PROJECTIVE SURFACES OF DEGREE $r+1$ IN PROJECTIVE r-SPACE AND ALMOST NON-SINGULAR PROJECTIONS

MARKUS BRODMANN AND PETER SCHENZEL

ABSTRACT. We study projective surfaces of degree $r+1$ in projective r-space, more precisely (non-conic) irreducible non-degenerate surfaces $X \subset \mathbb{P}^r$ of degree $r+1$. We may divide up the class of these surfaces in surfaces whose affine cone satisfies the second Serre property S_2 and surfaces which occur as almost non-singular projections of either a smooth rational sroll or else of a del Pezzo surfaces which is arithmetically Cohen Macaulay. We focus on those surfaces which occur as almost non-singular projections and study their geometric and cohomological properties.

1. Introduction

Let $r \geq 5$ be an integer, let $\mathbb{P}^r_K$ denote the projective r-space over the algebraically closed field K and let $X \subset \mathbb{P}^r_K$ be a non-degenerate irreducible projective variety. Then, the invariant

$$\Delta'(X) := \deg(X) - \operatorname{codim}(X) - 1 = \Delta(X) + h^1(X, \mathcal{J}_X(1)),$$

where $\Delta(X) = \deg(X) - h^0(X, \mathcal{O}_X(1)) + \dim(X)$ denotes the Δ-genus of the polarized pair $(X, \mathcal{O}_X(1))$ in the sense of Fujita [F]. Note that $\Delta(X)$ is always non-negative. We allow ourselves to call $\Delta'(X)$ the *Δ'-genus of X*.

If $\Delta'(X) = 0$, the variety X is called a *variety of minimal degree*. It is classical that these varieties are (cones over) smooth rational normal srolls or over the Veronese surface. The structure of them is well understood.

If $\Delta'(X) = 1$, we say that X is a *variety of almost minimal degree*. These varieties may been understood up to isomorphic projections by Fujita's classification of polarized pairs of Δ-genus ≤ 1 (see [F]). In [BS4] we have shown that they are either arithmetically normal or else simple linear exterior birational projections of a variety $\tilde{X} \subset \mathbb{P}^{r+1}_K$ of minimal degree. In the latter case the singular locus of the induced morphism $f : \tilde{X} \twoheadrightarrow X$ is of codimension > 1 if and only if X is normal.

An obvious next step is to study varieties satisfying $\Delta'(X) = 2$. Again, these varieties may be understood up to isomorphic projections by Fujita's classification of varieties of Δ-genus ≤ 2 (see [F]). One might try to study these varieties by an approach similar to what we did in our investigation of varieties of almost minimal degree, namely: to distinguish *varieties with "good arithmetic properties"* and varieties which are birational linear outer projections of *"varieties which are already classified"*. In the case of curves, we did pursue this idea in [BS1]. In the present paper we consider the case of surfaces.

Date: Zürich, 3. December 2011.

2000 Mathematics Subject Classification. Primary: 14H45, 13D02.

Key words and phrases. projective surface, Δ-genus 2, outer projection.

So, from the mentioned point of view our aim is to study non-degenerate irreducible projective surfaces $X \subset \mathbb{P}^r$ which satisfy the equality

$$\deg(X) = r + 1.$$

In this situation, the above distinction of two types of varieties still takes a comparatively simple form. We shall see in the current of this paper, that any surface $X \subset \mathbb{P}^r$ with $\deg(X) = r + 1$ belongs to one of the following two classes:

(A) *Surfaces with "good arithmetic properties", i.e. projective surfaces whose affine cone satisfies the second Serre property S_2.*

(B) *Surfaces which are almost non-singular projections – i.e. outer linear projections with finitely many mapping singularities – of a surface of strictly lower Δ'-genus.*

This allows to pursue the general approach mentioned above. In the spirit of what we did in the case of varieties of almost minimal degree and in the case of curves of Δ'-genus 2, we shall not focus on the *"generic class"* (A) any more and restrict ourselves to study those surfaces of Δ'-genus 2, which belong to the class (B). We start from the *"classification by cohomological invariants"* of surfaces of Δ'-genus 2 which is given in [B2] and which leaves us with twelve different cases $1 - 12$, which might occur.

In the cases 1 and 2 we have surfaces which belong to the above class (A), hence surfaces which are arithmetically S_2. In the 9 cases $3, 4, \ldots, 11$ we have surfaces which belong to the class (B). We also shall see, that the case 12 does not occur at all.

As the structure of curves of Δ'-genus 2 is known by [BS1], it suffices to consider only non-conic surfaces $X \subset \mathbb{P}^r$ of degree $r + 1$. In this situation we shall get the following detailed subdivision of the classes (A) and (B):

- Case 1: *The surface X is arithmetically Cohen-Macaulay* (CM).
- Case 2: *The surface X is arithmetically S_2 but not arithmetically* CM.
- Cases 3 and 4: *The surface X is an almost non-singular projection from a point of a* maximal del Pezzo *surface, i.e. of a surface $\tilde{X} \subset \mathbb{P}^{r+1}$ of degree $r+1$ which is arithmetically* CM. *The distinction of the two cases is given according to whether the projection map $X' \twoheadrightarrow X$ is an isomorphism or not.*
- Cases $5 - 11$: *The surface X is an almost non-singular projection from a line of a rational normal surface scroll $\tilde{X} \subset \mathbb{P}^{r+2}$.*

Our aim is to establish this subdivision and to discuss in detail the cases 3–11. We also shall characterize these cases in terms of simple almost non-singular projections of certain surfaces of almost minimal degree and in terms of double projections of smooth rational normal surface scrolls.

In Section 2 Theorem 2.7 we prove the basic fact, that our surfaces always posses *(arithmetic) S_2-covers.* That is, they are almost non-singular outer projections from varieties which are arithmetically S_2 (see Theorem 2.7). We approach this result in a purely algebraic way and consider S_2-covers of homogeneous coordinate rings. As a first consequence we may identify the surfaces which belong to the class (A), exclude the case 12 of [B2] and get some restriction results for embedding dimensions and Hartshorne-Rao numbers (see Theorem 2.8). In Section 2, we shall make heavy use of the classification results of [B2] and so we list these in a number of tables.

In Section 3 we first give results on the second Hartshorne-Rao number $h^1(\mathbb{P}^r_K, \mathcal{J}_X(2))$ of our surfaces X. This leads to a much simpler procedure to distinguish the cases 3 – 11 than the one used in [B2] (see Corollary 3.1). Namely, one does not have to know the *sectional regularity* (i.e. the lowest possible Castelnuovo-Mumford regularity of a hyperplane section) of X any more. From the computational point of view, this is a considerable advantage, as in general the sectional regularity can be calculated only by a generic hyperplane section. In a concrete situation it will be difficult to describe "genericity" of a hyperplane. We illustrate the suggested method at some particular types of surfaces (see Remark 3.3 and Examples 3.4, 3.5). In this section we also determine the structure of the *second deficiency module* $K^2(A)$ of the homogeneous coordinate ring A of our surfaces X in all 11 cases (see Theorem 3.6).

In Section 4 we show that the (non-conic) surfaces X which fall under case 3 or 4 are precisely the almost non-singular projections of a (non-conic) *maximal del Pezzo surface* $\tilde{X}$ from a point. In this situation, the surface $\tilde{X}$ is uniquely determined by X up to projective equivalence (see Theorem 4.7). In particular we shall see that the projecting maximal del Pezzo surface $\tilde{X}$ is non-normal if and only if X has infinitely many singularities or – equivalently – more than one non-normal point (see Corollary 4.8).

In Section 5 we show that the (non-conic) surfaces which fall under the cases 5 – 11 are precisely the almost non-singular projections of a smooth rational surface scroll $\tilde{X}$ from a line and that the type of the projecting scroll $\tilde{X}$ is uniquely determined by X in this case (see Theorem 5.7). As a consequence we get that these (non-conic) surfaces are precisely the almost non-singular projections of a non-linearly normal surface $\tilde{X}$ of almost minimal degree from a point (see Corollary 5.8 (b)). In this situation, the projective equivalence class of the projecting surface $\tilde{X}$ is not determined by X. We also characterize those of our surfaces X which are outer linear projections of smooth rational surface scrolls $\tilde{X}$ from lines not contained in the secant locus of $\tilde{X}$ (see Corollaries 5.10, 5.11).

2. S_2-Covers

We first fix a few notations which we keep for the rest of our paper. By $\mathbb{N}_0$ and $\mathbb{N}$ we denote the set of non-negative respectively of positive integers.

2.1. **Notation.** (A) Let K be an algebraically closed field, let $r \geq 5$ be an integer, consider the polynomial ring $S := K[x_0, \ldots, x_r]$ and let $X \subset \mathbb{P}^r_K = \mathrm{Proj}(S)$ be a reduced, irreducible and non-degenerate projective surface. Let $\mathcal{J} = \mathcal{J}_X \subset \mathcal{O}_{\mathbb{P}^r_K}$ denote the sheaf of vanishing ideals of X, let $I = I_X = \oplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^r_K, \mathcal{J}(n)) \subset S$ be the homogeneous vanishing ideal of X and let $A = A_X = S/I$ denote the coordinate ring of X.
(B) If M is a finitely generated graded S-module we write $H^i(M) = H^i_{S_+}(M)$ and $D(M) = D_{S_+}(M)$ for the i-th local cohomology module of M with respect to $S_+ := \oplus_{n\in\mathbb{N}} S_n$ and the S_+-transform $\varinjlim \mathrm{Hom}((S_+)^n, M)$ of M. We also write

$$h^i_M(n) := \dim_K H^i(M)_n,$$

where $H^i(M)_n$ denotes the n-th graded component of $H^i(M)$. Keep in mind that

$$H^i(A)_n = H^i(\mathbb{P}^r_K, \mathcal{J}(n)) \text{ for all } i < r \text{ and all } n \in \mathbb{Z}.$$

(C) By $\sigma(X)$ we shall denote the *sectional genus* of X, so that $\sigma(X) = p_a(\mathrm{Proj}(A/fA)) = h^2_{A/fA}(0)$ for a generic linear form $f \in S_1$. We also introduce the invariant

$$e(X) := \sum\nolimits_{x\in X,\mathrm{closed}} \text{length } (H^1_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x}))$$

which counts the *non-Cohen-Macaulay points* of X in a weighted way (see also 3.7 for the relation to local duality). Keep in mind that

$$e(X) = h^2_A(n) = h^1(X, \mathcal{O}_X(n)) \text{ for all } n \ll 0.$$

The *arithmetic depth* of X shall be denoted by depth X.
(D) For a closed subscheme $X \subset \mathbb{P}^r_K$ let $\mathrm{Reg}(X), \mathrm{Nor}(X), \mathrm{CM}(X)$ denote respectively the locus of smooth, normal and Cohen-Macaulay points $x \in X$.

