CASTELNUOVO-MUMFORD REGULARITY, POSTULATION NUMBERS AND RELATION TYPES

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Dedicated to Prof. Ngo Viet Trung on the occasion of his 60-th Birthday

ABSTRACT. We establish a bound for the Castelnuovo-Mumford regularity of the associated graded ring $G_I(A)$ of an $\mathfrak{m}$-primary ideal I of a local Noetherian ring $(A, \mathfrak{m})$ in terms of the dimension of A, the relation type and the number of generators of I. As a consequence, we obtain that the existence of uniform bounds for the regularity of the associated graded ring, and the relation type of parameter ideals in A, are equivalent conditions. In addition, we establish an equation for the postulation number and the Castelnuovo-Mumford regularity of the associated graded ring $G_{\mathfrak{q}}(A)$ of a parameter ideal $\mathfrak{q}$, which holds under certain conditions on the depths of the occurring rings. We also show, that the regularity of the ring $G_{\mathfrak{q}}(A)$ is bounded in terms the dimension of A, the length of $A/\mathfrak{q}$ and the postulation number of $G_{\mathfrak{q}}(A)$.

1. INTRODUCTION

Let $(A, \mathfrak{m})$ be a Noetherian local ring and let $I \subset A$ be an $\mathfrak{m}$-primary ideal. We denote by $G_I(A) := \bigoplus_{n \geq 0} I^n/I^{n+1}$ the associated graded ring of A with respect to I. The Castelnuovo-Mumford regularity $\mathrm{reg}(G_I(A))$ of the associated graded ring $G_I(A)$ provides upper bounds for some other important invariants of I such as the reduction number $r(I)$, the relation type $\mathrm{reltype}(I)$, and the postulation number $n(I)$ of I. More precisely, it holds that

$$\max\{n(I), r(I), \mathrm{reltype}(I) - 1\} \leq \mathrm{reg}(G_I(A)).$$

A most important issue is the question, whether the previously mentioned invariants have uniform upper bounds as I runs through all parameter ideals of A. More precisely, if there exists a number N such that $\mathrm{reg}(G_{\mathfrak{q}}(A)) \leq N$ (or $\mathrm{reltype}(\mathfrak{q}) \leq N$, or $n(\mathfrak{q}) \leq N$) for all parameter ideals $\mathfrak{q}$ of A, then we say that *A has a uniform bound for the regularity of the associated graded ring* (respectively *the relation type*, or *the postulation number*) *of parameter ideals.* The search for such uniform bounds was initiated by Huneke's *Relation Type Conjecture* (see [1, Question 1.1]), that is the question:

Question 0. *Does there exist a uniform bound for the relation type of parameter ideals of a complete equidimensional Noetherian local ring?*

The Relation Type Conjecture has attracted a lot of attention. So, for example Lai [7] showed that this conjecture holds for generalized Cohen-Macaulay rings under the assumption that the residue field is infinite. Later, Wang [14] proved that the conjecture holds for generalized Cohen-Macaulay rings without any restriction on the residue field. In [15], Wang showed that the above conjecture holds for all 2-dimensional Noetherian local rings. In [1], Aberbach, Ghezzi and Ha provided an example showing that the conjecture need not hold for local rings with two-dimensional non-Cohen-Macaulay locus. Moreover, they showed that the conjecture holds in formally unmixed Noetherian local rings whose formal homology-multiplier ideal is a prime of dimension one. This answers Huneke's Relation Type Conjecture affirmatively in the formally unmixed case and in a situation which implies that the non-Cohen-Macaulay locus is of dimension 1. It is still an open problem, whether Huneke's conjecture holds for arbitrary formally unmixed Noetherian local rings with one-dimensional non-Cohen-Macaulay locus.

In [9], Linh and Trung established a uniform bound for the regularity of parameter ideals in generalized Cohen-Macaulay rings. Clearly, by the previously mentioned estimate, a uniform bound for the regularity of associated graded rings of parameter ideals implies a uniform bound for the relation type and also for the postulation number of such ideals. In relation with this observation, Ngo Viet Trung raised the following two questions:

Question 1. *Are the existence of uniform bounds for the regularity of the associated graded ring and for the relation type of parameter ideals equivalent conditions?*

Question 2. *Which local rings do have uniform bounds for the regularity of the associated graded ring or the relation type of paramater ideals?*

One aim of this paper is to give an affirmative answer to Question 1. Before we do so, we shall establish our first main result which relates the regularity $\mathrm{reg}(G_{\mathfrak{q}}(A))$ of the associated graded ring $G_{\mathfrak{q}}(A)$ and the postulation number $n(\mathfrak{q})$ of a parameter ideal $\mathfrak{q}$ of A under certain assumptions on the depths of A and $G_{\mathfrak{q}}(A)$:

Theorem 3.5 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and let $\mathfrak{q}$ be a parameter ideal of A. If $\mathrm{depth}(A) = d - 1$ and $d - 2 \leq \mathrm{depth}(G_{\mathfrak{q}}(A)) \leq d - 1$, then*

$$\mathrm{reg}(G_{\mathfrak{q}}(A)) = n(\mathfrak{q}) + d.$$

Using Theorem 3.5 and Wang's existence result for uniform bounds for the relation type of parameter ideals in 2-dimensional local rings, we obtain a uniform bound for the regularity of associated graded rings of parameter ideals in such rings (see Corollary 3.9).

