M, N)) \leq$$

$$\Big[\mathrm{w}(M) + \mathrm{w}(N) - 1 + \Big(\binom{\mathrm{w}(M)+r}{r-1}^{i+1} \lambda^{i+1} \mu\nu + 1\Big)\lambda\Big]^{2^{r-1}} + \mathrm{beg}(N) - \mathrm{beg}(M) - i.$$

Proof. This is clear by Theorem 4 as $\mathrm{reg}(R) = 0$. $\square$

In [17], Hoa and Hyry did give upper bounds for the Castelnuovo–Mumford regularity of deficiency modules of graded ideals in polynomial rings over a field. In [9], Brodmann, Jahangiri, and Linh took up this idea and gave upper bounds for the Castelnuovo–Mumford regularity of deficiency modules of finitely generated graded modules over a standard graded Noetherian ring R with local Artinian base ring $(R_0, \mathfrak{m}_0)$. We aim to take up this theme again.

Remark and Notation 6. We set $r := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1)$. Then there is a surjective homomorphism of graded R_0-algebras $R' := R'_0[x_1, x_2, \ldots, x_r] \twoheadrightarrow R$ where $(R'_0, \mathfrak{m}'_0)$ is an Artinian Gorenstein ring. Let M be a finitely generated graded R-module, which we also consider as R'-module by means of the above homomorphism. Then, for each $i \in \mathbb{N}_0$, the *ith deficiency module of M* is given by

$$K^i(M) = \operatorname{Ext}^{r-i}_{R'}(M, R'(-r)).$$

We write $\lambda := \operatorname{length}(R_0)$ and λ' for the minimum length of all local Artinian Gorenstein rings R'_0 such that R_0 is a homomorphic image of R'_0. We may write R_0 as a homomorphic image of a complete regular local ring S_0 of dimension $\mathrm{e} := \operatorname{edim}(R_0) = \operatorname{length}_{R_0}(\mathfrak{m}/\mathfrak{m}^2)$. Let $a_1, a_2, \ldots, a_{\mathrm{e}}$ be a regular system of parameters of S_0. Then $(a_j)^\lambda$ is mapped to 0 under the canonical map $S_0 \twoheadrightarrow R_0$ for all $j \in \{1, 2, \ldots, \mathrm{e}\}$. Therefore R_0 is a homomorphic image of the Artinian Gorenstein ring

$$R'_0 := S_0/\langle (a_1)^\lambda, (a_2)^\lambda, \ldots, (a_{\mathrm{e}})^\lambda \rangle$$

But this means that we have

$$\lambda \leq \lambda' \leq 1 + \lambda \operatorname{edim}(R_0).$$

Now, as an application of Theorem 4 and with the above notations, we get the following bounding result on the regularity of deficiency modules. Observe in particular that the estimates given in statements (a) and (c) allow to bound the regularity of the ith deficiency module of a finitely generated graded R-module M only in terms of i, the initial degree of M, the regularity of M and invariants of R.

Corollary 7. *Let r and ρ be as in Theorem 4 and let M be a nonzero graded R-module which is generated by μ homogeneous elements. Let $t := \operatorname{reg}^2(M)$, let $p_M(n)$ denote the Hilbert polynomial of M, and let λ' be defined as in Remark and Notation 6. Then the following statements hold:*

(a) $\operatorname{reg}(K^0(M)) \leq -\operatorname{beg}(M)$.
(b) $\operatorname{reg}(K^1(M)) \leq \max\{0, 1 + t - \operatorname{beg}(M)\} + (d-1)p_M(t) - t$.
(c) *For all $i \in \mathbb{N}$ we have*

$$\operatorname{reg}(K^i(M)) \leq i - \operatorname{beg}(M) +$$

$$[\mathrm{w}(M) + (r-i-1)\rho - 1 + (\lambda')^{i+1}\mu\tbinom{\mathrm{w}(M)+r+(r-i)\rho}{r-1}\prod_{j=1}^{r-i}(\tbinom{\mathrm{w}(M)+j\rho+r}{r-1}+1)]^{2^{r-1}}.$$

Proof. (a) and (b) were proved in [9, Theorem 4.2]. (c) is implied directly by Theorem 4 as in the notations of Remark and Notation 6 we may replace R by $R' := R'_0[x_1, x_2, \ldots, x_r]$, where R'_0 is an Artinian Gorenstein ring of length λ'. $\square$

In case R is a polynomial ring over a field, statement (c) of the above result takes a particularly simple form.

