BOUNDING COHOMOLOGY OF COHERENT SHEAVES OVER PROJECTIVE SCHEMES

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Introduction

In this survey we approach the question *"What bounds the cohomology of a projective scheme with coefficients in an arbitrary coherent sheaf?"* Before we present some details of our exposition, we set up the general framework of our approach and give some ideas which form the motivating background of our investigation.

One of the most fascinating aspects of Projective Algebraic Geometry is the interplay between *discrete* and *continuous* data of a projective algebraic variety, or more generally of a closed subscheme X of a given projective space $\mathbb{P}^r_K$ over some (algebraically closed) field K. Indeed, one approach to study projective schemes consists in looking at classes of closed subschemes $X \subseteq \mathbb{P}^r_K = \mathrm{Proj}(K[\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_r])$ with a given *"discrete skeleton"*.

A most prominent occurrence of this concept comes up in the theory of *Hilbert schemes*, where the class of all closed subschemes $X \subseteq \mathbb{P}^r_K$ is "sliced up" into classes of subschemes X having a given *Hilbert-Serre polynomial* $p = P_X = P_{\mathcal{O}_X}$. In this particular case, the objects in each single slice are parametrized by a projective scheme, the Hilbert scheme $\mathrm{Hilb}^p_{\mathbb{P}^r}$ with respect to the polynomial p. Here, the discrete skeleton is given by the Hilbert-Serre polynomial p and the continuous data are encoded in the corresponding Hilbert scheme $\mathrm{Hilb}^p_{\mathbb{P}^r}$.

Another occurrence of the same principle is to use the *Betti numbers*

$$\beta_{i,j}(I_X) := \dim_K\big(\mathrm{Tor}_i^{K[\mathbf{x}_0,\mathbf{x}_1\ldots,\mathbf{x}_r]}(K, I_X)_{j+i}\big)$$

of X (more precisely of the homogeneous vanishing ideal $I_X \subseteq K[\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_r]$ of X) to define the discrete skeleton – and to study all closed subschemes $X \subseteq \mathbb{P}^r_K$ with a given *Betti table*

$$\beta(X) := \big(\beta_{i,j}(I_X)\big)_{(i,j)\in\{0,1,\ldots,r\}\times\mathbb{N}} \quad .$$

As the Betti table of $X \subseteq \mathbb{P}^r_K$ determines the Hilbert-Serre polynomial $p = P_X = P_{\mathcal{O}_X}$ of X, the "slices" now are are "thinner" than in the previous case – and form indeed locally closed strata of the corresponding Hilbert scheme $\mathrm{Hilb}^p_{\mathbb{P}^r}$. Moreover, in this situation the "thick slice" $\mathrm{Hilb}^p_{\mathbb{P}^r}$ contains only finitely many of the "thin slices" which belong to a fixed the Betti table $\beta(X)$ of X.

Clearly, one also could use *sheaf cohomology* to define the discrete skeleton. In the previously described situation, this would mean to slice up the scheme $\mathrm{Hilb}^p_{\mathbb{P}^r}$ into strata consisting of all closed subschemes $X \subseteq \mathbb{P}^r_K$ (with Hilbert-Serre polynomial p) and given *cohomology table*

$$h_{X,\mathcal{O}_X} = h_{\mathcal{O}_X} := \big(h^i(X, \mathcal{O}_X(n))\big)_{(i,n)\in\{0,1,\ldots,r\}\times\mathbb{Z}} \quad .$$

This point of view is supported by the fact that (if K is algebraically closed and of characteristic 0), imposing an arbitrary lower bound on the cohomology table $h_{\mathcal{O}_X}$, always leaves us with a locally closed and (rationally) connected stratum in $\mathrm{Hilb}^p_{\mathbb{P}^r}$ (see [22]). Moreover, in this case too, the "thick slice" $\mathrm{Hilb}^p_{\mathbb{P}^r}$ contains only finitely many of the "thin slices" given by fixing the cohomology table $h_{\mathcal{O}_X}$ of X. In addition, the cohomology table $h_{\mathcal{O}_X}$ determines the Hilbert-Serre polynomial $P_{\mathcal{O}_X}$ of the scheme X, so that each of our thin slices is contained in a unique thick slice.

One finally could consider a more general situation, which exceeds the framework of Hilbert schemes and just slice up the class $\mathcal{S}^t$ of all pairs $(X, \mathcal{F})$ in which X is a projective scheme over some field K and $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules with $\dim(\mathcal{F}) = t$, by means of the *cohomology table*

$$h_{X,\mathcal{F}} = h_{\mathcal{F}} := \big(h^i(X, \mathcal{F}(n))\big)_{(i,n)\in\{0,1,\ldots,t\}\times\mathbb{Z}}$$

of X with respect to the coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. Clearly, in this situation one cannot expect that the class $\mathcal{S}^t$ splits up into finitely many of the "slices" which are given by fixing the cohomology table $h_{\mathcal{F}}$. Nevertheless one would like to know in this more general setting, whether the single slices are not "too thin" or – equivalently – the family of all slices is "not too large".

A way of understanding this "thickness problem for slices obtained by fixing cohomology tables" would be to describe all possible cohomology tables $h_{\mathcal{F}}$, if $(X, \mathcal{F})$ runs through the full class $\mathcal{S}^t$.

A more realistic approach would be to prove some "finiteness results" which hint, that the class of occurring slices is not too large. This is actually the basic aim of the present paper. In particular, we shall see that prescribing the entries of the cohomology tables $h_{\mathcal{F}}$ along certain finite patterns $\Sigma \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ - which we call *quasi-diagonal sets* - leaves us with only finitely many possible cohomology tables. The mentioned quasi-diagonal sets are meant to be the sets of the shape $\{(i, n_i) \mid i = 0, 1, \ldots, t\}$ with $n_t < n_{t-1} < \ldots < n_1 < n_0$.

The main ingredient for the proof of this finiteness result is a bounding result on the so called *cohomological postulation numbers*. To be more precise, let $i \in \mathbb{N}_0$. Then the i-th cohomological postulation number $\nu^i_{\mathcal{F}}$ of $\mathcal{F}$ is defined to be the ultimate place $n \in \mathbb{Z}$ at which $h^i(X, \mathcal{F}(n))$ does not take the same value as the "i-th cohomological Serre polynomial" $p^i_{\mathcal{F}}$ of $\mathcal{F}$, which latter is characterized by the property that $h^i(X, \mathcal{F}(m)) = p^i_{\mathcal{F}}(m)$ for all $m \ll 0$. Our bounding result says that the numbers $\nu^i_{\mathcal{F}}$ find a lower bound in terms of the "cohomology diagonal"

$$\Delta_{\mathcal{F}} := \left(h^i(X, \mathcal{F}(-i)\right)_{i=0,1,\ldots,t} \quad .$$

This bounding result on its turn will follow from another result, which says that the *Castelnuovo-Mumford regularity* $\operatorname{reg}(K^i(M))$ of the *i-th deficiency module* $K^i(M)$ of a finitely generated graded module M over a Noetherian homogeneous K-algebra R is bounded in terms of the *beginning* or *initial degree* $\operatorname{beg}(M) := \inf\{n \in \mathbb{Z} \mid M_n \neq 0\}$ of M and the *geometric cohomology diagonal* $\Delta_{\widetilde{M}}$ of M, that is the cohomology diagonal of the coherent sheaf $\widetilde{M}$ of $\mathcal{O}_{\operatorname{Proj}(R)}$-modules induced by M.

Besides of these main results, we shall also discuss a number of related subjects and their history. Already here, we recommend the lecture notes [5] for a more complete and fairly self-contained presentation of the subject. The prerequisites which are needed to follow the notes [5] are contained in [3], in [8] or in [13].

The present paper is divided into six sections. In Section 1 we introduce the basic notions and concepts which we shall need later. We freely use the necessary background from Graded Local Cohomology Theory and from Sheaf Cohomology Theory over Projective Schemes. In particular we define the notion of "subclass $\mathcal{D} \subseteq \mathcal{S}^t$ of finite cohomology (on a subset $\mathbb{S} \subseteq \{0, 1, 2 \ldots, t\} \times \mathbb{Z}$)", which will be of fundamental meaning in this paper. We also present examples of subclasses $\mathcal{D} \subseteq \mathcal{S}^t$ which are of finite cohomology and subclasses of $\mathcal{S}^t$ which are not of finite cohomology. It is important to observe, that the class $\mathcal{D}$ consisting of of all pairs $(X, \mathcal{O}_X) \in \mathcal{S}^{\deg(p)}$ where $X \subseteq \mathbb{P}^r$ runs through the closed subschemes parametrized by the Hilbert scheme $\operatorname{Hilb}^p_{\mathbb{P}^r}$ is indeed of finite cohomology.

In Section 2 we are interested in supporting degrees of cohomology and in the related notion of "cohomological pattern" $\mathcal{P}(X, \mathcal{F})$ of a pair $(X, \mathcal{F}) \in \mathcal{S}^t$. This pattern is the set of pairs $(i, n) \in \{0, 1, \ldots, t\} \times \mathbb{Z}$ for which the entry $h^i(X, \mathcal{F}(n))$ of the cohomology table $h_{\mathcal{F}}$ at the place (i, n) does not vanish. Without proof we state a combinatorial characterization of these cohomological patterns. In Section 2, we also recall the notion of "Castelnuovo-Mumford regularity" and its basic properties. In particular we emphasize the sheaf theoretic aspect of this invariant. In addition, we give a brief (and partially historic) account on the "Vanishing Theorem of Severi-Enriques-Zariski-Serre" and its algebraic generalizations which are due to Grothendieck and Faltings.

In Section 3 we introduce the "deficiency modules" $K^i(M)$ of a finitely generated graded module M over a Noetherian homogeneous K-algebra R. As we restrict ourselves to work over Noetherian homogeneous algebras over a field, we can do this in a "narrow-gauge" manner. This has the advantage to encode a fortiori the graded form of Grothendieck's Local Duality Theorem or - equivalently - the Serre Duality Theorem. We first calculate the deficiency modules of a polynomial ring over a field and then prove the basic properties of such modules in general. In a next step we use deficiency modules to introduce the notion of "cohomological Hilbert polynomial", of "cohomological Serre polynomial" and of "cohomological postulation number". We also briefly consider the special case of "canonical modules" and prove that these latter satisfy (a weak form of) the "second Serre property S_2".

In Section 4 we use the tools developed in Section 3 to establish a number of bounding results for the Castelnuovo-Mumford regularity of finitely generated graded modules over Noetherian homogeneous algebras over a field. The most prominent of these is the main result we already announced above: an upper bound on the regularity of the deficiency modules $K^i(M)$ of a finitely generated graded R-module M over a Noetherian homogeneous K-algebra R in terms of the beginning and the geometric cohomology diagonal of M. This result has in fact a number of further applications, which are beyond the reach of this expository paper and for which the interested reader is recommended to consult [10].

In Section 5 we first prove the previously announced bounding result on the cohomological postulation numbers $\nu^i_{\mathcal{F}}$, where $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules and X is a projective scheme over some field K. As an application we show that a class $\mathcal{D} \subseteq \mathcal{S}^t$ which is of finite cohomology on an arbitrary quasi-diagonal subset $\Sigma \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ is of finite cohomology at all. Our final conclusion is, that a subset $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ bounds cohomology (which means that each subclass $\mathcal{D} \subseteq \mathcal{S}^t$ which is of finite cohomology on $\mathbb{S}$ is of finite cohomology at all) if and only if it contains a quasi-diagonal set Σ. We conclude this section with a number of remarks and open questions. To readers, who aim to see a complete account and further consequences of the results of Section 5, we recommend to consult [11].

In Section 6 we focus on *vector bundles over projective spaces*, more precisely on the class $\mathcal{V}^t_K$ of all locally free coherent sheaves over the projective space $\mathbb{P}^t_K$ over the field K. This class can be considered in a canonical way as a subclass of $\mathcal{S}^t$. So a set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ which contains a quasi-diagonal subset Σ bounds cohomology on the class $\mathcal{V}^t_K$. The question arises, whether this condition is necessary. We will prove, that this is indeed the case, so that the set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ bounds cohomology on the class $\mathcal{V}^t_K$ of all vector bundles on $\mathbb{P}^t$ if and only if it contains a quasi-diagonal set Σ. We also provide an example which shows that this is not true anymore if one replaces the class $\mathcal{V}^t_K$ by its subclass $\mathrm{ind}\mathcal{V}^t_K$ of *indecomposable vector bundles*.

This paper is a modified, extended and updated version of the lecture notes for a course taught in the framework of the International CIMPA-Tubitak Summer School *Commutative Algebra and Applications to Combinatorics and Algebraic Geometry* at 12-25 September 2010 in Istanbul.

1. Preliminaries

The central problem hidden behind our investigation is a basic question of projective algebraic geometry, namely:

1.1. **Question.** *What bounds cohomology of a projective scheme?*

We first shall make more precise this question. To do so, we have to introduce a few notions. We shall define the basic concepts and formulate our main results primarily in the geometric language of schemes and coherent sheaves. But we also shall recall the necessary tools from local cohomology theory which allow to translate our results to the purely algebraic language of graded rings and modules - and vice versa.

Our main results are contained in our joint work with Hellus [9] and Jahangiri-Linh [10],[11] and in the lecture notes [5]. To simplify matters we shall content ourselves in this paper to consider projective schemes over fields instead of projective schemes over local Artinian rings. A complete and fairly self contained exposition of our results is given in [5]. As this set of lecture notes is available on-line under www.math.uzh.ch/brodmann (click: Publications) we allow ourselves to quote this source repeatedly. For those readers, who wish to consult a self-contained introduction to the foundations of local cohomology and sheaf cohomology over projective schemes, we recommend [8], which is also available under the previous URL. To readers, who like to start from a more extended background we recommend to consult [13] and Chapters II and III of [24]. To illustrate the significance of the whole subject we also shall present a few classical results and discuss their relationship with the topic of the present paper.

As announced above, we now shall make precise the question asked at the beginning of this section. Throughout, we use $\mathbb{N}$ to denote the set of strictly positive integers and $\mathbb{N}_0$ to denote the set of non-negative integers. We first introduce the basic notations we shall use in these lectures

1.2. **Notation.** *(Two Basic Classes)* A) Let $d \in \mathbb{N}$. By $\mathcal{M}^d$ we shall denote the class of all pairs (R, M) such that

$$R = K \oplus R_1 \oplus R_2 \oplus \ldots = K[R_1]$$

is a Noetherian homogeneous K-algebra over some field K (hence R is $\mathbb{Z}$-graded and generated as a K-algebra by finitely many elements of degree 1) and

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

is a finitely generated graded R-module with Krull dimension $\dim_R(M) = d$.

Sometimes we like to fix the base field K and write $\mathcal{M}_K^d$ for the class of all pairs $(R, M) \in \mathcal{M}^d$ for which $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ is a Noetherian homogeneous algebra over the given field K.

B) Let $t \in \mathbb{N}_0$. By $\mathcal{S}^t$ we shall denote the class of all pairs $(X, \mathcal{F})$ such that X is a projective scheme over some field K and $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules whose Krull dimension satisfies

$$\dim(\mathcal{F}) := \dim\big(\mathrm{Supp}(\mathcal{F})\big) = \sup\{\dim_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \mid x \in X\} = t.$$

Sometimes, also here, we like to fix the base field K and write $\mathcal{S}^t_K$ for the class of all pairs $(X, \mathcal{F}) \in \mathcal{S}^t$ in which X is a projective K-scheme.

We prefer to formulate our final results in the language of Algebraic Geometry, thus terms of sheaf cohomology over projective schemes and hence in the framework of the classes introduced in part B) of Notation 1.2. Nevertheless, as many readers may prefer to think on the whole subject in algebraic terms, hence in terms of the classes defined in part A) of Notation 1.2, we shall introduce our basic notions in both formalisms. We consider this as a way of encouraging "Pure Algebraists" to throw a glance to the rich and appealing geometric phenomena related to the algebraic formalisms we shall use. Moreover, the subjects we are speaking about, have there roots in Algebraic Geometry.

1.3. **Remark.** A) *(Relating the Two Basic Classes)* Let $t \in \mathbb{N}_0$. Then, by the well known relation between graded modules over homogeneous rings and coherent sheaves over projective schemes we may write

$$\mathcal{S}^t = \{(\mathrm{Proj}(R), \widetilde{M}) \mid (R, M) \in \mathcal{M}^{t+1}),$$

where $\mathrm{Proj}(R)$ denotes the projective scheme induced by the Noetherian homogeneous ring R and $\widetilde{M}$ denotes the coherent sheaf of $\mathcal{O}_{\mathrm{Proj}(R)}$-modules induced by the finitely generated graded R-module M (see [24] Chapters II and III, [13] Chapter 20, or [8] Sections 11 and 12). It should be noted, that the assignment

$$(R, M) \mapsto (\mathrm{Proj}(R), \widetilde{M})$$

does not define a bijection between the classes $\mathcal{M}^{t+1}$ and $\mathcal{S}^t$. Indeed, for two finitely generated graded R-modules $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and $N = \bigoplus_{n \in \mathbb{Z}} N_n$ with $M \subseteq N$ one has $\widetilde{M} = \widetilde{N}$ if and only if $M_n = N_n$ for all $n \gg 0$.

B) *(Shifting and Twisting)* Keep the above notations and hypotheses. Let $(R, M) \in \mathcal{M}^d$, let $X := \mathrm{Proj}(R)$ and let $\mathcal{F} = \widetilde{M}$ be the coherent sheaf of $\mathcal{O}_X$-modules induced by M, so that $(X, \mathcal{F}) \in \mathcal{S}^{d-1}$. For each $n \in \mathbb{Z}$ let $M(n)$ denote the *n-th shift of* M, hence the module M endowed with the grading

$$\big(M(n)_i\big)_{i \in \mathbb{Z}} \text{ defined by } M(n)_i := M_{i+n} \text{ for all } i \in \mathbb{Z}.$$

In addition, let

$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \text{ (with } \mathcal{O}_X(n) := \widetilde{R(n)})$$

be the *n-th twist of* $\mathcal{F}$. Then we have (see [24])

$$\mathcal{F}(n) = \widetilde{M(n)}, \text{ for all } n \in \mathbb{Z}.$$

In particular, we can say that the classes $\mathcal{M}^d$ and $\mathcal{S}^t$ are closed under shifting respectively twisting:

a) If $(R, M) \in \mathcal{M}^d$, then $(R, M(n)) \in \mathcal{M}^d$ for all $n \in \mathbb{Z}$.
b) If $(X, \mathcal{F}) \in \mathcal{S}^t$, then $(X, \mathcal{F}(n)) \in \mathcal{S}^t$ for all $n \in \mathbb{Z}$.

1.4. **Notation and Reminder.** A) *(Local Cohomology and Algebraic Cohomological Hilbert Functions)* Let $d \in \mathbb{N}$, let K be a field and let $(R, M) \in \mathcal{M}^d_K$, with $R = K \oplus R_1 \oplus R_2 \oplus \dots$ be a Noetherian homogeneous algebra over the field K. In this situation we always shall write

$$R_+ := \bigoplus_{n \in \mathbb{N}} R_n$$

for the *irrelevant ideal of* R. Moreover, for each integer $i \in \mathbb{N}_0$ let

$$H^i_{R_+}(M) = \big(\mathcal{R}^i \Gamma_{R_+}\big)(M) = \lim_{\underset{n}{\longrightarrow}} \operatorname{Ext}^i_R(R/(R_+)^n, M)$$

denote the *i-th local cohomology module of M with respect to* R_+, that is the i-th right derived functor of the (covariant left-exact) *R_+-torsion-functor*

$$\Gamma_{R_+}(\bullet) = \lim_{\underset{n}{\longrightarrow}} \operatorname{Hom}_R(R/(R_+)^n, \bullet)$$

with respect to R_+ evaluated at the object M. Keep in mind the well known fact that the R-modules $H^i_{R_+}(M)$ carry a natural grading and that for the corresponding graded components we have (see [13], [8], [3] or [5])

a) $h^i_M(n) := \dim_K(H^i_{R_+}(M)_n) < \infty$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.
b) $h^i_M(n) = 0$ for all $i \in \mathbb{N}_0$ and all $n \gg 0$.
c) $h^i_M(n) = 0$ for all $i > d$ and all $n \in \mathbb{Z}$.
d) $h^d_M(n) \neq 0$ for all $n \ll 0$.

In particular, for each $i \in \mathbb{N}_0$, we may define the *i-th algebraic cohomological Hilbert function of* M, that is the right-vanishing function

$$h^i_M : \mathbb{Z} \to \mathbb{N}_0, \quad n \mapsto h^i_M(n), \quad \forall n \in \mathbb{Z}.$$

B) *(Ideal Transforms and Geometric Cohomological Hilbert Functions)* Keep the notations and hypotheses of part A). We consider the *i-th R_+-transform of* M, that is the R-module

$$D^i_{R_+}(M) := \big(\mathcal{R}^i D_{R_+}\big)(M) = \lim_{\underset{n}{\longrightarrow}} \operatorname{Ext}^i_R((R_+)^n, M),$$

obtained by evaluating the i-th right derived of the *R_+-transform functor* $D_{R_+}(\bullet) = \lim_{\underset{n}{\longrightarrow}} \operatorname{Hom}_R((R_+)^n, \bullet)$ at the object M. Keep in mind, that the R-modules $D^i_{R_+}(M)$ carry a natural grading and moreover (see [13])

a) There is a natural short exact sequence of graded R-modules

$$0 \to H^0_{R_+}(M) \to M \to D^0_{R_+}(M) \to H^1_{R_+}(M) \to 0.$$

b) For all $i \in \mathbb{N}$ there is a natural isomorphism of graded R-modules

$$D^i_{R_+}(M) \cong H^{i+1}_{R_+}(M).$$

In particular it follows from statements a)-d) of part A) that

c) $d^i_M(n) := \dim_K(D^i_{R_+}(M)_n) < \infty$, for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.
d) $d^0_M(n) = \dim_K(M_n) + h^1_M(n) - h^0_M(n)$ for all $n \in \mathbb{Z}$.
e) $d^0_M(n) = \dim_K(M_n)$ for all $n \gg 0$.
f) $d^i_M(n) = h^{i+1}_M(n)$ for all $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$.
g) $d^i_M(n) = 0$ for all $i \in \mathbb{N}$ and all $n \gg 0$.
h) $d^i_M(n) = 0$ for all $i \geq d$ and all $n \in \mathbb{Z}$.
i) $d^{d-1}_M(n) \neq 0$ for all $n \ll 0$.

Now, for each $i \in \mathbb{N}_0$ we may define the *i-th geometric cohomological Hilbert function of M* as the function

$$d^i_M : \mathbb{Z} \to \mathbb{N}_0, \quad n \mapsto d^i_M(n), \quad \forall n \in \mathbb{Z}.$$

C) *(Behaviour in Short Exact Sequences)* Let K be a field, let R be a Noetherian homogeneous K-algebra and let

$$\mathbb{S} : 0 \to L \to M \to N \to 0$$

be an exact sequence of finitely generated graded R-modules. Then, on use of the right derived exact sequences of the functors $\Gamma_{R_+}(\bullet)$ and $D_{R_+}(\bullet)$ one obtains the following inequalities for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ (and observing the notational convention that $h^j_M = d^j_M = 0$ for all integers $j < 0$):

a) $h^i_M(n) \leq h^i_L(n) + h^i_N(n)$.
b) $h^i_L(n) \leq h^i_M(n) + h^{i-1}_N(n)$.
c) $h^i_N(n) \leq h^i_M(n) + h^{i+1}_L(n)$.
d) $d^i_M(n) \leq d^i_L(n) + d^i_N(n)$.
e) $d^i_L(n) \leq d^i_M(n) + d^{i-1}_N(n)$.
f) $d^i_N(n) \leq d^i_M(n) + d^{i+1}_L(n)$.

Now, we aim to link the above algebraic concepts to sheaf theory.

1.5. **Notation and Reminder.** A) *(Serre Cohomology of Projective Schemes with Coefficients in Coherent Sheaves)* Let $t \in \mathbb{N}_0$, let K be a field and let $(X, \mathcal{F}) \in \mathcal{S}^t_K$, so that X is a projective scheme over the field K. For each $i \in \mathbb{N}_0$ and each $n \in \mathbb{Z}$ let

$$H^i(X, \mathcal{F}(n)) := \big(\mathcal{R}^i\Gamma(X, \bullet)\big)(\mathcal{F}(n))$$

denote the *i-th Serre (or sheaf) cohomology group of (X with coefficients in) the n-th twist $\mathcal{F}(n)$ of $\mathcal{F}$* – that is the i-th right derived of the functor $\Gamma(X, \bullet)$ of taking global sections, evaluated at the object $\mathcal{F}(n)$. Keep in mind, that the cohomology groups $H^i(X, \mathcal{F}(n))$ all carry a natural structure of K-vector space.

