GEOMETRIC DOMAINS WITH SPECIFIED PSEUDO-DIMENSIONS

MARKUS P. BRODMANN AND RODNEY Y. SHARP

Abstract. **This paper pursues the study of the pseudo-supports of a finitely generated module over a finite-dimensional commutative Noetherian ring, and is particularly concerned with the properties of the pseudo-supports of a geometric normal local domain. It is shown that these pseudo-supports have certain properties that can be summarized by the so-called occurrence diagram of the ring, which is a certain subset of $\mathbb{N}_0 \times \mathbb{N}_0$. Standard properties of local cohomology modules enable one to conclude that there are obvious restrictions on the type of subset of $\mathbb{N}_0 \times \mathbb{N}_0$ that can be an occurrence diagram in this way, and much of the paper is devoted to consideration of whether a subset of $\mathbb{N}_0 \times \mathbb{N}_0$ that meets those obvious restrictions is an occurrence diagram of some geometric normal local domain. These occurrence diagrams are also used to assist with the production of examples that show that a question raised by C. Huneke, about certain sets related to Grothendieck's Finiteness Theorem for local cohomology, has a negative answer.**

0. Introduction

Let M be a finitely generated module over a finite-dimensional commutative Noetherian ring R. This paper is concerned with the pseudo-supports of M, which are defined as follows. For each integer i, the *i-th pseudo-support* $\operatorname{Psupp}^i(M)$ of M is defined by

$$\operatorname{Psupp}^i(M) = \operatorname{Psupp}^i_R(M) := \left\{ \mathfrak{p} \in \operatorname{Spec}(R) : H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0 \right\};$$

the *i-th pseudo-dimension* $\operatorname{psd}^i(M)$ of M is defined by

$$\operatorname{psd}^i(M) = \operatorname{psd}^i_R(M) = \sup\left\{ \dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Psupp}^i(M) \right\},$$

interpreted as -1 when $\operatorname{Psupp}^i(M)$ is empty. (For a module N over a local ring A with maximal ideal $\mathfrak{m}$, the notation $H^i_{\mathfrak{m}}(N)$ denotes the i-th local cohomology module of N with respect to $\mathfrak{m}$.)

In a recent paper [2], we were led to consider such pseudo-supports in the case when R is local: we showed that when the local ring R is catenary, all such pseudo-supports are closed under specialization ([2, Lemma 2.2]), and that when the local ring R is universally catenary and has the property that all its formal fibres are Cohen–Macaulay, then all such pseudo-supports are closed in the Zariski topology. We also gave some examples in [2, §3] which exhibit bad behaviour of pseudo-supports: [2, Example 3.1] is a universally catenary 3-dimensional local Noetherian domain R_1 such that $\operatorname{Psupp}^2(R_1)$ is not closed in the Zariski topology; and [2, Example 3.2] is a 3-dimensional local Noetherian domain R_2 all of whose formal fibres are Cohen–Macaulay but for which $\operatorname{Psupp}^3(R_2)$ is not closed under specialization.

In this paper, we are concerned with the properties of pseudo-supports of geometric local domains, that is, local domains which are essentially of finite type over an algebraically closed field. Because we shall be working with localizations of affine domains, we have extended the definition of pseudo-support from that given in [2] so that it applies over any finite-dimensional commutative Noetherian ring.

We shall see that the pseudo-supports of such geometric local domains have certain properties that can be summarized either by what we call an occurrence diagram or by a sequence of pseudo-dimensions.

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We make these ideas precise next. Throughout, we use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers.

0.1. **Definitions.** Let M be a non-zero finitely generated module over a finite-dimensional commutative Noetherian ring R. The *occurrence diagram for* M is the subset Δ of $\mathbb{N}_0 \times \mathbb{N}_0$ given by

$$\begin{aligned}\Delta &= \{(x,y) \in \mathbb{N}_0 \times \mathbb{N}_0 : \text{ there exists } \mathfrak{p} \in \operatorname{Psupp}^x(M) \text{ with } \dim R/\mathfrak{p} = y\} \\ &= \left\{(x,y) \in \mathbb{N}_0 \times \mathbb{N}_0 : \text{ there exists } \mathfrak{p} \in \operatorname{Spec}(R) \text{ with } \dim R/\mathfrak{p} = y \text{ and } H^{x-y}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\right\}.\end{aligned}$$

The *sequence of pseudo-dimensions of* M is the sequence

$$(\operatorname{psd}^0(M), \operatorname{psd}^1(M), \ldots, \operatorname{psd}^{\dim M}(M)),$$

although in a case where, say, $\operatorname{Psupp}^i(M) = \emptyset$ for an integer i with $0 \leq i < \dim M$, we shall often leave a space at the appropriate place in the sequence rather than write the '-1' for $\operatorname{psd}^i(M)$.

We now list some properties of occurrence diagrams which can be deduced easily from known theory of local cohomology modules.

0.2. *Remark.* Let the situation be as in 0.1.

(i) If the occurrence diagram for M has a node at (x,y), then $H^{x-y}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$ and $\dim R/\mathfrak{p} = y$, so that $x \geq y \geq 0$. Thus all nodes in the occurrence diagram for M have to lie in the first quadrant and on or below the line $y = x$.

(ii) Let $j \in \mathbb{N}_0$. Then the occurrence diagram for M has a node at (j,j) if and only if there exists $\mathfrak{p} \in \operatorname{Supp}(M)$ with $\dim R/\mathfrak{p} = j$ and $H^0_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$, that is, with $\dim R/\mathfrak{p} = j$ and $\mathfrak{p} \in \operatorname{Ass}(M)$. Since there exists an associated prime $\mathfrak{p}$ of M with $\dim R/\mathfrak{p} = \dim M$, it follows that the occurrence diagram for M must have a node at $(\dim M, \dim M)$.