Corollary 8. *Let K be a field and let M be a finitely generated graded module over the polynomial ring $K[x_1, x_2 \ldots, x_r]$. Let $i \in \mathbb{N}$. Then it holds*

$$\operatorname{reg}(K^i(M)) \leq i - \operatorname{beg}(M) + [\mathrm{w}(M) - 1 + \mu \binom{\mathrm{w}(M)+r}{r-1}^{r-i+1} + 1)]^{2^{r-1}}.$$

Proof. Observe that in our situation we have $\rho = 0$ and $\lambda' = 1$. ☐

5 Regularity Bounds for Tor-Modules

For two finitely generated graded R-modules M and N, the modules $\operatorname{Tor}_i^R(M, N)$ are finitely generated and carry a natural grading for all $i \in \mathbb{N}_0$. The aim of this section is to give an upper bound for the Castelnuovo–Mumford regularity of the modules $\operatorname{Tor}_i^R(M, N)$ in terms of the same bounding invariants as in Sect. 4. As in Sect. 4 we begin with the case $i = 0$ and give a regularity bound for the graded R-module $M \otimes_R N$.

Lemma 1. *Let $r := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1) > 0$, $\lambda = \operatorname{length}(R_0)$ and let*

$$U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

be an exact sequence of finitely generated graded R-modules. If $V \neq 0$ is generated by μ homogeneous elements, then we have

$$\operatorname{reg}(W) \leq \max\{\operatorname{reg}(V), \big[\max\{\operatorname{gendeg}(U), \operatorname{reg}(V) + 1\} + \lambda(\mu + 1) - \operatorname{beg}(V)\big]^{2^{r-1}} + \operatorname{beg}(V) - 1\}.$$

Proof. In view of the short exact sequence of graded R-modules

$$0 \longrightarrow \operatorname{Im}(f) \longrightarrow V \xrightarrow{g} W \longrightarrow 0,$$

we have $\operatorname{reg}(W) \leq \max\{\operatorname{reg}^1(\operatorname{Im}(f)) - 1, \operatorname{reg}(V)\}$. So it suffices to show that

$$\operatorname{reg}(\operatorname{Im}(f)) \leq \big[\max\{\operatorname{gendeg}(U), \operatorname{reg}(V) + 1\} + \lambda(\mu + 1) - \operatorname{beg}(V)\big]^{2^{r-1}} + \operatorname{beg}(V).$$

As in the proof of Lemma 1, we may assume that $R = R_0[x_1, x_2, \ldots, x_r]$ is a polynomial ring, so that R is CM of dimension r of multiplicity λ and satisfies $\operatorname{reg}(R) = 0$. Now, we get the requested inequality if we apply once more Lemma 4 to the homomorphism $U \xrightarrow{f} V$. ☐

Lemma 2. *Let* $r := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1) > 0$, $\lambda := \mathrm{length}(R_0)$, *let* M *and* N *be two non-zero finitely generated graded* R*-modules, and suppose that there is an exact sequence of graded* R*-modules*

$$\bigoplus_{j=1}^{\sigma} R(-b_j) \longrightarrow \bigoplus_{i=1}^{\mu} R(-a_i) \longrightarrow M \longrightarrow 0$$

with integers $b_1 \leq b_2 \leq \cdots \leq b_\sigma$ *and* $a_1 \leq a_2 \leq \cdots \leq a_\mu$. *Suppose in addition that* N *is generated by* ν *homogeneous elements and set*

$$\delta := \max\{\mathrm{gendeg}(N) + b_\sigma, \mathrm{reg}(N) + a_\mu + 1\}, \quad \varepsilon := \mathrm{beg}(N) + a_1.$$

Then we have

$$\mathrm{reg}(M \otimes_R N) \leq \max\{\mathrm{reg}(N) + a_\mu, \big[\delta + \lambda(\mu\nu + 1) - \varepsilon\big]^{2^{r-1}} + \varepsilon - 1\}.$$

Proof. Apply Lemma 1 to the induced exact sequence

$$\bigoplus_{j=1}^{\sigma} N(-bj) \longrightarrow \bigoplus_{i=1}^{\mu} N(-a_i) \longrightarrow M \otimes_R N \longrightarrow 0$$

and observe that

$$\mathrm{gendeg}\big(\bigoplus_{j=1}^{\sigma} N(-b_j)\big) = \mathrm{gendeg}(N) + b_\sigma, \quad \mathrm{reg}\big(\bigoplus_{i=1}^{\mu} N(-a_i)\big) = \mathrm{reg}(N) + a_\mu,$$

$$\mathrm{beg}\big(\bigoplus_{i=1}^{\mu} N(-a_i)\big) = \mathrm{beg}(N) + a_1$$

and that $\bigoplus_{i=1}^{\mu} N(-a_i)$ is generated by $\mu\nu$ homogeneous elements. This gives the requested bound. □