The second main result of the present paper gives a bound for the regularity of the associated graded ring $G_I(A)$ of A with respect to an $\mathfrak{m}$-primary ideal I in terms of the relation type, the number of generators of I, and the dimension of A:

Theorem 4.2 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and let $I = (x_1, ..., x_r)$ be an $\mathfrak{m}$-primary ideal of A. Then*

$$\mathrm{reg}(G_I(A)) \leq [(r - d + 1)\,\mathrm{reltype}(I)^{r-d+1}]^{2^{d-2}}.$$

Applying this theorem, we get that the existence of a uniform bound for the regularity of the associated graded ring and the relation type of parameter ideals are indeed equivalent conditions. This implies in particular that there exists a uniform bound for the regularity of the associated graded ring of parameter ideals of a formally unmixed local Noetherian ring whose formal homology-multiplier ideal is a prime of dimension one, an analogue of the previously mentioned uniform bounding result for the relation type of [1] (see Corollary 4.6).

We conclude our paper with two results inspired by the open question as to whether a uniform bound for the postulation number of parameter ideals implies a uniform bound on the relation type (and hence on the Castelnuovo-Mumford regularity of the associated graded ring) of such ideals (see Proposition 4.9 and Corollary 4.10).

2. Preliminaries

Throughout this paper, let $\mathbb{N}_0$ and $\mathbb{N}$ denote the sets of non-negative and positive integers respectively. For a subset $\mathbb{M} \subseteq \mathbb{Z}$ we form the *supremum* $\sup(\mathbb{M})$ and the *infimum* $\inf(\mathbb{M})$ in the set $\mathbb{Z} \cup \{\infty, -\infty\}$, with the usual convention that $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = \infty$.

We first recall a few elementary facts on *arithmetic functions*, that is on functions $f : \mathbb{Z} \longrightarrow \mathbb{Z}$. An arithmetic function f is said to be *of polynomial type* if there is a polynomial $p_f = p_f(X) \in \mathbb{Q}[X]$ such that

$$f(n) = p_f(n) \text{ for all } n \gg 0.$$

Keep in mind, that the polynomial $p_f = p_f(X)$ is unique and has rational coefficients. Assume now, that f is an arithmetic function of polynomial type. Then, the same holds for the *difference function* Δf of f, that is the function given by $\Delta f(n) := f(n) - f(n-1)$, and in addition the polynomial $p_{\Delta f}(X)$ is equal to the *difference polynomial* $\Delta p_f(X) := p_f(X) - p_f(X-1)$ of p_f. Moreover, for each $m \in \mathbb{Z}$, the *m-th shift* $f[m]$ of f, that is the function defined by $f[m](n) := f(n+m)$, is again of polynomial type and $p_{f[m]}(X) = p_f(X+m)$.

The *postulation number* of an arithmetic function f of polynomial type is defined by

$$p(f) := \sup\{n \in \mathbb{Z} \mid f(n) \neq p_f(n)\}.$$

We say that the arithmetic function f is *left-vanishing* if $f(n) = 0$ for all $n \ll 0$. Observe that for a non-zero left-vanishing arithmetic function of polynomial type we have $p(f) \in \mathbb{Z}$. We later will have to use the following auxiliary result.

Lemma 2.1. *Let f be a function of polynomial type and let $m \in \mathbb{Z}$. Then*

$$p(f[m]) = p(f) - m \quad \text{and} \quad p(\Delta f) = p(f) + 1.$$

Proof. The first equality is obvious. To prove the second inequality, we write $s := p(f) \quad (\in \mathbb{Z} \cup \{-\infty\})$ and observe that

$$\Delta f(n) = f(n) - f(n-1) \begin{cases} = p_f(n) - p_f(n-1) = p_{\Delta f}(n) & \text{for all } n \geq s+2 \\ \neq p_f(s+1) - p_f(s) = p_{\Delta f}(s+1) & \text{for } n = s+1. \end{cases}$$

$\square$

Next, we recall a few basic facts on the numerical invariants we are interested in. So, let $R = \bigoplus_{n \geq 0} R_n$ be a finitely generated standard graded algebra over a Noetherian commutative ring R_0. Let $R_+ = \bigoplus_{n \in \mathbb{N}} R_n$ be the *irrelevant ideal* of R, that is the ideal generated by all homogeneous elements of positive degree in R. Let E be a finitely generated graded R-module with $\dim(E) = d$. For each $i \in \mathbb{N}_0$ we set

$$a_i(E) := \sup\{n \in \mathbb{Z} \mid H^i_{R_+}(E)_n \neq 0\},$$

where $H^i_{R_+}(E)$ denotes the i-th local cohomology module of E with respect to the irrelevant ideal R_+, furnished with its natural grading. The *Castelnuovo-Mumford regularity* of E is defined as the invariant

$$\text{reg}(E) := \max\{a_i(E) + i \mid i \geq 0\} \quad (\in \mathbb{Z} \cup \{-\infty\}).$$

Assume now that the base ring R_0 of R is Artinian and let

$$h_E : \mathbb{Z} \longrightarrow \mathbb{N}_0, \quad n \mapsto h_E(n) := \text{length}(E_n)$$

denote the *Hilbert function* of E, a left-vanishing arithmetic function of polynomial type. The polynomial

$$p_E(X) := p_{h_E}(X), \text{ hence with } h_E(n) = p_E(n) \text{ for all } n \gg 0$$

is called the *Hilbert polynomial* of E. The *postulation number* $p(E)$ of E is defined by

$$p(E) := p(h_E) = \sup\{n \in \mathbb{Z} \mid h_E(n) \neq p_E(n)\}.$$

Observe that $p(E) = -\infty$ if and only if $E = 0$.