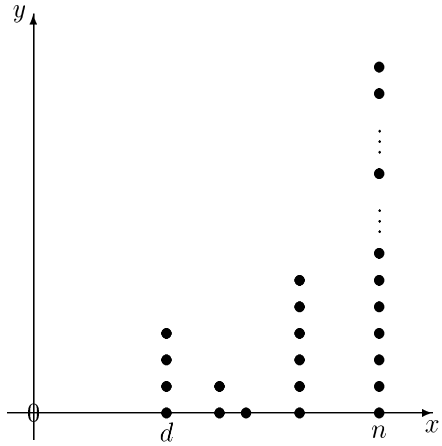
(iii) It follows from (ii) that, if R is an integral domain, then the occurrence diagram for R (as a module over itself) intersects the line $y = x$ at the point $(\dim R, \dim R)$ and no other.

(iv) If R is a catenary normal local domain, then the statement of (iii) above can be strengthened, as follows. Let $n := \dim R$. Suppose that, for some $j \in \mathbb{N}$ with $j < n$, the occurrence diagram for R (as a module over itself) has a node at $(j, j-1)$. Then there exists $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $H^1_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \neq 0$ and $\dim R/\mathfrak{p} = j - 1 < j < n$. Since R satisfies Serre's criterion (S_2), it follows that $\operatorname{ht}\mathfrak{p} = 1$. However, this provides a contradiction to the catenarity of R, since $\dim R/\mathfrak{p} \leq n-2$. Also, there does exist a prime $\mathfrak{q}$ of R of height 1 with $\dim R/\mathfrak{q} = n-1$. Thus, in this case, we can say that the occurrence diagram for R has (n,n) as its only point of intersection with the line $y = x$, and has $(n, n-1)$ as its only point of intersection with the line $y = x - 1$.

(v) Assume that R is a catenary local ring. Then, by [2, Lemma 2.2], $\operatorname{Psupp}^x_R(M)$ is closed under specialization for all $x \in \mathbb{N}_0$. Therefore, if there is a node in the occurrence diagram for M at (x,y), then there must be a vertical column of nodes at (x,y), $(x,y-1)$, $(x,y-2), \ldots, (x,0)$; in particular, it follows from (ii) above that the occurrence diagram for M must have a vertical column of $\dim M + 1$ nodes stretching from $(\dim M, \dim M)$ to $(\dim M, 0)$. Note that, in this situation, the information conveyed by the occurrence diagram for M can also be conveyed, rather more concisely, by the sequence of pseudo-dimensions of M.

(vi) Assume that R is a catenary local ring. If $x \in \mathbb{N}_0$ is such that there is a node of the occurrence diagram for M at $(x,0)$, then $H^x_{\mathfrak{m}}(M) \neq 0$, and so x must lie between $d := \operatorname{depth} M$ and $n := \dim M$; moreover, there are nodes at $(d,0)$ and $(n,0)$. It follows that, if there is a vertical column of nodes in the occurrence diagram for M, then that column must lie on a line $x = t$

for some integer t between d and n. The diagram



illustrates these properties and those described in part (v). Although such diagrams are helpful, we shall not use them much in this paper, and shall instead use sequences of pseudo-dimensions (see 0.1).

The points set out in 0.1 show that the occurrence diagram for a catenary local domain (over itself) is subject to certain constraints. This paper grew out of attempts to determine which subsets of $\mathbb{N}_0 \times \mathbb{N}_0$ that satisfy those constraints can actually be achieved as occurrence diagrams of geometric local domains. We formulate our questions more precisely in terms of the following terminology.

0.3. **Definitions.** Let $n \in \mathbb{N}$.

(i) An *n-diagram* is a subset Δ of $\{(x, y) \in \mathbb{N}_0 \times \mathbb{N}_0 : y \leq x \leq n\}$ such that
  (a) $(n, n) \in \Delta$;
  (b) if $(x, y) \in \Delta$, then $(x, j) \in \Delta$ for all $j = y - 1, \ldots, 0$;
  (c) $(j, j) \in \Delta$ only when $j = n$.

(ii) Let Δ be an n-diagram, as in (i). The *height sequence* $d(\Delta)$ of Δ is the sequence

$$(d_0, d_1, \ldots, d_{n-1}, n),$$

where $d_i = \sup\{j \in \mathbb{N}_0 : (i, j) \in \Delta\}$ (interpreted as -1 if the set whose supremum is taken is empty) for each $i = 0, \ldots, n-1$.

Observe that $d_0 = -1$, and that each of Δ and $d(\Delta)$ determine the other.

(iii) A *normal n-diagram* is an n-diagram Δ such that $(j, j-1) \in \Delta$ only when $j = n$.

Observe that, if Δ is a normal n-diagram and $n \geq 2$, then $d(\Delta)$ has the form

$$(-1, -1, d_2, \ldots, d_{n-1}, n).$$

Note that, by 0.2, if R is an n-dimensional catenary local domain (such as, for example, a geometric local domain), then the occurrence diagram for R is an n-diagram $\Delta(R)$ for which $d(\Delta(R))$ is the sequence of pseudo-dimensions of R, and that, if R is, in addition, normal, then its occurrence diagram $\Delta(R)$ is a normal n-diagram. These points raise the following.

0.4. **Question.** Let $n \in \mathbb{N}$, and let Δ be an n-diagram, as in 0.3(i).

Does there exist an n-dimensional geometric local domain R such that Δ is the occurrence diagram for R?

0.5. **Question.** Let $n \in \mathbb{N}$, and let Δ be a normal n-diagram, as in 0.3(iii).

Does there exist an n-dimensional normal geometric local domain R such that Δ is the occurrence diagram for R?

In the next section, we shall establish results which show that, when the normal n-diagram Δ has a certain rather simple form, the answers to both Questions 0.4 and 0.5 are affirmative. In §2, we shall indicate how to use Segre products to construct geometric normal local domains with more complicated occurrence diagrams, but we shall give a normal 4-diagram for which we have not so far been able to resolve Question 0.5.