Proposition 3. *Let* $r := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1) > 0$, *let* $\lambda = \mathrm{length}(R_0)$. *If* M *and* N *are two non-zero graded* R*-modules which are generated respectively by* μ *and* ν *homogeneous elements, then we have*

$$\mathrm{reg}(M \otimes_R N) \leq \big[\mathrm{w}(M) + \mathrm{w}(N) + \lambda(\mu\nu + 1) - 1\big]^{2^{r-1}} + \mathrm{beg}(M) + \mathrm{beg}(N) - 1.$$

Proof. Again, there is a surjective homomorphism $\phi : S = R_0[x_1, x_2, \ldots, x_r] \twoheadrightarrow R$ of homogeneous R_0-algebras and the graded S-modules $M \otimes_S N$ and $M \otimes_R N$ are isomorphic. So none of the numerical invariants which occur in our statement is affected if we consider M and N as graded S-modules by means of ϕ. Therefore we may again assume that $R = R_0[x_1, x_2, \dots, x_r]$ is a polynomial ring and chose a minimal graded free presentation

$$\bigoplus_{j=1}^{\sigma} R(-b_j) \longrightarrow \bigoplus_{i=1}^{\mu} R(-a_i) \longrightarrow M \longrightarrow 0$$

of M with $b_1 \leq b_2 \leq \cdots \leq b_\sigma$ and $a_1 \leq a_2 \cdots \leq a_\mu$. Then, as

$$a_1 = \operatorname{beg}(M), \quad a_\mu \leq \operatorname{reg}(M), \quad \operatorname{gendeg}(N) \leq \operatorname{reg}(N), \quad b_\sigma \leq \operatorname{reg}(M) + 1$$

we get the relations

$$\operatorname{reg}(N) + a_\mu \leq \operatorname{reg}(M) + \operatorname{reg}(N),$$

$$\delta := \max\{\operatorname{gendeg}(N) + b_\sigma, \operatorname{reg}(N) + a_\mu + 1\} \leq \operatorname{reg}(M) + \operatorname{reg}(N) + 1,$$

$$\varepsilon := \operatorname{beg}(N) + a_1 = \operatorname{beg}(M) + \operatorname{beg}(N).$$

Now, it follows by Lemma 2 that

$$\operatorname{reg}(M \otimes_R N) \leq$$

$$\leq \max\{\operatorname{reg}(M)+\operatorname{reg}(N), \left[\mathrm{w}(M)+\mathrm{w}(N)+\lambda(\mu\nu+1)-1\right]^{2^{r-1}}+\operatorname{beg}(M)+\operatorname{beg}(N)-1\}.$$

As

$$\operatorname{reg}(M) + \operatorname{reg}(N) = [\mathrm{w}(M) + \mathrm{w}(N) - 1] + \operatorname{beg}(M) + \operatorname{beg}(N) - 1 \leq$$

$$\leq \left[\mathrm{w}(M) + \mathrm{w}(N) + \lambda(\mu\nu + 1) - 1\right]^{2^{r-1}} + \operatorname{beg}(M) + \operatorname{beg}(N) - 1,$$

we finally get our claim. □

As an application we get the following estimate for the regularity of Tor-modules, which is not symmetric in the two occurring modules. So, to get out the best of it, one should apply the result after eventually exchanging M and N such that $\mathrm{w}(M) \leq \mathrm{w}(N)$. Observe also that the case $r = 1$ is omitted in this result.

Theorem 4. *Let $r := \dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0 R_1) > 1$, let $\lambda := \operatorname{length}(R_0)$, let $\rho = \operatorname{reg}(R)$ and let M and N be two non-zero finitely generated graded R modules which are generated respectively by μ and ν homogeneous elements. Then, for all $i \in \mathbb{N}_0$ we have*

$$\operatorname{reg}(\operatorname{Tor}_i^R(M, N)) \leq$$

$$\left[\mathrm{w}(M) + \mathrm{w}(N) + i\rho - 1 + \left(\lambda^i \mu\nu \prod_{j=1}^{i} \tbinom{\mathrm{w}(M)+r+j\rho}{r-1} + 1\right)\lambda\right]^{2^{r-1}}$$

$$+ \operatorname{beg}(N) + \operatorname{reg}(M) + i\rho.$$

Proof. We proceed by induction on i. The case $i = 0$ is clear by Proposition 3. We first treat the case $i = 1$. Consider a graded short exact sequence

$$0 \longrightarrow M' \longrightarrow \bigoplus_{i=1}^{\mu} R(-a_i) \xrightarrow{\pi} M \longrightarrow 0.$$

