NOTES ON WEYL ALGEBRAS AND D-MODULES

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Abstract. Weyl algebras, sometimes called algebras of differential operators, are a fascinating and important subject, which relates Non-Commutative and Commutative Algebra, Algebraic Geometry and Analysis in very appealing way. The theory of modules over Weyl algebras, sometimes called D-modules, finds application in the theory of partial differential equations, and thus has a great impact to many fields of Mathematics. In our course, we shall give a short introduction to the subject, using only prerequisites from Linear Algebra, Basic Abstract Algebra, and Basic Commutative Algebra. In addition, in the last two sections, we present a few recent results.

1. Introduction

The present notes base on two short courses:

(1) *Introduction to Weyl Algebras*: 5 Twin Lessons, Thai Nguyen University of Science TNUS (Thai Nguyen, Vietnam), November 1 - 10, 2013.
(2) *Weyl Algebras, Universal Gröbner Bases, Filtration Deformations and Characteristic Varieties of D-Modules*: 4 Twin Lessons, Vietnam Institute for Advanced Study in Mathematics VIASM (Hanoi, Vietnam), November 12 - 26, 2013.

They were also the base for a third course:

(3) *Introduction to Weyl Algebras and D-Modules*: 4 Lessons and 2 Tutorial Sessions, "Workshop on Local Cohomology", St. Joseph's College Irinjalakuda, Kerala (India), June 20 - July 2, 2016.

These notes aim to give an approach to Boldini's stability and deformation results for characteristic varieties [11], [12] and to the bounding result [16] for the degrees of defining equations of characteristic varieties, including a self-contained introduction to the needed background on Weyl Algebras and D-modules. In particular, these notes should not be understood as an independent or complete introduction to the field of Weyl algebras and D-modules, which could replace one of the existing textbooks or monographs like [9], [8], [13], [24], [29], [37] or [38]: Too many core subjects are not treated at all or only marginally in these notes, as they are not needed on the way to our final results.

So, a few basic topics which are lacking in these notes – and which ought to be considered as indispensable in a complete introduction to the field – are:
- a systematic study of Bernstein's Inequality and holonomic D-modules (we treat these subjects only briefly in Exercise and Remark 14.3),

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- Bernstein's result on singularities of generalized Γ-functions and Bernstein-Sato polynomials,
- weighted filtrations with negative weights,
- the sheaf theoretic and cohomological aspect,
- the analytic aspect.

Another subject which is not treated in our notes are Lyubeznik's finiteness results for local cohomology modules of regular local rings in characteristic 0 (see [33] and also [34]), which brought a break-through in Commutative Algebra, as they base on the use of (holonomic) D-modules – and hence present a very important link between these two fields.

These notes are divided up in 14 Sections:

– *1. Introduction*
– *2. Filtered Algebras*
– *3. Associated Graded Rings*
– *4. Derivations*
– *5. Weyl Algebras*
– *6. Arithmetic in Weyl Algebras*
– *7. The Standard Basis*
– *8. Weighted Degrees and Filtrations*
– *9. Weighted Associated Graded Rings*
– *10. Filtered Modules*
– *11. D-Modules*
– *12. Gröbner Bases*
– *13. Weighted Orderings*
– *14. Standard Degree and Hilbert Polynomials*

Sections 1 - 9 were the subject of the introductory course (1) at TNUS. In our course (2) at the VIASM we gave an account on all 14 sections, and discussed a few applications (to the Gelfand-Kirillow dimension of D-modules for example), which are not contained in these notes. In the course (3) at St. Joseph's College we treated Sections 1 - 9 and 14.

Our suggested basic reference is Coutinho's introduction [24], although we do not follow that introduction and we partly use our own terminology and notations. We start in a slightly more general setting, than Coutinho, and so also we recommend the references [4], [10], [11], [32] and [35]. Files of [10] and [11] are available on request at the author. For readers who have already some background in the subject, we recommend as possible references [8], [9], [13], [29], [37], or the first part of the PhD thesis [11].

Finally, we aim to fix a few notations and make a few conventions which we shall keep throughout these notes. We do this on a fairly elementary level, according to the original intention of the short course (1): To give a first introduction to the subject to an audience having only some background in Linear Algebra and basic Abstract Algebra. Only in Section 14 we will need some background from Commutative Algebra, notably Hilbert functions and -polynomials, Local Cohomology and Castelnuovo-Mumford regularity. We shall give brief reminders on these more advanced preliminaries in Section 14.

1.1. **Conventions, Reminders and Notations.** (A) *(General Notations)* By $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ we respectively denote the set of integers, of rationals and of real numbers. We also write

$$\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \mid x \geq 0\} \text{ and } \mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$$

for the set of non-negative respectively of positive real numbers. Moreover, we use the following notations for the set of non-negative respectively the set of positive integers:

$$\mathbb{N}_0 := \mathbb{Z} \cap \mathbb{R}_{\geq 0} \text{ and } \mathbb{N} := \mathbb{Z} \cap \mathbb{R}_{>0} = \mathbb{N}_0 \setminus \{0\}.$$

If $\mathbb{S} \subset \mathbb{R}$ we form the supremum and infimum $\sup(\mathbb{S})$ resp. $\inf(\mathbb{S})$ within the set $\mathbb{R} \cup \{-\infty, \infty\}$, using the convention that $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = \infty$.
Empty sums and empty products are respectively understood to be 0 or 1. We thus set

$$\sum_{i=0}^{-1} x_i := 0 \text{ and } \prod_{i=0}^{-1} x_i := 1 \text{ with } x_1, x_2, \dots \in \mathbb{R}.$$

(B) *(Rings)* All rings R are understood to be associative, non-trivial and unital, so that they have a unit-element $1 = 1_R \in R \setminus \{0\}$ and the following properties hold

(a) $0x = x0 = 0$ and $1x = x1 = x$ for all $x \in R$;
(b) $x(yz) = (xy)z$, $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$ for all $x, y, z \in R$.

Rings need not be commutative.
If R is a ring, a *subring* of R is a subset $R_0 \subseteq R$, such that $1_R \in R_0$ and $x+y, xy \in R_0$ whenever $x, y \in R_0$. If $R_0 \subseteq R$ is a subring and $S \subseteq R$ is an arbitrary subset, we write $R_0[S]$ for the *subring of R generated by R_0 and S*, hence for the smallest subring of R which contains R_0 and S. Thus, $R_0[S]$ is the intersection of all subrings of R which contain R_0 and S, and may be written in the form

$$R_0[S] = \{\sum_{i=1}^{r} \prod_{j=1}^{k_r} a_{i,j} \mid r, k_1, \dots, k_r \in \mathbb{N}, a_{i,j} \in R_0 \cup S, \forall i \leq r, \forall j \leq k_i\}.$$

If $a_1, a_2, \dots, a_r$ is a finite collection of elements of R, we set

$$R_0[a_1, a_2, \dots, a_r] := R_0[\{a_1, a_2, \dots, a_r\}].$$

(C) *(Homomorphisms of Rings)* All *homomorphisms of rings* are understood to be unital, and hence are maps $h : R \longrightarrow S$, with R and S rings, such that

(a) $h(x+y) = h(x) + h(y)$ and $h(xy) = h(x)h(y)$ for all $x, y \in R$;
(b) $h(1_R) = 1_S$.

Clearly, the identity map $\mathrm{Id}_R : R \longrightarrow R$ is a homomorphism of rings, and the composition of homomorphisms of rings is again a homomorphism of rings. An *isomorphism of rings* is a homomorphism of rings admitting an inverse homomorphism. A homomorphism of rings is an isomorphism, if and only if it is bijective.

(D) *(K-Algebras)* All fields are considered as commutative. If K is a field, a K-algebra is understood to be a ring A together with a homomorphism of rings $\epsilon : K \longrightarrow A$ such that

$$\epsilon(c)a = a\epsilon(c) \text{ for all } c \in K \text{ and all } a \in A.$$

As the ring A is non-trivial, the homomorphism $\epsilon : K \longrightarrow A$ is injective. So, we can and do always embed K into A by means of ϵ and thus identify c with $\epsilon(c)$ for all $c \in K$. Hence we have

$$c := \epsilon(c) = c1_A = 1_A c \text{ and } ca = ac \text{ for all } c \in K \text{ and all } a \in A.$$

Keep in mind, that a K-algebra A is a K-vector space in a natural way.

(E) *(Homomorphisms of K-Algebras)* Let K be a field. A homomorphism of K-algebras $h : A \longrightarrow B$ is a map with K-algebras A and B such that:

(a) $h : A \longrightarrow B$ is a homomorphism of rings;
(b) $h(c) = c$ for all $c \in K$.

Observe, that a homomorphism of K-algebras is also a homomorphism of K-vector spaces.

(F) *(Modules)* We usually shall consider unital left-modules, hence modules M over a ring R, such that

$$x(m+n) = xm + xn, \quad (x+y)m = xm + ym, \quad (xy)m = x(ym) \text{ and } 1m = m$$

for all $x, y \in R$ and all $m, n \in M$. We shall refer to left-modules just as modules.
By a *homomorphism of R-modules* we mean a map $h : M \longrightarrow N$, with M and N both R-modules, such that

(a) $h(m+n) = h(m) + h(n)$ for all $m, n \in M$.
(b) $h(xm) = xh(m)$ for all $x \in R$ and all $m \in M$.

A *submodule* of a R-module M is a subset $N \subseteq M$, such that $m + n \in N$ and $xm \in N$ whenever $m, n \in N$ and $x \in R$. Clearly $0 := \{0\}$ and M are submodules of M.
If $h : M \longrightarrow N$ is a homomorphism of R-modules, the the *kernel* $\mathrm{Ker}(h) := \{m \in M \mid h(m) = 0\}$ of h is a submodule of M and the *image* $\mathrm{Im}(h) := h(M)$ of h is a submodule of N.
A sequence of (homomorphisms of) R-modules

$$M_0 \xrightarrow{h_0} M_1 \xrightarrow{h_1} M_2 \cdots M_{i-1} \xrightarrow{h_{i-1}} M_i \xrightarrow{h_i} M_{i+1} \cdots M_{r-1} \xrightarrow{h_{r-1}} M_r$$

is said to be *exact* if $\mathrm{Ker}(h_i) = \mathrm{Im}(h_{i-1})$ for all $i = 1, 2, \ldots, r-1$. A *short exact sequence* of R-modules is an exact sequence of the form $0 \longrightarrow M \xrightarrow{h} N \xrightarrow{l} P \longrightarrow 0$, meaning that h is injective, l is surjective and $\mathrm{Ker}(l) = \mathrm{Im}(h)$.
The *annihilator* of an R-module M is defined as the left ideal of R consisting of all elements which annihilate M, thus:

$$\mathrm{Ann}_R(M) := \{x \in R \mid xM = 0\}.$$

(G) *(Noetherian Modules and Rings)* Let R be a ring. A left R-module is said to be *Noetherian*, if it satisfies the following equivalent conditions

(i) Each left submodule $N \subseteq M$ if finitely generated, and hence of the form $N = \sum_{i=1}^{r} Rn_i$ with $r \in \mathbb{N}_0$ and $n_1, n_2, \ldots, n_r \in N$.
(ii Each ascending sequence $N_0 \subseteq N_1 \subseteq \cdots N_i \subseteq N_{i+1} \subseteq \cdots$ of left submodules $N_i \subseteq M$ ultimately becomes stationary and thus statisfies $N_{i_0} = N_{i_0+1} = N_{i_0+2} = \ldots$ for some $i_0 \in \mathbb{N}_0$.

We say that the ring R is *left Noetherian* if it is Noetherian as a left module.
Keep in mind the following facts:

(a) If $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ is an exact sequence of left R-modules then M is Noetherian if and only N and P are both Noetherian.
(b) If M and N are two Noetherian left R-modules, then their direct sum $M \oplus N$ is Noetherian, too.
(c) If R is left Noetherian, a left R-module M is Noetherian if and only if it is finitely generated.

(H) *(Modules of Finite Presentation)* Let R be a ring. A left R-module M is said to be of *finite presentation* if there is an exact sequence of left R-modules

$$R^s \xrightarrow{h} R^r \longrightarrow M \longrightarrow 0 \quad \text{with } r, s \in \mathbb{N}_0.$$

In this situation, the above exact sequence is called a *(finite) presentation* of M and $R^s \xrightarrow{h} R^r$ is called a *presenting homomorphism* for M.
Keep in mind, that the presenting homomorphism is given by a matrix with entries in R, more precisely: There is a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1r} \\ a_{21} & a_{22} & \ldots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \ldots & a_{sr} \end{pmatrix} \in R^{s \times r} \text{ such that}$$

$$h(x_1, x_2, \ldots, x_s) = (x_1, x_2, \ldots, x_s)A = \big(\sum_{i=1}^{s} x_i a_{i1}, \sum_{i=1}^{s} x_i a_{i2}, \ldots, \sum_{i=1}^{s} x_i a_{ir}\big)$$

for all $(x_1, x_2, \ldots, x_s) \in R^s$. This matrix A is called a *presentation matrix* for M.
Note the following facts:

(a) A left R-module M of finite presentation is finitely generated.
(b) If R is left Noetherian, then each finitely generated left R-module is of finite presentation.

(I) *(Graded Rings and Modules)* A *(positively) graded ring* is a ring R together with a family $(R_i)_{i \in \mathbb{N}_0}$ of additive subgroups $R_i \subseteq R$ such that

(1) $R = \bigoplus_{i \in \mathbb{N}_0} R_i$;
(2) $1 \in R_0$;
(3) for all $i, j \in \mathbb{N}_0$ and all $a \in R_i$ and all $b \in R_j$ it holds $ab \in R_{i+j}$.

In this situation we also refer to $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ as (positively) *graded R_0-algebra*. If $a \in R_i \setminus \{0\}$, we call a a *homogeneous element of degree i*.
Let $R' = \bigoplus_{i \in \mathbb{N}_0} R'_i$ be a second graded ring. A *homomorphism of graded rings* is a homomorphism $f : R \longrightarrow R'$ of rings which *respects gradings*, hence such that $f(R_i) \subseteq R'_i$

for all $i \in \mathbb{N}_0$. Clearly, the identity map $\mathrm{Id}_R : R \longrightarrow R$ of a graded ring as well as the composition of two homomorphisms of graded rings is a homomorphism of graded rings. An *isomorphism of graded rings* is a homomorphism of graded rings which admits an inverse which is a homomorphism of graded rings – or, equivalently – a bijective homomorphism of graded rings.
The (positively) graded ring $R = \bigoplus_{i\in\mathbb{N}_0} R_i$ is called a *homogeneous ring* if it is generated over R_0 by homogeneous elements of degree 1, hence if (in the notation introduced in part (B)) we have $R = R_0[R_1]$.
A *graded (left) module* over the graded ring R is a left R-module together with a family $(M_j)_{j\in\mathbb{Z}}$ of additive subgroups $M_j \subseteq M$ such that

(1) $M = \bigoplus_{j\in\mathbb{Z}} M_j$;
(2) For all $i \in \mathbb{N}_0$, all $j \in \mathbb{Z}$, all $a \in R_i$ and all $m \in M_j$ it holds $am \in M_{i+j}$.

A *homomorphism of graded (left) modules* is a homomorphism $h : M \longrightarrow N$ of R-modules, in which $M = \bigoplus_{j\in\mathbb{Z}} M_j$ and $N = \bigoplus_{j\in\mathbb{Z}} N_j$ are both graded and $h(M_j) \subseteq N_j$ for all $j \in \mathbb{Z}$. Clearly, the identity map of a graded R-module and the composition of two homomorphisms of graded R-modules are again homomorphisms of graded R-modules. An *isomorphism of graded R-modules* is a homomorphism of graded R-modules which admits an inverse which is a homomorphism of graded R-modules – or, equivalently – a bijective homomorphism of graded R-modules.

(K) *(Prime Varieties)* Let R be a commutative ring. We denote the *prime spectrum* of R, hence the set of all prime ideals in R, by $\mathrm{Spec}(R)$. If $\mathfrak{a} \subseteq R$ is an ideal, we denote by $\mathrm{Var}(\mathfrak{a})$ the *prime variety* of $\mathfrak{a}$, thus

$$\mathrm{Var}(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

Let

$$\sqrt{\mathfrak{a}} := \{a \in R \mid \exists n \in \mathbb{N} : a^n \in \mathfrak{a}\}.$$

denote the *radical ideal* of $\mathfrak{a}$. Keep in mind the following facts:

(a) $\mathrm{Var}(\mathfrak{a}) = \mathrm{Var}(\sqrt{\mathfrak{a}})$.
(b) If $\mathfrak{a}, \mathfrak{b} \subseteq R$ are ideals, then $\mathrm{Var}(\mathfrak{a}) = \mathrm{Var}(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

(L) *(Krull Dimension)* Let R be as in part (K) and let M be a finitely generated R-module. Then, the *(Krull) dimension* $\dim_R(M)$ of M is defined as the supremum of the lengths of chains of prime ideals which can be found in the prime variety of the annihilator of M:

$$\dim_R(M) := \sup\{r \in \mathbb{N}_0 \mid \exists \mathfrak{p}_0, \ldots, \mathfrak{p}_r \in \mathrm{Var}\big(\mathrm{Ann}_R(M)\big) \text{ with } \mathfrak{p}_{i-1} \subsetneq \mathfrak{p}_i \text{ for } i = 1, \ldots, r\}.$$

In particular, the (Krull) dimension $\dim(R)$ of R is the dimension of the R-module R:

$$\dim(R) = \sup\{r \in \mathbb{N}_0 \mid \exists \mathfrak{p}_0, \ldots, \mathfrak{p}_r \in \mathrm{Spec}(R) \text{ with } \mathfrak{p}_{i-1} \subsetneq \mathfrak{p}_i \text{ for } i = 1, \ldots, r\}.$$

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2. Filtered Algebras

We begin with a few general preliminaries, which will pave our way to introduce and to treat Weyl algebras and D-modules. Our first preliminary theme are filtered algebras over a field. It will turn out later, that this concept is of basic significance for the theory of Weyl algebras.

2.1. **Definition and Remark.** (A) Let K be a field and let A be K-algebra (see Conventions, Reminders and Notations 1.1 (D)). By a *filtration* of A we mean a family

$$A_\bullet = (A_i)_{i\in\mathbb{N}_0}$$

such that the following conditions hold:

(a) Each A_i is a K-vector subspace of A;
(b) $A_i \subseteq A_{i+1}$ for all $i \in \mathbb{N}_0$;
(c) $1 \in A_0$;
(d) $A = \bigcup_{i\in\mathbb{N}_0} A_i$;
(e) $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N}_0$.

In requirement (e) we have used the standard notation

$$A_iA_j := \sum_{(f,g)\in A_i\times A_j} Kfg \text{ for all } i, j \in \mathbb{N}_0,$$

which we shall use from now on without further mention. To simplify notation, we also often set

$$A_i = 0 \text{ for all } i < 0$$

and then write our filtration in the form

$$A_\bullet = (A_i)_{i\in\mathbb{Z}}.$$

If a filtration of A is given, we say that $(A, A_\bullet)$ or – by abuse of language – that A is a *filtered K-algebra.*

(B) Keep the notations and hypotheses of part (A) and let $A_\bullet = (A_i)_{i\in\mathbb{Z}}$ be a filtered K-algebra. Observe that we have the following statements:

(a) A_0 is a K-subalgebra of A.
(b) For all $i \in \mathbb{Z}$ the K-vector space A_i is a left- and a right- A_0-submodule of A.

2.2. **Example.** *(The degree filtration of a commutative polynomial ring)* Let $n \in \mathbb{N}$ and let $A = K[X_1, X_2, \ldots, X_n]$ be the commutative polynomial algebra over the field K in the indeterminates $X_1, X_2, \ldots, X_n$. Then clearly A is a K-space over its *monomial basis*:

$$A = K[X_1, X_2, \ldots, X_n] = \bigoplus_{\nu_1,\nu_2,\ldots,\nu_n\in\mathbb{N}_0} KX_1^{\nu_1}X_2^{\nu_2}\ldots X_n^{\nu_n} = \bigoplus_{\underline{\nu}\in\mathbb{N}_0^n} K\underline{X}^{\underline{\nu}},$$

where we have used use the standard notation

$$\underline{X}^{\underline{\nu}} := X_1^{\nu_1}X_2^{\nu_2}\ldots X_n^{\nu_n}, \text{ for } \underline{\nu} := (\nu_1, \nu_2 \ldots \nu_n) \in \mathbb{N}_0^n.$$

So, each $f \in A$ can be written as

$$f = \sum_{\underline{\nu}\in\mathbb{N}_0^n} c_{\underline{\nu}}^{(f)}\underline{X}^{\underline{\nu}}$$

with a unique family

$$\big(c_{\underline{\nu}}^{(f)}\big)_{\underline{\nu}\in\mathbb{N}_0^n} \in \prod_{\underline{\nu}\in\mathbb{N}_0^n} K = K^{\mathbb{N}_0^n},$$

whose *support*

$$\operatorname{supp}(f) = \operatorname{supp}\big((c_{\underline{\nu}}^{(f)})_{\underline{\nu}\in\mathbb{N}_0^n}\big) := \{\underline{\nu} \in \mathbb{N}_0^n \mid c_{\underline{\nu}}^{(f)} \neq 0\}$$

is finite. We also introduce the notation

$$|\underline{\nu}| = \sum_{i=1}^{n} \nu_i, \text{ for } \underline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}_0^n.$$

Then, with the usual convention of Conventions, Reminders andNotations 1.1 (A) we may describe the *degree* of the polynomial $f \in A$ by

$$\deg(f) := \sup\{|\underline{\nu}| \mid c_{\underline{\nu}}^{(f)} \neq 0\} = \sup\{|\underline{\nu}| \mid \underline{\nu} \in \operatorname{supp}(f)\}.$$

Now, for each $i \in \mathbb{N}_0$ we introduce the K-subspace A_i of A which is given by

$$A_i := \{f \in A \mid \deg(f) \le i\} = \bigoplus_{\underline{\nu}\in\mathbb{N}_0^n \text{ with } |\underline{\nu}|\le i} K\underline{X}^{\underline{\nu}}.$$

With the usual convention that $u + (-\infty) = -\infty$ for all $u \in \mathbb{Z} \cup \{-\infty\}$, we have the obvious relation

$$\deg(fg) = \deg(f) + \deg(g) \text{ for all } f, g \in A = K[X_1, X_2, \ldots, X_n].$$

From this it follows easily:

$$\text{The family } A_\bullet = \big(A_i := \{f \in A \mid \deg(f) \le i\}\big)_{i\in\mathbb{N}_0}$$

is a filtration of A. This filtration is called the *degree filtration* of the polynomial algebra $A = K[X_1, X_2, \ldots, X_n]$.

Clearly filtrations also may occur in non-commutative algebras. The next example presents somehow the "generic occurrence" of this.

2.3. **Example.** *(The degree filtration of a free associative algebra)* Let $n \in \mathbb{N}$, let K be a field and let $A = K\langle X_1, X_2, \ldots, X_n\rangle$ be the free associative algebra over K in the indeterminates $X_1, X_2, \ldots, X_n$. We suppose in particular that (see Conventions, Reminders and Notations 1.1 (D))

$$cX_i = X_i c \text{ for all } c \in K \text{ and all } i = 1, 2, \ldots, n,$$

and hence

$$cf = fc \text{ for all } c \in K \text{ and all } f \in A.$$

Let $i \in \mathbb{N}_0$. If

$$\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_i) \in \{1, 2, \ldots, n\}^i$$

is a sequence of length i with values in the set $\{1, 2, \ldots, n\}$ we write

$$\underline{X}_{\underline{\sigma}} := \prod_{j=1}^{i} X_{\sigma_j} = X_{\sigma_1} X_{\sigma_2} \ldots X_{\sigma_i}.$$

Then, with the usual convention that the product $\prod_{j\in\emptyset} X_j$ of an empty family of factors equals 1 and using the notation

$$\mathbb{S}_n := \bigl\{\{1, 2, \ldots, n\}^i \mid i \in \mathbb{N}_0\bigr\}$$

we can write A as a K-space over its *monomial basis* as follows:

$$\begin{aligned} A = K\langle X_1, X_2, \ldots, X_n\rangle &= \\ &= \bigoplus_{i\in\mathbb{N}_0} \bigoplus_{(\sigma_1,\sigma_2\ldots\sigma_i)\in\{1,2,\ldots,n\}^i} KX_{\sigma_1}X_{\sigma_2}\ldots X_{\sigma_i} = \\ &= \bigoplus_{i\in\mathbb{N}_0} \bigoplus_{\underline{\sigma}\in\{1,2,\ldots,n\}^i} K\underline{X}_{\underline{\sigma}} = \\ &= \bigoplus_{\underline{\sigma}\in\mathbb{S}_n} K\underline{X}_{\underline{\sigma}}. \end{aligned}$$

Clearly, as in the case of a commutative polynomial ring, each $f \in A$ may be written in the form

$$f = \sum_{\underline{\sigma}\in\mathbb{S}_n} c_{\underline{\sigma}}^{(f)} \underline{X}_{\underline{\sigma}}$$

with a unique family

$$\bigl(c_{\underline{\sigma}}^{(f)}\bigr)_{\underline{\sigma}\in\mathbb{S}_n} \in \prod_{\underline{\sigma}\in\mathbb{S}_n} K = K^{\mathbb{S}_n},$$

whose *support*

$$\operatorname{supp}(f) = \operatorname{supp}\bigl((c_{\underline{\sigma}}^{(f)})_{\underline{\sigma}\in\mathbb{S}_n}\bigr) := \{\underline{\sigma} \in \mathbb{S}_n \mid c_{\underline{\sigma}}^{(f)} \neq 0\}$$

is finite. We also introduce the notion of *length* of a sequence $\underline{\sigma} \in \mathbb{S}_n$ by setting

$$\lambda(\underline{\sigma}) := i, \text{ if } \underline{\sigma} \in \{1, 2, \ldots, n\}^i.$$

Now, we may define the *degree* of an element $f \in A$ by

$$\deg(f) := \sup\{\lambda(\underline{\sigma}) \mid c_{\underline{\sigma}}^{(f)} \neq 0\} = \sup\{\lambda(\underline{\sigma}) \mid \underline{\sigma} \in \operatorname{supp}(f)\}.$$

For each $i \in \mathbb{N}_0$ we introduce a K-subspace A_i of A, by setting

$$A_i := \{f \in A \mid \deg(f) \leq i\} = \bigoplus_{\underline{\sigma} \in \mathbb{S}_n \text{ with } \lambda(\underline{\sigma}) \leq i} K\underline{X}_{\underline{\sigma}}.$$

We obviously have the relation

$$\deg(fg) \leq \deg(f) + \deg(g) \text{ for all } f, g \in A = K\langle X_1, X_2, \ldots, X_n\rangle.$$

Moreover, it is easy to see:

$$\text{The family } A_\bullet = \big(A_i = \{f \in A \mid \deg(f) \leq i\}\big)_{i \in \mathbb{N}_0}$$

is a filtration of A. This filtration is called the *degree filtration* of the free associative K-algebra $A = K\langle X_1, X_2, \ldots, X_n\rangle$.

Later, our basic filtered algebras will be Weyl algebras. These are non-commutative too, but they also admit the notion of degree and of degree filtration. From the point of view of filtrations, these algebras will turn out to be "close to commutative", as we shall see later. To make this more precise, we will introduce the notion of associated graded ring with respect to a filtration in the next Section.

3. Associated Graded Rings

3.1. **Remark and Definition.** (A) Let K be a field and let $A = (A, A_\bullet)$ be a filtered K-algebra. We consider the K-vector space

$$\operatorname{Gr}(A) = \operatorname{Gr}_{A_\bullet}(A) = \bigoplus_{i \in \mathbb{N}_0} A_i/A_{i-1}.$$

For all $i \in \mathbb{N}_0$ we also use the notation

$$\operatorname{Gr}(A)_i = \operatorname{Gr}_{A_\bullet}(A)_i := A_i/A_{i-1},$$

so that we may write

$$\operatorname{Gr}(A) = \operatorname{Gr}_{A_\bullet}(A) = \bigoplus_{i \in \mathbb{N}_0} \operatorname{Gr}_{A_\bullet}(A)_i.$$

(B) Let $i, j \in \mathbb{N}_0$, let $f, f' \in A_i$ and let $g, g' \in A_j$ such that

$$h := f - f' \in A_{i-1} \text{ and } k := g - g' \in A_{j-1}.$$

It follows that

$$\begin{aligned} fg - f'g' &= fg - (f-h)(g-k) = fk + hg - hk \\ &\in A_iA_{j-1} + A_{i-1}A_j + A_{i-1}A_{j-1} \subseteq \\ &\subseteq A_{i+(j-1)} + A_{j+(i-1)} + A_{(i-1)+(j-1)} \subseteq A_{i+j-1}. \end{aligned}$$

So in $A_{i+j}/A_{i+j-1} = \mathrm{Gr}_{A_\bullet}(A)_{i+j} \subset \mathrm{Gr}_{A_\bullet}(A)$ we get the relation

$$fg + A_{i+j-1} = f'g' + A_{i+j-1}.$$

This allows to define a *multiplication* on the K-space $\mathrm{Gr}_{A_\bullet}(A)$ which is induced by

$$(f + A_{i-1})(g + A_{j-1}) := fg + A_{i+j-1} \text{ for all } i, j \in \mathbb{N}_0, \text{ all } f \in A_i \text{ and all } g \in A_j.$$

With respect to this multiplication, the K-vector space $\mathrm{Gr}_{A_\bullet}(A)$ acquires a structure of K-algebra.
Observe that, if $r, s \in \mathbb{N}_0$ and

$$\overline{f} = \sum_{i=0}^{r} \overline{f_i}, \text{ with } f_i \in A_i \text{ and } \overline{f_i} = (f_i + A_{i-1}) \in \mathrm{Gr}_{A_\bullet}(A)_i \text{ for all } i = 0, 1, \ldots, r,$$

and, moreover

$$\overline{g} = \sum_{j=0}^{s} \overline{g_j}, \text{ with } g_j \in A_j \text{ and } \overline{g_j} = (g_j + A_{j-1}) \in \mathrm{Gr}_{A_\bullet}(A)_j \text{ for all } j = 0, 1, \ldots, s,$$

then

$$\overline{f}\,\overline{g} = \sum_{k=0}^{r+s} \sum_{i+j=k} \overline{f_i}\,\overline{g_j} = \sum_{k=0}^{r+s} \sum_{i+j=k} (f_i g_j + A_{i+j-1}).$$

(C) Keep the above notations and hypotheses. Observe in particular, that $\mathrm{Gr}_{A_\bullet}(A)_0$ is a K-subalgebra of $\mathrm{Gr}_{A_\bullet}(A)$, and that there is an isomorphism of K-algebras

$$\mathrm{Gr}_{A_\bullet}(A)_0 \cong A_0.$$

Moreover, with respect to our multiplication on $\mathrm{Gr}_{A_\bullet}(A)$ we have the relations

$$\mathrm{Gr}_{A_\bullet}(A)_i \mathrm{Gr}_{A_\bullet}(A)_j \subseteq \mathrm{Gr}_{A_\bullet}(A)_{i+j} \text{ for all } i, j \in \mathbb{N}_0.$$

So, the K-vector space $\mathrm{Gr}_{A_\bullet}(A)$ is turned into a (positively) graded ring

$$\mathrm{Gr}_{A_\bullet}(A) = \big(\mathrm{Gr}_{A_\bullet}(A), (\mathrm{Gr}_{A_\bullet}(A)_i)_{i \in \mathbb{N}_0}\big) = \bigoplus_{i \in \mathbb{N}_0} \mathrm{Gr}_{A_\bullet}(A)_i$$

by means of the above multiplication. We call this ring the *associated graded ring* of A with respect to the filtration $A_\bullet$. From now on, we always furnish $\mathrm{Gr}_{A_\bullet}(A)$ with this multiplication.

3.2. **Example and Exercise.** (A) Let $n \in \mathbb{N}$, let K be a field and consider the commutative polynomial ring $A = K[X_1, X_2, \ldots, X_n]$. Show that A has the following *universal property* within the category of all commutative K-algebras:

> If B is a commutative K-algebra and $\phi : \{X_1, X_2, \ldots, X_n\} \longrightarrow B$ is a map, then there is a unique homomorphism of K-algebras $\widetilde{\phi} : A \longrightarrow B$ such that $\widetilde{\phi}(X_i) = \phi(X_i)$ for all $i = 1, 2, \ldots, n$.

Show also, that A has the following *relational universal property* within the category of all associative K-algebras:

If B is an associative K-algebra and $\phi : \{X_1, X_2, \ldots, X_n\} \longrightarrow B$ is a map such that $\phi(X_i)\phi(X_j) = \phi(X_j)\phi(X_i)$ for all $i, j \in \{1, 2, \ldots, n\}$, then there is a unique homomorphism of K-algebras $\widetilde{\phi} : A \longrightarrow B$ such that $\widetilde{\phi}(X_i) = \phi(X_i)$ for all $i = 1, 2, \ldots, n$.

(B) Now, furnish $A = K[X_1, X_2, \ldots, X_n]$ with its degree filtration (see Example 2.2). Then, on use of the above universal property of A it is not hard to show that there is an isomorphism of K-algebras

$$K[X_1, X_2, \ldots, X_n] \xrightarrow{\cong} \mathrm{Gr}_{A_\bullet}(A),$$

given by $X_i \mapsto (X_i + A_0) \in A_1/A_0 = \mathrm{Gr}_{A_\bullet}(A)_1 \subset \mathrm{Gr}_{A_\bullet}(A)$ for all $i = 1, 2 \ldots, n$.

We now introduce a class of filtrations, which will be of particular interest for our lectures.

3.3. **Definition.** Let K be a field and let $A = (A, A_\bullet)$ be a filtered K-algebra. The filtration $A_\bullet$ is said to be *commutative* if

$$fg - gf \in A_{i+j-1} \text{ for all } i, j \in \mathbb{N}_0 \text{ and for all } f \in A_i \text{ and all } g \in A_j.$$

It is equivalent to say that the associated graded ring $\mathrm{Gr}_{A_\bullet}(A)$ is commutative. In this situation, we also say that $(A, A_\bullet)$ is a *commutatively filtered K-algebra.*

Later, in the case of Weyl algebras, we shall meet various interesting commutative filtrations - and precisely this makes these algebras to a subject which is intimately tied to Commutative Algebra. We now shall define three special types of commutative filtrations, which will play a particularly important rôle in Weyl algebras.

3.4. **Definition and Remark.** (A) Let $(A, A_\bullet)$ be a filtered K-algebra. The filtration $A_\bullet$ is said to be *very good* if it satisfies the following conditions:

(a) The filtration $A_\bullet$ is commutative;
(b) $A_0 = K$;
(c) $\dim_K(A_1) < \infty$;
(d) $A_i = A_1 A_{i-1}$ for all $i \in \mathbb{N}$.

Under these circumstances and on use of the notation introduced in Conventions, Reminders and Notations 1.1 (B) we clearly have

$$\dim_K(A_1/A_0) = \dim_K\big(\mathrm{Gr}_{A_\bullet}(A)_1\big) = \dim_K(A_1) - 1 < \infty \text{ and } \mathrm{Gr}_{A_\bullet}(A) = K[\mathrm{Gr}_{A_\bullet}(A)_1].$$

So, in this situation, the associated graded ring $\mathrm{Gr}_{A_\bullet}(A)$ is a commutative homogeneous (thus standard graded) Noetherian K-algebra (see Conventions, Reminders and Notations 1.1 (I)). If $A_\bullet$ is a very good filtration of A, we say that $(A, A_\bullet)$ – or briefly A – is a *very well-filtered* K-algebra.

(B) Let $(A, A_\bullet)$ be a filtered K-algebra. The filtration $A_\bullet$ is said to be *good* if it satisfies the following conditions:

(a) The filtration $A_\bullet$ is commutative;
(b) A_0 is a K-algebra of finite type;
(c) A_1 is finitely generated as a (left-)module over A_0;

(d) $A_i = A_1 A_{i-1}$ for all $i \in \mathbb{N}$.

Under these circumstances we clearly have

$$\begin{aligned} A_0 &\cong \mathrm{Gr}_{A_\bullet}(A)_0 \text{ is commutative and Noetherian} \\ A_1/A_0 &= \mathrm{Gr}_{A_\bullet}(A)_1 \text{ is a finitely generated } A_0\text{-module, and} \\ \mathrm{Gr}_{A_\bullet}(A) &= \mathrm{Gr}_{A_\bullet}(A)_0[\mathrm{Gr}_{A_\bullet}(A)_1]. \end{aligned}$$

So, in this situation, the associated graded ring $\mathrm{Gr}_{A_\bullet}(A)$ is a commutative homogeneous Noetherian A_0-algebra (see Conventions, Reminders and Notations 1.1 (I)). If $A_\bullet$ is a good filtration of A, we say that $(A, A_\bullet)$ – or briefly A – is a *well-filtered* K-algebra. Clearly, a very well-filtered K-algebra is also well-filtered.

(C) Let $(A, A_\bullet)$ be a filtered K-algebra. The filtration $A_\bullet$ is said to be *of finite type* if it satisfies the following conditions:

(a) The filtration $A_\bullet$ is commutative;
(b) A_0 is a K-algebra of finite type;
(c) There is an integer $\delta \in \mathbb{N}$ such that A_j is finitely generated as a (left-)module over A_0 for all $j \leq \delta$ and
(d) $A_i = \sum_{j=1}^{\delta} A_j A_{i-j}$ for all $i > \delta$.