B) *(The Serre-Grothendieck Correspondence and Cohomological Hilbert Functions of Coherent Sheaves)* Let the notations and hypotheses be as in part A). Then, according to Remark 1.3 A), there is a (not necessarily unique) pair $(R, M) \in \mathcal{M}_K^{t+1}$ such that

$$(X, \mathcal{F}) = (\mathrm{Proj}(R), \widetilde{M})$$

and in this situation the *Serre-Grothendieck Correspondence* (see [13](20.4.4)) yields that for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ there is an isomorphism of K-vector spaces

$$H^i(X, \mathcal{F}(n)) \cong D^i_{R_+}(M)_n.$$

In particular we have

$$h^i_{\mathcal{F}}(n) = h^i(X, \mathcal{F}(n)) := \dim_K\big(H^i(X, \mathcal{F}(n))\big) = d^i_M(n), \quad \forall i \in \mathbb{N}_0, \forall n \in \mathbb{Z}.$$

This allows to define for each $i \in \mathbb{N}_0$ the *i-th cohomological Hilbert function of (X with coefficients in)* $\mathcal{F}$, that is the function

$$h^i_{\mathcal{F}} : \mathbb{Z} \to \mathbb{N}_0, \quad n \mapsto h^i_{\mathcal{F}}(n) = h^i(X, \mathcal{F}(n)), \quad \forall n \in \mathbb{Z}.$$

C) *(First Properties of Cohomological Hilbert Functions of Coherent Sheaves)* Let the notations and hypotheses be as in parts A) and B). In particular, let $(R, M) \in \mathcal{M}^{t+1}$ with $(X, \mathcal{F}) = (\mathrm{Proj}(R), \widetilde{M})$. Then, by the observations made in part B) we have

$$h^i_{\mathcal{F}} = d^i_M, \quad \forall i \in \mathbb{N}_0.$$

This explains why we called the functions d^i_M the *geometric* cohomological Hilbert functions of M: these functions actually describe the geometric object associated to the algebraic object M. By statements g), h) and i) of Notation and Reminder 1.4 B) we now respectively obtain

a) $h^i_{\mathcal{F}}(n) = 0$ for all $i \in \mathbb{N}$ and all $n \gg 0$.
b) $h^i_{\mathcal{F}}(n) = 0$ for all $i > t = \dim(\mathcal{F})$ and all $n \in \mathbb{Z}$.
c) $h^t_{\mathcal{F}}(n) \neq 0$ for all $n \ll 0$.

D) *(Relating Cohomological Hilbert Functions of Sheaves and Modules)* Let the notations and hypotheses be as in part C). Then, the equalities observed at the beginning of part C), together with the relations d),f) of Notation and Reminder 1.4 B) imply that

a) $h^i_{\mathcal{F}}(n) = h^{i+1}_M(n)$ for all $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$.
b) $h^0_{\mathcal{F}}(n) = h^1_M(n)$ for all $n \ll 0$.

In order to collect all cohomological Hilbert functions of a pair in $\mathcal{M}^d$ respectively in $\mathcal{S}^t$, we give the following definition.

1.6. **Definition and Remark.** A) *(Cohomology Tables of Graded Modules)* Let $d \in \mathbb{N}$ and let $(R, M) \in \mathcal{M}^d$. We define the *algebraic cohomology table of* M as the family of non-negative integers

$$h_M := \left(h^i_M(n)\right)_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}} \quad .$$

The *geometric cohomology table of* M is defined as the family of non-negative integers

$$d_M := \left(d^i_M(n)\right)_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}} \quad .$$

For a subset $\mathbb{S} \subseteq \mathbb{N}_0 \times \mathbb{Z}$ we also aim to consider the *restricted cohomology tables of* M, that is the restricted families of non-negative integers

$$h_M \upharpoonright_{\mathbb{S}} := \left(h^i_M(n)\right)_{(i,n)\in\mathbb{S}} \quad ,$$

$$d_M \upharpoonright_{\mathbb{S}} := \left(d^i_M(n)\right)_{(i,n)\in\mathbb{S}} \quad .$$

B) *(Cohomology Tables of Coherent Sheaves)* Let $t \in \mathbb{N}_0$ and let $(X, \mathcal{F}) \in \mathcal{S}^t$. We define the *cohomology table of the (scheme X with coefficients in the) coherent sheaf* $\mathcal{F}$ as the family of non-negative integers

$$h^i_{\mathcal{F}} := \left(h^i_{\mathcal{F}}(n)\right)_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}} \quad .$$

Correspondingly, for any subset $\mathbb{S} \subseteq \mathbb{N}_0 \times \mathbb{Z}$, we define the *restricted cohomology table of* $\mathcal{F}$ as the restricted family of non-negative integers

$$h_{\mathcal{F}} \upharpoonright_{\mathbb{S}} := \left(h^i_{\mathcal{F}}(n)\right)_{(i,n)\in\mathbb{S}} \quad .$$

C) *(Identifying Cohomology Tables of Sheaves and of Modules)* Let the notations be as in part B), and let $(R, M) \in \mathcal{M}^{t+1}$ with $X = \mathrm{Proj}(R)$ and $\mathcal{F} = \widetilde{M}$. Then, it follows from Notation and Reminder 1.5 C) that

$$h_{\mathcal{F}} \upharpoonright_{\mathbb{S}} = d_M \upharpoonright_{\mathbb{S}} .$$

This tells us, that instead of (restricted) cohomology tables of coherent sheaves we may content ourselves to consider (restricted) geometric cohomology tables of graded modules-and vice versa.

1.7. **Definition and Remark.** A) *(Classes of Finite Cohomology: the Case of Modules)* Let $d \in \mathbb{N}$ and let $\mathbb{S} \subseteq \mathbb{N}_0 \times \mathbb{Z}$. We say that a subclass $\mathcal{C} \subseteq \mathcal{M}^d$ is *of finite cohomology on* $\mathbb{S}$ if the set of cohomology tables

$$\{d_M \upharpoonright_{\mathbb{S}} \quad | \ (R, M) \in \mathcal{C}\} = \{\left(d^i_M(n)\right)_{(i,n)\in\mathbb{S}} \mid (R, M) \in \mathcal{C}\}$$

is finite. Clearly in view of Notation and Reminder 1.4 B)g) it suffices to consider this conditions only for sets

$$\mathbb{S} \subseteq \{0, 1, \ldots, d-1\} \times \mathbb{Z}.$$

We say that the class $\mathcal{C} \subseteq \mathcal{M}^d$ *is of finite cohomology (at all)* if it is of finite cohomology on the set $\{0, 1, \ldots, d-1\} \times \mathbb{Z}$ or, equivalently, on an arbitrary set $\mathbb{S} \subseteq \mathbb{N}_0 \times \mathbb{Z}$ containing the former.

B) *(Classes of Finite Cohomology: the Case of Sheaves)* Let $t \in \mathbb{N}_0$ and let $\mathbb{S} \subseteq \mathbb{N}_0 \times \mathbb{Z}$. We say that a subclass $\mathcal{D} \subseteq \mathcal{S}^t$ is *of finite cohomology on* $\mathbb{S}$ if the set of cohomology tables

$$\{h_{\mathcal{F}} \upharpoonright_{\mathbb{S}} \ \mid (X, \mathcal{F}) \in \mathcal{D}\} = \{\big(h^i_{\mathcal{F}}(n)\big)_{(i,n)\in\mathbb{S}} \mid (R, M) \in \mathcal{C}\}$$

is finite. Similarly as in part A), it suffices to consider sets $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$. Also, similarly as in part A), we say that the class $\mathcal{D} \subseteq \mathcal{S}^t$ *is of finite cohomology (at all)* if it is of finite cohomology on the set $\{0, 1, \ldots, t\} \times \mathbb{Z}$ or, again equivalently, on an arbitrary set $\mathbb{S} \subseteq \mathbb{N}_0 \times \mathbb{Z}$ containing the former.

C) *(Relating the Two Notions of Part A) and B))* Let t, $\mathbb{S}$, and $\mathcal{D} \subseteq \mathcal{S}^t$ be as in part B). Then, according to Remark 1.3 A) there is a subclass $\mathcal{C} \subseteq \mathcal{M}^{t+1}$ such that

$$\mathcal{D} = \{(\mathrm{Proj}(R), \widetilde{M}) \mid (R, M) \in \mathcal{C}\}.$$

In this situation, it follows from Definition and Remark 1.6 C) that $\mathcal{D}$ is of finite cohomology on $\mathbb{S}$ if and only if $\mathcal{D}$ is. So, the study of classes of finite cohomology for modules and sheaves are equivalent features.

We new recall a few basic facts on *Hilbert polynomials* and *characteristic functions*, which we shall frequently use later.

1.8. **Reminder and Remark.** A) *(Hilbert Functions and Postulation Numbers of Graded Modules)* Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated R-module of dimension $d \in \mathbb{N}_0 \cup \{-\infty\}$. In this situation we denote the *Hilbert polynomial* of M by P_M, so that

a) $P_M \in \mathbb{Q}[\mathbf{x}]$ with $\deg(P_M) = d - 1$ if $d > 0$ and $P_M = 0$ if $d \leq 0$.
b) $\dim_K(M_n) = P_M(n)$ for all $n \gg 0$.

In view of statement b) we may define the *postulation number of the graded R-module M* by

$$P(M) := \sup\{n \in \mathbb{Z} \mid \dim_K(M_n) \neq P_M(n)\} \in \mathbb{Z} \cup \{-\infty\}.$$

B) *(Characteristic Functions of Graded Modules)* Let R and M be as above. The *characteristic function* of M is defined by

$$\chi_M : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \chi_M(n) := \sum_{i=0}^{d-1} (-1)^i d^i_M(n) = \sum_{i\in\mathbb{N}_0} (-1)^i d^i_M(n), \quad \forall n \in \mathbb{Z}.$$

Clearly, by the observations made in Notation and Reminder 1.4 B) d),f) for all $n \in \mathbb{Z}$, we may write

a) $\chi_M(n) = \dim_K(M_n) - \sum_{i=0}^{d} (-1)^i h^i_M(n) = \dim_K(M_n) - \sum_{i\in\mathbb{N}_0} (-1)^i h^i_M(n)$.

A most important fact is the so called *(Algebraic) Serre Formula* which relates the characteristic function and the Hilbert polynomial of a finitely generated graded R-module (see [8](9.17),(9.18) or [13] (17.1.7) for example):

b) $P_M(n) = \chi_M(n)$ for all $n \in \mathbb{Z}$.

As an easy consequence of this, one gets the following estimate for the postulation number of the graded R-module M:

c) $P(M) \leq \sup\{n \in \mathbb{Z} \mid \exists i \in \mathbb{N}_0 : h^i_M(n) \neq 0\}$.

C) *(Hilbert-Serre Polynomials and Characteristic Functions of Sheaves)* Let X be a projective scheme over some field K and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules with $\dim(\mathcal{F}) = t$. Then we may define the *characteristic function of* $\mathcal{F}$ by

$$\chi_{\mathcal{F}} : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \chi_{\mathcal{F}}(n) := \sum_{i=0}^{t} (-1)^i h^i_{\mathcal{F}}(n) = \sum_{i \in \mathbb{N}_0} (-1)^i h^i_{\mathcal{F}}(n), \quad \forall n \in \mathbb{Z}.$$

We now find R and M as in part A) such that $X = \mathrm{Proj}(R)$ and $\mathcal{F} = \widetilde{M}$. In this situation it follows from Notation and Reminder 1.5 C) that

$$\chi_{\mathcal{F}} = \chi_M.$$

Hence by the Algebraic Serre Formula and by statement B)b), there is a unique polynomial $P_{\mathcal{F}} \in \mathbb{Q}[\mathbf{x}]$ (namely $P_{\mathcal{F}} := P_M$) such that

a) $\deg(P_{\mathcal{F}}) = \dim(\mathcal{F})$,
b) $\chi_{\mathcal{F}}(n) = P_{\mathcal{F}}(n)$ for all $n \in \mathbb{Z}$,
c) $P_{\mathcal{F}}(n) = h^0(X, \mathcal{F}(n)) = h^0_{\mathcal{F}}(n)$ for all $n \in \mathbb{Z}$ such that $h^i_{\mathcal{F}}(n) = 0$ for all $i \in \mathbb{N}$.

This polynomial is called the *Hilbert-Serre polynomial of* $\mathcal{F}$. Now, again, we may define the *postulation number of* $\mathcal{F}$ as

$$P(\mathcal{F}) := \sup\{n \in \mathbb{Z} \mid h^0(X, \mathcal{F}(n)) \neq P_{\mathcal{F}}(n)\} \in \mathbb{Z} \cup \{-\infty\}$$

and similarly as in statement B)c) we get

d) $P(\mathcal{F}) \leq \sup\{n \in \mathbb{Z} \mid \exists i \in \mathbb{N} : h^i_{\mathcal{F}}(n) \neq 0\}$.

One of our principal aims is to prove finiteness results for classes $\mathcal{D} \subseteq \mathcal{S}^t$ or, equivalently, for classes $\mathcal{C} \subseteq \mathcal{M}^d$. The following example can be considered as being classical. In geometric terms it says that the pairs $(X, \mathcal{O}_X)$ in which $X \subseteq \mathbb{P}^r_K$ runs through all closed subschemes with a given Hilbert polynomial p (hence the pairs $(X, \mathcal{O}_X) \in \mathcal{S}^{\deg(p)}$ parametrized by the Hilbert scheme $\mathrm{Hilb}^p_{\mathbb{P}^r}$) form a class of finite cohomology.

1.9. **Example.** Let $r \in \mathbb{N}$, let $p \in \mathbb{Q}[\mathbf{x}]$ be a polynomial with $\deg(p) < r$, let K be an algebraically closed field and consider the class

$$\mathcal{C} := \{(R, R) \in \mathcal{M}_K^{\deg(p)+1} \mid \dim_K(R_1) = r + 1; \quad P_R = p\}$$

of all pairs (R, R) in which $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ is a Noetherian homogeneous K-algebra, having Hilbert polynomial p and being generated by $r + 1$ linear forms. Then the class $\mathcal{C}$ is indeed of finite cohomology. We shall derive this fact from a much more general result which we will treat later. As the class

$$\mathcal{D} := \{(\mathrm{Proj}(R), \widetilde{R}) \mid (R, R) \in \mathcal{C}\} \subseteq \mathcal{S}^{\deg(p)}$$

is parametrized by the Hilbert scheme $\mathrm{Hilb}^p_{\mathbb{P}^r}$, Definition and Remark 1.7 C) implies, that there are only finitely many cohomology tables $h_{\mathcal{O}_X}$ if X runs through $\mathrm{Hilb}^p_{\mathbb{P}^r}$. This fact was mentioned already in the Introduction.

1.10. **Example and Remark.** A) Let $r > 1$ be an integer, and consider the polynomial ring

$$R := K[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r].$$

over the field K, furnished with its standard $\mathbb{Z}$-grading. Let

$$\mathcal{C} := \{(R, R(m)) \mid m \in \mathbb{Z}\} \subseteq \mathcal{M}^r.$$

It is well known, that the local cohomology modules of R with respect to R_+ satisfy the following requirements (see [13](3.3.3),(6.2.7),(13.5.3) or [5](9.6),(9.4)C)b), for example)

a) $H^i_{R_+}(R) = 0$ for all $i \neq r$,

b) $H^r_{R_+}(R)_n \cong R_{-r-n}$ for all $n \in \mathbb{Z}$.

Keeping in mind that local cohomology with respect to R_+ commutes with shifting and in view of statements d) and e) of Notation and Reminder 1.4 B) it follows, with the notational convention that $\binom{k}{r-1} = 0$ for all $k < r - 1$

a) $d^0_{R(m)}(n) = \binom{r+m+n-1}{r-1}$ for all $m, n \in \mathbb{Z}$,

b) $d^{r-1}_{R(m)}(n) = \binom{-m-n-1}{r-1}$ for all $m, n \in \mathbb{Z}$,

c) $d^i_{R(m)}(n) = 0$ for all $i \neq 0, r-1$ and all $m, n \in \mathbb{Z}$.

This clearly shows that the class $\mathcal{C}$ is not of finite cohomology.

B) Keep the notations and hypotheses of part A). For each $m \in \mathbb{N}$ chose a form $f_m = R_m \setminus \{0\}$ and consider the Noetherian homogeneous graded K-algebra $R^{[m]} := R/f_m R$ with Hilbert-polynomial

$$P_{R^{[m]}} = \binom{r + \mathbf{x} - 1}{r - 1} - \binom{r - m + \mathbf{x} - 1}{r - 1}.$$

Observe that for each $m \in \mathbb{N}$ there is an exact sequence of graded R-modules

$$0 \to R \xrightarrow{f_m} R(m) \to R^{[m]} \to 0.$$

On use of the observations made in part A) it follows easily that the class $\{(R, R^{[m]}) \mid m \in \mathbb{N}\}$ is not of finite cohomology. But then, on use of the Base-Ring Independence Property of Local Cohomology (see [13](4.2.1), or [5] Section 1) it follows immediately that the family

$$\mathcal{D} := \{(R^{[m]}, R^{[m]}) \mid m \in \mathbb{N}\} \subseteq \mathcal{M}^{r-1}_K$$

is not of finite cohomology either. It is noteworthy to figure out the difference of the above class $\mathcal{D}$ and the classes presented in Example 1.9.

C) Let $R = K[\mathbf{x}, \mathbf{y}]$ be a polynomial ring in two indeterminates over a field K, furnished with its standard grading. For each positive integer $m \in \mathbb{N}$ let $M^{[m]} := R(-m) \oplus R(m)$. It is easy to compute the Hilbert polynomials $P_{M^{[m]}}$ for all $m \in \mathbb{N}$ and to conclude that the class

$$\mathcal{E} := \{(R, M^{[m]}) \mid m \in \mathbb{N}\} \subseteq \mathcal{M}_K^2$$

is not of finite cohomology. One should compare this situation with the one described in Example 1.9.

2. Supporting Degrees of Cohomology

Let $d \in \mathbb{N}$ and let $(R, M) \in \mathcal{M}^d$. For each $i \in \mathbb{N}_0$ we now aim to look at the *supporting degrees* of the local cohomology modules $H^i_{R_+}(M)$ and the R_+-transform modules $D^i_{R_+}(M)$ of M with respect to R_+, hence at the integers $n \in \mathbb{Z}$ for which $h^i_M(n)$ respectively $d^i_M(n)$ does not vanish. Similarly, if $t \in \mathbb{N}_0$ and $(X, \mathcal{F}) \in \mathcal{S}^t$ we aim to focus at the integers n for which $h^i_{\mathcal{F}}(n)$ does not vanish. We start with a few basic notions concerning graded rings and modules.

2.1. **Notation and Reminder.** A) *(Generating Degrees of Graded Modules)* Let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra, where K is a field. Keep in mind that $R = K[x_1, x_2, ..., x_r]$ for finitely many homogeneous elements $x_1, x_2, ..., x_r \in R_1$. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be an arbitrary graded R-module. Keep in mind that $\dim_K(M_n) < \infty$ for all $n \in \mathbb{Z}$ and $M_n = 0$ for all $n \ll 0$, provided that M is in addition finitely generated. Let us recall the notion of *generating degree* of M (see [8] (9.6)D)), which is defined by:

$$\text{gendeg}(M) := \inf\{t \in \mathbb{Z} \mid M = \sum_{n \leq t} RM_n\}.$$

Keep in mind the following facts:

a) If M is finitely generated, then $\text{gendeg}(M) < \infty$.
b) If $\text{gendeg}(M) \leq n \in \mathbb{Z}$, then $M_{n+k} = R_k M_n$ for all $k \in \mathbb{N}$.

B) *(Beginnings and Ends of Graded Modules)* Keep the notations and hypotheses of part A) and let us introduce the *beginning* or *initial degree* and the *end* of M, which are defined respectively by

$$\text{beg}(M) := \inf\{n \in \mathbb{Z} \mid M_n \neq 0\},$$
$$\text{end}(M) := \sup\{n \in \mathbb{Z} \mid M_n \neq 0\}.$$

Observe the following facts:

a) If $M \neq 0$ is a graded R-module, then $\text{beg}(M) \leq \text{gendeg}(M) \leq \text{end}(M)$.
b) If $M \neq 0$ is a finitely generated graded R-module, then

$$-\infty < \text{beg}(M) \leq \text{gendeg}(M) < \infty.$$

Now, we remind the notion of Castelnuovo-Mumford regularity which plays a fundamental rôle in our investigation.

2.2. **Reminder and Definition.** A) *(Castelnuovo-Mumford Regularity of Graded Modules)* Let $d \in \mathbb{N}$, let $(R, M) \in \mathcal{M}^d$ and let $l \in \mathbb{N}_0$. We define the *(Castelnuovo-Mumford) regularity of M at and above level l* by

$$\text{reg}^l(M) := \sup\{\text{end}(H^i_{R_+}(M)) + i \mid i \geq l\}.$$

The *(Castelnuovo-Mumford) regularity of M (at all)* is defined as

$$\text{reg}(M) := \text{reg}^0(M).$$

Now, on use of statements a)-d) of Notation and Reminder 1.4 A) one verifies immediately that

a) $\operatorname{reg}^l(M) < \infty$.
b) $\operatorname{reg}^l(M) = -\infty$ if and only if $l > d$.
c) $\operatorname{reg}^l(M(n)) = \operatorname{reg}^l(M) - n$ for all $n \in \mathbb{Z}$.
d) If $k \in \{0, 1, \ldots, l\}$, then $\operatorname{reg}^l(M) \leq \operatorname{reg}^k(M)$.

Let us recall the following most important fact, whose proof may be found in [5] (Proposition 3.4) or in [13] (16.3.1) for example.

e) $\operatorname{gendeg}(M) \leq \operatorname{reg}(M)$.

Finally, let us mention the following statements on the behaviour of regularity in short exact sequences, which follow easily from the corresponding observations made in Notation and Reminder 1.4 C). So, let

$$\mathbb{S} : 0 \to L \to M \to N \to 0$$

be an exact sequence of finitely generated graded R-modules. Then

f) $\operatorname{reg}(L) \leq \max\{\operatorname{reg}(M), \operatorname{reg}(N) + 1\}$.
g) $\operatorname{reg}^{l+1}(L) \leq \max\{\operatorname{reg}^{l+1}(M), \operatorname{reg}^l(N) + 1\}$.
h) $\operatorname{reg}^l(M) \leq \max\{\operatorname{reg}^l(L), \operatorname{reg}^l(N)\}$.
i) $\operatorname{reg}^l(N) \leq \max\{\operatorname{reg}^{l+1}(L) - 1, \operatorname{reg}^l(M)\}$.

B) *(Castelnuovo-Mumford Regularity of Sheaves)* Let $t \in \mathbb{N}_0$, let $(X, \mathcal{F}) \in \mathcal{S}^t$ and let $k \in \mathbb{N}_0$. We define the *(Castelnuovo-Mumford) regularity of $\mathcal{F}$ above level k* by

$$\operatorname{reg}_k(\mathcal{F}) := \inf\{r \in \mathbb{Z} \mid H^i(X, \mathcal{F}(r-i)) = 0, \quad \forall i > k\}.$$

The *(Castelnuovo-Mumford) regularity of $\mathcal{F}$ (at all)* is defined as

$$\operatorname{reg}(\mathcal{F}) := \operatorname{reg}_0(\mathcal{F}).$$

Now, on use of statements a), b) and c) of Notation and Reminder 1.5 C) it follows at once that

a) $\operatorname{reg}_k(\mathcal{F}) < \infty$.
b) $\operatorname{reg}_k(\mathcal{F}) = -\infty$ if and only if $k \geq t$.
c) $\operatorname{reg}_k(\mathcal{F}(n)) = \operatorname{reg}_k(\mathcal{F}) - n$ for all $n \in \mathbb{Z}$.
d) If $m \in \{0, 1, \ldots, k\}$ then $\operatorname{reg}_k(\mathcal{F}) \leq \operatorname{reg}_m(\mathcal{F})$.