As in the proof of Theorem 4 we see that

$$\operatorname{reg}(M') \le \operatorname{reg}(M) + \rho + 1, \quad \mathrm{w}(M') \le \mathrm{w}(M) + \rho$$

and that the minimal number μ' of homogeneous generators of M' satisfies

$$\mu' \le \tbinom{\mathrm{w}(M)+\rho+r}{r-1}\lambda\mu.$$

By Proposition 3 and as $\operatorname{beg}(M') \le \operatorname{reg}(M')$, we thus have

$$\operatorname{reg}(M' \otimes_R N) \le$$

$$\left[\mathrm{w}(M) + \mathrm{w}(N) + \rho + \lambda(\tbinom{\mathrm{w}(M)+\rho+r}{r-1}\lambda\mu\nu + 1) - 1\right]^{2^{r-1}} + \operatorname{reg}(M) + \operatorname{beg}(N) + \rho.$$

Next, look at the induced exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(M, N) \longrightarrow M' \otimes_R N \xrightarrow{f} \bigoplus_{i=1}^{\mu} N(-a_i) \longrightarrow M \otimes_R N \longrightarrow 0$$

and the two resulting short exact sequences

$$0 \longrightarrow \operatorname{Im}(f) \longrightarrow \bigoplus_{i=1}^{\mu} N(-a_i) \longrightarrow M \otimes_R N \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Tor}_1^R(M, N) \longrightarrow M' \otimes_R N \longrightarrow \operatorname{Im}(f) \longrightarrow 0.$$

If follows (see [7] Exercise 15.2.15) that

$$\operatorname{reg}(\operatorname{Im}(f)) \le \max\{\operatorname{reg}(\bigoplus_{i=1}^{\mu} N(-a_i)), \operatorname{reg}(M \otimes_R N) + 1\},$$

$$\operatorname{reg}(\operatorname{Tor}_1^R(M, N)) \le \max\{\operatorname{reg}(M' \otimes_R N), \operatorname{reg}(\operatorname{Im}(f)) + 1\}.$$

But $\operatorname{reg}(\bigoplus_{i=1}^{\mu} N(-a_i)) + 1 = \operatorname{gendeg}(M) + \operatorname{reg}(N) + 1 \le \operatorname{reg}(M) + \operatorname{beg}(N) + \mathrm{w}(N)$ as well as $\operatorname{reg}(M \otimes_R N) + 2$ cannot exceed the previously given upper bound for $\operatorname{reg}(M' \otimes_R N)$ (see also Proposition 3 and observe that $r > 1$). Therefore we end up with the estimate

$$\operatorname{reg}(\operatorname{Tor}_1^R(M, N)) \le$$

$$\left[\mathrm{w}(M) + \mathrm{w}(N) + \rho + \lambda(\tbinom{\mathrm{w}(M)+\rho+r}{r-1}\lambda\mu\nu + 1) - 1\right]^{2^{r-1}} + \operatorname{reg}(M) + \operatorname{beg}(N) + \rho.$$

This proves the case $i = 1$. Now, assume that $i > 1$. Then the isomorphism of graded R-modules

$$\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_{i-1}^R(M', N)$$

allow to proceed by induction as in the proof of Theorem 4. □

In case R is a polynomial ring, the upper bound of the previous theorem takes a simpler form. It follows because in this case $\rho = 0$.

Corollary 5. *Let $r > 1$, let R_0 be a local Artinian ring of length λ, and let M and N be two non-zero graded modules generated, respectively, by μ and ν homogeneous elements over the polynomial ring $R = R_0[x_1, \dots, x_r]$. Then we have*

$$\mathrm{reg}(\mathrm{Tor}_i^R(M, N)) \leq \left[\mathrm{w}(M) + \mathrm{w}(N) - 1\left(\lambda^i \mu\nu \binom{\mathrm{w}(M)+r}{r-1}^i + 1\right)\lambda\right]^{2^{r-1}} + \mathrm{beg}(N) + \mathrm{reg}(M).$$

Remark 6. As already observed above, the case $r = 1$ is not included in the previous two bounding results. But a look at the proof of Theorem 4 shows that for $r = 1$ we have the estimate

$$\mathrm{reg}(\mathrm{Tor}_i^R(M, N)) \leq \mathrm{w}(M) + \mathrm{reg}(M) + \mathrm{reg}(N) + 2i\rho + (\lambda^i \mu\nu + 1)\lambda + 1.$$

Up to now, the bounding results of this section where of *a priori type*, for example, valid without any further conditions on the Noetherian homogeneous ring R and the finitely generated graded R-modules M and N. We now follow the direction pointed out by earlier work of Caviglia and Eisenbud–Huneke–Ulrich and give a bound for the regularity of the modules $\mathrm{Tor}_k^R(M, N)$ under the additional condition that one of the modules M or N has finite projective dimension and that the modules $\mathrm{Tor}_i^R(M, N)$ are of dimension ≤ 1 for all $i \in \mathbb{N}$. We end up by generalizing the corresponding results of the mentioned authors (proved by them in case R is a polynomial ring over a field) to the case of homogeneous Noetherian rings R with Artinian base ring R_0 such that the singular locus of $\mathrm{Proj}(R)$ is a finite set (see Theorem 10).