In this situation, we call the number δ a *generating degree* of the filtration $A_\bullet$. Under these circumstances we clearly have

$$\begin{aligned} A_i &= \sum_{1 \leq j_1, \ldots, j_s \leq \delta : j_1 + \cdots + j_s = i} A_{j_1} \cdots A_{j_s}, \quad (\forall i \in \mathbb{N}) \\ A_0 &\cong \mathrm{Gr}_{A_\bullet}(A)_0 \text{ is commutative and Noetherian} \\ A_1/A_0 &= \mathrm{Gr}_{A_\bullet}(A)_1 \text{ is a finitely generated } A_0\text{-module, and} \\ \mathrm{Gr}_{A_\bullet}(A) &= \mathrm{Gr}_{A_\bullet}(A)_0[\sum_{i=1}^{\delta} \mathrm{Gr}_{A_\bullet}(A)_i]. \end{aligned}$$

So, in this situation, the associated graded ring $\mathrm{Gr}_{A_\bullet}(A)$ is a commutative Noetherian graded A_0-algebra, which is generated by finitely many homogeneous elements of degree $\leq \delta$. If $A_\bullet$ is a filtration of A, which is of finite type, we say that $(A, A_\bullet)$ is a *filtered algebra of finite type.*

Clearly, a well-filtered K-algebra is also finitely filtered. Moreover, if $A_\bullet$ is of finite type and $\delta = 1$, the filtration $A_\bullet$ is good.

3.5. **Example and Exercise.** (A) Let $n \in \mathbb{N}$, let K be a field and consider the commutative polynomial ring $A = K[X_1, X_2, \ldots, X_n]$, furnished with its degree filtration. Then, it is easy to see, that $A = K[X_1, X_2, \ldots, X_n]$ is a very well filtered K-algebra.

(B) Let $n \in \mathbb{N}$, let K be a field and consider the commutative polynomial ring $A = K[X_1, X_2, \ldots, X_n]$. Let $m \in \{0, 1, \ldots, n-1\}$ and consider the subring $B := K[X_1, X_2, \ldots, X_m] \subset A$, so that $A = B[X_{m+1}, X_{m+2}, \ldots, X_n]$. For each polynomial $f = \sum_{\underline{\nu}} c_{\underline{\nu}}^{(f)} \underline{X}^{\underline{\nu}} \in A$ we denote by $\deg_B(f)$ the degree of f with respect to the indeterminates $X_{m+1}, X_{m+2}, \ldots, X_n$, hence the degree of f considered as a polynomial in these indeterminates with coefficients in B. Thus we may write

$$\deg_B(f) = \sup\{\sum_{i=1}^{n} w_i\nu_i \mid (\nu_1, \nu_2, \ldots, \nu_n) \in \operatorname{supp}(f)\}$$

where

$$w_1 = w_2 = \cdots = w_m = 0 \text{ and } w_{m+1} = w_{m+2} = \cdots = w_n = 1.$$

Show, that by

$$A_i := \{f \in A \mid \deg_B(f) \leq i\} \text{ for all } i \in \mathbb{N}_0$$

a good filtration $A_\bullet$ on A is defined and that there is a canonical isomorphism of graded B-algebras

$$A = B[X_{m+1}, X_{m+2}, \ldots, X_n] \cong \operatorname{Gr}_{A_\bullet}(A),$$

where $A = B[X_{m+1}, X_{m+2}, \ldots, X_n]$ is endowed with the standard grading with respect to the indeterminates $X_{m+1}, \ldots, X_n$, hence with the grading given by $\deg(X_i) = 0$ if $1 \leq i \leq m$ and $\deg(X_i) = 1$ for $m < i \leq n$.

(C) Let $n \in \mathbb{N}$, with $n > 1$, let K be a field and consider the free associative K-algebra $A = K\langle X_1, X_2, \ldots, X_n\rangle$, furnished with its degree filtration $A_\bullet$. For each $i \in \{1, 2, \ldots, n\}$, let

$$\overline{X}_i := (X_i + A_0) \in A_1/A_0 = \operatorname{Gr}_{A_\bullet}(A)_1 \subset \operatorname{Gr}_{A_\bullet}(A).$$

Show that

$$\overline{X}_i\overline{X}_j = \overline{X}_j\overline{X}_i \text{ if and only if } i = j.$$

(D) Let the notations and hypotheses be as in part (C). Show that $A = K\langle X_1, X_2, \ldots, X_n\rangle$ has the following universal property in the category of K-algebras:

> If B is a K-algebra and $\phi : \{X_1, X_2, \ldots, X_n\} \longrightarrow B$ is a map, there is a unique homomorphism of K-algebras $\widetilde{\phi} : A \longrightarrow B$ such that $\widetilde{\phi}(X_i) = \phi(X_i)$ for all $i = 1, 2, \ldots, n$.

Use this to show, that there is a unique homomorphism of (graded) K-algebras (which must be in addition surjective)

$$\widetilde{\phi} : A \twoheadrightarrow \operatorname{Gr}_{A_\bullet}(A), \text{ such that } X_i \mapsto \overline{X}_i := (X_i + A_0) \in A_1/A_0 = \operatorname{Gr}_{A_\bullet}(A)_1.$$

(E) Let $(A, A_\bullet)$ be a filtered K-algebra, let $r \in \mathbb{N}$ and let $i_1, i_2, \ldots, i_r \in \mathbb{N}_0$. We define inductively

$$A_{i_1}A_{i_2}\ldots A_{i_r} = \prod_{j=1}^{r} A_{i_j} := \begin{cases} A_{i_1}, & \text{if } r = 1,\\ \big(\prod_{j=1}^{r-1} A_{i_j}\big)A_{i_r}, & \text{if } r > 1.\end{cases}$$

In particular, if $i \in \mathbb{N}_0$ we set

$$(A_i)^r := \prod_{j=1}^{r} A_i.$$

Assume now, that the filtration $A_\bullet$ is good and prove that

$$A_r = (A_1)^r \text{ and } A_iA_j = A_{i+j} \text{ for all } r \in \mathbb{N} \text{ and all } i, j \in \mathbb{N}_0.$$

Assume that the filtration $A_\bullet$ is of finite type and has generating degree δ. Prove that

$$A_i = \sum_{\nu_0,\nu_1,\ldots,\nu_\delta\in\mathbb{N}_0 : i=\sum_{j=0}^{\delta} j\nu_j} \prod_{j=0}^{\delta} A_j^{\nu_j} \text{ for all } i \in \mathbb{N}_0.$$

4. Derivations

Filtered K-algebras and their associated graded rings are one basic ingredient of the theory of Weyl algebras. Another basic ingredient are derivations (or derivatives). The present section is devoted to this subject.

4.1. **Definition and Remark.** (A) Let K be a field, let A be a commutative K-algebra and let M be an A-module. A *K-derivation* (or *K-derivative*) on A with values in M is a map $d : A \longrightarrow M$ such that:

(a) d is K-linear: $d(\alpha a + \beta b) = \alpha d(a) + \beta d(b)$ for all $\alpha, \beta \in K$ and all $a, b \in A$.
(b) d satisfies the *Leibniz Product Rule*: $d(ab) = ad(b) + bd(a)$ for all $a, b \in A$.

We denote the set of all K-derivations on A with values in M by $\mathrm{Der}_K(A, M)$, thus:

$$\mathrm{Der}_K(A, M) := \{d \in \mathrm{Hom}_K(A, M) \mid d(ab) = ad(b) + bd(a) \text{ for all } a, b \in A\}.$$

To simplify notations, we also write

$$\mathrm{Der}_K(A, A) =: \mathrm{Der}_K(A).$$

(B) Keep in mind, that $\mathrm{Hom}_K(A, M)$ carries a natural structure of A-module, with scalar multiplication given by

$$(ah)(x) := a(h(x)) \text{ for all } a \in A,\ \text{all } h \in \mathrm{Hom}_K(A, M) \text{ and all } x \in A.$$

It is easy to verify:

$$\mathrm{Der}_K(A, M) \text{ is a submodule of the } A\text{-module } \mathrm{Hom}_K(A, M).$$

With our usual convention (suggested in Conventions, Reminders and Notations 1.1 (D)) that we identify $c \in K$ with $c1_A \in A$, the rules (a) and (b) of part (A) imply $d(c) = d(c1) = cd(1)$ and $d(c1) = 1d(c) + cd(1)$, hence $d(c) = d(c) + cd(1) = d(c) + d(c)$, thus

$$d(c) = 0 \text{ for all } c \in K \text{ and all } d \in \mathrm{Der}_K(A, M) : \text{ “Derivations vanish on constants.”}$$

Next, we shall look at the arithmetic properties of derivations and gain an important embedding procedure for modules of derivations of K-algebras of finite type.

4.2. **Exercise and Definition.** (A) Let K be a field, let A be a commutative K-algebra and let M be an A-module. Let $d \in \mathrm{Der}_K(A, M)$, let $r \in \mathbb{N}$, let $\nu_1, \nu_2, \ldots, \nu_r \in \mathbb{N}$ and let $a_1, a_2, \ldots, a_r \in A$. Use induction on r to prove the *Generalized Product Rule*

$$d\Big(\prod_{j=1}^{r} a_j^{\nu_j}\Big) = \sum_{i=1}^{r} \nu_i a_i^{\nu_i - 1}\Big(\prod_{j\neq i} a_j^{\nu_j}\Big) d(a_i)$$

and the resulting *Power Rule*

$$d(a^r) = ra^{r-1}d(a) \text{ for all } a \in A.$$

(B) Let the notations and hypotheses be as in part (A). Assume in addition that $A = K[a_1, a_2, \ldots, a_r]$. Let $e \in \mathrm{Der}_K(A, M)$. Use what you have shown in part (A) together with the fact that e and d are K-linear to prove that the following uniqueness statement holds:

$$e = d \text{ if and only if } e(a_i) = d(a_i) \text{ for all } i = 1, 2, \ldots, r.$$

(C) Yet assume that $A = K[a_1, a_2, \ldots, a_r]$. Prove that there is a monomorphism (thus an injective homomorphism) of A-modules

$$\Theta^M_{\underline{a}} = \Theta^M_{(a_1, a_2, \ldots, a_r)} : \mathrm{Der}_K(A, M) \longrightarrow M^r, \text{ given by } d \mapsto \big(d(a_1), d(a_2), \ldots, d(a_r)\big).$$

This monomorphism $\Theta^M_{\underline{a}}$ is called the *embedding* of $\mathrm{Der}_K(A, M)$ in M^r with respect to $\underline{a} := (a_1, a_2, \ldots, a_r)$.

(D) Let the notations and hypotheses be as in part (C). Assume that M is finitely generated. Prove, that the A-module $\mathrm{Der}_K(A, M)$ is finitely generated.

Now, we turn to derivatives in polynomial algebras, a basic ingredient of Weyl algebras.

4.3. **Exercise and Definition.** *(Partial Derivatives in Polynomial Rings)* (A) Let $n \in \mathbb{N}$, let K be a field and consider the polynomial algebra $K[X_1, X_2, \ldots, X_n]$. Fix $i \in \{1, 2, \ldots, n\}$. Then, using the monomial basis of $K[X_1, X_2, \ldots, X_n]$ we see that there is a unique K-linear map

$$\partial_i = \frac{\partial}{\partial X_i} : K[X_1, X_2, \ldots, X_n] \longrightarrow K[X_1, X_2, \ldots, X_n]$$

such that for all $\underline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}_0^n$ we have

$$\partial_i(\underline{X}^{\underline{\nu}}) = \frac{\partial}{\partial X_i}\big(\prod_{j=1}^{n} X_j^{\nu_j}\big) = \begin{cases} \nu_i X_i^{\nu_i - 1} \prod_{j \neq i} X_j^{\nu_j}, & \text{if } \nu_i > 0 \\ 0, & \text{if } \nu_i = 0. \end{cases}$$

(B) Keep the notations and hypotheses of part (A). Let

$$\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_n), \quad \underline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}_0^n$$

and prove that

$$\partial_i\big(\underline{X}^{\underline{\mu}}\underline{X}^{\underline{\nu}}\big) = \underline{X}^{\underline{\mu}}\partial_i\big(\underline{X}^{\underline{\nu}}\big) + \underline{X}^{\underline{\nu}}\partial_i\big(\underline{X}^{\underline{\mu}}\big).$$

Use the K-linearity of ∂_i to conclude that

$$\partial_i = \frac{\partial}{\partial X_i} \in \mathrm{Der}_K\big(K[X_1, X_2, \ldots, X_n]\big) \text{ for all } i = 1, 2 \ldots, n.$$

The derivation $\partial_i = \frac{\partial}{\partial X_i}$ is called the *i-th partial derivative* in $K[X_1, X_2, \ldots, X_n]$.

As we shall see in the proposition below, the embedding introduced in Exercise and Definition 4.2 (C) takes a particularly favorable shape in the case of polynomial algebras. The exercice to come is aimed to prepare the proof this.

4.4. **Exercise.** (A) Let the notations and hypotheses be as in Exercise and Definition 4.3. For all $i, j \in \mathbb{Z}$ let $\delta_{i,j}$ denote the *Kronecker symbol*, so that

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Check that

$$\partial_i(X_j) = \delta_{i,j}, \text{ for all } i, j \in \{1, 2 \ldots, n\}.$$

(B) Keep the above notations and hypotheses. Show that

(a) For each $i \in \{1, 2, \ldots, n\}$ it holds $K[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n] \subseteq \mathrm{Ker}(\partial_i)$ with equality if and only if $\mathrm{Char}(K) = 0$.

(b) $K \subseteq \bigcap_{i=1}^n \mathrm{Ker}(\partial_i)$ with equality if and only if $\mathrm{Char}(K) = 0$.

4.5. **Proposition.** ***(The Canonical Basis for the Derivations of a Polynomial Ring)*** *Let $n \in \mathbb{N}$, let K be a field and consider the polynomial algebra $K[X_1, X_2, \ldots, X_n]$. Then the canonical embedding of $\mathrm{Der}_K\big(K[X_1, X_2, \ldots, X_n]\big)$ into $K[X_1, X_2, \ldots, X_n]^n$ with respect to $X_1, X_2, \ldots, X_n$ (see Exercise and Definition 4.2 (C)) yields an isomorphism of $K[X_1, X_2, \ldots, X_n]$-modules*

$$\Theta := \Theta_{X_1, X_2, \ldots, X_n} : \mathrm{Der}_K\big(K[X_1, X_2, \ldots, X_n]\big) \xrightarrow{\cong} K[X_1, X_2, \ldots, X_n]^n,$$

given by

$$d \mapsto \Theta(d) := \Theta_{X_1, X_2, \ldots, X_n}(d) = \big(d(X_1), d(X_2), \ldots, d(X_n)\big),$$
$$\textit{for all } d \in \mathrm{Der}_K\big(K[X_1, X_2, \ldots, X_n]\big).$$

In particular, the n partial derivatives $\partial_1, \partial_2, \ldots, \partial_n$ form a free basis of the $K[X_1, X_2, \ldots, X_n]$-module $\mathrm{Der}_K\big(K[X_1, X_2, \ldots, X_n]\big)$, hence

$$\mathrm{Der}_K\big(K[X_1, X_2, \ldots, X_n]\big) = \bigoplus_{i=1}^n K[X_1, X_2, \ldots, X_n]\partial_i.$$

Proof. According to Exercise and Definition 4.2 (C), the map Θ is a monomorphism of $K[X_1, X_2, \ldots, X_n]$-modules. By what we have seen in Exercise 4.4 (A) we have

$$\Theta(\partial_i) = \big(\delta_{i,1}, \delta_{i,2}, \ldots, \delta_{i-1,i}, \delta_{i,i}, \delta_{i,i+1}, \ldots, \delta_{i,n}\big) = \big(\delta_{i,j}\big)_{j=1}^n =: \mathrm{e}_i$$

for all $i = 1, 2, \ldots, n$. As the n elements

$$\mathrm{e}_i = \big(\delta_{i,j}\big)_{j=1}^n \in K[X_1, X_2, \ldots, X_n]^n \text{ with } i = 1, 2, \ldots, n$$

form the canonical free basis of the $K[X_1, X_2, \ldots, X_n]$-module $K[X_1, X_2, \ldots, X_n]^n$ our claims follow immediately. □

5. Weyl Algebras

Now, we are ready to introduce Weyl algebras. We first remind a few facts on endomorphism rings of commutative K-algebras and relate these to modules of derivations.

5.1. **Reminder and Remark.** (A) Let K be a field and let A be a commutative K-algebra and let M be an A-module. Keep in mind, that the A-module

$$\mathrm{End}_K(M) := \mathrm{Hom}_K(M, M)$$

carries a natural structure of K-algebra, whose multiplication is given by composition of maps, thus:

$$fg := f \circ g, \text{ hence } (fg)(m) := f(g(m)) \text{ for all } f, g \in \mathrm{End}_K(M) \text{ and all } m \in M.$$

The module $\mathrm{End}_K(M)$ endowed with this multiplication is called the *K-endomorphism ring* of M. Observe, that this endomorphism ring is not commutative in general.

(B) Keep the above notations and hypothesis. Then, we have a *canonical homomorphism* of rings

$$\epsilon_M : A \longrightarrow \mathrm{End}_K(M) \text{ given by } a \mapsto \epsilon_M(a) := a\mathrm{id}_M \text{ for all } a \in A,$$

where $\mathrm{id}_M : M \longrightarrow M$ is the *identity map* on M, so that

$$\epsilon_M(a)(m) = am \text{ for all } a \in A \text{ and all } m \in M.$$

It is immediate to verify that this canonical homomorphism is injective if $M = A$:

$$\text{The canonical homomorphism } \epsilon_A : A \longrightarrow \mathrm{End}_K(A) \text{ is injective.}$$

We therefore call the map $\epsilon_A : A \longrightarrow \mathrm{End}_K(A)$ the *canonical embedding* of A into its K-endomorphism ring and we consider A as a subalgebra of $\mathrm{End}_K(A)$ by means of this canonical embedding. So, for all $a \in A$ we identify a with $\epsilon_A(a)$.

5.2. **Remark and Definition.** (A) Let K be a field and let A be a commutative K-algebra. By the convention made in Reminder and Remark 5.1 we may consider A as a subalgebra of the endomorphism ring $\mathrm{End}_K(A)$. We obviously also have $\mathrm{Der}_K(A) \subseteq \mathrm{End}_K(A)$. So using the notation introduced in Conventions, Reminders and Notations 1.1 (B), we have may consider the K-subalgebra

$$W_K(A) := K[A \cup \mathrm{Der}_K(A)] = A[\mathrm{Der}_K(A)] \subseteq \mathrm{End}_K(A).$$

of the K-endomorphism ring of A which is generated by A and all derivations on A with values in A. We call $W_K(A)$ the *Weyl algebra of the K-algebra A*.

(B) Keep the hypotheses and notations of part (A). Assume in addition, that the commutative K-algebra A is of finite type, so that we find some $r \in \mathbb{N}_0$ and elements $a_1, a_2, \ldots, a_r \in A$ such that

$$A = K[a_1, a_2, \ldots, a_r].$$

Then according to Exercise and Definition 4.2 (D), the A-module $\mathrm{Der}_K(A)$ is finitely generated. We thus find some $s \in \mathbb{N}_0$ and derivations $d_1, d_2, \ldots, d_s \in \mathrm{Der}_K(A)$ such that

$$\mathrm{Der}_K(A) = \sum_{i=1}^{s} Ad_i.$$

A straight forward computation now allows to see, that

$$W_K(A) = K[a_1, a_2 \ldots, a_r, d_1, d_2, \ldots, d_s] \subseteq \mathrm{End}_K(A).$$

In particular we may conclude, that the K-algebra $W_K(A)$ is finitely generated.

(C) Keep the above notations and let $n \in \mathbb{N}$. The *n-th standard Weyl algebra* $\mathbb{W}(K,n)$ over the field K is defined as the Weyl algebra of the polynomial ring $K[X_1, X_2, \ldots, X_n]$, thus

$$\mathbb{W}(K,n) := W_K\big(K[X_1, X_2, \ldots, X_n]\big) \subseteq \operatorname{End}_K\big(K[X_1, X_2, \ldots, X_n]\big).$$

Observe, that by Propsition 4.5 and according to the observations made in part (B) we may write

$$\mathbb{W}(K,n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_1, \partial_2, \ldots, \partial_n] \subseteq \operatorname{End}_K\big(K[X_1, X_2, \ldots, X_n]\big).$$

The elements of $\mathbb{W}(K,n)$ are called *polynomial differential operators* in the indeterminates $X_1, X_2, \ldots, X_n$ over the field K. They are all K-linear combinations of products of indeterminates X_i and partial derivatives ∂_j.
The differential operators of the form

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} := X_1^{\nu_1} \ldots X_n^{\nu_n}\partial_1^{\mu_1} \ldots \partial_n^{\mu_n} = \prod_{i=1}^{n} X^{\nu_i} \prod_{j=1}^{n} \partial^{\mu_j} \in \mathbb{W}(K,n)$$

with

$$\underline{\nu} := (\nu_1, \ldots, \nu_n), \quad \underline{\mu} := (\mu_1, \ldots, \mu_n) \in \mathbb{N}_0^n$$

are called *elementary differential operators* in the indeterminates $X_1, X_2, \ldots, X_n$ over the field K.

We now aim to study the structure of standard Weyl algebras. One of the main goals we are heading for is to find an appropriate "monomial basis" in each of these algebras. We namely shall see later that the previously introduced elementary differential operators form a K-basis of the standard Weyl algebra $\mathbb{W}(K,n)$, provided K is of characteristic 0. To pave our way to this fundamental result, we first of all have to prove that in standard Weyl algebras certain commutation relations hold: the so-called Heisenberg relations. To establish these relations, we begin with the following preparations.

5.3. **Remark and Exercise.** (A) If K is a field and B is a K-algebra, we introduce the *Poisson operation*, that is the map

$$[\bullet, \bullet] : B \times B \longrightarrow B, \text{ defined by } [a,b] := ab - ba \text{ for all } a, b \in B.$$

Show, that the Poisson operation has the following properties:

(a) $[a,b] = -[b,a]$ for all $a, b \in B$.
(b) $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$ for all $a, b, c \in B$.
(c) $[\alpha a + \alpha' a', \beta b + \beta' b'] = \alpha\beta[a,b] + \alpha\beta'[a,b'] + \alpha'\beta[a',b] + \alpha'\beta'[a',b']$ for all $\alpha, \alpha', \beta, \beta' \in K$ and all $a, a', b, b' \in B$.

Observe in particular, that statement (a) says that the Poisson operation is *anti-commutative*, whereas statement (c) says that this operation is *K-bilinear*. We call $[a,b]$ the *commutator* of a and b.

(B) Now, let K be a field, let A be a commutative K-algebra and consider the Weyl algebra $W_K(A) := A[\operatorname{Der}_K(A)]$. Show that the following relations hold:

(a) $[a, b] = 0$ for all $a, b \in A$.
(b) $[a, d] = -d(a)$ for all $a \in A$ and all $d \in \mathrm{Der}_K(A)$.
(c) $[d, e] \in \mathrm{Der}_K(A)$ for all $d, e \in \mathrm{Der}_K(A)$.

(C) Let the notations and hypotheses be as in part (B). Let $d, e \in \mathrm{Der}_K(A)$, let $r \in \mathbb{N}$, let $\nu_1, \nu_2, \ldots, \nu_r \in \mathbb{N}$ and let $a_1, a_2, \ldots, a_r \in A$. Use statement (c) of part (B) and the Generalized Product Rule of Exercise and Definition 4.2 (A) to prove that

$$[d, e]\big(\prod_{j=1}^{r} a_j^{\nu_j}\big) = \sum_{i=1}^{r} \nu_i a_i^{\nu_i - 1}\big(\prod_{j \neq i} a_j^{\nu_j}\big)[d, e](a_i).$$

5.4. **Proposition.** ***(The Heisenberg Relations)*** *Let $n \in \mathbb{N}$, and let K be a field. Then, in the standard Weyl algebra*

$$\mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n]$$

the following relations hold:
(a) $[X_i, X_j] = 0,$ *for all $i, j \in \{1, 2, \ldots, n\}$;*

(b) $[X_i, \partial_j] = -\delta_{i,j}$, *for all $i, j \in \{1, 2, \ldots, n\}$;*

(c) $[\partial_i, \partial_j] = 0,$ *for all $i, j \in \{1, 2, \ldots, n\}$.*

Proof. (a): This is clear on application of Remark and Exercise 5.3 (B)(a) with $a = X_i$ and $b = X_j$.

(b): If we apply Remark and Exercise 5.3 (B)(b) with $a = X_i$ and $d = \partial_j$, and observe that $\partial_j(X_i) = \delta_{j,i} = \delta_{i,j}$ we get our claim.

(c): Observe that for all $i, k \in \{1, 2, \ldots, n\}$ we have $\partial_i(X_k) \in \{0, 1\} \subseteq K$. So for all $i, j, k \in \{1, 2, \ldots, n\}$ we obtain (see Definition and Remark 4.1 (B)):

$$[\partial_i, \partial_j](X_k) = \partial_i\big(\partial_j(X_k)\big) - \partial_j\big(\partial_i(X_k)\big) \in \partial_i(K) + \partial_j(K) = \{0\} + \{0\} = \{0\}.$$

Now, we get our claim by Exercise and Definition 4.2 (B) and Remark and Exercise 5.3 (B) (c) and (C). □

The Heisenberg relations are of basic significance for the arithmetic in standard Weyl algebras. Before we show that the elementary differential operators provide a basis for a standard Weyl algebra we shall study the arithmetic of these algebras. In particular, in the next section we shall prove a product formula for elementary differential operators, which will be of basic significance. We shall do this in a slightly more general setting, namely just for K-algebras "mimicking" the Heisenberg relations. The next exercise is aimed to prepare this.

5.5. **Exercise.** (A) Let $n \in \mathbb{N}$, let K be a field, let B be a K-algebra and let

$$a_1, a_1, \ldots, a_n, d_1, d_2, \ldots, d_n \in B$$

be elements *mimicking the Heisenberg relations*, which means:
(1) $[a_i, a_j] = 0,$ for all $i, j \in \{1, 2, \ldots, n\}$;

(2) $[a_i, d_j] = -\delta_{i,j}$, for all $i, j \in \{1, 2, \ldots, n\}$;
(3) $[d_i, d_j] = 0$, for all $i, j \in \{1, 2, \ldots, n\}$.

Let $\mu, \nu \in \mathbb{N}_0$. To simplify notations, we set

$$0b^k := 0 \text{ for all } b \in B \text{ and all } k \in \mathbb{Z}.$$

prove the following statements (using induction on μ and ν):

(a) $a_i^\mu a_j^\nu = a_j^\nu a_i^\mu$;
(b) $d_i^\mu d_j^\nu = d_j^\nu d_i^\mu$;
(c) $d_i^\mu a_j^\nu = a_j^\nu d_i^\mu$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.
(d) $d_i a_i^\nu = a_i^\nu d_i + \nu a_i^{\nu-1}$ for all $i \in \{1, 2, \ldots, n\}$.

(B) Keep the notations and hypotheses of part (A). For all $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}_0^n$ and each sequence $(b_1, b_2, \ldots, b_n) \in B^n$ we use again our earlier standard notation

$$\underline{\lambda} := (\lambda_1, \lambda_2, \ldots, \lambda_n) \text{ and } \underline{b}^{\underline{\lambda}} := b_1^{\lambda_1} b_2^{\lambda_2} \ldots b_n^{\lambda_n} = \prod_{i=1}^{n} b_i^{\lambda_i}.$$

Now, let

$$\underline{\mu} := (\mu_1, \mu_1, \ldots, \mu_n), \quad \underline{\nu} := (\nu_1, \nu_2, \ldots, \nu_n), \text{ and}$$
$$\underline{\mu'} := (\mu'_1, \mu'_1, \ldots, \mu'_n), \quad \underline{\nu'} := (\nu'_1, \nu'_2, \ldots, \nu'_n) \in \mathbb{N}_0^n.$$

Prove that the following relations hold

(a) $\underline{a}^{\underline{\nu}} \underline{d}^{\underline{\mu}} = \prod_{i=1}^n a_i^{\nu_i} \prod_{j=1}^n d_j^{\mu_j} = \prod_{i=1}^n a_i^{\nu_i} d_i^{\mu_i}$.
(b) $(\underline{a}^{\underline{\nu}} \underline{d}^{\underline{\mu}})(\underline{a}^{\underline{\nu'}} \underline{d}^{\underline{\mu'}}) = \big(\prod_{i=1}^n a_i^{\nu_i} \prod_{j=1}^n d_j^{\mu_j}\big)\big(\prod_{i=1}^n a_i^{\nu'_i} \prod_{j=1}^n d_j^{\mu'_j}\big) = \prod_{i=1}^n a_i^{\nu_i} d_i^{\mu_i} a_i^{\nu'_i} d_i^{\mu'_i}$.

6. Arithmetic in Weyl Algebras

As announced above, we now aim to do some Arithmetic in standard Weyl algebras. This means in particular, that we make explicit a number of computations in the hope that readers who up to now were mainly faced with commutative rings, get fascinated by the complexity of the arithmetic in Weyl algebras.

The following arithmetical Lemma is formulated in a more general framework, namely in a situation, which "mimicks" the Heisenberg relation. If we specialize this Lemma to standard Weyl algebras, we get a most important formula, which expresses the product of two elementary differential operators as a $\mathbb{Z}$-linear combination of elementary differential operators. This will also give us an explicit presentation of the commutator $[d, e]$ (see Remark and Exercise 5.3 (A)) of two elementary differential operators d and e. Asa further application we will get the Reduction Principle for arbitrary products of elementary differential operators and thus pave our way to the standard basis presentation of Weyl algebras, which we shall introduce in the next section.

We prove the announced Lemma in a setting which is more general than just the framework of Weyl algebras, because in this form it will help us to prove the universal property of Weyl algebras formulated in Corollary 7.5. This property is an analogue of the (relational) universal property of commutative polynomial algebras (see Example and Exercise 3.2 (A)) or of free associative algebras (see Example and Exercise 3.5 (D)).

6.1. **Lemma.** *Let $n \in \mathbb{N}$, let K be a field, let B be a K-algebra and let*

$$a_1, a_2, \ldots, a_n, d_1, d_2, \ldots, d_n \in B$$

such that:

(1) $[a_i, a_j] = 0,$ *for all* $i, j \in \{1, 2, \ldots, n\}$;
(2) $[a_i, d_j] = -\delta_{i,j},$ *for all* $i, j \in \{1, 2, \ldots, n\}$;
(3) $[d_i, d_j] = 0,$ *for all* $i, j \in \{1, 2, \ldots, n\}$.

Then, the following statements hold:

(a) *For all $\mu, \nu \in \mathbb{N}_0$ and all $i \in \{1, 2, \ldots, n\}$ we have*

$$d_i^\mu a_i^\nu = \sum_{k=0}^{\min\{\mu,\nu\}} \binom{\mu}{k} \prod_{p=0}^{k-1} (\nu - p) a_i^{\nu-k} d_i^{\mu-k}.$$

(b) *Let*

$$\underline{\mu} := (\mu_1, \mu_1, \ldots, \mu_n), \quad \underline{\nu} := (\nu_1, \nu_2, \ldots, \nu_n), \text{ and}$$
$$\underline{\mu'} := (\mu'_1, \mu'_1, \ldots, \mu'_n), \quad \underline{\nu'} := (\nu'_1, \nu'_2, \ldots, \nu'_n) \in \mathbb{N}_0^n.$$

Set

$$\mathbb{I} := \{\underline{k} := (k_1, k_2, \ldots, k_n) \in \mathbb{N}_0^n \mid k_i \leq \min\{\mu_i, \nu'_i\} \text{ for } i = 1, 2, \ldots, n\},$$

and let

$$\lambda_{\underline{k}} := [\prod_{i=1}^n \binom{\mu_i}{k_i}] \times [\prod_{i=1}^n \prod_{p=0}^{k_i-1} (\nu'_i - p)].$$

Then, we have the relation

$$(\underline{a}^{\underline{\nu}} \underline{d}^{\underline{\mu}})(\underline{a}^{\underline{\nu'}} \underline{d}^{\underline{\mu'}}) := (\prod_{i=1}^n a_i^{\nu_i} \prod_{j=1}^n d_j^{\mu_j}) \prod_{i=1}^n a_i^{\nu'_j} \prod_{j=1}^n d_i^{\mu'_j}) =$$
$$= \prod_{i=1}^n a_i^{\nu_i+\nu'_i} \prod_{i=1}^n d_i^{\mu_i+\mu'_i} + \sum_{\underline{k} \in \mathbb{I} \setminus \{\underline{0}\}} \lambda_{\underline{k}} \prod_{i=1}^n a_i^{\nu_i+\nu'_i-k_i} \prod_{i=1}^n d_i^{\mu_i+\mu'_i-k_i} =$$
$$= \underline{a}^{\underline{\nu}+\underline{\nu'}} \underline{d}^{\underline{\mu}+\underline{\mu'}} + \sum_{\underline{k} \in \mathbb{I} \setminus \{\underline{0}\}} \lambda_{\underline{k}} \underline{a}^{\underline{\nu}+\underline{\nu'}-\underline{k}} \underline{d}^{\underline{\mu}+\underline{\mu'}-\underline{k}}.$$

Proof. (a): To simplify matters we use the notation

$$0b^k := 0 \text{ for all } b \in B \text{ and all } k \in \mathbb{Z}$$

already introduced in the previous Exercise 5.5 (A). Then, it suffices to show that

$$d_i^\mu a_i^\nu = \sum_{k=0}^{\mu} \binom{\mu}{k} \prod_{p=0}^{k-1} (\nu - p) a_i^{\nu-k} d_i^{\mu-k}.$$

We proceed by induction on μ. The case $\mu = 0$ is obvious. The case $\mu = 1$ is clear by Exercise 5.5 (A)(d). So, let $\mu > 1$. By induction we have

$$d_i^{\mu-1} a_i^\nu = \sum_{k=0}^{\mu-1} \binom{\mu-1}{k} \prod_{p=0}^{k-1} (\nu - p) a_i^{\nu-k} d_i^{\mu-1-k}.$$

It follows on use of Exercise 5.5 (A)(d) and the Pascal formulas for the sum of binomial coefficients, that

$$
\begin{aligned}
d_i^\mu a_i^\nu &= d_i(d_i^{\mu-1}a_i^\nu) = d_i(\sum_{k=0}^{\mu-1}\binom{\mu-1}{k}\prod_{p=0}^{k-1}(\nu-p)a_i^{\nu-k}d_i^{\mu-1-k}) = \\
&= \sum_{k=0}^{\mu-1}\binom{\mu-1}{k}\prod_{p=0}^{k-1}(\nu-p)(d_i a_i^{\nu-k})d_i^{\mu-1-k} = \\
&= \sum_{k=0}^{\mu-1}\binom{\mu-1}{k}\prod_{p=0}^{k-1}(\nu-p)\big(a_i^{\nu-k}d_i + (\nu-k)a_i^{\nu-k-1}\big)d_i^{\mu-1-k} = \\
&= \sum_{k=0}^{\mu-1}[\binom{\mu-1}{k}\prod_{p=0}^{k-1}(\nu-p)a_i^{\nu-k}d_i^{\mu-k} + \binom{\mu-1}{k}\prod_{p=0}^{k-1}(\nu-p)(\nu-k)a_i^{\nu-k-1}d_i^{\mu-1-k}] = \\
&= \sum_{k=0}^{\mu-1}\binom{\mu-1}{k}\prod_{p=0}^{k-1}(\nu-p)a_i^{\nu-k}d_i^{\mu-k} + \sum_{k=0}^{\mu-1}\binom{\mu-1}{k}\prod_{p=0}^{k}(\nu-p)a_i^{\nu-k-1}d_i^{\mu-1-k} = \\
&= \sum_{k=0}^{\mu-1}\binom{\mu-1}{k}\prod_{p=0}^{k-1}(\nu-p)a_i^{\nu-k}d_i^{\mu-k} + \sum_{k=1}^{\mu}\binom{\mu-1}{k-1}\prod_{p=0}^{k-1}(\nu-p)a_i^{\nu-k}d_i^{\mu-k} = \\
&= a_i^\nu d_i^\mu + \sum_{k=1}^{\mu-1}\binom{\mu-1}{k}\prod_{p=0}^{k-1}(\nu-p)a_i^{\nu-k}d_i^{\mu-k} + \\
&\quad + \sum_{k=1}^{\mu-1}\binom{\mu-1}{k-1}\prod_{p=0}^{k-1}(\nu-p)a_i^{\nu-k}d_i^{\mu-k} + \prod_{p=0}^{\mu-1}(\nu-p)a_i^{\nu-\mu} = \\
&= a_i^\nu d_i^\mu + \sum_{k=1}^{\mu-1}[\binom{\mu-1}{k} + \binom{\mu-1}{k-1}]\prod_{p=0}^{k-1}(\nu-p)a_i^{\nu-k}d_i^{\mu-k} + \prod_{p=0}^{\mu-1}(\nu-p)a_i^{\nu-\mu} = \\
&= a_i^\nu d_i^\mu + \sum_{k=1}^{\mu-1}\binom{\mu}{k}\prod_{p=0}^{k-1}(\nu-p)a_i^{\nu-k}d_i^{\mu-k} + \prod_{p=0}^{\mu-1}(\nu-p)a_i^{\nu-\mu} = \\
&= \sum_{k=0}^{\mu}\binom{\mu}{k}\prod_{p=0}^{k-1}(\nu-p)a_i^{\nu-k}d_i^{\mu-k}.
\end{aligned}
$$

(b): According to Exercise 5.5 (B)(a),(b), the previous statement (a) and Exercise 5.5 (A)(a),(b) and (c) we may write

$$(\underline{a}^{\underline{\nu}}\underline{d}^{\underline{\mu}})(\underline{a}^{\underline{\nu}'}\underline{d}^{\underline{\mu}'}) := (\prod_{i=1}^{n} a_i^{\nu_i} \prod_{j=1}^{n} d_j^{\mu_j})(\prod_{i=1}^{n} a_i^{\nu_i'} \prod_{j=1}^{n} d_j^{\mu_j'}) = \prod_{i=1}^{n} a_i^{\nu_i} d_i^{\mu_i} a_i^{\nu_i'} d_i^{\mu_i'} =$$

$$= \prod_{i=1}^{n} a_i^{\nu_i} (d_i^{\mu_i} a_i^{\nu_i'}) d_i^{\mu_i'} = \prod_{i=1}^{n} a_i^{\nu_i} [\sum_{k=0}^{\min\{\mu_i,\nu_i'\}} \binom{\mu_i}{k} \prod_{p=0}^{k-1} (\nu_i' - p) a_i^{\nu_i'-k} d_i^{\mu_i-k}] d_i^{\mu_i'} =$$

$$= \prod_{i=1}^{n} (\sum_{k=0}^{\min\{\mu_i,\nu_i'\}} \binom{\mu_i}{k} \prod_{p=0}^{k-1} (\nu_i' - p) a_i^{\nu_i+\nu_i'-k} d_i^{\mu_i+\mu_i'-k}] =$$

$$= \sum_{\underline{k}:=(k_1,k_2,\ldots,k_n)\in\mathbb{I}} \prod_{i=1}^{n} (\binom{\mu_i}{k_i} \prod_{p=0}^{k_i-1} (\nu_i' - p) a_i^{\nu_i+\nu_i'-k_i} d_i^{\mu_i+\mu_i'-k_i}) =$$

$$= \sum_{\underline{k}\in\mathbb{I}} (\prod_{i=1}^{n} \binom{\mu_i}{k_i})(\prod_{i=1}^{n} \prod_{p=0}^{k_i-1} (\nu_i' - p)) \prod_{i=1}^{n} a_i^{\nu_i+\nu_i'-k_i} d_i^{\mu_i+\mu_i'-k_i} =$$

$$= \sum_{\underline{k}\in\mathbb{I}} (\prod_{i=1}^{n} \binom{\mu_i}{k_i})(\prod_{i=1}^{n} \prod_{p=0}^{k_i-1} (\nu_i' - p)) \prod_{i=1}^{n} a_i^{\nu_i+\nu_i'-k_i} \prod_{i=1}^{n} d_i^{\mu_i+\mu_i'-k_i} =$$

$$= \prod_{i=1}^{n} a_i^{\nu_i+\nu_i'} \prod_{i=1}^{n} d_i^{\mu_i+\mu_i'} + \sum_{\underline{k}\in\mathbb{I}\setminus\{\underline{0}\}} \lambda_{\underline{k}} \prod_{i=1}^{n} a_i^{\nu_i+\nu_i'-k_i} \prod_{i=1}^{n} d_i^{\mu_i+\mu_i'-k_i} =$$

$$= \underline{a}^{\underline{\nu}+\underline{\nu}'} \underline{d}^{\underline{\mu}+\underline{\mu}'} + \sum_{\underline{k}\in\mathbb{I}\setminus\{\underline{0}\}} \lambda_{\underline{k}} \underline{a}^{\underline{\nu}+\underline{\nu}'-\underline{k}} \underline{d}^{\underline{\mu}+\underline{\mu}'-\underline{k}}.$$

□

As an application we now get the announced product formula for elementary differential operators.