In the final section, we shall present some connections between occurrence diagrams and certain sets that are related to Grothendieck's Finiteness Theorem for local cohomology. To explain these briefly, we revert to the situation where M is a non-zero finitely generated module over a commutative Noetherian ring R. Let $\mathfrak{a}$ be an arbitrary proper ideal of R; set

$$F_{\mathfrak{a}}(M) := \left\{i \in \mathbb{N} : H^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\right\}$$

and

$$\Lambda_{\mathfrak{a}}(M) := \{\operatorname{depth} M_{\mathfrak{p}} + \operatorname{ht}(\mathfrak{a}+\mathfrak{p})/\mathfrak{p} : \mathfrak{p} \in \operatorname{Supp}(M) \setminus \operatorname{Var}(\mathfrak{a})\}.$$

(We use $\operatorname{Var}(\mathfrak{a})$ to denote the variety of $\mathfrak{a}$.)

Grothendieck's Finiteness Theorem (see [1, 9.5.2]) asserts that, when R is a homomorphic image of a regular ring, the infima of $F_{\mathfrak{a}}(M)$ and $\Lambda_{\mathfrak{a}}(M)$ are equal. C. Huneke [7, Problem 3.3] asked whether it is always the case that the sets $F_{\mathfrak{a}}(M)$ and $\Lambda_{\mathfrak{a}}(M)$ are equal. M. Hellus [6, §4] provided an easy example of an ideal $\mathfrak{a}$ in a regular local ring R for which $F_{\mathfrak{a}}(R) \neq \Lambda_{\mathfrak{a}}(R)$. However, it seems to us to be rather harder to produce an example of a local domain $(R, \mathfrak{m})$ which is a homomorphic image of a regular ring and for which $F_{\mathfrak{m}}(R) \neq \Lambda_{\mathfrak{m}}(R)$. In §3 below, we shall provide (many) such examples: it turns out that it is always the case that $\Lambda_{\mathfrak{m}}(M) \subseteq F_{\mathfrak{m}}(M)$, but, in certain circumstances, it is possible to deduce from the appearance of the occurrence diagram for M that the inclusion is strict.

1. Catenary local domains with certain simple occurrence diagrams

Our first lemma will enable us, in some circumstances, to obtain a geometric local domain with a specified occurrence diagram from an affine algebra (over an algebraically closed field) with that same specified occurrence diagram.

1.1. **Lemma.** *Let A be a finite-dimensional commutative Noetherian ring such that every maximal saturated chain of prime ideals of A has length $\dim A$. Let N be a finitely generated A-module, let $i \in \mathbb{Z}$, and let $\mathfrak{p} \in \operatorname{Spec}(A)$. Then*

$$\operatorname{Psupp}^i_{A_{\mathfrak{p}}}(N_{\mathfrak{p}}) = \left\{\mathfrak{q}A_{\mathfrak{p}} : \mathfrak{q} \in \operatorname{Psupp}^{i+\dim A/\mathfrak{p}}_A(N) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\right\}.$$

Proof. Let $\mathfrak{q} \in \operatorname{Spec}(A)$ with $\mathfrak{q} \subseteq \mathfrak{p}$. Then $\mathfrak{q}A_{\mathfrak{p}} \in \operatorname{Psupp}^i_{A_{\mathfrak{p}}}(N_{\mathfrak{p}})$ if and only if $\left(H^{i-\operatorname{ht}\mathfrak{p}/\mathfrak{q}}_{\mathfrak{q}A_{\mathfrak{p}}}(N_{\mathfrak{p}})\right)_{\mathfrak{q}A_{\mathfrak{p}}} \neq 0$. The assumption about saturated chains of prime ideals of A ensures that

$$\begin{aligned}\operatorname{ht}\mathfrak{p}/\mathfrak{q} &= \operatorname{ht}\mathfrak{p} - \operatorname{ht}\mathfrak{q}\\ &= \dim A - \dim A/\mathfrak{p} - (\dim A - \dim A/\mathfrak{q})\\ &= \dim A/\mathfrak{q} - \dim A/\mathfrak{p}.\end{aligned}$$

We can now use the canonical ring isomorphism $(A_{\mathfrak{p}})_{\mathfrak{q}A_{\mathfrak{p}}} \cong A_{\mathfrak{q}}$ to complete the proof. □

1.2. **Definition.** Let $n \in \mathbb{N}$, and let $h, j \in \mathbb{N}_0$ with $0 \leq h < n$. We say that an n-diagram Δ, as in 0.3(i), is *h-level* if and only if its height sequence $d(\Delta) = (d_0, d_1, \ldots, d_{n-1}, n)$ is such that the *set* $\{d_0, d_1, \ldots, d_{n-1}\}$ (as opposed to the *sequence* $(d_0, d_1, \ldots, d_{n-1})$) is equal to $\{-1, h\}$.

Note that an h-level n-diagram Δ contains a node (a, b) with $a < n$, and that, whenever $(a, b) \in \Delta$ with $a < n$, then $d_a = h$.

E. G. Evans and P. Griffith have produced in [4, Theorem 4.13] an example which quickly provides affirmative answers to Questions 0.4 and 0.5 in the case of 0-level n-diagrams.

1.3. **Example.** Let $n \in \mathbb{N}$ and let Δ be a 0-level n-diagram. Thus

$$\Delta = \{(m_1, 0), \ldots, (m_s, 0), (n, 0), (n, 1), \ldots, (n, n)\}$$

for some positive integers $m_1, \ldots, m_s$, where $0 < m_1 < \cdots < m_s < n$. Note that $n \geq 2$ and, if Δ is normal, then $m_1 > 1$.