Lemma 7. *Let $\rho = \mathrm{reg}(R)$. If M and N are finitely generated graded R-modules such that $p := \mathrm{pdim}_R(M) < \infty$ and $\dim_R(\mathrm{Tor}_i^R(M, N)) \leq 1$ for all $i > 0$, then it holds*

$$\mathrm{reg}(M \otimes_R N) \leq \mathrm{reg}(M) + \mathrm{reg}(N) + p\rho.$$

Proof. We proceed by induction on p. If $p = 0$, the graded R-module M is free, and our claim is obvious. So, let $p > 0$ and consider a graded short exact sequence

$$0 \longrightarrow M' \longrightarrow \bigoplus_{i=1}^{\mu} R(-a_i) \xrightarrow{\pi} M \longrightarrow 0$$

in which the homomorphism π is minimal. Look at the induced exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(M,N) \longrightarrow M' \otimes_R N \xrightarrow{f} \bigoplus_{i=1}^{\mu} N(-a_i) \longrightarrow M \otimes_R N \longrightarrow 0$$

and the two resulting short exact sequences

$$0 \longrightarrow \mathrm{Im}(f) \longrightarrow \bigoplus_{i=1}^{\mu} N(-a_i) \longrightarrow M \otimes_R N \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Tor}_1^R(M,N) \longrightarrow M' \otimes_R N \longrightarrow \mathrm{Im}(f) \longrightarrow 0.$$

The first of these two sequences implies

$$\mathrm{reg}(M \otimes_R N) \le \max\{\mathrm{reg}^1(\mathrm{Im}(f)) - 1, \mathrm{reg}(\bigoplus_{i=1}^{\mu} N(-a_i))\},$$

whereas the second of these sequences implies

$$\mathrm{reg}^1(\mathrm{Im}(f)) \le \max\{\mathrm{reg}^2(\mathrm{Tor}_1^R(M,N)) - 1, \mathrm{reg}(M' \otimes_R N)\}.$$

As $\dim_R(\mathrm{Tor}_1^R(M,N)) \le 1$, we have $\mathrm{reg}^2(\mathrm{Tor}_1^R(M,N)) = -\infty$. Hence, we obtain

$$\mathrm{reg}^1(\mathrm{Im}(f)) \le \mathrm{reg}(M' \otimes_R N))$$

Therefore, we deduce that

$$\mathrm{reg}(M \otimes_R N) \le \max\{\mathrm{reg}(M' \otimes_R N) - 1, \mathrm{reg}(\bigoplus_{i=1}^{\mu} N(-a_i))\}.$$

As $\mathrm{pdim}_R(M') = p - 1$ and $\mathrm{Tor}_j^R(M',N) \cong \mathrm{Tor}_{j+1}^R(M,N)$ for all $j \in \mathbb{N}$, the inductive hypothesis implies that $\mathrm{reg}(M' \otimes_R N) \le \mathrm{reg}(M') + \mathrm{reg}(N) + (p-1)\rho$. Our initial graded short exact sequence yields that $\mathrm{reg}(M') \le \mathrm{reg}(M) + \mathrm{reg}(R) + 1$. Moreover, $\mathrm{reg}(\bigoplus_{i=1}^{\mu} N(-a_i)) = a_\mu + \mathrm{reg}(N) \le \mathrm{reg}(M) + \mathrm{reg}(N)$. Therefore

$$\begin{aligned}
\mathrm{reg}(M \otimes_R N) &\le \max\{\mathrm{reg}(M' \otimes_R N) - 1, \mathrm{reg}(\bigoplus_{i=1}^{\mu} N(-a_i))\} \\
&\le \max\{\mathrm{reg}(M') + \mathrm{reg}(N) + (p-1)\rho - 1, \mathrm{reg}(\bigoplus_{i=1}^{\mu} N(-a_i))\} \\
&\le \mathrm{reg}(M) + \rho + 1 + \mathrm{reg}(N) + (p-1)\rho - 1 \\
&= \mathrm{reg}(M) + \mathrm{reg}(N) + p\rho.
\end{aligned}$$

□

Proposition 8. *Let $\rho = \operatorname{reg}(R)$. If M and N are finitely generated graded R-modules such that $p := \operatorname{pdim}_R(M) < \infty$ and $\dim_R(\operatorname{Tor}_i^R(M,N)) \leq 1$ for all $i > 0$. Then it holds*

$$\operatorname{reg}(\operatorname{Tor}_k^R(M,N)) \leq \operatorname{reg}(M) + \operatorname{reg}(N) + (k+1)p\rho + k \text{ for all } k \in \mathbb{N}_0.$$