2.2. **Reminder.** (A) Keep the above notations and hypotheses. In particular let $r \geq 5$. If $\deg(X) = r+1$ and $f \in S_1$ is a generic linear form we have $h^1_{A/fA}(n) = 0$ for all $n \neq 1, 2$ and

$$\text{sct } X := \big(h^1_{A/fA}(1), h^1_{A/fA}(2)\big) \in \{(0,0), (1,0), (2,0), (2,1)\}$$

(see [B2, Remark 4.1], [BS1, (3.9)]). We call the pair

$$\text{sct } X = \big(h^1_{A/fA}(1), h^1_{A/fA}(2)\big) \in \mathbb{N}_0 \times \mathbb{N}_0$$

the *sectional cohomology type* of X.
(B) If $r \geq 5$ and $\deg(X) = r + 1$, then at most the following 12 cases can be expected to occur (see [B2, Propositions (4.2), (4.11), (5.5), (5.6)])

Case		sct X	depth X	$\sigma(X)$	$e(X)$	$h^1_A(1)$
1	I	$(0,0)$	3	2	0	0
2	IIA	$(1,0)$	2	1	0	0
3	IIA'	$(1,0)$	2	1	1	0
4	IIB	$(1,0)$	1	1	0	1
5	IIIA	$(2,0)$	2	0	2	0
6	IIIB	$(2,0)$	1	0	1	1
7	IIIC	$(2,0)$	1	0	0	2
8	IVAI	$(2,1)$	2	0	3	0
9	IVC	$(2,1)$	1	0	0	2
10	IVBO	$(2,1)$	1	0	1	1
11	IVAO	$(2,1)$	1	0	2	0
12	IVB1	$(2,1)$	1	0	2	1

Column 2 shows that the labeling of cases as given in [B2]. Here we shall use mostly the simpler labeling given in column 1.
(C) In each of the above cases much more can be said about the numerical invariants of X (cf [B2]). We first remind a useful fact concerning the Hartshorne-Rao module

$$H^1(A) = \oplus_{n\in\mathbb{Z}} H^1(\mathbb{P}^r_K, \mathcal{J}(n))$$

of X, which is proved in the previously quoted results of [B2]: In all cases except the case 11, the S-module $H^1(A)$ is generated by homogeneous elements of degree 1. In particular $D := D(A) = A + SD_1 = K[D_1]$ with $\dim_K(D_1) = r + 1 + h^1_A(1)$. Here D_1 denotes the degree one component of the global transform D. Concerning the Hartshorne-Rao function

$$h^1_A : \mathbb{Z} \to \mathbb{N}_0,\ n \mapsto h^1_A(n)$$

of X we have the following table, in which $a_1(n) := \sup\{n \,|\, h^1_A(n) \neq 0\}$.

Case		$n < 0$	$n = 1$	$n = 2$	$3 \leq n$	$a_1(A)$
1	$h^1_A(n) =$	0	0	0	0	$-\infty$
2	$h^1_A(n) =$	0	0	0	0	$-\infty$
3	$h^1_A(n) =$	0	0	0	0	$-\infty$
4	$h^1_A(n) =$	0	1	0	0	1
5	$h^1_A(n) =$	0	0	0	0	$-\infty$
6	$h^1_A(n) =$	0	1	≤ 1	0	≤ 2
7	$h^1_A(n) =$	0	2	≤ 2	$\leq \max\{h^1_A(n-1)-1, 0\}$	≤ 3
8	$h^1_A(n) =$	0	0	0	0	$-\infty$
9	$h^1_A(n) =$	0	2	≤ 3	$\leq \max\{h^1_A(n-1)-1, 0\}$	≤ 4
10	$h^1_A(n) =$	0	1	≤ 2	$\leq \max\{h^1_A(n-1)-1, 0\}$	≤ 3
11	$h^1_A(n) =$	0	0	1	0	2
12	$h^1_A(n) =$	0	1	1	$\leq \max\{n-2, 0\}$	≤ 3

(D) Finally in the situation where $r \geq 5$ and $\deg(X) = r + 1$ we list the values of the second cohomological Hilbert function

$$h^2_A : \mathbb{Z} \to \mathbb{N}_0,\ n \mapsto h^2_A(n) = h^2(\mathbb{P}^r_K, \mathcal{J}(n)) = h^1(X, \mathcal{O}_X(n))$$

as they are found in the previously quoted results of [B2].

Case		$n \leq -1$	$n = 0$	$n = 1$	$2 \leq n$
1	$h^2_A(n) =$	0	0	0	0
2	$h^2_A(n) =$	0	1	0	0
3	$h^2_A(n) =$	1	1	0	0
4	$h^2_A(n) =$	0	0	0	0
5	$h^2_A(n) =$	2	2	0	0
6	$h^2_A(n) =$	1	1	0	0
7	$h^2_A(n) =$	0	0	0	0
8	$h^2_A(n) =$	3	3	1	0
9	$h^2_A(n) =$	0	0	0	0
10	$h^2_A(n) =$	1	1	0	0
11	$h^2_A(n) =$	2	2	0	0
12	$h^2_A(n) =$	2	1	1	0

2.3. **Remark and Definition.** (A) Let $\mathfrak{a} \subseteq A_+$ be the graded radical ideal which defines the non-Cohen-Macaulay locus $X \setminus \mathrm{CM}(X)$ of X. Note that $\mathfrak{a} = A_+$ if and only if $X = \mathrm{CM}(X)$. Observe that height $\mathfrak{a} \geq 2$, so that the ideal transform

$$B(A) := D_{\mathfrak{a}}(A) = \varinjlim \mathrm{Hom}_A(\mathfrak{a}^n, A) = \oplus_{n\in\mathbb{Z}}\Gamma(\mathrm{CM}(X), \mathcal{O}_X(n))$$

of A with respect to $\mathfrak{a}$ is a positively graded finite birational integral extension domain of A. Moreover $B(A)$ has the second Serre-property S_2 (see e.g. [BS4, Proposition 5.2]). As $\mathrm{Proj}(B(A))$ is of dimension 2, it thus is a CM-scheme. We call $B(A)$ the *S_2-cover of A*.

We can also describe $B(A)$ as the endomorphism ring $\operatorname{End}(K(A), K(A))$ of the canonical module $K(A) = K^3(A) = \operatorname{Ext}_S^{r-2}(A, S(-r-1))$ of A (cf [BS4, Proposition 5.2]).
(B) As $\mathfrak{a} \subseteq A_+$ we have $D(A) = D_{A_+}(A) \subseteq D_{\mathfrak{a}}(A) = B(A)$. As $X \simeq \operatorname{Proj}(D(A))$ we have equality if and only if $\mathfrak{a} = A_+$, hence if and only if $X = \operatorname{CM}(X)$. Finally, if B' is an S_2-ring with $A \subseteq B' \subseteq B(A)$, then $B' = B(A)$.
(C) Let the natations be as above and let $\nu : \widetilde{X} = \operatorname{Proj}(B) \to X = \operatorname{Proj}(A)$ be the finite birational morphism induced by the inclucion map $A \to B$ and let $C := B/D$. Then, the short exact sequence of graded S-modules

$$0 \to D \to B \to C \to 0$$

induces the short exact sequence

$$0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{\widetilde{X}} \to \nu_* \mathcal{O}_{\widetilde{X}} / \mathcal{O}_X \to 0.$$

In particular we have

$$\operatorname{Supp}(\widetilde{B/A}) = \operatorname{Supp}(\widetilde{B/D}) = \operatorname{Supp}(\nu_* \mathcal{O}_{\widetilde{X}} / \mathcal{O}_X) = X \setminus \operatorname{CM}(X).$$

2.4. **Lemma.** *With the previous notation let $D := D(A)$ and $B = B(A)$. Then*

(a) $e(X) = 0$ *if and only if* $B = D$.
(b) *Suppose that* $e(X) > 0$. *Then*
   (i) $h_B^2(n) \leq h_A^2(n)$ *and* $\dim_K(B_n/D_n) = e(X) - h_A^2(n) + h_B^2(n)$ *for all* $n \in \mathbb{Z}$.
   (ii) $a(X) := \sup\{n \in \mathbb{Z} | h_B^2(n) \neq h_A^2(n)\} \geq 0$.
   (iii) *For a generic linear form* $f \in S_1$, *the multiplication map*

$$f : B_n/D_n \to B_{n+1}/D_{n+1}$$

   is an injection for all $n \in \mathbb{N}_0$ *and an isomorphism for all* $n > a(X)$.
   (vi) *The graded* A-*module* B/D *is generated in degrees* $\leq a(X) + 1$.

Proof. Let $C := B/D$. We shall repeatedly use the short exact sequence of graded S-modules $0 \to D \to B \to C \to 0$ of Remark and Definition 2.3 (C).
(a) As $e(X) = 0$ is equivalent to the fact that X is Cohen-Macaulay, we conclude by Remark and Definition 2.3 (B).
(b) Let let $f \in S_1$ be a generic linear form. The short exact sequence $0 \to D \to B \to C \to 0$ together with the isomorphism $H^2(A) \simeq H^2(D)$ gives rise to exact sequences of K-vector spaces

$$(*) \qquad 0 \to H^1(C)_n \to H^2(A)_n \to H^2(B)_n \to 0$$

for all $n \in \mathbb{Z}$. In particular we have $h_A^2(n) \geq h_B^2(n)$ for all $n \in \mathbb{Z}$.

As the A-module B is S_2, we have $h^2(B)_n = 0$ for all $n \ll 0$. As $H^1(C)_n \simeq D(C)_n$ and $h_A^2(n) = e(X)$ for all $n \ll 0$ it follows that $\dim_K D(C)_n = e(X)$ for all $n \ll 0$. As $\dim_A(C) \leq 1$ and by the genericity of $f \in S_1$ the multiplication map

$$f : D(C)_n \to D(C)_{n+1}$$

is an isomorphism for all $n \in \mathbb{Z}$. Therefore $\dim_K D(C)_n = e(X)$ for all $n \in \mathbb{Z}$.

As $H^1(D) = 0$, the sequence $0 \to D \to B \to C \to 0$ yields that $H^0(C) = 0$. In particular we have

$$\dim_K C_n = \dim_K D(C)_n - h_C^1(n) = e(X) - h_C^1(n) \text{ for all } n \in \mathbb{Z}.$$

Now, the sequences $(*)$ imply that $\dim_K C_n = e(X) - h^2_A(n) + h^2_B(n)$ for all $n \in \mathbb{Z}$ and so statement (i) is proved completely.

Finally observe that $C_0 = 0$, so that

$$h^1_C(0) = \dim_K D(C)_0 = e(X) > 0.$$

If we apply the sequence $(*)$ with $n = 0$, we thus get $h^2_B(0) \neq h^2_A(0)$, whence $a(X) \geq 0$. This proves statement (ii).

Now, statement (ii) implies hat $\dim_K C_n = e(X)$ for all $n > a(X)$. As $H^0(C) = 0$, the genericity of f yields that the multiplication map $f : C_n \to C_{n+1}$ is injective for all $n \in \mathbb{Z}$ and hence bijective for all $n > a(X)$. This proves statement (iii). But now statement (vi) follows immediately. □

2.5. **Proposition.** *Let $r \geq 5$ and let $X \subset \mathbb{P}^r_K$ be of degree $r+1$ such that X is not* CM. *Then the ring $B = B(A)$ is* CM *and satisfies $B = K[B_1]$ with*

$$\dim_K B_1 = r + 1 + e(X) + h^1_A(1) - h^2_A(1).$$

Proof. Set $r' := \dim_K B_1 - 1$, so that $W := \mathrm{Proj}(K[B_1]) \subset \mathbb{P}^{r'}_K$ is a non-degenerate surface of degree $\deg(X) = r+1$. In the table given in Reminder 2.2 (B) the surface X falls under one of the following 7 cases: $3, 5, 6, 8, 10, 11, 12$. Consulting the two tables of Reminder 2.2 (B) and (D), we see that

$$e(X) + h^1_A(1) - h^2_A(1) = 2$$

in the cases $5 - 12$. So, Lemma 2.4 (b)(i) implies

$$r' + 1 = \dim_K B_1 = \dim_K D_1 + e(X) - h^2_A(1) + h^2_B(1) \geq$$
$$\dim_K A_1 + h^1_A(1) + e(X) - h^2_A(1) = r + 1 + e(X) + h^1_A(1) - h^2_A(1) = r + 3.$$

It follows that $\Delta'(W) = 0$, so that $W \subset \mathbb{P}^{r'}_K$ is of minimal degree and hence arithmetically CM. Therefore $K[B_1] = B$ and $h^2_B(1) = 0$ and hence $\dim_K B_1 = \dim_K D_1 + e(X) - h^2_A(1)$ in the 6 dases $5, 6, 8, 10, 11, 12$.