6.2. **Proposition.** ***(The Product Formula for Elementary Differential Operators)*** *Let $n \in \mathbb{N}$, let K be a field and consider the standard Weyl algebra*

$$\mathbb{W}(K, n) = K[X_1, X_2, \ldots X_n, \partial_1, \partial_2 \ldots, \partial_n].$$

Moreover, let

$$\underline{\mu} := (\mu_1, \mu_1, \ldots, \mu_n), \quad \underline{\nu} := (\nu_1, \nu_2, \ldots, \nu_n) \text{ and}$$

$$\underline{\mu}' := (\mu_1', \mu_1', \ldots, \mu_n'), \quad \underline{\nu}' := (\nu_1', \nu_2', \ldots, \nu_n') \in \mathbb{N}_0^n.$$

Set

$$\mathbb{I} := \{\underline{k} := (k_1, k_2, \ldots, k_n) \in \mathbb{N}_0^n \mid k_i \leq \min\{\mu_i, \nu_i'\} \text{ for } i = 1, 2, \ldots, n\},$$

and let

$$\lambda_{\underline{k}} := (\prod_{i=1}^{n} \binom{\mu_i}{k_i})(\prod_{i=1}^{n} \prod_{p=0}^{k_i-1} (\nu_i' - p)).$$

Then, we have the equality

$$(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})(\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}) := \big(\prod_{i=1}^{n} X_i^{\nu_i} \prod_{j=1}^{n} \partial_j^{\mu_j}\big)\big(\prod_{i=1}^{n} X_i^{\nu'_i} \prod_{j=1}^{n} \partial_j^{\mu'_j}\big) =$$
$$= \prod_{i=1}^{n} X_i^{\nu_i+\nu'_i} \prod_{i=1}^{n} \partial_i^{\mu_i+\mu'_i} + \sum_{\underline{k}\in\mathbb{I}\setminus\{\underline{0}\}} \lambda_{\underline{k}} \prod_{i=1}^{n} X_i^{\nu_i+\nu'_i-k_i} \prod_{i=1}^{n} \partial_i^{\mu_i+\mu'_i-k_i} =$$
$$= \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'} + \sum_{\underline{k}\in\mathbb{I}\setminus\{\underline{0}\}} \lambda_{\underline{k}} \underline{X}^{\underline{\nu}+\underline{\nu}'-\underline{k}}\underline{\partial}^{\underline{\mu}+\underline{\mu}'-\underline{k}}.$$

Proof. It suffices to apply Lemma 6.1 (b) with $a_i := X_i$ and $d_i := \partial_i$ for $i = 1, 2 \ldots, n$. □

Now, we can prove the main result of the present section. To formulate it, we introduce another notation and suggest a further exercise.

6.3. **Notation and Remark.** (A) Let $n \in \mathbb{N}$ and let

$$\underline{\kappa} := (\kappa_1, \kappa_2, \ldots, \kappa_n) \text{ and } \underline{\lambda} := (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}_0^n.$$

We write

$$\underline{\kappa} \leq \underline{\lambda} \text{ if and only if } \kappa_i \leq \lambda_i \text{ for } i = 1, 2, \ldots, n$$

and

$$\underline{\kappa} < \underline{\lambda} \text{ if and only if } \underline{\kappa} \leq \underline{\lambda} \text{ and } \underline{\kappa} \neq \underline{\lambda}.$$

(B) Keep the notations of part (A). Observe that

$$\underline{\kappa} \leq \underline{\lambda} \text{ if and only if } \underline{\lambda} - \underline{\kappa} \in \mathbb{N}_0^n$$

and

$$\underline{\kappa} < \underline{\lambda} \text{ if and only if } \underline{\lambda} - \underline{\kappa} \in \mathbb{N}_0^n \setminus \{\underline{0}\}.$$

(C) We now introduce a few notations, which we will have to use later very frequently. Namely, for

$$\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n), \underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}_0^n$$

we set

$$\mathbb{M}(\underline{\alpha}, \underline{\beta}) := \{(\underline{\alpha} - \underline{k}, \underline{\beta} - \underline{k}) \mid \underline{k} \in \mathbb{N}_0^n \setminus \{\underline{0}\} \text{ with } \underline{k} \leq \underline{\alpha}, \underline{\beta}\}$$

and

$$\overline{\mathbb{M}}(\underline{\alpha}, \underline{\beta}) := \{(\underline{\alpha} - \underline{k}, \underline{\beta} - \underline{k}) \mid \underline{k} \in \mathbb{N}_0^n \text{ with } \underline{k} \leq \underline{\alpha}, \underline{\beta}\} = \mathbb{M}(\underline{\alpha}, \underline{\beta}) \cup \{(\underline{\alpha}, \underline{\beta})\}.$$

Moreover, we write

$$\mathbb{M}_{\leq}(\underline{\alpha}, \underline{\beta}) := \{(\underline{\lambda}, \underline{\kappa}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid \underline{\lambda} \leq \underline{\nu} \text{ and } \underline{\kappa} \leq \underline{\mu} \text{ for some } (\underline{\nu}, \underline{\mu}) \in \mathbb{M}(\underline{\alpha}, \underline{\beta})\}.$$

Observe that

$$\mathbb{M}(\underline{\alpha}, \underline{\beta}) \subseteq \mathbb{M}_{\leq}(\underline{\alpha}, \underline{\beta}).$$

6.4. **Exercise.** (A) Let $n \in \mathbb{N}$, let K be a field and consider the standard Weyl algebra

$$\mathbb{W}(K,n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n].$$

In addition, let

$$\underline{\mu} := (\mu_1, \mu_1, \ldots, \mu_n), \quad \underline{\nu} := (\nu_1, \nu_2, \ldots, \nu_n) \text{ and}$$
$$\underline{\mu}' := (\mu_1', \mu_1', \ldots, \mu_n'), \quad \underline{\nu}' := (\nu_1', \nu_2', \ldots, \nu_n') \in \mathbb{N}_0^n.$$

Moreover, let the sets

$$\mathbb{M}(\underline{\nu}+\underline{\nu}', \underline{\mu}+\underline{\mu}') \subset \overline{\mathbb{M}}(\underline{\nu}+\underline{\nu}', \underline{\mu}+\underline{\mu}') \subset \mathbb{N}_0^n \times \mathbb{N}_0^n$$

be defined according to Notation and Remark 6.3 (C). Prove that

$$(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})(\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}) - \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'} \in \sum_{(\underline{\lambda},\underline{\kappa}) \in \mathbb{M}(\underline{\nu}+\underline{\nu}', \underline{\mu}+\underline{\mu}')} \mathbb{Z}\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}}.$$

and

$$(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})(\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}) \in \sum_{(\underline{\lambda},\underline{\kappa}) \in \overline{\mathbb{M}}(\underline{\nu}+\underline{\nu}', \underline{\mu}+\underline{\mu}')} \mathbb{Z}\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}}.$$

(B) Let the notations be as in part (A) and let the set

$$\mathbb{M}(\underline{\nu}+\underline{\nu}', \underline{\mu}+\underline{\mu}') \subset \mathbb{N}_0^n \times \mathbb{N}_0^n$$

be defined according to Notation and Remark 6.3 (C). Prove that

$$\big[\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}, \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big] \in \sum_{(\underline{\lambda},\underline{\kappa}) \in \mathbb{M}(\underline{\nu}+\underline{\nu}', \underline{\mu}+\underline{\mu}')} \mathbb{Z}\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}}.$$

(C) To give a more precise statement than what was just said in part (B), keep the notations of Proposition 6.2 and set in addition

$$\mathbb{I}' := \{\underline{k}' := (k_1', k_2', \ldots, k_n') \in \mathbb{N}_0^n \mid k_i' \leq \min\{\mu_i', \nu_i\} \text{ for } i = 1, 2, \ldots, n\}.$$

Use the product formula of Proposition 6.2 to show that

$$\big[\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}, \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big] = \sum_{\underline{k} \in \mathbb{I}\setminus\{\underline{0}\}} \lambda_{\underline{k}} \underline{X}^{\underline{\nu}+\underline{\nu}'-\underline{k}}\underline{\partial}^{\underline{\mu}+\underline{\mu}'-\underline{k}} - \sum_{\underline{k}' \in \mathbb{I}'\setminus\{\underline{0}\}} \lambda_{\underline{k}'} \underline{X}^{\underline{\nu}+\underline{\nu}'-\underline{k}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'-\underline{k}'}.$$

(D) Let $i \in \{1,,2,\ldots,n\}$ and consider the n-tuple $\underline{e}_i := (\delta_{i,j})_{j=1}^n = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{N}_0^n$. Use what you have shown in part (C) to prove the following statements

(a) $$\big[X_i, \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big] = \begin{cases} -\mu_i \underline{X}^{\underline{v}}\underline{\partial}^{\underline{\mu}-\underline{e}_i} & , \text{ if } \mu_i > 0; \\ 0 & , \text{ if } \mu_i = 0. \end{cases}$$

(b) $$\big[\partial_i, \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big] = \begin{cases} \nu_i \underline{X}^{\underline{v}-\underline{e}_i}\underline{\partial}^{\underline{\mu}} & , \text{ if } \nu_i > 0; \\ 0 & , \text{ if } \nu_i = 0. \end{cases}$$

6.5. **Theorem.** ***(The Reduction Principle)*** *Let $n \in \mathbb{N}$, let K be a field and consider the standard Weyl algebra*

$$\mathbb{W}(K,n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n].$$

Let $r \in \mathbb{N}$, let

$$\underline{\nu}^{(i)} := (\nu_1^{(i)}, \nu_2^{(i)}, \ldots, \nu_n^{(i)}) \text{ and } \underline{\mu}^{(i)} := (\mu_1^{(i)}, \mu_2^{(i)}, \ldots, \mu_n^{(i)}) \in \mathbb{N}_0^n, \text{ for } i = 1, 2, \ldots, r$$

and abbreviate

$$\underline{\nu} := \sum_{i=1}^{r} \underline{\nu}^{(i)}, \quad \underline{\mu} := \sum_{i=1}^{r} \underline{\mu}^{(i)}.$$

Moreover, let the set

$$\mathbb{M} := \mathbb{M}_{\leq}\big(\underline{\nu}, \underline{\mu}\big) \subset \mathbb{N}_0^n \times \mathbb{N}_0^n$$

be defined according to Notation and Remark 6.3 (C). Then, we have

$$\prod_{i=1}^{r} \underline{X}^{\underline{\nu}^{(i)}} \underline{\partial}^{\underline{\mu}^{(i)}} - \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \in \sum_{(\underline{\kappa}, \underline{\lambda}) \in \mathbb{M}} \mathbb{Z} \underline{X}^{\underline{\lambda}} \underline{\partial}^{\underline{\kappa}}.$$

Proof. We proceed by induction on r. The case $r = 1$ is obvious. The case $r = 2$ follows from Proposition 6.2, more precisely from its consequence proved in Exercise 6.4 (A) (see also Notation and Remark 6.3 (C)) . So, let $r > 2$. We set

$$\underline{\nu}' := \sum_{i=1}^{r-1} \underline{\nu}^{(i)}, \quad \underline{\mu}' := \sum_{i=1}^{r-1} \underline{\mu}^{(i)} \text{ and } \mathbb{M}' := \mathbb{M}_{\leq}\big(\underline{\nu}', \underline{\mu}'\big).$$

By induction we have

$$\varrho := \prod_{i=1}^{r-1} \underline{X}^{\underline{\nu}^{(i)}} \underline{\partial}^{\underline{\mu}^{(i)}} - \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'} \in \sum_{(\underline{\lambda}', \underline{\kappa}') \in \mathbb{M}'} \mathbb{Z} \underline{X}^{\underline{\lambda}'} \underline{\partial}^{\underline{\kappa}'} =: N.$$

By the case $r = 2$ we have (see once more Notation and Remark 6.3 (C) and Exercise 6.4 (A))

$$\sigma := \big(\underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}\big) \underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}} - \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \in \sum_{(\underline{\lambda}, \underline{\kappa}) \in \mathbb{M}} \mathbb{Z} \underline{X}^{\underline{\lambda}} \underline{\partial}^{\underline{\kappa}} =: M.$$

As

$$\prod_{i=1}^{r} \underline{X}^{\underline{\nu}^{(i)}} \underline{\partial}^{\underline{\mu}^{(i)}} - \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} = \sigma + \varrho \underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}},$$

it remains to show that

$$\varrho \underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}} \in M.$$

Observe that

$$\varrho \underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}} \in N \underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}} = \sum_{(\underline{\lambda}', \underline{\kappa}') \in \mathbb{M}'} \mathbb{Z} \underline{X}^{\underline{\lambda}'} \underline{\partial}^{\underline{\kappa}'} \underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}}.$$

Observe also that

$$(\underline{\lambda}' + \underline{\nu}^{(r)}, \underline{\kappa}' + \underline{\mu}^{(r)}) \in \mathbb{M} \text{ for all } (\underline{\lambda}', \underline{\kappa}') \in \mathbb{M}',$$

so that in the notation introduced in Notation and Remark 6.3 (C) we have

$$\overline{\mathbb{M}}(\underline{\lambda}' + \underline{\nu}^{(r)}, \underline{\kappa}' + \underline{\mu}^{(r)}) \subseteq \mathbb{M} \text{ for all } (\underline{\lambda}', \underline{\kappa}') \in \mathbb{M}'.$$

Hence, on application of Exercise 6.4 (A) it follows that

$$\underline{X}^{\underline{\lambda}'}\underline{\partial}^{\underline{\kappa}'}\underline{X}^{\underline{\nu}^{(r)}}\underline{\partial}^{\underline{\mu}^{(r)}} \in \sum_{(\underline{\lambda},\underline{\kappa})\in\overline{\mathbb{M}}(\underline{\lambda}'+\underline{\nu}^{(r)},\underline{\kappa}'+\underline{\mu}^{(r)})} \mathbb{Z}\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}} \subseteq \sum_{(\underline{\lambda},\underline{\kappa})\in\mathbb{M}} \mathbb{Z}\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}} = M,$$

and this shows that indeed $\varrho\underline{X}^{\underline{\nu}^{(r)}}\underline{\partial}^{\underline{\mu}^{(r)}} \in M$. □

Now, in the next section, we can show that the elementary differential operators form a K-basis of the standard Weyl algebra $\mathbb{W}(K, n)$, provided the field K has characteristic 0. To prepare this, we add an additional exercise.

6.6. **Exercise.** (A) Let $n \in \mathbb{N}$ and consider the polynomial ring $K[X_1, X_2, \ldots, X_n]$. Moreover, let

$$\underline{\mu} := (\mu_1, \mu_1, \ldots, \mu_n), \text{ and } \underline{\nu} := (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}_0^n.$$

Fix $i \in \{1, 2, \ldots, n\}$ and prove by induction on μ_i, that

$$\partial_i^{\mu_i}(\underline{X}^{\underline{\nu}}) = \partial_i^{\mu_i}\Big(\prod_{j=1}^{n} X_j^{\nu_j}\Big) = \begin{cases} \prod_{k=0}^{\mu_i-1}(\nu_i - k)X_i^{\nu_i-\mu_i}\prod_{j\neq i} X_j^{\nu_j}, & \text{if } \nu_i \geq \mu_i; \\ 0, & \text{if } \nu_i < \mu_i. \end{cases}$$

(B) Let the notations and hypotheses be as in part (A) and use what you have shown there to prove that

$$\begin{aligned} \underline{\partial}^{\underline{\mu}}(\underline{X}^{\nu}) &= \prod_{i=1}^{n}\partial_i^{\mu_i}\Big(\prod_{j=1}^{n} X^{\nu_j}\Big) = \\ &= \begin{cases} \prod_{i=1}^{n}\prod_{k=0}^{\mu_i-1}(\nu_i - k)X_i^{\nu_i-\mu_i}, & \text{if } \nu_i \geq \mu_i \text{ for all } i \in \{1, 2, \ldots, n\}; \\ 0, & \text{if } \nu_i < \mu_i \text{ for some } i \in \{1, 2, \ldots, n\}. \end{cases} \\ &= \begin{cases} \prod_{i=1}^{n}\prod_{k=0}^{\mu_i-1}(\nu_i - k)\underline{X}^{\underline{\nu}-\underline{\mu}}, & \text{if } \underline{\nu} \geq \underline{\mu}; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

7. The Standard Basis

Now, we are ready to prove the fact that over a base field of characteristic 0 the elementary differential operators form a vector space basis of the standard Weyl algebra.

7.1. **Theorem.** ***(The Standard Basis)*** *Let $n \in \mathbb{N}$ and let K be a field of characteristic 0. Then, the elementary differential operators*

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} = \prod_{i=1}^{n} X_i^{\nu_i}\prod_{i=1}^{n}\partial_i^{\mu_i} \text{ with } \underline{\mu} := (\mu_1, \mu_2, \ldots, \mu_n) \text{ and } \underline{\nu} := (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}_0^n$$

form a K-vector space basis of the standard Weyl algebra

$$\mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n].$$

So, in particular we can say

(a) $\mathbb{W}(K,n) = \bigoplus_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} = \bigoplus_{\mu_1,\mu_2,\ldots,\mu_n,\nu_1,\nu_2,\ldots,\nu_n\in\mathbb{N}_0} K\prod_{i=1}^n X_i^{\nu_i}\prod_{i=1}^n \partial_i^{\mu_i}$.

(b) *Each differential operator* $d \in \mathbb{W}(K,n)$ *can be written in the form*

$$d = \sum_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n} c^{(d)}_{\underline{\nu},\underline{\mu}}\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}$$

with a unique family

$$\left(c^{(d)}_{\underline{\nu},\underline{\mu}}\right)_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n} \in \prod_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n} K = K^{\mathbb{N}_0^n\times\mathbb{N}_0^n},$$

whose support

$$\operatorname{supp}(d) = \operatorname{supp}\big((c^{(d)}_{\underline{\nu},\underline{\mu}})_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n}\big) := \{(\underline{\nu},\underline{\mu}) \in \mathbb{N}_0^n\times\mathbb{N}_0^n \mid c^{(d)}_{\underline{\nu},\underline{\mu}} \neq 0\}$$

is a finite set. We thus can write

$$d = \sum_{(\underline{\nu},\underline{\mu})\in\operatorname{supp}(d)} c^{(d)}_{\underline{\nu},\underline{\mu}}\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \quad .$$

Proof. We first show, that the elementary differential operators generate $\mathbb{W}(K,n)$ as a K-vector space, hence that

$$\mathbb{W}(K,n) = \sum_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} =: M.$$

Observe, that by definition each element d of $\mathbb{W}(K,n)$ is a K-linear combination of products of elementary differential operators. But by the Reduction Principle of Theorem 6.5 each product of elementary differential operators is contained in the K-vector space M. It remains to show, that the elementary differential operators are linearly independent among each other. Assume to the contrary, that there are linearly dependent elementary differential operators in $\mathbb{W}(K,n)$. Then, we find a positive integer $r \in \mathbb{N}$, families

$$\underline{\mu}^{(i)} := (\mu_1^{(i)}, \mu_2^{(i)}, \ldots, \mu_n^{(i)}), \quad \underline{\nu}^{(i)} := (\nu_1^{(i)}, \nu_2^{(i)}, \ldots, \nu_n^{(i)}) \in \mathbb{N}_0^n, \quad (i = 1,2,\ldots,r)$$

with

$$(\underline{\mu}^{(i)}, \underline{\nu}^{(i)}) \neq (\underline{\mu}^{(j)}, \underline{\nu}^{(j)}) \text{ for all } i,j \in \{1,2,\ldots,r\} \text{ with } i \neq j,$$

and elements

$$c^{(i)} \in K \setminus \{0\} \quad (i = 1,2,\ldots,r),$$

such that

$$d := \sum_{i=1}^r c^{(i)}\underline{X}^{\underline{\nu}^{(i)}}\underline{\partial}^{\underline{\mu}^{(i)}} = 0.$$

We may assume, that

$$|\underline{\mu}^{(r)}| = \max\{|\underline{\mu}^{(i)}| \mid i = 1,2,\ldots,r\}$$

and that for some $s \in \{1,2,\ldots,r\}$ we have

$$\underline{\mu}^{(i)} \neq \underline{\mu}^{(r)} \text{ for all } i < s \text{ and } \underline{\mu}^{(i)} = \underline{\mu}^{(r)} \text{ for all } i \geq s.$$

Then, it follows easily by what we have seen in Exercise 6.6 (B), that

$$\underline{X}^{\underline{\nu}^{(i)}}\underline{\partial}^{\underline{\mu}^{(i)}}\big(\underline{X}^{\underline{\mu}^{(r)}}\big) = \begin{cases} \prod_{j=1}^{n} \mu_j^{(r)}!\underline{X}^{\underline{\nu}^{(r)}}, & \text{if } s \leq i \leq r \\ 0, & \text{if } i < s. \end{cases}$$

So, we get

$$0 = d\big(\underline{X}^{\underline{\mu}^{(r)}}\big) = \sum_{i=1}^{r} c^{(i)}\underline{X}^{\underline{\nu}^{(i)}}\underline{\partial}^{\underline{\mu}^{(i)}}\big(\underline{X}^{\underline{\mu}^{(r)}}\big) = \sum_{i=s}^{r} c^{(i)}\prod_{j=1}^{n}\mu_j^{(r)}!\underline{X}^{\underline{\nu}^{(i)}}.$$

As $\mathrm{Char}(K) = 0$, and as the monomials $\underline{X}^{\underline{\nu}^{(i)}}$ are pairwise different for $i = s, s+1, \ldots, r$, the last sum does not vanish, and we have a contradiction. □

7.2. **Definition and Remark.** (A) Let the notations and hypotheses be as in Theorem 6.5. We call the basis of $\mathbb{W}(K,n)$ which consists of all elementary differential operators the *standard basis*. If we present a differential operator $d \in \mathbb{W}(K,n)$ with respect to the standard basis and write

$$d = \sum_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n} c_{\underline{\nu},\underline{\mu}}^{(d)}\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}$$

as in statement (b) of Theorem 6.5, we say that d is written in *standard form*. The support of a differential operator d in $\mathbb{W}(K,n)$ is always defined with respect to the standard form as in statement (b) of Theorem 7.1. We therefore call the support of d also the *standard support* of d.

(B) Keep the above notations and hypotheses. It is a fundamental task, to write an arbitrarily given differential operator $d \in \mathbb{W}(K,n)$ in standard form. This task actually is reduced by the Reduction Principle of Theorem 6.5 to make explicit the coefficients of the differences

$$\Delta_{\underline{\nu}^{(\bullet)}\underline{\mu}^{(\bullet)}} := \prod_{i=1}^{r}\underline{X}^{\underline{\nu}^{(i)}}\underline{\partial}^{\underline{\mu}^{(i)}} - \underline{X}^{\sum_{i=1}^{r}\underline{\nu}^{(i)}}\underline{\partial}^{\sum_{i=1}^{r}\underline{\mu}^{(i)}} \in \sum_{(\underline{\lambda},\underline{\kappa})\in\mathbb{M}} \mathbb{Z}\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}}.$$

This task can be solved by a repeated application of the Product Formula of Proposition 6.2 or – directly – by a repeated application of the Heisenberg relations. Clearly, this is a task which usually is performed by means of Computer Algebra systems.

We now prove the following application, a result on supports, which will turn out to be useful in the next section.

7.3. **Proposition.** ***(Behavior of Supports)*** *Let $n \in \mathbb{N}$, let K be a field of characteristic 0 and consider the differential operators*

$$d, e \in \mathbb{W}(K,n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n].$$

For all $(\underline{\alpha}, \underline{\beta}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$, let the sets

$$\mathbb{M}(\underline{\alpha}, \underline{\beta}) \subset \overline{\mathbb{M}}(\underline{\alpha}, \underline{\beta}) \subset \mathbb{N}_0^n \times \mathbb{N}_0^n$$

be defined according to Notation and Remark 6.3 (C). Then, we have

(a) $\big(\mathrm{supp}(d) \cup \mathrm{supp}(e)\big) \setminus \big(\mathrm{supp}(d) \cap \mathrm{supp}(e)\big) \subseteq \mathrm{supp}(d+e) \subseteq \mathrm{supp}(d) \cup \mathrm{supp}(e)$.

(b) $\mathrm{supp}(cd) = \mathrm{supp}(d)$ *for all $c \in K \setminus \{0\}$.*

(c) $\mathrm{supp}(de) \subseteq \bigcup_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} \overline{\mathbb{M}}(\underline{\nu}+\underline{\nu}',\underline{\mu}+\underline{\mu}')$.
(d) $\mathrm{supp}\big([d,e]\big) \subseteq \bigcup_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} \mathbb{M}(\underline{\nu}+\underline{\nu}',\underline{\mu}+\underline{\mu}')$.

Proof. (a), (b): These statements follow in a straight forward way from our definition of support, and we leave it as an exercise to perform their proof.

(c): In the notations of Theorem 7.1 we write

$$d = \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d)} c^{(d)}_{\underline{\nu},\underline{\mu}} \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \text{ and } e = \sum_{(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(e)}_{\underline{\nu}',\underline{\mu}'} \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}.$$

it follows that

$$de = \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(d)}_{\underline{\nu},\underline{\mu}} c^{(e)}_{\underline{\nu}',\underline{\mu}'} \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}.$$

But according to Exercise 6.4 (A) we have

$\mathrm{supp}\big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) \subseteq \overline{\mathbb{M}}(\underline{\nu}+\underline{\nu}',\underline{\mu}+\underline{\mu}')$ for all $(\underline{\nu},\underline{\mu}) \in \mathrm{supp}(d)$ and all $(\underline{\nu}',\underline{\mu}') \in \mathrm{supp}(e)$.
Now, our claim follows easily on repeated application of statements (a) and (b).

(d): As in the proof of statement (c) we can write

$$de = \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(d)}_{\underline{\nu},\underline{\mu}} c^{(e)}_{\underline{\nu}',\underline{\mu}'} \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}$$

and, similarly

$$ed = \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(d)}_{\underline{\nu},\underline{\mu}} c^{(e)}_{\underline{\nu}',\underline{\mu}'} \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}.$$

It follows that

$$\begin{aligned}
\big[de, ed\big] &= de - ed = \\
&= \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(d)}_{\underline{\nu},\underline{\mu}} c^{(e)}_{\underline{\nu}',\underline{\mu}'} \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} \quad - \\
&- \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(d)}_{\underline{\nu},\underline{\mu}} c^{(e)}_{\underline{\nu}',\underline{\mu}'} \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \quad = \\
&= \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(d)}_{\underline{\nu},\underline{\mu}} c^{(e)}_{\underline{\nu}',\underline{\mu}'} \big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} - \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big) = \\
&= \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(d)}_{\underline{\nu},\underline{\mu}} c^{(e)}_{\underline{\nu}',\underline{\mu}'} \big[\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}, \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big].
\end{aligned}$$

By Exercise 6.4 (B) we have

$$\mathrm{supp}\big(\big[\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}, \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big]\big) \subseteq \mathbb{M}(\underline{\nu}+\underline{\nu}',\underline{\mu}+\underline{\mu}')$$

for all $(\underline{\nu},\underline{\mu}) \in \mathrm{supp}(d)$ and all $(\underline{\nu}',\underline{\mu}') \in \mathrm{supp}(e)$.

Now, statement (d) follows easily on repeated application of statements (a) and (b). □

7.4. **Exercise.** (A) Let $n \in \mathbb{N}$, let K be a field of characteristic 0 and consider the standard Weyl algebra

$$\mathbb{W} = \mathbb{W}(K,n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n].$$

Prove in detail statements (a) and (b) of Proposition 7.3.

(B) Let the notations and hypotheses be as in part (A). Present in standard form the following differential operators:

$$\partial_1^2 X_1^2 - X_1\partial_1 X_1 - 1, \quad \partial_1^2 X_1^2 \partial_1^2 - \partial_1 X_1^2, \quad \partial_2 X_1 X_2 \partial_1 + \partial_1 X_1 X_2 \in \mathbb{W}(K,n).$$

(C) Keep the notations of part (A), but assume that $n = 1$ and $\mathrm{Char}(K) = 2$. Compute $\partial_1(X_1^\nu)$ for all $\nu \in \mathbb{N}_0$ and comment your findings in view of the Standard Basis Theorem.

(D) Keep the notations of part (A), let

$$d = \sum_{(\underline{\nu},\underline{\mu})\in \mathrm{supp}(d)} c^{(d)}_{\underline{\nu},\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \in \mathbb{W}, \quad \big(c^{(d)}_{\underline{\nu},\underline{\mu}} \in K \setminus \{0\}, \forall (\underline{\nu},\underline{\mu}) \in \mathrm{supp}(d)\big)$$

(see Theorem 7.1) and let $i \in \{1,2,\ldots,n\}$. Use Exercise 6.4 (D) to prove the following equalities:

(a) $\big[X_i, d\big] = -\sum_{(\underline{\nu},\underline{\mu})\in \mathrm{supp}(d):\mu_i>0} \mu_i c^{(d)}_{\underline{\nu},\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}-\underline{e}_i}$.

(b) $\big[\partial_i, d\big] = \sum_{(\underline{\nu},\underline{\mu})\in \mathrm{supp}(d):\nu_i>0} \nu_i c^{(d)}_{\underline{\nu},\underline{\mu}} \underline{X}^{\underline{\nu}-\underline{e}_i} \underline{\partial}^{\underline{\mu}}$.

Conclude that

(c) $d = 0 \Leftrightarrow \forall i \in \{1,2,\ldots,n\} : \big[X_i, d\big] = \big[\partial_i, d\big] = 0$.

As another application of the Standard Basis Theorem we now can prove

7.5. **Corollary.** ***(The Universal Property of Weyl Algebras)*** *Let $n \geq 2$ and let the notations and hypotheses be as in Theorem 7.1. Let B be a K-algebra and let*

$$\phi : \{X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n\} \longrightarrow B$$

be a map "which respects the Heisenberg relations" and hence satisfies the requirements

(1) $[\phi(X_i), \phi(X_j)] = 0,$ *for all $i, j \in \{1,2,\ldots,n\}$;*
(2) $[\phi(X_i), \phi(\partial_j)] = -\delta_{i,j},$ *for all $i, j \in \{1,2,\ldots,n\}$;*
(3) $[\phi(\partial_i), \phi(\partial_j)] = 0,$ *for all $i, j \in \{1,2,\ldots,n\}$.*

Then, there is a unique homomorphism of K-algebras

$$\widetilde{\phi} : \mathbb{W}(K,n) \longrightarrow B$$

such that

$$\widetilde{\phi}(X_i) = \phi(X_i) \text{ and } \widetilde{\phi}(\partial_i) = \phi(\partial_i) \text{ for all } i = 1,2,\ldots,n.$$

Proof. According to Theorem 7.1 there is a K-linear map

$$\widetilde{\phi} : \mathbb{W}(K,n) \longrightarrow B \text{ given by}$$

$$\widetilde{\phi}\big(\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}\big) = \prod_{i=1}^n \phi(X_i)^{\nu_i} \prod_{i=1}^n \phi(\partial_i)^{\mu_i} \text{ for all}$$

$$\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_n) \text{ and } \underline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}_0^n.$$

Next, we show, that the previously defined K-linear map $\widetilde{\phi}$ is multiplicative, and hence satisfies the condition that

$$\widetilde{\phi}(de) = \widetilde{\phi}(d)\widetilde{\phi}(e) \text{ for all } d, e \in \mathbb{W}(K, n).$$

As the multiplication maps

$$\mathbb{W}(K,n) \times \mathbb{W}(K,n) \longrightarrow \mathbb{W}(K,n), (d,e) \mapsto de \quad \text{ and } \quad B \times B \longrightarrow B, (a,b) \mapsto ab$$

are both K-bilinear, it suffices to verify the above multiplicativity condition in the special case where

$$d := \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \text{ and } e := \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}$$

with

$$\underline{\mu} := (\mu_1, \mu_2, \ldots, \mu_n), \quad \underline{\nu} := (\nu_1, \nu_2, \ldots, \nu_n) \text{ and}$$
$$\underline{\mu}' := (\mu_1', \mu_2', \ldots, \mu_n'), \quad \underline{\nu}' := (\nu_1', \nu_2', \ldots, \nu_n') \in \mathbb{N}_0^n.$$

But this can be done by a straight forward computation, on use of the Product Formula of Proposition 6.2 and on application of Lemma 6.1 with

$$a_i : \phi(X_i) \text{ and } d_i := \phi(\partial_i) \text{ for all } i = 1, 2, \ldots, n.$$

It remains to show, that $\widetilde{\phi} : \mathbb{W}(K,n) \longrightarrow B$ is the only homomorphism of K-algebras which satisfies the requirement that

$$\widetilde{\phi}(X_i) = \phi(X_i) \quad \text{ and } \quad \widetilde{\phi}(\partial_i) = \phi(\partial_i) \text{ for all } i = 1, 2, \ldots, n.$$

But indeed, if a map $\widetilde{\phi}$ satisfies this requirement and is multiplicative, it must be defined on the elementary differential operators as suggested above. This proves the requested uniqueness. □

7.6. **Exercise.** (A) Let $n \in \mathbb{N}$, let K be a field of characteristic 0. Show, that there is a unique automorphism of K-algebras

$$\alpha : \mathbb{W}(K,n) \xrightarrow{\cong} \mathbb{W}(K,n) \text{ with } \alpha(X_i) = \partial_i \text{ and } \alpha(\partial_i) = -X_i \text{ for all } i = 1, 2, \ldots, n.$$

(B) Keep the notations and hypotheses of part (A). Present in standard form all elements $\alpha(X_i^{\nu}\partial_i^{\mu}) \in \mathbb{W}(K,n)$ with $\mu, \nu \in \mathbb{N}_0$.

8. Weighted Degrees and Filtrations

In this Section we introduce and investigate a particularly nice class of filtrations of the standard Weyl algebras, the so-called weighted filtrations. To do so, we first will introduce the related notion of weighted degree of a differential operator.