Proof. The case $k = 0$ is clear by Lemma 7. To treat the cases with $k > 0$, we choose a short exact sequence of graded R-modules

$$0 \longrightarrow M' \longrightarrow \oplus_{i=1}^{\mu} R(-a_i) \xrightarrow{\pi} M \longrightarrow 0$$

in which π is minimal, such that $\operatorname{beg}(M) = a_1 \leq a_2 \leq \cdots \leq a_\mu = \operatorname{gendeg}(M)$. We proceed by induction on p. If $p = 0$ the module M is free and hence our claim is obvious. So, let $p > 0$ and consider the induced exact sequence of graded R-modules

$$0 \longrightarrow \operatorname{Tor}_1^R(M,N) \longrightarrow M' \otimes_R N \xrightarrow{f} \oplus_{i=1}^{\mu} N(-a_i) \longrightarrow M \otimes_R N \longrightarrow 0$$

and the induced isomorphisms of graded R-modules

$$\operatorname{Tor}_k^R(M,N) \cong \operatorname{Tor}_{k-1}^R(M',N) \text{ for } k > 1.$$

As $\operatorname{pdim}_R(M') = p - 1$, these isomorphisms and the inductive hypotheses imply that

$$\operatorname{reg}(\operatorname{Tor}_k^R(M,N,)) \leq \operatorname{reg}(M') + \operatorname{reg}(N) + k(p-1)\rho + k - 1 \text{ for all } k > 1.$$

Our initial short exact sequence yields that $\operatorname{reg}(M') \leq \operatorname{reg}(M) + \rho + 1$. From this our claim follows for all $k > 1$. It thus remains to treat the case $k = 1$. The above exact sequence, induces two short exact sequences:

(1) $0 \longrightarrow \operatorname{Im}(f) \longrightarrow \bigoplus_{i=1}^{\mu} N(-a_i) \longrightarrow M \otimes_R N \longrightarrow 0$
(2) $0 \longrightarrow \operatorname{Tor}_1^R(M,N) \longrightarrow M' \otimes_R N \longrightarrow \operatorname{Im}(f) \longrightarrow 0$

As $\operatorname{reg}(\bigoplus_{i=1}^{\mu} N(-a_i)) = \operatorname{reg}(N) + \mu \leq \operatorname{reg}(M) + \operatorname{reg}(N)$, sequence (1) implies that

$$\operatorname{end}(H^0_{R_+}(\operatorname{Im}(f)) \leq \operatorname{reg}(M) + \operatorname{reg}(N).$$

As $\operatorname{pdim}_R(M') = p - 1$ and $\operatorname{reg}(M') \leq \operatorname{reg}(M) + \rho + 1$, it follows by Lemma 7 that

$$\operatorname{reg}(M' \otimes_R N) \leq \operatorname{reg}(M) + \operatorname{reg}(N) + p\rho + 1.$$

As $\dim(\operatorname{Tor}_1^R(M,N)) \leq 1$, sequence (2) yields an epimorphism of graded R-modules $H^1_{R_+}(M' \otimes_R N) \twoheadrightarrow H^1_{R_+}(\operatorname{Im}(f))$. Therefore

$$\operatorname{end}(H^1_{R_+}(\operatorname{Im}(f)) \leq \operatorname{reg}(M' \otimes_R N) - 1 \leq \operatorname{reg}(M) + \operatorname{reg}(N) + p\rho.$$

But now by sequence (2), we get

$$\mathrm{end}(H^1_{R_+}(\mathrm{Tor}^R_1(M,N)) \leq \max\{\mathrm{end}(H^0_{R_+}(\mathrm{Im}(f))), \mathrm{end}(H^1_{R_+}(M' \otimes_R N))\} \leq$$

$$\leq \max\{\mathrm{reg}(M)+\mathrm{reg}(N), \mathrm{reg}(H^1(M' \otimes_R N)) - 1\} \leq \mathrm{reg}(M)+\mathrm{reg}(N)+p\rho.$$

Another use of sequence (2) yields that

$$\mathrm{end}(H^0_{R_+}(\mathrm{Tor}^R_1(M,N))) \leq \mathrm{reg}(M' \otimes_R N) \leq \mathrm{reg}(M)+\mathrm{reg}(N)+p\rho+1.$$