Let K be an infinite field, let C denote the polynomial ring $K[X_1,\ldots,X_{n+2}]$ in $n+2$ indeterminates over K, and let $\mathfrak{n} := (X_1,\ldots,X_{n+2})$. Then there exists a graded prime ideal $\mathfrak{q}$ of height 2 of C such that $C/\mathfrak{q}$ is normal if $m_1 > 1$ and, for $i \in \mathbb{Z}$,

$$\operatorname{Psupp}^i_C(C/\mathfrak{q}) = \begin{cases} \operatorname{Var}(\mathfrak{q}) & \text{if } i = n, \\ \{\mathfrak{n}\} & \text{if } i \in \{m_1,\ldots,m_s\}, \\ \emptyset & \text{if } i \notin \{m_1,\ldots,m_s,n\}. \end{cases}$$

Proof. By [4, Theorem 4.13] (see also Migliore, Nagel and Peterson [10, Theorem 2.3 and Remark 2.4(i)]), there exists a graded prime ideal $\mathfrak{q}$ of height 2 of C such that the local cohomology module $H^i_{\mathfrak{n}/\mathfrak{q}}(C/\mathfrak{q})$ is zero unless $i \in \{m_1,\ldots,m_s,n\}$, and there exist an integer t and homogeneous isomorphisms

$$H^i_{\mathfrak{n}/\mathfrak{q}}(C/\mathfrak{q}) \cong (C/\mathfrak{n})(t) \quad \text{for all } i = m_1,\ldots,m_s.$$

(Here, (t) denotes the t-th shift functor.) Furthermore, $C/\mathfrak{q}$ is normal if $m_1 > 1$.

Set $A := C/\mathfrak{q}$, a graded integral domain that is an affine K-algebra. Observe that $A_\mathfrak{p}$ is Cohen–Macaulay for each non-maximal graded prime ideal $\mathfrak{p}$ of A (by [1, 13.1.17], for example); it therefore follows (from [3, 1.5.8 and 1.5.9], for example) that $A_{\mathfrak{p}'}$ is Cohen–Macaulay for all $\mathfrak{p}' \in \operatorname{Spec}(A) \setminus \{\mathfrak{n}/\mathfrak{q}\}$. All the claims now follow from these observations used in conjunction with the Independence Theorem for local cohomology. □

1.4. **Corollary.** *Let $n \in \mathbb{N}$ and let Δ be a 0-level n-diagram. With the notation of* 1.3*, let $R := (C/\mathfrak{q})_{\mathfrak{n}/\mathfrak{q}}$. Then it follows from the Independence Theorem for local cohomology and* 1.1 *that the local domain $(R,\mathfrak{m})$ (which is normal if Δ is normal and geometric when K is chosen to be algebraically closed) satisfies*

$$\operatorname{Psupp}^i_R(R) = \begin{cases} \operatorname{Spec}(R) & \textit{if } i = n, \\ \{\mathfrak{m}\} & \textit{if } i \in \{m_1,\ldots,m_s\}, \\ \emptyset & \textit{if } i \notin \{m_1,\ldots,m_s,n\}. \end{cases}$$

Therefore the occurrence diagram for R (as a module over itself) is Δ, and so Questions 0.4 *and* 0.5 *have been answered affirmatively when Δ is a 0-level n-diagram.*

1.5. **Lemma.** *Let K be a field, let A be an affine K-algebra which is an integral domain, and let $B := A[Y_1,\ldots,Y_h]$, the polynomial ring over A in h indeterminates $Y_1,\ldots,Y_h$. Let N be a finitely generated A-module. Let $\mathfrak{P} \in \operatorname{Spec}(B)$, and set $\mathfrak{p} := \mathfrak{P} \cap A$; let $i \in \mathbb{Z}$. Then*

$$\mathfrak{P} \in \operatorname{Psupp}^{i+h}_B(N \otimes_A B) \quad \textit{if and only if} \quad \mathfrak{p} \in \operatorname{Psupp}^i_A(N).$$

Proof. Let $g : A_\mathfrak{p} \longrightarrow B_\mathfrak{P}$ be the (flat local) ring homomorphism induced by the inclusion map $\iota : A \longrightarrow B$. Now the fibre ring S of g over the maximal ideal $\mathfrak{p}A_\mathfrak{p}$ of $A_\mathfrak{p}$ is a localization of the fibre ring of ι over $\mathfrak{p}$, and the latter is isomorphic to the polynomial ring $k(\mathfrak{p})[Y_1,\ldots,Y_h]$, where $k(\mathfrak{p})$ is the residue field of the local ring $A_\mathfrak{p}$. Thus S is a Cohen–Macaulay local ring: let s denote its dimension. Now

$$\left(H^{i+h-\dim B/\mathfrak{P}}_{\mathfrak{P}}(N \otimes_A B)\right)_{\mathfrak{P}} \neq 0 \quad \text{if and only if} \quad H^{i+h-\dim B/\mathfrak{P}}_{\mathfrak{P}B_\mathfrak{P}}\left(N_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{P}\right) \neq 0;$$

by [2, Theorem 2.1], this is the case if and only if $H^{i+h-s-\dim B/\mathfrak{P}}_{\mathfrak{p}A_\mathfrak{p}}(N_\mathfrak{p}) \neq 0$. It is therefore sufficient for us to show that $h - s - \dim B/\mathfrak{P} = -\dim A/\mathfrak{p}$.

To achieve this, note that, since both A and B are affine K-algebras which are domains, we have

$$\operatorname{ht}\mathfrak{p} + \dim A/\mathfrak{p} = \dim A \quad \text{and} \quad \operatorname{ht}\mathfrak{P} + \dim B/\mathfrak{P} = \dim B = \dim A + h.$$

Also, $\operatorname{ht}\mathfrak{P} = \operatorname{ht}\mathfrak{p} + \dim S = \operatorname{ht}\mathfrak{p} + s$. Therefore

$$\begin{aligned} h - s - \dim B/\mathfrak{P} &= h + \operatorname{ht}\mathfrak{p} - \operatorname{ht}\mathfrak{P} - \dim B/\mathfrak{P} \\ &= h + \operatorname{ht}\mathfrak{p} - (\dim A + h) \\ &= -\dim A/\mathfrak{p}, \end{aligned}$$

as required. □

1.6. **Example.** Let $n \in \mathbb{N}$, let $h \in \mathbb{N}_0$, and let Δ be an h-level n-diagram. Thus

$$\Delta = \left(\bigcup_{i=1}^{s} \{(m_i, j) : j = 0, \ldots, h\} \right) \bigcup \{(n, j) : j = 0, \ldots, n\}$$

for some positive integers $m_1, \ldots, m_s$, where $0 \leq h < m_1 < \cdots < m_s < n$. Note that $n - h \geq 2$ and, if Δ is normal, then $m_1 - h > 1$.