8.1. **Convention.** Throughout this section we fix a positive integer n, a field K of characteristic 0 and we consider the standard Weyl algebra

$$\mathbb{W} := \mathbb{W}(K,n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n]$$

8.2. **Definition and Remark.** (A) By a *weight* we mean a pair

$$(\underline{v}, \underline{w}) = \big((v_1, v_2, \dots, v_n), (w_1, w_2, \dots, w_n)\big) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$$

such that

$$(v_i, w_i) \neq (0,0) \text{ for all } i = 1, 2, \dots, n.$$

For

$$\underline{a} := (a_1, a_2, \dots, a_n), \quad \underline{b} := (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$$

we frequently shall use the *scalar product*

$$\underline{a} \cdot \underline{b} := \sum_{i=1}^{n} a_i b_i.$$

(B) Fix a weight $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$. We define the *degree associated to the weight* $(\underline{v}, \underline{w})$ (or just the *weighted degree*) of a differential form $d \in \mathbb{W}$ by

$$\deg^{\underline{vw}}(d) := \sup\{\underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} \mid (\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(d)\}.$$

with the usual convention that $\sup(\emptyset) = -\infty$.
Observe that by our definition of weight, for all $d \in \mathbb{W}$ and all $\underline{\mu}, \underline{\nu} \in \mathbb{N}_0$ – and using the notations of Notation and Remark 6.3 (C)– we can say:

(a) $\deg^{\underline{vw}}(d) \in \mathbb{N}_0 \cup \{-\infty\}$ with $\deg^{\underline{vw}}(d) = -\infty$ if and only if $d = 0$.
(b) If $\underline{\lambda} \leq \underline{\nu}$ and $\underline{\kappa} \leq \underline{\mu}$ for all $(\underline{\lambda}, \underline{\kappa}) \in \operatorname{supp}(d)$, then

$$\deg^{\underline{vw}}(d) \leq \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu}.$$

(c) If $\operatorname{supp}(d) \subseteq \mathbb{M}_{\leq}(\underline{\nu}, \underline{\mu})$, then

$$\deg^{\underline{vw}}(d) < \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu}.$$

(C) Keep the notations and hypotheses of part (B). We fix some non-negative integer $i \in \mathbb{N}_0$ and set

$$\mathbb{W}_i^{\underline{vw}} := \{d \in \mathbb{W} \mid \deg^{\underline{vw}}(d) \leq i\}.$$

Observe, that we also may write

$$\mathbb{W}_i^{\underline{vw}} = \bigoplus_{\underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n : \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} \leq i} K \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}.$$

8.3. **Lemma.** *Let $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ be a weight and let $d, e \in \mathbb{W}$. Then we have*

(a) $\deg^{\underline{vw}}(d + e) \leq \max\{\deg^{\underline{vw}}(d), \deg^{\underline{vw}}(e)\}$, *with equality if* $\deg^{\underline{vw}}(d) \neq \deg^{\underline{vw}}(e)$;
(b) $\deg^{\underline{vw}}(cd) = \deg^{\underline{vw}}(d)$ *for all* $c \in K \setminus \{0\}$.
(c) $\deg^{\underline{vw}}(de) \leq \deg^{\underline{vw}}(d) + \deg^{\underline{vw}}(e)$;
(d) $\deg^{\underline{vw}}\big([d, e]\big) < \deg^{\underline{vw}}(d) + \deg^{\underline{vw}}(e)$.

Notice: In statement (c) actually equality holds. We shall prove this later (see Corollary 9.5).

Proof. (a): The stated inequality is clear by the second inclusion of the following relation (see Proposition 7.3 (a)):

$$\big(\mathrm{supp}(d) \cup \mathrm{supp}(e)\big) \setminus \big(\mathrm{supp}(d) \cap \mathrm{supp}(e)\big) \subseteq \mathrm{supp}(d+e) \subseteq \mathrm{supp}(d) \cup \mathrm{supp}(e).$$

It remains to establish the stated equality if $\deg^{\underline{vw}}(d) \neq \deg^{\underline{vw}}(e)$. It suffices to treat the case in which $\deg^{\underline{vw}}(d) < \deg^{\underline{vw}}(e)$. In this case, there is some

$$(\underline{\nu}, \underline{\mu}) \in \mathrm{supp}(e) \setminus \mathrm{supp}(d) \text{ with } \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} = \deg^{\underline{vw}}(e).$$

By the first of the previous inclusions we have $(\underline{\nu}, \underline{\mu}) \in \mathrm{supp}(d+e)$ and hence

$$\deg^{\underline{vw}}(d+e) \geq \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} = \deg^{\underline{vw}}(e).$$

By the already proved inequality $\deg^{\underline{vw}}(d+e) \leq \max\{\deg^{\underline{vw}}(d), \deg^{\underline{vw}}(e)\}$ it follows that $\deg^{\underline{vw}}(d+e) = \deg^{\underline{vw}}(e)$.

(b): This is obvious.

(c): This follows easily by Proposition 7.3 (c) and Definition and Remark 8.2 (B) (b).

(d): This follows in a straight forward manner by Proposition 7.3 (d) and Definition and Remark 8.2 (B) (c). □

8.4. **Theorem.** ***(Weighted Filtrations)*** *Let*

$$\big((v_1, v_2, \ldots, v_n), (w_1, w_2, \ldots, w_n)\big) = (\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$$

be a weight. Then, the family

$$\mathbb{W}_\bullet^{\underline{vw}} := \big(\mathbb{W}_i^{\underline{vw}} = \{d \in \mathbb{W} \mid \deg^{\underline{vw}}(d) \leq i\}\big)_{i \in \mathbb{N}_0}$$

is a commutative filtration of the the K-algebra $\mathbb{W} = \mathbb{W}(K, n)$.
Moreover, the following statements hold.

- (a) $\mathbb{W}_0^{\underline{vw}} = K[X_i, \partial_j \mid v_i = 0, w_j = 0]$, *so that* $\mathbb{W}_0^{\underline{vw}}$ *is a commutative polynomial algebra in the variables X_i and ∂_j for which either $v_i = 0$ or else $w_j = 0$.*
- (b) *Let $\delta = \delta(\underline{vw}) = \max\{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$. Then, for all $i > \delta$ it holds*

$$\mathbb{W}_i^{\underline{vw}} = \sum_{j=1}^{\delta} \mathbb{W}_j^{\underline{vw}} \mathbb{W}_{i-j}^{\underline{vw}}.$$

- (c) *The filtration* $\mathbb{W}_\bullet^{\underline{vw}} = \big(\mathbb{W}_i^{\underline{vw}}\big)_{i \in \mathbb{N}_0}$ *is of finite type.*

Proof. It is clear from our definitions, that

$$\mathbb{W}_i^{\underline{vw}} \subseteq \mathbb{W}_{i+1}^{\underline{vw}} \text{ for all } i \in \mathbb{N}_0, \quad 1 \in \mathbb{W}_0^{\underline{vw}} \quad \text{ and } \mathbb{W} = \bigcup_{i \in \mathbb{N}_0} \mathbb{W}_i^{\underline{vw}}.$$

On use of Lemma 8.3 (c) it follows immediately that

$$\mathbb{W}_i^{\underline{vw}} \mathbb{W}_j^{\underline{vw}} \subseteq \mathbb{W}_{i+j}^{\underline{vw}} \text{ for all } i, j \in \mathbb{N}_0.$$

So the family $\big(\mathbb{W}_i^{\underline{v}\underline{w}} := \{d \in \mathbb{W} \mid \deg^{\underline{v}\underline{w}}(d) \leq i\}\big)_{i\in\mathbb{N}_0}$ constitutes indeed a filtration on the K-algebra $\mathbb{W}$.

Now, let $i, j \in \mathbb{N}_0$, let $d \in \mathbb{W}_i^{\underline{v}\underline{w}}$ and let $e \in \mathbb{W}_j^{\underline{v}\underline{w}}$. Then by Lemma 8.3 (d) we have

$$\deg^{\underline{v}\underline{w}}\big(de - ed\big) = \deg^{\underline{v}\underline{w}}\big([d, e]\big) \leq \deg^{\underline{v}\underline{w}}(d) + \deg^{\underline{v}\underline{w}}(e) - 1 \leq i + j - 1,$$

so that

$$de - ed \in \mathbb{W}_{i+j-1}^{\underline{v}\underline{w}}.$$

This proves, that our filtration is commutative (see Definition 3.3).

(a): Set

$$\mathbb{S} := \{i = 1, 2, \ldots, n \mid v_i \neq 0\} \text{ and } \mathbb{T} := \{j = 1, 2, \ldots, n \mid w_j \neq 0\} \text{ and}$$

$$\overline{\mathbb{S}} := \{1, 2, \ldots, n\} \setminus \mathbb{S} \text{ and } \overline{\mathbb{T}} := \{1, 2, \ldots, n\} \setminus \mathbb{T}.$$

Let $\underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n$. Then

$$\underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} = 0 \text{ if and only if } \nu_i = 0 \text{ for all } i \in \mathbb{S} \text{ and } \mu_j = 0 \text{ for all } j \in \mathbb{T}.$$

But this means that

$$\mathbb{W}_0^{\underline{v}\underline{w}} = \sum_{(\nu_i)_{i\in\overline{\mathbb{S}}}, (\mu_j)_{j\in\overline{\mathbb{T}}}} K \prod_{i\in\overline{\mathbb{S}}, j\in\overline{\mathbb{T}}} X_i^{\nu_i}\partial_j^{\mu_j} =$$
$$= K[X_i, \partial_j \mid v_i = 0, w_j = 0].$$

It remains to show, that this latter ring is a commutative polynomial algebra in all the variables X_i and ∂_j for which either $v_i = 0$ or else $w_j = 0$. In view of Theorem 7.1 it suffices to show that $X_i\partial_j = \partial_j X_i$ for all i, j with $v_i = v_j = 0$. But as $(v_k, w_k) \neq (0, 0)$ for all $k = 1, 2, \ldots, n$ (see Definition and Remark 8.2 (A)), this is clear by the Heisenberg relations (see Proposition 5.4 (b)).

(b): Let $i > \delta$. Let

$$\underline{\nu} := (\nu_1, \nu_2, \ldots, \nu_n), \quad \underline{\mu} := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}_0^n \text{ with}$$

$$\sigma := \deg^{\underline{v}\underline{w}}\big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big) = \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} \leq i.$$

We aim to show that

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \in \sum_{j=1}^{\delta} \mathbb{W}_j^{\underline{v}\underline{w}}\mathbb{W}_{i-j}^{\underline{v}\underline{w}} =: M.$$

If $\sigma \leq 0$ this is clear as $i > 0$ implies $i \geq 1$, so that

$$\mathbb{W}_0^{\underline{v}\underline{w}} = \mathbb{W}_0^{\underline{v}\underline{w}}\mathbb{W}_0^{\underline{v}\underline{w}} \subseteq \mathbb{W}_1^{\underline{v}\underline{w}}\mathbb{W}_{i-1}^{\underline{v}\underline{w}} \subseteq M.$$

So, let $\sigma > 0$. Then either

1. there is some $p \in \{1, 2, \ldots, n\}$ with $v_p > 0$ and $\nu_p > 0$, or else,
2. there is some $q \in \{1, 2, \ldots, n\}$ with $w_q > 0$ and $\mu_q > 0$.

In the above case (1) we can write

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} = X_p d, \text{ with } d := \big(\prod_{k=1}^{n} X_k^{\nu_k - \delta_{k,p}}\big)\underline{\partial}^{\underline{\mu}}.$$

As $\deg^{\underline{vw}}(X_p) = v_p \leq \delta$ and $\deg^{\underline{vw}}(d) = \sigma - v_p$ it follows that

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} = X_p d \in \mathbb{W}^{\underline{vw}}_{v_p}\mathbb{W}^{\underline{vw}}_{\sigma - v_p} \subseteq \mathbb{W}^{\underline{vw}}_{v_p}\mathbb{W}^{\underline{vw}}_{i - v_p} \subseteq M.$$

In the above case (2) we may first assume, that we are not in the case (1). This means in particular that either $v_q = 0$ or $\nu_q = 0$, hence $v_q\nu_q = 0$, so that

$$\deg^{\underline{vw}}(X_q^{\nu_q}\partial_q) = w_q \leq \delta.$$

Now, in view of the Heisenberg relations, we may write

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} = X_q^{\nu_q}\partial_q e \text{ with } e := \prod_{s \neq q} X_s^{\nu_s} \prod_{k=1}^{n} \partial_k^{\mu_k - \delta_{k,q}}.$$

As $v_q\nu_q = 0$, we have $\deg^{\underline{vw}}(e) = \sigma - w_q$, and it follows that

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} = X_q^{\nu_q}\partial_q e \in \mathbb{W}^{\underline{vw}}_{w_q}\mathbb{W}^{\underline{vw}}_{\sigma - w_q} \subseteq \mathbb{W}^{\underline{vw}}_{w_q}\mathbb{W}^{\underline{vw}}_{i - w_q} \subseteq M.$$

But this shows, what we were aiming for, hence that

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \in M \text{ whenever } \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} \leq i.$$

But this means that

$$\mathbb{W}^{\underline{vw}}_{i} \subseteq M = \sum_{j=1}^{\delta} \mathbb{W}^{\underline{vw}}_{j}\mathbb{W}^{\underline{vw}}_{i-j}$$

and hence proves statement (b).

(c): This is an immediate consequence of statements (a) and (b) (see Definition and Remark 3.4 (C)). □

8.5. **Definition.** Let the notations and hypotheses be as in Theorem 8.4. In particular, let

$$\big((v_1, v_2, \ldots, v_n), (w_1, w_2, \ldots, w_n)\big) = (\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$$

be a weight. Then, the filtration

$$\mathbb{W}^{\underline{vw}}_{\bullet} = \big(\mathbb{W}^{\underline{vw}}_{i}\big)_{i \in \mathbb{N}_0} = \big(\{d \in \mathbb{W} \mid \deg^{\underline{vw}}(d) \leq i\}\big)_{i \in \mathbb{N}_0}$$

is called the *filtration induced by the weight* $(\underline{v}, \underline{w})$. Generally, we call *weighted filtrations* all fltrations which are induced in this way by a weight.

8.6. **Definition and Remark.** (A) We consider the strings

$$\underline{0} := (0, 0, \ldots, 0), \quad \underline{1} := (1, 1, \ldots, 1) \in \mathbb{N}_0^n$$

and a differential form $d \in \mathbb{W}$. We define the *standard degree* or just the *degree* $\deg(d)$ of d as the weighted degree with respect to the weight $(\underline{1}, \underline{1}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$, hence

$$\deg(d) := \deg^{\underline{11}}(d).$$

Observe that

$$\deg(d) := \sup\{|\underline{\nu}| + |\underline{\mu}| \mid (\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(d)\}.$$

The corresponding induced weighted filtration

$$\mathbb{W}_\bullet^{\deg} := \mathbb{W}_\bullet^{\underline{11}} = \big(\mathbb{W}_i^{\underline{11}}\big)_{i \in \mathbb{N}_0} = \big(\{d \in \mathbb{W} \mid \deg(d) \leq i\}\big)_{i \in \mathbb{N}_0}$$

is called the *standard degree filtration* or just the *degree filtration* of $\mathbb{W}$.

(B) Keep the notations and hypotheses of part (A). The *order* of the differential operator d is defined by

$$\operatorname{ord}(d) := \deg^{\underline{01}}(d).$$

Observe that

$$\operatorname{ord}(d) = \sup\{|\underline{\mu}| \mid (\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(d)\}.$$

The corresponding induced weighted filtration

$$\mathbb{W}_\bullet^{\operatorname{ord}} := \mathbb{W}_\bullet^{\underline{01}} = \big(\mathbb{W}_i^{\underline{01}}\big)_{i \in \mathbb{N}_0} = \big(\{d \in \mathbb{W} \mid \operatorname{ord}(d) \leq i\}\big)_{i \in \mathbb{N}_0}$$

is called the *order filtration* of $\mathbb{W}$.

Now, as an immediate application of Theorem 8.4 we obtain:

8.7. **Corollary.** *Let the notations be as in Convention 8.1. Then it holds*

(a) *The degree filtration* $\mathbb{W}_\bullet^{\deg}$ *is very good.*
(b) *The order filtration* $\mathbb{W}_\bullet^{\operatorname{ord}}$ *is good and* $\mathbb{W}_0^{\operatorname{ord}} = K[X_1, X_2, \ldots, X_n]$.

Proof. In the notations of Theorem 8.4 (b) we have

$$\delta(\underline{1}, \underline{1}) = 1 \text{ and } \delta(\underline{0}, \underline{1}) = 1.$$

Moreover, by Theorem 8.4 (a) we have

$$\mathbb{W}_0^{\underline{11}} = K \text{ and } \mathbb{W}_0^{\underline{01}} = K[X_1, X_2, \ldots, X_n]$$

This proves our claim (see Definition and Remark 3.4 (C)). □

8.8. **Exercise.** (A) Show that the degree filtration is the only very good filtration on $\mathbb{W}$.

(B) Write down all weights $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ for which the induced filtration $\mathbb{W}_\bullet^{\underline{vw}}$ is good.

9. Weighted Associated Graded Rings

This Section is devoted to the study of the associated graded rings of weighted filtrations of standard Weyl algebras. We shall see, that these are all naturally isomorphic to polynomial rings.

9.1. **Convention.** Again, throughout this section we fix a positive integer n, a field K of characteristic 0 and consider the standard Weyl algebra

$$\mathbb{W} := \mathbb{W}(K,n) = K[X_1, X_2, \dots, X_n, \partial_1, \partial_2, \dots, \partial_n].$$

In addition, we introduce the polynomial ring

$$\mathbb{P} := K[Y_1, Y_2, \dots, Y_n, Z_1, Z_2, \dots, Z_n]$$

in the indeterminates $Y_1, Y_2, \dots, Y_n, Z_1, Z_2, \dots, Z_n$ with coefficients in the field K.

9.2. **Definition and Remark.** (A) Fix a weight $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ and consider the induced weighted filtration $\mathbb{W}_\bullet^{\underline{v}\underline{w}}$. To write down the corresponding associated graded ring, we introduce the following notation:

$$\mathbb{G}^{\underline{v}\underline{w}} = \bigoplus_{i \in \mathbb{N}_0} \mathbb{G}_i^{\underline{v}\underline{w}} := \mathrm{Gr}_{\mathbb{W}_\bullet^{\underline{v}\underline{w}}}(\mathbb{W}) = \bigoplus_{i \in \mathbb{N}_0} \mathrm{Gr}_{\mathbb{W}_\bullet^{\underline{v}\underline{w}}}(\mathbb{W})_i.$$

(B) Keep the above notations and hypotheses. For each $j \in \mathbb{Z}$ we introduce the notations:

$$\mathbb{I}_{\leq j}^{\underline{v}\underline{w}} := \{(\underline{\nu}, \underline{\mu}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} \leq j\};$$
$$\mathbb{I}_{= j}^{\underline{v}\underline{w}} := \{(\underline{\nu}, \underline{\mu}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} = j\}.$$

Fix some $i \in \mathbb{N}_0$. Observe that

$$\begin{aligned}\mathbb{G}_i^{\underline{v}\underline{w}} &= \mathbb{W}_i^{\underline{v}\underline{w}} / \mathbb{W}_{i-1}^{\underline{v}\underline{w}} = \\ &= \Big(\bigoplus_{(\underline{\nu},\underline{\mu}) \in \mathbb{I}_{\leq i}^{\underline{v}\underline{w}}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\Big) / \Big(\bigoplus_{(\underline{\nu},\underline{\mu}) \in \mathbb{I}_{\leq i-1}^{\underline{v}\underline{w}}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\Big) = \\ &= \Big[\Big(\bigoplus_{(\underline{\nu},\underline{\mu}) \in \mathbb{I}_{\leq i-1}^{\underline{v}\underline{w}}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\Big) \oplus \Big(\bigoplus_{(\underline{\nu},\underline{\mu}) \in \mathbb{I}_{= i}^{\underline{v}\underline{w}}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\Big)\Big] / \Big(\bigoplus_{(\underline{\nu},\underline{\mu}) \in \mathbb{I}_{\leq i-1}^{\underline{v}\underline{w}}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\Big).\end{aligned}$$

As a consequence, we get an isomorphism of K-vector spaces

$$\epsilon_i^{\underline{v}\underline{w}} : \bigoplus_{(\underline{\nu},\underline{\mu}) \in \mathbb{I}_{= i}^{\underline{v}\underline{w}}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \xrightarrow{\cong} \mathbb{G}_i^{\underline{v}\underline{w}}$$

such that

$$\epsilon_i^{\underline{v}\underline{w}}\big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big) = \big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} + \mathbb{W}_{i-1}^{\underline{v}\underline{w}}\big) \in \mathbb{W}_i^{\underline{v}\underline{w}} / \mathbb{W}_{i-1}^{\underline{v}\underline{w}} = \mathbb{G}_i^{\underline{v}\underline{w}} \text{ for all } (\underline{\nu}, \underline{\mu}) \in \mathbb{I}_{= i}^{\underline{v}\underline{w}}.$$

In particular we can say:

The family $\big((\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})^* := \epsilon_i^{\underline{v}\underline{w}}(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})\big)_{(\underline{\nu},\underline{\mu}) \in \mathbb{I}_{= i}^{\underline{v}\underline{w}}}$ is a K-basis of $\mathbb{G}_i^{\underline{v}\underline{w}}$.

We call this basis the *standard basis* of $\mathbb{G}_i^{\underline{v}\underline{w}}$. Its elements are called *standard basis elements* of the associated graded ring $\mathbb{G}^{\underline{v}\underline{w}}$.

(C) Keep the previously introduced notation. We add a few more useful observations on standard basis elements. First, observe that we may write

(a) $(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})^* \in \mathbb{G}^{\underline{vw}}_{\underline{v}\cdot\underline{\nu}+\underline{w}\cdot\underline{\mu}}$ for all $(\underline{\nu},\underline{\mu}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$.

(b) $X_i^* \in \mathbb{G}^{\underline{vw}}_{v_i}$ and $\partial_j^* \in \mathbb{G}^{\underline{vw}}_{w_j}$ for all $i, j \in \{1, 2, \ldots, n\}$.

Moreover, by the observations made in part (B) we also can say that all standard basis elements form a K-basis of the whole associated graded ring, thus:

(c) The family $\big((\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})^*\big)_{(\underline{\nu},\underline{\mu})\in\mathbb{N}_0^n\times\mathbb{N}_0^n}$ is a K-basis of $\mathbb{G}^{\underline{vw}}$.

Finally, as the associated graded ring is commutative, and keeping in mind how the multiplication in this ring is defined (see Remark and Definition 3.1 (B)) we get the following product formula

(d) $(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})^* = \big(\prod_{i=1}^n X_i^{\nu_i}\prod_{j=1}^n \partial^{\mu_j}\big)^* = \prod_{i=1}^n \big(X_i^*\big)^{\nu_i}\prod_{j=1}^n \big(\partial_j^*\big)^{\mu_j} =: \big(\underline{X}^*\big)^{\underline{\nu}}\big(\underline{\partial}^*\big)^{\underline{\mu}}$.

9.3. **Exercise and Definition.** (A) We fix a weight $(\underline{v},\underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$. As in Definition and Remark 9.2 (A) we use again the notation

$$\mathbb{I}^{\underline{vw}}_{=i} := \{(\underline{\nu},\underline{\mu}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid \underline{v}\cdot\underline{\nu} + \underline{w}\cdot\underline{\mu} = i\}$$

and consider the K-subspace

$$\mathbb{P}^{\underline{vw}}_i := \bigoplus_{(\underline{\nu},\underline{\mu})\in\mathbb{I}^{\underline{vw}}_{=i}} K\underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \subseteq \mathbb{P} \text{ for all } i \in \mathbb{N}_0.$$

of our polynomial ring $\mathbb{P} = K[Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n]$. Prove the following statements:

(a) $K \subseteq \mathbb{P}^{\underline{vw}}_0$;

(b) $\mathbb{P}^{\underline{vw}}_i\mathbb{P}^{\underline{vw}}_j \subseteq \mathbb{P}^{\underline{vw}}_{i+j}$ for all $i, j \in \mathbb{N}_0$.

(c) $\mathbb{P} = \bigoplus_{i\in\mathbb{N}_0}\mathbb{P}^{\underline{vw}}_i$.

(B) Let the hypotheses and notations be as in part (A). Conclude that

$$\text{the family } \big(\mathbb{P}^{\underline{vw}}_i\big)_{i\in\mathbb{N}_0} \text{ defines a grading of the ring } \mathbb{P}.$$

We call this grading the *grading induced by the weight* $(\underline{v},\underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$. If we endow our polynomial ring with this grading we write it as $\mathbb{P}^{\underline{vw}}$, thus

$$\mathbb{P} = \mathbb{P}^{\underline{vw}} = \bigoplus_{i\in\mathbb{N}_0}\mathbb{P}^{\underline{vw}}_i.$$

9.4. **Theorem.** ***(Structure of Weighted Associated Graded Rings)*** *Let* $(\underline{v},\underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ *be a weight. Then there exists an isomorphism of K-algebras, which preserves gradings (see Convention, Reminders and Notations 1.1 (I)).*

$$\eta^{\underline{vw}} : \mathbb{P} = \mathbb{P}^{\underline{vw}} \xrightarrow{\cong} \mathbb{G}^{\underline{vw}}$$

given by

$$Y_i \mapsto \eta^{\underline{vw}}(Y_i) := X_i^*, \textit{ for all } i = 1, 2, \ldots, n;$$
$$Z_j \mapsto \eta^{\underline{vw}}(Z_j) := \partial_j^*, \textit{ for all } j = 1, 2, \ldots, n.$$

Proof. According to the universal property of the polynomial ring $\mathbb{P}$ there is a unique homomorphism of K-algebras

$$\eta^{\underline{vw}} : \mathbb{P} \longrightarrow \mathbb{G}^{\underline{vw}}$$

such that

$$Y_i \mapsto \eta^{\underline{vw}}(Y_i) := X_i^*, \text{ for all } i = 1, 2, \ldots, n;$$
$$Z_j \mapsto \eta^{\underline{vw}}(Z_j) := \partial_j^*, \text{ for all } j = 1, 2, \ldots, n.$$

In view of the product formula of Definition and Remark 9.2 (C) we obtain

$$\eta^{\underline{vw}}\big(\underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}}\big) = \big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big)^* \text{ for all } \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n.$$

In particular $\eta^{\underline{vw}}$ yields a bijection between the monomial basis of the polynomial ring $\mathbb{P}$ and the standard basis of the associated graded ring $\mathbb{G}^{\underline{vw}}$. So, $\eta^{\underline{vw}}$ is indeed an isomorphism. But moreover, for each $i \in \mathbb{N}_0$ it also follows that $\eta^{\underline{vw}}$ yields an bijection between the monomial basis of the subspace $\mathbb{P}_i^{\underline{vw}} \subseteq \mathbb{P}$ and the standard basis of $\mathbb{G}_i^{\underline{vw}}$. But this means, that $\eta^{\underline{vw}}$ preserves the gradings. □

In Lemma 8.3 (c) we have seen that weighted degrees are *sub-additive*, which means that $\deg^{\underline{vw}}(de) \leq \deg^{\underline{vw}}(d) + \deg^{\underline{vw}}(e)$ for all $d, e \in \mathbb{W}$. As an application of Theorem 9.4 we now shall improve on this and show, that weighted degrees are indeed *additive*, which means that the above inequality is in fact always an equality.

9.5. **Corollary.** ***(Additivity of Weighted Degrees)*** *Let $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ be a weight and let $d, e \in \mathbb{W}$. Then*

$$\deg^{\underline{vw}}(de) = \deg^{\underline{vw}}(d) + \deg^{\underline{vw}}(e).$$

Proof. If $d = 0$ or $e = 0$ our claim is clear. So let $d, e \neq 0$. We have

$$i := \deg^{\underline{vw}}(d) \in \mathbb{N}_0 \text{ and } j := \deg^{\underline{vw}}(e).$$

We use again the notation

$$\mathbb{I}_{=k}^{\underline{vw}} := \{(\underline{\nu}, \underline{\mu}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} = k\} \text{ for all } k \in \mathbb{N}_0$$

and set

$$M := \bigoplus_{(\underline{\nu}, \underline{\mu}) \in \mathbb{I}_{=i}^{\underline{vw}}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \quad \text{and } N := \bigoplus_{(\underline{\nu}, \underline{\mu}) \in \mathbb{I}_{=j}^{\underline{vw}}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}.$$

We then may write

$$d = a + r \text{ with } a \in M \setminus \{0\} \text{ and } \deg^{\underline{vw}}(r) < i;$$
$$e = b + s \text{ with } a \in N \setminus \{0\} \text{ and } \deg^{\underline{vw}}(s) < j.$$

We thus have

$$de = ab + (as + rb + rs)$$

By what we know already about degrees we have $\deg^{\underline{vw}}(as+rb+rs) < i+j$ (see Lemma 8.3 (a), (c)). So, in view of Lemma 8.3 (a) it suffices to show that

$$\deg^{\underline{vw}}(ab) = i + j.$$

To do so, we write

$$a = \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a)} c^{(a)}_{\underline{\nu},\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}, \text{ with } c^{(a)}_{\underline{\nu},\underline{\mu}} \in K\setminus\{0\} \text{ for all } (\underline{\nu},\underline{\mu})\in\mathrm{supp}(a) \text{ and}$$

$$b = \sum_{(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(b)}_{\underline{\nu}',\underline{\mu}} \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}, \text{ with } c^{(b)}_{\underline{\nu}',\underline{\mu}'} \in K\setminus\{0\} \text{ for all } (\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b).$$

It follows that

$$ab = \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a) \text{ and } (\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}.$$

By Exercise 6.4 (A) and in the notation of Notation and Remark 6.3 (C), it follows that

$$\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'} - \underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'} \in \sum_{(\underline{\lambda},\underline{\kappa})\in\mathbb{M}(\underline{\nu}+\underline{\nu}',\underline{\mu}+\underline{\mu}')} K\underline{X}^{\underline{\lambda}} \underline{\partial}^{\underline{\kappa}}$$

for all $(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a)$ and all $(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)$. Observe that

$$(\underline{\nu}+\underline{\nu}',\underline{\mu}+\underline{\mu}')\in\mathbb{I}^{\underline{vw}}_{=i+j} \text{ for all } (\underline{\nu},\underline{\mu})\in\mathrm{supp}(a) \text{ and all } (\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b).$$

So, by Definition and Remark 8.2 (B)(c) it follows that

$$\deg^{\underline{vw}}\left(\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'} - \underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'}\right) < i+j$$

for all $(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a)$ and all $(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)$. If we set

$$h := \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'}.$$

and on repeated use of Lemma 8.3 (a) and (b) we thus get

$$\deg^{\underline{vw}}(ab-h) =$$
$$\deg^{\underline{vw}}[\sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}'} (\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'} - \underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'})] < i+j.$$

So, we may write

$$ab = h+u \text{ with } \deg^{\underline{vw}}(u) < i+j.$$

By Lemma 8.3 (a) it thus suffices to show that $\deg^{\underline{vw}}(h) = i+j$. As

$$h = \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'} \in \bigoplus_{(\underline{\nu},\underline{\mu})\in\mathbb{I}^{\underline{vw}}_{=i+j}} K\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}$$

It suffices to show that $h\neq 0$. To do so, we consider the two polynomials

$$f := \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a)} c^{(a)}_{\underline{\nu},\underline{\mu}} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}} \in \mathbb{P}^{\underline{vw}}_{i} \text{ and}$$

$$g := \sum_{(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \underline{Y}^{\underline{\nu}'} \underline{Z}^{\underline{\mu}'} \in \mathbb{P}^{\underline{vw}}_{j}.$$

As supp(a) and supp(b) are non-empty, and all coefficients of f and g are non-zero, we have $f \neq 0$ and $g \neq 0$. As $\mathbb{P}$ is an integral domain. it follows that $fg \neq 0$. We set

$$h^* := (h + \mathbb{W}^{\underline{vw}}_{i+j-1}) \in \mathbb{W}^{\underline{vw}}_{i+j}/\mathbb{W}^{\underline{vw}}_{i+j-1} = \mathbb{G}^{\underline{vw}}_{i+j},$$

so that

$$h^* = \sum_{(\underline{\nu},\underline{\mu})\in \text{supp}(a),(\underline{\nu}',\underline{\mu}')\in \text{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \big(\underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'}\big)^*.$$

Applying the isomorphism

$$\eta^{\underline{vw}} : \mathbb{P} = \mathbb{P}^{\underline{vw}} \xrightarrow{\cong} \mathbb{G}^{\underline{vw}}$$

of Theorem 9.4, we now get

$$\begin{aligned}
0 \neq \eta^{\underline{vw}}(fg) &= \eta^{\underline{vw}}\big(\big[\sum_{(\underline{\nu},\underline{\mu})\in \text{supp}(a)} c^{(a)}_{\underline{\nu},\underline{\mu}} \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}}\big]\big[\sum_{(\underline{\nu}',\underline{\mu}')\in \text{supp}(b)} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \underline{Y}^{\underline{\nu}'}\underline{Z}^{\underline{\mu}'}\big]\big) = \\
&= \eta^{\underline{vw}}\big(\sum_{(\underline{\nu},\underline{\mu})\in \text{supp}(a),(\underline{\nu}',\underline{\mu}')\in \text{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \underline{Y}^{\underline{\nu}+\underline{\nu}'}\underline{Z}^{\underline{\mu}+\underline{\mu}'}\big) = \\
&= \sum_{(\underline{\nu},\underline{\mu})\in \text{supp}(a),(\underline{\nu}',\underline{\mu}')\in \text{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \eta^{\underline{vw}}\big(\underline{Y}^{\underline{\nu}+\underline{\nu}'}\underline{Z}^{\underline{\mu}+\underline{\mu}'}\big) = \\
&= \sum_{(\underline{\nu},\underline{\mu})\in \text{supp}(a),(\underline{\nu}',\underline{\mu}')\in \text{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \big(\underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'}\big)^* = h^*.
\end{aligned}$$

But this clearly implies that $h \neq 0$. □

9.6. **Corollary.** ***(Integrity of Standard Weyl Algebras)*** *The standard Weyl algebra* $\mathbb{W}$ *is an integral domain:*

$$\textit{If } d, e \in \mathbb{W} \setminus \{0\}, \textit{ then } de \neq 0.$$

Proof. Apply Theorem 9.4 and keep in mind that an element of $\mathbb{W}$ vanishes if and only if its degree (with respect to any weight) equals $-\infty$. □

9.7. **Exercise.** (A) We fix a weight $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ and set

$$\Gamma^{\underline{v},\underline{w}} := \{\underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} \mid \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n\}.$$

Prove the following statements

(a) $0 \in \Gamma^{\underline{vw}} \subseteq \mathbb{N}_0$.
(b) If $i, j \in \Gamma^{\underline{vw}}$, then $i + j \in \Gamma^{\underline{vw}}$.
(c) $\mathbb{G}^{\underline{vw}}_i \neq 0 \Leftrightarrow \mathbb{P}^{\underline{vw}}_i \neq 0 \Leftrightarrow i \in \Gamma^{\underline{vw}}$.

$\Gamma^{\underline{v},\underline{w}}$ is called the *degree semigroup* associated to the weight $(\underline{v}, \underline{w})$.

(B) Let $n = 1$, $\underline{v} = (p)$ and $\underline{w} = (q)$, where $p, q \in \mathbb{N}$ are two distinct prime numbers. Determine $\Gamma^{\underline{v},\underline{w}}$ and the standard bases of all K-vector spaces

$$\mathbb{P}^{\underline{vw}}_i \text{ and } \mathbb{G}^{\underline{vw}}_i \text{ for } i \in \Gamma^{\underline{vw}},$$

at least for some specified pairs like $(p, q) = (2, 3), (2, 5), (5, 7), \ldots$

(C) Show, that the ring $\operatorname{End}_K\big(K[X_1, X_2, \ldots, X_n]\big)$ is not an integral domain.

10. Filtered Modules

Now, we aim to consider finitely generated left-modules over standard Weyl algebras: the so-called D-modules. Our basic aim is to endow such modules with appropriate filtrations, which are compatible with a given weighted filtration of the underlying Weyl algebra. This will allow us to define associated graded modules over the corresponding associated graded ring of the Weyl algebra - hence over a weight graded polynomial ring. We approach the subject in a more general setting.

10.1. **Definition and Remark.** (A) Let K be a field and let $A = (A, A_\bullet)$ be a filtered K-algebra. Let U be a left-module over A. By a *filtration of* U *compatible with* $A_\bullet$ or just an $A_\bullet$*-filtration* of U we mean a family

$$U_\bullet = (U_i)_{i\in\mathbb{Z}}$$

such that the following conditions hold:

(a) Each U_i is a K-vector subspace of U;
(b) $U_i \subseteq U_{i+1}$ for all $i \in \mathbb{Z}$;
(c) $U = \bigcup_{i\in\mathbb{Z}} U_i$;
(d) $A_i U_j \subseteq U_{i+j}$ for all $i \in \mathbb{N}_0$ and all $j \in \mathbb{Z}$.

In requirement (d) we have used the standard notation

$$A_i U_j := \sum_{(f,u)\in A_i\times U_j} K f u \text{ for all } i \in \mathbb{N}_0 \text{ and all } j \in \mathbb{Z},$$

which we shall use from now on without further mention. If an $A_\bullet$-filtration $U_\bullet$ of U is given, we say that $(U, U_\bullet)$ or – by abuse of language – that U is a $A_\bullet$ *filtered* A*-module* or just that U is a *filtered* A*-module.*

(B) Keep the notations and hypotheses of part (A) and let $U_\bullet = (U_i)_{i\in\mathbb{Z}}$ be a filtered A-module. Observe that

For all $i \in \mathbb{Z}$ the K-vector space U_i is a left A_0-submodule of U.

(C) We say that two $A_\bullet$-filtrations $U_\bullet^{(1)}$, $U_\bullet^{(2)}$ are *equivalent* if there is some $r \in \mathbb{N}_0$ such that

(a) $U_{i-r}^{(1)} \subseteq U_i^{(2)} \subseteq U_{i+r}^{(1)}$ for all $i \in \mathbb{Z}$.

Later, we shall use the following observation.

Assume that the obove condition (a) holds, let $i \in \mathbb{N}$ and let $a \in A_i$. Then we have

(b) $aU_j^{(1)} \subseteq U_{j+i-1}^{(1)}$ for all $j \in \mathbb{Z}$ $\Rightarrow$ $a^k U_j^{(1)} \subseteq U_{j+k(i-1)}^{(1)}$ for all $j \in \mathbb{Z}$ and all $k \in \mathbb{N}_0$.
(c) $aU_j^{(1)} \subseteq U_{j+i-1}^{(1)}$ for all $j \in \mathbb{Z}$ $\Rightarrow$ $a^{2r+1} U_j^{(2)} \subseteq U_{j+(2r+1)i-1}^{(2)}$ for all $j \in \mathbb{Z}$.