As $\dim_R(\mathrm{Tor}^R_1(M,N)) \leq 1$, it follows that

$$\mathrm{reg}(\mathrm{Tor}^R_1(M,N)) \leq \mathrm{reg}(M)+\mathrm{reg}(N)+p\rho+1$$

and this proves our claim. □

Lemma 9. *Let $i \in \mathbb{N}$, $d \in \mathbb{N}_0$ and assume that the local ring $R_\mathfrak{p}$ is regular for all graded primes $\mathfrak{p} \subset R$ with $\dim(R/\mathfrak{p}) > d$. Let M and N be finitely generated graded R-modules such that $\dim_R(\mathrm{Tor}^R_i(M,N)) \leq d$. Then it holds*

$$\dim_R(\mathrm{Tor}^R_j(M,N)) \leq d \textit{ for all } j \geq i.$$

Proof. Let $\mathfrak{p} \subset R$ be a graded prime with $\dim(R/\mathfrak{p}) > d$. As $\dim_R(\mathrm{Tor}^R_i(M,N)) \leq d$, it follows

$$\mathrm{Tor}^{R_\mathfrak{p}}_i(M_\mathfrak{p}, N_\mathfrak{p}) \cong \mathrm{Tor}^R_i(M,N)_\mathfrak{p} = 0.$$

The regular local ring $R_\mathfrak{p}$ contains the field $R_0/\mathfrak{m}_0$ and hence is unramified. So, by Auslander's Rigidity Theorem (see [1] Corollary 2.2), we have

$$\mathrm{Tor}^R_j(M,N)_\mathfrak{p} \cong \mathrm{Tor}^{R_\mathfrak{p}}_j(M_\mathfrak{p}, N_\mathfrak{p}) = 0 \text{ for all } j \geq i.$$

Therefore $\mathfrak{p} \notin \mathrm{Supp}_R(\mathrm{Tor}^R_j(M,N))$ for all $j \geq i$ and all graded primes $\mathfrak{p} \subset R$ with $\dim(R/\mathfrak{p}) > d$. As the R-modules $\mathrm{Tor}^R_j(M,N)$ are graded, our claim follows. □

Theorem 10. *Let $\rho := \mathrm{reg}(R)$ and assume that the local ring $R_\mathfrak{p}$ is regular for all graded primes $\mathfrak{p} \subset R$ with $\dim(R/\mathfrak{p}) \geq 2$. Let M and N be finitely generated graded R-modules such that $p = \mathrm{pdim}_R(M) < \infty$ and $\dim_R(\mathrm{Tor}^R_1(M,N)) \leq 1$. Then it holds*

$$\mathrm{reg}(\mathrm{Tor}^R_k(M,N)) \leq \mathrm{reg}(M)+\mathrm{reg}(N)+(k+1)p\rho+k \textit{ for all } k \in \mathbb{N}_0.$$

Proof. If we apply Lemma 9 with $d=1$ and $i=1$, we obtain that $\dim_R(\mathrm{Tor}^R_i(M,N)) \leq 1$ for all $i > 0$. Now, our claim follows by Proposition 8. □

Corollary 11. *Let $r > 0$ and let $R = K[x_1, x_2, \ldots, x_r]$ be a polynomial ring over the field K. Let M and N be finitely generated graded R-modules such that $\dim_R(\mathrm{Tor}_1^R(M, N)) \leq 1$. Then it holds*

$$\mathrm{reg}(\mathrm{Tor}_k^R(M, N)) \leq \mathrm{reg}(M) + \mathrm{reg}(N) + k \text{ for all } k \in \{0, 1, \ldots, r\}.$$

Proof. This is clear from Theorem 10 as $\mathrm{reg}(R) = 0$ and R is a regular ring. □

Remark 12. Corollary 11 has been proved by Eisenbud–Huneke–Ulrich (see[15] Corollary 3.1). The special case with $k = 0$ has been proved by Caviglia [11].
The conclusion of Theorem 10 need not hold if $\dim_R(\mathrm{Tor}_1^R(M, N)) > 1$, even in the special case where $R = K[x_1, x_2, \ldots, x_r]$ is a polynomial ring over the field K and for $k = 0$. Indeed Caviglia has constructed in this situation an example with $\dim_R(\mathrm{Tor}_1^R(M, N)) = 2$ and $\mathrm{reg}(M \otimes_R N) > \mathrm{reg}(M) + \mathrm{reg}(N)$.

Finally, we aim to conclude this section with slightly more geometric formulations of Theorem 10 and Corollary 11. To do so, we write

$$\mathrm{Sing}(X) := \{x \in X \mid \mathcal{O}_{X,x} \text{ is not regular}\}$$

for the *singular locus* of the Noetherian scheme X. If $\mathcal{H}$ is a coherent sheaf of $\mathcal{O}_X$-modules, we write

$$\mathrm{Sing}(\mathcal{H}) := \{x \in X \mid \mathcal{H}_x \text{ is not free over } \mathcal{O}_{X,x}\}$$

for the set of all points $x \in X$ at which the stalk $\mathcal{H}_x$ of $\mathcal{H}$ in x is not free.