Let K be an infinite field, let C denote the polynomial ring $K[X_1, \ldots, X_{n+2}]$ in $n+2$ indeterminates over K, and let $\mathfrak{n} := (X_1, \ldots, X_{n+2})$ and $\mathfrak{t} := (X_1, \ldots, X_{n-h+2})$. Then there exists a graded prime ideal $\mathfrak{q}$ of height 2 of C with $\mathfrak{q} \subseteq \mathfrak{t}$ and such that $C/\mathfrak{q}$ is normal if $m_1 - h > 1$ and, for $i \in \mathbb{Z}$,

$$\operatorname{Psupp}^i_C(C/\mathfrak{q}) = \begin{cases} \operatorname{Var}(\mathfrak{q}) & \text{if } i = n, \\ \operatorname{Var}(\mathfrak{t}) & \text{if } i \in \{m_1, \ldots, m_s\}, \\ \emptyset & \text{if } i \notin \{m_1, \ldots, m_s, n\}. \end{cases}$$

Proof. Let Δ' be the 0-level $(n-h)$-diagram given by

$$\Delta' = \{(m_1 - h, 0), \ldots, (m_s - h, 0), (n-h, 0), (n-h, 1), \ldots, (n-h, n-h)\}.$$

By Example 1.3 applied to Δ', there exists a graded prime ideal $\mathfrak{q}'$ of height 2 of the polynomial ring $C' := K[X_1, \ldots, X_{n-h+2}]$ such that $C'/\mathfrak{q}'$ is normal if $m_1 - h > 1$ and, for $i \in \mathbb{Z}$,

$$\operatorname{Psupp}^i_{C'}(C'/\mathfrak{q}') = \begin{cases} \operatorname{Var}(\mathfrak{q}') & \text{if } i = n-h, \\ \{(X_1, \ldots, X_{n-h+2})\} & \text{if } i \in \{m_1 - h, \ldots, m_s - h\}, \\ \emptyset & \text{if } i \notin \{m_1 - h, \ldots, m_s - h, n-h\}. \end{cases}$$

Note that $C = C'[X_{n-h+3}, \ldots, X_{n+2}]$; set $\mathfrak{q} = \mathfrak{q}'C$. Observe that $C/\mathfrak{q} \cong (C'/\mathfrak{q}')[X_{n-h+3}, \ldots, X_{n+2}]$ is normal if $C'/\mathfrak{q}'$ is, and so $C/\mathfrak{q}$ is normal if $m_1 - h > 1$.

Let $\mathfrak{P} \in \operatorname{Spec}(C)$. Then, by 1.5, we have $\mathfrak{P} \in \operatorname{Psupp}^i_C(C/\mathfrak{q})$ if and only if $\mathfrak{P} \cap C' \in \operatorname{Psupp}^{i-h}_{C'}(C'/\mathfrak{q}')$. This is the case if and only if $i - h = n - h$ and $\mathfrak{P} \cap C' \in \operatorname{Var}(\mathfrak{q}')$ or $i - h \in \{m_1 - h, \ldots, m_s - h\}$ and $\mathfrak{P} \cap C' = (X_1, \ldots, X_{n-h+2})C'$. The claims all follow from these observations. $\square$

1.7. **Corollary.** *Let $n \in \mathbb{N}$, let $h \in \mathbb{N}_0$, and let Δ be an h-level n-diagram. With the notation of* 1.6*, let $R := (C/\mathfrak{q})_{\mathfrak{n}/\mathfrak{q}}$. Then it follows from the Independence Theorem for local cohomology and* 1.1 *that the local domain $(R, \mathfrak{m})$ (which is essentially of finite type over K, normal if Δ is normal, and geometric if K is algebraically closed) has the following properties:*

(i) $\mathfrak{m}$ *can be generated by $n+2$ elements;*

(ii) $R/\mathfrak{t}R \cong K[Y_1, \ldots, Y_h]_{(Y_1, \ldots, Y_h)}$*, where $Y_1, \ldots, Y_h$ are indeterminates over K;*

(iii) $$\operatorname{Psupp}^i_R(R) = \begin{cases} \operatorname{Spec}(R) & \text{if } i = n; \\ \operatorname{Var}(\mathfrak{t}R) & \text{if } i \in \{m_1, \ldots, m_s\}; \\ \emptyset & \text{if } i \notin \{m_1, \ldots, m_s, n\}; \end{cases}$$

It follows that Questions 0.4 *and* 0.5 *have been answered affirmatively when Δ is an h-level n-diagram.*

2. Use of Segre products

In this section, we show how to produce examples of normal geometric local domains with occurrence diagrams that are more complicated than the 0-level and h-level n-diagrams produced in 1.4 and 1.7, including ones in which the heights of the non-empty columns fail to form a non-decreasing sequence as one moves from left to right. To produce such examples, we shall use Segre products of standard normal graded Noetherian domains defined over an algebraically closed field.

2.1. **Reminders.** Let K be an algebraically closed field. Recall that a positively graded commutative Noetherian K-algebra $A = \bigoplus_{n \in \mathbb{N}_0} A_n$ is said to be *standard* if $A = A_0[A_1]$; the *irrelevant ideal* of A is $\bigoplus_{n \in \mathbb{N}} A_n$, and this will be denoted by A_+.

Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ and $S = \bigoplus_{n\in\mathbb{N}_0} S_n$ be positively graded standard commutative Noetherian K-algebra domains, with $R_0 = K = S_0$. Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded R-module and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ be a graded S-module.