It remains to prove our claim, if X falls under the case 3. In this situation $h^1_A(1) = 1$ and $h^2_A(n) = e(X) = 1$ for all $n \leq 0$. As $B_n/D_n = 0$ for all $n \leq 0$, it follows by Lemma 2.4 (b)(i) that $h^2_B(n) = 0$ for all $n \leq 0$. Moreover, by the table given in Reminder 2.2 (D), we see that $h^2_A(n) = 0$ for all $n \geq 1$. So, another use of Lemma 2.4 (b) (i) shows that $H^2(B) = 0$ and $a(X) = 0$. Therefore B is a CM-Ring, $\dim_K(B_1/D_1) = e(X) = 1$ and the A-module B/D is generated in degree 1. In particular the D-module B/D is generated in degree 1, so that we can write $B = D[B_1]$. According to the quoted results of [B2], the K-algebra D is generated in degree one, so that $B = K[D_1][B_1] = K[B_1]$. As $h^2_A(1) = 0$, $\dim_K(B_1/D_1) = e(X) = 1$ and $\dim_K(D_1) = r + 1 + h^1_A(1) = r + 1$ the stated equality follows immediately. So our claim is true if X falls under the case 3. □

2.6. **Proposition.** *Let $r \geq 5$ and let $X \subset \mathbb{P}^r_K$ be of degree $r+1$ such that X is* CM. *Then*

$$B := B(A) = D(A) = K[B_1] \text{ with } \dim_K B_1 = r + 1 + h^1_A(1).$$

Moreover B is CM, *except in the case* 2, *where $A = B$ is of depth* 2.

Proof. This follows immediately from Reminder 2.2 (B), (C) and (D). □

Now we may summarize the previous results, in order to get an algebraically disguised form of the fact that our surfaces always admit arithmetic S_2-covers, and to describe these in all 12 cases.

2.7. **Theorem.** *Let $r \geq 5$, let $X \subset \mathbb{P}^r_K$ be of degree $r+1$, let $B := B(A)$ and $D := D(A)$. Then $B = K[B_1]$. In all cases except case 11 we have $D = K[D_1]$, and A, D and B are presented below*

Case	Inclusions	$\dim_K D_1$	$\dim_K B_1$	depth D	depth B
1	$A = D = B$	$r+1$	$r+1$	3	3
2	$A = D = B$	$r+1$	$r+1$	2	2
3	$A = D \subset B$	$r+1$	$r+2$	2	3
4	$A \subset D = B$	$r+2$	$r+2$	3	3
5	$A = D \subset B$	$r+1$	$r+3$	2	3
6	$A \subset D \subset B$	$r+2$	$r+3$	2	3
7	$A \subset D = B$	$r+3$	$r+3$	3	3
8	$A = D \subset B$	$r+1$	$r+3$	2	3
9	$A \subset D = B$	$r+3$	$r+3$	3	3
10	$A \subset D \subset B$	$r+2$	$r+3$	2	3
11	$A \subset D \subset B$	$r+1$	$r+3$	2	3
12	$A \subset D \subset B$	$r+2$	$r+3$	2	3

Proof. We may conclude by Propositions 2.5 and 2.6 and Reminder 2.2 (B), (C), (D). ☐

We now may formulate and prove the conclusive result of this section.

2.8. **Theorem.** *Let $r \geq 5$ and let $X \subset \mathbb{P}^r_K$ be of degree $r+1$. Then:*

(a) *The surface X falls under the case 1 if and only if it is arithmetically* CM.
(b) *The surface X falls under the case 2 if and only if it is arithmetically S_2 but not arithmetically* CM.
(c) *The cases $9, 10$ and 11 cannot occur if $r \geq 6$.*
(d) *In the case 9 we have $h^1_A(2) = 3$ and in the case 10 we have $h^1_A(2) = 2$.*
(e) *The case 12 cannot occur at all.*

Proof. (a), (b): This is easily read off from the table presented in Theorem 2.7.

(c): In the cases $9, 10, 11$ and 12 we have $\operatorname{sct}(X) = (2, 1)$. This implies that the generic hyperplane section curve $\mathcal{C}$ of X has Castelnuovo-Mumford regularity $4 = \deg(\mathcal{C}) - \operatorname{codim}(\mathcal{C}) + 1$. So, X is of maximal sectional regularity in the sense of [BS5, Definition 5.1].

Moreover, according to Reminder 2.2 (C) we respectively have $h^1_A(n) = 0$ for all $n > 2$ in the cases 9 and 12 and $h^1_A(n) \leq \max\{h^1_A(n-1) - 1, 0\}$ for all $n > 2$ in the cases 10 and 11. This means that the invariant

$$\delta(X) := \inf\{m \in \mathbb{Z} \,|\, h^1_A(n) \leq \max\{h^1_A(n-1) - 1, 0\} \text{ for all } n > m\}$$

of [BS5, Notation and Remark 4.4] takes a value $\leq 2 = \deg(X) - r + 1$ in the cases $9, 10, 11$ and 12. So, by [BS5, Theorem 5.10] we have $r + 1 = \deg(X) > 2r - 5$ and hence $r \leq 5$ in the four cases $9, 10, 11$ and 12 and hence in particular in the three cases $9, 10$ and 11.

(d), (e): Let $f \in S_1$ be a generic linear form and consider the hyperplane section curve $Y = \mathrm{Proj}(A/fA) \subset \mathbb{P}^{r-1}_K = \mathrm{Proj}(S/fS)$. In the four cases $9, 10, 11$ and 12 the surface X is of sectional cohomology type $(2, 1)$ so that $H^1(A/fA)_2 \simeq K$ (see Reminder 2.2 (B)). Consider the short exact sequence of graded S-modules

$$0 \to U \to H^1(A)(-1) \xrightarrow{f} H^1(A) \to H^1(A/fA) \to H^2(A)(-1).$$

as $r = 5$ we have $\deg(X) = 6 = 2r - 4$, and so by [BS5, Proposition 5.9] the module U is generated by homogeneous elements of degrees 3 and 4. Moreover by Reminder 2.2 (D) we have $H^2(A)_1 = 0$. So, we end up with a short exact sequence

$$0 \to H^1(A)_1 \to H^1(A)_2 \to K \to 0.$$

It follows that $h^1_A(2) = h^1_A(1) + 1$ in all the four cases $9, 10, 11$ and 12. So in view of the table given in Reminder 2.2 (C) we have $h^1_A(2) = 3$ in the case 9, $h^1_A(2) = 2$ in the case 10, which proves statement (b). By the same table we get a contradiction in the case 12, so that this case cannot occur. □

3. 2nd Hartshorne-Rao Numbers and 2nd Deficiency Modules

Keep all notations and hypotheses of the previous section. Observe first, that in statement (d) of Theorem 2.8 we have improved on the values of the second Hartshorne-Rao numbers $h^1_A(2)$ of X as they were originally given in the table in Reminder 2.2 (C) for the cases 9 and 10. This improvement now allows to distinguish the two cases 7 and 9 only by means of the values of the first and second Hartshorne-Rao numbers $h^1_A(1)$ and $h^1_A(2)$ of X. More precisely, we can say:

3.1. **Corollary.** *Let $r \geq 5$ and let $X \subset \mathbb{P}^r_K$ be of degree $r + 1$. Then, the cases $1, 2$ and $3 - 11$ may be distinguished only by the three invariants $h^1_A(1), h^2_A(2)$ and $e(X)$ according to the following table*

Case	$1,2$	3	4	5	6	7	8	9	10	11
$h^1_A(1)$	0	0	1	0	1	2	0	2	1	0
$h^1_A(2)$	0	0	0	0	≤ 1	≤ 2	0	3	2	1
$e(X)$	0	1	0	2	1	0	3	0	1	2

Proof. Immediate by Reminder 2.2 (C), Corollary 2.7 and Theorem 2.8 (b). □

3.2. **Remark.** (A) The statements given in 3.1 present an essential gain in effectiveness, in particular concerning the distinction of the two cases 7 and 9. Without the distinction of these two cases by means of $h^1_A(2)$ only a direct calculation of the sectional cohomology type would be available. A direct calculation of this latter invariant, for example on use of the program Singular (see [GrP]) is indeed much more demanding than the calculation of $h^1_A(2)$. We shall illustrate the emphasized use of the Hartshorne-Rao number $h^1_A(2)$ in 3.4.

(B) In the cases 7 and 9 we have $D = B$ and B is the homogeneous coordinate ring of a surface $\tilde{X} \subset \mathbb{P}^{r+2}_K$ of minimal degree (see Corollary 2.7). Moreover, if $y \in B_1 \setminus A_1$, the A-algebra $A[y_1]$ is the homogeneous coordinate ring of a surface $Y \subset \mathbb{P}^{r+1}_K$ of almost minimal degree and arithmetic depth 1. The fact that $D = B$ yields isomorphisms $\tilde{X} \simeq Y \simeq X$ induced by simple outer linear projections. In particular $\tilde{X}$ must be smooth (see [BS1, Proposition 3.4]) and so X is smooth in these cases. So, if $\mathrm{Char}(K) = 0$, the surface $X \subset \mathbb{P}^r_K$ satisfies the Eisenbud-Goto inequality (see [P]) and hence $\mathrm{reg}(X) \leq 4$, so that $h^1_A(n) = 0$ for all $n \geq 3$.

It seems natural to ask whether in the cases 6 and 7 the inequalities $h^1_A(2) \leq 1$ resp. $h^1_A(2) \leq 2$ may be strict. We now present a few examples showing that in the case 7 the invariant $h^1_A(2)$ may take all possible values $0, 1, 2$. With the same construction we also shall give an example which falls under the case 9, in order to illustrate what we said in Remark 3.2 (A). We begin with the following remark, which is aimed to pave the way for the construction we wish to perform.