Moreover, the concepts of regularity for graded modules and sheaves may be easily related on use of the observation made in Notation and Reminder 1.4 B)f) and the consequence of the Serre-Grothendieck Correspondence observed in Notation and Reminder 1.5 C).

e) If $(R, M) \in \mathcal{M}^{t+1}$ with $X = \mathrm{Proj}(R)$ and $\mathcal{F} = \widetilde{M}$, then
$$\mathrm{reg}_k(\mathcal{F}) = \mathrm{reg}^{k+2}(M).$$

C) *(Regularity and Global Generation of Sheaves)* On may wonder, whether statement e) of part A) has some analogue in the sheaf theoretic context. This is indeed true, and we briefly recall the corresponding facts. For readers who want to get a detailed and self-contained approach to the subject, we recommend to consult [5] (3.8)-(3.13) or [13] Chapter 20. Let the notations and hypotheses be as in Part B). We say, that the sheaf $\mathcal{F}$ is *generated by global sections* if for each point $x \in X$ the stalk $\mathcal{F}_x$ of the sheaf $\mathcal{F}$ at x is generated over the local ring $\mathcal{O}_{X,x}$ by germs of global sections $\gamma \in \Gamma(X, \mathcal{F})$. So if
$$\bullet_x : \Gamma(X, \mathcal{F}) \to \mathcal{F}_x, \quad \gamma \mapsto \gamma_x, \quad \forall \gamma \in \Gamma(X, \mathcal{F})$$
denotes the map of taking germs at the point $x \in X$, the sheaf $\mathcal{F}$ is generated by its global sections if and only if for each $x \in X$ we have
$$\mathcal{F}_x = \sum_{\gamma \in \Gamma(X,\mathcal{F})} \mathcal{O}_{X,x}\gamma_x.$$
Then, the announced analogue of statement A)e) says:

a) For each $n \geq \mathrm{reg}(\mathcal{F})$ the n-fold twist $\mathcal{F}(n)$ of $\mathcal{F}$ is generated by its global sections.

We now define the notion of cohomological pattern of a coherent sheaf, which is naturally related to the supporting degrees of cohomology.

2.3. **Definition.** *(Cohomological Patterns)* Let $t \in \mathbb{N}_0$ and let $(X, \mathcal{F}) \in \mathcal{S}^t$. We define the *cohomological pattern of the pair* $(X, \mathcal{F})$ (or simply) *of the sheaf* $\mathcal{F}$ as the set
$$\mathcal{P}_{\mathcal{F}} = \mathcal{P}(X, \mathcal{F}) := \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid H^i(X, \mathcal{F}(n)) \neq 0\}$$
of all pairs $(i, n) \in \mathbb{N}_0 \times \mathbb{Z}$ such that the cohomology table $h_{\mathcal{F}}$ of $\mathcal{F}$ has a non-zero entry at (i, n).

We now formulate the following *Structure Theorem for Cohomological Patterns*, whose proof is given in [9].

2.4. **Theorem.** *Let $t \in \mathbb{N}_0$. Then, a set $\mathcal{P} \subseteq \mathbb{N}_0 \times \mathbb{Z}$ is the cohomological pattern of a pair $(X, \mathcal{F}) \in \mathcal{S}^t$ if and only if the following six requirements are satisfied:*

a) $\sup\{i \in \mathbb{N}_0 \mid \exists n \in \mathbb{Z} : (i, n) \in \mathcal{P}\} = t$;
b) $\exists n \in \mathbb{Z} : (0, n) \in \mathcal{P}$;
c) $\forall (i, n) \in \mathcal{P} : \exists k \geq i : (k, n - k + i - 1) \in \mathcal{P}$;
d) $\forall (i, n) \in \mathcal{P} : \exists l \leq i : (l, n - l + i + 1) \in \mathcal{P}$;
e) $\forall i \in \mathbb{N}, \quad \forall n \gg 0 : (i, n) \notin \mathcal{P}$;
f) $\forall i \in \mathbb{N}_0 : \#\{n \in \mathbb{Z} \mid (i, n) \in \mathcal{P}, (i, n - 1) \notin \mathcal{P}\} < \infty$.

2.5. **Remark.** A) *(Around Cohomological Patterns)* Let the notations be as in Definition2.3 and Theorem 2.4. One might present the cohomological pattern $\mathcal{P}_{\mathcal{F}} = \mathcal{P}(X, \mathcal{F})$ of the pair $(X, \mathcal{F}) \in \mathcal{S}^t$ (resp. of the sheaf of $\mathcal{O}_X$-modules $\mathcal{F} \neq 0$) in a diagram with horizontal n-axis and vertical i-axis, marking the place $(i, n) \in \mathbb{N}_0 \times \mathbb{Z}$ by $\bullet$ if $(i, n) \in \mathcal{P}$ and by $\circ$ otherwise. Then, the five statements a)-f) of Theorem 2.4 respectively say:

a) One finds a $\bullet$ on the row at level t and no $\bullet$ on a row with level strictly higher than t.
b) One finds a $\bullet$ on the bottom row.
c) If there is a diagonal consisting entirely of $\circ$'s above a certain level i, there are no $\bullet$'s right of this diagonal above level i.
d) If there is a diagonal consisting entirely of $\circ$'s below a certain level i, there are no $\bullet$'s left of this diagonal below level i.
e) Except on the bottom row one finds only $\circ$'s far out to the right.
f) At no level i there are infinitely many $\bullet$'s and infinitely many $\circ$'s left of the i-axis.

Observe in particular, that as a consequence of these properties of $\mathcal{P}$ we get:

g) If there is a $\bullet$ on the bottom level 0, then right of this $\bullet$ there are only $\bullet$'s at level 0.
h) If there is a $\bullet$ on the top level t, then left of this $\bullet$ there are only $\bullet$'s at level t.

$$\begin{array}{ccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \circ \\
\circ & \bullet & \circ & \bullet & \bullet & \circ & \bullet & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\
\circ & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}$$

B)*(Cohomological Tameness)* For the moment, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be an arbitrary Noetherian homogeneous ring (so that R_0 is Noetherian and R is generated over R_0 by finitely many elements of degree 1). Let M be a finitely generated graded R-module and let $\mathcal{F} = \widetilde{M}$ be the coherent sheaf of $\mathcal{O}_X$-modules induced by M. Again, for each $n \in \mathbb{Z}$, let $\mathcal{F}(n) = \widetilde{M(n)}$ denote the n-th twist of $\mathcal{F}$. We say that the sheaf $\mathcal{F}$ is *cohomologically tame at level* i, if one of the following requirements is satisfied:

a) $H^i(X, \mathcal{F}(n)) \neq 0$ for all $n \ll 0$;
b) $H^i(X, \mathcal{F}(n) = 0$ for all $n \ll 0$.

By the Serre-Grothendieck Correspondence and by Notation and Reminder 1.4 B)a),b) this is equivalent to the fact, that one of the following two requirements is satisfied:

c) $H^{i+1}_{R_+}(M))_n \neq 0$ for all $n \ll 0$,
d) $H^{i+1}_{R_+}(M))_n = 0$ for all $n \ll 0$,

where $H^{i+1}_{R_+}(M)_n$ denotes the n-th graded component of the $(i+1)$-st local cohomology module of M with respect to the irrelevant ideal $R_+ := \bigoplus_{n \in \mathbb{N}} R_n$ of R. We also say in this situation, that the graded R-module M is *cohomologically tame* at level $i+1$. We say that the coherent sheaf $\mathcal{F}$ is *cohomologically tame* at all, if it is tame at all levels $i \in \mathbb{N}_0$. Correspondingly we say that the finitely generated graded R-module M is *cohomologically tame*, if is tame levels $j \in \mathbb{N}$. Now, statement f) of Theorem 2.4 says

e) If $(X, \mathcal{F}) \in \mathcal{S}^t$ (so that $R_0 = K$ is a field), then $\mathcal{F}$ is cohomologically tame. In particular, at each level i there are either only finitely many $\bullet$'s or finitely many $\circ$' with negative n-coordinate.

C)*(The Tameness Problem)* For a while it was an open problem, whether all coherent sheaves $\mathcal{F} = \widetilde{M}$ over a projective scheme $X = \mathrm{Proj}(R)$ defined by an arbitrary Noetherian homogeneous ring $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ are are cohomologically tame at all levels or - equivalently - whether all finitely generated graded modules M over a Noetherian homogeneous ring R are cohomologically tame at all levels (see [1], [2]). There are indeed many results, proving tameness of a finitely generated graded module M over a Noetherian homogeneous ring R at particular levels or under certain assumptions on R - or else on M (see [2], [4], [7], or also [9], [27], [32] for example). Nevertheless in [16], a striking counter-example is constructed. Namely, it is shown there:

a) There exists a Noetherian homogeneous domain $R = \bigoplus_{n \in \mathbb{N}_0} R_n$, of finite type over the complex field $\mathbb{C}$ with $\dim(R) = 4$ and $\dim(R_0) = 3$ such that $M = R$ is not cohomologically tame at level 2 (or - equivalently - $\mathcal{O}_{\mathrm{Proj}(R)}$ is not cohomologically tame at level 1).

D)*(The Realization Problem for Smooth Complex Projective Varieties)* Let $X = \mathrm{Proj}(R)$ be a smooth connected complex projective variety of dimension at least 2, so that R is a Noetherian homogeneous integral $\mathbb{C}$-algebra such that the local ring $\mathcal{O}_{X,x} = R_{(\mathfrak{p})}$ is regular for all $x = \mathfrak{p} \in X = \mathrm{Proj}(R)$. Then, by the Vanishing Theorem of Kodaira [26] one has $H^i(X, \mathcal{O}_X(n)) = 0$ for all $i < \dim(X) = \dim(R) - 1$ and all $n < 0$. By another result of Mumford and Ramanujam [31] one has the same vanishing statement for $i = 1$ under the weaker assumption that X is normal. So, one is naturally lead to ask the following realization question:

a) Let $t \geq 2$ be an integer and let $\mathcal{P} \subseteq \{0, 1, \cdots t\} \times \mathbb{Z}$ be a set which satisfies the pattern requirements (i)-(v) of Theorem 2.4 and the additional *positivity condition* that $(i, n) \notin \mathcal{P}$ if $i < t$ and $n < 0$. Does there exist a smooth (or only normal) complex projective variety X (of dimension t) such that $\mathcal{P}_X(X, \mathcal{O}_X) = \mathcal{P}$?

We do not know the answer to this question, even in the surface case, that is in the case $t = 2$. In [28] a method is given, which allows to realize by smooth surfaces a great variety of positive patters as discussed above. We also should mention that by the *Non-Rigidity Theorem* of Evans-Griffiths [19] (see also [29]) there are realization results of the above type in which indeed more than the cohomological pattern is described. Nevertheless, these results allow a realization only up to an eventual twist – but they do not allow to control the last supporting degree of the top cohomology group. Therefore they do not answer our question. Another, local realization result, similar to those just quoted, is given in [14].

We now return to cohomological patterns of pairs $(X, \mathcal{F}) \in \mathcal{S}^t$, hence we concentrate again to the case where $X = \mathrm{Proj}(R)$ is a projective scheme over some field K and thus induced by some Noetherian homogeneous K-algebra. A natural (and fundamental) question is to ask for the lowest level i at which the cohomological pattern $\mathcal{P}_{\mathcal{F}} = \mathcal{P}(X, \mathcal{F})$ of $\mathcal{F}$ has infinitely many entries (i, n) with $n < 0$. By the tameness-property (vi) of Theorem 2.4 it is equivalent to ask for the lowest level $i \leq t$ such that there are only finitely many negative integers $n < 0$ with $(i, n) \notin \mathcal{P}_{\mathcal{F}}$.

2.6. **Definition and Remark.** A) *(Cohomological Finiteness Dimension of a Coherent Sheaf)* Let $t \in \mathbb{N}_0$ and let $(X, \mathcal{F}) \in \mathcal{S}^t$. Then, the *(cohomological) finiteness dimension of (X with respect to)* $\mathcal{F}$ is defined as

$$\mathrm{fdim}(\mathcal{F}) := \inf\{i \in \mathbb{N}_0 \mid \#\{n < 0 \mid h^i_{\mathcal{F}}(n) \neq 0\} = \infty\}$$

$$= \inf\{i \in \mathbb{N}_0 \mid \#\{n < 0 \mid h^i_{\mathcal{F}}(n) = 0\} < \infty\}.$$

So $\mathrm{fdim}(\mathcal{F})$ is the lowest level on which there are infinitely many $\bullet$'s at places with negative n-coordinate or - equivalently - only finitely many $\circ$'s with negative n-coordinate.

B)*(Algebraic Characterization of the Cohomological Finiteness Dimension)* Keep the notations and hypotheses of part A) and let $(R, M) \in \mathcal{M}^{t+1}$ such that $(X, \mathcal{F}) = (\mathrm{Proj}(R), \widetilde{M})$. In [13] (9.1.3) the *$R_+$-finiteness dimension of M* is introduced as the invariant

a) $f_{R_+}(M) := \inf\{j \in \mathbb{N} \mid H^j_{R_+}(M) \text{ not finitely generated}\}$.

As the K-vector spaces $H^j_{R_+}(M)_n$ are finitely generated and vanish for all $n \gg 0$, it follows, that

b) $f_{R_+}(M) = \inf\{j \in \mathbb{N} \mid \#\{n < 0 \mid h^j_M(n) \neq 0\} = \infty\}$.

Now on use of the relations a),b) of Notation and Reminder 1.5 D) it follows that

c) $\operatorname{fdim}(\mathcal{F}(M)) = f_{R_+}(M) - 1$.

The finiteness dimension of a coherent sheaf $\mathcal{F}$ of modules over a projective scheme X over some field K is by its very definition a "global invariant" as it is defined by means of the vanishing and non-vanishing of the cohomology groups $H^i(X, \mathcal{F}(n))$ – which at their turn are global invariants of the twisted sheaf $\mathcal{F}(n)$. But nevertheless these global invariants are intimately related to a local invariant of the sheaf $\mathcal{F}$, which we shall define now.

2.7. **Definition and Remark.** A) *(Subdepth of a Coherent Sheaf).* Let $t \in \mathbb{N}_0$ and let $(X, \mathcal{F}) \in \mathcal{S}^t$. Let $(R, M) \in \mathcal{M}^{t+1}$ with $(X, \mathcal{F}) = (\operatorname{Proj}(R), \widetilde{M})$. We consider the *set of closed points of* $X = \operatorname{Proj}(R)$, that is the set

$$\mathrm{m}X = \mathrm{mProj}(R) := \{x = \mathfrak{p} \in X = \operatorname{Proj}(R) \mid \dim(R/\mathfrak{p}) = 1\}$$

and define the *subdepth of (X with respect to)* $\mathcal{F}$ by

$$\delta(\mathcal{F}) := \inf\{\operatorname{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \mid x \in \mathrm{m}X\}.$$

B) *(Algebraic Description of the Subdepth)* Keep the above notations and hypotheses. If $x = \mathfrak{p} \in X = \operatorname{Proj}(R)$ we have

$$\mathcal{O}_{X,x} = R_{(\mathfrak{p})}, \quad \mathcal{F}_x = M_{(\mathfrak{p})}$$

where $\bullet_{(\mathfrak{p})}$ denotes homogeneous localization at $\mathfrak{p}$. Therefore, we also may write

$$\delta(\mathcal{F}) = \inf\{\operatorname{depth}_{R_{(\mathfrak{p})}}(M_{(\mathfrak{p})}) \mid \mathfrak{p} \in \mathrm{mProj}(R)\}.$$

Now, the two previously introduced invariants are related be the *Vanishing Theorem of Severi-Enriques-Zariski-Serre*:

2.8. **Theorem.** *Let $t \in \mathbb{N}_0$ and let $(X, \mathcal{F}) \in \mathcal{S}^t$. Then*

$$\operatorname{fdim}(\mathcal{F}) = \delta(\mathcal{F}).$$

2.9. **Remark.** A) *(On the Proof of the Vanishing Theorem of Severi-Enriques-Zariski-Serre).* One approach to prove Theorem 2.8 is to use Serre-Duality (see [24] for example). This approach corresponds essentially to Serre's original proof in [33]. Another approach (which leads indeed even to a quantitative version of the requested result) for projective schemes over algebraically closed fields is found in [8] (see Chapters 10 for an algebraic version and Chapter 12 for the translation to sheaf theory). It is easy, to drop the hypothesis of algebraically closed ground field (see [5] (7.11) D)). Another approach is to use a much more general algebraic result: the *Graded Version of Grothendieck's Finiteness Theorem* or - in fact even more general - the *Graded Version of Falting's Annihilator Theorem* (for both see [13] (14.3.10)). Under the hypotheses and in the notations of (2.6) and (2.7), one has only to show that

a) $\delta(\mathcal{F})+1=\inf\{\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p})+\mathrm{ht}\big((R_+ +\mathfrak{p})/\mathfrak{p}\big)\mid \mathfrak{p}\in\mathrm{Proj}(R)\}$,

Indeed, from this the mentioned graded version of Grothendieck's Finiteness Theorem allows to deduce the relation $f_{R_+}(M)=\delta(\mathcal{F})$, and by Definition and Remark 2.6 B)c) one obtains Theorem 2.8. But statement a) follows easily from the well known fact that

b) $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p})=\mathrm{depth}_{R_{(\mathfrak{p})}}(M_{(\mathfrak{p})}),\quad \forall\mathfrak{p}\in\mathrm{Proj}(R)$

B) *(Around the History of the Vanishing Theorem of Severi-Enriques-Zariski-Serre).* We briefly aim to explain the string of names associate to this theorem. There are three preceding results to the mentioned vanishing theorem, shown by Severi [34] 1942, Enriques [18] 1949 and Zariski [35] 1952 - thus all three of them in the "pre-cohomological aera" of Algebraic Geometry - and formulated in the language of linear systems of divisors. If one translates these results to our cohomological language, they respectively correspond to the following special cases of Theorem 2.8.

a) (Severi 1942) $X\subseteq\mathbb{P}^3_\mathbb{C}$ is a smooth surface in complex projective 3-space and $\mathcal{F}:=\omega_X$ is the canonical bundle of X
b) (Enriques 1949) $X\subseteq\mathbb{P}^r_\mathbb{C}$ is a smooth hypersurface in complex projective r-space and $\mathcal{F}:=\omega_X$ is the canonical bundle of X.
c) (Zariski 1952) X is a normal projective variety over an algebraically closed field and $\mathcal{F}=\mathcal{L}$ is a line bundle on X.

C) *(Sixty Years of Modern Algebraic Geometry).* In his seminal work [33] in 1955, Serre proved Theorem 2.8 for arbitrary projective varieties over algebraically closed fields and arbitrary coherent sheaves over such varieties. In that same paper he actually introduced sheaf theory and sheaf cohomology for arbitrary algebraic varieties over algebraically closed fields. As observed in part B), the Vanishing Theorems of Severi, of Enriques and of Zariski are very special cases of Theorem 2.8 and they all were formulated in the language of divisors, which is only available for particular classes of projective varieties. So, Serre's cohomological approach to the mentioned Vanishing Theorems was a fundamental break through in Algebraic Geometry.
Indeed, the basic motivation for Serre's work [33] was the aim to introduce a sheaf theory and a sheaf cohomology theory for algebraic varieties which were as powerful as the corresponding theories for smooth complex analytic varieties, which already were available at that time. The main results of [33] prove, that Serre's approach was very successful, and Theorem 2.8 is one of the most convincing instances of this. Another celebrated result of [33] was the

relation between the global generation of a coherent sheaf and the vanishing of its cohomology, as expressed by Reminder and Definition 2.2 C).
So, with his paper [33], Serre "opened the door" to a sheaf theoretic and cohomological approach to Algebraic Geometry. Moreover, a number of basic arguments occurring in [33] base on systematic application of homological and cohomological methods to commutative rings – and hence very successfully "imported" these methods to Commutative Algebra. Altogether, this finally paved the way to "Modern Algebraic Geometry" – namely Grothendiek's scheme theoretic and functorial approach to this theory [23] – and to a systematic use of Commutative Algebra to study local properties of the occurring geometric objects.
In particular, a number of geometric results now turned out as special cases of more general algebraic results. For example, as mentioned already above, Theorem 2.8 found an algebraic generalization: Grothendieck's Finiteness Theorem for Local Cohomology. A next step of generalization is Falting's Annihilator Theorem for Local Cohomology (see [20] or [13] (9.5.1)), whose graded version we already mentioned in part A).

D) *(Cohomological Characterization of Algebraic Vector Bundles).* One of the early successful applications of homological methods to Commutative Algebra was the *homological characterization of regular local rings*. We recall this result, which is essentially due to Serre and Auslander-Buchsbaum in the following form:

(a) A Noetherian local ring $(R, \mathfrak{m})$ is regular if and only if each finitely generated R-module $M \neq 0$ has a minimal finite free resolution

$$0 \longrightarrow R^{b_p} \xrightarrow{\partial_p} R^{b_{p-1}} \cdots R^{b_1} \xrightarrow{\partial_1} R^{b_0} \xrightarrow{\partial_0} M \longrightarrow 0 \quad (b_0, \ldots, b_p \in \mathbb{N}).$$

In this situation, the numbers $b_i = b_i(M)$ are determined by M and called the *local Betti numbers* of M. The number $p \in \mathbb{N}_0$ is called the *projective dimension* of M and denoted by $\mathrm{pdim}_R(M)$. The *Formula of Auslander-Buchsbaum-Serre* says:

(b) If M is a finitely generated module over the regular local ring $(R, \mathfrak{m})$, then

$$\mathrm{pdim}_R(M) + \mathrm{depth}_R(M) = \dim(R).$$

One of the basic and most typical applications of the Vanishing Theorem of Severi-Enriques-Zariski-Serre is a cohomological characterization of algebraic vector bundles. To formulate this application, we assume that X is a regular irreducible projective scheme of dimension $t > 0$ over a field K, so that $X = \mathrm{Proj}(R)$, where $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ is an Noetherian homogeneous integral domain with the property that $R_{(\mathfrak{p})} = \mathcal{O}_{X,x}$ is a regular local ring of dimension t for all $x = \mathfrak{p} \in \mathrm{m}X = \mathrm{mProj}(R)$. Now, let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then, by the Formula of Auslander-Buchsbaum-Serre it follows easily, that for every $x \in \mathrm{m}X$ the finitely generated $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is free if and only if $\operatorname{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = t$. By definition, the coherent sheaf $\mathcal{F}$ is an *(algebraic) vector bundle over* X if and only if the stalk $\mathcal{F}_x$ of $\mathcal{F}$ is a free module over the local ring $\mathcal{O}_{X,x}$ for all $x \in \mathrm{m}X$. So, by Theorem 2.8 we can say:

(c) If X is a regular irreducible projective scheme over a field K, a non-zero coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$- modules is a vector bundle if and only if the cohomological finiteness dimension $\operatorname{fdim}(\mathcal{F})$ of $\mathcal{F}$ takes the (maximally possible) value $\dim(X)$.

Therefore, we can say:

(d) If X is a regular irreducible projective scheme over a field K, a non-zero coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$- modules is a vector bundle if and only if the cohomological pattern $\mathcal{P}(X, \mathcal{F}) = \mathcal{P}_{\mathcal{F}}$ of $\mathcal{F}$ is *asymptotically left-vanishing* below the level $\dim(X)$, thus if and only if

$$(i, n) \notin \mathcal{P}_{\mathcal{F}} \text{ for all } i < \dim(X) \text{ and all } n \ll 0.$$

A more detailed presentation of the relation between sheaf cohomology and algebraic vector bundles may be found in [5] (7.11)-(7.13) or [13] Chapter 20.

3. Modules of Deficiency

In this section we introduce an important tool for the treatment of local cohomology modules, the so called *Modules of Deficiency* (or just *Deficiency Modules* for short). More precisely, for each each finitely generated graded module M over a Noetherian homogeneous algebra R over a field K we introduce a family of finitely generated graded R-modules $\left(K^i(M)\right)_{i\in\mathbb{N}_0}$, such that for each $i \in \mathbb{N}_0$ and each $n \in \mathbb{Z}$, the n-th graded component $K^i(M)_n$ of $K^i(M)$ is K-dual to the $-n$-th component $H^i_{R_+}(M)_{-n}$ of the corresponding local cohomology module $H^i_{R_+}(M)$. In our *Main Theorem on Modules of Deficiency* we collect all the relevant properties of deficiency modules. As an application, we shall be able to introduce the concept of the *i-th Cohomological Hilbert Polynomial p^i_M of M* and the notion of the *i-th Cohomological Postulation Number ν^i_M of M* of a finitely generated graded module over a Noetherian homogeneous K-algebra R. A basic issue is the fact, that the Castelnuovo-Mumford regularity of the i-th deficiency module gives a lower bound for the i-th cohomological postulation number. We also shall consider the most important class of deficiency modules: the *Canonical Modules* $K(M) := K^{\dim_R(M)}$. We shall establish one property of these modules, which has to be used to prove the main result of Section 4. A more detailed and complete treatment of all these and further related results may be found in [5].