We also recall the following basic fact:

Lemma 2.2. [Serre's Formula] *For all $n \in \mathbb{Z}$ we have*

$$h_E(n) - p_E(n) = \sum_{i=0}^{d} (-1)^i \text{length}(H^i_{R_+}(E)_n).$$

Proof. See e.g. [10, Lemma 1.3], [3, Theorem 17.1.7]. $\square$

As an immediate application of this formula we get:

Corollary 2.3.

    (1) $p(E) \leq \max\{a_0(E), a_1(E), ..., a_d(E)\}$.
    (2) *If there exists i such that $a_j(E) < a_i(E)$ for all $j \neq i$, then $p(E) = a_i(E)$.*
    (3) $p(E) + \text{depth}(E) \leq \text{reg}(E)$.

We denote by $\text{NZD}(E)$ the set of *non-zero divisors* in R with respect to E or equivalently, the set of E-regular elements of R. Later we shall have to use repeatedly the following auxiliary result:

Lemma 2.4. *Assume that $x \in R_1 \cap \mathrm{NZD}(E)$. Then*
$$p(E/xE) = p(E) + 1.$$

Proof. Since $x \in R_1 \cap \mathrm{NZD}(E)$ we have a short exact sequence of graded R-modules
$$0 \longrightarrow E(-1) \overset{x}{\longrightarrow} E \longrightarrow E/xE \longrightarrow 0,$$
so that, by the additivity of length we get
$$h_{E/xE}(n) = h_E(n) - h_E(n-1) = \Delta h_E(n) \text{ for all } n \in \mathbb{Z}.$$
Now, we conclude by Lemma 2.1. $\qquad\square$

In the course of our arguments we shall have to apply the previous lemma to the graded R-module E under the assumption that $\mathrm{depth}(E) > 0$. For this purpose, we need to find elements $x \in R_1 \cap \mathrm{NZD}(E)$ which avoid finitely many proper R_0-submodules of R_1. If the base ring $(R_0, \mathfrak{m}_0)$ is local with infinite residue field $R_0/\mathfrak{m}_0$, such elements always exist by the Homogeneous Prime Avoidance Lemma.

If $(R_0, \mathfrak{m}_0)$ is a local ring with finite residue field $R_0/\mathfrak{m}_0$, we will reduce the situation to the previous case as follows: we consider the localization $R_0' := R_0[X]_{\mathfrak{m}_0 R_0[X]}$, of the polynomial ring $R_0[X]$ at the prime ideal $\mathfrak{m}_0 R_0[X]$, which is an Artinian local ring with maximal ideal $\mathfrak{m}_0' = \mathfrak{m}_0 R'$ and infinite residue field $R_0'/\mathfrak{m}_0' \cong R_0/\mathfrak{m}_0(X)$. Observe that now $R' := R \otimes_{R_0} R_0'$ is a Noetherian standard graded ring over the local Artinian ring R_0' with infinite residue field, and that $E' := E \otimes_{R_0} R_0'$ is a finitely generated graded R'-module.

By [3, Example 16.2.4] we have $H^i_{R_+}(E)_n \otimes_{R_0} R_0' \cong H^i_{R_+'}(E')_n$ for all $i \geq 0$ and all $n \in \mathbb{Z}$. As R_0' is faithfully flat over R_0, this implies that
$$\mathrm{depth}(E') = \mathrm{depth}(E) \quad \text{and} \quad \mathrm{reg}(E') = \mathrm{reg}(E).$$
Moreover, we have
$$h_{E'}(n) = \mathrm{length}_{R_0'}(E_n') = \mathrm{length}_{R_0'}(R_0' \otimes_{R_0} E_n) = \mathrm{length}_{R_0}(E_n) = h_E(n)$$
for all $n \in \mathbb{Z}$, so that
$$h_{E'} = h_E, \quad p_{E'}(X) = p_E(X) \quad \text{and} \quad p(E') = p(E).$$
In our context, these observations will allow us to replace R and E respectively by R' and E'.

3. Castelnuovo-Mumford regularity and postulation numbers

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and let I be an $\mathfrak{m}$-primary ideal of A. We use the convention that $I^0 = A$, and we denote by
$$G := G_I(A) = \bigoplus_{n \geq 0} I^n/I^{n+1} = \bigoplus_{n \in \mathbb{Z}} I^{\max\{0,n\}}/I^{\max\{0,n+1\}}$$
the *associated graded ring* of A with respect to I. Let
$$H_I : \mathbb{Z} \longrightarrow \mathbb{N}_0, \quad n \mapsto H_I(n) := \mathrm{length}(A/I^{\max\{0,n+1\}}) = \sum_{k \leq n} h_G(k)$$
denote the *Hilbert-Samuel function* of I, which is a non-zero left vanishing arithmetic function of polynomial type. The *Hilbert-Samuel polynomial* of I is defined as the polynomial $P_I(X)$ such that
$$H_I(n) = P_I(n) \text{ for all } n \gg 0.$$

Observe that
$$h_G = \Delta H_I \quad \text{and} \quad p_G(X) = \Delta P_I(X) = P_I(X) - P_I(X - 1).$$
We define the *postulation number* of I by
$$n(I) := p(H_I) = \sup\{n \in \mathbb{Z} | H_I(n) \neq P_I(n)\} \quad (\in \mathbb{Z}).$$

The postulation numbers $p(G_I(A))$ and $n(I)$ are related as follows (see also Strunk, proof of [12, Theorem 3.10]).