Corollary 13. *Let $\rho := \mathrm{reg}(R)$, and set $X := \mathrm{Proj}(R)$. Let M and N be finitely generated graded R-modules such that $p = \mathrm{pdim}_R(M) < \infty$. Let $\mathcal{F} := \widetilde{M}$ and $\mathcal{G} := \widetilde{N}$ be the coherent sheaves of $\mathcal{O}_X$-modules induced, respectively, by M and N. Assume that the sets $\mathrm{Sing}(X)$ and $\mathrm{Sing}(\mathcal{F}) \cap \mathrm{Sing}(\mathcal{G})$ are finite. Then it holds*

$$\mathrm{reg}(\mathrm{Tor}_k^R(M, N)) \leq \mathrm{reg}(M) + \mathrm{reg}(N) + (k + 1)p\rho + k \text{ for all } k \in \mathbb{N}_0.$$

Proof. The finiteness of the singular locus of X implies that $R_\mathfrak{p}$ is a regular local ring for all graded primes $\mathfrak{p} \subset R$ with $\dim(R/\mathfrak{p}) \geq 2$. Our hypothesis on the stalks of $\mathcal{F}$ and $\mathcal{G}$ imply that at least one of the two finitely generated $R_\mathfrak{p}$-modules $M_\mathfrak{p}$ or $N_\mathfrak{p}$ is free for each graded prime $\mathfrak{p} \subset R$ with $\dim(R/\mathfrak{p}) \geq 2$. Therefore $\mathrm{Tor}_1^R(M, N)_\mathfrak{p} \cong \mathrm{Tor}_1^{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$ for all such $\mathfrak{p}$—and hence $\dim_R(\mathrm{Tor}_1^R(M, N)) \leq 1$. Now, we get our claim by Theorem 10. □

To formulate Corollary 11 in geometric terms, we recall a few notions from sheaf cohomology.

Reminder 14. (See Chap. 20 of [7] for example.) Let $X := \mathrm{Proj}(R)$ and let $\mathcal{H}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then, the *(Castelnuovo–Mumford) regularity* of $\mathcal{H}$ is defined as

$$\text{reg}(\mathcal{H}) := \inf\{r \in \mathbb{Z} \mid H^i(X, \mathcal{H}(r-i)) = 0 \text{ for all } i > 0\},$$

where $H^i(X, \mathcal{H}(n))$ denotes the ith *sheaf cohomology* group of (X with coefficients in) the *nth twist* $\mathcal{H}(n) := \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{O}(n)$ of $\mathcal{H}$.
The *total group of sections* of $\mathcal{H}$ is defined by

$$\Gamma_*(\mathcal{H}) := \bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{H}(n))$$

and carries a natural structure of graded R-module. Moreover, the sheaf $\widetilde{\Gamma_*(\mathcal{H})}$ of $\mathcal{O}_X$-modules induced by the graded R-module $\Gamma_*(\mathcal{H})$ coincides with $\mathcal{H}$. Finally, the R-module $\Gamma_*(\mathcal{H})$ is finitely generated and only if the set $\text{Ass}_X(\mathcal{H})$ contains no closed points of X—and if this is the case, we have

$$\text{reg}(\mathcal{H}) = \text{reg}(\Gamma_*(\mathcal{H})).$$

Corollary 15. *Let $r \in \mathbb{N}$, let K be a field, and let $\mathcal{F}$ and $\mathcal{G}$ be two coherent sheaves of $\mathcal{O}_{\mathbb{P}^r_K}$-modules such that the set* $\text{Ass}_{\mathbb{P}^r_K}(\mathcal{F}) \cup \text{Ass}_{\mathbb{P}^r_K}(\mathcal{G})$ *contains no closed points and the set* $\text{Sing}(\mathcal{F}) \cap \text{Sing}(\mathcal{G})$ *is finite. Then it holds*

$$\text{reg}\big(\text{Tor}_k^{\Gamma_*(\mathcal{O}_{\mathbb{P}^r_K})}(\Gamma_*(\mathcal{F}), \Gamma_*(\mathcal{G}))\big) \leq \text{reg}(\mathcal{F}) + \text{reg}(\mathcal{G}) + k \textit{ for all } k \in \{0, 1, \ldots, r+1\}.$$

Proof. Consider the polynomial ring $R := K[x_0, x_1, \ldots, x_r]$ and write $\mathbb{P}^r_K = \text{Proj}(R)$. As $r > 0$ we have $\Gamma_*(\mathcal{O}_{\mathbb{P}^r_K}) = R$. According to Reminder 14, the graded R-modules $\Gamma_*(\mathcal{F})$ and $\Gamma_*(\mathcal{G})$ are finitely generated and induce, respectively, the coherent sheaves $\mathcal{F}$ and $\mathcal{G}$. Now, we get our claim by Corollary 13 and Reminder 14. $\square$

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