3.1. **Construction and Remark.** A) *(Graded Dual Modules)* Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded R-module. We consider the K-dual of M, that is the K-vector space

$$M^\vee := \operatorname{Hom}_K(M, K)$$

of all K-linear maps $h : M \to K$. By means of the scalar multiplication defined by

$$xh := h \circ x\mathrm{Id}_M, \quad \forall x \in R, \quad \forall h \in M^\vee,$$

the K-vector space $M^\vee$ is turned into an R-module. We consider the subset

$$D(M) := \{h \in M^\vee \mid \#\{n \in \mathbb{Z} \mid h(M_n) \neq 0\} < \infty\} \subseteq M^\vee$$

consisting of all K-linear maps $h : M \to K$ which vanish on almost all graded components of M. Moreover, for each $t \in \mathbb{Z}$ we define the subset

$$D(M)_t := \{h \in M^\vee \mid h(M_n) = 0, \quad \forall n \neq -t\}.$$

Now, one may easily verify the following statements:

a) $D(M) \subseteq M^\vee$ is an R-submodule.

b) For all $t \in \mathbb{Z}$ the set $D(M)_t \subseteq D(M)$ is a K-subspace.

c) The family $(D(M)_t)_{t\in\mathbb{Z}}$ of K-subspaces $D(M)_t \subseteq D(M)$ defines a grading on the R-module $D(M)$.

d) For all $t \in \mathbb{Z}$ there is an isomorphism of K-vector spaces

$$\tau^M_t : (M_{-t})^\vee := \operatorname{Hom}_K(M_{-t}, K) \xrightarrow{\cong} D(M)_t$$

given by

$$\tau_t^M(h)(m) := h(m_{-t}), \quad \forall h \in (M_{-t})^\vee, \quad \forall m := (m_n)_{n\in\mathbb{Z}} \in M = \bigoplus_{n\in\mathbb{Z}} M_n.$$

e) For all $r, t \in \mathbb{Z}$ we have $D(M(r))_t = D(M)_{t-r}$.

From now on, we always furnish the R-module $D(M)$ with the grading mentioned in statement c). Hence we write

$$D(M) = \bigoplus_{t\in\mathbb{Z}} D(M)_t,$$

and call $D(M)$ the *graded (K-) dual of M*. Observe that by statement e) we have

f) $D(M(r)) = D(M)(-r), \quad \forall r \in \mathbb{Z}.$

B) *(The Graded Duality Functor)* Keep the notations and hypotheses of part A) and let $h : M \to N$ be a homomorphism of graded R-modules. It is easy to see, that there is a homomorphism of graded R-modules

$$D(h) : D(N) \to D(M), \quad f \mapsto f \circ h, \quad \forall f \in D(N).$$

This homomorphisms of graded R-modules is called the *graded (K-) dual of h*. Now, we obtain a *contravariant, R-linear, exact functor of graded R-modules*

$$D(\bullet) : (M \xrightarrow{h} N) \leadsto \big(D(N) \xrightarrow{D(h)} D(M)\big),$$

the *functor of taking graded (K-)duals* or the *graded duality functor (with respect to K)*.

C) *(First Properties of Graded Duality Functors)* Keep the notations and hypotheses of parts A) and B). It is easy to verify the following claims:

a) For all $t \in \mathbb{Z}$ there is a natural equivalence of contravariant functors from graded R-modules to K-vector spaces

$$\tau_t^M : (\bullet_{-t})^\vee \xrightarrow{\cong} D(\bullet)_t : M \leadsto \big((M_{-t})^\vee \xrightarrow{\tau_t^M} D(M)_t\big),$$

where τ_t^M is defined as in statement A)d).

b) There is a natural transformation of covariant functors of graded R-modules

$$\gamma : \bullet \to D(D(\bullet)) : M \leadsto \big(M \xrightarrow{\gamma^M} D(D(M))\big),$$

where the homomorphism $\gamma^M : M \to D(D(M))$ is given by

$$\gamma^M(m)(f) = f(m), \quad \forall m \in M, \quad \forall f \in D(M).$$

D) *(Base Ring Independence of Graded Duals)* Keep the notations of part A) and assume that $\mathfrak{a} \subsetneq R$ is a proper graded ideal such that $\mathfrak{a}M = 0$. Then, it is easy to see that the graded R-module $D(M)$ satisfies $\mathfrak{a}D(M) = 0$ and is independent on whether we consider M as an R-module or an $R/\mathfrak{a}$-module.

We now establish a few basic facts about graded duals.

3.2. **Remark and Definition.** A) *(Modules with Finite Components)* Let the notations and hypotheses be as in Construction and Remark 3.1. We say that a graded R-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$ has *finite components* if

$$\dim_K(M_n) < \infty, \quad \forall n \in \mathbb{Z}.$$

We denote the class of graded R-modules with finite components by $\mathbb{F}_R$. On use of the well known properties of taking duals of finite-dimensional vector spaces and the natural equivalences of Construction and Remark 3.1 C)a) one easily gets the following statements:

a) If $M \in \mathbb{F}_R$, then $\dim_K(D(M)_t) = \dim_K(M_{-t})$ for all $t \in \mathbb{Z}$.
b) If $M \in \mathbb{F}_R$, then $D(M) \in \mathbb{F}_R$.
c) If $M \in \mathbb{F}_R$, the canonical map $\gamma^M : M \to D(D(M))$ (see Construction and Remark 3.1 C)b)) is an isomorphism of graded R-modules.

B) *(Equihomogeneous Ideals)* Keep the above notations and hypotheses. An ideal $\mathfrak{a} \subseteq R$ is said to be *equihomogeneous* if it is generated by homogeneous elements of the same degree. We now are interested in finitely generated equihomogeneous ideals. So, let $s \in \mathbb{Z}$, let $r \in \mathbb{N}$, let $x_1, x_2, \ldots, x_r \in R_s$, let M be a graded R-module and consider the multiplication maps given by these elements x_i, that is the homomorphisms of graded R-modules

$$x_i = x_i \mathrm{Id}_M : M \to M(s), \quad m \mapsto x_i m, (i = 1, 2, \ldots, r).$$

On use of the properties of kernels and cokernels of K-linear maps with respect to taking K-duals, it is straightforward to prove the following facts:

a) $(0 :_M \langle x_1, x_2, \ldots, x_r\rangle)_{-t} = \bigcap_{i=1}^{r} \mathrm{Ker}(x_i \restriction_{M_{-t}})$ for all $t \in \mathbb{Z}$.
b) There is an isomorphism of graded R-modules

$$D(M)/\langle x_1, x_2, \ldots, x_r\rangle D(M) \xrightarrow{\cong} D(0 :_M \langle x_1, x_2, \ldots, x_r\rangle)$$

defined by

$$u + \langle x_1, x_2, \ldots, x_r\rangle D(M) \mapsto u\restriction_{(0:_M\langle x_1,x_2,\ldots,x_r\rangle)}, \quad \forall u \in D(M).$$

Now, we shall introduce the notions of *Deficiency Functors and Deficiency Modules.*

3.3. **Remark and Definition.** A) *(Deficiency Functors and -Modules)* Let K be a field and let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra. For each $i \in \mathbb{N}_0$ we define the *i-th deficiency functor $K^i = K^i(\bullet)$ (over R)* as the contravariant linear functor of graded R-modules obtained by composing the (graded) local cohomology functor $H^i_{R_+}(\bullet)$ with the graded duality functor $D = D(\bullet)$, thus the functor of graded R-modules given by the assignment

$$(M \xrightarrow{h} N) \rightsquigarrow \big(K^i(M) := D(H^i_{R_+}(N)) \xrightarrow{K^i(h):=D(H^i_{R_+}(h))} D(H^i_{R_+}(M)) =: K^i(M)\big).$$

For each graded R-module M, the graded R-module $K^i(M)$ is called the *i-th deficiency module of M.*

B) *(First Properties of Deficiency Functors)* Keep the notations and hypotheses of part A). Let $i \in \mathbb{N}_0$. Then, it is easy to verify the following facts:

a) *(Duals of Deficiency Modules)* There is a natural transformation of covariant functors of graded R-modules

$$\kappa^i : H^i_{R+}(\bullet) \to D(K^i(\bullet)) : M \rightsquigarrow \big(H^i_{R_+}(M) \xrightarrow{\kappa^{i,M}:=\gamma^{H^i_{R_+}(M)}} D(K^i(M))\big),$$

where the homomorphism

$$\gamma^{H^i_{R_+}(M)} : H^i_{R_+}(M) \to D\big(D(H^i_{R_+}(M))\big) = D(K^i(M))$$

is defined according to Construction and Remark 3.1 C)b).

b) *(Base Ring Independence of Deficiency Modules)* If M is a graded R-module and $\mathfrak{a} \subsetneq R$ is a proper graded ideal with $\mathfrak{a}M = 0$ we have $\mathfrak{a}K^i(M) = 0$. In addition (up to isomorphism of graded R-modules) the module $K^i(M)$ remains the same if we consider M as as a graded $R/\mathfrak{a}$-module.

C) *(Deficiency Modules of Finitely Generated Modules)* Let the notations be as in parts A) and B) and assume that the graded R-module M is finitely generated. One easily can prove the following facts:

a) $H^i_{R_+}(M)$ and $K^i(M)$ belong to the class $\mathbb{F}_R$ (see Remark and Definition 3.2 A)).
b) $\dim_K(K^i(M))_n = h^i_M(-n)$ for all $n \in \mathbb{Z}$.
c) $\mathrm{beg}(K^i(M)) = -\mathrm{end}(H^i_{R_+}(M)) > -\infty$.
d) $\sup\{i \in \mathbb{N}_0 \mid K^i(M) \neq 0\} = \dim_R(M)$.
e) The natural homomorphism of graded R-modules of statement B)a) becomes an isomorphism

$$\kappa^{i,M} : H^i_{R_+}(M) \xrightarrow{\cong} D(K^i(M)).$$

D) *(The Deficiency Sequence)* Keep the above notations and hypothesis and let

$$\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$$

be an exact sequence of graded R-modules. We apply the *exact graded cohomology sequence with respect to R_+ and associated to* $\mathbb{S}$ (see [8](8.26)A) for example) and then apply the exact contravariant graded duality functor $D(\bullet)$ to the resulting sequence. In doing so, we end up with a natural exact sequence of graded R-modules

$$\begin{array}{l}
\dots \longrightarrow K^{i+1}(M) \xrightarrow{K^{i+1}(h)} K^{i+1}(N) \xrightarrow{\varepsilon^{i}_{\mathbb{S}}} \\
K^{i}(P) \xrightarrow{K^{i}(l)} K^{i}(M) \xrightarrow{K^{i}(h)} K^{i}(N) \xrightarrow{\varepsilon^{i-1}_{\mathbb{S}}} \\
K^{i-1}(P) \longrightarrow \dots \\
\dots \longrightarrow K^{1}(M) \xrightarrow{K^{1}(h)} K^{1}(N) \xrightarrow{\varepsilon^{0}_{\mathbb{S}}} \\
K^{0}(P) \xrightarrow{K^{0}(l)} K^{0}(M) \xrightarrow{K^{0}(h)} K^{0}(N) \longrightarrow 0
\end{array}$$

in which the maps $\varepsilon^{i}_{\mathbb{S}}$ are induced by the corresponding connecting homomorphism in the cohomology sequence associated to $\mathbb{S}$. We call this sequence the *deficiency sequence associated to* $\mathbb{S}$.

In the next exercise we prepare some arguments which will be used repeatedly later.

3.4. **Remark.** A) *(Deficiency Modules and Torsion).* Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \dots$ be a Noetherian homogeneous K-algebra and let M be a graded R-module. Let us recall, that an R-module T is said to be R_+-*torsion*, if $T = \Gamma_{R_+}(T)$. On use of the corresponding statements on induced homomorphisms between local cohomology modules (see [8] (3.18) for example) it is easy to see that:

a) If M is R_+-torsion, then $K^i(M) = 0$ for all $i \in \mathbb{N}$.
b) If M is finitely generated, then $K^0(M)$ is R_+-torsion, finitely generated and satisfies $\dim_K(K^0(M)) = \dim_K(H^0_{R_+}(M)) < \infty$.
c) If $N \subseteq M$ is a graded submodule which is R_+-torsion and $p : M \to M/N$ is the canonical homomorphism, then the induced homomorphism

$$K^i(p) : K^i(M/N) \to K^i(M)$$

is an isomorphism if $i > 0$ and a monomorphism if $i = 0$.

B) *(Deficiency Modules and Non-Zero Divisors)* Let the notations and hypotheses be as in part A). For any R-module N let

$$\mathrm{NZD}_R(N) := \{x \in R \mid xn \neq 0. \quad \forall n \in N \setminus \{0\}\}$$

denote the set of *non-zero divisors of* R *with respect to* N. Now, let $t \in \mathbb{N}$ and let $x \in R_t \cap \mathrm{NZD}_R(M)$. If we form the deficiency sequence associated to the short exact sequence of graded R-modules

$$\$: 0 \to M(-t) \xrightarrow{x} M \xrightarrow{p} M/xM \to 0$$

and write $\varepsilon^i_{M,x} := \varepsilon^i_{\$}$ for all $i \in \mathbb{N}_0$ (see Remark and Definition 3.3 D)), we can say:

a) For each $i \in \mathbb{N}_0$ there is an exact sequence of graded R-modules

$$K^{i+1}(M) \xrightarrow{x} K^{i+1}(M)(t) \xrightarrow{\varepsilon^i_{M,x}} K^i(M/xM) \xrightarrow{K^i(p)} K^i(M) \xrightarrow{x} K^i(M)(t).$$

Consequently

b) For each $i \in \mathbb{N}_0$ there is a short exact sequence of graded R-modules
$$0 \to \big(K^{i+1}(M)/xK^{i+1}(M)\big)(t) \to K^i(M/xM) \to (0 :_{K^i(M)} x) \to 0.$$

Now we are ready to give a first result on the structure of deficiency modules.

3.5. **Proposition.** *Let K be a field, let $R \in \mathbb{N}_0$ and let $R := K[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r]$ be a polynomial ring.*

a) *If $i \neq r$, then $K^i(R) = 0$.*
b) *$K^r(R) \cong R(-r)$.*

Proof. As R_+ is generated by the R-sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$, we have $H^i_{R_+}(R) = 0$ for all $i \neq r$ (see [8] (6.7)A) for example). So, statement a) follows from Remark and Definition 3.3 C)b).

We prove statement b) by induction on r. If $r = 0$, we have $R = K = H^0_{R_+}(R)$. If we apply Remark and Definition 3.3 C)b) with $i = 0$ it follows that $K^0(M) = K = R = R(-0)$.

So let $r > 0$. We consider the polynomial ring

$$R' := K[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{r-1}].$$

By induction we have $K^{r-1}(R') \cong R'(-r+1)$. Observe that there is an isomorphism of graded R-modules $R' \cong R/\mathbf{x}_r R$. So, by the Base Ring Independence of Deficiency Modules Remark and Definition 3.3 B)b) we get an isomorphism of graded R-modules

$$K^{r-1}(R/\mathbf{x}_r R) \cong (R/\mathbf{x}_r R)(-r+1).$$

If we apply the short exact sequence Reark 3.4 B)b) with $i = r-1$, $x = \mathbf{x}_r$, $M = R$ and keep in mind that $K^{r-1}(R) = 0$ we therefore get isomorphisms of graded R-modules

$$K^r(R)/\mathbf{x}_r K^r(R) \cong K^{r-1}(R/\mathbf{x}_r R)(-1) \cong (R/\mathbf{x}_r R)(-r).$$

As a consequence

$$K^r(R)/(R_+)K^r(R) \cong R/(\mathbf{x}_r R)(-r)/(R_+)(R/\mathbf{x}_r R)(-r) \cong$$
$$\cong \big((R/\mathbf{x}_r R)/(R_+)(R/\mathbf{x}_r)\big)(-r) \cong (R/R_+)(-r).$$

This shows that $K^r(R)/(R_+)K^r(R)$ is generated by a single element of degree r. As $\mathrm{beg}(K^r(M)) = -\mathrm{end}(H^r_{R_+}(R)) > -\infty$ (see Remark and Definition 3.3 C)c)), the Graded Nakayama Lemma implies that $K^r(R) = Ra$ for some $a \in K^r(R)_r$. So, there is an epimorphism of graded R-modules

$$R(-r) \xrightarrow{\pi} K^r(R) \to 0, \quad f \mapsto fa.$$

Now, let $x \in R_t \setminus \{0\}$ for some $t \in \mathbb{N}$. Then $\dim_R(R/xR) < r$ shows that the multiplication map $x : H^r_{R_+}(R)(-t) \to H^r_{R_+}(R)$ is surjective. Therefore the multiplication map $x : K^r(R) \to K^r(R)(t)$ is injective. This shows, that $K^r(R)$ is R-torsion-free and hence of dimension r. This proves, that the epimorphism π is indeed an isomorphism. □

Now, we are ready to prove the following *Main Theorem on Deficiency Modules.*

3.6. **Theorem.** *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra, let M be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. Then*

a) *$K^i(M)$ is a finitely generated graded R-module.*
b) *$\dim_K(K^i(M)_n) = h^i_M(-n)$ for all $n \in \mathbb{Z}$.*
c) *$\mathrm{beg}(K^i(M)) = -\mathrm{end}(H^i_{R_+}(M)) > -\infty$.*
d) *$K^i(M) = 0$ for all $i > \dim_R(M)$.*
e) *$\dim_R(K^i(M)) \leq i$ for all $i \leq \dim_R(M)$ with equality if $i = \dim_R(M)$.*

Proof. "a)“: We find a polynomial ring $S = K[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r]$ and a proper graded ideal $\mathfrak{a} \subsetneq S$ such that $R = S/\mathfrak{a}$. According to the Base Ring Independence of Deficiency Modules Remark and Definition 3.3 B)b) we may consider M as a graded S-module and hence assume that $R = K[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r]$. If $M = 0$ we have $K^i(M) = 0$. So, let $M \neq 0$. We show by induction on the homological dimension $h := \mathrm{hdim}(M) (\in \mathbb{N}_0)$ of M that $K^i(M)$ is finitely generated. If $h = 0$, we have an isomorphism of graded R-modules

$$M \cong \bigoplus_{k=1}^{s} R(-a_k), \quad a_k \in \mathbb{Z}, \quad \forall k \in \{1, 2, \ldots, s\}, \quad a_1 \leq a_2 \leq \ldots \leq a_s.$$

So, by Proposition 3.5 and the additivity of the contravariant functor of graded R-modules $K^i(\bullet)$ we get $K^i(M) = 0$ if $i \neq r$ and $K^r(M) \cong \bigoplus_{k=1}^{s} R(-r + a_k)$.

Now, let $h > 0$ and consider a minimal presentation

$$\mathbb{S} : 0 \to N \to F \to M \to 0, \quad F = \bigoplus_{k=1}^{s} R(-a_k), \quad a_1 \leq a_2 \leq \ldots \leq a_s$$

of M. As $\mathrm{hdim}(N) = \mathrm{hdim}(M) - 1 = h - 1$, by induction, $K^j(N)$ is finitely generated for all $j \in \mathbb{N}_0$. By the case $h = 0$ we have $K^j(F) = 0$ for all $j \neq r$, and $K^r(F)$ is a graded free R-module of finite rank. So, the deficiency sequence

Remark and Definition 3.3 D) associated to $ gives rise to isomorphisms of graded R-modules

$$K^{j+1}(N) \cong K^j(M), \quad \forall j \in \{0, 1, \ldots, r-2\},$$

an epimorphism of graded R-modules

$$K^r(N) \to K^{r-1}(M) \to 0,$$

and a short exact sequence of graded R-modules

$$K^{r+1}(N) \to K^r(M) \to K^r(F).$$

Hence, $K^i(M)$ is finitely generated if $i \leq r$. As $K^i(M) = 0$ if $i > \dim_R(M)$ (see (3.3)C)d)) and as $\dim_R(M) \leq r$, we get our claim.

"b)": This is nothing else than Remark and Definition 3.3 C)b).

"c)": This is a restatement of Remark and Definition 3.3 C)c).

"d)": This is clear by Remark and Definition 3.3 C)d).

"e)": We do not prove this here. Instead we refer to [5] (9.7). □

We now use the previous development to introduce *Cohomological Hilbert Polynomials* of graded modules and *Cohomological Serre Polynomials* of coherent sheaves and the related concepts of *Cohomological postulation Numbers*.

3.7. **Remark and Definition.** A) *(Cohomological Hilbert Polynomials)* Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Fix $i \in \mathbb{N}_0$ and consider the Hilbert polynomial $P_{K^i(M)}$ of the finitely generated graded R-module $K^i(M)$. Then, by the definition of $P_{K^i(M)}$ and by Theorem 3.6 b) we have

$$h^i_M(n) = \dim_K(K^i(M)_{-n}) = P_{K^i(M)}(-n), \quad \forall n \ll 0.$$

If we set

$$p^i_M(X) := P_{K^i(M)}(-X)$$

we thus have

$$h^i_M(n) = p^i_M(n), \quad \forall n \ll 0.$$

The polynomial $p^i_M \in \mathbb{Q}[X]$ is called the *i-th cohomological Hilbert polynomial of M*.

B) *(First Properties of Cohomological Hilbert Polynomials)* Let the notations and hypotheses be as in part A). Prove the following facts. (For statement c) see Reminder and Remark 1.8)

a) $\deg(p^i_M) \leq i-1$ with equality if $i = \dim_K(M) > 0$.
b) $p^i_{M(r)}(X) = p^i_M(r+X)$ for all $r \in \mathbb{Z}$.
c) $P_M(X) = \sum_{i=1}^{\dim_R(M)-1}(-1)^{i-1}p^i_M(X) = \sum_{n\in\mathbb{N}_0}(-1)^{i-1}p^i_M(X)$.

C) *(Cohomological Postulation Numbers of Graded Modules)* Let the notations and hypotheses be as in parts A) and B). Then clearly

$$\nu_M^i := \inf\{n \in \mathbb{Z} \mid p_M^i(n) \neq h_M^i(n)\} \in \mathbb{Z} \cup \{\infty\}.$$

The number ν_M^i is called the *i-th cohomological postulation number of M*. Prove the following statements:

a) $\nu_M^i = \infty$ if and only if $H_{R_+}^i(M) = 0$.

b) If $\nu_M^i < \infty$, then $\nu_M^i \leq \operatorname{end}(H_{R_+}^i(M))$.

c) $\nu_{M(r)}^i = \nu_M^i - r$ for all $r \in \mathbb{Z}$.

D) *(Cohomological Serre Polynomials)* Let (R, M) be as in parts A) and B), set $X := \operatorname{Proj}(R)$ and let $\mathcal{F} = \widetilde{M}$ be the coherent sheaf induced by M. Then, it follows by Notation and Reminder 1.4 B)d),f) and Notation and Reminder 1.5 C) that for all $i \in \mathbb{N}_0$ we have

a) $p_M^{i+1}(n) = h_{\mathcal{F}}^i(n)$ for all $n \ll 0$.

So, in particular, for each $i \in \mathbb{N}_0$ there is a unique polynomial (namely p_M^{i+1})

$$p_{\mathcal{F}}^i \in \mathbb{Q}[X] : \quad h_{\mathcal{F}}^i(n) = p_{\mathcal{F}}^i(n), \quad \forall n \ll 0,$$

the *i-th cohomological Serre polynomial of $\mathcal{F}$*. It follows easily that

b) $\deg(p_{\mathcal{F}}^i) \leq i$, with equality if $i = \dim(\mathcal{F}) \geq 0$.

c) If $i > \dim(\mathcal{F})$, then $p_{\mathcal{F}}^i = 0$.

d) $P_{\mathcal{F}} = \sum_{i=0}^{\dim(\mathcal{F})} (-1)^i p_{\mathcal{F}}^i = \sum_{i \in \mathbb{N}_0} (-1)^i p_{\mathcal{F}}^i$.

E) *(Cohomological Postulation Numbers of Sheaves)* Let the notations be as in part D). Then clearly

$$\nu_{\mathcal{F}}^i := \inf\{n \in \mathbb{Z} \mid p_{\mathcal{F}}^i(n) \neq h_{\mathcal{F}}^i(n)\} \in \mathbb{Z} \cup \{\infty\}.$$

The number $\nu_{\mathcal{F}}^i$ is called the *i-th cohomological postulation number of $\mathcal{F}$*. The following statements are easy to prove:

a) If $i \in \mathbb{N}$, then $\nu_{\mathcal{F}}^i = \nu_M^{i+1}$.

b) $\nu_{\mathcal{F}}^0 \geq \min\{\nu_M^1, \operatorname{beg}(M)\}$.

c) If $\mathcal{F} \neq 0$, then $\nu_{\mathcal{F}}^{\dim(\mathcal{F})} \in \mathbb{Z}$.

d) If $i > \dim(\mathcal{F})$, then $\nu_{\mathcal{F}}^i = \infty$.