Lemma 3.1. $\qquad\qquad p(G_I(A)) = n(I) + 1.$

Proof. As $h_G = \Delta H_I$, we may conclude by Lemma 2.1. $\qquad\qquad\square$

An ideal $J \subseteq I$ is called a *reduction* of I if
$$I^{n+1} = JI^n \text{ for some } n \in \mathbb{N}_0.$$
The least of these non-negative integers n is called the *reduction number* of I with respect to J and denoted by $r_J(I)$, thus
$$r_J(I) := \inf\{n \in \mathbb{N}_0 \mid I^{n+1} = JI^n\}.$$
If $J \subseteq I$ is a reduction of I and no other reduction of I is strictly contained in J, then J is called a *minimal reduction* of I. The *reduction number* of I, denoted by $r(I)$, is defined by
$$r(I) := \inf\{r_J(I) \mid J \subseteq I \text{ is a minimal reduction of } I\}.$$
Let us recall here the following result of Ngo Viet Trung [13, Proposition 3.2]

Lemma 3.2. $\quad a_d(G_I(A)) + d \leq r(I) \leq \text{reg}(G_I(A)).$

The aim of this section is to study the relation between $\text{reg}(G)$ and $p(G)$ if $I = \mathfrak{q}$ is a parameter ideal of A. We begin with the case $\dim(A) = 1$.

Lemma 3.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension one and let $\mathfrak{q} = (x)$ be a parameter ideal of A. Then*
$$\text{reg}(G_\mathfrak{q}(A)) = p(G_\mathfrak{q}(A)) + \text{depth}(A).$$

Proof. If $\text{depth}(A) = 1$, then $x \in \text{NZD}(R)$ so that $G_\mathfrak{q}(A) \cong A/\mathfrak{q}[X] = A/xA[X]$, whence $\text{reg}(G_\mathfrak{q}(A)) = 0$ and $p(G_\mathfrak{q}(A)) = -1$.
So assume that $\text{depth}(A) = 0$. Then $L := H^0_\mathfrak{m}(A) \subset A$ is a non-zero ideal of finite length. Moreover x is A/L-regular so that $L : x = L$ and hence $(x^n) \cap L = x^n L$ for all $n \geq 0$. Therefore
$$G_\mathfrak{q}(A/L) = \bigoplus_{n \geq 0} (x^n, L)/(x^{n+1}, L) = \bigoplus_{n \geq 0} (x^n)/(x^{n+1}, x^n L),$$
so that we have an exact sequence of the form
$$0 \longrightarrow K \longrightarrow G_\mathfrak{q}(A) \longrightarrow G_\mathfrak{q}(A/L) \longrightarrow 0,$$
where
$$K = \bigoplus_{n \geq 0} (x^{n+1}, x^n L)/(x^{n+1}) = \bigoplus_{n \geq 0} x^n L/x^{n+1} L.$$
Since A/L is Cohen-Macaulay and $\mathfrak{q}$ is a parameter ideal of A/L, we have $\text{reg}(G_\mathfrak{q}(A/L)) = 0$, and so it follows that
$$\text{reg}(G_\mathfrak{q}(A)) = \text{reg}(K) = \max\{n \in \mathbb{N}_0 \mid x^n L \neq 0\}.$$

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As $x \in \mathrm{NZD}(A/L)$ we have $H_{x(A/L)}(m) = P_{x(A/L)}(m)$ for all $m \geq -1$, so that for all $n \geq 0$ we get

$$H_{\mathfrak{q}}(n-1) = \mathrm{length}(A/\mathfrak{q}^n) = \mathrm{length}(A/(x^n A + L)) + \mathrm{length}((x^n A + L)/\mathfrak{q}^n)$$
$$= \mathrm{length}((A/L)/x^n(A/L)) + \mathrm{length}(L/x^n A \cap L)$$
$$= P_{x(A/L)}(n-1) + \mathrm{length}(L) - \mathrm{length}(x^n L).$$

This shows that $P_{\mathfrak{q}}(x) = P_{x(A/L)}(x) + \mathrm{length}(L)$, and it follows by Lemma 2.1 and Lemma 3.1 that

$$\max\{n \in \mathbb{N}_0 \mid x^n L \neq 0\} = p(H_{\mathfrak{q}}[-1]) = p(H_{\mathfrak{q}}(A)) + 1 = n(\mathfrak{q}) + 1 = p(G_{\mathfrak{q}}(A)).$$

This completes our proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Now, we can prove the following extension of the previous Lemma.

Proposition 3.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and let $\mathfrak{q}$ be a parameter ideal of A. If $\mathrm{depth}(A) = d - 1$ and $\mathrm{depth}(G_{\mathfrak{q}}(A)) = d - 1$, then*

$$\mathrm{reg}(G_{\mathfrak{q}}(A)) = p(G_{\mathfrak{q}}(A)) + d - 1 = n(\mathfrak{q}) + d.$$

Proof. First, let $d = 1$. Then we have $\mathrm{depth}(A) = 0$. Now, the assertion follows by Lemma 3.1.