3.3. **Remark.** (A) Let a, b be integers with $1 < a \leq b$ and $a + b + 1 = r + 2$, let $\tilde{X} = S(a, b) \subset \mathbb{P}^{r+2}_K$ be the smooth rational normal scroll of type (a, b) with homogeneous coordinate ring $B = K[y_0, y_1, \ldots, y_{r+2}] \subset K[s, t, u, v]$ with $y_i = u^b t^i s^{a-i}$ for $0 \leq i \leq a$ and $y_i = v^a t^{i-a-1} s^{a+b+1-i}$ for $a + 1 \leq i \leq r + 2$, where s, t, u, v are indeterminates. Keep in mind the obvious relations

$$y_i y_j = y_{i-1} y_{j+1} \text{ if } \begin{cases} \text{either } 0 < i \leq j < a \text{ and } a + 1 < i \leq j < r + 2 \\ \text{or } 0 \leq i \leq a < j < r + 2. \end{cases}$$

Now, let

$$\mathbb{I} := \{(0, j) \mid 0 \leq j \leq r + 2\} \cup \{(i, r + 2) \mid 1 \leq i \leq r + 2\}$$
$$\cup \{(i, a) \mid 1 \leq i \leq a\} \cup \{(a + 1, j) \mid a + 1 \leq j \leq r + 1\}.$$

Then, the family $\{y_i y_j\}_{(i,j) \in \mathbb{I}}$ is a K-basis of B_2 as easily seen.
(B) Now fix two integers $k, \ell \in \{1, \ldots, r + 1\} \setminus \{a, a + 1\}$ with $k < \ell$ and consider the K-algebra

$$A = A^{[k,\ell]} := K[y_i \mid i \in \{0, \ldots, r + 2\} \setminus \{k, \ell\}] \subset B.$$

Observe that $B = A[y_k, y_\ell]$. Furthermore, observe that the only pairs $(i, j) \in \mathbb{I}$ for which $y_i y_j$ does not obviously belong to A_2 are

$$(0, k), (0, \ell), (k, r + 2), (\ell, r + 2), \text{ and } (k, a), (\ell, a) \text{ if } \ell < a,$$
$$(k, a), (a + 1, \ell) \text{ if } k < a + 1 < \ell, (a + 1, k), (a + 1, \ell) \text{ if } a + 1 < k.$$

3.4. **Examples.** (A) Keep the notations and hypotheses of Remark 3.3 and fix k and ℓ such that $\ell \neq k + 1, k \neq 1, a - 1, a + 2$ and $\ell \neq a - 1, a + 2, r + 1$. Then, the relations of Remark 3.3 (A) show that $y_i y_j \in A_2$ also for the "missing pairs" (i, j) listed at the end of Remark 3.3 (B). So $y_i y_j \in A_2$ for all pairs $(i, j) \in \mathbb{I}$ and as these pairs span B_2 it follows that $A_2 = B_2$ and hence $A_t = B_t$ for all $t > 1$. As B is CM we thus get $B = D, h^1_A(1) = \dim_K(B_1/A_1) = 2, h^1_A(2) = \dim_K(B_2/A_2) = 0$ and $H^2(A) = H^2(B) = 0$. So, according to Corollary 3.1, the surface

$$X = X^{[k,\ell]} := \mathrm{Proj}(A) \subset \mathbb{P}^r_K$$

falls under the case 7 with $h^1_A(2) = 0$. Observe that this applies particularly if $1 < k < a-1$ and $a + 2 < \ell < r + 1 = a + b$.
(B) Next let $1 < a < r - 4$ and choose $k = a + 3, \ell = a + 4$. Then, it follows as above that $y_i y_j \in A_2$ for all pairs $(i, j) \in \mathbb{I}$ different from $(a + 1, \ell)$, so that $\dim_K(B_2/A_2) = 1$.

Another use of the relations of Remark 3.3 (A) now yields that $y_iy_jy_m \in A_t = B_t$ for all $i, j, m \in \{0, \dots, r+2\}$, so that $A_t = B_t$ for all $t > 2$. As $\dim_K(B_1/A_1) = 2$ it now follows as in part (A) that the surface $X := \mathrm{Proj}(A) \subset \mathbb{P}^r_K$ falls under the case 7 with $h^1_A(2) = 1$.
(C) Now, let $1 < a = r - 4$, $k := a + 3$ and $\ell = a + 4 = r$. Then, the relations of Remark 3.3 (A) show that the only missing pairs $(i, j) \in \mathbb{I}$ of Remark 3.3 (B) for which $y_iy_j \notin A_2$ are precisely the pairs $(k, r+2)$ and $(a+1, \ell)$, so that $\dim_K(B_2/A_2) = 2$, Now, it follows along the lines of the previous examples that the surface $X := \mathrm{Proj}(A) \subset \mathbb{P}^r_K$ falls under the case 7 with $h^1_A(2) = 2$.
(D) Finally, let $r = 5, a = 1, k = 4$ and $\ell = 5$. Then the relations of Remark 3.3 (A) show that precisely the three pairs $(0, 5), (4, 7), (2, 5) \in \mathbb{I}$ have the property that $y_iy_j \notin A_2$, whereas $y_iy_jy_m \in A_3$ for all $i, j, m \in \{0, \dots, 7\}$. Now, as above it follows that $D = B, h^1_A(1) = 2, h^1_A(2) = 3$, so that by Corollary 3.1 the surface $X := \mathrm{Proj}(A) \subset \mathbb{P}^5_K$ falls under the type 9 this time.

Our next aim is to show by an example that in the case 6 the second Hartshorne-Rao number $h^1_A(2)$ of X may indeed take the value 0.

3.5. **Example.** We keep the notations and hypotheses of Remark 3.3 and choose $k = 1, 4 \leq \ell < a - 1$. Then clearly $y_0^s y_1 \notin A^{[1,\ell]} = A$ for all $s \in \mathbb{N}_0$. Moreover the relations of Remark 3.3 (A) show immediately that $y_iy_1 \in A_2$ for all $i \neq 0$ and $y_iy_\ell \in A_2$ for all $i \in \{0, \dots, r+2\}$. From this it follows that $A[y_\ell] \subseteq D$ and $B_t = A[y_\ell]_t \oplus Ky_0^{t-1}y_1$ for all $t > 0, A[y_\ell]_t = A_t$ for all $t > 1$ and $y_1 \neq D$. From this it follows that $D = A[y_\ell]$ with $D_t = A_t$ for all $t > 1$ so that $h^1_A(1) = 1$ and $h^1_A(n) = 0$ for all $n > 1$. As $B_t/D_t \simeq K$ for all $t > 0$ Lemma 2.4 (b) yields $e(X) = 1$. So according to Corollary 3.1 we are in the case 6 with $h^1_A(2) = 0$.

Our next aim is to get deeper information concerning the invariant $e(X)$. Observe that by duality the cohomological invariant $e(X)$ is precisely the multiplicity of the second deficiency module

$$K^2(A) = \mathrm{Ext}^{r-1}_S(A, S(-r-2)) \simeq \mathrm{Hom}_K(H^2(A), K)$$

where A is the homogeneous coordinate ring of our surface $X \subset \mathbb{P}^r_K$, whereas the function $n \mapsto h^2_A(-n)$ is the Hilbert function of $K^2(A)$. So, to improve on the table of Reminder 2.2 (D) we now make explicit the structure of $K^2(A)$.

3.6. **Theorem.** *Let $r \geq 5$ and let $X \subset \mathbb{P}^r_K$ be of degree $r + 1$. Then:*

(a) *In the cases $1, 4, 7$ and 9 we have $K^2(A) = 0$.*
(b) *In the case 2 we have $K^2(A) = S/S_+$.*
(c) *In the cases $3, 6$ and 10 we have $K^2(A) \simeq S/L$, where $L \subset S$ is an ideal generated by r independent linear forms.*
(d) *In the cases 5 and 11 we have $K^2(A) \simeq S/L \oplus S/L'$, where $L, L' \subset S$ are ideals generated by r independent linear forms.*
(e) *In the case 8 we have $K^2(A) \simeq (S/J)(1)$, where $J \subsetneq S$ is an ideal minimally generated by $r - 2$ linear forms and 3 quadrics, and $\mathrm{Proj}(S/J)$ is a scheme of length 3.*

Proof. Statement (a) and (b) follow immediately by Reminder 2.2 (D) on use of local duality. In the remaining cases $3, 5, 6, 8, 10, 11$ the ring $B := B(A)$ is CM (see Propositions 2.5 and 2.6), so that $H^2(B) = 0$. We write $D = D(A)$ and $C = B/A$. It follows $H^2(A) \simeq H^1(C)$. In particular we get an epimorphism of graded S-modules

$$(*) \qquad D(C) \twoheadrightarrow H^2(A).$$

Moreover, as $H^1(C)_n = D(C)_n$ for all $n \ll 0$,

$$(**) \qquad \dim_K D(C)_n = e(X) \text{ for all } n \in \mathbb{Z}.$$

In view of Reminder 2.2 (D) we have $h^2_A(n) = e(X) = 1$ for all $n \leq 0$ and $h^2_A(n) = 0$ for all $n \geq 1$ in the three cases 3, 6 and 10. As $f : D(C) \to D(C)(1)$ is an isomorphism for a generic linear form $f \in S_1$, the homomorphism $f : H^2(A)_{n-1} \to H^2(A)_n$ becomes an isomorphism for all $n \geq 0$. By duality it follows that $K^2(A)_n = 0$ for all $n < 0, K^2(A)_0 \simeq K$ and $f : K^2(A)_n \to K^2(A)_{n+1}$ is an isomorphism for all $n \geq 0$ and for a generic linear form $f \in S_1$ in the cases 3, 6 and 10. So, in these cases we must indeed have $K^2(A) \simeq S/L$, where $L \subset S$ is an ideal minimally generated by r linear forms. This proves statement (c).

In the cases 5 and 11 we have $h^2_A(n) = e(X) = 2$ for all $n \leq 0$ and $h^2_A(n) = 0$ for all $n > 0$ by Reminder 2.2 (D). In particular $f : H^2(A)_{n-1} \to H^2(A)_n$ is again an isomorphism for all $n \leq 0$ in the cases 5 and 11. So $K^2(A)_n = 0$ for all $n < 0, K^2(A)_0 \simeq K^2$ and $f : K^2(A)_n \to K^2(A)_{n+1}$ is an isomorphism for all $n \geq 0$ for a generic linear form $f \in S_1$ in these two cases. This proves statement (d).

Finally by Reminder 2.2 (D) we have $h^2_A(n) = e(X) = 3$ for all $n \leq 0, h^2_A(1) = 1$ and $h^2_A(n) = 0$ for all $n > 1$. For a generic linear form $f \in S_1$, the hyperplane section curve $Y = \operatorname{Proj}(A/fA) \subset \mathbb{P}^{r-1}_K = \operatorname{Proj}(S/fS)$ is of degree $r + 1$ and of regularity 4. So the socle of the Harthshorne-Rao module $H^1(A/fA)$ of Y satisfies $\operatorname{soc}(H^1(A/fA)) = H^1(A/fA)_2 \simeq K$ (see [BS1, Theorem 3.9]). Hence, by the exact sequence

$$0 \to H^1(A/fA)(1) \xrightarrow{\delta} H^2(A) \xrightarrow{f} H^2(A)(1)$$

it follows that $\operatorname{soc} H^2(A) = \operatorname{Im}(f) = H^2(A)_1 \simeq K$. So, by duality $K^2(A)$ is generated by one homogeneous element of degree -1. Moreover $\dim_K(K^2(A)_n) = h^2_A(-n) = 3$ for all $n \geq 0$. Therefore $K^2(A) \simeq S/J$, where J is as in statement (e). □

Later, we shall reconsider the deficiency modules $K^2(A)$ in the geometric context. Then, it shall be useful to bear in mind the following observation.