An important fact is the following observation.

3.8. **Proposition.** *Let $i \in \mathbb{N}_0$, let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Then*

$$\nu_M^i \geq -\operatorname{reg}(K^i(M)).$$

Proof. This follows easily from the definition of ν_M^i and by Reminder and Remark 1.8 B)c). □

Next, we introduce the notion of *Canonical Module.*

3.9. **Definition.** Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let $M \neq 0$ be a finitely generated graded R-module. We define the *canonical module* $K(M)$ *of* M as the highest order non vanishing module of deficiency of M, thus:

$$K(M) := K^{\dim_R(M)}(M).$$

Moreover we set $K(0) := 0$

Next, we aim to prove a basic result canonical modules. We begin with a statement on the *Grade of Canonical Modules.* This result hints an important property of the operation of taking canonical modules: namely its "improving effect on grade“. We start by recalling the notion of *Grade.*

3.10. **Reminder.** Let M be a finitely generated module over the Noetherian ring R and let $\mathfrak{a} \subseteq R$ be an ideal of R. Then the *grade*

$$\mathrm{grade}_M(\mathfrak{a})$$

of $\mathfrak{a}$ *with respect to* M is defined as the supremum of lengths r of M-sequences

$$x_1, x_2, \ldots, x_r \in \mathfrak{a} : \quad x_i \in \mathrm{NZD}_R(M/\sum_{j=1}^{i-1} x_j M), \quad \forall i \in \{1, 2, \ldots, r\},$$

in $\mathfrak{a}$. Keep in mind the well known fact (see [8](4.4)(4.6) for example)

$$\mathrm{grade}_M(\mathfrak{a}) = \inf\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M) \neq 0\}.$$

3.11. **Proposition.** *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Then* $\dim_R(K(M)) = \dim_R(M)$ *and moreover*

$$\mathrm{grade}_{K(M)}(R_+) \geq \min\{2, \dim_R(M)\}.$$

Proof. Let $d := \dim_R(M)$. If $d \leq 0$ our claim is obvious. So, let $d > 0$. By Theorem 3.6 e) we know that $K(M)$ is of dimension d. Now set $\bar{M} := M/\Gamma_{R_+}(M)$. Then $\dim_R(\bar{M}) = d$ and hence $K(\bar{M}) = K^d(\bar{M}) \cong K^d(M) = K(M)$ (see Remark 3.4 A)c)). This allows us to replace M by $\bar{M}$ and hence to assume that $\Gamma_{R_+}(M) = 0$. So, by the Homogeneous Prime Avoidance Principle we find some $t \in \mathbb{N}$ and some $x \in R_t \cap \mathrm{NZD}_R(M)$. Now, by the exact sequence Remark 3.4 B)b), applied with $i = d$, we get an epimorphism

$$K^d(M/xM) \to (0 :_{K^d(M)} x) \to 0.$$

As $x \in R_+ \cap \mathrm{NZD}_R(M)$ we also have $\dim_R(M/xM) = d - 1$ and hence $K^d(M/xM) = 0$ (see Theorem 3.6 d)). It follows that $(0 :_{K^d(M)} x) = 0$ and hence $x \in \mathrm{NZD}_R(K^d(M))$. Thus, if $d = 1$, we get our claim. So, let $d > 1$.

Another use of the sequence Remark 3.4 B)b), this time applied with $i = d-1$, yields a monomorphism

$$0 \to \big(K^d(M)/xK^d(M)\big)(t) \to K^{d-1}(M/xM).$$

As $\dim_R(M/xM) = d-1 > 0$ we have $K^{d-1}(M/xM) = K(M/xM)$ and hence by induction we get $\mathrm{grade}_{K^{d-1}(M/xM)}(R_+) > 0$, hence $\Gamma_{R_+}(M/xM) = 0$. Now, the above monomorphism shows that $\Gamma_{R_+}\big((K^d(M)/xK^d(M))(t)\big) = 0$ and hence $\Gamma_{R_+}(K^d(M)/xK^d(M)) = 0$, so that $\mathrm{grade}_{K^d(M)/xK^d(M)}(R_+) \geq 1$. As $x \in R_+ \cap \mathrm{NZD}_R(K^d(M))$ it follows that $\mathrm{grade}_{K^d(M)}(R_+) \geq 2$ and this proves our claim. $\square$

4. Regularity of Modules of Deficiency

Already in Mumford's Lecture Notes [30] the study of the regularity of deficiency modules is called to be of basic significance. In this section, we are precisely concerned with this issue. Our main result is, that the regularity of the deficiency modules $K^i(M)$ of a given finitely generated graded module M over a Noetherian homogeneous K-algebra R is bounded in terms of the cohomology diagonal of M and the beginning of M. We rephrase this in more precise terms: Let $d \in \mathbb{N}$ and let $i \in \mathbb{N}_0$. Then, there is a function

$$G_d^i : \mathbb{N}_0^d \times \mathbb{Z} \to \mathbb{Z}$$

such that for each pair $(R, M) \in \mathcal{M}^d$ we have the estimate

$$\operatorname{reg}\bigl(K^i(M)\bigr) \leq G_d^i\bigl(d_M^0(0), d_M^1(-1), \ldots, d_M^{d-1}(1-d), \operatorname{beg}(M)\bigr),$$

where the number $d_M^j(-j)$ is the j-th geometric cohomological Hilbert function of M evaluated at the argument $-j$, (see Notation and Reminder 1.4 B)).

We begin with some preparations. First we recall the Notion of *Filter-Regular Element*.

4.1. **Definition and Remark.** A) *(Filter-Regular Elements)* Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Let $t \in \mathbb{N}$ and let $x \in R_t$. Then x is said to be *filter-regular with respect to M* if the following equivalent conditions are satisfied:

(i) $x \in \mathrm{NZD}_R(M/\Gamma_{R+}(M))$.
(ii) $x \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M) \cap \mathrm{Proj}(R)} \mathfrak{p}$.
(iii) $(0 :_R x) \subseteq \Gamma_{R_+}(M)$.
(iv) $\operatorname{end}(0 :_R x) < \infty$.
(v) The multiplication map $x : M_n \to M_{n+1}$ is injective for all $n \gg 0$.

B) *(Existence of Filter-Regular Elements)* Let the hypotheses and notations be as in part A). Then, on use of the *Homogeneous Prime Avoidance Principle* we can say:

a) There is some $t_0 \in \mathbb{N}$ such that for each integer $t \geq t_0$ there is an element $x \in R_t$ which is filter-regular with respect to M.
b) If K is infinite, the number t_0 of statement a) may be chosen to be 1.

Here comes a first result about filter-regular elements.

4.2. **Lemma.** *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra, let M be a finitely generated graded R-module and let $x \in R_1$ be a filter-regular element with respect to M. Then*

$$\operatorname{reg}^1(M) \leq \operatorname{reg}(M/xM) \leq \operatorname{reg}(M).$$

Proof. We have two short exact sequences of graded R-modules

$$0 \to (0 :_M x) \to M \to M/(0 :_M x) \to 0,$$

$$0 \to (M/(0 :_M x))(-1) \to M \to M/xM \to 0.$$

As $(0 :_M x)$ is R_+-torsion we get an isomorphism of graded R-modules

$$H^1_{R_+}(M) \cong H^1_{R_+}(M/(0 :_M x)),$$

so that $\operatorname{reg}^1(M/(0 :_M x)) = \operatorname{reg}^1(M)$. Now, if we apply cohomology to the second exact sequence it follows by Reminder and Definition 2.2 A)c),g) that

$$\operatorname{reg}^1(M) = \operatorname{reg}^1\big(M/(0 :_M x)\big) = \operatorname{reg}\big((M/(0 :_M x))(-1)\big) - 1 \leq$$

$$\leq \max\{\operatorname{reg}^1(M), \operatorname{reg}(M/xM) + 1\} - 1,$$

whence $\operatorname{reg}^1(M) \leq \operatorname{reg}(M/xM)$.

By another application of cohomology to the second sequence and observing Reminder and Definition 2.2 A)c),i) we similarly get

$$\operatorname{reg}(M/xM) \leq \max\{\operatorname{reg}^1\big((M/(0 :_M x))(-1)\big) - 1, \operatorname{reg}(M)\} =$$

$$= \max\{\operatorname{reg}^1(M), \operatorname{reg}(M)\} = \operatorname{reg}(M),$$

whence $\operatorname{reg}(M/xM) \leq \operatorname{reg}(M)$. □

Here comes a first application of the previous lemma, which will be of use in the proof of the main result of this section.

4.3. **Proposition.** *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \dots$ be a Noetherian homogeneous K-algebra, let M be a finitely generated graded R-module, let $x \in R_1$ be filter-regular with respect to M and let $m \in \mathbb{Z}$ be such that* $\operatorname{reg}(M/xM) \leq m$ *and* $\operatorname{gendeg}((0 :_M x)) \leq m$*. Then*

$$\operatorname{reg}(M) \leq m + h^0_M(m).$$

Proof. By Lemma 4.2 we have $\operatorname{reg}^1(M) \leq \operatorname{reg}(M/xM) \leq m$. So, it remains to show that

$$\operatorname{end}(H^0_{R_+}(M)) \leq m + h^0_M(m).$$

The short exact sequence of graded R-modules

$$0 \to (M/(0 :_M x))(-1) \to M \to M/xM \to 0$$

induces exact sequences of K-vector spaces

$$0 \to H^0_{R_+}(M/(0 :_M x))_n \to H^0_{R_+}(M)_{n+1} \to H^0_{R_+}(M/xM)_{n+1}$$

for all $n \in \mathbb{Z}$. As $H^0_{R_+}(M/xM)_{n+1} = 0$ for all $n \geq m$, we therefore obtain

$$H^0_{R_+}(M/(0 :_M x))_n \cong H^0_{R_+}(M)_{n+1}, \quad \forall n \geq m.$$

The short exact sequence of graded R-modules

$$0 \to (0 :_M x) \to M \to M/(0 :_M x) \to 0$$

and the facts that

$$H^0_{R_+}\big((0:_M x)\big) = (0:_M x), \text{ and } H^1_{R_+}\big((0:_M x)\big) = 0$$

induce short exact sequences of K-vector spaces

$$0 \to (0:_M x)_n \to H^0_{R_+}(M)_n \to H^0_{R_+}(M/(0:_M x))_n \to 0, \quad \forall n \in \mathbb{Z}.$$

So, for all $n \geq m$ we get an exact sequence of K-vector spaces

$$0 \to (0:_M x)_n \to H^0_{R_+}(M)_n \xrightarrow{\pi_n} H^0_{R_+}(M)_{n+1} \to 0.$$

To prove our claim we may assume that $\operatorname{end}(H^0_{R_+}(M)) > m$. As

$$\operatorname{end}((0:_M x)) = \operatorname{end}\big(H^0_{R_+}(M)\big), \text{ and } \operatorname{gendeg}((0:_M x)) \leq m$$

it follows that

$$(0:_M x)_n \neq 0, \quad \forall n \in \{m, m+1, \ldots, \operatorname{end}(H^0_{R_+}(M))\}.$$

Hence for all these values of n the homomorphism π_n is surjective but not injective. Therefore

$$h^0_M(n) > h^0_M(n+1), \quad \forall n \in \{m, m+1, \ldots, \operatorname{end}(H^0_{R_+}(M))\}.$$

So, in the range $n \geq m$ the function $n \mapsto h^0_M(n)$ is strictly decreasing until it reaches the value 0. Therefore $h^0_M(n) = 0$ for all $n > m + h^0_M(m)$. This proves our claim. $\square$

In the proof of our main result we have to perform a number of induction arguments, which use filter-regular elements of degree 1. In general, such elements only exist if the base field K of our Noetherian homogeneous ring R is infinite (see Definition and Remark 4.1 B)b)). So, we must be able to replace R by an appropriate Noetherian homogeneous algebra over an infinite field. The following remark is aimed to prepare this. For a more detailed presentation of the subject we recommend to consult [5] (10.7).

4.4. **Remark.** A) *(Base Field Extensions of Homogeneous Algebras)* Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let K' be an extension field of K. Then, the K' algebra $R' := K' \otimes_K R$ carries a natural grading, given by

$$R' = K' \otimes_K R = K' \oplus (K' \otimes_K R_1) \oplus (K' \otimes_K R_2) \oplus \ldots,$$

which turns R' into a Noetherian homogeneous K'-algebra with irrelevant ideal $R'_+ = R_+R'$.

B) *(Base Field Extensions and Graded Modules)* Let the notations and hypotheses be as in part A) and assume in addition, that $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a graded R-module. Then, the R'-module $M' := R' \otimes_R M = K' \otimes_K M$ carries a natural grading given by

$$M' = R' \otimes_R M = K' \otimes_K M = \bigoplus_{n\in\mathbb{Z}} K' \otimes_K M_n.$$

Moreover we can say:

a) $\operatorname{gendeg}(M') = \operatorname{gendeg}(M)$.
b) $\operatorname{beg}(M') = \operatorname{beg}(M)$.
c) $\operatorname{end}(M') = \operatorname{end}(M)$.
d) $\dim_{K'}(M'_n) = \dim_K(M_n)$ for all $n \in \mathbb{Z}$.
e) M' is finitely generated over R' if and only if M is finitely generated.

C) *(Base Field Extensions and Local Cohomology)* Keep the notations and hypotheses of parts A) and B). Then, the *Graded Flat Base Change Property of Local Cohomology* (see [13] (14.1.9) or [5] (1.15)) gives rise to isomorphisms of graded R'-modules

$$H^i_{R'_+}(M') \cong R' \otimes_R H^i_{R_+}(M) = K' \otimes_K H^i_{R_+}(M), \quad \forall i \in N_0.$$

If the graded R-module M is finitely generated, we thus can say (see statements a)-e) of part B)):

a) $h^i_{M'}(n) = h^i_M(n)$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.
b) $d^i_{M'}(n) = d^i_M(n)$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.
c) $\operatorname{reg}^l(M') = \operatorname{reg}^l(M)$ for all $l \in \mathbb{N}_0$.

D) *(Base Field Extensions and Deficiency Modules)* Keep the above notations and hypotheses. Then by the fact that taking vector-space duals naturally commutes with field extensions the observations made in part C) imply that there are isomorphisms of graded R-modules

$$K^i(M') \cong R' \otimes_R K^i(M) = K' \otimes_K K^i(M), \quad \forall i \in \mathbb{N}_0.$$

Consequently, for each finitely generated graded R-module M, statement b) of part C) implies

a) $\operatorname{reg}(K^i(M')) = \operatorname{reg}(K^i(M))$ for all $i \in \mathbb{N}_0$.

4.5. **Lemma.** *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Then, for all $i \in \mathbb{N}_0$ and all $n \geq i$ we have*

$$\dim_K\big(K^{i+1}(M)_n\big) \leq \sum_{j=0}^{i} \binom{n-j-1}{i-j} \Big[\sum_{l=0}^{i-j} \binom{i-j}{l} d^{i-l}_M(l-i)\Big].$$

Proof. We only sketch this proof. For more details see [5] (10.4). If K is finite, we chose an infinite extension field K' of K. Then, the observations made in Remark 4.4 allow to replace R and M respectively by $R' := K' \otimes_K R$ and $M' := R' \otimes_R M = K' \otimes_K M$. So we may assume at once that the base field K is infinite. Our first aim is to show the following statement (see [5](8.12)):

a) $d^i_M(-n) \leq \sum_{j=0}^{i} \binom{n-j-1}{i-j} \big[\sum_{l=0}^{i-j} \binom{i-j}{l} d^{i-l}_M(l-i)\big]$, $\forall i \in \mathbb{N}_0, \quad \forall n \geq i.$

By Definition and Remark 4.1 B)b) we find an element $x \in R_1$ which is filter-regular with respect to M. In view of the natural isomorphisms of graded R-modules $D^i_{R_+}(M) \cong D^i_{R_+}(M/\Gamma_{R_+}(M))$ for all $i \in \mathbb{N}_0$ we may replace M by $M/\Gamma_{R_+}(M)$ and hence assume the $x \in \mathrm{NZD}(M)$. The right derived sequence of the functor $D_{R+}(\bullet)$ associated to the short exact sequence of graded R-modules

$$\mathbb{S}: \quad 0 \to M(-1) \xrightarrow{x} M \to M/xM \to 0$$

gives rise to a monomorphism of graded R-modules

$$0 \to D^0_{R_+}(M)(-1) \xrightarrow{x} D^0_{R_+}(M)$$

and exact sequences of graded R-modules

$$D^{i-1}_{R_+}(M) \to D^{i-1}_{R_+}(M/xM) \to D^i_{R_+}(M)(-1) \xrightarrow{x} D^i_{R_+}(M), \quad \forall i \in \mathbb{N}.$$

Consequently we get

b) $d^0_M(-n) \leq d^0_M(-(n-1)), \quad \forall n \in \mathbb{Z}$

and $d^i_M(-n) \leq d^i_M(-(n-1)) + d^{i-1}_{M/xM}(-(n-1))$ for all $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$ and hence

c) $d^i_M(-n) \leq d^i_M(-i) + \sum_{m=i-1}^{n-1} d^{i-1}_{M/xM}(-m), \quad \forall i \in \mathbb{N}, \forall n \in \mathbb{Z}.$

d) $d^j_{M/xM}(-j) \leq d^j_M(-j) + d^{j+1}_M(-(j+1)), \quad \forall j \in \mathbb{N}_0.$

Now, inequality b) proves statement a) if $i = 0$. The inequalities c) and d) together with the Pascal equalities for binomial coefficients allow to prove statement a) by induction on i.

Finally, we show that statement a) implies our lemma. If $i > 0$, by Notation and Reminder 1.4 B)f) we have

$$d^i_M(-n) = h^{i+1}_M(-n).$$

Moreover $h^0_M(-n) \leq \dim_K(M_n)$, whence

$$h^1_M(-n) \leq \dim_K(M_{-n}) - h^0_M(-n) + h^1_M(-n) = d^0_M(-n).$$

As $h^{i+1}_M(-n) = \dim_K(K^{i+1}(M)_n)$ (see Theorem 3.6 b)) our claim follows. $\square$

Now, we define the bounding functions $G^i_d : N_0 \times \mathbb{Z} \to \mathbb{Z}$, which were mentioned already at the beginning of this section.

4.6. **Definition.** *(A Class of Bounding Functions)* For all $d \in \mathbb{N}$ and all $i \in \{0, 1, \ldots, d\}$ we define the functions

$$G^i_d : \mathbb{N}^d_0 \times \mathbb{Z} \to \mathbb{Z}$$

recursively as follows. In the case $i = 0$ we define

(i) $\quad G^0_d(x_0, x_1, \ldots, x_{d-1}, y) := -y.$

In the case $i = 1$ we set:

(ii) $G^1_1(x_0, y) := y - 1$;
(iii) $G^1_d(x_0, x_1, \ldots, x_{d-1}, y) := \max\{0, 1 - y\} + \sum_{i=0}^{d-2} \binom{d-1}{i} x_{d-i-2}$, if $d \geq 2$.

In the case $i = d = 2$ we define

(iv) $G^2_2(x_0, x_1, y) := G^1_2(x_0, x_1, y) + 2$.

Now, assume that $d \geq 3$ and that the functions G^{i-1}_{d-1}, G^i_{d-1} and G^{i-1}_d are already defined. In order to define the function G^i_d we first intermediately introduce the following notation:

(v) $m_i := \max\{G^{i-1}_{d-1}(x_0+x_1, \ldots, x_{d-2}+x_{d-1}, y), G^{i-1}_d(x_0, \ldots, x_{d-1}, y)+1\}+1$.
(vi) $n_i := G^i_{d-1}(x_0 + x_1, ..., x_{d-2} + x_{d-1}, y)$,
(vii) $t_i := \max\{m_i, n_i\}$,
(viii) $\Delta_{ij} := \sum_{l=0}^{i-j-1} \binom{i-j-1}{l} x_{i-l-1}$.

Using these notational conventions, we define

(ix) $G^i_d(x_0, \ldots, x_{d-1}, y) := t_i + \sum_{j=0}^{i-1} \Delta_{ij}, \quad \forall i \in \{2, 3, \ldots, d-1\}$.

Finally, if $d \geq 3$ and G^{d-1}_{d-1} and G^{d-1}_d are already defined, we set (see (v))

(x) $G^d_d(x_0, \ldots, x_{d-1}, y) := m_d$.

In order to prove our main result, we need a particular property of the previously defined bounding function

4.7. **Remark.** *(Monotonicity of the Bounding Functions G^i_d)* Let $d \in \mathbb{N}_0$, let $i \in \{0, 1, \ldots, d\}$ and let

$$(x_0, x_1, \ldots, x_{d-1}, y), \quad (x'_0, x'_1, \ldots, x'_{d-1}, y') \in \mathbb{N}_0^d \times \mathbb{Z}$$

such that

$$x_j \leq x'_j, \quad \forall j \in \{0, 1, \ldots, d-1\}, \quad y' \leq y.$$

It is easy to see by induction on i and d, that under these circumstances we have

$$G^i_d(x_0, x_1, \ldots, x_{d-1}, y) \leq G^i_d(x'_0, x'_1, \ldots, x'_{d-1}, y').$$

The following remark generalizes an argument made in the proof of Lemma 4.5.

4.8. **Remark.** Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra, let M be a finitely generated graded R-module, let $t \in \mathbb{N}$ and let $x \in R_t$ be filter-regular with respect to M. Then, it is easy to verify, that for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ we have the inequality

$$d^i_{M/xM}(n) \leq d^i_M(n) + d^{i+1}_M(n-t).$$

Now, we are ready to formulate and to prove the announced main result.

4.9. **Theorem.** *Let $d \in \mathbb{N}$, let $i \in \{0, 1, \dots, d\}$, let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \dots$ be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module with* $\dim_R(M) = d$. *Then*

$$\operatorname{reg}\big(K^i(M)\big) \leq G_d^i\big(d_M^0(0), d_M^1(-1), \dots, d_M^{d-1}(1-d), \operatorname{beg}(M)\big).$$

Proof. We proceed by induction on i. By Theorem 3.6 e) we have $\dim_R(K^0(M)) \leq 0$. So, in view of Theorem 3.6 b) we get

$$\operatorname{reg}(K^0(M)) = \operatorname{end}(K^0(M)) = -\operatorname{beg}(H^0_{R_+}(M)) \leq -\operatorname{beg}(M) =$$

$$G_d^0\big(d_M^0(0), d_M^1(-1), \dots, d_M^{d-1}(1-d), \operatorname{beg}(M)\big).$$

This clearly proves the case $i = 0$.

So let $i > 0$. As in the proof of Lemma 4.5 we may use the observations made in Remark 4.4 to assume that K is infinite.

Let $\bar{M} := M/\Gamma_{R_+}(M)$. Then $\dim_R(\bar{M}) = d$, and in view of the natural isomorphisms of graded R-modules $D^i_{R_+}(M) \cong D^i_{R_+}(\bar{M})$ we have $d^j_{\bar{M}}(n) = d^j_M(n)$ for all $j \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. In addition $\operatorname{beg}(M) \leq \operatorname{beg}(\bar{M})$, whence by Remark 4.7 we get

$$G_d^i\big(d_{\bar{M}}^0(0), d_{\bar{M}}^1(-1) \dots, d_{\bar{M}}^{d-1}(1-d), \operatorname{beg}(\bar{M})\big) \leq$$

$$\leq G_d^i\big(d_M^0(0), d_M^1(-1), \dots, d_M^{d-1}(1-d), \operatorname{beg}(M)\big).$$

As moreover we have an isomorphism of graded R-modules $K^i(\bar{M}) \cong K^i(M)$ (see Remark 3.4 A)c)), we thus may replace M by $\bar{M}$ and hence assume that $\Gamma_{R_+}(M) = 0$. Therefore we find some element $x \in R_1 \cap \operatorname{NZD}_R(M)$. By Homogeneous Prime Avoidance we may assume in addition, that x is filter-regular with respect to the modules $K^0(M), K^1(M), \dots K^d(M)$. By Remark 3.4 B)b) there is an exact sequence of graded R-modules

a) $0 \to \big(K^{j+1}(M)/xK^{j+1}(M)\big)(+1) \to K^j(M/xM) \to (0 :_{K^j(M)} x) \to 0,$

for all $j \in \mathbb{N}_0$. Since $H^0_{R_+}(M) = 0$ we have $K^0(M) = 0$ (see Theorem 3.6 b) for example), so that the sequence a) gives rise to an isomorphism of graded R-modules

b) $\big(K^1(M)/xK^1(M)\big)(+1) \cong K^0(M/xM).$

As $\dim_R(K^0(M/xM)) \leq 0$ (see Theorem 3.6 e)), the above isomorphism shows that $K^1(M)/xK^1(M)$ is R_+-torsion, so that (see Theorem 3.6 b))

$$\operatorname{reg}\big(K^1(M)/xK^1(M)\big) = \operatorname{reg}\big(K^0(M/xM)\big) + 1 = \operatorname{end}\big(K^0(M/xM)\big) + 1 =$$

$$1 - \operatorname{beg}\big(H^0_{R_+}(M/xM)\big) \leq 1 - \operatorname{beg}(M/xM) \leq 1 - \operatorname{beg}(M).$$

It follows that

c) $\operatorname{reg}\big(K^1(M)/xK^1(M)\big) \leq 1 - \operatorname{beg}(M).$

We first assume that $d = 1$. Then clearly $i = 1$, whence $K^i(M) = K^1(M) = K(M)$ so that by Proposition 3.11 we get $\mathrm{grade}_{K^1(M)}(R_+) = 1$ hence $H^0_{R_+}(K^1(M)) = 0$, so that $\mathrm{reg}(K^1(M)) = \mathrm{reg}^1(K^1(M))$. It follows that (see Lemma 4.2)

$$\mathrm{reg}\big(K^1(M)\big) \leq \mathrm{reg}\big(K^1(M)/xK^1(M)\big) \leq 1 - \mathrm{beg}(M) = G^1_1(d^0_M(0), \mathrm{beg}(M)\big).$$

This proves our claim if $d = 1$.