So, let $d \geq 2$. Then $\mathrm{depth}(G_{\mathfrak{q}}(A)) = d - 1 > 0$. Assume first, that the residue field $A/\mathfrak{m}$ of A is finite and consider the Noetherian local ring $A' := A[X]_{\mathfrak{m}A[X]}$ with maximal ideal $\mathfrak{m}' = \mathfrak{m}A'$ and infinite residue field $A'/\mathfrak{m}' \cong A/\mathfrak{m}(X)$. Observe that

$$\dim(A') = d, \mathrm{depth}(A') = \mathrm{depth}(A) \text{ and } \mathfrak{q}' := \mathfrak{q}A' \text{ is a parameter ideal of } A'.$$

Moreover, with $G_{\mathfrak{q}}(A)'_0 := (G_{\mathfrak{q}}(A)_0)[X]_{\mathfrak{m}(G_{\mathfrak{q}}(A)_0)[X]}$ we have

$$G_{\mathfrak{q}'}(A') \cong G_{\mathfrak{q}}(A) \otimes_{G_{\mathfrak{q}}(A)_0} G_{\mathfrak{q}}(A)'_0.$$

According to our preliminary observations in Section 2 it follows that

$$\mathrm{depth}(G_{\mathfrak{q}'}(A)) = \mathrm{depth}(G_{\mathfrak{q}}(A)),$$
$$\mathrm{reg}(G_{\mathfrak{q}'}(A)) = \mathrm{reg}(G_{\mathfrak{q}}(A)),$$
$$p(G_{\mathfrak{q}'}(A)) = p(G_{\mathfrak{q}}(A)).$$

This allows to replace A and $\mathfrak{q}$ respectively by A' and $\mathfrak{q}'$ and hence to assume hat the residue field $A/\mathfrak{m}$ of A is infinite. But then, by the Homogeneous Prime Avoidance Lemma we find some element $x \in \mathfrak{q} \setminus \mathfrak{m}\mathfrak{q}$ with leading form

$$x^* := x + \mathfrak{q}^2 \in \mathfrak{q}/\mathfrak{q}^2 = G_{\mathfrak{q}}(A)_1$$

such that

$$x \in \mathrm{NZD}(A) \text{ and } x^* \in \mathrm{NZD}(G_{\mathfrak{q}}(A)).$$

In particular we have

$$\mathrm{depth}(A/xA) = d - 2 = \dim(A/xA) - 1.$$

Moreover, we may assume that $\mathfrak{q} = (x_1, ..., x_d)$ with $x := x_1$, so that $\mathfrak{q}(A/xA)$ is a parameter ideal of A/xA. In addition, as the initial form x^* is a non-zero divisor with respect to $G_{\mathfrak{q}}(A)$, it is well known, that there is an isomorphism of graded rings

$$G_{\mathfrak{q}}(A)/x^* G_{\mathfrak{q}}(A) \cong G_{\mathfrak{q}(A/xA)}(A/xA).$$

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Now, by induction, we have $\text{reg}(G_{\mathfrak{q}(A/xA)}(A/xA)) = p(G_{\mathfrak{q}(A/xA)}(A/xA)) + d - 2$, and it follows that

$$\text{reg}(G_{\mathfrak{q}}(A)/x^*G_{\mathfrak{q}}(A)) = p(G_{\mathfrak{q}}(A)/x^*G_{\mathfrak{q}}(A)) + d - 2.$$

As $x^* \in G_{\mathfrak{q}}(A)_1 \cap \text{NZD}(G_{\mathfrak{q}}(A))$ we have

$$\text{reg}(G_{\mathfrak{q}}(A)/x^*G_{\mathfrak{q}}(A)) = \text{reg}(G_{\mathfrak{q}}(A))$$

and moreover Lemma 2.4 implies that

$$p(G_{\mathfrak{q}}(A)/x^*G_{\mathfrak{q}}(A)) = p(G_{\mathfrak{q}}(A)) + 1.$$

From this we get our claim. $\qquad\square$

Now, we are ready to prove the promised first main result of our paper.

Theorem 3.5. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and let $\mathfrak{q}$ be a parameter ideal of A. If $\text{depth}(A) = d - 1$ and $d - 2 \leq \text{depth}(G_{\mathfrak{q}}(A)) \leq d - 1$, then we have the equality*

$$\text{reg}(G_{\mathfrak{q}}(A)) = p(G_{\mathfrak{q}}(A)) + d - 1 = n(\mathfrak{q}) + d.$$

Proof. If $\text{depth}(G_{\mathfrak{q}}(A)) = d - 1$ the assertion holds by Proposition 3.4.

So, let us assume that $\text{depth}(G_{\mathfrak{q}}(A)) = d - 2$. Suppose first that $d = 2$. Then $\text{depth}(A) = 1$ and $\text{depth}(G_{\mathfrak{q}}(A)) = 0$. By [5, Theorem 5.2] we have

$$0 \leq a_0(G_{\mathfrak{q}}(A)) < a_1(G_{\mathfrak{q}}(A)).$$

On the other hand, Lemma 3.2 yields that

$$a_2(G_{\mathfrak{q}}(A)) + 2 \leq r(\mathfrak{q}) \leq \text{reg}(G_{\mathfrak{q}}(A)).$$

Since $\mathfrak{q}$ is a parameter ideal, we have $r(\mathfrak{q}) = 0$, and it follows that $a_1(G_{\mathfrak{q}}(A)) > a_2(G\mathfrak{q}(A))$. This implies that

$$\text{reg}(G_{\mathfrak{q}}(A)) = a_1(G_{\mathfrak{q}}(A)) + 1 \quad \text{and} \quad p(G_{\mathfrak{q}}(A)) = a_1(G_{\mathfrak{q}}(A))$$

(by Corollary 2.3(2)) and hence shows that the assertion of our theorem holds if $d = 2$.