3.7. **Remark.** Let the notations be as in Notation 2.1 and let

$$\mathcal{K}^2_X := \widetilde{K^2(A)}$$

be the coherent sheaf of $\mathcal{O}_X$-modules induced by the second deficiency module of the homogeneous coordinate ring of the surface $X \subset \mathbb{P}^r_K$. Let $x \in X$ be a closed point. Then the stalk of $\mathcal{K}^2_X$ at x coincides with the first deficiency module of the local ring of X at x, thus

$$\mathcal{K}^2_{X,x} \simeq \operatorname{Ext}^{r-1}_{\mathcal{O}_{\mathbb{P}^r,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{\mathbb{P}^r,x}) = K^1(\mathcal{O}_{X,x}).$$

By Local Duality and the fact that taking Matlis duals preserves lengths we thus obtain

$$\operatorname{length}(\mathcal{K}^2_{X,x}) = \operatorname{length}(H^1_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x})).$$

In particular we can say

$$\operatorname{length}(\mathcal{K}^2_X) = e(X), \qquad \operatorname{Supp}(\mathcal{K}^2_X) = X \setminus \operatorname{CM}(X).$$

4. Almost Non-singular Projections of Maximal Del Pezzo Surfaces

4.1. **Definition.** (A) A surjective morphism $f : \tilde{X} \twoheadrightarrow X$ between non-degenerate irreducible projective varieties $X \subset \mathbb{P}^r_K$ and $\tilde{X} \subset \mathbb{P}^{r+t}_K$ is called a *projection of* $\tilde{X}$ *onto* X from the linear subspace $\Lambda = \mathbb{P}^{t-1}_K \subset \mathbb{P}^{r+t}_K$ if $\Lambda \cap \tilde{X} = \emptyset$ and f is induced by a linear projection $\mathbb{P}^{r+t}_K \setminus \Lambda \twoheadrightarrow \mathbb{P}^r_K$ with center Λ.
(B) By $\operatorname{Sing}(f)$ we denote the singular locus of a morphism $f : \tilde{X} \to X$ of algebraic varieties, that is the least closed set $Z \subseteq X$ such that the induced morphism

$$f \upharpoonright : \tilde{X} \setminus f^{-1}(Z) \to X \setminus Z$$

is an isomorphism. If $\operatorname{Sing}(f) = \emptyset$, the morphism $f : \tilde{X} \to X$ is an isomorphism and said to be *non-singular*. If the set $\operatorname{Sing}(f)$ is finite (including the case of $\operatorname{Sing}(f) = \emptyset$), we say that f is *almost non-singular*.

4.2. **Remark.** Assume that $f : \widetilde{X} \twoheadrightarrow X$ is a finite dominant morphism of irreducible algebraic varieties. Then the singular locus of f is the support of the cokernel of the induced monomorphism of sheaves $0 \to \mathcal{O}_X \to f_*\mathcal{O}_{\widetilde{X}}$, thus

$$\operatorname{Sing}(f) = \operatorname{Supp}(f_*\mathcal{O}_{\widetilde{X}}/\mathcal{O}_X).$$

4.3. **Reminder.** According to [BS4] a *maximal del Pezzo variety* is a variety of almost minimal degree which is arithmetically CM.

The aim of this section is to characterize those surfaces $X \subset \mathbb{P}^r$, which admit an almost non-singular projection $\tilde{X} \twoheadrightarrow X$ from a closed point $p \in \mathbb{P}^{r+1} \setminus \tilde{X}$, where $\tilde{X} \subset \mathbb{P}^{r+1}$ is a maximal del Pezzo surface. We begin with a few preparations on almost non-singular projections.

4.4. **Lemma.** *Let A be the homogeneous coordinate ring of the non-degenerate irreducible surface $X \subset \mathbb{P}^r_K$, let $\tilde{X} \subset \mathbb{P}^{r+t}_K$ be a non-degenerate irreducible surface with coordinate ring $B' \supset A$ such that the inclusion $A \hookrightarrow B'$ yields a projection $f : \tilde{X} \twoheadrightarrow X$ of X onto $\tilde{X}$ from some linear subspace $\Lambda = \mathbb{P}^{t-1}_K \subset \mathbb{P}^{r+t}_K$ disjoint to $\tilde{X}$. Then, in the notation of Remark and Definition 2.3 the following statements are equivalent:*

(i) *The projection $f : \tilde{X} \to X$ is almost non-singular and B' has the property S_2.*
(ii) *$B' = B(A)$.*

Proof. "(i) $\Rightarrow$ (ii)": Assume that f is almost non-singular and B' has the second Serre property S_2. Let $\mathfrak{b} \subseteq A_+$ be the reduced graded ideal which defines the finite set $\operatorname{Sing}(f) \subseteq X$ and let $\mathfrak{a} \subseteq A_+$ be the reduced graded ideal which defines the finite set $X \setminus \operatorname{CM}(X)$. Then, in the notation of Remark and Definition 2.3 we have $A \subseteq B' \subseteq D_{\mathfrak{b}}(A)$.

As $\tilde{X} = \operatorname{Proj}(B')$ is CM, we also have $X \setminus \operatorname{CM}(X) \subseteq \operatorname{Sing}(f)$, so that $\mathfrak{a} \supseteq \mathfrak{b}$, whence $B(A) = D_{\mathfrak{a}}(A) \subseteq D_{\mathfrak{b}}(A)$. As $B(A)$ is a finitely generated A-module which satisfies the property S_2 (see Remark and Definition 2.3 (A)) and as height $(\mathfrak{b}) = 2$ we have

$D_{\mathfrak{b}}(B(A)) = B(A)$, whence $D_{\mathfrak{b}}(A) \subseteq D_{\mathfrak{b}}(B(A)) = B(A)$. Therefore $B(A) = D_{\mathfrak{b}}(A)$. As B' is an S_2-ring, it follows $B' = B(A)$ (see Remark and Definition 2.3 (B)).

"(ii) $\Rightarrow$ (i)": If $B' = B(A)$, then B' is S_2. By Remark 4.2 and Remark and Definition 2.3 (C) it follows that $\operatorname{Sing}(f) = \operatorname{Supp}(f_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X) = X \setminus \operatorname{CM}(X)$, a finite set. □

In the next lemma we use the notations introduced in Notation 2.1 (D).

4.5. **Lemma.** *Let the notations and hypotheses be as in Lemma 4.4 and assume that B' has the property S_2. Then:*

(a) *The following statements are equivalent:*
   (i) *$f : \tilde{X} \twoheadrightarrow X$ is almost non-singular;*
   (ii) $\operatorname{Sing}(f) = X \setminus \operatorname{CM}(X)$*;*
   (iii) $\operatorname{Sing}(f) \subseteq X \setminus \operatorname{CM}(X)$.

(b) *If $f : \tilde{X} \twoheadrightarrow X$ is almost non-singular, then*
   (1) *If $\tilde{X}$ is smooth, then* $\operatorname{Sing}(f) = X \setminus \operatorname{Reg}(X) = X \setminus \operatorname{Nor}(X) = X \setminus \operatorname{CM}(X)$.
   (2) *If $x \in X$ is a closed point with $f^{-1}(x) \subseteq \operatorname{CM}(\tilde{X})$, then $H^1_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x}) \simeq \left(f_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_{\tilde{X}}\right)_x$.*
   (3) *If $\tilde{X}$ is normal, it is determined up to X-isomorphism.*

Proof. (a): "(i) $\Rightarrow$ (ii)": Assume that statement (i) holds. As the homogeneous coordinate ring B' of $\tilde{X}$ is S_2, Lemma 4.4 implies that $B' = B(A)$. From this it follows as in the proof of the implication "(ii) $\Rightarrow$ (i)" of Lemma 4.4 that $\operatorname{Sing}(f) = X \setminus \operatorname{CM}(R)$. The implication "(ii) $\Rightarrow$ (iii)" is obvious and the implication "(iii) $\Rightarrow$ (i)" is clear as the set $X \setminus \operatorname{CM}(X)$ is finite.

(b): Assume that $f : \tilde{X} \twoheadrightarrow X$ is almost non-singular.

(1): As X is a surface, we have $\operatorname{Nor}(X) \subseteq \operatorname{CM}(X)$. In view of statement (a) it follows $X \setminus \operatorname{Reg}(X) \supseteq X \setminus \operatorname{Nor}(X) \supseteq X \setminus \operatorname{CM}(X) \supseteq \operatorname{Sing}(f)$. As $\tilde{X}$ is smooth we also have $X \setminus \operatorname{Reg}(X) \subseteq \operatorname{Sing}(f)$.

(2): By our assumption, the fiber $f^{-1}(x)$ consists of finitely many closed CM-points of $\tilde{X}$ and so we have a short exact sequence of $\mathcal{O}_{X,x}$-modules

$$0 \to \mathcal{O}_{X,x} \to \left(f_*\mathcal{O}_{\tilde{X}}\right)_x \to C \to 0$$

in which C is of finite length and $\left(f_*\mathcal{O}_{\tilde{X}}\right)_x$ is CM. Applying local cohomology with respect to $\mathfrak{m}_{X,x}$ we get our claim.

(3): If $\tilde{X}$ is normal, the finite birational morphism $f : \tilde{X} \twoheadrightarrow X$ is a normalization of X. □

4.6. **Remark.** Let $r \geq 5$ and let $X \subset \mathbb{P}^r_K$ be our non-degenerate irreducible projective surface of degree $r+1$. If X is a cone, there is a hyperplane $\mathbb{P}^{r-1}_K \subset \mathbb{P}^r_K$ and a non-degenerate irreducible curve $\mathcal{C} \subset \mathbb{P}^{r-1}_K$ of degree $r+1 = (r-1)+2$ such that X is a cone over $\mathcal{C}$. In this case X may be understood directly by means of the curve $\mathcal{C}$. In [BS1] we have studied the curves $\mathcal{C} \subset \mathbb{P}^{r-1}_K$ of degree $r+1$ for all $r \geq 5$, and hence can well understand the surfaces $X \subset \mathbb{P}^r_K$ of degree $r+1$ which are cones. We therefore shall not consider this case anymore.

4.7. **Theorem.** *Let $r \geq 5$ and let $X \subset \mathbb{P}^r_K$ be an irreducible non-degenerate surface.*

(a) *The following statements are equivalent:*
   (i) X *is of degree* $r+1$*, not a cone and falls under one of the two cases* 3 *or* 4*.*
   (ii) *There is an maximal del Pezzo surface* $\tilde{X} \subset \mathbb{P}_K^{r+1}$ *which is not a cone and an almost non-singular projection* $f : \tilde{X} \twoheadrightarrow X$ *from a point* $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$*.*

(b) *If* X *is as in statement* (a)(i) *and* $\tilde{X}$ *and* f *are as in* (a)(ii)*, then the maximal del Pezzo surface* $\tilde{X} \subset \mathbb{P}_K^{r+1}$ *is uniquely determined by* X *up to projective equivalence and*
   (1) X *falls under the case* 4 *if and only if* $f : \tilde{X} \twoheadrightarrow X$ *is an isomorphism.*
   (2) *If* X *falls under the case* 3 *and* $\mathcal{K}_X^2$ *is defined as in Remark 2.7 , then* $\mathrm{Sing}(f)$ *consist of a single closed point* $x \in X$ *and* $H^1_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x}) \simeq (f_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)_x \simeq \mathcal{K}^2_{X,x} \simeq K$*.*

Proof. (a): Let X be as in statement (i) and let A be the homogeneous coordinate ring of X. Then, according to Theorem 2.7 the homogeneous integral K-algebra $B = B(A)$ is CM and satisfies $\dim_K B_1 = r+2$. So, B is the homogeneous coordinate ring of an irreducible non-degenerate variety $\tilde{X} \subset \mathbb{P}_K^{r+1}$ which is arithmetically CM and of degree $r+1$ and hence maximally del Pezzo. Moreover, the inclusion map $A \to B$ induces an almost non-singular projection of $\tilde{X}$ from a point $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$. As X is not a cone $\tilde{X}$ cannot be a cone either. This proves the implication "(i) $\Rightarrow$ (ii)".