So, assume from now on, that $d \geq 2$. We first treat the case $i = 1$. To do so, we consider the sequence a) for $j = 1$, hence

d) $0 \to \big(K^2(M)/xK^2(M)\big)(+1) \to K^1(M/xM) \to (0 :_{K^1(M)} x) \to 0.$

If $d = 2$, we have $\dim_R(M/xM) = 1$ and so by the already treated case $d = 1$ we get

$$\mathrm{reg}\big(K^1(M/xM)\big) \leq 1 - \mathrm{beg}(M/xM) \leq 1 - \mathrm{beg}(M).$$

Consequently by Reminder and Definition 2.2 A)e) we have

$$\mathrm{gendeg}\big((0 :_{K^1(M)} x)\big) \leq \mathrm{gendeg}\big(K^1(M/xM)\big) \leq$$
$$\leq \mathrm{reg}\big(K^1(M/xM)\big) \leq 1 - \mathrm{beg}(M).$$

Assume first that $m_0 := 1 - \mathrm{beg}(M) \leq 0$. Then, by Proposition 4.3 (applied with $m = 0$) we obtain (see Theorem 3.6 b))

$$\mathrm{reg}\big(K^1(M)\big) \leq 0 + h^0_{K^1(M)}(0) \leq \dim_K\big(K^1(M)_0\big) = h^1_M(0) \leq d^0_M(0).$$

Now, assume that $m_0 := 1 - \mathrm{beg}(M) > 0$. Then $d^0_M(-m_0) \leq d^0_M(0)$ (see statement b) in the proof of Lemma 4.5). So by statement c), by Proposition 4.3 and by Theorem 3.6 b) we get

$$\mathrm{reg}\big(K^1(M)\big) \leq m_0 + h^0_{K^1(M)}(m_0) \leq m_0 + \dim_K(K^1(M)_{m_0}) =$$
$$= 1 - \mathrm{beg}(M) + h^1_M(-m_0) \leq 1 - \mathrm{beg}(M) + d^0_M(-m_0) \leq 1 - \mathrm{beg}(M) + d^0_M(0).$$

Therefore, bearing in mind Definition 4.6 (iii) we finally obtain

$$\mathrm{reg}\big(K^1(M)\big) \leq \max\{d^0_M(0), 1 - \mathrm{beg}(M) + d^0_M(0)\} \leq$$
$$\leq \max\{0, 1 - \mathrm{beg}(M)\} + d^0_M(0) = G^1_2\big(d^0_M(0), d^1_M(-1), \mathrm{beg}(M)\big).$$

This proves the case in which $d = 2$ and $i = 1$.

Now, let $d \geq 3$, but still let $i = 1$. Then, by induction on d we may write (see Definition 4.6 (iii))

$$\mathrm{reg}\big(K^1(M/xM)\big) \leq G^1_{d-1}\big(d^0_{M/xM}(0), \ldots, d^{d-2}_{M/xM}(2-d), \mathrm{beg}(M/xM)\big) =$$
$$= \max\{0, 1 - \mathrm{beg}(M/xM)\} + \sum_{i=0}^{d-3} \binom{d-2}{i} d^{d-i-3}_{M/xM}(i+3-d).$$

According to Remark 4.8 we have

$$d^{d-i-3}_{M/xM}(i+3-d) \leq d^{d-i-3}_M(i+3-d) + d^{d-i-2}_M(i+2-d), \quad \forall i \in \{0, 1, \ldots, d-3\}.$$

Therefore we obtain

$$\mathrm{reg}\big(K^1(M/xM)\big) \leq$$

$$\le \max\{0, 1-\mathrm{beg}(M)\} + \sum_{i=0}^{d-3} \binom{d-2}{i} \big[d_M^{d-i-3}(i+3-d) + d_M^{d-i-2}(i+2-d)\big] =: t_0.$$

By the exact sequence d) and Reminder and Definition 2.2 A)e) we now get

$$\mathrm{gendeg}\big((0 :_{K^1(M)} x)\big) \le \mathrm{reg}\big(K^1(M/xM)\big) \le t_0.$$

By the above inequality c) and the definition of t_0 we have

$$\mathrm{reg}\big(K^1(M)/xK^1(M)\big) \le t_0.$$

As $t_0 \ge 0$ we also have $d_M^0(-t_0) \le d_M^0(0)$. So, by Proposition 4.3 and Theorem 3.6 b) we obtain the inequalities

$$\mathrm{reg}\big(K^1(M)\big) \le t_0 + h^0_{K^1(M)}(t_0) \le t_0 + \dim_K\big(K^1(M)_{t_0}\big) =$$

$$= t_0 + h^1_M(-t_0) \le t_0 + d_M^0(-t_0) \le t_0 + d_M^0(0) =$$

$$= \max\{0, 1-\mathrm{beg}(M)\} + \sum_{i=0}^{d-3} \binom{d-2}{i} \big[d_M^{d-i-3}(i+3-d) + d_M^{d-i-2}(i+2-d)\big] + d_M^0(0) =$$

$$= \max\{0, 1-\mathrm{beg}(M)\} + d_M^{d-2}(2-d) + \sum_{i=1}^{d-3} \Big[\binom{d-2}{i-1} + \binom{d-2}{i}\Big] d_M^{d-i-2}(i+2-d) +$$

$$+ (d-2)d_M^0(0) + d_M^0(0) =$$

$$= \max\{0, 1-\mathrm{beg}(M)\} + \sum_{i=0}^{d-3} \binom{d-1}{i} d_M^{d-i-2}(i+2-d) + (d-1)d_M^0(0) =$$

$$= \max\{0, 1-\mathrm{beg}(M)\} + \sum_{i=0}^{d-2} \binom{d-1}{i} d_M^{d-i-2}(i+2-d).$$

In view of Definition 4.6 (iii) this means that

$$\mathrm{reg}\big(K^1(M)\big) \le G_d^1\big(d_M^0(0), d_M^1(-1), \ldots, d_M^{d-1}(1-d), \mathrm{beg}(M)\big).$$

So, we have settled the case $i = 1$ for all $d \in \mathbb{N}$.

We now attack the cases with $i \ge 2$. We begin with the case in which $d = 2$ and hence $i = 2$. In view of the exact sequence d) we obtain (see Reminder and Definition 2.2 A)c),f))

$$\mathrm{reg}\big(K^2(M)/xK^2(M)\big) \le \max\{\mathrm{reg}\big(K^1(M/xM)\big), \mathrm{reg}\big((0 :_{K^1(M)} x)\big) + 1\} + 1.$$

Observe that $\dim_R(M/xM) = 1$, so that by what we know from the already treated case $i = d = 1$ we get

$$\mathrm{reg}\big(K^1(M/xM)\big) \le G_1^1\big(d^0_{M/xM}(0), \mathrm{beg}(M/xM)\big) =$$

$$= \mathrm{beg}(M/xM) - 1 \le \mathrm{beg}(M) - 1.$$

As x is filter-regular with respect to $K^1(M)$, we have $(0 :_{K^1(M)} x) \subseteq H^0_{R_+}(M)$, so that

$$\mathrm{reg}\big((0 :_{K^1(M)} x)\big) = \mathrm{end}\big((0 :_{K^1(M)} x)\big) \le \mathrm{end}\big(H^0_{R_+}(K^1(M))\big) \le \mathrm{reg}\big(K^1(M)\big).$$

By what we know from the already treated case with $i = 1$ and $d = 2$ we have
$\operatorname{reg}\big(K^1(M)\big) \leq G_2^1\big(d_M^0(0), d_M^1(-1), \operatorname{beg}(M)\big) = \max\{0, 1-\operatorname{beg}(M)\} + d_M^0(0)$.
Therefore we get

$$\operatorname{reg}\big(K^2(M)/xK^2(M)\big) \leq \max\{1-\operatorname{beg}(M), \max\{0, 1-beg(M)\}+d_M^0(0)+1\}+1$$
$$\leq \max\{0, 1-\operatorname{beg}(M)\} + d_M^0(0) + 2.$$

As $\operatorname{grade}_{K^2(M)}(R_+) = \operatorname{grade}_{K(M)}(R_+) \geq \min\{2, d\} = 2 = d$ (see Proposition 3.11) we have $\operatorname{grade}_{K^2(M)}(R_+) = 2$, whence $H_{R_+}^j(K^2(M)) = 0$ for $j = 0, 1$. This means that $\operatorname{reg}(K^2(M)) = \operatorname{reg}^1(K^2(M))$. So by Lemma 4.2 we obtain

$$\operatorname{reg}\big(K^2(M)\big) \leq \operatorname{reg}\big(K^2(M)/xK^2(M)\big) \leq$$
$$\leq \max\{0, 1-\operatorname{beg}(M)\} + d_M^0(0) + 2 = G_2^2\big(d_M^0(0), d_M^1(-1), \operatorname{beg}(M)\big).$$

This completes our proof in the cases with $i \geq 2$ and $d = 2$.

So, let $d > 2$ and $i \geq 2$. By Remark 4.8 we have

$$d_{M/xM}^j(-j) \leq d_M^j(-j) + d_M^{j+1}(-j-1), \quad \forall j \in \mathbb{N}_0.$$

Let $k \in \{0, 1, \ldots, d-1\}$. Then, by induction on d and in view of Remark 4.7 we have

$$\operatorname{reg}\big(K^k(M/xM)\big) \leq G_{d-1}^k\big(d_{M/xM}^0(0), \ldots, d_{M/xM}^{d-2}(2-d), \operatorname{beg}(M/xM)\big) \leq$$
$$\leq G_{d-1}^k\big(d_M^0(0) + d_M^1(-1), \ldots, d_M^{d-2}(2-d) + d_M^{d-1}(1-d), \operatorname{beg}(M)\big) =: n_k.$$

Therefore

e) $\operatorname{reg}\big(K^k(M/xM)\big) \leq n_k$ for all $k \in \{0, 1, \ldots, d-1\}$.

Clearly, by induction on i we have

f) $\operatorname{reg}\big(K^{i-1}(M)\big) \leq G_d^{i-1}\big(d_M^0(0), d_M^1(-1), \ldots, d_M^{d-1}(1-d), \operatorname{beg}(M)\big) =: v_{i-1}$.

If we apply the exact sequence a) with $j = i-1$ we get (see Reminder and Definition 2.2 A)c),f))

$$\operatorname{reg}\big(K^i(M)/xK^i(M)\big) \leq \max\{\operatorname{reg}\big(K^{i-1}(M/xM)\big), \operatorname{reg}\big((0:_{K^{i-1}(M)} x)\big)+1\}+1.$$

By the inequality e) we have

$$\operatorname{reg}\big(K^{i-1}(M/xM)\big) \leq n_{i-1}.$$

Moreover, as x is filter-regular with respect to $K^{i-1}(M)$ we have once more $\operatorname{reg}\big((0:_{K^{i-1}(M)} x)\big) \leq \operatorname{end}\big(H_{R_+}^0(K^{i-1}(M))\big) \leq \operatorname{reg}\big(K^{i-1}(M)\big)$, so that by the inequality f) we have

$$\operatorname{reg}\big((0:_{K^{i-1}(M)} x)\big) \leq v_{i-1}.$$

Thus, gathering together, we we obtain (see Definition 4.6 (v)):

g) $\operatorname{reg}\big(K^i(M)/xK^i(M)\big) \leq \max\{n_{i-1}, v_{i-1}+1\} + 1 = m_i$.

Assume first, that $2 \leq i \leq d-1$. By the definitions of v_i and n_i (see Definition 4.6 (v),(vi)) it follows easily

$$t_i := \max\{m_i, n_i\} \geq i.$$

Moreover, if we apply the sequence a) with $j = i$ and keep in mind the inequality e) we get (see also Reminder and Definition 2.2 A)e))

$$\operatorname{gendeg}\big((0 :_{K^i(M)} x)\big) \leq \operatorname{reg}\big(K^i(M/xM)\big) \leq n_i.$$

So, by Proposition 4.3, applied to the graded R-module $K^i(M)$ with $m := t_i$ and with Lemma 4.5 applied with $n = t_i$ and with $i-1$ instead of i we obtain

$$\operatorname{reg}\big(K^i(M)\big) \leq t_i + h^0_{K^i(M)}(t_i) \leq t_i + \dim_K\big(K^i(M)_{t_i}\big) \leq$$

$$\leq t_i + \sum_{j=0}^{i-1} \binom{t_i - j - 1}{i - j - 1} \big[\sum_{l=0}^{i-j-1} \binom{i-j-1}{l} d_M^{i-l-1}(l - i + 1) \big].$$

In view of Definition 4.6 (viii),(ix) this means that

$$\operatorname{reg}\big(K^i(M)\big) \leq G_d^i\big(d_M^0(0), d_M^1(-1), \ldots, d_M^{d-1}(1-d), \operatorname{beg}(M)\big).$$

This completes our proof in the cases with $i \leq d-1$.

It remains to treat the cases with $i = d > 2$. Observe that by Proposition 3.11 we have $\operatorname{grade}_{K^d(M)}(R_+) = 2$, so that again $\operatorname{reg}\big(K^d(M)\big) = \operatorname{reg}^1\big(K^d(M)\big)$. Keep in mind, that x is filter-regular with respect to $K^d(M)$. So, if we apply Lemma 4.2 to this latter module and bear in mind the previous inequality g) we obtain

$$\operatorname{reg}\big(K^d(M)\big) \leq \operatorname{reg}\big(K^d(M)/xK^d(M)\big) \leq m_d.$$

In view of Definition 4.6 (x) this means that

$$\operatorname{reg}\big(K^d(M)\big) \leq G_d^d\big(d_M^0(0), d_M^1(-1), \ldots, d_M^{d-1}(1-d), \operatorname{beg}(M)\big).$$

This completes our proof. $\square$

4.10. **Corollary.** *Let $d \in \mathbb{N}$, and let $x_0, x_1, \ldots, x_{d-1} \in \mathbb{N}_0$ and $y \in \mathbb{Z}$. Then for each pair $(R, M) \in \mathcal{M}^d$ such that*

$$d_M^j(-j) \leq x_j \quad \forall j \in \{0, 1, \ldots, d-1\}, \quad \operatorname{beg}(M) \geq y$$

it holds

$$\operatorname{reg}\big(K^i(M)\big) \leq G_d^i(x_0, x_1, \ldots, x_{d-1}, y), \quad \forall i \in \{0, 1, \ldots, d\}.$$

Proof. This is immediate by Theorem 4.9 and Remark 4.7. $\square$

4.11. **Remark.** A) *(Around Regularity of Deficiency Modules)* (see [10]) The main result Theorem 4.9 of the present section and its consequence Corollary 4.10 may be proved in a more general context. These results namely hold over all Noetherian homogeneous rings $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ with Artinian local base Ring R_0. Clearly, in this general setting, the notion of deficiency module has to be defined in a different way. Moreover, these results have a number of further applications, which we quote here in simplified form. We always assume that R is as above, that M is a finitely generated graded R-module and that $i \in \mathbb{N}_0$.

a) The invariant $\operatorname{reg}(K^i(M))$ is bounded in terms of the three invariants $\operatorname{beg}(M)$, $\operatorname{reg}^2(M)$ and $P_M(\operatorname{reg}^2(M))$.
b) If $\mathfrak{a} \subseteq R$ is a graded ideal, then $\operatorname{reg}(K^i(\mathfrak{a}))$ and $\operatorname{reg}(K^i(R/\mathfrak{a}))$ can be bounded in terms of the three invariants $\operatorname{reg}^2(\mathfrak{a})$, $\operatorname{length}(R_0)$, $\operatorname{reg}^1(R)$ and the number of generating one-forms of R.

B) *(The Case of Polynomial Rings)* In the particular case, where

$$R = R_0[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d]$$

is a standard graded polynomial ring over the Artinian local ring R_0, we have in addition the following statements.

c) If $U \neq 0$ is a finitely generated graded R-module and $M \subseteq U$ is a graded submodule, then $\operatorname{reg}(K^i(M))$ and $\operatorname{reg}(K^i(U/M))$ are bounded in terms of d, $\operatorname{length}(R_0)$, $\operatorname{beg}(U)$, $\operatorname{reg}(U)$, the number $\dim_{R_0/\mathfrak{m}_0}(U/(\mathfrak{m}_0 + R_+)U)$ of generators of U and the generating degree $\operatorname{gendeg}(M)$ of M.
d) If U and M are as in statement c), then $\operatorname{reg}(K^i(M))$ and $\operatorname{reg}(K^i(U/M))$ are bounded in terms of $\operatorname{length}(R_0)$, $\operatorname{beg}(U)$, $\operatorname{reg}(U)$, the Hilbert polynomial P_U of U and the Hilbert polynomial P_M of M.
e) It $p : F \to M \to 0$ is an epimorphism of graded R-modules such that F is free and of finite rank r, then $\operatorname{reg}(K^i(M))$ is bounded in terms of d, r, $\operatorname{length}(R_0)$, $\operatorname{beg}(F)$, $\operatorname{gendeg}(F)$ and $\operatorname{gendeg}(\operatorname{Ker}(p))$.

C) *(Presentation Matrices)* Keep the notations and hypotheses of part B), but assume in addition, that the base ring R_0 is a field. Let

$$\bigoplus_{j=1}^{s} R(-\beta_j) \xrightarrow{q} \bigoplus_{i=1}^{r} R(-\alpha_i) \longrightarrow M \longrightarrow 0$$

$$\alpha_1 \leq \alpha_2 \leq \cdots, \leq \alpha_r, \quad \beta_1 \leq \beta_2 \leq \cdots, \leq \beta_s$$

$$\big(\text{with } r, s \in \mathbb{N}, \text{ and } \alpha_i, \beta_j \in \mathbb{Z} \text{ for all } i = 1, 2, \ldots r \text{ and all } j = 1, 2, \ldots s\big)$$

be a *free graded presentation* of the finitely generated graded R-module M with *presentation matrix* A. So, we have

$$A = [a_{ij} \mid i = 1, 2, \ldots, r; j = 1, 2, \ldots, s] \in R^{s,r},$$

$$\text{with } a_{i,j} \in R_{\beta_j - \alpha_i}, \big(i = 1, 2, \ldots, r, j = 1, 2, \ldots, s\big)$$

– and the map q is given by

$$\begin{pmatrix} f_1 \\ f_2 \\ \\ \\ f_s \end{pmatrix} \mapsto \begin{pmatrix} \sum_{j=1}^{s} a_{1,j} f_j \\ \sum_{j=1}^{s} a_{2,j} f_j \\ \\ \\ \sum_{j=1}^{s} a_{r,j} f_j \end{pmatrix} \qquad \big(f_1, f_2, \ldots, f_s \in R\big).$$

Now, it follows from statement e) of part B), that the numbers $\operatorname{reg}(K^i(M))$ are bounded in terms of the number d of indeterminates of our polynomial ring, the number of rows r of the matrix A and the two degrees α_1, α_r and the maximal degree of all entries of A. This is particularly important from the point of view of *Computational Algebraic Geometry*. Namely, in the setting of this theory, graded modules over the polynomial ring R are given by the strings of numbers

$$\underline{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{Z}^r \text{ and } \underline{\beta} := (\beta_1, \beta - 2, \ldots, \beta_s) \in \mathbb{Z}^s$$

and the presentation matrix A. Our result says in particular, that the invariants $\operatorname{reg}\big(K^i(M)\big)$ are bounded in terms of d and of the two strings $\underline{\alpha}$ and $\underline{\beta}$. Actually, to bound the regularities in questions, only the invariants d, r, α_1, α_r and β_s must be known. The strings $\underline{\alpha}$ and $\underline{\beta}$ fix a discrete skeleton for the module M, whereas the entries $a_{i,j}$ encode the continuous data of M. So, the invariants $\operatorname{reg}\big(K^i(M)\big)$ are bounded by the discrete skeleton of M.
What we have just observed is particularly appealing, if M is a cyclic R-module generated in degree 0 and hence of the form $M = R/\mathfrak{a}$, where $\mathfrak{a} \subsetneq R$ is a (non-zero) proper graded ideal. In this case, we namely may chose

$$r = 1, \alpha_1 = 0 \text{ and } A = (a_1, a_2, \ldots, a_s),$$

where $(a_j)_{j=1}^s$ is a (minimal) homogeneous system of generators of the ideal $\mathfrak{a}$. Now, the invariants $\operatorname{reg}\big(K^i(R/\mathfrak{a})\big)$ are bounded only in terms of d and the generating degree

$$\operatorname{gendeg}(\mathfrak{a}) = \max\{\deg(a_j) \mid j = 1, 2, \ldots, s\}$$

of $\mathfrak{a}$.

5. Bounding Cohomology

Now, we return to our original aim, which was to give criteria allowing to specify classes $\mathcal{D} \subseteq \mathcal{S}^t$ of finite cohomology for a given positive integer t. We begin with a bound for the cohomological postulation numbers

$$\nu_M^i := \inf\{n \in \mathbb{Z} \mid p_M^i(n) \neq h_M^i(n)\}$$

of a finitely generated graded module M over a Noetherian homogeneous K-algebra R, as they were introduced in Remark and Definition 3.7 C). The notations are the same as in Section 4.

5.1. **Proposition.** *Let $d \in \mathbb{N}$, let $i \in \{0, 1, \dots, d-1\}$, let $x_0, x_1 \dots, x_{d-1} \in \mathbb{N}_0$, let $y \in \mathbb{Z}$, let $(R, M) \in \mathcal{M}^d$ such that*

$$d_M^j(-j) \leq x_j \quad \forall i \in \{0, 1, \dots, d-1\}, \quad \operatorname{beg}(M) \geq y.$$

Then

$$\nu_M^i \geq -G_d^i(x_0, x_1, \dots, x_{d-1}, y).$$

Proof. This is immediate by Corollary 4.10 and Proposition 3.8. $\square$

We first aim to apply this result in she sheaf theoretic context using the concept of postulation number of sheaf as defined in Remark and Definition 3.7 E). To this end we now introduce some appropriate notation.

5.2. **Notation.** Let $t \in \mathbb{N}_0$ and let $i \in \{0, 1, \dots, t\}$. We define the *bounding function*

$$L_t^i : \mathbb{N}_0^{t+1} \to \mathbb{Z}$$

by setting

$$L_t^i(x_0, x_1, \dots, x_t) := -G_{t+1}^{i+1}(x_0, x_1, \dots, x_t, 0), \quad \forall x_0, x_1, \dots, x_t \in \mathbb{N}_0,$$

where the function

$$G_{t+1}^{i+1} : \mathbb{N}_0^t \times \mathbb{Z} \to \mathbb{Z}$$

is defined according to Definition 4.6.