So, let $d > 2$. Then $\text{depth}(G_{\mathfrak{q}}(A)) > 0$. As in the proof of Proposition 3.4 we may assume that the residue field $A/\mathfrak{m}$ of A is infinite, so that we again find an element

$$x \in \text{NZD}(A) \cap (\mathfrak{q} \setminus \mathfrak{m}\mathfrak{q}) \text{ with } x^* := x + \mathfrak{q}^2 \in \mathfrak{q}/\mathfrak{q}^2 = G_{\mathfrak{q}}(A)_1 \cap \text{NZD}(G_{\mathfrak{q}}(A)).$$

Now, $\text{depth}(A/xA) = d - 2 = \dim(A/xA) - 1$ and $\mathfrak{q}(A/xA)$ is a parameter ideal of A/xA. We also have an isomorphism

$$G_{\mathfrak{q}}(A)/x^*G_{\mathfrak{q}}(A) \cong G_{\mathfrak{q}(A/xA)}(A/xA).$$

Hence

$$d - 3 \leq \text{depth}(G_{\mathfrak{q}(A/xA)}(A/xA)) \leq d - 2.$$

So, by induction on d we may assume that

$$\text{reg}(G_{\mathfrak{q}/xA}(A/xA)) = p(G_{\mathfrak{q}/xA}(A/xA)) + d - 2.$$

By Lemma 2.4, we get

$$\text{reg}(G_{\mathfrak{q}}(A)) = \text{reg}(G_{\mathfrak{q}}(A)/x^*G_{\mathfrak{q}}(A)) \quad \text{and} \quad p(G_{\mathfrak{q}}(A)) = p(G_{\mathfrak{q}}(A)/x^*G_{\mathfrak{q}}(A)) - 1.$$

Altogether, this allows us to complete the proof. $\qquad\square$

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We aim to draw a conclusion from this result, which involves the notion of relation type. We first recall a few facts about this invariant. Let $I \subset A$ be an arbitrary $\mathfrak{m}$-primary ideal of A and let

$$R_I(A) := \bigoplus_{n \in \mathbb{N}_0} I^n = A[IX]$$

be the *Rees algebra* of A with respect to I. Ooishi [11, Lemma 4.8] has proved that

$$\mathrm{reg}(G_I(A)) = \mathrm{reg}(R_I(A)).$$

Let $x_1, \ldots, x_r$ be a minimal system of generators of I. Then we can present the Rees algebra of I in the form

$$R_I(A) = A[X_1, \ldots, X_r]/J,$$

where $A[X_1, \ldots, X_t]$ is a polynomial ring and $J \subset A[X_1, \ldots, X_r]$ is a homogeneous ideal. The *relation type* reltype(I) of the ideal I is defined as the largest degree occurring in a minimal homogeneous system of generators of J. By [13, Corollary 3.3 and Proposition 4.1], we have the following basic estimate

Proposition 3.6. reltype$(I) \leq \mathrm{reg}(G_I(A)) + 1$.

In the special case of a local ring A of dimension 2 and depth 1 and if I is a parameter ideal of A, one has the following equality:

Lemma 3.7. [14, Theorem 5.2] *Let $(A, \mathfrak{m})$ be a two-dimensional Noetherian local ring with* $\mathrm{depth}(A) = 1$. *Let $\mathfrak{q}$ be a parameter ideal of A. Then*

$$\mathrm{reltype}(\mathfrak{q}) = p(G_\mathfrak{q}(A)) + 2.$$

Applying Theorem 3.5 and Lemma 3.7 in the special situation of a parameter ideal in a two-dimensional local ring of depth 1, we therefore get equality in Proposition 3.6:

Corollary 3.8. *Let $(A, \mathfrak{m})$ be a two-dimensional Noetherian local ring with* $\mathrm{depth}(A) = 1$ *and let $\mathfrak{q}$ be a parameter ideal of A. Then*

$$\mathrm{reltype}(\mathfrak{q}) = \mathrm{reg}(G_\mathfrak{q}(A)) + 1.$$

If we combine Corollary 3.8 with a bounding result of Wang [14], we obtain a uniform bound for the regularity of the associated graded ring of parameter ideals:

Corollary 3.9. *Let $(A, \mathfrak{m})$ be a two-dimensional Noetherian local ring. Then, there exists an integer $N \geq 0$ such that the regularity of the associated graded ring of A with respect to each parameter ideal is bounded by N, thus*

$$\mathrm{reg}(G_\mathfrak{q}(A)) \leq N \text{ for all parameter ideals } \mathfrak{q} \text{ of } A.$$

Proof. Let $\mathfrak{q}$ be an parameter ideal of A. Set $L := H_\mathfrak{m}^0(A)$. By [8, Lemma 4.3] we have

$$\mathrm{reg}(G_\mathfrak{q}(A)) \leq \mathrm{reg}(G_\mathfrak{q}(A/L)) + \mathrm{length}(L).$$