To prove the converse inclusion, let X, $\tilde{X}$ and p be as in statement (ii). Then, the homogeneous coordinate ring B' is CM and hence coincides with $B = B(A)$ by Lemma 4.4. In particular, we must have $\dim_K B_1 = \dim_K B'_1 = r+2$. So according to Theorem 2.7, the surface X falls under one of the two cases 3 or 4.

(b): Let X, $\tilde{X}$ and f be as in statement (a) and keep in mind that we just have seen that $B = B(A)$ is the homogeneous coordinate ring of $\tilde{X}$. This proves the stated uniqueness of $\tilde{X}$.

Moreover in the notations of Theorem 2.7 the map f is an isomorphism if and only if $D = B$, and according to this same theorem this equality holds in the case 4 but not in the case 3. This proves sub-statement (1).

Substatement (2) now follows from statement (b)(2) of Lemma 4.5, Remark 3.7 and the fact that $e(X) = 1$ (see Reminder 2.2 (B)).

□

4.8. **Corollary.** *Let* $r \geq 5$ *and* $X \subset \mathbb{P}_K^r$ *be an irreducible non-degenerate surface of degree* $r+1$ *which falls under case* 3 *or* 4*, so that* X *is an almost non-singular projection of an essentially unique maximal del Pezzo surface* $\tilde{X} \subset \mathbb{P}_K^{r+1}$*. Then, the following statements are equivalent:*

(i) *The del Pezzo surface* $\tilde{X}$ *is not normal.*
(ii) $X \setminus \mathrm{Nor}(X) = \mathbb{L} \cup (X \setminus \mathrm{CM}(X))$*, for some line* $\mathbb{L} = \mathbb{P}_K^1 \subset \mathbb{P}_K^r$*.*
(iii) *The non-normal locus* $X \setminus \mathrm{Nor}(X)$ *is infinite.*
(iv) *The non normal locus* $X \setminus \mathrm{Nor}(X)$ *contains two distinct points.*
(v) $\mathrm{CM}(X) \neq \mathrm{Nor}(X)$*.*

Proof. (i) $\Rightarrow$ (ii): Assume that $\tilde{X}$ is not normal. Then, according to [BS4, Theorem 1.3] the non-normal locus of $\tilde{X}$ is a line. Moreover by Theorem 4.7(b) and Lemma 4.5(a) we have $\mathrm{Sing}(f) = X \setminus \mathrm{CM}(X)$. This implies statement (ii).

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. The implication (iv) $\Rightarrow$ (v) follows from the fact that X contains at most one non-CM point. The implication (v) $\Rightarrow$ (i) follows easily from the inclusions $\mathrm{Nor}(X) \subseteq \mathrm{CM}(X)$ and $\mathrm{Sing}(f) \subseteq X \setminus \mathrm{CM}(X)$ (see Lemma 4.5(a)). □

4.9. **Remark.** According to Theorem 1.2 of [BS4] a non-normal del Pezzo variety $\tilde{X} \subset \mathbb{P}^{r+1}_K$ is obtained by projecting a variety $W \subset \mathbb{P}^{r+1}_K$ of minimal degree from a closed point $p \in \mathbb{P}^{r+2}_K \setminus W$. So, if $r \geq 5$ and the surface $X \subset \mathbb{P}^r_K$ falls under cases 3 or 4 and satisfies the equivalent conditions (ii)-(v) of Corollary 4.8, the surface X is obtained by projecting a (possibly singular) surface sroll $W \subset \mathbb{P}^{r+2}_K$ from a line $\Lambda = \mathbb{P}^1_K \subset \mathbb{P}^{r+2}_K \setminus W$. If X is not a cone, then W is not a cone either and hence must be smooth.

5. Almost Non-singular Projections of Smooth Surface Scrolls

The aim of this section is to characterize those surfaces $X \subset \mathbb{P}^r_K$ which admit an almost non-singular projection $\tilde{X} \twoheadrightarrow X$ from a line $\mathbb{L} \subset \mathbb{P}^{r+2}_K \setminus \tilde{X}$, where $\tilde{X} \subset \mathbb{P}^{r+2}_K$ is a smooth rational normal surface scroll. We therefore recall a few facts on rational normal surface scrolls.

5.1. **Reminder.** (A) Let a and b be two positive integers with $a \leq b$, consider the polynomial ring

$$R := K[x_0, x_1, \ldots, x_{a+b+1}]$$

and the smooth rational normal surface scroll (cf. [H])

$$S(a,b) \subset \mathbb{P}^{a+b+1}_K = \mathrm{Proj}(R)$$

of type (a,b). So $S(a,b)$ is the non-degenerate irreducible surface of degree $a+b$ whose vanishing ideal

$$I_{S(a,b)} = I_{a,b} \subset R$$

is generated by the 2×2-minors of the $2 \times (a+b)$-matrix

$$M_{a,b} := \begin{pmatrix} x_0 & x_1 & \cdots & x_{a-1} & \vdots & x_{a+1} & x_{a+2} & \cdots & x_{a+b} \\ x_1 & x_2 & \cdots & x_a & \vdots & x_{a+2} & x_{a+3} & \cdots & x_{a+b+1} \end{pmatrix}$$

(B) Keep the notations of part (A). Then, according to [C], the vanishing ideal $I_{\mathrm{Sec}(\tilde{X})} \subset R$ of the secant variety

$$\mathrm{Sec}(\tilde{X}) := \bigcup \{\mathbb{L} = \mathbb{P}^1_K \subset \mathbb{P}^{a+b+1}_K \,\big|\, \mathrm{length}(\mathbb{L} \cap \tilde{X}) > 1\}$$

of the scroll $\tilde{X} := S(a,b)$ is generated by the 3×3-minors of the $3 \times (a+b-2)$-matrix

$$M'_{a,b} := \begin{pmatrix} x_0 & x_1 & \cdots & x_{a-2} & \vdots & x_{a+1} & x_{a+2} & \cdots & x_{a+b-1} \\ x_1 & x_2 & \cdots & x_{a-1} & \vdots & x_{a+2} & x_{a+3} & \cdots & x_{a+b} \\ x_2 & x_3 & \cdots & x_a & \vdots & x_{a+3} & x_{a+4} & \cdots & x_{a+b+1} \end{pmatrix}.$$

If $N = (f_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ is a matrix whose entries are linear forms in R and if $p = (\alpha_0 : \alpha_1 : \cdots : \alpha_{a+b+1}) \in \mathbb{P}_K^{a+b+1}$ we denote by $\operatorname{rank} N(p)$ the rank of the matrix

$$N(\alpha_0, \alpha_1, \cdots, \alpha_{a+b+1}) = (f_{ij}(\alpha_0, \alpha_1, \cdots, \alpha_{a+b+1}))_{1\leq i\leq m, 1\leq j\leq n}.$$

Using this notation we have

$$S(a,b) = \{p \in \mathbb{P}_K^{a+b+1} \,|\, \operatorname{rank}(M_{a,b}(p)) \leq 1\},$$
$$\operatorname{Sec} S(a,b) = \{p \in \mathbb{P}_K^{a+b+1} \,|\, \operatorname{rank}(M'_{a,b}(p)) \leq 2\}.$$

(C) Keep the above notations and consider the subspaces

$$\mathbb{P}_K^a := \operatorname{Proj}(R/(x_{a+1}, \cdots, x_{a+b+1})R) \subset \mathbb{P}_K^{a+b+1}$$
$$\mathbb{P}_K^b := \operatorname{Proj}(R/(x_0, \cdots, x_a)R) \subset \mathbb{P}_K^{a+b+1}$$

and the corresponding subscrolls

$$S(a) := S(a,b) \cap \mathbb{P}_K^a \subset \mathbb{P}_K^a = \langle S(a) \rangle,$$
$$S(b) := S(a,b) \cap \mathbb{P}_K^b \subset \mathbb{P}_K^b = \langle S(b) \rangle,$$

which are rational normal curves. In addition, we consider the Veronese embeddings

$$\nu_a : \mathbb{P}_K^1 \to \mathbb{P}_K^a, \quad (s:t) \mapsto (s^a : s^{a-1}t : \cdots : st^{a-1} : t^a)$$
$$\nu_b : \mathbb{P}_K^1 \to \mathbb{P}_K^b, \quad (s:t) \mapsto (s^b : s^{b-1}t : \cdots : st^{b-1} : t^b)$$

and the generating lines

$$\mathbb{L}(x) := \langle \nu_a(x), \nu_b(x) \rangle, \quad x \in \mathbb{P}_K^1.$$

Then $S(a,b) = \dot{\bigcup}_{x\in\mathbb{P}^1} \mathbb{L}(x)$ and there is a canonical projection $\varphi : S(a,b) \twoheadrightarrow \mathbb{P}_K^1$ such that $\varphi^{-1}(x) = \mathbb{L}(x)$ for all $x \in \mathbb{P}_K^1$.

5.2. **Remark.** (A) Keep the above notations and consider the secant cone

$$\operatorname{Sec}_p(\tilde{X}) := \{p\} \cup \bigcup \{\mathbb{L} = \mathbb{P}_K^1 \subset \mathbb{P}_K^{a+b+1} \,|\, p \in \mathbb{L}, \operatorname{length}(\mathbb{L} \cap \tilde{X}) > 1\}$$

and the secant locus

$$\Sigma_p(\tilde{X}) := \operatorname{Sec}_p(\tilde{X}) \cap \tilde{X} = \{\tilde{x} \in \tilde{X} \,|\, \operatorname{length}(\langle p, \tilde{x} \rangle \cap \tilde{X}) > 1\}$$

of the scroll $\tilde{X} := S(a,b)$ with respect to the point $p \in \mathbb{P}_K^{a+b+1} \setminus \tilde{X}$. Observe that

$$\operatorname{Sec}_p(\tilde{X}) = \{p\} \text{ and } \Sigma_p(\tilde{X}) = \emptyset \text{ if and only if } p \notin \operatorname{Sec}(\tilde{X}).$$

Moreover, by the rank formula of [BPS] we have

$$\dim \Sigma_p(\tilde{X}) = \dim(\operatorname{Sec}_p(\tilde{X})) - 1 = 4 - \operatorname{rank}(M'_{a,b}(p))$$

for all points $p \in \mathbb{P}_K^{a+b+1} \setminus \tilde{X}$.