Now, we are ready to formulate and to prove our first main application of Proposition 5.10, which says that the *cohomology diagonal of a coherent sheaf $\mathcal{F}$ over a projective K-scheme X bounds the cohomological postulation numbers of the sheaf $\mathcal{F}$.*

5.3. **Theorem.** *Let $t \in \mathbb{N}_0$, let $i \in \{0, 1, \dots, t\}$, let $x_0, x_1, \dots, x_t \in \mathbb{N}_0$, and let $(X, \mathcal{F}) \in \mathcal{S}^t$ such that*

$$h_{\mathcal{F}}^j(-j) = h^i(X, \mathcal{F}(-j)) \leq x_j \quad \forall j \in \{0, 1, \dots, t\}.$$

Then

$$\nu_{\mathcal{F}}^i \geq L_t^i(x_0, x_1, \dots, x_t).$$

Proof. We chose a pair $(R, M) \in \mathcal{M}^{t+1}$, such that $(X, \mathcal{F}) = (\mathrm{Proj}(R), \widetilde{M})$. As $\widetilde{M} = \widetilde{M_{\geq 0}}$ we may replace M by $M_{\geq 0}$ and hence assume that

$$\mathrm{beg}(M) \geq 0.$$

Keep in mind that $\dim_R(M) = t + 1$ and that by Notation and Reminder 1.5 C) we have

$$d_M^j(-j) = h^j(X, \mathcal{F}(-j)) \leq x_j, \quad \forall j \in \{0, 1, \ldots, t\}.$$

So, we may apply Proposition 5.10 with $y = 0$ and with $i + 1$ instead of i and obtain

$$\nu_M^{i+1} \geq -G_{t+1}^{i+1}(x_0, x_1, \ldots, x_t, 0) = L_t^i(x_0, x_1 \ldots, x_t).$$

By Remark and Definition 3.7 E)a) we have in addition that $\nu_{\mathcal{F}}^i = \nu_M^{i+1}$ provided that $i > 0$. In these cases we therefore have our claim. So, it remains to consider the case $i = 0$. By Remark and Definition 3.7 E)b) and the previous estimate we have

$$\nu_{\mathcal{F}}^0 \geq \min\{\nu_M^1, 0\} \geq \min\{L_s^0(x_0, x_1, \ldots, x_s), 0\}.$$

According to Definition 4.6 (iii) we have

$$L_s^0(x_0, x_1, \ldots, x_s) = -G_{s+1}^1(x_0, x_1, \ldots, x_s, 0) < 0,$$

so that indeed $\nu_{\mathcal{F}}^0 \geq L_s^0(x_0, x_1, \ldots, x_s)$, as requested. □

In order to draw conclusions from the previous estimate we need a further result, which was originally shown in [12]. In these lectures, we will not prove it. For a complete and self-contained presentation we recommend to consider Section 8 of [5].

5.4. **Theorem.** *Let $t \in \mathbb{N}_0$ and let $(X, \mathcal{F}) \in \mathcal{S}^t$. Then*

a) $\mathrm{reg}(\mathcal{F}) \leq \left(2 \sum_{i=1}^t \binom{t-1}{i-1} h_{\mathcal{F}}^i(-i)\right)^{2^{t-1}} =: B$.

b) $\sum_{i=1}^t \binom{t-1}{i-1} h_{\mathcal{F}}^i(n - i) \leq \frac{B}{2}$ *for all* $n \in \mathbb{N}_0$.

Now, combining Theorem 5.3 and Theorem 5.4 and using the terminology introduced in Section 1, we get the following finiteness result:

5.5. **Theorem.** *Let $t \in N_0$, let $r \in \mathbb{Z}$ and let $x_0, x_1, \ldots, x_t \in \mathbb{N}_0$. Then the class*

$$\mathcal{D} = \mathcal{D}_{x_0, x_1, \ldots, x_t} := \{(X, \mathcal{F}) \in \mathcal{S}^t \mid h_{\mathcal{F}}^i(r - i) \leq x_i, \quad \forall i \leq t\}$$

is of finite cohomology.

Proof. After twisting we may assume that $r = 0$. Let B as in Theorem 5.4 a) and set

$$C := \min\{-t, \min\{-L_t^i(x_0, x_1, \ldots, x_t) \mid i = 0, 1, \ldots, t\}\}.$$

According to Theorem 5.4 a),b) the class $\mathcal{D}$ is of finite cohomology on the set

$$\mathbb{S} := \{(i, n) \mid 1 \leq i \leq t, \quad n \geq -i\}.$$

Moreover, the inequality a) proved in the current of the proof of Lemma 4.5 together with the observations made in Notation and Reminder 1.5 C) yields that

a) $h^i_{\mathcal{F}}(n) \leq \sum_{j=0}^{i} \binom{-n-j-1}{i-j} \big[\sum_{k=j}^{i-j} \binom{i-j}{k-j} x_k \big]$ for all $i \in \mathbb{N}_0$, all $n \leq -i$ and all pairs $(X, \mathcal{F}) \in \mathcal{D}$.

This implies that the class $\mathcal{D}$ is of finite cohomology on the set

$$\mathbb{T} := \{(i,n) \mid 0 \leq i \leq t, C - t - 2 \leq n \leq -i\}.$$

By Theorem 5.4 we have $\nu^i_{\mathcal{F}} \geq C$ and hence

b) $p^i_{\mathcal{F}}(n) = h^i_{\mathcal{F}}(n)$ for all $(X, \mathcal{F}) \in \mathcal{D}$, for all $i \in \mathbb{N}_0$ and for all $n \leq C - 1$.

In particular, for all $(i, n) \in \mathbb{T}$ we have $p^i_{\mathcal{F}}(n) = h^i_{\mathcal{F}}(n)$. As $\mathcal{D}$ is of finite cohomology on $\mathbb{T}$ and all polynomials $p^i_{\mathcal{F}}$ are of degree at most t, it follows:

c) The set $\{p^i_{\mathcal{F}} \mid 0 \leq i \leq t, (X, \mathcal{F}) \in \mathcal{D}\}$ is finite.

But now another use of the previously observed coincidence b) of cohomological Hilbert functions $h^i_{\mathcal{F}}$ and cohomological Serre polynomials $p^i_{\mathcal{F}}$ in the range $n \leq C - 1$ for all $(X, \mathcal{F}) \in \mathcal{D}$, it follows that the family $\mathcal{D}$ is also of finite cohomology on the set

$$\mathbb{U} := \{(i,n) \mid 0 \leq i \leq t, \quad n \leq -i\}.$$

It thus remains to show that the class $\mathcal{D}$ is of finite cohomology on the set

$$\mathbb{V} := \{0\} \times \mathbb{N}.$$

Observe first, that by statement c) and by Remark and Definition 3.7 D)d) the set of Serre-polynomials $\{P_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{D}\}$ is finite. So, from Reminder and Remark 1.8 C)c) and Theorem 5.4 a) it follows that the set of restricted functions

$$\{h^0_{\mathcal{F}} \restriction_{\mathbb{Z}_{>B}} : \mathbb{Z}_{>B} \to \mathbb{N}_0 \mid (X, \mathcal{F}) \in \mathcal{D}\}$$

is finite. But this implies, that the class $\mathcal{D}$ is of finite cohomology on the set

$$\mathbb{W} := \{0\} \times \mathbb{Z}_{>B}.$$

It remains to be shown that the class $\mathcal{D}$ is of finite cohomology on the finite set

$$\mathbb{I} := \mathbb{V} \setminus \mathbb{W} = \{0\} \times \{1, 2, \ldots, B\}.$$

This means to show that the set of restricted functions

$$\mathcal{H} := \{h^0_{\mathcal{F}} \restriction_{\{1,2,\ldots,B\}} : \{1, 2, \ldots\} \to \mathbb{N}_0 \mid (X, \mathcal{F}) \in \mathcal{D}\}$$

is finite. As in statement b) in the proof of Lemma 4.5 we see that $d^0_M(n-1) \leq d^0_M(n)$ for all $n \in \mathbb{Z}$ and all $(R, M) \in \mathcal{M}^{t+1}$. So, by Notation and Reminder 1.5 C) we get

$$h^0_{\mathcal{F}}(n-1) \leq h^0_{\mathcal{F}}(n), \quad \forall n \in \mathbb{Z}, \quad \forall (X, \mathcal{F}) \in \mathcal{S}^t.$$

As the class $\mathcal{D}$ is of finite cohomology on the singleton set $\{(0, B+1)\}$ it follows immediately, that the set $\mathcal{H}$ is finite. $\square$

Rephrasing the previous result we may say:

5.6. **Corollary.** *Let $t \in \mathbb{N}_0$ and let $r \in \mathbb{Z}$. Then, a subclass $\mathcal{D} \subseteq \mathcal{S}^t$ is of finite cohomology if and only if it is of finite cohomology on the diagonal subset*

$$\Delta = \Delta_r = \Delta_r^t := \{(i.r - i) \mid i = 0, 1, \ldots, t\}.$$

Proof. This is immediate by Theorem 5.5. □

5.7. **Definition and Remark.** A) *(Bounding Sets for Cohomology)* Let $t \in \mathbb{N}_0$ and let $\mathcal{D} \subseteq \mathcal{S}^t$. A subset

$$\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$$

is called a *bounding set for cohomology with respect to the class* $\mathcal{D}$, if each subclass $\mathcal{E} \subseteq \mathcal{D}$ which is of finite cohomology on $\mathbb{S}$ is of finite cohomology at all. If $\mathbb{S}$ is a bounding set with respect to the full class $\mathcal{S}^t$ it is called a *bounding set for cohomology (at all)*.

B) *(Rephrasing Corollary 5.6)* Keep the notations and hypotheses of part A). We now may rephrase Corollary 5.6 as follows:

a) Each set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ which contains a diagonal subset $\Delta = \Delta_r^t$ is a bounding set for cohomology.

The reformulation of Corollary 5.6 suggested above gives rise to the question, whether there is a combinatorial characterization of all bounding sets. In order to deal with this problem, we define the notion of *quasi-diagonal set.*

5.8. **Definition.** *(Quasi-Diagonal Sets*) Let $t \in \mathbb{N}_0$. A subset

$$\Sigma \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$$

is said to be a *quasi-diagonal (subset)* if there are integers $n_t < n_{t-1} < \ldots < n_0$ such that

$$\Sigma = \{(i, n_i) \mid i = 0, 1, \ldots, t\}.$$

Observe, that each diagonal subset $\Delta = \Delta_r^t = \{(i, r - i) \mid i = 0, 1 \ldots, t\} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ is a quasi-diagonal subset.

We next prove the following auxiliary result.

5.9. **Lemma.** *Let $t \in \mathbb{N}_0$, let $n_t < n_{t-1} < \ldots < n_0$ be integers and let $\mathcal{D} \subseteq \mathcal{S}^t$ be a subclass which is of finite cohomology on the quasi-diagonal subset*

$$\Sigma := \{(i, n_i) \mid i = 0, 1, \ldots, t\} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}.$$

Then, the class $\mathcal{D}$ is of finite cohomology on the diagonal set

$$\Delta = \Delta_{t+n_t}^t := \{(i, t + n_t - i) \mid i = 0, 1, \ldots, t\}.$$

Proof. . After twisting, we may assume that $n_t = -t$, so that

$$\Delta = \Delta_0^t = \{(i, -i) \mid i = 0, 1, \ldots, t\}.$$

The statement a) made in the proof of Theorem 5.5 yields the estimate

a) $h^i_{\mathcal{F}}(n) \leq \sum_{j=0}^{i} \binom{-n-j-1}{i-j} \big[\sum_{k=j}^{i-j} \binom{i-j}{k-j} h^k_{\mathcal{F}}(-k) \big]$, for all $i \in \mathbb{N}_0$, for all $n \leq -i$ and for all pairs $(X, \mathcal{F}) \in \mathcal{S}^t$.

Now, we prove our claim by induction on the number

$$\sigma = \sigma(\Sigma) := n_0 - t \quad (\geq t).$$

If $\sigma = t$ we have $\Sigma = \Delta$ and our claim is obvious.

So, let $\sigma > t$. Then there is some $i \in \{0, 1, \ldots, t-1\}$ such that $n_i - n_{i+1} > 1$. We chose $i := i(\Sigma)$ minimal with this property and proceed by induction on $i = i(\Sigma)$. Assume first, that $i = 0$. Then $n_1 + 1 < n_0$ and it follows by the above statement a) applied with $i = 0$ that

$$h^0_{\mathcal{F}}(n_1 + 1) = h^0_{\mathcal{F}(n_0)}(n_1 + 1 - n_0) \leq h^0_{\mathcal{F}(n_0)}(0) = h^0_{\mathcal{F}}(n_0), \quad \forall (X, \mathcal{F}) \in \mathcal{D}.$$

But this implies that the class $\mathcal{D}$ is of finite cohomology on the set

$$\Sigma' := \{(0, n_1 + 1)\} \cup \{(j, n_j) \mid j = 1, 2 \ldots, t\}.$$

But for this set we also have $\sigma(\Sigma') < \sigma(\Sigma) = \sigma$. Therefore, by induction the class $\mathcal{D}$ is of finite cohomology on the set Δ.

Now, let $i > 0$. Then clearly $n_j - 1 - n_0 = -j - 1$ for all $j = 0, 1, \ldots, i$, hence $n_k = n_0 - k$ for all $k = 0, 1, \ldots, i$. Therefore the class $\mathcal{D}$ is of finite cohomology on the non-empty set

$$\{(k, n_0 - k) \mid k = 0, 1, \ldots, i\} = \{(k, n_k) \mid k = 0, 1, \ldots, i\} \subseteq \Sigma.$$

So, there is some $h \in \mathbb{N}_0$ such that

$$h^k_{\mathcal{F}(n_0)}(-k) = h^k_{\mathcal{F}}(n_0 - k) \leq h, \quad \forall k \in \{0, 1, \ldots, i\}, \quad \forall (X, \mathcal{F}) \in \mathcal{D}.$$

By the above statement a) – applied with $\mathcal{F}(n_0)$ instead of $\mathcal{F}$ – it follows that there is some $h' \in \mathbb{N}_0$ such that

$$h^i_{\mathcal{F}}(n_i - 1) = h^i_{\mathcal{F}(n_0)}(n_i - 1 - n_0) = h^i_{\mathcal{F}(n_0)}(-i - 1) \leq h', \quad \forall (X, \mathcal{F}) \in \mathcal{D}.$$

From this we obtain that the class $\mathcal{D}$ is of finite cohomology on the set

$$\Sigma'' := \{(j, n_j) \mid j = 0, 1, \ldots, i-1\} \cup \{(i, n_i - 1)\} \cup \{(k, n_k) \mid k = i+1, i+2, \ldots, t\}.$$

As $i(\Sigma'') = i(\Sigma) - 1 = i - 1$, we may conclude by induction. $\square$

Now, we can deduce the following result.

5.10. **Proposition.** *Let $t \in \mathbb{N}_0$ let $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ be a subset which contains a quasi-diagonal subset*

$$\Sigma = \{(i, n_i) \mid i = 0, 1, \ldots, t\}, \quad n_t < n_{t-1} < \ldots < n_0$$

and let $\mathcal{D} \subseteq \mathcal{S}^t$ a subclass which is of finite cohomology on $\mathbb{S}$. Then, the class $\mathcal{D}$ is of finite cohomology at all.

Proof. This is clear by Corollary 5.6 and Lemma 5.9. $\square$

In particular, we can say.

5.11. **Corollary.** *Let $t \in N_0$. Then each set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ which contains a quasi-diagonal subset Σ is a bounding set for cohomology.*

Proof. This is immediate by Proposition 5.10. □

So, we have seen, that for a set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ a sufficient condition for being a bounding set for cohomology is to contain a quasi-diagonal. It is natural to ask whether this condition is also necessary. This is indeed the case, as stated by the following result.

5.12. **Theorem.** *Let $t \in \mathbb{N}_0$. Then, a set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ is a bounding set for cohomology, if and only if it contains a quasi-diagonal set.*

Proof. The sufficiency of the condition to contain a quasi-diagonal is stated in Corollary 5.11. The necessity needs some extra work: one supposes that the set $\mathbb{S}$ does not contain a quasi-diagonal. Then, using appropriate vector bundles on certain Segre products, one constructs families of pairs $(X, \mathcal{F}) \in \mathcal{S}^t$ which are of finite cohomology on $\mathbb{S}$ but not on the set $\{0, 1, \ldots, t\} \times \mathbb{Z}$. For a detailed proof see [11] (4.6). □

We now give a number of applications of the previous results, which generalize what we said in Example 1.9 about Hilbert schemes. We begin with linking classes of finite cohomology to classes of bounded regularity.

5.13. **Remark.** A) *(Specifying classes of Finite Cohomology)* Let $t \in \mathbb{N}_0$ and let $\mathcal{D} \subseteq \mathcal{S}^t$ be a subclass. Fix a quasi-diagonal subset

$$\Sigma = \{(i, n_i) \mid i = 0, 1, \ldots, t\} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}, \quad n_t < n_{t-1} < \ldots < n_0.$$

Then Proposition 5.10 says that the class $\mathcal{D}$ is of finite cohomology if and only if the set

$$\{h^i(X, \mathcal{F}(n_i)) \mid (X, \mathcal{F}) \in \mathcal{D}\} = \{h^i_{\mathcal{F}}(n_i) \mid (X, \mathcal{F}) \in \mathcal{D}\}$$

is finite for all $i \in \{0, 1, \ldots, t\}$. So, the $t + 1$ numerical invariants $h^i_{\mathcal{F}}(n_i)$ with $i = 0, 1, \ldots, t$ may be used to specify subclasses $\mathcal{D} \subseteq \mathcal{S}^t$ of finite cohomology. Indeed, specifying such classes by subjecting numerical invariants to some conditions, is a basic issue. In this spirit we add the following statements, whose proves are straightforward:

a) The class $\mathcal{D} \subseteq \mathcal{S}^t$ is of finite cohomology if and only if there are integers $r \in \mathbb{Z}$ and $h \in \mathbb{N}_0$ such that $\mathrm{reg}(\mathcal{F}) \leq r$ and $h^0(X, \mathcal{F}(r)) \leq h$ for all pairs $(X, \mathcal{F}) \in \mathcal{D}$.

B) *(Classes of Bounded Regularity)* We say that the class $\mathcal{D} \subseteq \mathcal{S}^t$ is *of bounded regularity* if the set of integers

$$\{\mathrm{reg}(\mathcal{F}) \mid (X, \mathcal{F}) \in \mathcal{D}\}$$

has an upper bound in $\mathbb{Z}$. On use of our previously shown results, it follows easily:

b) The class $\mathcal{D} \subseteq \mathcal{S}^t$ is of finite cohomology if and only if it is of bounded regularity and the set of Serre polynomials $\{P_\mathcal{F} \mid (X, \mathcal{F}) \in \mathcal{D}\}$ is finite.

C) *(Regularity and Classes of Subsheaves and Quotient Sheaves)* Let $t \in \mathbb{N}_0$. we consider the class

$$\mathcal{S}^{\leq t} := \bigcup_{i=0}^{t} \mathcal{S}^i$$

of all pairs $(X, \mathcal{F})$ in which X is a projective scheme over some field K and $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules with $\dim(\mathcal{F}) \leq t$. The notions of *subclass* $\mathcal{D} \subseteq \mathcal{S}^{\leq t}$ *of finite cohomology* and of subclass *of bounded regularity* are defined in the obvious way as previously. Now, let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}^{\leq t}$. We say that $\mathcal{D}$ is *a class of subsheaves with respect to* $\mathcal{C}$ if for all pairs $(X, \mathcal{F}) \in \mathcal{D}$ there is a monomorphism of sheaves $0 \to \mathcal{F} \xrightarrow{h} \mathcal{G}$ with $(X, \mathcal{G}) \in \mathcal{C}$. Again, on use of our previous results, one may verify the following claim:

a) Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}^{\leq t}$ be such that $\mathcal{C}$ is of finite cohomology and $\mathcal{D}$ is a class of subsheaves with respect to $\mathcal{C}$. Then the class $\mathcal{D}$ is of finite cohomology if and only if it is of bounded regularity.

If X is a projective scheme over some field K and $\mathcal{F}, \mathcal{G}$ are two coherent sheaves of $\mathcal{O}_X$-modules we say that $\mathcal{F}$ is *a quotient of* $\mathcal{G}$ if there is an epimorphism of sheaves $\mathcal{G} \xrightarrow{h} \mathcal{F} \to 0$. Accordingly we say that $\mathcal{D}$ is *a class of quotient sheaves with respect to* $\mathcal{C}$ if for each pair $(X, \mathcal{F}) \in \mathcal{D}$ there is a pair $(X, \mathcal{G}) \in \mathcal{C}$ such that $\mathcal{F}$ is a quotient of $\mathcal{G}$. In this setting it follows from our previous results:

b) Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}^{\leq t}$ be such that $\mathcal{C}$ is of finite cohomology and $\mathcal{D}$ is a class of quotient sheaves with respect to $\mathcal{C}$. Then the class $\mathcal{D}$ is of finite cohomology if and only if it is of bounded regularity.

D) *(Serre Polynomials and Classes of Subsheaves and Quotient Sheaves)* We now generalize what was said about Hilbert schemes in Example 1.9. Keep the notations and hypotheses of part C). Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}^t$ be subclasses. We recall the following fact, which generalizes Mumford's basic Bounding Result given in [30] (see [5] or [13] (20.4.18)).

a) For each polynomial $p \in \mathbb{Q}[X]$ of degree t and each integer $\rho \in \mathbb{Z}$ there is a number $\beta = \beta_r(\rho)$ such that for each projective scheme X and each epimorphism of coherent sheaves of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{H} \to 0$ with $\mathrm{reg}(\mathcal{F}) \leq \rho$ and $P_\mathcal{H} = p$ one has $\mathrm{reg}(\mathcal{H}) \leq \beta$.

Using this, it is not hard to prove the following statement

b) Let $\mathcal{D}$ be a class of subsheaves (resp. of quotient sheaves) with respect to $\mathcal{C}$ and assume that $\mathcal{C}$ is of finite cohomology. Then, class $\mathcal{D}$ is of finite cohomology if and only if the set of Serre polynomials $\{P_\mathcal{F} \mid (X, \mathcal{F}) \in \mathcal{D}\}$ is finite.

The following special case of the previous statement covers most closely our observation on Hilbert schemes made in Example 1.9. Fix a pair $(X, \mathcal{G}) \in \mathcal{S}^{\leq t}$ and let $\mathcal{D}$ be a class of subsheaves or of quotient sheaves of $\mathcal{G}$. Then, it is not hard to verify that the following statements are equivalent.

(i) $\mathcal{D}$ is a class of finite cohomology.
(ii) $\mathcal{D}$ is a class of bounded regularity.
(iii) The set $\{P_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{D}\}$ is finite.

Now we give another remark, which concerns bounding sets for cohomology with respect to specific subclasses of $\mathcal{S}^t$.

5.14. **Remark.** A) *(Bounding Sets for Classes of Vector Bundles)* Let $t \in \mathbb{N}$. It is natural to ask, whether for appropriate subclasses of $\mathcal{D} \subseteq \mathcal{S}^t$ there are more bounding sets for cohomology than those specified by Theorem 5.12 . A particularly interesting setting for this question is given as follows: Let K be a field, let

$$\mathcal{V}_K^t \subseteq \mathcal{S}^t$$

be the family of all algebraic vector bundles over the projective space $\mathbb{P}_K^t = \mathrm{Proj}(K[\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_t])$ and let $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$. We say that $\mathbb{S}$ *bounds cohomology of vector bundles (over* $\mathbb{P}_K^t$*)*, if each subclass $\mathcal{D} \subseteq \mathcal{V}$ which is of finite cohomology on $\mathbb{S}$ is of finite cohomology at all. One could a fortiori expect, that for the class $\mathcal{V}_K^t$ of vector bundles – which is considerably smaller than the class of coherent sheaves $\mathcal{S}^t$ – a weaker condition could suffice that $\mathbb{S}$ bounds cohomology. But indeed, this is not the case, as we shall prove in Section 6.

B) *(Counting Cohomology Tables)* Fix an arbitrary (quasi-)diagonal subset

$$\Sigma = \{(i, n_i) \mid i = 0, 1, \ldots, t\} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}, \quad (n_t < n_{t-1} < \ldots < n_0).$$

and fix a family of non-negative integers

$$\bar{h} := (h^i)_{i=0}^t.$$

Then clearly we know by Theorem 5.12, that the number of cohomology tables

$$N_{\Sigma, \bar{h}} := \#\{h_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{S}^t : h^i(X, \mathcal{F}(n_i)) = h_i, \quad i = 0, 1 \ldots, t\}$$

is finite. Going tediously through our arguments on could indeed get some upper bound for this number, at least in the case where Σ is the standard diagonal subset $\{(i, -i) \mid i = 0, 1, \ldots, t\}$. So, one could get stuck to the idea of counting all possible cohomology tables with a given standard cohomology diagonal, or at least to bound there number in a satisfactory way. Clearly, one cannot expect, that a bound which is obtained on use of the arguments of our proves will be satisfactory. The enormous discrepancy between the expected and the actual number of cohomology tables is made evident in the Master thesis [15].