To prove statement (b), we assume that $aU_j^{(1)} \subseteq U_{j+i-1}^{(1)}$ for all $j \in \mathbb{Z}$ and proceed by induction on k. If $k = 0$ our claim is obvious. If $k > 0$, we may assume by induction that $a^{k-1}U_j^{(1)} \subseteq U_{j+(k-1)(i-1)}^{(1)}$ for all $j \in \mathbb{Z}$, so that indeed

$$a^k U_j^{(1)} = aa^{k-1}U_j^{(1)} \subseteq aU_{j+(k-1)(i-1)}^{(1)} \subseteq U_{j+(k-1)(i-1)+(i-1)}^{(1)} = U_{j+k(i-1)}^{(1)} \text{ for all } j \in \mathbb{Z},$$

and this proves statement (b). If we apply statement (b) with $k = 2r + 1$ and observe condition (a), we get

$$\begin{aligned} a^{2r+1}U_j^{(2)} &\subseteq a^{2r+1}U_{j+r}^{(1)} \subseteq U_{j+r+(2r+1)(i-1)}^{(1)} \subseteq U_{j+2r+(2r+1)(i-1)}^{(2)} \\ &= U_{j+2r+2ri-2r+i-1}^{(1)} = U_{j+2ri+i-1}^{(2)} = U_{j+(2r+1)i-1}^{(2)} \text{ for all } j \in \mathbb{Z}, \end{aligned}$$

and this proves statement (c).

10.2. **Remark and Definition.** (A) Let K be a field and let $A = (A, A_\bullet)$ be a filtered K-algebra and let $U = (U, U_\bullet)$ be an $A_\bullet$-filtered A-module. We consider the corresponding associated graded ring

$$\operatorname{Gr}(A) = \operatorname{Gr}_{A_\bullet}(A) = \bigoplus_{i \in \mathbb{N}_0} A_i/A_{i-1}.$$

and the K-vector space

$$\operatorname{Gr}(U) = \operatorname{Gr}_{U_\bullet}(U) = \bigoplus_{i \in \mathbb{Z}} U_i/U_{i-1}.$$

For all $i \in \mathbb{Z}$ we also use the notation

$$\operatorname{Gr}(U)_i = \operatorname{Gr}_{U_\bullet}(U)_i := U_i/U_{i-1},$$

so that we may write

$$\operatorname{Gr}(U) = \operatorname{Gr}_{U_\bullet}(U) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Gr}_{U_\bullet}(U)_i.$$

(B) Let $i \in \mathbb{N}_0$, let $j \in \mathbb{Z}$ let $f, f' \in A_i$ and let $g, g' \in U_j$ such that

$$h := f - f' \in A_{i-1} \text{ and } k := g - g' \in U_{j-1}.$$

It follows that

$$\begin{aligned} fg - f'g' &= fg - (f-h)(g-k) = fk + hg - hk \\ &\in A_iU_{j-1} + A_{i-1}U_j + A_{i-1}U_{j-1} \subseteq \\ &\subseteq U_{i+(j-1)} + U_{j+(i-1)} + U_{(i-1)+(j-1)} \subseteq U_{i+j-1}. \end{aligned}$$

So in $U_{i+j}/U_{i+j-1} = \operatorname{Gr}_{U_\bullet}(U)_{i+j} \subset \operatorname{Gr}_{U_\bullet}(U)$ we get the relation

$$fg + U_{i+j-1} = f'g' + U_{i+j-1}.$$

This allows to define a $\operatorname{Gr}_{A_\bullet}(A)$-*scalar multiplication* on the K-space $\operatorname{Gr}_{U_\bullet}(U)$ which is induced by

$$(f + A_{i-1})(g + U_{j-1}) := fg + U_{i+j-1}$$

for all $i \in \mathbb{N}_0$, all $j \in \mathbb{Z}$, all $f \in A_i$ $g \in U_j$. More generally, if $r, s \in \mathbb{N}_0$, $t \in \mathbb{Z}$,

$$\overline{f} = \sum_{i=0}^{r} \overline{f_i}, \text{ with } f_i \in A_i \text{ and } \overline{f_i} = (f_i + A_{i-1}) \in \mathrm{Gr}_{A_\bullet}(A)_i \text{ for all } i = 0, 1, \dots, r,$$

and

$$\overline{g} = \sum_{j=t}^{t+s} \overline{g_j}, \text{ with } g_j \in U_j \text{ and } \overline{g_j} = (g_j + U_{j-1}) \in \mathrm{Gr}_{U_\bullet}(U)_j \text{ for all } j = t, t+1, \dots, t+s,$$

then

$$\overline{f}\overline{g} = \sum_{k=t}^{r+t+s} \sum_{i+j=k} \overline{f_i}\overline{g_j} = \sum_{k=t}^{r+t+s} \sum_{i+j=k} (f_i g_j + U_{i+j-1}).$$

(C) Keep the above notations and hypotheses. With respect to our scalar multiplication on $\mathrm{Gr}_{U_\bullet}(U)$ we have the relations

$$\mathrm{Gr}_{A_\bullet}(A)_i \mathrm{Gr}_{U_\bullet}(U)_j \subseteq \mathrm{Gr}_{U_\bullet}(U)_{i+j} \text{ for all } i, j \in \mathbb{Z}.$$

So, the K-vector space $\mathrm{Gr}_{U_\bullet}(U)$ is turned into a graded $\mathrm{Gr}_{A_\bullet}(A)$-module

$$\mathrm{Gr}_{U_\bullet}(U) = \big(\mathrm{Gr}_{U_\bullet}(U), (\mathrm{Gr}_{U_\bullet}(U)_i)_{i \in \mathbb{Z}}\big) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Gr}_{U_\bullet}(U)_i$$

by means of the above multiplication. We call this $\mathrm{Gr}_{A_\bullet}(A)$-module $\mathrm{Gr}_{U_\bullet}(U)$ the *associated graded module* of U with respect to the filtration $U_\bullet$. From now on, we always furnish $\mathrm{Gr}_{U_\bullet}(U)$ with this structure of graded $\mathrm{Gr}_{A_\bullet}(A)$-module.

10.3. **Definition.** Let K be a field and let $A = (A, A_\bullet)$ be a filtered K-algebra. Assume that the filtration $A_\bullet$ is commutative, so that the corresponding associated graded ring

$$\mathrm{Gr}(A) = \mathrm{Gr}_{A_\bullet}(A) = \bigoplus_{i \in \mathbb{N}_0} A_i / A_{i-1}$$

is commutative.
Moreover, let $U = (U, U_\bullet)$ be an $A_\bullet$-filtered A-module and consider the corresponding associated graded module

$$\mathrm{Gr}(U) = \mathrm{Gr}_{U_\bullet}(U) = \bigoplus_{i \in \mathbb{Z}} U_i / U_{i-1}.$$

in addition, consider the *annihilator* ideal

$$\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet}(U)\big) := \{f \in \mathrm{Gr}_{A_\bullet}(A) \mid f\mathrm{Gr}_{U_\bullet}(U) = 0\}$$

of the $\mathrm{Gr}_{A_\bullet}(A)$-module $\mathrm{Gr}_{U_\bullet}(U)$. We define the *characteristic variety* $\mathbb{V}_{U_\bullet}(U)$ of the $A_\bullet$-filtered A-module $U = (U, U_\bullet)$ as the *prime variety* of the annihilator ideal of $\mathrm{Gr}_{U_\bullet}(U)$, hence

$$\mathbb{V}_{U_\bullet}(U) := \mathrm{Var}\big(\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}(\mathrm{Gr}_{U_\bullet}(U))\big) \subseteq \mathrm{Spec}(\mathrm{Gr}_{A_\bullet}(A)).$$

We also call this variety the *characteristic variety of the left A-module U with respect to the $A_\bullet$-filtration $U_\bullet$* or just the *characteristic variety of U with respect to $U_\bullet$*.

10.4. **Proposition.** ***(Equality of Characteristic Varieties for Equivalent Filtrations)*** *Let K be a field and let $A = (A, A_\bullet)$ be a filtered K-algebra. Assume that the filtration $A_\bullet$ is commutative (see Definition 3.3). Let U be an A-module which is endowed with two equivalent $A_\bullet$-filtrations $U_\bullet^{(1)}$ and $U_\bullet^{(2)}$. Then*

$$\mathbb{V}_{U_\bullet^{(1)}}(U) = \mathbb{V}_{U_\bullet^{(2)}}(U).$$

Proof. We have to show that

$$\sqrt{\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(1)}}(U)\big)} = \sqrt{\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(2)}}(U)\big)}.$$

By symmetry, it suffices to show that

$$\sqrt{\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(1)}}(U)\big)} \subseteq \sqrt{\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(2)}}(U)\big)}.$$

In view of the fact that the formation of radicals of ideals is idempotent, it suffices even to show that

$$\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(1)}}(U)\big) \subseteq \sqrt{\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(2)}}(U)\big)}.$$

As $\mathrm{Gr}_{U_\bullet^{(1)}}(U)$ is a graded $\mathrm{Gr}_{A_\bullet}(A)$-module, its annihilator is a graded ideal of $\mathrm{Gr}_{A_\bullet}(A)$. So, it finally is enough to show, that

$$\overline{a} \in \sqrt{\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(2)}}(U)\big)} \text{ for all } i \in \mathbb{N}_0 \text{ and all } \overline{a} \in \mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(1)}}(U)\big)_i.$$

So, fix some $i \in \mathbb{N}_0$ and some

$$\overline{a} \in \mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(1)}}(U)\big)_i \subseteq \mathrm{Gr}_{A_\bullet}(A)_i = A_i/A_{i-1}.$$

We chose some $a \in A_i$ with $\overline{a} = a + A_{i-1} \in A_i/A_{i-1}$.. For all $j \in \mathbb{Z}$ we have in $\mathrm{Gr}_{U_\bullet^{(1)}}(U)$ the relation

$$aU_j^{(1)} + U_{j+i-1}^{(1)} = (a + A_{i-1})(U_j^{(1)}/U_{j-1}^{(1)}) = \overline{a}(U_j^{(1)}/U_{j-1}^{(1)}) = \overline{a}\mathrm{Gr}_{U_\bullet^{(1)}}(U)_j = 0,$$

and hence

$$aU_j^{(1)} \subseteq U_{j+i-1}^{(1)} \text{ for all } j \in \mathbb{Z}.$$

According to our hypotheses we find some $r \in \mathbb{N}_0$ such that $U_{k-r}^{(1)} \subseteq U_k^{(2)} \subseteq U_{k+r}^{(1)}$ for all $k \in \mathbb{Z}$. By Definition and Remark 10.1 (C)(c) we therefore have

$$a^{2r+1}U_j^{(2)} \subseteq U_{j+(2r+1)i-1}^{(2)} \text{ for all } j \in \mathbb{Z}.$$

So, for all $j \in \mathbb{Z}$ we get in $U_{j+(2r+1)i}^{(2)}/U_{j+(2r+1)i-1}^{(2)} = \mathrm{Gr}_{U_\bullet^{(2)}}(U)_{j+(2r+1)i}$ the relation:

$$\overline{a}^{2r+1}\mathrm{Gr}_{U_\bullet}(U)_j = (a^{2r+1} + A_{(2r+1)i-1})(U_j^{(2)}/U_{j-1}^{(2)}) \subseteq a^{2r+1}U_j^{(2)}/U_{j+(2r+1)i-1}^{(2)} = 0.$$

This shows that $\overline{a}^{2r+1} \in \mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(2)}}(U)\big)$ and hence that indeed

$$\overline{a} \in \sqrt{\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(2)}}(U)\big)}.$$

□

So, provided $(A, A_\bullet)$ is a commutatively filtered K-algebra (see Definition 3.3), the characteristic variety of an $A_\bullet$-graded A-module $(U, U_\bullet)$ depends only on the equivalence class of the filtration $U_\bullet$. This allows us to define in an intrinsic way the notion of characteristic variety of a finitely generated (left-) module over the filtered ring A. We work this out in the following combined exercise and definition.

10.5. **Exercise and Definition.** (A) Let $(A, A_\bullet)$ be a filtered K-algebra and let U be a (left) module over A.

$$\text{Let } V \subseteq U \text{ be a } K\text{-subspace such that } U = AV.$$

Prove the following claims:

(a) $A_iV = 0$ for all $i < 0$.
(b) The family $A_\bullet V := \big(A_iV\big)_{i\in\mathbb{Z}}$ is an $A_\bullet$-filtration of U.

The above filtration $A_\bullet V$ is called the $A_\bullet$-filtration of U *induced* by the subspace V.

(B) Let the notations and hypotheses be as in part (A). Assume in addition that

$$s := \dim_K(V) < \infty.$$

Prove that

(a) U is finitely generated as an A-module;
(b) A_iV is a finitely generated (left-) module over A_0.
(c) The graded $\mathrm{Gr}_{A_\bullet}(A)$-module $\mathrm{Gr}_{A_\bullet V}(U)$ is generated by finitely many elements $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_s \in \mathrm{Gr}_{A_\bullet V}(U)_0$.

Keep in mind that we can always find a vector space $V \subseteq U$ of finite dimension with $AV = U$ if the A-module U is finitely generated.

(C) Let the notations and hypotheses be as above. Let $V^{(1)}, V^{(2)} \subseteq U$ be two K-subspaces such that

$$AV^{(1)} = AV^{(2)} = U \text{ and } \dim_K(V^{(1)}), \dim_K(V^{(2)}) < \infty.$$

Prove that

(a) The two induced $A_\bullet$-filtrations $A_\bullet V^{(1)}$ and $A_\bullet V^{(2)}$ are equivalent.
(b) If the filtration $A_\bullet$ is commutative, it holds

$$\mathbb{V}_{A_\bullet V^{(1)}}(U) = \mathbb{V}_{A_\bullet V^{(2)}}(U).$$

(D) Keep the above notations and hypotheses. Assume that the filtration $A_\bullet$ is commutative and that the (left) A-module U is finitely generated. By what we have learned by the previous considerations, we find a K-subspace $V \subseteq U$ of finite dimension such that $AV = U$, and the characteristic variety $\mathbb{V}_{A_\bullet V}(U)$ of U with respect to the induced filtration $A_\bullet V$ is independent of the choice of V. So, we may just write

$$\mathbb{V}_{A_\bullet}(U) := \mathbb{V}_{A_\bullet V}(U),$$

and we call $\mathbb{V}_{A_\bullet}(U)$ the *characteristic variety of U with respect to the (commutative !) filtration $A_\bullet$ of A*. This is the announced notion of *intrinsic characteristic variety.*

(E) Keep the above notations. Assume that the filtration $A_\bullet$ is of finite type (see Definition and Remark 3.4 (C)) and that the (left) A-module U is finitely generated. The $A_\bullet$ filtration $U_\bullet$ of U is said to be *of finite type* if

(a) There is some $j_0 \in \mathbb{Z}$ such that $U_j = 0$ for all $j \leq j_0$;
(b) There is an integer σ such that:
  (1) U_j is finitely generated as a (left) A_0-module for all $j \leq \sigma$ and
  (2) $U_i = \sum_{j \leq \sigma} A_j U_{i-j}$ for all $i > \sigma$.

In this situation σ is again called a *generating degree* of the $A_\bullet$-filtration $U_\bullet$ (compare Definition and Remark 3.4 (C)). Prove that in this situation, we have

$$A_{i-\sigma} U_\sigma \subseteq U_i = \sum_{j=j_0}^{\sigma} A_{i-j} U_j \subseteq A_{i-j_0} U_\sigma \text{ for all } i > \sigma.$$

As U_σ is a finitely generated A_0-module, we may chose a K-subspace $V \subseteq U$ such that

$$\dim_K(V) < \infty \text{ and } A_0 V = U_\sigma.$$

Prove that for this choice of V we have:

$$U = AV \text{ and the filtrations } U_\bullet \text{ and } A_\bullet V \text{ are equivalent.}$$

As a consequence it follows by Proposition 10.4 and the observations made in part (D), that

$$\mathbb{V}_{U_\bullet}(U) = \mathbb{V}_{A_\bullet}(U) \text{ for each } A_\bullet\text{-filtration } U_\bullet \text{ which is of finite type.}$$

11. D-Modules

11.1. **Convention.** (A) As in section 9, we fix a positive integer n, a field K of characteristic 0 and consider the standard Weyl algebra

$$\mathbb{W} := \mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n].$$

In addition, we consider the polynomial ring

$$\mathbb{P} := K[Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n]$$

in the indeterminates $Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n$ with coefficients in the field K.

(B) Let $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ be a weight. We consider the induced weighted filtration $\mathbb{W}_\bullet^{\underline{vw}}$ and also the corresponding associated graded ring.

$$\mathbb{G}^{\underline{vw}} = \bigoplus_{i \in \mathbb{N}_0} \mathbb{G}_i^{\underline{vw}} := \mathrm{Gr}_{\mathbb{W}_\bullet^{\underline{vw}}}\left(\mathbb{W}^{\underline{vw}}\right) = \bigoplus_{i \in \mathbb{N}_0} \mathrm{Gr}_{\mathbb{W}_\bullet^{\underline{vw}}}\left(\mathbb{W}^{\underline{vw}}\right)_i.$$

(see Definition and Remark 9.2 (A)).

(C) Moreover, we shall consider the polynomial ring

$$\mathbb{P} = \mathbb{P}^{\underline{vw}} = \bigoplus_{i \in \mathbb{N}_0} \mathbb{P}_i^{\underline{vw}}.$$

furnished with the grading induced by our given weight $(\underline{v}, \underline{w})$ (see Exercise and Definition 9.3 (B)), as well as the canonical isomorphism of graded rings (see Theorem 9.4):

$$\eta^{\underline{v}\underline{w}} : \mathbb{P} = \mathbb{P}^{\underline{v}\underline{w}} \xrightarrow{\cong} \mathbb{G}^{\underline{v}\underline{w}}.$$

11.2. **Definition and Remark.** (A) By a *D-module* we mean a finitely generated left module over the standard Weyl algebra $\mathbb{W}$.

(B) Let U be a D-module. If $U_\bullet$ is a $\mathbb{W}^{\underline{v}\underline{w}}_\bullet$-filtration of U, we may again introduce the corresponding *associated graded module* of U with respect to the filtration $U_\bullet$ (see Definition 10.3):

$$\mathrm{Gr}_{U_\bullet}(U) = \bigoplus_{i \in \mathbb{Z}} U_i/U_{i-1},$$

which is indeed a graded module over the associated graded ring $\mathbb{G}^{\underline{v}\underline{w}}$. But, in fact, we prefer to consider $\mathrm{Gr}_{U_\bullet}(U)$ as a graded $\mathbb{P}^{\underline{v}\underline{w}}$-module by means of the canonical isomorphism $\eta^{\underline{v}\underline{w}} : \mathbb{P} = \mathbb{P}^{\underline{v}\underline{w}} \xrightarrow{\cong} \mathbb{G}^{\underline{v}\underline{w}}$.

(C) Keep the notations and hypotheses of part (B). Then, we may again consider the *characteristic variety* of U with respect to the filtration $U_\bullet$, but under the previous view, that $\mathrm{Gr}_{U_\bullet}(U)$ is a graded module over the graded polynomial ring $\mathbb{P} = \mathbb{P}^{\underline{v}\underline{w}}$. So, we define this characteristic variety by

$$\mathbb{V}_{U_\bullet}(U) := \mathrm{Var}\big(\mathrm{Ann}_{\mathbb{P}^{\underline{v}\underline{w}}}(\mathrm{Gr}_{U_\bullet}(U))\big) = \mathrm{Var}\big((\eta^{\underline{v}\underline{w}})^{-1}\big[\mathrm{Ann}_{\mathbb{G}^{\underline{v}\underline{w}}}(\mathrm{Gr}_{U_\bullet}(U))\big]\big) \subseteq \mathrm{Spec}(\mathbb{P}).$$

Observe in particular, that the ideal

$$\mathrm{Ann}_{\mathbb{P}^{\underline{v}\underline{w}}}\big(\mathrm{Gr}_{U_\bullet}(U)\big) = (\eta^{\underline{v}\underline{w}})^{-1}\big[\mathrm{Ann}_{\mathbb{G}^{\underline{v}\underline{w}}}\big(\mathrm{Gr}_{U_\bullet}(U)\big)\big] \subseteq \mathbb{P}^{\underline{v}\underline{w}}$$

is graded.

(D) Finally, as U is finitely generated, we may again chose a finite dimensional K-subspace $V \subseteq U$ such that $\mathbb{W}V = U$, and then consider the induced filtration $\mathbb{W}^{\underline{v}\underline{w}}_\bullet V$ of U and the corresponding *intrinsic characteristic variety* (see Exercise and Definition 10.5 (D)) of U with respect to the weight $(\underline{v}, \underline{w})$, hence:

$$\mathbb{V}^{\underline{v}\underline{w}}(U) := \mathbb{V}_{\mathbb{W}^{\underline{v}\underline{w}}_\bullet}(U) = \mathbb{V}_{\mathbb{W}^{\underline{v}\underline{w}}_\bullet V}(U).$$

11.3. **Example.** (A) Keep the above notations and let

$$d := \sum_{(\underline{\nu},\underline{\mu}) \in \mathrm{supp}(d)} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \in \mathbb{W} \setminus \{0\} \text{ and } \delta := \deg^{\underline{v}\underline{w}}(d),$$

with $c^{(d)}_{\underline{\nu}\underline{\mu}} \in K \setminus \{0\}$ for all $(\underline{\nu}, \underline{\mu}) \in \mathrm{supp}(d)$. We also consider the so-called *leading differential* form of d with respect to the weight $(\underline{v}, \underline{w})$, which is given by

$$h^{\underline{v}\underline{w}} := \sum_{(\underline{\nu},\underline{\mu}) \in \mathrm{supp}(d) : \underline{v}\cdot\underline{\nu} + \underline{w}\cdot\underline{\mu} = \delta} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \in \mathbb{W} \setminus \{0\}.$$

Moreover, we introduce the polynomial

$$f^{\underline{v}\underline{w}} := \sum_{(\underline{\nu},\underline{\mu})\in \mathrm{supp}(d):\underline{v}\cdot\underline{\nu}+\underline{w}\cdot\underline{\mu}=\delta} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}} \in \mathbb{P}\setminus\{0\}.$$

Now, consider the cyclic left $\mathbb{W}$-module

$$U := \mathbb{W}/\mathbb{W}d, \text{ the element } \overline{1} := (1+\mathbb{W}_d)/\mathbb{W}_d \in U \text{ and the } K\text{-subspace } K\overline{1} \subseteq U.$$

Endow U with the $\mathbb{W}^{\underline{v}\underline{w}}_\bullet$-filtration (see Exercise and Definition 10.5 (A)):

$$U_\bullet := \mathbb{W}^{\underline{v}\underline{w}}_\bullet K\overline{1} = \big(U_i := (\mathbb{W}^{\underline{v}\underline{w}}_i + \mathbb{W}d)/\mathbb{W}d)_{i\in\mathbb{Z}}.$$

(B) Keep the above notations and hypotheses. Observe first, that for all $i \in \mathbb{Z}$ we may write

$$U_i/U_{i-1} = \mathbb{W}^{\underline{v}\underline{w}}_i/(\mathbb{W}^{\underline{v}\underline{w}}_{i-1} + (\mathbb{W}d \cap \mathbb{W}^{\underline{v}\underline{w}}_i)).$$

By the additivity of weighted degrees (see Corollary 9.5) we have

$$\mathbb{W}d \cap \mathbb{W}^{\underline{v}\underline{w}}_i = \mathbb{W}^{\underline{v}\underline{w}}_{i-\delta}d \text{ for all } i \in \mathbb{Z}.$$

So, we obtain

$$\mathrm{Gr}_{U_\bullet}(U)_i = U_i/U_{i-1} = \mathbb{W}^{\underline{v}\underline{w}}_i/\big(\mathbb{W}^{\underline{v}\underline{w}}_{i-1} + \mathbb{W}^{\underline{v}\underline{w}}_{i-\delta}d\big) \text{ for all } i \in \mathbb{N}_0.$$

Consequently, there is a surjective homomorphism of graded $\mathbb{G}^{\underline{v}\underline{w}}$-modules

$$\pi : \mathbb{G}^{\underline{v}\underline{w}} = \bigoplus_{i\in\mathbb{Z}} \mathbb{W}^{\underline{v}\underline{w}}_i/\mathbb{W}^{\underline{v}\underline{w}}_{i-1} \twoheadrightarrow \mathrm{Gr}_{U_\bullet}(U) = \bigoplus_{i\in\mathbb{Z}} \mathbb{W}^{\underline{v}\underline{w}}_i/\big(\mathbb{W}^{\underline{v}\underline{w}}_{i-1} + \mathbb{W}^{\underline{v}\underline{w}}_{i-\delta}d\big).$$

If we set

$$\overline{h}^{\underline{v}\underline{w}} := h^{\underline{v}\underline{w}} + \mathbb{W}^{\underline{v}\underline{w}}_{\delta-1} \in \mathbb{W}^{\underline{v}\underline{w}}_\delta/\mathbb{W}^{\underline{v}\underline{w}}_{\delta-1} = \mathbb{G}^{\underline{v}\underline{w}}_\delta$$

it follows that

$$\begin{aligned}\mathrm{Ann}_{\mathbb{G}^{\underline{v}\underline{w}}}\big(\mathrm{Gr}_{U_\bullet}(U)\big) = \mathrm{Ker}(\pi) &= \bigoplus_{i\in\mathbb{Z}} \big(\mathbb{W}^{\underline{v}\underline{w}}_{i-1} + \mathbb{W}^{\underline{v}\underline{w}}_{i-\delta}d\big)/\mathbb{W}^{\underline{v}\underline{w}}_{i-1} = \\ &= \bigoplus_{i\in\mathbb{Z}} \big(\mathbb{W}^{\underline{v}\underline{w}}_{i-1} + \mathbb{W}^{\underline{v}\underline{w}}_{i-\delta}h^{\underline{v}\underline{w}}\big)/\mathbb{W}^{\underline{v}\underline{w}}_{i-1} = \mathbb{G}^{\underline{v}\underline{w}}\overline{h}^{\underline{v}\underline{w}}.\end{aligned}$$

Consequently we get

$$\mathrm{Gr}_{U_\bullet}(U) \cong \mathbb{G}^{\underline{v}\underline{w}}/\mathbb{G}^{\underline{v}\underline{w}}\overline{h}^{\underline{v}\underline{w}}.$$

As $\eta^{\underline{v}\underline{w}}(f^{\underline{v}\underline{w}}) = \overline{h}^{\underline{v}\underline{w}}$ and if we consider $\mathrm{Gr}_{U_\bullet}(U)$ as a graded $\mathbb{P}^{\underline{v}\underline{w}}$-module by means of $\eta^{\underline{v}\underline{w}}$, we thus may write

$$\mathrm{Gr}_{U_\bullet}(U) \cong \mathbb{P}^{\underline{v}\underline{w}}/\mathbb{P}^{\underline{v}\underline{w}}f^{\underline{v}\underline{w}} \text{ and } \mathrm{Ann}_{\mathbb{P}}\big(\mathrm{Gr}_{U_\bullet}(U)\big) = \mathbb{P}f^{\underline{v}\underline{w}}.$$

In particular we obtain:

$$\mathbb{V}_{U_\bullet}(U) = \mathbb{V}^{\underline{v}\underline{w}}(U) = \mathbb{V}^{\underline{v}\underline{w}}\big(\mathbb{W}/\mathbb{W}d\big) = \mathrm{Var}(\mathbb{P}f^{\underline{v}\underline{w}}) \subseteq \mathrm{Spec}(\mathbb{P}).$$

11.4. **Exercise.** (A) Let $n = 1$, $K = \mathbb{R}$ and let $d := X_1^4 + \partial_1^2 - X_1^2\partial_1^2$. Determine the two characteristic varieties

$$\mathbb{V}^{\underline{v}\underline{w}}(\mathbb{W}/\mathbb{W}d) \text{ for } (\underline{v}, \underline{w}) = (1,1) \text{ and } (\underline{v}, \underline{w}) = (0,1).$$

(B) To make more apparent what you have done in part (A), determine and sketch the *real traces*

$$\mathbb{V}^{\underline{v}\underline{w}}_{\mathbb{R}}(\mathbb{W}/\mathbb{W}d) := \{(y,z) \in \mathbb{R}^2 \mid (Y_1 - y, Z_1 - z)K[Y_1, Z_1] \in \mathbb{V}^{\underline{v}\underline{w}}(\mathbb{W}/\mathbb{W}d)\}$$

for $(\underline{v}, \underline{w}) = (1,1)$ and $(\underline{v}, \underline{w}) = (0,1)$. Comment your findings.

Now, we shall establish the fact that D-modules are finitely presentable. To do so we first will show that standard Weyl algebras are left Noetherian (see Conventions, Reminders and Notations 1.1 (G) and (H)). We begin with the following preparation.

11.5. **Definition and Remark.** (A) Let $I \subseteq \mathbb{W}$ be a left ideal. We consider the following K-subspace of $\mathbb{G}^{\underline{v}\underline{w}}$:

$$\mathbb{G}^{\underline{v}\underline{w}}(I) := \bigoplus_{i \in \mathbb{N}_0} \big(I \cap \mathbb{W}^{\underline{v}\underline{w}}_i + \mathbb{W}^{\underline{v}\underline{w}}_{i-1}\big)/\mathbb{W}^{\underline{v}\underline{w}}_{i-1} \subseteq \bigoplus_{i \in \mathbb{N}_0} \mathbb{W}^{\underline{v}\underline{w}}_i/\mathbb{W}^{\underline{v}\underline{w}}_{i-1} = \mathbb{G}^{\underline{v}\underline{w}}.$$

It is immediate to see, that $\mathbb{G}^{\underline{v}\underline{w}}(I) \subseteq \mathbb{G}^{\underline{v}\underline{w}}$ is graded ideal. We call this ideal the *graded ideal induced by I* in $\mathbb{G}^{\underline{v}\underline{w}}$.

(B) Let the notations and hypotheses as in part (A). It is straight forward to see, that the family

$$I^{\underline{v}\underline{w}}_\bullet := \big(I \cap \mathbb{W}^{\underline{v}\underline{w}}_i\big)_{i \in \mathbb{Z}}$$

is a filtration of the (left) $\mathbb{W}$-module I, which we call the *filtration induced by* $\mathbb{W}^{\underline{v}\underline{w}}_\bullet$. Observe, that for all $i \in \mathbb{Z}$ we have a canonical isomorphism of K-vector spaces

$$\mathbb{G}^{\underline{v}\underline{w}}(I)_i := \big(I \cap \mathbb{W}^{\underline{v}\underline{w}}_i + \mathbb{W}^{\underline{v}\underline{w}}_{i-1}\big)/\mathbb{W}^{\underline{v}\underline{w}}_{i-1} \cong I \cap \mathbb{W}^{\underline{v}\underline{w}}_i / I \cap \mathbb{W}^{\underline{v}\underline{w}}_{i-1} = I^{\underline{v}\underline{w}}_i / I^{\underline{v}\underline{w}}_{i-1} = \mathrm{Gr}_{I^{\underline{v}\underline{w}}_\bullet}(I)_i.$$

It is easy to see, that these isomorphisms of K-vector spaces actually give rise to a canonical isomorphism of graded $\mathbb{G}^{\underline{v}\underline{w}}$-modules

$$\mathbb{G}^{\underline{v}\underline{w}}(I) := \bigoplus_{i \in \mathbb{Z}} \big((I \cap \mathbb{W}^{\underline{v}\underline{w}}_i) + \mathbb{W}^{\underline{v}\underline{w}}_{i-1}\big)/\mathbb{W}^{\underline{v}\underline{w}}_{i-1} \cong \bigoplus_{i \in \mathbb{Z}} I^{\underline{v}\underline{w}}_i / I^{\underline{v}\underline{w}}_{i-1} = \mathrm{Gr}_{I^{\underline{v}\underline{w}}_\bullet}(I).$$

So, by means of this canonical isomorphism we may identify

$$\mathbb{G}^{\underline{v}\underline{w}}(I) = \mathrm{Gr}_{I^{\underline{v}\underline{w}}_\bullet}(I).$$

11.6. **Lemma.** *Let $I, J \subseteq \mathbb{W}$ be two left ideals with $I \subseteq J$. Then we can say:*

(a) *There is an inclusion of graded ideals $\mathbb{G}^{\underline{v}\underline{w}}(I) \subseteq \mathbb{G}^{\underline{v}\underline{w}}(J)$ in the graded ring $\mathbb{G}^{\underline{v}\underline{w}}$.*
(b) *If $\mathbb{G}^{\underline{v}\underline{w}}(I) = \mathbb{G}^{\underline{v}\underline{w}}(J)$, then $I = J$.*

Proof. (a): This is immediate by Definition and Remark 11.5 (A).

(b): Assume that $I \subsetneq J$. Then, there is a least integer $i \in \mathbb{N}_0$ such that

$$I^{\underline{v}\underline{w}}_i = I \cap \mathbb{W}^{\underline{v}\underline{w}}_i \subsetneq J^{\underline{v}\underline{w}}_i = J \cap \mathbb{W}^{\underline{v}\underline{w}}_i.$$

As $I^{\underline{vw}}_{i-1} = J^{\underline{vw}}_{i-1}$ it follows that

$$\mathbb{G}^{\underline{vw}}(I)_i \cong I^{\underline{vw}}_i / I^{\underline{vw}}_{i-1} \text{ is not isomorphic to } I^{\underline{vw}}_i / I^{\underline{vw}}_{i-1} \cong \mathbb{G}^{\underline{vw}}(J)_i,$$

so that indeed

$$\mathbb{G}^{\underline{vw}}(I) \neq \mathbb{G}^{\underline{vw}}(J).$$

□

11.7. **Theorem.** ***(Noetherianness of Weyl Algebras)*** *The Weyl algebra* $\mathbb{W}$ *is left Noetherian.*

Proof. Otherwise $\mathbb{W}$ would contain an infinite strictly ascending chain of left ideals $I^{(1)} \subsetneq I^{(2)} \subsetneq I^{(3)} \subsetneq \cdots$. But then, by Lemma 11.6 we would have an infinite strictly ascending chain $\mathbb{G}^{\underline{vw}}(I^{(1)}) \subsetneq \mathbb{G}^{\underline{vw}}(I^{(2)}) \subsetneq \mathbb{G}^{\underline{vw}}(I^{(3)}) \subsetneq \cdots$ of ideals in the Noetherian ring $\mathbb{G}^{\underline{vw}} \cong \mathbb{P}^{\underline{vw}} = \mathbb{P}$, a contradiction. □

11.8. **Corollary.** ***(Finite Presentability of D-Modules)*** *Each* D*-module* U *admits a finite presentation*

$$\mathbb{W}^s \longrightarrow \mathbb{W}^r \longrightarrow U \longrightarrow 0.$$

Proof. This follows immediately by Theorem 11.7 and the observations made in Conventions, Reminders and Notations 1.1 (H). □

11.9. **Example.** (A) Consider the polynomial ring $U := K[X_1, X_2, \ldots, X_n]$. As

$$\mathbb{W} \subseteq \operatorname{End}_K\big(K[X_1, X_2, \ldots, X_n]\big) = \operatorname{End}_K(U),$$

this polynomial ring can be viewed in a canonical way as a left module over $\mathbb{W}$, the scalar being multiplication given by

$$d \cdot f := d(f) \text{ for all } d \in \mathbb{W} \text{ and all } f \in U.$$

As $f \cdot 1 = f$ for all $f \in U$ it follows that

$$U = \mathbb{W}1_U.$$

So, the $\mathbb{W}$-module $U := K[X_1, X_2, \ldots, X_n]$ is generated by a single element, and hence in particular a D-module.