(B) If $a + b \geq 5$ and $p \in \operatorname{Sec}(\tilde{X}) \setminus \tilde{X}$, by [BP, Theorem 4.2] we can say the following:

(1) If $a = 1$ and $p \in \operatorname{Join}(S(1), \tilde{X})$, then $\operatorname{Sec}_p(\tilde{X}) = \mathbb{P}_K^2$ and $\Sigma_p(\tilde{X}) = S(1) \cup \mathbb{L}(x)$ for some $x \in \mathbb{P}_K^1$.

(2) If $a = 2$ and $p \in \langle S(2) \rangle$, then $\operatorname{Sec}_p(\tilde{X}) = \mathbb{P}_K^2$ and $\Sigma_p(\tilde{X}) \subset \mathbb{P}_K^2$ is a smooth conic.

(3) In all other cases $\operatorname{Sec}_p(\tilde{X}) = \mathbb{P}_K^1$ and $\Sigma_p(\tilde{X}) \subset \mathbb{P}_K^1$ is a subscheme of length 2.

5.3. **Lemma.** *Let the notations and hypotheses be as in Reminder 5.1. Let* $a+b>r$*, let* $\Lambda=\mathbb{P}_K^{a+b+1-r}\subset\mathbb{P}_K^{a+b+1}$ *be a linear subspace disjoint to* $\tilde{X}:=S(a,b)$ *and let* $f:\tilde{X}\twoheadrightarrow X$ *be a projection of* $\tilde{X}$ *from* Λ *onto the surface* $X\subset\mathbb{P}_K^r$*. Then*

$$f^{-1}(\operatorname{Sing}(f))=\bigcup_{p\in\Lambda}\Sigma_p(\tilde{X}).$$

Proof. We may assume that $\mathbb{P}_K^r\subset\mathbb{P}_K^{a+b+1}$ is a subspace disjoint to Λ and f is induced by the canonical projection $\pi_\Lambda:\mathbb{P}_K^{a+b+1}\setminus\Lambda\twoheadrightarrow\mathbb{P}_K^r$. A closed point $x\in X$ belongs to $\operatorname{Sing}(f)$ if and only if $f^{-1}(x)=\langle\Lambda,x\rangle\cap\tilde{X}$ is of length >1. This latter condition is equivalent to the fact that $\langle\Lambda,x\rangle$ contains a secant or tangent line $\mathbb{L}$ to $\tilde{X}$. As each such line intersects Λ in precisely one point p, we get our claim. □

5.4. **Lemma.** *Let the notations and hypotheses as in Lemma 5.3 and assume that* $r\geq 4$*. Then, the following statements are equivalent:*

(i) f *is almost non-singular.*
(ii) *For all* $p\in\Lambda$ *we have* $\operatorname{rank}(M'_{a,b}(p))\geq 2$ *with equality for only finitely many* p*.*
(iii) $\sharp(\Lambda\cap\operatorname{Sec}(S(a,b)))<\infty$ *and either*
  (1) $a>2$*;*
  (2) $a=2$ *and* $\Lambda\cap\langle S(2)\rangle=\emptyset$*;*
  (3) $a=1$ *and* $\Lambda\cap\operatorname{Join}(S(1),S(1,b))=\emptyset$*.*

Proof. This is immediate by Lemma 5.3 and Remark 5.2(B). □

5.5. **Lemma.** *Let the notations and hypotheses be as in Lemma 5.4. Assume that the projection* $f:\tilde{X}=S(a,b)\twoheadrightarrow X$ *is almost non-singular, with* $0<a\leq b$*. Then, the numbers* a *and* b *are uniquely determined by* X*.*

Proof. As f is a birational projection, we have $a+b=\deg(\tilde{X})=\deg(X)$. It thus remains to show that $b-a$ is uniquely determined by X. To do so, let $\tilde{X}'=S(a',b')\subset\mathbb{P}_K^{a+b+1}$ (with $a\leq a'\leq b'=a+b-a'$) be another smooth rational surface scroll and let $f':\tilde{X}'\twoheadrightarrow X$ be an almost non-singular projection of $\tilde{X}'$ onto X from a linear subspace $\Lambda'=\mathbb{P}_K^{a+b+1-r}\subset\mathbb{P}_K^{a+b+1}$ disjoint to $\tilde{X}'$. If $a'=1$, our claim is clear. So, let $a'>1$.

Writing $Z:=f^{-1}(\operatorname{Sing}(f)\cup\operatorname{Sing}(f'))$ and $Z':=(f')^{-1}(\operatorname{Sing}(f)\cup\operatorname{Sing}(f'))$ we thus get a commutative diagram

$$\begin{array}{ccccc}\tilde{X}\setminus Z & & \xrightarrow[\simeq]{\;g\;} & & \tilde{X}'\setminus Z' \\ \big\downarrow & & & & \big\downarrow \\ \tilde{X} & \xrightarrow{\;f\;} & X & \xleftarrow{\;f'\;} & \tilde{X}'\end{array}.$$

Now, let $\mathbb{L}\subseteq\tilde{X}$ be a ruling disjoint to the finite set Z. Then as f and f' are linear projections, the set $\mathbb{L}':=g(\mathbb{L})$ is a line contained in $\tilde{X}'$ which avoids the finite set Z'. As $a'>1$, the line $\mathbb{L}'$ is a ruling of $\tilde{X}'$. Therefore $U:=\tilde{X}\setminus\mathbb{L}$ and $U':=\tilde{X}'\setminus\mathbb{L}'$ are smooth open neighborhoods of the finite sets Z and Z' respectively and hence the isomorphism $U\setminus Z\xrightarrow{\simeq}U'\setminus Z'$ given by g may be extended to an isomorphism $U\xrightarrow{\simeq}U'$. This means that g may be extended to an isomorphism $h:\tilde{X}\xrightarrow{\simeq}\tilde{X}'$. Now, as f and f' are projections, h

maps each ruling $\mathbb{L}$ of $\tilde{X}$ onto a line contained in $\tilde{X}'$ and hence to a ruling of $\tilde{X}$. This means that after a linear automorphism in the base $\mathbb{P}^1_K$ of $\tilde{X}'$, the map $h : \tilde{X} \xrightarrow{\cong} \tilde{X}'$ becomes an isomorphism of ruled surfaces, so that $b' - a' = b - a$ (see [Ha, V. Proposition 2.2]). □

5.6. **Remark.** Let $r \geq 5$ and let $X \subset \mathbb{P}^r_K$ be our non-degenerate irreducible projective surface of degree $r+1$. If X is a cone, we understand its structure by what we said in Remark 4.6. We therefore shall not consider this case anymore.

Now, we are ready to formulate and to prove the main result of this section.

5.7. **Theorem.** *Let $r \geq 5$ and let $X \subset \mathbb{P}^r_K$ be an irreducible non-degenerate surface.*

(a) *The following statements are equivalent:*
  (i) *X is of degree $r+1$, not a cone and falls under one of the seven cases 5-11.*
  (ii) *There is a positive integer $a \leq \frac{r+1}{2}$ and an almost non-singular projection $f : S(a, r+1-a) \twoheadrightarrow X$ from a line $\mathbb{P}^1_K = \mathbb{L}$ disjoint to $S(a, r+1-a)$.*

(b) *If X is as in statement* (a)(i) *and a, f and $\mathbb{L}$ are as in statement* (a)(ii)*, then the number $a \in \mathbb{N}$ is uniquely determined by X and*
  (1) *$\mathbb{L} = \Lambda$ satisfies the equivalent conditions* (ii) *and* (iii) *of Lemma 5.4.*
  (2) *If $x \in X$ is a closed point and $\mathcal{K}^2_X$ is defined as in Remark 3.7, the $\mathcal{O}_{X,x}$-modules $H^1_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x})$ and $\left(f_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X\right)_x$ are isomorphic and of the same length as the $\mathcal{O}_{X,x}$-module $\mathcal{K}^2_{X,x}$.*
  (3) *If $\mathcal{K}^2_X$ is as in statement* (2)*, then* $\text{length}(\mathcal{K}^2_X) = \text{length}(f_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X) = e(X)$ *and* $X \setminus \text{CM}(X) = X \setminus \text{Nor}(X) = X \setminus \text{Reg}(X) = \text{Sing}(f)$.

Proof. (a): Let X be as in statement (i) and let A be the homogeneous coordinate ring of X. Then, according to Theorem 2.7 the homogeneous integral K-algebra $B = B(A)$ satisfies $\dim_K B_1 = r+3$. So B is the homogeneous coordinate ring of an irreducible non-degenerate surface $\tilde{X} \subset \mathbb{P}^{r+2}_K$ and the inclusion $A \hookrightarrow B$ yields an almost non-singular projection $f : \tilde{X} \twoheadrightarrow X$ of $\tilde{X}$ onto X from a line $\mathbb{L}$ disjoint to $\tilde{X}$. In particular, f is birational, so that $\deg(\tilde{X}) = \deg(X) = r+1 = \text{codim}(\tilde{X}) + 1$. So $\tilde{X} \subset \mathbb{P}^{r+2}_K$ is a surface of minimal degree. As X is not a cone, $\tilde{X}$ cannot be a cone either. So $\tilde{X}$ is a smooth surface of minimal degree. As $r+1 > 5$ clearly $\tilde{X}$ cannot not be the Veronese surface and hence must be some smooth surface scroll. After a linear coordinate transformation we thus can write $\tilde{X} = S(a, r+1-a)$ where $a \leq \frac{r+1}{2}$ is a positive integer. This proves the implication “(i) ⇒ (ii)”. The converse implication follows immediately by Lemma 4.4 and the fact that $S(a, r+1-a)$ is arithmetically Cohen-Macaulay.

(b): The uniqueness of the number a follows from Lemma 5.4. Statement (1) follows from Lemma 5.4. Statement (2) is immediate by Lemma 4.5(b)(2) and Remark 3.7. Statement (3) now is immediate by statement (2), Lemma 4.5 and by Remark 3.7. □

Now we may characterize all non-conic surfaces of Δ-genus 2 which fall under the cases 3 – 11 as almost non-singular projections from surfaces $\tilde{X}$ of almost minimal degree, which are either non-conic and maximally del Pezzo or non-linearly normal.

5.8. **Corollary.** *Let $r \geq 5$ and let $X \subset \mathbb{P}^r_K$ be an irreducible non-degenerate surface which is not a cone. Then*

(a) *The following statements are equivalent*
   (i) X *is of degree* $r+1$ *and falls under the cases* 3 *or* 4.
   (ii) X *is an almost non-singular projection of a non-conic maximal del Pezzo surface* $\tilde{X} \subset \mathbb{P}_K^{r+1}$ *from a point* $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$.

(b) *The following statements are equivalent*
   (i) X *is of degree* $r+1$ *and falls under one of the cases* 5 – 11.
   (ii) X *is an almost non-singular projection of a non-linearly normal (and hence smooth) surface* $\tilde{X} \subset \mathbb{P}_K^{r+1}$ *of almost minimal degree from a point* $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$.