So, our bounding results are not appropriate to perform quantitative arguments in the sense of counting cohomology tables. On the other hand our results furnish at least the equivalence of the following statements, which also follows easily from the properties of cohomological patterns (see Remark 2.5).

(i) $\mathcal{F} = 0$.
(ii) $h^i(X, \mathcal{F}(-i)) = 0$ for all $i \in \{0, 1, \ldots, t\}$.
(iii) There is some $r \in \mathbb{Z}$ such that $H^i(X, \mathcal{F}(r-i)) = 0$ for all $i \in \{0, 1, \ldots, t\}$.
(iv) $h^i_{\mathcal{F}} = 0$ for all $i \in \{0, 1, \ldots, t\}$.

C) *(Characterizing Cohomology Tables)* Refining what we presented in Section 2 and pushing further the idea of counting cohomology tables, one could try to characterize all families

$$\left(h^i_n\right)_{(i,n)\in\{0,1,\ldots,t\}\times\mathbb{Z}} \in \Pi_{(i,n)\in\{0,1,\ldots,t\}\times\mathbb{Z}}\mathbb{N}_0$$

of non-negative numbers h^i_n which occur as cohomology table $h_{\mathcal{F}}$ of some pair $(X, \mathcal{F}) \in \mathcal{S}^t$. We do not know the answer to this problem. Clearly it would be much more interesting and more challenging to answer this question for some specific classes $\mathcal{D} \in \mathcal{S}^t$. So, one could think to choose $\mathcal{D}$ to be the class of all pairs $(X, \mathcal{O}_X)$, where $X \subseteq \mathbb{P}^r_K$ runs through all closed subschemes with a given Serre polynomial $P_{\mathcal{O}_X} = p$ with $\deg(p) = t$, hence through the class of closed subschemes parametrized by $\mathrm{Hilb}^p_{\mathbb{P}^r}$. Another challenge would be to attack this problem in the case where $\mathcal{D}$ is the class of all vector bundles $\mathcal{E}$ of given rank $\mathrm{rank}(\mathcal{E}) = r$ over a fixed projective space $\mathbb{P}^t_K$ or with given Serre polynomial $P_{\mathcal{E}}$.

6. Bounding Cohomology of Vector Bundles

We now want to characterize in purely combinatorial terms the sets $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ which bound cohomology on the class of all algebraic vector bundles over the projective t-space $\mathbb{P}^t_K$ over an arbitrarily given field K. Our main result will show, that these bounding sets are indeed the same as the ones which bound cohomology on the full class $\mathcal{S}^t$. We begin with a few preparations.

6.1. **Notation and Remark.** A) *(Vector Bundles over Projective Space)* Throughout this section we fix some field K and a positive integer t. We write $\mathcal{V}^t_K$ for the class of all *(algebraic) vector bundles* $\mathcal{E}$ over the projective t-space $\mathbb{P}^t = \mathrm{Proj}(K[\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_t])$ over K. So, $\mathcal{V}^t_K$ is the class of all locally free coherent sheaves of $\mathcal{O}_{\mathbb{P}^t_K}$-modules.
Moreover, we shall write $\mathrm{ind}\mathcal{V}^t_K$ for the class of all *indecomposable (algebraic) vector bundles* over $\mathbb{P}^t_K$, hence, the class of all coherent locally free sheaves of $\mathcal{O}_{\mathbb{P}^t_K}$-modules $\mathcal{E}$ which are not the sum of two non-zero sheaves of $\mathcal{O}_{\mathbb{P}^t_K}$-modules.

B) *(Embedding into the Class of Coherent Sheaves)* There is a canonical injective map

$$\mathcal{V}^t_K \longrightarrow \mathcal{S}^t \text{ given by } \mathcal{E} \mapsto (\mathbb{P}^t_K, \mathcal{E}) \text{ for all } \mathcal{E} \in \mathcal{V}^t_K.$$

By means of this map, we always consider $\mathcal{V}^t_K$ as a subclass of $\mathcal{S}^t$ and hence write

$$\mathrm{ind}\mathcal{V}^t_K \subsetneq \mathcal{V}^t_K \subsetneq \mathcal{S}^t.$$

The basic ingredient we need to prove the main result of this section, is actually of combinatorial nature, and concerns the structure of certain sets $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$. We begin with some preparations. We will use repeatedly the *diagonal projection*

$$\varrho : \{0, 1, \ldots, t\} \times \mathbb{Z} \longrightarrow \mathbb{Z}; \quad (i, n) \mapsto \varrho(i, n) := i + n.$$

Moreover, if $\mathbb{U} \subseteq \mathbb{Z}$ we form the supremum $\sup(\mathbb{U})$ and the infimum $\inf(\mathbb{U})$ of $\mathbb{U}$ in $\mathbb{Z} \cup \{\pm\infty\}$ with the usual convention that $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) := +\infty$.

6.2. **Definition.** A) *(Beginning and Ends of Sets)* Let $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$. Let $i \in \{0, \ldots, t\}$. We define the *beginning* and the *end* of the set $\mathbb{S}$ *at level* i respectively by:

$$\mathrm{beg}^i(\mathbb{S}) := \inf\{n \in \mathbb{Z} \mid (i, n) \in \mathbb{S}\};$$
$$\mathrm{end}^i(\mathbb{S}) := \sup\{n \in \mathbb{Z} \mid (i, n) \in \mathbb{S}\}.$$

B) *(Heights, Depths and Widths of Sets)* The *height* and the *depth* of the set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$. are defined respectively by:

$$\mathrm{height}(\mathbb{S}) := \sup\{i \in \mathbb{Z} \mid \exists n \in \mathbb{Z} : (i, n) \in \mathbb{S}\};$$
$$\mathrm{depth}(\mathbb{S}) := \inf\{i \in \mathbb{Z} \mid \exists n \in \mathbb{Z} : (i, n) \in \mathbb{S}\}.$$

The *width* of $\mathbb{S}$ is defined by

$$\mathrm{width}(\mathbb{S}) := \mathrm{height}(\mathbb{S}) - \mathrm{depth}(\mathbb{S}).$$

6.3. **Reminder.** A) *(Combinatorial Patterns)* According to [9] (1.2), the set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ is called a *combinatorial pattern*, if:

a) $\mathrm{depth}(\mathbb{S}) = 0$;
b) $(i, n) \in \mathbb{S} \Rightarrow \exists j \leq i : (j, n + i - j + 1) \in \mathbb{S}$;
c) $(i, n) \in \mathbb{S} \Rightarrow \exists k \geq i : (k, n + i - k - 1) \in \mathbb{S}$;
d) $0 < i \leq t \Rightarrow \mathrm{end}^i(\mathbb{S}) < \infty$.

Observe, that according to Theorem 2.4, the cohomological pattern $\mathcal{P}(X, \mathcal{F})$ of a pair $(X, \mathcal{F}) \in \mathcal{S}^t$ is a combinatorial pattern of width t, which is in addition *tame*, which means that meaning that it satisfies the requirement (f) of Theorem 2.4.

B) *(Minimal Combinatorial Patterns)* The set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ is called a *minimal combinatorial pattern*, if it is a combinatorial pattern and if there is no combinatorial pattern $\mathbb{S}' \subsetneq \mathbb{S}$ strictly contained in $\mathbb{S}$.

C) *(Diagonal Projections of Combinatorial Patterns)* Let $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ be a combinatorial pattern. Then, according to [9] (2.1) and (2.2), the following two statements hold for the restriction $\varrho : \mathbb{S} \to \mathbb{Z}$ of the diagonal projection:

a) The map $\varrho : \mathbb{S} \to \mathbb{Z}$ is surjective.
b) The map $\varrho : \mathbb{S} \to \mathbb{Z}$ is bijective if and only if the combinatorial pattern $\mathbb{S}$ is minimal.

D) *(A Characterization of Minimal Combinatorial Patterns)* Let

$$\mathbb{F}^t := \{\underline{a} := (a_j)_{j=1}^p \mid p \in \mathbb{N}_0 \text{ and } 1 \leq a_1 \leq a_2 \leq \cdots \leq a_p \leq t - 1\}$$

be the set of all monotonically increasing sequences in $\{1, \ldots, t-1\}$. According to the properties of cohomological patterns a)–e) of part A) and statement b) of part C) there is a bijection between the set $\mathbb{Z} \times \mathbb{F}^t$ and the set of all minimal combinatorial patterns of width t. Indeed, for each $b \in \mathbb{Z}$ and each $\underline{a} = (a_1, \ldots, a_p) \in \mathbb{F}^t$ (with $p \in \mathbb{N}_0$), the set

$$\mathbb{M}^t_{b,\underline{a}} := \big(\{0\} \times \mathbb{Z}_{\geq b}\big) \cup \{(a_j, b - j - a_j) \mid 1 \leq j \leq p\} \cup \big(\{t\} \times \mathbb{Z}_{\leq b-t-p-1}\big)$$

is a minimal combinatorial pattern of width t. Conversely, each minimal combinatorial pattern of width t can be written as $\mathbb{M}^t_{b,\underline{a}}$ with uniquely determined $b \in \mathbb{Z}$ and $\underline{a} \in \mathbb{F}^t$.

Our next result is an *Avoidance Principle for Minimal Combinatorial Patterns.* It is the basic combinatorial ingredient for the proof of the main result we are heading for in this section. This Avoidance Principle has been shown in the Master Thesis [25]. The proof we give below may be found in [6].

6.4. **Proposition.** *Assume that the set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ contains no quasi-diagonal of width t, that $\operatorname{beg}^t(\mathbb{S}) \neq -\infty$ and $\operatorname{end}^0(\mathbb{S}) \neq \infty$. Then, there is a minimal combinatorial pattern $\mathbb{M}$ of width t such that $\mathbb{M} \cap \mathbb{S} = \emptyset$.*

Proof. Assume that $\operatorname{end}^0(\mathbb{S}) = -\infty$, so that $(0, n) \notin \mathbb{S}$ for all $n \in \mathbb{Z}$. Chose an integer $c \leq \operatorname{beg}^t(\mathbb{S})$. Then, the minimal combinatorial pattern (see Reminder 6.3 B),D))

$$\mathbb{M}^t_{c+t,\emptyset} = \big(\{0\} \times \mathbb{Z}_{\geq c+t}\big) \cup \big(\{t\} \times \mathbb{Z}_{\leq c-1}\big)$$

is disjoint to $\mathbb{S}$.
Assume now that $\operatorname{end}^0(\mathbb{S}) \neq -\infty$, so that $\operatorname{end}^0(\mathbb{S}) \in \mathbb{Z}$. Set

$$b := \operatorname{end}^0(\mathbb{S}) + 1 \text{ and } c := \operatorname{beg}^t(\mathbb{S}).$$

First, assume that $t = 1$. As $\mathbb{S}$ contains no quasi-diagonal of width t, we then have $c \geq b - 1$ so that $\mathbb{M}^1_{b,\emptyset} = \big(\{0\} \times \mathbb{Z}_{\geq b}\big) \cup \big(\{1\} \times \mathbb{Z}_{\leq b-2}\big)$ is disjoint to $\mathbb{S}$.
So, assume from now on that $t > 1$. Our aim is to construct a sequence of integers $(a_1, a_2, \ldots, a_p) = \underline{a} \in \mathbb{F}^t$ with $p \in \mathbb{N}_0$ such that $\mathbb{M}^t_{b,\underline{a}} \cap \mathbb{S} = \emptyset$. To achieve this, we construct a sequence of integers $0 =: a_0 < a_1 \leq a_2 \leq \cdots \leq a_i \leq a_{i+1} \leq \cdots \leq a_p < t$ with the following properties:

(1) $(a_i, b - i - a_i) \notin \mathbb{S}$ for all $i \in \{0, \ldots, p\}$.
(2) If $0 \leq i < p$ and $a_i < a_{i+1}$, then $(j, b - i - j - 1) \in \mathbb{S}$ for all j with $a_i \leq j < a_{i+1}$.
(3) $p \geq b - c - t$.

By our choice of b and a_0, condition (1) is satisfied with 0 instead of p. If $b - c - t \leq 0$ we can set $p = 0$ and the requested sequence is constructed.
Thus, assume from now on that $b - c - t > 0$, so that $c < b - t$. As $(0, b - 1), (t, c) \in \mathbb{S}$ and $\mathbb{S}$ contains no quasi-diagonal of width t, we have

$$1 \leq a_1 := \inf\{k \in \mathbb{N}_0 \mid (k, b - k - 1) \notin \mathbb{S}\} \leq t - 1.$$

For this choice of a_1, condition (2) is satisfied with 1 instead of p.
Assume now, that the integers $a_0 < a_1 \leq \ldots \leq a_s < t$ $(s \in \mathbb{N})$ are already constructed such that conditions (1) and (2) are satisfied with s instead of p. If $s \geq b - c - t$, we set $p := s$, and condition (3) is satisfied. So the requested sequence is constructed in this case. Therefore, it remains to define a_{s+1} if $s < b - c - t$. To do so, we distinguish two cases.
If $(a_s, b - s - a_s - 1) \notin \mathbb{S}$, we set $a_{s+1} := a_s$. Then, clearly, the sequence $a_0, a_1, \ldots, a_{s+1}$ satisfies the requirements (1) and (2) with $s + 1$ instead of p and we have extended our sequence in the requested way.
So, finally assume that $(a_s, b - s - a_s - 1) \in \mathbb{S}$. We find integers $0 = i_0 < i_1 < \ldots < i_r < p$ $(r \in \mathbb{N}_0)$ such that

$$\{i_0, \ldots, i_r\} = \{i \mid 0 \leq i < s \text{ and } a_i < a_{i+1}\}.$$

Set

$$\mathbb{T}_1 := \bigcup_{k=0}^{r} \{(j, b - i_k - j - 1) \mid a_{i_k} \leq j < a_{i_k+1}\} \text{ and}$$
$$\mathbb{T}_2 := \{(j, b - s - j - 1) \mid a_s \leq j < t\} \cup \{(t, c)\}.$$

As $s < b - c - t$ it follows that $\mathbb{T}_1 \cup \mathbb{T}_2$ is a quasi-diagonal of width t. Keep in mind that by condition (1) we have $\mathbb{T}_1 \subseteq \mathbb{S}$. As $\mathbb{S}$ does not contain a quasi-diagonal of width t, the set $\mathbb{T}_2$ cannot be contained in $\mathbb{S}$. As

$$(a_s, b - s - a_s - 1), (t, c) \in \mathbb{T}_2 \cap \mathbb{S}$$

it follows that $(k, b - k - 1) \notin \mathbb{S}$ for some k with $a_s < k < t$. Choose k minimal with this property and set $a_{s+1} := k$. Then the sequence $a_0, \ldots, a_s, a_{s+1}$ satisfies the properties (1) and (2) with $s + 1$ instead of p and again we have extended our sequence in the requested way.
So altogether, the requested sequence is constructed. Setting $\underline{a} = (a_1, \ldots, a_p)$ we easily see by conditions (1) and (3) that indeed $\mathbb{M}^t_{b,\underline{a}} \cap \mathbb{S} = \emptyset$. $\square$

The next ingredient needed to prove of our main result is the following *Realization Result for Cohomological Patterns of Irreducible Vector Bundles*:

6.5. **Proposition.** *Let K be a field and let $\mathbb{M} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ be a minimal combinatorial pattern of width t. Then, there is an indecomposable vector bundle $\mathcal{E} \in \mathrm{ind}\mathcal{V}^i_K$, whose cohomological pattern $\mathcal{P}(\mathcal{E})$ coincides with $\mathbb{M}$.*

Proof. See Proposition 4.5 of [9]. $\square$

Now, we are ready to prove the crucial result we need to reach our goal.

6.6. **Proposition.** *Assume that the set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ contains no quasi-diagonal of width t and let K be a field. Then, $\mathbb{S}$ does not bound cohomology in the class $\mathcal{V}^t_K$ of algebraic vector bundles over $\mathbb{P}^t_K$.*

Proof. Assume first, that $\mathrm{beg}^t(\mathbb{S}) > -\infty$ and $\mathrm{end}^0(\mathbb{S}) < \infty$. Then, by Proposition 6.4 there is a minimal combinatorial pattern $\mathbb{M}$ of width t such that $\mathbb{M} \cap \mathbb{S} = \emptyset$. According to Proposition 6.5, there is a non-zero locally free indecomposable sheaf $\mathcal{E} \in \mathcal{V}^t_K$ such that

$$\mathcal{P}(\mathcal{E}) := \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid h^i_{\mathcal{E}}(n) \neq 0\} = \mathbb{M}.$$

As $\mathcal{E}^{\oplus^r} \in \mathcal{V}^t_K$ and $h^i_{\mathcal{E}^{\oplus r}}(n) = r h^i_{\mathcal{E}}(n)$ for all $r \in \mathbb{N}$ and all $(i, n) \in \mathbb{N}_0 \times \mathbb{Z}$ it follows that the set of cohomology tables $\{h_{\mathcal{E}^{\oplus r}} \mid r \in \mathbb{N}\}$ is an infinite subset of

$$\{h_{\mathcal{F}} \mid \mathcal{F} \in \mathcal{V}^t_K : h^i_{\mathcal{F}}(n) = 0 \text{ for all } (i, n) \in \mathbb{S}\}.$$

Therefore $\mathbb{S}$ does not bound cohomology in the class $\mathcal{V}^t_K$ in this case.
Assume now, that $\mathrm{beg}^t(\mathbb{S}) = -\infty$ or $\mathrm{end}^0(\mathbb{S}) = \infty$. For each $r \in \mathbb{N}_0$, set

$$\mathbb{S}_{[r]} := \mathbb{S} \cap \big(\{0, \ldots, t\} \times \{-r, -r + 1, \ldots, r - 1, r\}\big).$$

Observe that

$$\mathbb{S} = \bigcup_{r\in\mathbb{N}_0} \mathbb{S}_{[r]}.$$

Clearly, for each $r \in \mathbb{N}_0$ the set $\mathbb{S}_{[r]}$ contains no quasi-diagonal of width t and moreover satisfies $\mathrm{beg}^t(\mathbb{S}_{[r]}) \geq -r$ and $\mathrm{end}^0(\mathbb{S}_{[r]}) \leq r$. So, according to Proposition 6.4 for each $r \in \mathbb{N}_0$, there is a minimal combinatorial pattern $\mathbb{M}_{[r]}$ of width t such that

$$\mathbb{M}_{[r]} \cap \mathbb{S}_{[r]} = \emptyset.$$

According to Proposition 6.5 for each $r \in \mathbb{N}_0$ we find a locally free sheaf $\mathcal{E}_{[r]} \in \mathcal{V}_K^t$ such that

$$\mathcal{P}(\mathcal{E}_{[r]}) := \{(i,n) \in \mathbb{N}_0 \times \mathbb{Z} \mid h^i_{\mathcal{E}_{[r]}}(n) \neq 0\} = \mathbb{M}_{[r]}.$$

Now, fix a pair $(i,n) \in \mathbb{S}$. Then, there is an integer $s(i,n) \in \mathbb{N}_0$ such that $(i,n) \in \mathbb{S}_{[r]}$ for all $r \geq s(i,n)$. Consequently $h^i_{\mathcal{E}_{[r]}}(n) = 0$ for all $r \geq s(i,n)$. Therefore

$$h^{(i,n)} := \sup\{h^i_{\mathcal{E}_{[r]}}(n) \mid r \in \mathbb{N}_0\} \in \mathbb{N}_0$$

and hence

$$h^i_{\mathcal{E}_{[r]}}(n) \leq h^{(i,n)}, \text{ for all } (i,n) \in \mathbb{S} \text{ and all } r \in \mathbb{N}_0.$$

Our next aim is to show that the set of minimal combinatorial patterns

$$\mathbb{T} := \{\mathbb{M}_{[r]} \mid r \in \mathbb{N}_0\}$$

is not finite. Observe first, that for each $r \in \mathbb{N}_0$ there is some $n(r) \in \mathbb{N}_0$ such that $(t,-n),(0,n) \in \mathbb{M}_{[r]}$ for all $n \geq n(r)$. Assume now that $\mathbb{T}$ is finite, so that there are finitely many integers $r_1,\ldots,r_k$ with $\mathbb{T} = \{\mathbb{M}_{[r_1]},\ldots,\mathbb{M}_{[r_k]}\}$. As $\mathbb{S} = \cup_{r\in\mathbb{N}_0}\mathbb{S}_{[r]}$ and by our choice of the patterns $\mathbb{M}_{[r]}$, we then obtain

$$\mathbb{S} \cap \bigcap_{j=1}^{k} \mathbb{M}_{[r_k]} = \emptyset.$$

It follows that $(t,-n),(0,n) \notin \mathbb{S}$ for all $n \geq m := \max\{n(r_j) \mid j = 1,\ldots,k\}$. But this implies that $\mathrm{beg}^0(\mathbb{S}) \geq -m$ and $\mathrm{end}^0(\mathbb{S}) \leq m$, a contradiction!
So, the set $\mathbb{T}$ is infinite, as requested. As a consequence, $\{h_{\mathcal{E}_{[r]}} \mid r \in \mathbb{N}_0\}$ is an infinite subset of

$$\{h_{\mathcal{E}} \mid \mathcal{E} \in \mathcal{V}_K^t : h^i_{\mathcal{E}}(n) \leq h^{(i,n)} \text{ for all } (i,n) \in \mathbb{S}\}.$$

This shows that $\mathbb{S}$ does not bound cohomology in $\mathcal{V}_K^t$. $\square$

Now, we are ready to prove our main result.

6.7. **Theorem.** *Let K be a field. Then the set $\mathbb{S} \subseteq \{0,\ldots,t\} \times \mathbb{Z}$ bounds cohomology in the class $\mathcal{V}_K^t$ of algebraic vector bundles over $\mathbb{P}_K^t$ if and only if $\mathbb{S}$ contains a quasi-diagonal of width t.*

Proof. This is clear by Theorem 6.6, Theorem 5.12 and by Notation and Remark 6.1 B). $\square$

6.8. **Corollary.** *Let K be a field and let $\mathbb{S} \subseteq \{0, \ldots, t\} \times \mathbb{Z}$. Then, the following statements are equivalent:*

(i) $\mathbb{S}$ *contains a quasi-diagonal of width t.*
(ii) $\mathbb{S}$ *bounds cohomology in the class $\mathcal{S}^t$.*
(iii) $\mathbb{S}$ *bounds cohomology in the class $\mathcal{V}^t_K$.*

Proof. This is clear by Theorem 5.12 and Corollary 6.7. □

One could ask, whether the previous result still remains true, if one replaces the class $\mathcal{V}^t_K$ by the smaller class $\mathrm{ind}\mathcal{V}^t_K$ of indecomposable vector bundles over $\mathbb{P}^t_K$. This is not the case, as shown by the following example.

6.9. **Example.** Let $t = 1$ and consider the set $\mathbb{S} := \{(0, -1), (1, -1)\} \subseteq \{0, 1\} \times \mathbb{Z}$. Clearly, $\mathbb{S}$ does not contain a quasi-diagonal. According to the algebraic form of Grothendieck's Splitting Theorem for vector bundles over the projective line (see [13] (20.5.9) for example), (up to isomorphism) the line bundles $\mathcal{O}_{\mathbb{P}^1_K}(n)$ with $n \in \mathbb{Z}$ are precisely the indecomposable algebraic vector bundles over $\mathbb{P}^1_K$. So, (up to isomorphism) for each choice of $h^{(0,-1)}, h^{(1,-1)} \in \mathbb{N}_0$ the $h^{(0,-1)} + h^{(1,-1)} + 1$ bundles

$$\mathcal{O}_{\mathbb{P}^1_K}(n) \text{ with } -h^{(1,-1)} \leq n \leq h^{(0,-1)}$$

are precisely the indecomposable algebraic vector bundles $\mathcal{E}$ over the projective line $\mathbb{P}^1_K$ which satisfy $h^i_{\mathcal{E}}(n) \leq h^{(i,n)}$ for all $(i, n) \in \mathbb{S}$. Hence

$$\#\{h_{\mathcal{E}} \mid \mathcal{E} \in \mathrm{ind}\mathcal{V}^1_K : \forall (i, n) \in \mathbb{S} : h^i_{\mathcal{E}}(n) \leq h^{(i,n)}\} \leq h^{(0,-1)} + h^{(1,-1)} + 1.$$

This clearly shows that the set $\mathbb{S}$ does indeed bound cohomology in the class $\mathrm{ind}\mathcal{V}^1_K$, although it does not contain a quasi-diagonal.

6.10. **Question.** We do not know a purely combinatorial characterization of the sets $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ which bound cohomology on the class $\mathrm{ind}\mathcal{V}^t_K$. This leaves us with the question, whether such a combinatorial characterization can be made explicit. A partial answer to this question is given in [6].

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