(B) Keep the previous notations and hypotheses. Observe that

$$\sum_{i=1}^{n} \mathbb{W}\partial_i = \bigoplus_{\underline{\nu},\underline{\mu} \in \mathbb{N}_0^n : \underline{\mu} \neq \underline{0}} K \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}$$

and hence

$$\mathbb{W} = K[X_1, X_2, \ldots, X_n] \oplus \sum_{i=1}^{n} \mathbb{W}\partial_i = U \oplus \sum_{i=1}^{n} \mathbb{W}\partial_i.$$

We thus have an exact sequence of K-vector spaces

$$0 \longrightarrow \sum_{i=1}^{n} \mathbb{W}\partial_i \longrightarrow \mathbb{W} \xrightarrow{\pi} U \longrightarrow 0,$$

in which $\mathbb{W} \xrightarrow{\pi} U$ is the *canonical projection* map given by

$$\pi\big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big) = \begin{cases} \underline{X}^{\underline{\nu}}, & \text{if } \underline{\mu} = \underline{0}, \\ 0, & \text{if } \underline{\mu} \neq \underline{0} \end{cases}.$$

Our aim is to show:

$$\mathbb{W} \xrightarrow{\pi} U \text{ is a homomorphism of left } \mathbb{W}\text{-modules.}$$

To do so, it suffices to show that for all $\underline{\nu}, \underline{\mu}, \underline{\nu}', \underline{\mu}' \in \mathbb{N}_0^n$ it holds

$$\pi(dd') = d\pi(d'), \text{ where } d := \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \text{ and } d' := \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}.$$

If $\underline{\mu} = \underline{\mu}' = \underline{0}$, we have

$$\pi(dd') = \pi\big(\underline{X}^{\underline{\nu}}\underline{X}^{\underline{\nu}'}\big) = \pi\big(\underline{X}^{\underline{\nu}+\underline{\nu}'}\big) = \underline{X}^{\underline{\nu}+\underline{\nu}'} = \underline{X}^{\underline{\nu}}\underline{X}^{\underline{\nu}'} = \underline{X}^{\underline{\nu}}\pi\big(\underline{X}^{\underline{\nu}'}\big) = d\pi(d').$$

If $\underline{\mu} = 0$ and $\underline{\mu}' \neq 0$ we have

$$\pi(dd') = \pi\big(\underline{X}^{\underline{\nu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) = \pi\big(\underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) = 0 = \underline{X}^{\underline{\nu}}\pi\big(\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) = d\pi(d').$$

So, let $\underline{\mu} \neq \underline{0}$. By the Product Formula of Proposition 6.2 we have

$$dd' = \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} = \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'} + s,$$

with

$$s := \sum_{\underline{k} \in \mathbb{N}_0^n : \underline{0} < \underline{k} \leq \underline{\mu}, \underline{\nu}'} \lambda_{\underline{k}} \underline{X}^{\underline{\nu}+\underline{\nu}'-\underline{k}}\underline{\partial}^{\underline{\mu}+\underline{\mu}'-\underline{k}}$$

and

$$\lambda_{\underline{k}} = \Big(\prod_{i=1}^{n} \binom{\mu_i}{k_i}\Big)\Big(\prod_{i=1}^{n}\prod_{p=0}^{k_i-1}(\nu_i' - p)\Big).$$

Assume first, that $\underline{\mu}' \neq 0$. Then we have

$$\pi\big(\underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'}\big) = 0 \text{ and } \pi\big(\underline{X}^{\underline{\nu}+\underline{\nu}'-\underline{k}}\underline{\partial}^{\underline{\nu}+\underline{\nu}'-\underline{k}}\big) = 0 \text{ for all } \underline{k} \in \mathbb{N}_0^n \text{ with } \underline{0} < \underline{k} \leq \underline{\mu}, \underline{\nu}'.$$

It thus follows, that

$$\pi(dd') = 0 = d0 = d\pi\big(\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) = d\pi(d').$$

So, finally let $\underline{\mu}' = 0$. Then $dd' = \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}} + s$, and

$$s = \begin{cases} \prod_{i=1}^{n}\prod_{p=0}^{\mu_i-1}(\nu_i' - p)\underline{X}^{\underline{\nu}+\underline{\nu}'-\underline{\mu}}, & \text{if } \underline{\mu} \leq \underline{\nu}'; \\ 0, & \text{otherwise.} \end{cases}$$

So, by what we have learned in Exercise 6.6 (B), we have

$$s = \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big(\underline{X}^{\underline{\nu}'}\big).$$

As s is a K-multiple of a monomial in the X_i's we have $\pi(s) = s$. It thus follows

$$\pi(dd') = \pi\big(\underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) + \pi(s) = s = \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big(\underline{X}^{\underline{\nu}'}\big) = \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'} = d\pi(d').$$

This proves, that π is indeed a homomorphism of left $\mathbb{W}$-modules.

(C) Keep the previous notations and hypotheses. Then, according the above observations, we have an exact sequence of left $\mathbb{W}$-modules

$$0 \longrightarrow \mathbb{W}^n \xrightarrow{h} \mathbb{W} \xrightarrow{\pi} U \longrightarrow 0,$$

in which h is given by

$$(d_1, d_2, \ldots, d_n) \mapsto h(d_1, d_2, \ldots, d_n) = \sum_{i=1}^{n} d_i \partial_i.$$

This sequence clearly constitutes a presentation of the left $\mathbb{W}$-module U (see Conventions, Reminders and Notations 1.1 (H)) and the corresponding presentation matrix for U is the row

$$\partial := \begin{pmatrix} \partial_1 \\ \partial_2 \\ \vdots \\ \partial_n \end{pmatrix} \in \mathbb{W}^{n \times 1}.$$

11.10. **Exercise.** (A) We consider the polynomial ring $U = K[X_1, X_2, \ldots, X_n]$ canonically as a D-module, as done in Example 11.9. Fix a weight $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$. Consider the K-subspace $K \subset U$, observe that $\mathbb{W}K = U$ and endow U with the induced filtration

$$U_\bullet := \mathbb{W}_\bullet^{\underline{v}\underline{w}} K.$$

Show, that there is an isomorphism of graded $\mathbb{P}$-modules

$$\mathrm{Gr}_{U_\bullet}(U) = \mathrm{Gr}_{\mathbb{W}^{\underline{v}\underline{w}}_\bullet K}(U) \cong U^{\underline{v}},$$

where

$$U^{\underline{v}} := \bigoplus_{i \in \mathbb{N}_0} U_i^{\underline{v}} \text{ with } U_i^{\underline{v}} := \sum_{\underline{v} \cdot \underline{\nu} = i} K \underline{X}^{\underline{\nu}} \text{ for all } i \in \mathbb{N}_0$$

is the polynomial ring U endowed with the grading associated to the weight $\underline{v} \in \mathbb{N}_0^n$. Determine the characteristic variety

$$\mathbb{V}^{\underline{v}\underline{w}}(U) \subseteq \mathrm{Spec}(\mathbb{P}).$$

(B) Keep the notations and hypotheses of part (A). Show, the left $\mathbb{W}$-module U is simple: If $V \subsetneq U$ is a proper left $\mathbb{W}$-submodule, then $V = 0$. (Hint: Let $f \in U \setminus \{0\}$ be of degree r and assume that $\underline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathrm{supp}(f)$ with $\sum_{i=1}^n \nu_i = r$ and show that $\underline{\partial}^{\underline{\nu}} \in K \setminus \{0\}$. Conclude that $\mathbb{W}f = U$.)

11.11. **Remark and Definition.** (A) We furnish the polynomial ring $K[X_1, X_2, \ldots, X_n]$ with its *canonical structure of D-module* (see Example 11.9). We now consider a ring $\mathcal{A}$ with the following properties

(1) $\mathcal{A}$ is commutative;
(2) $\mathcal{A}$ is a left $\mathbb{W}$-module;
(3) $K[X_1, X_2, \ldots, X_n] \subseteq \mathcal{A}$ is a left submodule.

In this situation, we call $\mathcal{A}$ a ring of *good functions* in $X_1, X_2, \ldots, X_n$ over K.
The idea covered by this concept is that for all $d \in \mathbb{W}$ and all $f \in \dashv$ the product $df \in \mathcal{A}$

should be viewed as the result of the application of the differential operator d to the function f. Therefore, one often writes

$$d(f) := df \text{ for all } d \in \mathbb{W} \text{ and all } f \in \mathcal{A}.$$

(B) Let the notations and hypotheses be as in part (A). By a *system of polynomial differential equations* in $\mathcal{A}$ we mean a system of equations

$$\begin{gathered} d_{11}(f_1) + d_{12}(f_2) + \ldots + d_{1r}(f_r) = 0 \\ d_{21}(f_1) + d_{22}(f_2) + \ldots + d_{2r}(f_r) = 0 \\ \vdots \\ d_{s1}(f_1) + d_{s2}(f_2) + \ldots + d_{sr}(f_r) = 0 \end{gathered}$$

with $r, s \in \mathbb{N}$ such that

$$d_{ij} \in \mathbb{W} \text{ and } f_j \in \mathcal{A} \text{ for all } i, j \in \mathbb{N} \text{ with } i \leq s \text{ and } j \leq r.$$

The above system of differential equations can be understood as a linear system of equations over the ring $\mathcal{A}$. We namely may consider the matrix

$$\mathcal{D} := \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1r} \\ d_{21} & d_{22} & \ldots & d_{2r} \\ \vdots & \vdots & & \vdots \\ d_{s1} & d_{s2} & \ldots & d_{sr} \end{pmatrix} \in \mathbb{W}^{s \times r}.$$

Then, the above system may be written in matrix form as

$$\mathcal{D} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We call $\mathcal{D}$ the *matrix of differential operators* associated to our system of linear differential equations. So, systems of differential equations correspond to matrices with entries in a standard Weyl algebra.

(C) Keep the previous notations and hypotheses, then the matrix of differential operators $\mathcal{D} \in \mathbb{W}^{s \times r}$ gives rise to an exact sequence of left $\mathbb{W}$-modules

$$0 \longrightarrow \mathbb{W}^s \xrightarrow{h_{\mathcal{D}}} \mathbb{W}^r \xrightarrow{\pi_{\mathcal{D}}} U_{\mathcal{D}} \longrightarrow 0.$$

In particular $U_{\mathcal{D}}$ is a D-module and the previous sequence is a finite presentation of $U_{\mathcal{D}}$. We call this presentation the *presentation induced by the matrix* $\mathcal{D}$ and we call $U_{\mathcal{D}}$ the D-module *defined* by the matrix $\mathcal{D}$ – or the D-module associated with our system of differential equations. So, each system of differential equations defines a D-module. Obviously, one is particularly interested in the *solution space* of our system of differential equations, hence in the K-vector space

$$\mathbb{S}_{\mathcal{D}}(\mathcal{A}) := \{(f_1, f_2, \ldots, f_r) \in \mathcal{A}^r \mid \mathcal{D}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\}.$$

Observe, that $\mathbb{S}_{\mathcal{D}}(\mathcal{A})$ is a K-subspace of $\mathcal{A}^r$.

11.12. **Proposition.** *Let $r, s \in \mathbb{N}$, let*

$$\mathcal{D} = \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1r} \\ d_{21} & d_{22} & \ldots & d_{2r} \\ \vdots & \vdots & & \vdots \\ d_{s1} & d_{s2} & \ldots & d_{sr} \end{pmatrix} \in \mathbb{W}^{s \times r}$$

be a matrix of differential operators, consider the induced presentation

$$0 \longrightarrow \mathbb{W}^s \stackrel{h=h_{\mathcal{D}}}{\longrightarrow} \mathbb{W}^r \stackrel{\pi=\pi_{\mathcal{D}}}{\longrightarrow} U_{\mathcal{D}} \longrightarrow 0$$

and the corresponding solution space $\mathbb{S}_{\mathcal{D}}(\mathcal{A})$.
For all $i = 1, 2, \ldots, r$ let $\mathrm{e}_i := (\delta_{i,j})_{j=1}^r \in \mathbb{W}^r$ be the i-th canonical basis element. Then, there is an isomorphism of K-vector spaces

$$\varepsilon_{\mathcal{D}} : \mathrm{Hom}_{\mathbb{W}}\big(U_{\mathcal{D}}, \mathcal{A}\big) \stackrel{\cong}{\longrightarrow} \mathbb{S}_{\mathcal{D}}(\mathcal{A}),$$

given by

$$m \mapsto \varepsilon_{\mathcal{D}}(m) := \big(m(\pi(\mathrm{e}_1)), m(\pi(\mathrm{e}_2)), \ldots, m(\pi(\mathrm{e}_r))\big) \text{ for all } m \in \mathrm{Hom}_{\mathbb{W}}\big(U_{\mathcal{D}}, \mathcal{A}\big).$$

Proof. Observe, that there is indeed a K-linear map

$$\varepsilon := \varepsilon_{\mathcal{D}} : \mathrm{Hom}_{\mathbb{W}}\big(U_{\mathcal{D}}, \mathcal{A}\big) \longrightarrow \mathcal{A}^r$$

given by

$$m \mapsto \varepsilon_{\mathcal{D}}(m) := \big(m(\pi(\mathrm{e}_1)), m(\pi(\mathrm{e}_2)), \ldots, m(\pi(\mathrm{e}_r))\big) \text{ for all } m \in \mathrm{Hom}_{\mathbb{W}}\big(U_{\mathcal{D}}, \mathcal{A}\big).$$

If $\varepsilon(m) = 0$, then $m(\pi(\mathrm{e}_i)) = 0$ for all $i = 1, 2, \ldots, r$. As π is surjective, the elements $\pi(\mathrm{e}_i)$ $(i = 1, 2, \ldots, r)$ generate the left $\mathbb{W}$-module $U = U_{\mathcal{D}}$. So, it follows that $m = 0$ and this proves, that the map ε is injective.
It remains to show that

$$\varepsilon\big(\mathrm{Hom}_{\mathbb{W}}\big(U_{\mathcal{D}}, \mathcal{A}\big)\big) = \mathbb{S}_{\mathcal{D}}(\mathcal{A}).$$

To do so, let

$$\mathrm{b}_j := (\delta_{j,k})_{k=1}^s \in \mathbb{W}^s \quad (j = 1, 2, \ldots, s)$$

be the canonical basis elements of $\mathbb{W}^s$.
First, let $m \in \mathrm{Hom}_{\mathbb{W}}\big(U_{\mathcal{D}}, \mathcal{A}\big)$. We aim to show, that $\varepsilon(m) \in \mathbb{S}_{\mathcal{D}}(\mathcal{A})$. We have to show, that the column

$$\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_s \end{pmatrix} := \mathcal{D}\begin{pmatrix} m(\mathrm{e}_1) \\ m(\mathrm{e}_2) \\ \vdots \\ m(\mathrm{e}_r) \end{pmatrix}$$

vanishes. For each $i = 1, 2, \dots, s$ we can write $\sum_{j=1}^{r} d_{ij}\mathrm{e}_j = \mathrm{b}_i\mathcal{D} = h(\mathrm{b}_i)$, and hence get indeed

$$g_i = \sum_{j=1}^{r} d_{ij} m(\pi(\mathrm{e}_j)) = m\big(\sum_{j=1}^{r} d_{ij}\pi(\mathrm{e}_j)\big) = m\big(\pi(\sum_{j=1}^{r} d_{ij}\mathrm{e}_j)\big) = m\big(\pi(h(\mathrm{b}_i))\big) = m(0) = 0.$$

Conversely, let $(f_1, f_2, \dots, f_r) \in \mathbb{S}_{\mathcal{D}}(\mathcal{A})$, so that $\sum_{j=1}^{r} d_{ij} f_j = 0$. We aim to show that $(f_1, f_2, \dots, f_r) \in \varepsilon\big(\mathrm{Hom}_{\mathbb{W}}(U, \mathcal{A})\big)$.
To this end, we consider the homomorphism of left $\mathbb{W}$-modules

$$k : \mathbb{W}^r \longrightarrow \mathcal{A}, \text{ given by } (u_1, u_2, \dots, u_r) \mapsto \sum_{j+1}^{r} u_j f_j.$$

Observe that

$$k(h(\mathrm{b}_i)) = k(\mathrm{b}_i\mathcal{D}) = k(d_{i1}, d_{i2}, \dots, d_{ir}) = \sum_{j=1}^{r} d_{ij} f_j = 0 \text{ for all } i = 1, 2, \dots, s.$$

It follows that $k \circ h = 0$. Therefore k induces a homomorphism of left $\mathbb{W}$-modules

$$m : U \longrightarrow \mathcal{A}, \text{ such that } m \circ \pi = k.$$

It follows that $m(\pi(\mathrm{e}_j)) = k(\mathrm{e}_j) = f_j$ for all $j = 1, 2, \dots, r$. But this means that $(f_1, f_2, \dots, f_r) = \varepsilon(m) \in \varepsilon\big(\mathrm{Hom}_{\mathbb{W}}(U, \mathcal{A})\big)$. □

11.13. **Exercise.** (A) Let $n = 1$, $K = \mathbb{R}$ and let $\mathcal{A} := \mathcal{C}^\infty(\mathbb{R})$ be set of smooth functions on $\mathbb{R}$. Fix $d \in \mathbb{W} = \mathbb{W}(\mathbb{R}, 1) = \mathbb{R}[X, \partial]$ and consider the matrix $\mathcal{D} = (d) \in \mathbb{W}^{1\times 1}$. Determine

$$U_{\mathcal{D}}, \quad \mathbb{S}_{\mathcal{D}}(\mathcal{A}) \text{ and } \mathbb{V}^{\underline{v},\underline{w}}(U_{\mathcal{D}})$$

for all weights $(\underline{v}, \underline{w}) = (v, w) \in \mathbb{N}_0 \times \mathbb{N}_0 \setminus \{(0, 0)\}$ and for

$$d = \partial, \quad d = \partial^2 - 1, \quad d = \partial - x^2 \text{ and } d = \partial^2 + c\partial - b \text{ with } c, b \in \mathbb{R} \setminus \{0\}.$$

(B) Let $n, m \in \mathbb{N}$, $\mathcal{A} := K[X_1, X_2, \dots, X_n]$ and consider the matrix

$$\mathcal{D} := \begin{pmatrix} \partial_1^m \\ \partial_2^m \\ \vdots \\ \partial_n^m \end{pmatrix} \in \mathbb{W}^{n\times 1}.$$

Determine

$$U_{\mathcal{D}}, \quad \mathbb{S}_{\mathcal{D}}(\mathcal{A}) \text{ and } \mathbb{V}^{\underline{11}}(U_{\mathcal{D}}).$$

12. Gröbner Bases

In this section, we introduce and treat Gröbner bases of left ideals in standard Weyl algebras with respect to so-called admissible orderings of the set of elementary differential operators. What we get is a theory very similar to the theory of Gröbner bases of ideals in polynomial rings. A theory many readers may be familiar with already. Indeed a great deal of what we shall present in the sequel could also be deduced from the theory of Gröbner in polynomial rings. Nevertheless, we prefer to introduce the subject in a self

contained way so that readers who are not familiar with Gröbner in polynomial rings can follow our approach without further prerequisites. As for Gröbner bases in (commutative) polynomial rings and their applications, there are indeed many introductory and advanced textbooks and monograph. So, we mention only a sample of possible references for this subject, namely [1], [6], [19], [25], [26], [30], [36] and [42].
In general, Gröbner bases are intimately related to Division Theorems, which generalize Euclid's Division Theorem for univariate polynomial rings over a field. Gröbner bases and Division Theorems for rngs of linear differential operators were introduced by Briançon and Maisonobe [14] in the univariate case and by Castro-Jiménez [21] in the multivariate case. Two more recent basic references in the field of are the textbook of Bueso,Gómez-Torricellas and Verschoren [20] and the PhD thesis [31] of Levandovskyy.
The main goal of the present section is to prove that left ideals in Weyl algebras admit so-called universal Gröbner bases. This existence result can actually be proved in the more general setting of admissible algebras. Readers, who are interested in this, should consult for example Boldini's thesis [10] or else [38], [41] or [43].

12.1. **Convention.** (A) As previously, we fix a positive integer n, a field K of characteristic 0 and consider the standard Weyl algebra

$$\mathbb{W} := \mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n].$$

Moreover, we consider the polynomial ring

$$\mathbb{P} := K[Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n]$$

in the indeterminates $Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n$ with coefficients in the field K.

(B) In addition, we fix the isomorphism of K-vector spaces

$$\Phi : \mathbb{W} \xrightarrow{\cong} \mathbb{P} \text{ given by } \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \mapsto \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \text{ for all all } \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n.$$

Moreover we respectively consider the set $\mathbb{E}$ of all elementary differential operators in $\mathbb{W}$ and the set $\mathbb{M}$ of all monomials in $\mathbb{P}$, thus:

$$\mathbb{E} := \{\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \mid \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n\} \text{ and } \mathbb{M} := \{\underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \mid \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n\} = \Phi(\mathbb{E}).$$

In a first step we now introduce some basic notions of our subject, namely: admissible orderings (of the set $\mathbb{E}$ of elementary differential operators, leading (elementary) differential operators and (in the polynomial ring $\mathbb{P}$) leading monomials and leading terms. Mainly for those readers who have not met these concepts in the framework of polynomial rings, we shall add below a number of examples and exercises on these new notions.

12.2. **Definition, Reminder and Exercise.** (A) *(Total Orderings)* Let S be any set. A *total ordering* of S is a binary relation $\leq \subseteq S \times S$ such that for all $a, b, c \in S$ the following requirements are satisfied:

(a) *(Reflexivity)* $a \leq a$.
(b) *(Antisymmetry)* If $a \leq b$ and $b \leq a$, then $a = b$.
(c) *(Transitivity)* If $a \leq b$ and $b \leq c$, then $a \leq c$.
(b) *(Totality)* Either $a \leq b$ or $b \leq a$.

We write $\mathrm{TO}(S)$ for the set of total orderings on S.
If $\leq \in \mathrm{TO}(S)$ and $a, b \in S$, we write

$$a < b \text{ if } a \leq b \text{ and } a \neq b, \quad b \geq a \text{ if } a \leq b, \quad b > a \text{ if } a < b.$$

(B) *(Well Orderings)* Keep the above notations and hypotheses. A total ordering $\leq \in \mathrm{TO}(S)$ is said to be a *well ordering* of S, if it satisfies the following additional requirement:

(e) *(Existence of Least Elements)* For each non-empty subset $T \subseteq S$ there is an element $t \in T$ such that $t \leq t'$ for all $t' \in T$.

In the situation mentioned in statement (e), the element $t \in T$ – if it exists at all – is uniquely determined by T and called the *least element* or the *minimum* of T with respect to $\leq$ and denoted by $\min_{\leq}(T)$, thus

$$t = \min_{\leq}(T) \text{ if } t \in T \text{ and } t \leq t' \text{ for all } t' \in T.$$

We write $\mathrm{WO}(S)$ for the set of all well orderings of S.

(C) *(Admissible Orderings)* A total ordering $\leq \in \mathrm{TO}(\mathbb{E})$ of the set of all elementary differential operators is called an *admissible ordering* of $\mathbb{E}$ if it satisfies the following requirements:

(a) *(Foundedness)* $1 \leq \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}$ for all $\underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n$

(b) *(Compatibility)* For all $\underline{\lambda}, \underline{\lambda}', \underline{\kappa}, \underline{\kappa}', \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n$ we have the implication:

$$\text{If } \underline{X}^{\underline{\lambda}} \underline{\partial}^{\underline{\kappa}} \leq \underline{X}^{\underline{\lambda}'} \underline{\partial}^{\underline{\kappa}'}, \text{ then } \underline{X}^{\underline{\lambda}+\underline{\nu}} \underline{\partial}^{\underline{\kappa}+\underline{\mu}} \leq \underline{X}^{\underline{\lambda}'+\underline{\nu}} \underline{\partial}^{\underline{\kappa}'+\underline{\mu}}.$$

We write $\mathrm{AO}(\mathbb{E})$ for the set of all admissible orderings of $\mathbb{E}$.
Prove the following facts:

(c) If $\underline{\nu}, \underline{\nu}', \underline{\mu}, \underline{\mu}', \underline{\lambda}, \underline{\lambda}', \underline{\kappa}, \underline{\kappa}', \in \mathbb{N}_0^n$ with $\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \leq \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}$ and $\underline{X}^{\underline{\lambda}} \underline{\partial}^{\underline{\kappa}} < \underline{X}^{\underline{\lambda}'} \underline{\partial}^{\underline{\kappa}'}$, then

$$\underline{X}^{\underline{\lambda}+\underline{\nu}} \underline{\partial}^{\underline{\kappa}+\underline{\mu}} < \underline{X}^{\underline{\lambda}'+\underline{\nu}'} \underline{\partial}^{\underline{\kappa}'+\underline{\mu}'}.$$

(d) $\mathrm{AO}(\mathbb{E}) \subseteq \mathrm{WO}(\mathbb{E})$.

(D) *(Leading Elementary Differential Operators and Related Concepts)* From now on, for all $d \in \mathbb{W}$, we use the notation

$$\mathrm{Supp}(d) := \{\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \mid (\underline{\nu}, \underline{\mu}) \in \mathrm{supp}(d)\}.$$

Keep the above notations and hypotheses. If $\leq \in \mathrm{AO}(\mathbb{E})$ and $d \in \mathbb{W} \setminus \{0\}$, we define the *leading elementary differential operator* of d with respect to $\leq$ by:

$$\mathrm{LE}_{\leq}(d) := \max_{\leq} \mathrm{Supp}(d),$$

so that

$$\mathrm{LE}_{\leq}(d) \in \mathrm{Supp}(d) \text{ and } e \leq \mathrm{LE}_{\leq}(d) \text{ for all } e \in \mathrm{Supp}(d).$$

Moreover, we define the *leading coefficient* $\mathrm{LC}_{\leq}(d)$ of d with respect to $\leq$ as the coefficient of d with respect to $\mathrm{LE}_{\leq}(d)$, and the *leading differential operator* $\mathrm{LD}_{\leq}(d)$ of d with respect to $\leq$ as the product of the leading elementary differential operator with the leading coefficient, so that:

(a) $\mathrm{LC}_{\leq}(d) \in K \setminus \{0\}$ with $\mathrm{LE}_{\leq}\big(d - \mathrm{LC}_{\leq}(d)\mathrm{LE}_{\leq}(d)\big) < \mathrm{LE}_{\leq}(d)$.

(b) $\mathrm{LD}_{\leq}(d) = \mathrm{LC}_{\leq}(d)\mathrm{LE}_{\leq}(d)$.

(c) $\mathrm{LE}_{\leq}\big(d - \mathrm{LD}_{\leq}(d)\big) < \mathrm{LE}_{\leq}(d)$.

Finally, we define the *leading monomial* and the *leading term* of d with respect to $\leq$ respectively by

$$\mathrm{LM}_{\leq}(d) := \Phi\big(\mathrm{LE}_{\leq}(d)\big) \text{ and } \mathrm{LT}_{\leq}(d) := \Phi\big(\mathrm{LD}_{\leq}(d)\big) = \mathrm{LC}_{\leq}(d)\mathrm{LM}_{\leq}(d).$$

Prove the following statements:

(d) If $d, e \in \mathbb{W} \setminus \{0\}$, with $d \neq -e$, then $\mathrm{LE}_{\leq}(d+e) \leq \max_{\leq}\{\mathrm{LE}_{\leq}(d), \mathrm{LE}_{\leq}(e)\}$, with equality if and only if $\mathrm{LD}_{\leq}(d) \neq -\mathrm{LD}_{\leq}(e)$.

The previously introduced notions are of basic significance for this and the next section. So, we hope to illuminate their meaning in the following series of examples and exercises, which were already announced prior to the definition of these concepts.

12.3. **Examples and Exercises.** (A) *(Well Orderings)* Keep the above notations and hypotheses. Prove the following statements:

(a) Let $\varphi : \mathbb{N}_0 \longrightarrow \mathbb{N}_0^n \times \mathbb{N}_0^n$ be a bijective map. Show that the binary relation $\leq_\varphi \subseteq \mathbb{E} \times \mathbb{E}$ defined by

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \leq_\varphi \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} \Leftrightarrow \varphi^{-1}(\underline{\nu}, \underline{\mu}) \leq \varphi^{-1}(\underline{\nu}, \underline{\mu})$$

for all $\underline{\nu}, \underline{\mu}, \underline{\nu}', \underline{\mu}' \in \mathbb{N}_0^n$ is a well ordering of $\mathbb{E}$.

(b) Show that in the notations of exercise (a) the well ordering $\leq_\varphi$ is *discrete*, which means that the set $\{e \in \mathbb{E} \mid e \leq_\varphi d\}$ is finite for all $d \in \mathbb{E}$.

(c) Show, that there uncountably many discrete well orderings of $\mathbb{E}$.

(d) Let $n = 1$, set $X_1 =: X, \partial_1 =: \partial$ and define the binary relation $\leq$ on the set of elementary differential operators $\mathbb{E} = \{X^\nu\partial^\mu \mid \nu, \mu \in \mathbb{N}_0\}$ by

$$X^\nu\partial^\mu \leq X^{\nu'}\partial^{\mu'} \text{ if either } \begin{cases} \nu < \nu' \text{ or else} \\ \nu = \nu' \text{ and } \mu < \mu' \end{cases}$$

for all $\nu, \mu \in \mathbb{N}_0$. Show, that $\leq$ is a non-discrete well ordering of $\mathbb{E}$.

(B) *(Admissible Orderings)* Keep the above notations and hypotheses.

(a) We define the binary relation $\leq_{\text{lex}} \subseteq \mathbb{E} \times \mathbb{E}$ by setting (again for all $\underline{\nu}, \underline{\mu}, \underline{\nu}', \underline{\mu}' \in \mathbb{N}_0^n$):

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \leq_{\text{lex}} \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} \text{ if either}$$

(1) $\underline{\nu} = \underline{\nu}'$ and $\underline{\mu} = \underline{\mu}'$, or

(2) $\underline{\nu} = \underline{\nu}'$ and $\exists j \in \{1, 2, \ldots, n\} : \big[\mu_j < \mu'_j \text{ and } \mu_k = \mu'_k, \forall k < j\big]$, or else

(3) $\exists i \in \{1, 2, \ldots, n\} : \big[\nu_i < \nu'_i \text{ and } \nu_k = \nu'_k, \forall k < i\big]$.

Prove that $\leq_{\text{lex}} \in \mathrm{AO}(\mathrm{E})$. The admissible ordering $\leq_{\text{lex}}$ is called the *lexicographic ordering* of the set of elementary differential operators.

(b) Set $n = 1$, $X_1 =: X$, $\partial_1 =: \partial$ and write down the first 20 elementary differential operators $d \in \mathbb{E} = \{X^\nu\partial^\mu \mid \nu, \mu \in \mathbb{N}_0\}$ with respect to the ordering $\leq_{\text{lex}}$.

(c) Solve the similar task as in exercise (b), but with $n = 2$ instead of $n = 1$ and with 30 instead of 20.

(d) We define another binary relation $\leq_{\text{deglex}} \subseteq \mathbb{E} \times \mathbb{E}$ by setting

$$d \leq_{\text{deglex}} e \text{ if either } \begin{cases} \deg(d) < \deg(e) \text{ or else} \\ \deg(d) = \deg(e) \text{ and } d \leq_{\text{lex}} e. \end{cases}$$

Show, that $\leq_{\text{deglex}} \in \text{AO}(\mathbb{E})$. This admissible ordering is called the *degree-lexicographic ordering* of the set of elementary differential operators.

(e) Solve the previous exercises (b) and (c) but this time with the ordering $\leq_{\text{deglex}}$.

(f) We introduce a further binary relation $\leq_{\text{degrevlex}} \subseteq \mathbb{E} \times \mathbb{E}$ by setting (again for all $\underline{\nu}, \underline{\mu}, \underline{\nu}', \underline{\mu}' \in \mathbb{N}_0^n$):

$$\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \leq_{\text{degrevlex}} \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'} \text{ if either}$$

(1) $\deg\left(\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}\right) < \deg\left(\underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}\right)$, or else

(2) $\deg\left(\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}\right) = \deg\left(\underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}\right)$ and either

(i) $\underline{\nu} = \underline{\nu}'$ and $\underline{\mu} = \underline{\mu}'$, or

(ii) $\underline{\mu} = \underline{\mu}'$ and $\exists i \in \{1, 2, \ldots, n\} : \left[\nu_i > \nu_i' \text{ and } \nu_k = \nu_k', \forall k > i\right]$, or else

(iii) $\exists j \in \{1, 2, \ldots, n\} : \left[\mu_j > \mu_j' \text{ and } \mu_k = \mu_k', \forall k > j\right]$.

Prove, that $\leq_{\text{degrevlex}} \in \text{AO}(\mathbb{E})$. This admissible ordering is called the *degree-reverse-lexicographic ordering* of the set of elementary differential operators.

(g) Solve the previous exercise (e) but with $\leq_{\text{degrevlex}}$ instead of $\leq_{\text{deglex}}$.

(h) An *admissible ordering* of the set $\mathbb{M} = \{\underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}} \mid \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n\}$ of all monomials in $\mathbb{P}$ is a total ordering of $\mathbb{M}$ which satisfies the requirements

(1) *(Foundedness)* $1 \leq m$ for all $m \in \mathbb{M}$.

(2) *(Compatibility)* For all m, m' and $t \in \mathbb{M}$ we have the implication:

$$\text{If } m \leq m', \text{ then } mt \leq m't.$$

For any $\leq \in \text{AO}(\mathbb{E})$ we define the binary relation $\leq_\Phi \subseteq \mathbb{M} \times \mathbb{M}$ by setting

$$m \leq_\Phi m' \Leftrightarrow \Phi^{-1}(m) \leq \Phi^{-1}(m') \text{ for all } m, m' \in \mathbb{M}.$$

Prove, that $\leq_\Phi \in \text{AO}(\mathbb{M})$ and that there is indeed a bijection

$$\bullet_\Phi : \text{AO}(\mathbb{E}) \xrightarrow{\cong} \text{AO}(\mathbb{M}), \text{ given by } \leq \mapsto \leq_\Phi .$$

The names given in the previous exercises (a), (d) and (f) to the three admissible orderings of $\mathbb{E}$ introduced in these exercises are "inherited" from the "classical" designations used in polynomial rings, via the above bijection.

(i) Prove, that $\leq_{\text{deglex}}$ and $\leq_{\text{degrevlex}}$ are both discrete in the sense of exercise (A) (b), where as $\leq_{\text{lex}}$ is not.

(C) *(Leading Elementary Differential Operators and Related Concepts)* Keep the previous notations and hypotheses.

(a) Let $n = 1$, set $X_1 =: X$, $\partial_1 =: \partial$, $Y_1 =: Y$ and $Z_1 =: Z$. Write down the leading elementary differential operator, the leading differential operator, the leading coefficient, the leading monomial and the leading term of each of the following differential operators, with respect to each of the admissible orderings $\leq_{\text{lex}}, \leq_{\text{deglex}}$ and $\leq_{\text{degrevlex}}$:

(1) $5X^6 + 4X^4\partial - 2X^2\partial^3 + X\partial^4 - 3\partial^6$.

(2) $\partial^4 - 4X\partial^3 + 6X^2\partial^2 - 4X\partial + X^4$.
(3) $\partial^{12} - X^5\partial^7 + X^7\partial^5 - X^9\partial^3 + X^{12}$.

(b) Let $n = 2$ solve the task corresponding to exercise (a) above for the differential operators
(1) $X_1^3X_2^2 + 2\partial_1^3\partial_2^2$.
(2) $X_1^2X_2^3\partial_1^2\partial_2^3 - \partial_1^4\partial_2^6$.
(3) $X_1^k + X_2^k + \partial_1^k + \partial_2^k$ with $k \in \mathbb{N}$.

The next proposition will play a crucial role for our further considerations. it tells us essentially, that "leading differential operators behave as leading terms of polynomials". It is precisely this property, which will allow us to introduce a fertile notion of Gröbner bases for left ideals in Weyl algebras.

12.4. **Proposition.** ***(Multiplicativity of Leading Terms)*** *Let* $\leq \in \mathrm{AO}(\mathbb{E})$ *and let* $d, e \in \mathbb{W} \setminus \{0\}$*. Then it holds*

(a) $\mathrm{LT}_{\leq}(de) = \mathrm{LT}_{\leq}(d)\mathrm{LT}_{\leq}(e)$.
(b) $\mathrm{LM}_{\leq}(de) = \mathrm{LM}_{\leq}(d)\mathrm{LM}_{\leq}(e)$.