(i) Recall that the *Segre product* of R and S is the positively graded subring $R \boxtimes_K S$ of the graded ring $R \otimes_K S$ given by $R \boxtimes_K S := \bigoplus_{n\in\mathbb{N}_0}(R \boxtimes_K S)_n$, where $(R \boxtimes_K S)_n = R_n \otimes_K S_n$ for all $n \in \mathbb{N}_0$. See S. Goto and K. Watanabe [5, Chapter 4]. Recall also that the *Segre product* of M and N is the subgroup $M \boxtimes_K N$ of $M \otimes_K N$ given by

$$M \boxtimes_K N = \bigoplus_{n\in\mathbb{Z}} M_n \otimes_K N_n,$$

and it is a graded $R \boxtimes_K S$-module with n-th component $M_n \otimes_K N_n$ (for all $n \in \mathbb{Z}$).

(ii) It follows from [12, Chapter III, §15, Theorem 40, Corollary 1] that, since K is algebraically closed and R and S are domains, $R \otimes_K S$ is again a domain, so that $R \boxtimes_K S$ is a domain. Also ([5, Proposition (4.0.2)]), the Segre product $R \boxtimes_K S$ is again Noetherian. Note that it is standard and that $(R \boxtimes_K S)_0 = K$.

(iii) Recall from [5, Remark (4.0.3)(v)] that, since K is algebraically closed, if R and S are normal, then so too is $R \boxtimes_K S$. Thus the Segre product of two positively graded standard commutative Noetherian K-algebra normal domains over K produces another one.

(iv) Suppose that R and S are both normal and of dimension at least 2. Thus $H^q_{R_+}(R) = 0$ and $H^q_{S_+}(S) = 0$ for $q = 0, 1$. Then it follows from [5, Theorem (4.1.5)] that, for all $q \in \mathbb{N}_0$, there is a homogeneous isomorphism

$$H^q_{(R\boxtimes_K S)_+}(R \boxtimes_K S) \cong \left(R \boxtimes_K H^q_{S_+}(S)\right) \bigoplus \left(H^q_{R_+}(R) \boxtimes_K S\right) \bigoplus \left(\bigoplus_{\substack{i,j\in\mathbb{N}_0 \\ i+j=q+1}} \left(H^i_{R_+}(R) \boxtimes_K H^j_{S_+}(S)\right)\right).$$

Also, $\dim(R \boxtimes_K S) = \dim R + \dim S - 1$ by [5, Theorem (4.2.3)(i)]

(v) Let $i \in \mathbb{N}_0$. It follows from [1, Proposition 15.1.5] that, for all $n \in \mathbb{Z}$, the n-th component $H^i_{R_+}(R)_n$ of the local cohomology module $H^i_{R_+}(R)$ is a finite-dimensional vector space over K; we denote this dimension by $h^i(n)$. By [1, Theorem 17.1.9], the function $h^i_R : \mathbb{Z} \longrightarrow \mathbb{N}_0$, called the *$i$-th cohomological Hilbert function of R*, is of reverse polynomial type of degree less than i (in the sense of [1, 17.1.1(vi)]); in other words, there is a polynomial $p^i_R \in \mathbb{Q}[X]$ of degree less than i such that

$$\dim_K(H^i_{R_+}(R)_n) = p^i_R(n) \quad \text{for all } n << 0.$$

The polynomial p^i_R is called the *i-th cohomological Hilbert polynomial of R.*

(vi) Since $R_0 = K$, we can express R as a homomorphic image of a polynomial ring over K. To be precise, there is a polynomial ring $R' = \bigoplus_{n\in\mathbb{N}_0} R'_n$ with $R'_0 = K$ in finitely many indeterminates over K (graded in the usual way), and a surjective ring homomorphism $f : R' \longrightarrow R$ which is homogeneous in the sense that $f(R'_n) \subseteq R_n$ for all $n \in \mathbb{N}_0$. Let $d := \dim R$ and $d' := \dim R'$. By [1, Corollary 13.3.8], there exists $a' \in \mathbb{Z}$ such that $R'(a')$ is a *canonical module for R'. Let $K^i_R = {}^*\operatorname{Ext}^{d'-i}_{R'}(R, R'(a'))$, a finitely generated graded R-module. One can now use the argument based on Graded Local Duality and Graded Matlis Duality in the proof of [1, Theorem 17.1.9] to see that $h^i_R(n) = \dim_K((K^i_R)_{-n})$ for all $n \in \mathbb{Z}$.

Thus, if $p_{K^i_R}$ denotes the ordinary Hilbert polynomial of K^i_R, then $p^i_R(X) = p_{K^i_R}(-X)$, and the degree of p^i_R is one less than $\dim_R K^i_R$ (provided that $K^i_R \neq 0$).

Finally, an argument based on the Local Duality Theorem like that used in the proof of [2, Proposition 1.2(iii)] will show that $\operatorname{Psupp}^i(R) = \operatorname{Supp}_R(K^i_R)$, so that $\operatorname{Psupp}^i(R) = \emptyset$ if and only if $H^i_{R_+}(R) = 0$; it also follows that $\operatorname{psd}^i(R) = \dim_R K^i_R$, and, provided these are non-negative, they are both equal to $\deg(p^i_R) + 1$ and $\deg(p_{K^i_R}) + 1$.

The next two lemmas will assist in our application of the ideas of 2.1 to the construction of examples.

2.2. **Lemma.** *Let K be an algebraically closed field. Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a positively graded standard commutative Noetherian K-algebra normal domain of dimension at least 2, with $R_0 = K$. Note that R is *local and that R_+ is its unique *maximal ideal. Then $\operatorname{psd}^i(R_{R_+}) = \operatorname{psd}^i(R)$ for all $i = 0, \ldots, \dim R$.*

Proof. With the notation of 2.1(vi), $\operatorname{Psupp}^i(R) = \operatorname{Supp}(K^i_R)$; since K^i_R is a finitely generated graded R-module, it is non-zero if and only if $R_+ \in \operatorname{Supp}(K^i_R)$, and, when this is the case, its dimension is equal to the supremum of lengths of chains of prime ideals in $\operatorname{Psupp}^i(R)$ that have R_+ as their largest term. The result therefore follows from Lemma 1.1. $\square$

2.3. **Lemma.** *Let K be an algebraically closed field. Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ and $S = \bigoplus_{n\in\mathbb{N}_0} S_n$ be positively graded standard commutative Noetherian K-algebra normal domains of dimension at least 2, with $R_0 = K = S_0$. Let q be an integer with $q \geq 2$.*

(i) *If there exists $i, j \in \mathbb{N}$ such that $i + j = q + 1$, $\operatorname{psd}^i(R) \geq 1$ and $\operatorname{psd}^j(S) \geq 1$, then*

$$\operatorname{Psupp}^q(R \boxtimes_K S) \neq \emptyset$$

and $\operatorname{psd}^q(R \boxtimes_K S) = \max\left\{\operatorname{psd}^i(R) + \operatorname{psd}^j(S) - 1 : i, j \in \mathbb{N} \text{ and } i + j = q + 1\right\}$.