Proof. (a): This equivalence is clear by Theorem 4.7

(b): "(i) $\Rightarrow$ (ii)": Let X be as in statement (b)(i). Then, according to Theorem 5.7 There is a smooth rational normal scroll $S(a, r+1-a) \subset \mathbb{P}^{r+2}$ and an almost non-singular morphism $f : S(a, r+1-a) \twoheadrightarrow X$ induced by a projection $\pi : \mathbb{P}_K^{r+1} \setminus \Lambda \twoheadrightarrow \mathbb{P}_K^r$ for a line $\Lambda \subset \mathbb{P}_K^{r+2}$ disjoint to $S(a, r+a-1)$ and such that the intersection $W := \mathrm{Sec}(S(a, r+1-a)) \cap \Lambda$ is finite. Chose a closed point $q \in \Lambda \setminus W$, a projection $\rho : \mathbb{P}_K^{r+2} \setminus \{q\} \twoheadrightarrow \mathbb{P}_K^{r+1}$ from q, and a second projection $\sigma : \mathbb{P}_K^{r+1} \setminus \{p\} \twoheadrightarrow \mathbb{P}_K^r$ from a point $p \in \rho(\Lambda \setminus \{q\})$ so that $\pi = \sigma \circ \rho$. Let $\tilde{X} := \rho(S(a, r+1-a)$. Let $g : S(a, r+1-a) \twoheadrightarrow \tilde{X}$ and $h : \tilde{X} \twoheadrightarrow X$ be the morphisms induced by ρ and σ respectively. Then $f = h \circ g$ and by our choice of q the map g is an isomorphism. Therefore $\tilde{X} \subset \mathbb{P}_K^{r+1}$ is a surface of almost minimal degree which is not linearly normal and hence not a cone. In particular $\tilde{X}$ is a non-singular projection of a surface $W \subset \mathbb{P}_K^{r+2}$ of minimal degree (see Theorem 1.2 and Theorem 1.3 (c) of [BS4]), which is not a cone either. So W is smooth and hence $\tilde{X}$ is smooth. Moreover the morphism h is almost non-singular.

"(ii) $\Rightarrow$ (i)": Let X, p and $\tilde{X} \subset \mathbb{P}_K^{r+1}$ be as in statement (b)(ii). Let $\tilde{X} \twoheadrightarrow X$ the corresponding almost non-singular projection from a point p. As $\tilde{X}$ is not linearly normal, there is an isomorphic projection $g : S(a, r+1-a) \xrightarrow{\cong} \tilde{X}$ of smooth rational surface scroll $S(a, r+1-a) \subset \mathbb{P}_K^{r+2}$ from a pont q. Now, clearly the composition $f := h \circ g$ is an almost non-singular projection of $S(a, r+1-a)$ from a line $\Lambda \subset \mathbb{P}_K^{r+2} \setminus S(a, r+1-a)$. So, by Theorem 5.7(a) the surface X is of degree $r+1$ and falls under one of the cases 5 – 11. □

5.9. **Remark.** According to Theorem 4.7 the maximal del Pezzo surface $\tilde{X} \subset \mathbb{P}_K^{r+1}$ of Corollary 5.8(a)(ii) is uniquely determined by X up to projective equivalence. On the other hand, the non-linearly normal surface $\tilde{X} \subset \mathbb{P}_K^{r+1}$ of almost-minimal degree of Corollary 5.8(b)(ii) is not uniquely determined by X up to projective equivalence. So, Corollary 5.8 is not of immediate use for a classification of surfaces of Δ-genus 2.

Now, we may precisely characterize those non-conic surfaces of Δ-genus 2 which are obtained as projected images of smooth rational normal scrolls from a line.

5.10. **Corollary.** *Let* $r \geq 5$ *let* $a \leq \frac{r+1}{2}$ *be a positive integer, consider the smooth rational normal scroll* $\tilde{X} := S(a, r+1-a) \subset \mathbb{P}_K^{r+2}$ *and let* $\Lambda \subset \mathbb{P}_K^{r+2} \setminus \tilde{X}$ *be a line not contained in the secant variety* $\mathrm{Sec}(\tilde{X})$. *Let* $X \subset \mathbb{P}_K^r$ *be the image of* $\tilde{X}$ *under a projection from the line* Λ.

(a) *The following statements are equivalent*
   (i) X *falls under one of the cases* 3 *or* 4 *and satisfies the equivalent conditions (i)–(v) of Corollary 4.8.*

(ii) *The line* Λ *and the number* a *satisfy either*
(1) $a = 1$ *and* $\Lambda \cap \mathrm{Join}(S(1), \tilde{X}) \neq \emptyset$*;*
(2) $a = 2$ *and* $\Lambda \cap \langle S(2) \rangle \neq \emptyset$*.*

(b) *The following statements are equivalent*
(i) X *falls under one of the cases* 5–11*.*
(ii) *The line* Λ *and the number* a *satisfy either*
(1) $a = 1$ *and* $\Lambda \cap \mathrm{Join}(S(1), \tilde{X}) = \emptyset$*;*
(2) $a = 2$ *and* $\Lambda \cap \langle S(2) \rangle = \emptyset$*;*
(3) $a > 2$*.*

Proof. Statement (b) is clear by Theorem 5.7.

(a): "(i) $\Rightarrow$ (ii)": This is clear from the implication "(ii) $\Rightarrow$ (i)" of statement (b) and the fact that our cases exclude each other pairwise.

"(ii) $\Rightarrow$ (i)": Assume that statement (a)(ii) holds. If condition (1) holds let $Z := \mathrm{Join}(S(1), \tilde{X})$, while $Z = \langle S(2) \rangle$ if condition (2) holds. Chose $q \in \Lambda \cap Z$. Let $Y \subset \mathbb{P}_K^{r+1}$ be a projected image of $\tilde{X}$ from q. Then, according to [BP, Theorem 4.2] the surface Y is a non-conic, non-normal maximal del Pezzo surface. Moreover the morphism $f : \tilde{X} \twoheadrightarrow X$ induced by the projection from Λ factorizes through a unique projection $h : Y \twoheadrightarrow X$ of Y from the point $p \in \mathbb{P}_K^{r+1} \setminus Y$ which is the projected image of $\Lambda \setminus \{p\}$ from p. according to Corollary 4.8 it remains to show that the projection morphism h is almost non-singular.

To show this, we have only to prove, that there are only finitely many planes $\mathbb{E} = \mathbb{P}_K^2 \subset \mathbb{P}_K^{r+1}$ which contain Λ and a secant or tangent line to $\tilde{X}$. On use of Remark 5.2(B) we see that each point $s \in \Lambda \setminus Z$ is contained in at most one secant or tangent line to $\tilde{X}$. As $\Lambda \cap \mathrm{Sec}(\tilde{X})$ is a finite set and contains Z, our claim follows. □

5.11. **Corollary.** *Let* $r \geq 5$ *and let* $X \subset \mathbb{P}_K^r$ *be an irreducible non-degenerate surface which is not a cone. Then, the following statements are equivalent*

(i) X *is a surface of degree* $r + 1$ *and either*
(1) X *falls under one of the two cases* 3 *or* 4 *and satisfies the equivalent conditions (i)–(v) of Corollary 4.8.*
(2) X *falls under one of the cases the* 5–11*.*

(ii) X *is obtained by projecting a smooth rational normal sroll* $\tilde{X} \subset \mathbb{P}_K^{r+2}$ *from a line* Λ *which avoids* $\tilde{X}$ *and is not contained in* $\mathrm{Sec}(\tilde{X})$*.*

Proof. This is clear by Corollary 5.10 and Remark 4.9. □

Acknowledgement. The authors thank the referee for pointing out a simplified proof of Proposition 2.5 and for several suggestions for improving the text.

References

[B1] Brodmann, M.: *Cohomology of surfaces* $X \subset \mathbb{P}_K^r$ *with degree* $\leq 2r - 2$, in: F. van Oystayen (Ed.) "Commutative Algebra and Algebraic Geometry", 15 - 33, M. Dekker Lect. Notes 206, 1999.

[B2] Brodmann, M.: *Cohomology of certain projective surfaces with low sectional genus and degree*, in: D. Eisenbud (Ed.) "Commutative Algebra, Algebraic Geometry and Computational Methods", 172 - 200, Springer, 1999.

[BP] Brodmann, M., Park, E.: *On varieties of almost minimal degree I: Secant loci of rational normal scrolls.* Journal of Pure and Applied Algebra 214 (2010), 2033-2043.

[BPS] Brodmann, M., Park, E., Schenzel, P.: *On varieties of almost minimal degree II: A rank-depth formula.* To appear in Proceedings of the AMS.

[BS1] Brodmann, M., Schenzel, P.: *Curves of degree* $r+2$ *in* $\mathbb{P}^r$*: Cohomological, geometric, and homological aspects.* J. of Algebra 242 (2001), 577 - 623.

[BS2] Brodmann, M., Schenzel, P.: *On projective curves of maximal regularity.* Math. Z. 244 (2003), 271 - 289.

[BS3] Brodmann, M., Schenzel, P.: *On varieties of almost minimal degree in small codimension.* J. Algebra 305 (2006), 789 - 801.

[BS4] Brodmann, M., Schenzel, P.: *Arithmetic properties of projective varieties of almost minimal degree.* J. Algebraic Geom. 16 (2007), 347 - 400.

[BS5] Brodmann, M., Schenzel, P.: *Projective curves with maximal regularity and applications to syzygies and surfaces.* Manuscripta Mathematica 135 (2011), 469-495.

[BSh] Brodmann, M., Sharp, R.Y.: *Local cohomolgy – an algebraic introduction with geometric applications.* Cambridge Studies in Advanced Mathematics Vol. 60, Cambridge University Press, Cambridge, UK, 1998.

[C] Catalano-Johnson, M.: *The possible dimensions of higher secant varieties.* American Journal of Mathematics 118 (1996), 355 - 361.

[E] Ein, L: *Nondegenerate surfaces of degree* $n+3$ *in* $\mathbb{P}^n_{\mathbb{C}}$. Journal für die reine und angewandte Mathematik 351 (1984), 1 - 11.

[EG] Eisenbud, D., Goto, S.: *Linear free resolutions and minimal multiplicity.* J. Algebra 88 (1984), 89 - 133.

[F] Fujita, T.: *Classification theories of polarized varieties*, London Mathematical Society Lecture Notes Series 155, Cambridge University Press, 1990.

[G] Green, M.: *Koszul homology and the geometry of projective varieties.* J. Differential Geometry 19 (1984), 125 - 171.

[GL] Green, M., Lazarsfeld, R.: *Some results on the syzygies of finite sets and algebraic curves.* Compositio Math. 67 (1988), 301 - 314.

[GrP] Greuel, G.M., Pfister, G. et al: *Singular* 3.1.2*, a computer algebra system for polynomial computations.* Center for Computer Algebra, University of Kaiserslautern (2007) (http://www.singular.uni-kl.de).

[GLP] Gruson, L., Lazarsfeld, R., Peskine, C.: *On a theorem of Castelnuovo and the equations defining space curves.* Inventiones Math. 72 (1983), 491 - 506.

[H] Harris, J.: *Algebraic geometry: A first course.* Graduate Texts in Mathematics, Vol. 133, Springer-Verlag, New York, 1992.

[Ha] Hartshorne, R.: *Algebraic Geometry.* Springer Verlag, Berlin, 1977.

[P] Pinkham, H.: *A Castelnuovo bound for smooth surfaces.* Inventiones Math. 83 (1986), 321 - 332.

Universität Zürich, Institut für Mathematik, Winterthurer Str. 190, CH – Zürich, Switzerland

E-mail address: brodmann@math.unizh.ch

Martin-Luther-Universität Halle-Wittenberg, Institut für Informatik, Von-Seckendorff-Platz 1, D – 06120 Halle (Saale), Germany

E-mail address: schenzel@informatik.uni-halle.de