Proof. The product formula for elementary differential operators of Proposition 6.2 yields that

$$\mathrm{LE}_{\leq}\big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) = \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}'+\underline{\mu}'} \text{ for all } \underline{\nu}, \underline{\nu}', \underline{\mu}, \underline{\mu}' \in \mathbb{N}_0^n.$$

We may write

$$d = \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d)} c^{(d)}_{\underline{\nu}\underline{\mu}}\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \text{ and } e = \sum_{(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(e)}_{\underline{\nu}'\underline{\mu}'}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}$$

with $c^{(d)}_{\underline{\nu}\underline{\mu}}, c^{(e)}_{\underline{\nu}'\underline{\mu}'} \in K \setminus \{0\}$ for all $(\underline{\nu}, \underline{\mu}) \in \mathrm{supp}(d)$ and all $(\underline{\nu}', \underline{\mu}') \in \mathrm{supp}(e)$. With appropriate pairs $(\underline{\nu}^{(0)}, \underline{\mu}^{(0)}) \in \mathrm{supp}(d)$ and $(\underline{\nu}'^{(0)}, \underline{\mu}'^{(0)}) \in \mathrm{supp}(e)$ we also may write

$$\mathrm{LE}_{\leq}(d) = \underline{X}^{\underline{\nu}^{(0)}}\underline{\partial}^{\underline{\mu}^{(0)}} \text{ and } \mathrm{LE}_{\leq}(e) = \underline{X}^{\underline{\nu}'^{(0)}}\underline{\partial}^{\underline{\mu}'^{(0)}}, \text{ hence also}$$

$$\mathrm{LC}_{\leq}(d) = c^{(d)}_{\underline{\nu}^{(0)}\underline{\mu}^{(0)}} \text{ and } \mathrm{LC}_{\leq}(e) = c^{(e)}_{\underline{\nu}'^{(0)}\underline{\mu}'^{(0)}}.$$

Now, bearing in mind the previous observation on leading elementary differential operators we may write

$$\begin{aligned} de &= \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(d)}_{\underline{\nu}\underline{\mu}}\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}c^{(e)}_{\underline{\nu}'\underline{\mu}'}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} = \\ &= \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(d)}_{\underline{\nu}\underline{\mu}}c^{(e)}_{\underline{\nu}'\underline{\mu}'}\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} = \\ &= \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} \big[c^{(d)}_{\underline{\nu}\underline{\mu}}c^{(e)}_{\underline{\nu}'\underline{\mu}'}\underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'} + r_{\underline{\nu}\underline{\nu}'\underline{\mu}\underline{\mu}'}\big], \end{aligned}$$

with $r_{\underline{\nu}\underline{\nu}'\underline{\mu}\underline{\mu}'} \in \mathbb{W}$, such that for all $(\underline{\nu}, \underline{\mu}) \in \mathrm{supp}(d)$ and all $(\underline{\nu}', \underline{\mu}') \in \mathrm{supp}(e)$ it holds

$$\mathrm{LE}_{\leq}(r_{\underline{\nu}\underline{\nu}'\underline{\mu}\underline{\mu}'}) < \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'}, \text{ whenever } r_{\underline{\nu}\underline{\nu}'\underline{\mu}\underline{\mu}'} \neq 0.$$

By Definition, Reminder and Exercise 12.2 (C)(c) we have

$$\underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'} < \underline{X}^{\underline{\nu}^{(0)}+\underline{\nu}'^{(0)}}\underline{\partial}^{\underline{\mu}^{(0)}+\underline{\mu}'^{(0)}}, \text{ for all}$$
$$\big((\underline{\nu},\underline{\mu}),(\underline{\nu}',\underline{\mu}')\big) \in \mathrm{supp}(d)\times\mathrm{supp}(e)\setminus\{\big((\underline{\nu}^{(0)},\underline{\mu}^{(0)}),(\underline{\nu}'^{(0)},\underline{\mu}'^{(0)})\big)\}.$$

By Definition, Reminder and Exercise 12.2 (D)(d) it now follows easily that

$$\mathrm{LE}_{\leq}(de) = \underline{X}^{\underline{\nu}^{(0)}+\underline{\nu}'^{(0)}}\underline{\partial}^{\underline{\mu}^{(0)}+\underline{\mu}'^{(0)}} \text{ and}$$
$$\mathrm{LC}_{\leq}(de) = c^{(d)}_{\underline{\nu}^{(0)}\underline{\mu}^{(0)}}c^{(e)}_{\underline{\nu}'^{(0)}\underline{\mu}'^{(0)}} = \mathrm{LC}_{\leq}(d)\mathrm{LC}_{\leq}(e).$$

We thus obtain

$$\mathrm{LM}_{\leq}(de) = \Phi\big(\underline{X}^{\underline{\nu}^{(0)}+\underline{\nu}'^{(0)}}\underline{\partial}^{\underline{\mu}^{(0)}+\underline{\mu}'^{(0)}}\big) = \underline{Y}^{\underline{\nu}^{(0)}+\underline{\nu}'^{(0)}}\underline{Z}^{\underline{\mu}^{(0)}+\underline{\mu}'^{(0)}} = \underline{Y}^{\underline{\nu}^{(0)}}\underline{Z}^{\underline{\mu}^{(0)}}\underline{Y}^{\underline{\nu}'^{(0)}}\underline{Z}^{\underline{\mu}'^{(0)}} =$$
$$= \Phi\big(\underline{X}^{\underline{\nu}^{(0)}}\underline{\partial}^{\underline{\mu}^{(0)}}\big)\Phi\big(\underline{X}^{\underline{\nu}'^{(0)}}\underline{\partial}^{\underline{\mu}'^{(0)}}\big) = \Phi\big(\mathrm{LE}_{\leq}(d)\big)\Phi\big(\mathrm{LE}_{\leq}(e)\big) = \mathrm{LM}_{\leq}(d)\mathrm{LM}_{\leq}(e).$$

But now it follows

$$\mathrm{LT}_{\leq}(de) = \mathrm{LC}_{\leq}(de)\mathrm{LM}_{\leq}(de) = \mathrm{LC}_{\leq}(d)\mathrm{LC}_{\leq}(e)\mathrm{LM}_{\leq}(d)\mathrm{LM}_{\leq}(e) =$$
$$= \mathrm{LC}_{\leq}(d)\mathrm{LM}_{\leq}(d)\mathrm{LC}_{\leq}(e)\mathrm{LM}_{\leq}(e) = \mathrm{LT}_{\leq}(d)\mathrm{LT}_{\leq}(e).$$

□

The next result may be understood as an extension of the classical division algorithms of Euclid for univariate polynomials to the case of differential operators. It was first proved in 1984 by Briançon-Maisonobe in the univariate case and by Castro-Jiménez in the multivariate case.
Those readers, who are familiar with the Buchberger algorithm in multivariate polynomial rings will realize that our result corresponds to the division algorithm in multi-variate polynomial rings. Observe in particular that – as in the case of multi-variate polynomials – we will divide "by a family of denominators“ and that the presented division procedure depends on an admissible ordering.

12.5. **Proposition.** ***(The Division Property, Briançon-Maisonobe*** [14] ***and Castro-Jiménez*** [21]***)*** *Let* $\leq \in \mathrm{AO}(\mathbb{E})$, *let* $d \in \mathbb{W}$ *and let* $F \subset \mathbb{W}$ *be a finite set. Then, there is an element* $r \in \mathbb{W}$ *and a family* $(q_f)_{f\in F} \in \mathbb{W}^F$ *such that (in the notations of Convention 12.1 (B) and Definition, Reminder and Exercise 12.2 (D))*

(a) $d = \sum_{f\in F} q_f f + r$;
(b) $\Phi(s) \notin \mathbb{P}\mathrm{LM}_{\leq}(f)$ *for all* $f \in F\setminus\{0\}$ *and all* $s \in \mathrm{Supp}(r)$.
(c) $\mathrm{LE}_{\leq}(q_f f) \leq \mathrm{LE}_{\leq}(d)$ *for all* $f \in F$ *with* $q_f f \neq 0$.

Proof. We clearly may assume that $F \subset \mathbb{W}\setminus\{0\}$. If $d = 0$, we choose $r = 0$ and $q_f = 0$ for all $f \in F$. Assume, that our claim is wrong, and let $U \subsetneq \mathbb{W}$ be the non-empty set of all differential operators $d \in \mathbb{W}$ which do not admit a presentation of the requested form. As $\leq \in \mathrm{WO}(\mathbb{E})$ and $U \subset \mathbb{W}\setminus\{0\}$, we find some $d \in U$ such that

$$\mathrm{LE}_{\leq}(d) = \min_{\leq}\{\mathrm{LE}_{\leq}(u) \mid u \in U\}.$$

We distinguish the following two cases:

(1) There is some $f \in F$ such that $\mathrm{LM}_{\leq}(d) \in \mathbb{P}\mathrm{LM}_{\leq}(f)$.
(2) $\mathrm{LM}_{\leq}(d) \notin \bigcup_{f\in F}\mathbb{P}\mathrm{LM}_{\leq}(f)$.

In the case (1) we find some $e \in \mathbb{E}$ such that $\mathrm{LM}_{\leq}(d) = \Phi(e)\mathrm{LM}_{\leq}(f)$ and so we can introduce the element

$$d' := d - \frac{\mathrm{LC}_{\leq}(d)}{\mathrm{LC}_{\leq}(f)} ef \in \mathbb{W}.$$

If $d' = 0$, we set

$$r = 0, \quad q_f := \frac{\mathrm{LC}_{\leq}(d)}{\mathrm{LC}_{\leq}(f)} e, \text{ and } q_{f'} = 0 \text{ for all } f' \in F \setminus \{f\}.$$

But then

$$d = \frac{\mathrm{LC}_{\leq}(d)}{\mathrm{LC}_{\leq}(f)} ef = q_f f + r$$

is a presentation of d with the requested properties.
So, let $d' \neq 0$. Observe, that by Proposition 12.4 (a) we can write

$$\mathrm{LT}_{\leq}\big(\frac{\mathrm{LC}_{\leq}(d)}{\mathrm{LC}_{\leq}(f)} ef\big) = \frac{\mathrm{LC}_{\leq}(d)}{\mathrm{LC}_{\leq}(f)} \mathrm{LT}_{\leq}(ef) = \frac{\mathrm{LC}_{\leq}(d)}{\mathrm{LC}_{\leq}(f)} \mathrm{LT}_{\leq}(e)\mathrm{LT}_{\leq}(f) =$$
$$\mathrm{LC}_{\leq}(d)\mathrm{LM}_{\leq}(e)\mathrm{LM}_{\leq}(f) = \mathrm{LC}_{\leq}(d)\Phi(e)\mathrm{LM}_{\leq}(f) = \mathrm{LC}_{\leq}(d)\mathrm{LM}_{\leq}(d) = \mathrm{LT}_{\leq}(d).$$

If follows that $\mathrm{LD}_{\leq}\big(\frac{\mathrm{LC}_{\leq}(d)}{\mathrm{LC}_{\leq}(f)} ef\big) = \mathrm{LD}_{\leq}(d)$, and hence by Definition, Reminder and Exercise 12.2 (D)(d) we obtain that

$$\mathrm{LE}_{\leq}(d') < \mathrm{LE}_{\leq}(d) = \min_{\leq}\{\mathrm{LE}_{\leq}(u) \mid u \in U\}.$$

Therefore, $d' \notin U$ and so we find an element $r' \in \mathbb{W}$ and a family $(q'_{f'})_{f' \in F} \in \mathbb{W}^F$ such that

(a)$'$ $d' = \sum_{f' \in F} q'_{f'} f' + r'$;
(b)$'$ $\Phi(s') \notin \mathbb{P}\mathrm{LM}_{\leq}(f')$ for all $f' \in F$ and all $s' \in \mathrm{Supp}(r')$.
(c)$'$ $\mathrm{LE}_{\leq}(q'_{f'} f') \leq \mathrm{LE}_{\leq}(d')$ for all $f' \in F$ with $q'_{f'} \neq 0$.

Now, we set

$$r := r' \text{ and } q_f := \begin{cases} q'_{f'} & \text{if } f' \neq f, \\ q'_f + \frac{\mathrm{LC}_{\leq}(d)}{\mathrm{LC}_{\leq}(f)} e & \text{if } f = f'. \end{cases}$$

As

$$\mathrm{LE}_{\leq}(q'_{f'} f') \leq \mathrm{LE}_{\leq}(d') < \mathrm{LE}_{\leq}(d) \text{ and } \mathrm{LE}_{\leq}\big(\frac{\mathrm{LC}_{\leq}(d)}{\mathrm{LC}_{\leq}(f)} e\big) = \mathrm{LE}_{\leq}(e) \leq \mathrm{LE}_{\leq}(d),$$

we get

$$\mathrm{LE}_{\leq}(q_f F) = \mathrm{LE}_{\leq}\big((q'_f + \frac{\mathrm{LC}_{\leq}(d)}{\mathrm{LC}_{\leq}(f)} e) f\big) \leq \mathrm{LE}_{\leq}(d).$$

Now, it follows easily, that the requirements (a),(b) and (c) of our proposition are satisfied in the case (1).

So, let us assume that we are in the case (2). We set

$$d' := d - \mathrm{LD}_{\leq}(d).$$

If $d' = 0$ we have $d' = \mathrm{LD}_{\leq}(d)$ and it suffices to choose $q_f := 0$ for all $f \in F$ and $r = d$.
So, let $d' \neq 0$. Then, we have $\mathrm{LE}_{\leq}(d') < \mathrm{LE}_{\leq}(d)$ (see Definition, Reminder and

Exercise‘12.2 (D)(c)), so that again $d' \notin U$. But this means once more, that we get elements r' and $q'_{f'} \in \mathbb{W}$ (for all $f' \in F$) such that the above conditions $(a)', (b)'$ and $(c)'$ are satisfied. Now, we set

$$r := r' + \mathrm{LD}_{\leq}(d) \text{ and } q_f := q'_f \text{ for all } f \in F.$$

As $\mathrm{supp}(r) \subseteq \mathrm{supp}(r') \cup \{\mathrm{LE}_{\leq}(d)\}$ and $\mathrm{LE}_{\leq}(q_f f) \leq \mathrm{LE}(d') \leq \mathrm{LE}_{\leq}(d)$ for all $f \in F$ with $q_f \neq 0$ the requirements (a),(b) and (c) are again satisfied for the suggested choice. □

Now, we are ready to introduce the basic notion of this section: the concept of Gröbner basis.

12.6. **Definition, Reminder and Exercise.** (A) *(Monomial Ideals)* An ideal $I \subseteq \mathbb{P}$ is called a *monomial ideal* if there is a set $S \subset \mathbb{M} = \{\underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \mid \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n\}$ such that

$$I = \sum_{s \in S} \mathbb{P}s.$$

Show that in this situation for all $m \in \mathbb{M} \setminus \{0\}$ we have

(a) If $m = \sum_{i=1}^{t} f_i s_i$ with $s_1, s_2, \ldots, s_t \in S$ and $f_1, f_2, \ldots, f_t \in \mathbb{P}$, then there is some $i \in \{1, 2, \ldots, t\}$ and some $n_i \in \mathrm{supp}(f_i)$ such that $m = n_i s_i$.

(b) $m \in I$ if and only if there are $n \in \mathbb{M}$ and some $s \in S$ such that $m = ns$.

(B) *(Leading Monomial Ideals)* Let $\leq \in \mathrm{AO}(\mathbb{E})$ and $T \subset \mathbb{W}$. Then, the ideal

$$\mathrm{LMI}_{\leq}(T) := \sum_{d \in T \setminus \{0\}} \mathbb{P}\mathrm{LM}_{\leq}(d)$$

is called the *leading monomial ideal of* T with respect to $\leq$.
Prove that for all $m \in \mathbb{M}$, we have the following statements.

(a) If $m = \sum_{i=1}^{s} f_i \mathrm{LM}_{\leq}(t_i)$ with $t_1, t_2, \ldots, t_s \in T$ and $f_1, f_2, \ldots, f_s \in \mathbb{P}$, then there is some $i \in \{1, 2, \ldots, s\}$ and some $n_i \in \mathrm{supp}(f_i)$ such that $t_i \neq 0$ and $m = n_i \mathrm{LM}_{\leq}(t_i)$.

(b) $m \in \mathrm{LMI}_{\leq}(T)$ if and only if there are elements $u \in \mathbb{E}$ and $t \in T$ such that $m = \mathrm{LM}_{\leq}(u)\mathrm{LM}_{\leq}(t)$.

(C) *(Gröbner Bases)* Let $\leq \in \mathrm{AO}(\mathbb{E})$ and let $L \subseteq \mathbb{W}$ be a left ideal. A *Gröbner basis* of L with respect to $\leq$ (or a $\leq$-*Gröbner basis* of L) is a subset $G \subseteq L$ such that

$$\#G < \infty \text{ and } \mathrm{LMI}_{\leq}(L) = \mathrm{LMI}_{\leq}(G).$$

Prove the following facts:

(a) If G is a $\leq$-Gröbner basis of L and $G \subseteq H \subseteq L$ with $\#H < \infty$, then H is a $\leq$-Gröbner basis of L.

(b) If G is a $\leq$-Gröbner basis of L, then for each $d \in L \setminus \{0\}$ there is some $u \in \mathbb{E}$ and some $g \in G \setminus \{0\}$ such that

$$\mathrm{LM}_{\leq}(d) = \mathrm{LM}_{\leq}(u)\mathrm{LM}_{\leq}(g) = \mathrm{LM}_{\leq}(ug).$$

(c) If G is a $\leq$-Gröbner basis of L, then for each $d \in L \setminus \{0\}$ there is some monomial $m = \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \in \mathbb{P}$ and some $g \in G \setminus \{0\}$ such that

$$\mathrm{LM}_{\leq}(d) = m\mathrm{LM}_{\leq}(g).$$

Now, we prove that Gröbner bases always exist, and that they deserve the name of "basis", as they generate the involved left ideal. Clearly, these statements correspond precisely to well known facts in multi-variate polynomial rings. After having established the announced existence and generating property of Gröbner bases, we shall add a few examples and exercises on the subject.

12.7. **Proposition.** ***(Existence and Generating Property of Gröbner Bases)*** *Let* $\leq \in \mathrm{AO}(\mathrm{E})$ *and let* $L \subseteq \mathbb{W}$ *be a left ideal. Then the following statements hold.*

(a) *L admits a $\leq$-Gröbner basis.*
(b) *If G is any $\leq$-Gröbner basis of L, then $L = \sum_{g \in G} \mathbb{W}g$.*

Proof. (a): This is clear as the ideal $\mathrm{LMI}_{\leq}(L)$ is generated by finitely many elements of the form $\mathrm{LM}_{\leq}(g)$ with $g \in L$.

(b): Let $G \subseteq L$ be a $\leq$-Gröbner basis of L and assume that $\sum_{g \in G} \mathbb{W}g \subsetneq L$. As $\leq \in \mathrm{WO}(\mathbb{E})$, we find some $e \in L \setminus \sum_{g \in G} \mathbb{W}g$ such that

$$\mathrm{LE}(e) = \min_{\leq}\{\mathrm{LE}_{\leq}(d) \mid d \in L \setminus \sum_{g \in G} \mathbb{W}g\}.$$

By Definition, Reminder and Exercise 12.6 (C)(b) we find some $u \in \mathbb{E}$ and some $g \in G$ such that

$$\mathrm{LM}_{\leq}(e) = \mathrm{LM}_{\leq}(u)\mathrm{LM}_{\leq}(g).$$

Setting

$$v := -\frac{\mathrm{LC}_{\leq}(e)}{\mathrm{LC}_{\leq}(g)}u$$

we now get on use of Proposition 12.4 (a) that

$$\begin{aligned}\mathrm{LT}_{\leq}(e) &= \mathrm{LC}_{\leq}(e)\mathrm{LM}_{\leq}(e) = \mathrm{LC}_{\leq}(e)\mathrm{LM}_{\leq}(u)\mathrm{LM}_{\leq}(g) = \\ &= \mathrm{LC}_{\leq}(e)\mathrm{LT}_{\leq}(u)\frac{1}{\mathrm{LC}_{\leq}(g)}\mathrm{LT}_{\leq}(g) = \frac{\mathrm{LC}_{\leq}(e)}{\mathrm{LC}_{\leq}(g)}\mathrm{LT}_{\leq}(u)\mathrm{LT}_{\leq}(g) = \\ &= -\mathrm{LT}_{\leq}(v)\mathrm{LT}_{\leq}(g) = -\mathrm{LT}_{\leq}(vg).\end{aligned}$$

As $e \notin \sum_{g \in G} \mathbb{W}g$ and $g \in G$, we have

$$e + vg \in L \setminus \sum_{g \in G} \mathbb{W}g.$$

In particular $e + vg \neq 0$. So by Definition, Reminder and Exercise 12.2 (D)(d) it follows that

$$\mathrm{LE}_{\leq}(e + vg) < \mathrm{LE}_{\leq}(e) = \min_{\leq}\{\mathrm{LE}_{\leq}(d) \mid d \in L \setminus \sum_{g \in G} \mathbb{W}g\}.$$

But this is a contradiction. □

Now, we add the previously announced examples and exercises.

12.8. **Examples and Exercises.** (A) *(Leading Monomial Ideals)* Keep the above notations and hypotheses. Prove the following statements:

(a) Let $d \in \mathbb{W} \setminus \{0\}$ and $\leq \in \mathrm{AO}(\mathbb{E})$. Prove that $\mathrm{LMI}_{\leq}(\mathbb{W}d)$ is a principal ideal.

(b) Let $n = 1$, $X_1 =: X$ and $\partial_1 =: \partial$. Set $L := \mathbb{W}(X^2 - \partial) + \mathbb{W}(X\partial)$ and determine $\mathrm{LMI}_{\leq}(L)$ for $\leq := \leq_{\mathrm{lex}}, \leq_{\mathrm{deglex}}$ and $\leq := \leq_{\mathrm{degrevlex}}$.

(B) *(Gröbner Bases)* Keep the above notations and hypotheses. Prove the following statements:

(a) Let the notations be as in exercise (a) of part (A) and prove that $\{cd\}$ is a $\leq$-Gröbner basis of $\mathbb{W}d$ for all $c \in K \setminus \{0\}$, and that any singleton $\leq$-Gröbner bases of $\mathbb{W}d$ is of the above form.

(b) Let the notations and hypotheses be as in exercise (b) of part (A) and compute a $\leq$-Gröbner basis for $\leq := \leq_{\mathrm{lex}}, \leq_{\mathrm{deglex}}$ and $\leq := \leq_{\mathrm{degrevlex}}$

We now head for another basic result on Gröbner bases, which says that these bases enjoy a certain restriction property. This will be an important ingredient in our treatment of Universal Gröbner bases. We begin with the following preparations.

12.9. **Notation.** (A) For any set $S \subseteq \mathbb{W}$ we write (see also Definition, Reminder and Exercise 12.2 (D)):

$$\mathrm{supp}(S) := \bigcup_{s \in S} \mathrm{supp}(s) \text{ and } \mathrm{Supp}(S) := \bigcup_{s \in S} \mathrm{Supp}(s).$$

(B) Let $\leq \in \mathrm{TO}(\mathbb{E})$ (see Definition, Reminder and Exercise 12.2 (A)) and let $T \subset \mathbb{E}$. We write $\leq\restriction_T$ for the *restriction* of $\leq$ to T, thus – if we interpret binary relations on a set S as subsets of $S \times S$:

$$\leq\restriction_T \quad := \quad \leq \cap (T \times T), \text{ so that } : d \leq\restriction_T e \Leftrightarrow d \leq e \text{ for all } d, e \in T.$$

12.10. **Proposition.** ***(The Restriction Property of Gröbner Bases)*** *Let $L \subseteq \mathbb{W}$ be a left ideal. Let $\leq, \leq' \in \mathrm{AO}(\mathbb{E})$ and let G be a $\leq$-Gröbner basis of L. Assume that*

$$\leq\restriction_{\mathrm{Supp}(G)} \quad = \quad \leq'\restriction_{\mathrm{Supp}(G)} .$$

Then G is also a $\leq'$-Gröbner basis of L.

Proof. Let $d \in L \setminus \{0\}$. We have to show that $\mathrm{LM}_{\leq'}(d) \in \mathrm{LMI}_{\leq'}(G)$. We may assume that $0 \notin G$. If we apply Proposition 12.5 to the ordering $\leq'$, we find an element r and a family $(q_g)_{g \in G} \in \mathbb{W}^G$ such that

(1) $d = \sum_{g \in G} q_g g + r$;
(2) $\Phi(s) \notin \mathbb{P}\mathrm{LM}_{\leq'}(g)$ for all $g \in G$ and all $s \in \mathrm{Supp}(r)$.
(3) $\mathrm{LE}_{\leq'}(q_g g) \leq' \mathrm{LE}_{\leq'}(d)$ for all $g \in G$ with $q_g \neq 0$.

Our immediate aim is to show that $r = 0$. Assume to the contrary that $r \neq 0$. As $r \in L$ and G is a $\leq$-Gröbner basis of L, we get $\mathrm{LM}_{\leq}(r) \in \mathrm{LMI}_{\leq}(G)$. So, there is some $g \in G$ such that $\mathrm{LM}_{\leq}(r) = m\mathrm{LM}_{\leq}(g)$ for some $m \in \mathbb{M}$ (see Definition, Reminder and Exercise 12.6 (C)(c)). As $\leq\restriction_{\mathrm{Supp}(G)} \;=\; \leq'\restriction_{\mathrm{Supp}(G)}$ it follows that

$$\Phi\big(\mathrm{LT}_{\leq}(r)\big) = \mathrm{LM}_{\leq}(r) \in \mathbb{P}\mathrm{LM}_{\leq'}(g).$$

As $\mathrm{LT}_{\leq}(r) \in \mathrm{Supp}(r)$, this contradicts the above condition (2). Therefore $r = 0$.
But now, we may write

$$d = \sum_{g \in G^*} q_g g, \text{ whith } G^* := \{g \in G \mid q_g \neq 0\}.$$

By the above condition (3) we have $\mathrm{LE}_{\leq'}(q_g g) \leq' \mathrm{LE}_{\leq'}(d)$ for all $g \in G^*$. So, there is some $g \in G^*$ such that $\mathrm{LE}_{\leq'}(d) = \mathrm{LE}_{\leq'}(q_g g)$ (see Definition, Reminder and Exercise 12.2 (D)(d)), and hence $\mathrm{LM}_{\leq'}(d) = \mathrm{LM}_{\leq'}(q_g g)$. Thus, on use of Proposition 12.4 (b) we get indeed

$$\mathrm{LM}_{\leq'}(d) = \mathrm{LM}_{\leq'}(q_g)\mathrm{LM}_{\leq'}(g) \in \mathrm{LMI}_{\leq'}(G).$$

□

Now, we shall introduce the central concept of this section.

12.11. **Definition.** *(Universal Gröbner Bases)* Let $L \subseteq \mathbb{W}$ be a left ideal. A *universal Gröbner basis* of L is a (finite) subset $G \subset \mathbb{W}$ which is a $\leq$-Gröbner basis for all $\leq \in \mathrm{AO}(\mathbb{E})$.

Universal Gröbner bases have been studied by Sturmfels [41] in the polynomial ring $K[X_1, X_2, \ldots, X_n]$ – and indeed this notion can be immediately extended to the Weyl algebra $\mathbb{W}$. Gröbner bases for left ideals in the Weyl algebra were introduced by Assi, Castro-Jiménez and Granger [3] and also by Saito, Sturmfels and Takayama [38].

Clearly, our next aim should be to show, that universal Gröbner bases always exist. There are indeed various possible ways to prove this. Here, we shall do this by a topological approach which relies on an idea of Sikora [40], and which can be found in greater generality in Boldini's thesis [11]. We approach the subject by first introducing a natural metric on the set of total orderings of all elementary differential operators. Then, we make the reader prove in a series of exercises, that we get a complete metric space in this way.

12.12. **Definition, Exercise and Convention.** (A) *(The Natural Metric on the Set* $\mathrm{TO}(\mathbb{E})$*)* For all $i \in \mathbb{Z}$ we introduce the notation

$$\mathbb{E}_i := \{e \in \mathbb{E} \mid \deg(e) \leq i\} = \{\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \mid |\underline{\nu}| + |\underline{\mu}| \leq i\}.$$

We define a map

$$\mathrm{dist} : \mathrm{TO}(\mathbb{E}) \times \mathrm{TO}(\mathbb{E}) \longrightarrow \mathbb{R}, \text{ given by for all } \leq, \leq' \in \mathrm{TO}(\mathbb{E}) \text{ by}$$

$$\mathrm{dist}(\leq, \leq') := \begin{cases} 2^{-\sup\{r \in \mathbb{N}_0 \mid \leq\restriction_{\mathbb{E}_r} = \leq'\restriction_{\mathbb{E}_r}\}}, & \text{if } \leq \neq \leq', \\ 0, & \text{if } \leq = \leq'. \end{cases}$$

Prove that

(a) For all $\leq, \leq' \in \mathrm{TO}(\mathbb{E})$ and all $r \in \mathbb{N}_0$ we have

$$\mathrm{dist}(\leq, \leq') < \frac{1}{2^r} \text{ if and only if } \leq\restriction_{\mathbb{E}_{r+1}} = \leq'\restriction_{\mathbb{E}_{r+1}}.$$

(b) The map $\mathrm{dist} : \mathrm{TO}(\mathbb{E}) \times \mathrm{TO}(\mathbb{E}) \longrightarrow \mathbb{R}$ is a *metric* on $\mathrm{TO}(\mathbb{E})$.

From now on, we always endow $\mathrm{TO}(\mathbb{E})$ with this metric and the induced *Hausdorff topology*.

(B) *(Completeness of the Metric Space* $\mathrm{TO}(\mathbb{E})$*)* Let $(\leq_i)_{i \in \mathbb{N}_0}$ be a *Cauchy sequence* in $\mathrm{TO}(\mathbb{E})$. This means:

For all $r \in \mathbb{N}_0$ there is some $n(r) \in \mathbb{N}_0$ such that $\mathrm{dist}(\leq_i, \leq_j) < \frac{1}{2^r}$ for all $i, j \geq n(r)$.

We introduce the binary relation $\leq \subseteq \mathbb{E} \times \mathbb{E}$ given for all $d, e \in \mathbb{E}$ by

$$d \leq e \text{ if and only if } d \leq_i e \text{ for all } i \gg 0.$$

Prove the following statements:

(a) If $r \in \mathbb{N}_0$, $d, e \in \mathbb{E}_{r+1}$, and $i, j \geq n(r)$, then $d \leq_i e$ if and only if $d \leq_j e$.
(b) If $r \in \mathbb{N}_0$, $d, e \in \mathbb{E}_{r+1}$, and $i \geq n(r)$, then $d \leq_i e$ if and only if $d \leq e$.
(c) $\leq \in \mathrm{TO}(\mathbb{E})$.
(d) If $r \in \mathbb{N}_0$, and $i \geq n(r)$, then $\mathrm{dist}(\leq_i, \leq) \leq \frac{1}{2^r}$.
(e) $\lim_{i\to\infty} \leq_i = \leq$.
(f) $\mathrm{TO}(\mathbb{E})$ is a *complete* metric space.

Now, we are ready to prove the basic ingredient of our existence proof for universal Gröbner bases.

12.13. **Proposition.** ***(Compactness of the Space of Total Orderings)*** *The space* $\mathrm{TO}(\mathbb{E})$ *is compact.*

Proof. Let $(\leq_i)_{i\in\mathbb{N}_0}$ be a sequence in $\mathrm{TO}(\mathbb{E})$. It suffices to show, that $(\leq_i)_{i\in\mathbb{N}_0}$ has a convergent subsequence. Bearing in mind Definition, Exercise and Convention 12.12 (B)(f) (or (e)), it suffices to find a subsequence of $(\leq_i)_{i\in\mathbb{N}_0}$ which is a Cauchy sequence. Observe that all the sets $\mathbb{E}_r$ are finite. We want to construct a sequence $(\mathbb{S}_r)_{r\in\mathbb{N}_0}$ of infinite subsets $\mathbb{S}_r \subseteq \mathbb{N}_0$ such that for all $s \in \mathbb{N}_0$ we have

(1) $\mathbb{S}_{s+1} \subseteq \mathbb{S}_s$.
(2) $\leq_j\restriction_{\mathbb{E}_{s+1}} \;=\; \leq_k\restriction_{\mathbb{E}_{s+1}}$ for all $j, k \in \mathbb{S}_s$.

We construct the members $\mathbb{S}_r$ of the sequence $(\mathbb{S}_r)_{r\in\mathbb{N}_0}$ by induction r. As $\mathbb{E}_1$ is finite, we can find an infinite set $\mathbb{S}_0 \subseteq \mathbb{N}_0$ such that requirement (2) is satisfied with $s = 0$. Now, let $r > 0$ and assume that the sets $\mathbb{S}_0, \mathbb{S}_1, \ldots, \mathbb{S}_r$ are already defined such that requirement (1) holds for all $s < r$ and requirement (2) holds for all $s \leq r$.
As $\mathbb{E}_{r+2}$ is finite, we find an infinite subset $\mathbb{S}_{r+1} \subseteq \mathbb{S}_r$ (which hence satisfies requirement (1) for $s = r$) such that requirement (2) is also satisfied with $s = r+1$. This completes the step of induction and hence proves that a sequence $(\mathbb{S}_r)_{r\in\mathbb{N}_0}$ with the requested properties exists.
Now, we may choose a sequence $(i_k)_{k\in\mathbb{N}_0}$ in $\mathbb{N}_0$, such that

$$i_r < i_{r+1} \text{ and } i_r \in \mathbb{S}_r \text{ for all } r \in \mathbb{N}_0.$$

In particular it follows that

$$\leq_{i_j}\restriction_{\mathbb{E}_{r+1}} \;=\; \leq_{i_k}\restriction_{\mathbb{E}_{r+1}} \text{ for all } j, k \geq r$$

and hence (see Definition, Exercise and Convention 12.12 (A)(a))

$$\mathrm{dist}(\leq_{i_j}, \leq_{i_k}) < \frac{1}{2^r} \text{ for all } j, k \geq r.$$

So, the constructed subsequence $(\leq_{i_k})_{k\in\mathbb{N}_0}$ of our original sequence $(\leq_i)_{i\in\mathbb{N}_0}$ is indeed a Cauchy sequence. □

What we need indeed to prove our main result, is the compactness of subspace of admissible orderings in the topological space of total orderings.

12.14. **Proposition.** ***(Compactness of the Space of Admissible Orderings)*** *The set* $\mathrm{AO}(\mathbb{E})$ *is a closed subset of* $\mathrm{TO}(\mathbb{E})$ *and hence compact.*

Proof. Let $(\leq_i)_{i\in\mathbb{N}_0}$ be sequence in $\mathrm{AO}(\mathbb{E})$, which converges in $\mathrm{TO}(\mathbb{E})$ and let

$$\lim_{i\to\infty} \leq_i \quad = \quad \leq .$$

We aim to show, that $\leq \in \mathrm{AO}(\mathbb{E})$. According to Definition, Reminder and Exercise 12.2 (C), we must show, that for all $\underline{\lambda}, \underline{\lambda}', \underline{\kappa}, \underline{\kappa}', \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n$ the following statements hold.

(1) $1 \leq \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}$.

(2) If $\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}} \leq \underline{X}^{\underline{\lambda}'}\underline{\partial}^{\underline{\kappa}'}$ then $\underline{X}^{\underline{\lambda}+\underline{\nu}}\underline{\partial}^{\underline{\kappa}+\underline{\mu}} \leq \underline{X}^{\underline{\lambda}'+\underline{\nu}}\underline{\partial}^{\underline{\kappa}'+\underline{\mu}}$.

So, fix $\underline{\lambda}, \underline{\lambda}', \underline{\kappa}, \underline{\kappa}', \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n$. Then we find some $r \in \mathbb{N}_0$ such that all the elementary differential operators which occur in (1) and (2) belong to $\mathbb{E}_{r+1}$. Now, we find some $i \in \mathbb{N}_0$ such that $\mathrm{dist}(\leq_i, \leq) < \frac{1}{2^r}$, hence such that $\leq\restriction_{\mathbb{E}_{r+1}} \; = \; \leq_i\restriction_{\mathbb{E}_{r+1}}$. As $\leq_i \in \mathrm{AO}(\mathbb{E})$ the required inequalities hold for $\leq_i$. But then, by the coincidence of $\leq$ and $\leq_i$ on $\mathbb{E}_{r+1}$, they hold also for $\leq$. □

Now, after having established the following auxiliary result, we are ready to prove the announced main result.

12.15. **Lemma.** *Let* $L \subset \mathbb{W}$ *be a left ideal and let* $G \subseteq L$ *be a finite subset. Then, the set*

$$\mathbb{U}_L(G) := \{\leq \in \mathrm{AO}(\mathbb{E}) \mid G \textit{ is a } \leq - \textit{ Gröbner basis of } L\}$$

is open in $\mathrm{AO}(\mathbb{E})$.

Proof. We may assume that $\mathbb{U}_L(G)$ is not empty and choose $\leq \in \mathbb{U}_L(G)$. We find some $r \in \mathbb{N}_0$ with $\mathrm{supp}(G) \subseteq \mathbb{E}_{r+1}$. Let $\leq' \in \mathrm{AO}(\mathbb{E})$ such that $\mathrm{dist}(\leq, \leq') < \frac{1}{2^r}$. So, we obtain that $\leq\restriction_{\mathbb{E}_{r+1}} \; = \; \leq'\restriction_{\mathbb{E}_{r+1}}$ and hence in particular that $\leq\restriction_{\mathrm{Supp}(G)} \; = \; \leq'\restriction_{\mathrm{Supp}(G)}$. By Proposition 12.10 it follows that G is a $\leq'$-Gröbner basis of L and hence that $\leq' \in \mathbb{U}_L(G)$. But this means, that the open neighborhood

$$\{\leq' \in \mathrm{AO}(\mathbb{E}) \mid \mathrm{dist}(\leq', \leq) < \frac{1}{2^r}\}$$

of $\leq$ belongs to $\mathbb{U}_L(G)$. □

12.16. **Theorem.** ***(Existence of Universal Gröbner Bases)*** *Each left ideal* L *of* $\mathbb{W}$ *admits a universal Gröbner basis.*

Proof. Let $L \subseteq \mathbb{W}$ be a left ideal. For each $\leq \in \mathrm{AO(E)}$ we choose a $\leq$-Gröbner basis $G_\leq$ of L. In the notations of Lemma 12.15 we have $\leq \in \mathbb{U}_L(G_\leq)$. So, by this same Lemma the family

$$\big(\mathbb{U}_L(G_\leq)\big)_{\leq\in\mathrm{AO}(\mathbb{E})}$$

is an open covering of $\mathrm{AO}(\mathbb{E})$. By Proposition 12.14 we thus find finitely many elements

$$\leq_1, \leq_2, \ldots, \leq_r \in \mathrm{AO}(\mathbb{E})$$

such that

$$\mathrm{AO}(\mathbb{E}) = \bigcup_{i=1}^{r} \mathbb{U}_L(G_{\leq_i}).$$

Let $\leq \in \mathrm{AO}(\mathbb{E})$. Then $\leq \in \mathbb{U}_L(G_{\leq_i})$ for some $i \in \{1, 2, \ldots, r\}$. Therefore $G_{\leq_i}$ is a $\leq$-Gröbner basis of L. So $\bigcup_{i=1}^r G_{\leq_i}$ is a Gröbner basis of L for all $\leq \in \mathrm{AO}(\mathbb{E})$. □

As a first application of the previous existence result we get the following finiteness result.

12.17. **Corollary.** ***(Finiteness of the Set of Leading Monomial Ideals)*** *Let $L \subseteq \mathbb{W}$ be a left ideal. Then the set*

$$\{\mathrm{LMI}_{\leq}(L) \mid \leq \in \mathrm{AO}(\mathbb{E})\}$$

of all leading monomial ideals of L with respect to admissible orderings of $\mathbb{E}$ is finite.

Proof. Let $G \subseteq L$ be a universal Gröbner basis of L. Then we have

$$\{\mathrm{LMI}_{\leq}(L) \mid \leq \in \mathrm{AO}(\mathbb{E})\} = \{\mathrm{LMI}_{\leq}(G) \mid \leq \in \mathrm{AO}(\mathbb{E})\}.$$

Therefore

$$\#\{\mathrm{LMI}_{\leq}(L) \mid \leq \in \mathrm{AO}(\mathbb{E})\} \leq \#\{\sum_{h \in H} \mathbb{P}\Phi(h) \mid H \subseteq \mathrm{supp}(G)\} \leq$$
$$\leq \#\{H \subseteq \mathrm{supp}(G)\} = 2^{\#\mathrm{supp}(G)}.$$

□

13. Weighted Orderings

This section is devoted to the study of admissible orderings which are compatible with a given weight and the related notion of weighted (admissible) ordering. Such weighted orderings were first studied by Assi, Castro-Jiménez and Granger [3] and by Saito, Sturmfels and Takayama [38].