(ii) *If there do not exist $i, j \in \mathbb{N}$ such that $i + j = q + 1$, $\operatorname{Psupp}^i(R) \neq \emptyset$ and $\operatorname{Psupp}^j(S) \neq \emptyset$, then $\operatorname{Psupp}^q(R \boxtimes_K S) \neq \emptyset$ if and only if $\max\{\operatorname{end}(H^q_{R_+}(R)), \operatorname{end}(H^q_{S_+}(S))\} \geq 0$; when this is the case, $\operatorname{psd}^q(R \boxtimes_K S) = 0$.*

Proof. (i) Since each of $H^q_{R_+}(R)$ and $H^q_{S_+}(S)$ has end that is finite or $-\infty$, each of the graded $R \boxtimes_K S$-modules $R \boxtimes_K H^q_{S_+}(S)$ and $H^q_{R_+}(R) \boxtimes_K S$ has only finitely many non-zero components. Also, for all $n \in \mathbb{Z}$, we have $\dim_K \left(H^i_{R_+}(R) \boxtimes_K H^j_{S_+}(S)\right)_n = h^i_R(n) h^j_S(n)$. Thus

$$\dim_K \left(H^i_{R_+}(R) \boxtimes_K H^j_{S_+}(S)\right)_n = p^i_R(n) p^j_S(n) \quad \text{for all } n << 0.$$

Now $p^i_R p^j_S$ is a rational polynomial of degree $\operatorname{psd}^i(R) + \operatorname{psd}^j(S) - 2$, a non-negative integer. Note also that the leading coefficient of $p^i_R(-X) p^j_S(-X)$ must be positive. Thus, if $i', j' \in \mathbb{N}$ are such that $i' + j' = q + 1$, $\operatorname{psd}^{i'}(R) \geq 1$, $\operatorname{psd}^{j'}(S) \geq 1$ and $\operatorname{psd}^{i'}(R) + \operatorname{psd}^{j'}(S) = \operatorname{psd}^i(R) + \operatorname{psd}^j(S)$, then the sum of the leading terms of $p^i_R(-X) p^j_S(-X)$ and $p^{i'}_R(-X) p^{j'}_S(-X)$ cannot be zero. The claim now follows from these observations and 2.1(iv),(vi).

(ii) The hypotheses in this case, together with 2.1(iv),(vi), yield that

$$H^q_{(R\boxtimes_K S)_+}(R \boxtimes_K S) \cong \left(R \boxtimes_K H^q_{S_+}(S)\right) \bigoplus \left(H^q_{R_+}(R) \boxtimes_K S\right),$$

and this is non-zero if and only if $\max\{\operatorname{end}(H^q_{R_+}(R)), \operatorname{end}(H^q_{S_+}(S))\} \geq 0$. When this is the case, since each of $H^q_{R_+}(R)$ and $H^q_{S_+}(S)$ has end that is finite or $-\infty$, each of the graded $R \boxtimes_K S$-modules $R \boxtimes_K H^q_{S_+}(S)$ and $H^q_{R_+}(R) \boxtimes_K S$ has only finitely many non-zero components. The claim now follows from 2.1(v),(vi). $\square$

2.4. **Example.** Let K be an algebraically closed field. We use Example 1.6 to construct 7-dimensional positively graded standard commutative Noetherian K-algebra normal domains $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ and $S = \bigoplus_{n\in\mathbb{N}_0} S_n$ with $R_0 = K = S_0$, such that the sequences of pseudo-dimensions of R and S are

$$(\ , \ , \ , \ , 2, \ , \ , 7) \quad \text{and} \quad (\ , \ , \ , \ , \ , \ , 1, 7)$$

respectively. By 2.1 and 2.3, the Segre product $R \boxtimes_K S$ is a positively graded standard commutative Noetherian K-algebra normal domain of dimension 13, and its sequence of pseudo-dimensions is

$$(\ , \ , \ , \ , p_4, \ , p_6, p_7, \ , 2, 8, \ , 7, 13),$$

where $p_4, p_6, p_7 \in \{-1, 0\}$. By Lemma 2.2, the localization of $R \boxtimes_K S$ at its unique maximal graded ideal is a 13-dimensional normal local domain which again has the sequence

$$(\ , \ , \ , \ , p_4, \ , p_6, p_7, \ , 2, 8, \ , 7, 13)$$

as its sequence of pseudo-dimensions. Note that the non-negative terms of this sequence do not form a non-decreasing sequence.

2.5. *Remark.* Even though we have explored various methods of construction of geometric normal local domains over an algebraically closed field K, we have not even been able to produce a 4-dimensional one that has sequence of pseudo-dimensions $(\quad, \quad, 0, 1, 4)$. Our investigations suggest that it might be quite difficult to settle Question 0.5!