Therefore, we may replace A by A/L and hence assume that $\mathrm{depth}(A) > 0$.
If $\mathrm{depth}(A) = 2$ the ring A is Cohen-Macaulay so that $\mathrm{reg}(G_\mathfrak{q}(A)) = 0$.
So, let us assume that $\mathrm{depth}(A) = 1$. Then, by Corollary 3.8 we have

$$\mathrm{reg}(G_\mathfrak{q}(A)) = \mathrm{reltype}(\mathfrak{q}) - 1.$$

Applying the uniform bounding result for the relation type of parameter ideals in a two-dimensional local Noetherian ring given by Wang [14, Corollary 5.3], we obtain the requested bound. $\qquad\square$

4. A Bound for the regularity in terms of the relation type

Let $(A, \mathfrak{m})$ be a Noetherian local ring and let $I = (x_1, \ldots, x_r)$ be an $\mathfrak{m}$-primary ideal of A. In this section, we shall give a bound for the regularity of $G_I(A)$ in terms of the relation type reltype(I) of I. We begin with the following auxiliary result, in which

$$\text{gendeg}(M) := \inf\{n \in \mathbb{Z} \mid M = R \sum_{k \leq n} M_k\}$$

is used to denote the *generating degree* of a graded module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ over a positively graded ring $R = \bigoplus_{n \in \mathbb{N}_0} R_n$.

Lemma 4.1. *Let $S = A_0[X_1, ..., X_r]$ be a polynomial ring over an Artinian local ring A_0 and let J be a homogeneous ideal of S such that $\dim(S/J) = d \geq 2$. Then*

$$\text{reg}(J) \leq [(r-d+1)\text{gendeg}(J)^{r-d+1}]^{2^{d-2}}.$$

Proof. See [4, Example 3.6]. $\qquad\square$

Now, we get the following bounding result:

Theorem 4.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and let $I = (x_1, \ldots, x_r)$ be an $\mathfrak{m}$-primary ideal of A. Then*

$$\text{reg}(G_I(A)) \leq [(r-d+1)\text{reltype}(I)^{r-d+1}]^{2^{d-2}} - 1.$$

Proof. We may write $R_I(A) \cong A[X_1, \ldots, X_r]/J$, where J is a homogeneous ideal of the polynomial ring $A[X_1, \ldots, X_r]$. Then, we have

$$\text{reltype}(I) = \text{gendeg}(J).$$

Moreover, we may write

$$G_I(A) = R_I(A)/IR_I(A) \cong A[X_1, \ldots, X_r]/(IA[X_1, \ldots, X_r] + J).$$

Since $A[X_1, \ldots, X_r]/IA[X_1, \ldots, X_r] \cong A/I[X_1, \ldots, X_r]$, we have

$$A[X_1, \ldots, X_r]/(IA[X_1, \ldots, X_r] + J) \cong A/I[X_1, \ldots, X_r]/\overline{J},$$

where $\overline{J} = J(A/I[X_1, \ldots, X_r])$. Hence, with $A_0 := A/I$, we finally get an isomorphism of graded rings

$$G_I(A) \cong A_0[X_1, ..., X_r]/\overline{J}.$$

On application of Lemma 4.1, we now obtain

$$\text{reg}(G_I(A)) \leq \text{reg}(\overline{J}) - 1 \leq [(r-d+1)\text{gendeg}(\overline{J})^{r-d+1}]^{2^{d-2}} - 1.$$

Since $\text{gendeg}(\overline{J}) \leq \text{gendeg}(J) = \text{reltype}(I)$, we finally get

$$\text{reg}(G_I(A)) \leq [(r-d+1)\text{reltype}(I)^{r-d+1}]^{2^{d-2}} - 1.$$

$\qquad\square$

If $I = \mathfrak{q} = (x_1, ..., x_d)$ is a parameter ideal of A, the previous result furnishes a bound for $\text{reg}(G_{\mathfrak{q}}(A))$ which depends only on d and the relation type of $\mathfrak{q}$.

Corollary 4.3. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring with* $\dim(A) = d \geq 2$ *and let* $\mathfrak{q}$ *be a parameter ideal of* A. *Then*

$$\mathrm{reg}(G_{\mathfrak{q}}(A)) \leq \mathrm{reltype}(\mathfrak{q})^{2^{d-2}} - 1.$$

If $\dim(A) = 2$, the above inequality implies that $\mathrm{reg}(G_I(A)) \leq \mathrm{reltype}(I) - 1$ and hence that $\mathrm{reg}(G_I(A)) + 1 \leq \mathrm{reltype}(I)$. Since $\mathrm{reltype}(I) \leq \mathrm{reg}(G_I(A)) + 1$, by Proposition 3.6, this furnishes again the equality $\mathrm{reltype}(I) = \mathrm{reg}(G_I(A)) + 1$ of Corollary 3.8, but this time without the hypotheses that $\mathrm{depth}(A) = 1$.

Corollary 4.4. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring with* $\dim(A) = 2$ *and let* $\mathfrak{q}$ *be a parameter ideal of* A. *Then*

$$\mathrm{reltype}(\mathfrak{q}) = \mathrm{reg}(G_{\mathfrak{q}}(A)) + 1.$$

From Corollary 4.3 we now obtain the promised equivalence of the existence of a uniform bound for the regularity of the associated graded ring and the existence of a uniform bound for the relation type of parameter ideals, and this is the desired affirmative answer to Question 1 mentioned in the introduction.