3. Connections with Grothendieck's Finiteness Theorem

In this section, we use some notation that was introduced in the Introduction: for a non-zero finitely generated module M over a commutative Noetherian ring R, and an arbitrary proper ideal $\mathfrak{a}$ of R, we set

$$F_{\mathfrak{a}}(M) := \left\{ i \in \mathbb{N} : H^i_{\mathfrak{a}}(M) \text{ is not finitely generated} \right\}$$

and

$$\Lambda_{\mathfrak{a}}(M) := \{\operatorname{depth} M_{\mathfrak{p}} + \operatorname{ht}(\mathfrak{a}+\mathfrak{p})/\mathfrak{p} : \mathfrak{p} \in \operatorname{Supp}(M) \setminus \operatorname{Var}(\mathfrak{a})\}.$$

3.1. **Lemma.** *Let M be a non-zero finitely generated module over the local ring $(R, \mathfrak{m})$. Then, with the notation introduced immediately above, $\Lambda_{\mathfrak{m}}(M) \subseteq F_{\mathfrak{m}}(M)$.*

Proof. Let $i \in \Lambda_{\mathfrak{m}}(M)$. Thus there exists $\mathfrak{p} \in \operatorname{Supp}(M) \setminus \{\mathfrak{m}\}$ such that

$$\operatorname{depth} M_{\mathfrak{p}} + \dim R/\mathfrak{p} = i.$$

Therefore $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$. Now $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is an Artinian $R_{\mathfrak{p}}$-module; it must therefore have an attached prime ideal. Thus there exists $\mathfrak{p}' \in \operatorname{Spec}(R)$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$ and $\mathfrak{p}'R_{\mathfrak{p}}$ is an attached prime ideal of $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. It follows from the Weak General Shifted Localization Principle [11, Theorem 4.8] that

$$\mathfrak{p}' \in \operatorname{Att}_R\left(H^{i-\dim R/\mathfrak{p}+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M)\right) = \operatorname{Att}_R(H^i_{\mathfrak{m}}(M)).$$

Since $\mathfrak{p}' \subseteq \mathfrak{p}$, we must have $\mathfrak{p}' \neq \mathfrak{m}$; hence $H^i_{\mathfrak{m}}(M)$ is not finitely generated, and $i \in F_{\mathfrak{m}}(M)$. $\square$

We point out next some possible conclusions about the sets $\Lambda_{\mathfrak{m}}(M)$ and $F_{\mathfrak{m}}(M)$ which can, in some circumstances, be drawn from the form of the sequence of pseudo-dimensions of M.

3.2. *Remark.* Assume that the local ring R is universally catenary, and that all formal fibres of all its localizations are Cohen–Macaulay. Assume also that the finitely generated R-module M is non-zero of dimension $n > 0$. Let $i \in \mathbb{N}_0$.

(i) The integer i lies in $F_{\mathfrak{m}}(M)$ if and only if $H^i_{\mathfrak{m}}(M)$ is not of finite length; by [2, Theorem 2.4(i)], this is the case if and only if $\mathfrak{m}$ is a member, but not a minimal member, of $\operatorname{Psupp}^i(M)$; this is the case if and only if $\operatorname{psd}^i(M) > 0$.

(ii) If the sequence $(d_0, d_1, \ldots, d_{n-1}, d_n)$ (where $d_n = n$) of pseudo-dimensions of M has some $d_i \geq y > 0$ (where y is an integer) and $d_j < y$ for all $j = 0, \ldots, i-1$, then it follows that $i \in \Lambda_{\mathfrak{m}}(M)$: this is because there must exist $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\dim R/\mathfrak{p} = y$ and $H^{i-y}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$, and $\operatorname{depth} M_{\mathfrak{p}}$ cannot be smaller than $i - y$.

(iii) Let $i \in \Lambda_{\mathfrak{m}}(M)$. Then there exists $\mathfrak{p} \in \operatorname{Supp}(M) \setminus \{\mathfrak{m}\}$ such that $\operatorname{depth} M_{\mathfrak{p}} = i - \dim R/\mathfrak{p}$. Then clearly $\mathfrak{p} \in \operatorname{Psupp}^i(M)$ but $\mathfrak{p} \notin \operatorname{Psupp}^j(M)$ for $j = 0, \ldots, i-1$.

(iv) The infima of $F_{\mathfrak{m}}(M)$ and $\Lambda_{\mathfrak{m}}(M)$ are equal by the extension of Grothendieck's Finiteness Theorem presented in [1, 9.6.7]. Since there is a minimal member $\mathfrak{p}$ of $\operatorname{Supp}(M)$ with $\operatorname{ht} \mathfrak{m}/\mathfrak{p} = n = \dim M$, it is clear that $\dim M$ is the supremum of $\Lambda_{\mathfrak{m}}(M)$. In fact, $F_{\mathfrak{m}}(M)$ also has $\dim M$ as its supremum, in view of Lemma 3.1 and the fact that $H^i_{\mathfrak{m}}(M) = 0$ for $i > \dim M$.

3.3. **Example.** Let $n, h \in \mathbb{N}$, and let Δ be a normal h-level n-diagram. Thus

$$\Delta = \left(\bigcup_{i=1}^{s} \{(m_i, j) : j = 0, \ldots, h\} \right) \bigcup \{(n, j) : j = 0, \ldots, n\}$$

for some positive integers $m_1, \ldots, m_s$, where $0 < h < m_1 < \cdots < m_s < n$, $n - h \geq 2$ and $m_1 - h > 1$. Let $(R, \mathfrak{m})$ be a geometric normal local domain, constructed as in Corollary 1.7, which has Δ as its occurrence diagram. Then it follows, on use of 3.1, 3.2 and property 1.7(iii), that

$$F_{\mathfrak{m}}(R) = \{m_1, \ldots, m_s, n\} \quad \text{and} \quad \Lambda_{\mathfrak{m}}(R) = \{m_1, n\}.$$

Therefore, provided s is chosen to be greater than 1, we have a strict inclusion $\Lambda_{\mathfrak{m}}(R) \subset F_{\mathfrak{m}}(R)$.

In this way, we can construct infinitely many geometric normal local domains that are counterexamples of the type sought for Huneke's problem mentioned in the Introduction.

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(Brodmann) Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland, *Fax number*: 0041-1-635-5706

E-mail address: Brodmann@math.unizh.ch

(Sharp) Department of Pure Mathematics, University of Sheffield, Hicks Building, Sheffield S3 7RH, United Kingdom, *Fax number*: 0044-114-222-3769

E-mail address: R.Y.Sharp@sheffield.ac.uk