Theorem 4.5. *Let* A *be a local ring with* $\dim(A) \geq 1$. *Then the following conditions are equivalent:*

> (i) *There exists a uniform bound for the relation type of parameter ideals of* A.
> (ii) *There exists a uniform bound for the regularity of the associated graded ring of parameter ideals of* A.

To formulate our last application, we denote by $\mathcal{A}(R)$ the ideal of a Noetherian local ring R generated by the *homology multipliers* as introduced in [1].

Corollary 4.6. *Let* $(A, \mathfrak{m})$ *be a formally unmixed local ring of dimension* ≥ 2 *such that the ideal of homology multipliers* $\mathcal{A}(\widehat{A})$ *in the completion* $\widehat{A}$ *of* A *is prime and satisfies* $\dim(\widehat{A}/\mathcal{A}(\widehat{A})) = 1$. *Then* A *admits a uniform bound for the regularity of the associated graded ring of parameter ideals.*

Proof. According to [1, Theorem 7.2] there is a uniform bound for the relation type of parameter ideals in A. Now, we conclude by Theorem 4.5. $\square$

It is well known that the reduction number $r(I)$ and the postulation number $n(I)$ are bounded by $\mathrm{reg}(G_I(A))$. Thus we get the following corollaries.

Corollary 4.7. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 2$ *and let* $I = (x_1, \ldots, x_r)$ *be an* $\mathfrak{m}$-*primary ideal of* A. *Then*

$$r(I) \leq [(r - d + 1)\,\mathrm{reltype}(I)^{r-d+1}]^{2^{d-2}}.$$

Corollary 4.8. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 2$ *and let* $I = (x_1, \ldots, x_r)$ *be an* $\mathfrak{m}$-*primary ideal of* A. *Then*

$$n(I) \leq [(r - d + 1)\,\mathrm{reltype}(I)^{r-d+1}]^{2^{d-2}}.$$

We finally return to uniform bounds for the postulation number of parameter ideals, more precisely to the question as to whether such bounds imply uniform bounds for the relation type or, equivalently for the regularity of the associated graded ring of such ideals. We do not know whether this question has an affirmative

answer in general. We first give the following result, which seems to be related to the mentioned question. It generalizes a result of B. Strunk (see [12, Theorem 2.9]) which applies to standard graded rings $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ whose base ring R_0 is a field.

Proposition 4.9. *There is a function*

$$B : \mathbb{N}^2 \times \mathbb{Z} \longrightarrow \mathbb{N}_0$$

such that for each standard graded Noetherian ring $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ with Artinian local base ring $(R_0, \mathfrak{m}_0, k)$ it holds

$$\mathrm{reg}(R) \leq B\big(\mathrm{length}(R_0), \dim_k(R_1 \otimes_{R_0} k), p(R)\big)$$

Proof. Let $q \in \mathbb{Q}[X]$ be a polynomial. According to [2, Proposition 5, pg. 212] there is a function

$$F_q : \mathbb{N}^2 \times \mathbb{Z} \longrightarrow \mathbb{Z}$$

such that for each standard graded ring $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ as above with Hilbert polynomial $p_R = q$ we have

$$\mathrm{reg}(R) \leq F_q\big(\mathrm{length}(R_0), \dim_k(R_1 \otimes_{R_0} k), p(R)\big).$$

Now, fix $\lambda, r \in \mathbb{N}$, $s \in \mathbb{Z}$ and set $t := \max\{0, s\} + 1$. Let $\mathcal{S}_{\lambda,r,s}$ be the set of all standard graded rings $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ as above which satisfy

$$\mathrm{length}(R_0) = \lambda, \dim_k(R_1 \otimes_{R_0} k) = r \text{ and } p(R) = s.$$

Then, for each $R \in \mathcal{S}_{\lambda,r,s}$ there is a surjective homomorphism of graded R_0-algebras

$$S := R_0[X_1, \dots, X_r] \longrightarrow R \longrightarrow 0,$$

where S is a polynomial ring in r indeterminates. It follows that $\deg(p_R) < r$. So, as $p(R) = s < t$, the r values

$$h_R(i) = \mathrm{length}(R_i) \leq \mathrm{length}(S_i) = \binom{r+i-1}{i}\lambda, \quad t \leq i \leq t+r-1$$

determine the Hilbert polynomial p_R for each $R \in \mathcal{S}_{\lambda,r,s}$. Consequently the set $\{p_R \mid R \in \mathcal{S}_{\lambda,r,s}\}$ is finite, and it suffices to set

$$B(\lambda, r, s) := \max\{F_{p_R}(\lambda, r, s) \mid R \in \mathcal{S}_{\lambda,r,s}\} \text{ for all } (\lambda, r, s) \in \mathbb{N}^2 \times \mathbb{Z}.$$

$\square$

As an immediate application we obtain

Corollary 4.10. *Let the function $B : \mathbb{N}^2 \times \mathbb{Z} \longrightarrow \mathbb{N}$ be as in Proposition 4.9. Then, for each local ring A and each parameter ideal $\mathfrak{q}$ of A it holds*

$$\mathrm{reg}\big(G_{\mathfrak{q}}(A)\big) \leq B\big(\mathrm{length}(A/\mathfrak{q}), \dim(A), p(G_{\mathfrak{q}}(A))\big).$$

So, one approach to the above question could be to try "to eliminate the length" from the bounding function B.

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