In relation to these weighted orderings, we shall introduce the fundamental notion of symbol of a differential operator with respect to a given weight. We will see, that these symbols, which are indeed polynomials, behave again multiplicatively. Moreover, we shall see that the symbols of all members of a Gröbner basis of a given left ideal generate the so-called induced ideal of the given left ideal. Our ultimate goal is to prove, that the number of characteristic varieties of given D-module with respect to all weights is finite. Moreover, we shall prove a certain stability result for characteristic varieties found in Boldini's thesis [11], which is published in [12].

13.1. **Notation.** (A) As previously, we fix a positive integer n, a field K of characteristic 0 and consider the standard Weyl algebra

$$\mathbb{W} := \mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n],$$

the polynomial ring

$$\mathbb{P} := K[Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n]$$

in the indeterminates $Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n$ with coefficients in the field K and the isomorphism of K-vector spaces

$$\Phi : \mathbb{W} \xrightarrow{\cong} \mathbb{P}, \quad \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \mapsto \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}} \text{ for all } \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n.$$

(B) We also write

$$\Omega := \{(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid (v_i, w_i) \neq (0, 0) \text{ for all } i = 1, 2, \ldots, n\} \subset \mathbb{N}_0^n \times \mathbb{N}_0^n$$

for the set of all weights. If

$$\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$$

we also use the suffix $\underline{\omega}$ instead of the suffix $\underline{vw}$ in all the previously introduced notations. So we write for example

$$\mathbb{W}_{\bullet}^{\underline{\omega}} := \mathbb{W}_{\bullet}^{\underline{vw}}, \quad \deg^{\underline{\omega}}(d) := \deg^{\underline{vw}}(d), \quad \mathbb{P}^{\underline{\omega}} := \mathbb{P}^{\underline{vw}}, \quad \ldots$$

Observe, that

$$\underline{\omega} + \underline{\alpha} \in \Omega \text{ and } s\underline{\omega} \in \Omega \text{ for all } \underline{\omega}, \underline{\alpha} \in \Omega \text{ and all } s \in \mathbb{N},$$

where the arithmetic operations are performed in $\mathbb{N}_0^{2n}$.

Now, we introduce the concept of admissible orderings which are compatible with a given weight.

13.2. **Definition and Exercise.** (A) *(Weight Compatible Orderings)* We fix a weight and an admissible ordering of the set $\mathbb{E}$ of elementary differential operators in $\mathbb{W}$ (see Definition, Reminder and Exercise 12.2 (C)):

$$\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega \text{ and } \leq \in \mathrm{AO}(\mathbb{E}).$$

We say that $\leq$ is *compatible* with the weight $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ (or $\underline{\omega}$*-compatible*), if for all $d, e \in \mathbb{E}$ we have:

$$\text{If } \deg^{\underline{\omega}}(d) < \deg^{\underline{\omega}}(e), \text{ then } d < e.$$

So, $\leq$ is compatible with $\underline{\omega} = (\underline{v}, \underline{w})$ if and only if for all $\underline{\nu}, \underline{\mu}, \underline{\nu}', \underline{\mu}' \in \mathbb{N}_0^n$ we have the following implication:

$$\text{If } \underline{\nu}\underline{v} + \underline{\mu}\underline{w} < \underline{\nu}'\underline{v} + \underline{\mu}'\underline{w}, \text{ then } \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} < \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}.$$

We set

$$\mathrm{AO}^{\underline{\omega}}(\mathbb{E}) = \mathrm{AO}^{\underline{vw}}(\mathbb{E}) := \{\leq \in \mathrm{AO}(\mathbb{E}) \mid \ \leq \text{ is compatible with } \underline{\omega} = (\underline{v}, \underline{w})\}.$$

(B) *(Weighted Admissible Orderings)* Keep the notations and hypotheses of part (A). We define a new binary relation

$$\leq^{\underline{\omega}} = \leq^{\underline{vw}} \subseteq \mathbb{E} \times \mathbb{E}$$

on $\mathbb{E}$, by setting, for all $d, e \in \mathbb{E}$:

$$d \leq^{\underline{\omega}} e \text{ if } \begin{cases} \text{either} & \deg^{\underline{\omega}}(d) < \deg^{\underline{\omega}}(e) \\ \text{or else} & \deg^{\underline{\omega}}(d) = \deg^{\underline{\omega}}(e) \text{ and } d < e. \end{cases}$$

Prove that for each weight $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ and each $\leq \in \mathrm{AO}(\mathbb{E})$ the following statements hold.

(a) $\leq^{\underline{\omega}} \in \mathrm{AO}^{\underline{\omega}}(\mathbb{E})$.
(b) $(\leq^{\underline{\omega}})^{\underline{\omega}} = \leq^{\underline{\omega}}$.
(c) $\leq \in \mathrm{AO}^{\underline{\omega}}(\mathbb{E})$ if and only if $\leq \ = \ \leq^{\underline{\omega}}$.

The admissible ordering $\leq^{\underline{\omega}} \in \mathrm{AO}(\mathbb{E})$ is called the $\underline{\omega}$*-weighted ordering associated to* $\leq$.

Another important concept, which was already mentioned in the introduction to this section, is the notion of symbol of a differential operator. We now will introduce this notion after a few preparatory steps.

13.3. **Definition and Exercise.** (A) Let $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$, let $i \in \mathbb{N}_0$ and let

$$d = \sum_{(\underline{\nu},\underline{\mu})\in \operatorname{supp}(d)} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \in \mathbb{W} \quad \text{with } c^{(d)}_{\underline{\nu}\underline{\mu}} \in K \setminus \{0\} \text{ for all } (\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(d).$$

We set

$$\operatorname{supp}^{\underline{\omega}}_i(d) := \{(\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(d) \mid \underline{\nu}\underline{v} + \underline{\mu}\underline{w} = i\}.$$

and

$$d^{\underline{\omega}}_i = d^{\underline{v}\underline{w}}_i := \sum_{(\underline{\nu},\underline{\nu})\in \operatorname{supp}^{\underline{\omega}}_i(d)} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}.$$

Prove that for all $d, e \in \mathbb{W}$, all $i, j \in \mathbb{N}_0$ and for all weights $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ the following statements hold:

(a) If $i > \deg^{\underline{\omega}}(d)$, then $d^{\underline{\omega}}_i = 0$.
(b) $d^{\underline{\omega}}_i = (d^{\underline{\omega}}_i)^{\underline{\omega}}_i$.
(c) $(d + e)^{\underline{\omega}}_i = d^{\underline{\omega}}_i + e^{\underline{\omega}}_i$.
(d) If $d, e \neq 0$, $i := \deg^{\underline{\omega}}(d)$ and $j := \deg^{\underline{\omega}}(e)$, then

$$\operatorname{supp}^{\underline{\omega}}_{i+j}(de) = \{(\underline{\nu} + \underline{\nu}', \underline{\mu} + \underline{\mu}') \mid (\underline{\nu}, \underline{\mu}) \in \operatorname{supp}^{\underline{\omega}}_i(d) \text{ and } (\underline{\nu}', \underline{\mu}') \in \operatorname{supp}^{\underline{\omega}}_j(e)\}.$$

(e) If $d, e \neq 0$, $i := \deg^{\underline{\omega}}(d)$ and $j := \deg^{\underline{\omega}}(e)$, then

$$(de)^{\underline{\omega}}_{i+j} = \sum_{(\underline{\nu},\underline{\mu})\in \operatorname{supp}^{\underline{\omega}}_i(d), (\underline{\nu}',\underline{\mu}')\in \operatorname{supp}^{\underline{\omega}}_j(e)} c^{(d)}_{\underline{\nu}\underline{\mu}} c^{(e)}_{\underline{\nu}'\underline{\mu}'} \underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'}.$$

(B) Keep the notations and hypotheses of part (A). We set

$$\sigma^{\underline{\omega}}_i(d) := \Phi\big(d^{\underline{\omega}}_i\big) = \sum_{(\underline{\nu},\underline{\nu})\in \operatorname{supp}^{\underline{\omega}}_i(d)} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}}.$$

Prove on use of statements (a)–(e) of part (A) that for all $d, e \in \mathbb{W}$, all $i, j \in \mathbb{N}_0$ and for all weights $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ the following statements hold:

(a) $\sigma^{\underline{\omega}}_i(d) := \sigma^{\underline{\omega}}_i(d^{\underline{\omega}}_i)$.
(b) If $i > \deg^{\underline{\omega}}(d)$, then $\sigma^{\underline{\omega}}_i(d) = 0$.
(c) $\sigma^{\underline{\omega}}_i(d) = \sigma^{\underline{\omega}}_i(d^{\underline{\omega}}_i)$.
(d) $\sigma^{\underline{\omega}}_i(d + e) = \sigma^{\underline{\omega}}_i(d) + \sigma^{\underline{\omega}}_i(e)$.

(C) *(The Symbol of a Differential operator with Respect to a Weight)* Keep the notations of part (A), (B). We define the $\underline{\omega} = (\underline{v}, \underline{w})$-*symbol* of the differential operator $d \in \mathbb{W}$ by

$$\sigma^{\underline{\omega}}(d) := \begin{cases} 0 & \text{if } d = 0, \\ \sigma^{\underline{\omega}}_{\deg^{\underline{\omega}}(d)}(d) & \text{if } d \neq 0. \end{cases}$$

Prove that for all $d, e \in \mathbb{W} \setminus \{0\}$ the following statements hold.

(a) $\sigma^{\underline{\omega}}(d) = \Phi(d^{\underline{\omega}}_{\deg^{\underline{\omega}}(d)}) = \sigma^{\underline{\omega}}\big(d^{\underline{\omega}}_{\deg^{\underline{\omega}}(d)}(d)\big)$.
(b) $\sigma^{\underline{\omega}}(d + e) = \begin{cases} \sigma^{\underline{\omega}}(d) + \sigma^{\underline{\omega}}(e) & \text{if } \deg^{\underline{\omega}}(d) = \deg^{\underline{\omega}}(e) = \deg^{\underline{\omega}}(d + e) \\ \sigma^{\underline{\omega}}(d) & \text{if } \deg^{\underline{\omega}}(d) > \deg^{\underline{\omega}}(e). \end{cases}$

First, we now prove that symbols behave well with respect to products of differential operators.

13.4. **Proposition.** ***(Multiplicativity of Symbols)*** *Let $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ and let $d, e \in \mathbb{W}$. Then*

$$\sigma^{\underline{\omega}}(de) = \sigma^{\underline{\omega}}(d)\sigma^{\underline{\omega}}(e).$$

Proof. If $d = 0$ or $e = 0$, our claim is obvious. So, let $d, e \neq 0$. We write $i := \deg^{\underline{\omega}}(d)$ and $j := \deg^{\underline{\omega}}(e)$. Observe that $\deg^{\underline{\omega}}(de) = i + j$. So, by Definition and Exercise 13.3 (A)(e) we have

$$\begin{aligned}
\sigma^{\underline{\omega}}(de) &= \sigma^{\underline{\omega}}_{i+j}(de) = \Phi\big((de)^{\underline{\omega}}_{i+j}\big) = \\
&= \Phi\Big(\sum_{(\underline{\nu},\underline{\mu})\in \mathrm{supp}^{\underline{\omega}}_i(d),(\underline{\nu}',\underline{\mu}')\in \mathrm{supp}^{\underline{\omega}}_j(e)} c^{(d)}_{\underline{\nu}\underline{\mu}} c^{(e)}_{\underline{\nu}'\underline{\mu}'} \underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'}\Big) = \\
&= \sum_{(\underline{\nu},\underline{\mu})\in \mathrm{supp}^{\underline{\omega}}_i(d),(\underline{\nu}',\underline{\mu}')\in \mathrm{supp}^{\underline{\omega}}_j(e)} c^{(d)}_{\underline{\nu}\underline{\mu}} c^{(e)}_{\underline{\nu}'\underline{\mu}'} \underline{Y}^{\underline{\nu}+\underline{\nu}'} \underline{Z}^{\underline{\mu}+\underline{\mu}'} = \\
&= \Big(\sum_{(\underline{\nu},\underline{\mu})\in \mathrm{supp}^{\underline{\omega}}_i(d)} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}}\Big)\Big(\sum_{(\underline{\nu}',\underline{\mu}')\in \mathrm{supp}^{\underline{\omega}}_j(e)} c^{(e)}_{\underline{\nu}'\underline{\mu}'} \underline{Y}^{\underline{\nu}'} \underline{Z}^{\underline{\mu}'}\Big) = \\
&= \Phi(d^{\underline{\omega}}_i)\Phi(e^{\underline{\omega}}_j) = \sigma^{\underline{\omega}}_i(d)\sigma^{\underline{\omega}}_j(e) = \sigma^{\underline{\omega}}(d)\sigma^{\underline{\omega}}(e).
\end{aligned}$$

□

In Definition and Remark 11.5 we have seen, that each left ideal L of the standard Weyl algebra $\mathbb{W}$ induces a graded ideal in the associated graded ring with respect to a given weight. These induced ideals will play a crucial role in our future considerations. We just revisit now these ideals.

13.5. **Reminder, Definition and Exercise.** (A) *(Induced Graded Ideals)* Let $L \subset \mathbb{W}$ be a left ideal, let $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ be a weight and let us consider the $\underline{\omega}$-graded ideal (see Definition and Remark 11.5)

$$\mathbb{G}^{\underline{\omega}}(L) := \bigoplus_{i\in\mathbb{Z}} \big((L \cap \mathbb{W}^{\underline{\omega}}_i) + \mathbb{W}^{\underline{\omega}}_{i-1}\big)/\mathbb{W}^{\underline{\omega}}_{i-1} \cong \bigoplus_{i\in\mathbb{Z}} L^{\underline{\omega}}_i / L^{\underline{\omega}}_{i-1} = \mathrm{Gr}_{L^{\underline{\omega}}_\bullet}(L) \subseteq \mathbb{G}^{\underline{\omega}}(\mathbb{W}),$$

where

$$L^{\underline{\omega}}_\bullet = L \cap \mathbb{W}^{\underline{\omega}}_\bullet := \big(L \cap \mathbb{W}^{\underline{\omega}}_i\big)_{i\in\mathbb{N}_0}$$

is the filtration induced on L by the weighted filtration $\mathbb{W}^{\underline{\omega}}_\bullet$. We now consider the $\underline{\omega}$-graded ideal of $\mathbb{P}^{\underline{\omega}} = \mathbb{P}$ given by

$$\overline{\mathbb{G}}^{\underline{\omega}}(L) := (\eta^{\underline{\omega}})^{-1}\big(\mathbb{G}^{\underline{\omega}}(L)\big),$$

where

$$\eta^{\underline{vw}} = \eta^{\underline{\omega}} : \mathbb{P} = \mathbb{P}^{\underline{\omega}} \xrightarrow{\cong} \mathbb{G}^{\underline{\omega}}.$$

is the canonical isomorphism of graded rings of Theorem 9.4. We call $\overline{\mathbb{G}}^{\underline{\omega}}(L)$ the ($\underline{\omega}$-graded) *ideal induced by* L in $\mathbb{P}$.

(B) Let the notations and hypotheses be as part (A). Fix $i \in \mathbb{N}_0$ and consider the i-th $\underline{\omega}$-graded part

$$\overline{\mathbb{G}}^{\underline{\omega}}(L)_i = \overline{\mathbb{G}}^{\underline{\omega}}(L) \cap \mathbb{P}^{\underline{\omega}}_i = (\eta^{\underline{\omega}})^{-1}\big(\mathbb{G}^{\underline{\omega}}_i\big)$$

of the ideal $\overline{\mathbb{G}}^{\underline{\omega}}(L) \subseteq \mathbb{P}$. Prove the following statements:

(a) Let $d \in L$ with $\deg^{\underline{\omega}}(d) = i$ and let $\overline{d} := d + \mathbb{W}^{\underline{\omega}}_{i-1} \in \mathbb{G}^{\underline{\omega}}(L)_i$. Then it holds

$$(\eta^{\underline{\omega}})^{-1}(\overline{d}) = \Phi(d^{\underline{\omega}}_i) = \sigma^{\underline{\omega}}(d) \in \overline{\mathbb{G}}^{\underline{\omega}}(L)_i.$$

(b) Each element $h \in \mathbb{G}^{\underline{\omega}}(L)_i \setminus \{0\}$ can be written as

$$h = \sigma^{\underline{\omega}}(d), \text{ with } d \in L \text{ and } \deg^{\underline{\omega}}(d) = i.$$

(C) *(The Induced Exact Sequence Associated to a Left Ideal with Respect to a Weight)* Keep the above notations and hypotheses. Prove the following statements:

(a) There is a short exact sequence of graded $\mathbb{P}^{\underline{\omega}}$-modules

$$0 \longrightarrow \overline{\mathbb{G}}^{\underline{\omega}}(L) \longrightarrow \mathbb{G}^{\underline{\omega}} \longrightarrow \mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_{\bullet} K\overline{1}}(\mathbb{W}/L) \longrightarrow 0,$$

where $\overline{1} := 1 + L \in \mathbb{W}/L$ and $\mathbb{W}^{\underline{\omega}}_{\bullet} K\overline{1}$ is the $\underline{\omega}$ -filtration induced on the cyclic D-module $\mathbb{W}/L$ by its subspace $K\overline{1}$.

(b) $\mathrm{Ann}_{\mathbb{P}}\big(\mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_{\bullet} K\overline{1}}(\mathbb{W}/L)\big) = \overline{\mathbb{G}}^{\underline{\omega}}(L)$.

(c) $\mathbb{V}^{\underline{\omega}}(\mathbb{W}/L) = \mathrm{Var}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big)$.

We call this sequence the *short exact sequence associated to the left ideal L with respect to the weight* $\underline{\omega}$.

Now, we are ready to formulate and to prove a result which we already announced in the introduction to this section. It relates the symbols of the members of a Gröbner bases of a left ideal with the induced ideal with respect to a given weight.

13.6. **Proposition.** ***(Generation of the Induced Ideal by the Symbols of a Gröbner Basis)*** *Let $\underline{\omega} \in \Omega$, let $L \subseteq \mathbb{W}$ be a left ideal, let $\leq \in \mathrm{AO}(\mathbb{E})$ and let G be a $\leq^{\underline{\omega}}$-Gröbner basis of L. Then it holds*

(a) $\overline{\mathbb{G}}^{\underline{\omega}}(L) = \sum_{g \in G} \mathbb{P}\sigma^{\underline{\omega}}(g)$.

(b) *For each $h \in \overline{\mathbb{G}}^{\underline{\omega}}(L) \setminus \{0\}$ there is some $g \in G \setminus \{0\}$ and some monomial $m = \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \in \mathbb{P}$ such that*

$$\mathrm{LM}_{\leq}\big(\Phi^{-1}(h)\big) = m\mathrm{LM}_{\leq}\big(\Phi^{-1}(\sigma^{\underline{\omega}}(g))\big).$$

Proof. (a): As the ideal $\overline{\mathbb{G}}^{\underline{\omega}}(L) \subseteq \mathbb{P}^{\underline{\omega}}$ is graded, it suffices to show, that for each $i \in \mathbb{N}_0$ and each $h \in \overline{\mathbb{G}}^{\underline{\omega}}(L)_i \setminus \{0\}$ we have $h \in \sum_{g \in G} \mathbb{P}\sigma^{\underline{\omega}}(g)$. So, fix $i \in \mathbb{N}_0$ and assume that $h \notin \sum_{g \in G} \mathbb{P}\sigma^{\underline{\omega}}(g)$ for some $h \in \overline{\mathbb{G}}^{\underline{\omega}}(L)_i \setminus \{0\}$. Then, by Reminder, Definition and Exercise 13.5 (B)(b), the set

$$\mathfrak{S} := \{e \in L \mid \deg^{\underline{\omega}}(e) = i \text{ and } \sigma^{\underline{\omega}}(e) \notin \sum_{g \in G} \mathbb{P}\sigma^{\underline{\omega}}(g)\}$$

is not empty. Choose $d \in \mathfrak{S}$ such that

$$\mathrm{LE}_{\leq^{\underline{\omega}}}(d) = \min_{\leq^{\underline{\omega}}}\{\mathrm{LE}_{\leq^{\underline{\omega}}}(e) \mid e \in \mathfrak{S}\}.$$

As G is a $\leq^{\underline{\omega}}$-Gröbner basis of L we find some $g \in G$ and some $u \in \mathbb{E}$ such that $\mathrm{LM}_{\leq^{\underline{\omega}}}(d) = \mathrm{LM}_{\leq^{\underline{\omega}}}(ug)$ (see Definition, Reminder and Exercise 12.6 (C)(b)). With

$$v := \frac{\mathrm{LC}_{\leq^{\underline{\omega}}}(d)}{\mathrm{LC}_{\leq^{\underline{\omega}}}(g)} u$$

it follows that $\mathrm{LE}_{\leq^{\underline{\omega}}}(d) = \mathrm{LE}_{\leq^{\underline{\omega}}}(vg)$, hence

$$\mathrm{LD}_{\leq^{\underline{\omega}}}(d) = \mathrm{LC}_{\leq^{\underline{\omega}}}(d)\mathrm{LE}_{\leq^{\underline{\omega}}}(d) = \mathrm{LC}_{\leq^{\underline{\omega}}}(d)\mathrm{LE}_{\leq^{\underline{\omega}}}(ug) = \mathrm{LD}_{\leq^{\underline{\omega}}}(vg) \text{ and } \deg^{\underline{\omega}}(vg) = i.$$

So, by Definition, Reminder and Exercise 12.2 (D)(d) we may conclude that either

(1) $\deg^{\underline{\omega}}(d - vg) < i$, or else
(2) $\deg^{\underline{\omega}}(d - vg) = i$ and $\mathrm{LE}_{\leq^{\underline{\omega}}}(d - vg) < \mathrm{LE}_{\leq^{\underline{\omega}}}(d)$.

In the case (1) we have (see Definition and Exercise 13.3 (C)(b) and Proposition 13.4)

$$\sigma^{\underline{\omega}}(d) = \sigma^{\underline{\omega}}(d - (d - vg)) = \sigma^{\underline{\omega}}(vg)) = \sigma^{\underline{\omega}}(v)\sigma^{\underline{\omega}}(g) \in \sum_{g\in G} \mathbb{P}\sigma^{\underline{\omega}}(g)$$

and hence get a contradiction.
So, assume that we are in the case (2). As $d - vg \in L$ it follows by our choice of d, that $\sigma^{\underline{\omega}}(d - vg) \in \sum_{g\in G} \mathbb{P}\sigma^{\underline{\omega}}(g)$. Observe that we have

$$i = \deg^{\underline{\omega}}(d - vg) = \deg^{\underline{\omega}}(vg) = \deg^{\underline{\omega}}(d) = \deg^{\underline{\omega}}\big((d - vg) + vg\big).$$

So, by Definition and Exercise 13.3 (C)(b) and by Proposition 13.4 we have

$$\sigma^{\underline{\omega}}(d) = \sigma^{\underline{\omega}}\big((d-vg)+vg\big) = \sigma^{\underline{\omega}}(d-vg)+\sigma^{\underline{\omega}}(vg) = \sigma^{\underline{\omega}}(d-vg)+\sigma^{\underline{\omega}}(v)\sigma^{\underline{\omega}}(g) \in \sum_{g\in G} \mathbb{P}\sigma^{\underline{\omega}}(g),$$

and this is again a contradiction.

(b): We find some $i \in \mathbb{N}_0$ such that $\mathrm{LM}_{\leq}\big(\Phi^{-1}(h)\big) = \mathrm{LM}_{\leq}\big(\Phi^{-1}(h_i^{\underline{\omega}}(h))\big)$. As the ideal $\overline{\mathbb{G}}^{\underline{\omega}}(L) \subseteq \mathbb{P}^{\underline{\omega}}$ is graded, we have $h_i^{\underline{\omega}}(h) \in \overline{\mathbb{G}}^{\underline{\omega}}(L)$. So we may assume, that $h \in \overline{\mathbb{G}}^{\underline{\omega}}(L)_i \setminus \{0\}$. Now, by Reminder, Definition and Exercise 13.5 (B), we find some $d \in L$ with $\deg^{\underline{\omega}}(d) = i$ and $\Phi^{-1}(h) = d_i^{\underline{\omega}}$, whence

$$\mathrm{LM}_{\leq}\big(\Phi^{-1}(h)\big) = \mathrm{LM}_{\leq}(d_i^{\underline{\omega}}) = \mathrm{LM}_{\leq^{\underline{\omega}}}(d).$$

As G is a $\leq^{\underline{\omega}}$-Gröbner basis of L, we find some $g \in G \setminus \{0\}$ with $\deg^{\underline{\omega}}(g) = j$ and some monomial $m = \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \in \mathbb{P}$ such that (see Definition, Reminder and Exercise 12.6 (C)(c) and also Definition and Exercise 13.3 (C)(a))

$$\mathrm{LM}_{\leq^{\underline{\omega}}}(d) = m\mathrm{LM}_{\leq^{\underline{\omega}}}(g) = m\mathrm{LM}_{\leq}(g_j^{\underline{\omega}}) = m\mathrm{LM}_{\leq}\big(\Phi^{-1}(\sigma_j^{\underline{\omega}}(g))\big),$$

and so we get our claim. □

Now, we are ready to prove our first basic finiteness result. It says that the set of all induced ideals of a given left ideal in the Weyl algebra is finite.

13.7. **Corollary.** ***(Finiteness of the Set of Induced Ideals)*** *Let $L \subseteq \mathbb{W}$ be a left ideal. Then, the following statements hold:*

(a) $\#\{\overline{\mathbb{G}}^{\underline{\omega}}(L) \mid \underline{\omega} \in \Omega\} < \infty$.
(b) $\#\{\mathbb{V}^{\underline{\omega}}(\mathbb{W}/L) \mid \underline{\omega} \in \Omega\} < \infty$.

Proof. (a): Let G be an universal Gröbner basis of L. Then, by Proposition 13.6, for each $\underline{\omega} \in \Omega$ we have $\overline{\mathbb{G}}^{\underline{\omega}}(L) = \sum_{g\in G} \mathbb{P}\sigma^{\underline{\omega}}(g)$. For each $g \in G$ we write

$$g = \sum_{(\underline{\nu},\underline{\mu})\in \mathrm{supp}(g)} c_{\underline{\nu}\underline{\mu}}^{(g)} \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}.$$

Then, for each $\underline{\omega} \in \Omega$ we have

$$\sigma^{\underline{\omega}}(g) = \Phi(g^{\underline{\omega}}_{\deg_{\underline{\omega}}(g)}) = \sum_{(\underline{\nu},\underline{\mu}) \in \operatorname{supp}^{\underline{\omega}}_{\deg^{\underline{\omega}}(g)}(g)} c^{(g)}_{\underline{\nu}\underline{\mu}} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}}.$$

Therefore

$$\#\{\sigma^{\underline{\omega}}(g) \mid \underline{\omega} \in \Omega\} \leq \#\{H \subseteq \operatorname{supp}(g)\} = 2^{\#\operatorname{supp}(g)}.$$

It follows that

$$\#\{\overline{\mathbb{G}}^{\underline{\omega}}(L) = \sum_{g \in G} \mathbb{P}\sigma^{\underline{\omega}}(g) \mid \underline{\omega} \in \Omega\} \leq \#\{\big(\sigma^{\underline{\omega}}(g)\big)_{g \in G} \in \mathbb{P}^G \mid \underline{\omega} \in \Omega\} \leq$$
$$\leq \prod_{g \in G} 2^{\#\operatorname{supp}(g)} = 2^{\#\operatorname{supp}(G)}.$$

(b): This follows immediately from statement (a) on use of Reminder, Definition and Exercise 13.5 (C)(c). ☐

The second statement of the previous result says that a given cyclic D-module has only finitely many characteristic varieties, if $\underline{\omega}$ runs through all weights. Our first main theorem says, that this finiteness statement holds indeed for arbitrary D-modules. To prove this, we first have to investigate the behavior of characteristic varieties in short exact sequences of D-modules. This needs some preparations.

13.8. **Exercise and Definition.** (A) Let $\underline{\omega} \in \Omega$ and let

$$0 \longrightarrow Q \xrightarrow{\iota} U \xrightarrow{\pi} P \longrightarrow 0$$

be an exact sequence of D-modules. Let $V \subseteq U$ be a finitely generated K-vector subspace such that $U = \mathbb{W}V$. We endow Q with the filtration

$$Q_\bullet := \big(\iota^{-1}(\mathbb{W}^{\underline{\omega}}_i V)\big)_{i \in \mathbb{N}_0}.$$

Prove the following statements:

(a) For each $i \in \mathbb{N}_0$ there is a K-linear map

$$\overline{\iota}_i : Q_i/Q_{i-1} \longrightarrow \mathbb{W}^{\underline{\omega}}_i V/\mathbb{W}^{\underline{\omega}}_{i-1} V, \quad q + Q_{i-1} \mapsto \iota(q) + \mathbb{W}^{\underline{\omega}}_{i-1} V.$$

(b) For each $i \in \mathbb{N}_0$ there is a K-linear map

$$\overline{\pi}_i : \mathbb{W}^{\underline{\omega}}_i V/\mathbb{W}^{\underline{\omega}}_{i-1} V \longrightarrow \mathbb{W}^{\underline{\omega}}_i \pi(V)/\mathbb{W}^{\underline{\omega}}_{i-1}\pi(V), \quad q + \mathbb{W}^{\underline{\omega}}_{i-1} V \mapsto \pi(q) + \mathbb{W}^{\underline{\omega}}_{i-1}\pi(V).$$

(c) For each $i \in \mathbb{N}_0$ it holds

$$\pi^{-1}\big(\mathbb{W}^{\underline{\omega}}_{i-1}\pi(V)\big) = \iota(Q) + \mathbb{W}^{\underline{\omega}}_{i-1} V.$$

(d) For each $i \in \mathbb{N}_0$ there is a short exact sequence of K-vector spaces

$$0 \longrightarrow Q_i/Q_{i-1} \xrightarrow{\overline{\iota}_i} \mathbb{W}^{\underline{\omega}}_i V/\mathbb{W}^{\underline{\omega}}_{i-1} V \xrightarrow{\overline{\pi}_i} \mathbb{W}^{\underline{\omega}}_i \pi(V)/\mathbb{W}^{\underline{\omega}}_{i-1}\pi(V) \longrightarrow 0.$$

(B) *(The Graded Exact Sequence associated to a Short Exact Sequence of D-Modules)* Keep the hypotheses and notations of part (A). Prove the following statements:

(a) For each $i \in \mathbb{N}_0$ there is a short exact sequence of K-vector spaces

$$0 \longrightarrow \operatorname{Gr}_{Q_\bullet}(Q)_i \xrightarrow{\overline{\iota}_i} \operatorname{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet V}(U)_i \xrightarrow{\overline{\pi}_i} \operatorname{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet \pi(V)}(P)_i \longrightarrow 0.$$

(b) There is an exact sequence of graded $\mathbb{P}^{\underline{\omega}}$-modules

$$0 \longrightarrow \mathrm{Gr}_{Q_\bullet}(Q) \xrightarrow{\bar{\iota}} \mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet V}(U) \xrightarrow{\bar{\pi}} \mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet \pi(V)}(P) \longrightarrow 0,$$

with $\bar{\iota} := \bigoplus_{i\in\mathbb{N}_0} \bar{\iota}_i$ and $\bar{\pi} := \bigoplus_{i\in\mathbb{N}_0} \bar{\pi}_i$.

The exact sequence of statement (b) is called the *exact sequence induced by the exact sequence* $0 \to Q \xrightarrow{\iota} U \xrightarrow{\pi} P \to 0$ and the generating vector space V of U.

(C) Keep the previous notations and hypotheses. Prove the following statements:

(a) For each finitely generated K-vector subspace $T \subseteq Q$ with $Q = \mathbb{W}T$ and $V \subseteq \iota(T)$, the two filtrations $Q_\bullet$ and $\mathbb{W}^{\underline{\omega}}_\bullet T$ of Q are equivalent.

(b) $\mathrm{Var}\big(\mathrm{Ann}_{\mathbb{P}}(\mathrm{Gr}_{Q_\bullet}(Q))\big) = \mathbb{V}^{\underline{\omega}}(Q)$.

Now, we can prove the crucial result, needed to extend the previous finiteness statement for characteristic varieties from cyclic to arbitrary D-modules.

13.9. **Proposition.** ***(Additivity of Characteristic Varieties)*** *Let* $\underline{\omega} \in \Omega$ *and let*

$$0 \longrightarrow Q \xrightarrow{\iota} U \xrightarrow{\pi} P \longrightarrow 0$$

be an exact sequence of D*-modules. Then it holds*

$$\mathbb{V}^{\underline{\omega}}(U) = \mathbb{V}^{\underline{\omega}}(Q) \cup \mathbb{V}^{\underline{\omega}}(P).$$

Proof. We fix a finitely generated K-vector subspace $V \subseteq U$ with $\mathbb{W}V = U$ and consider the corresponding induced short exact sequence (see Exercise and Definition 13.8 (B))

$$0 \longrightarrow \mathrm{Gr}_{Q_\bullet}(Q) \xrightarrow{\bar{\iota}} \mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet V}(U) \xrightarrow{\bar{\pi}} \mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet \pi(V)}(P) \longrightarrow 0.$$

On use of Exercise and Definition 13.8 (C)(b) we obtain

$$\mathbb{V}^{\underline{\omega}}(U) = \mathrm{Var}\big(\mathrm{Ann}_{\mathbb{P}}(\mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet V}(U))\big) =$$
$$= \mathrm{Var}\big(\mathrm{Ann}_{\mathbb{P}}(\mathrm{Gr}_{Q_\bullet}(Q))\big) \cup \mathrm{Var}\big(\mathrm{Ann}_{\mathbb{P}}(\mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet \pi(V)}(P))\big) = \mathbb{V}^{\underline{\omega}}(Q) \cup \mathbb{V}^{\underline{\omega}}(P).$$

□

Now, we are ready to prove the announced first main theorem of this section.

13.10. **Theorem.** ***(Finiteness of the Set of Characteristic Varieties)*** *Let* U *be a* D*-module. Then*

$$\#\{\mathbb{V}^{\underline{\omega}}(U) \mid \underline{\omega} \in \Omega\} < \infty.$$

Proof. We proceed by induction on the number r of generators of U. If $r = 1$ we have $U \cong \mathbb{W}/L$ for some left ideal $L \subseteq \mathbb{W}$. In this case, we may conclude by Corollary 13.7 (b). So, let $r > 1$. Then, we find a short exact of D-modules

$$0 \longrightarrow Q \xrightarrow{\iota} U \xrightarrow{\pi} P \longrightarrow 0$$

such that Q and P are generated by less than r elements. By induction, we have

$$\#\{\mathbb{V}^{\underline{\omega}}(Q) \mid \underline{\omega} \in \Omega\} < \infty \text{ and } \#\{\mathbb{V}^{\underline{\omega}}(P) \mid \underline{\omega} \in \Omega\} < \infty.$$

By Proposition 13.9 we also have

$$\{\mathbb{V}^{\underline{\omega}}(U) \mid \underline{\omega} \in \Omega\} = \{\mathbb{V}^{\underline{\omega}}(Q) \cup \mathbb{V}^{\underline{\omega}}(P) \mid \underline{\omega} \in \Omega\},$$

hence

$$\#\{\mathbb{V}^{\underline{\omega}}(U) \mid \underline{\omega} \in \Omega\} \leq \#\{\mathbb{V}^{\underline{\omega}}(Q) \mid \underline{\omega} \in \Omega\} + \#\{\mathbb{V}^{\underline{\omega}}(P) \mid \underline{\omega} \in \Omega\} < \infty.$$

□

As already announced in the introduction to this section, our ultimate goal is to establish a certain stability result for characteristic varieties of a given D-module. To pave the way for this, we perform a number of preparatory considerations, which are the subject of the exercises to come.

13.11. **Definition and Exercise.** (A) *(Leading Forms)* We consider the polynomial ring $\mathbb{P}$. Let

$$f = \sum_{(\underline{\nu},\underline{\mu})\in \mathrm{supp}(f)} c^{(f)}_{\underline{\nu}\underline{\mu}} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}} \in \mathbb{P} \quad \text{with } c^{(f)}_{\underline{\nu}\underline{\mu}} \in K \setminus \{0\} \text{ for all } (\underline{\nu},\underline{\mu}) \in \mathrm{supp}(f).$$

We set

$$\mathrm{supp}^{\underline{\omega}}_i(f) := \{(\underline{\nu},\underline{\mu}) \in \mathrm{supp}(f) \mid \underline{\nu}\underline{v} + \underline{\mu}\underline{w} = i\}$$

and consider the $i-th$ *homogeneous component* of f with respect to ω, thus the polynomial

$$f^{\underline{\omega}}_i = f^{\underline{v}\underline{w}}_i := \sum_{(\underline{\nu},\underline{\nu})\in \mathrm{supp}^{\underline{\omega}}_i(f)} c^{(f)}_{\underline{\nu}\underline{\mu}} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}}.$$

The *leading form* of f with respect to the weight $\underline{\omega}$ is defined by

$$\mathrm{LF}^{\underline{\omega}}(f) := \begin{cases} 0 & \text{if } f = 0, \\ f^{\underline{\omega}}_{\deg^{\underline{\omega}}(f)} & \text{if } f \neq 0. \end{cases}$$

Prove that for all $f, g \in \mathbb{P}$, all $i, j \in \mathbb{N}_0$ and for all weights $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ the following statements hold:

(a) If $i > \deg^{\underline{\omega}}(f)$, then $f^{\underline{\omega}}_i = 0$.
(b) $f^{\underline{\omega}}_i = f^{\underline{\omega}}_i(f^{\underline{\omega}}_i)$.
(c) $(f+g)^{\underline{\omega}}_i = f^{\underline{\omega}}_i + g^{\underline{\omega}}_i$.
(d) $(fg)^{\underline{\omega}}_i = \sum_{j+k=i} f^{\underline{\omega}}_j g^{\underline{\omega}}_k$.
(e) $\mathrm{LF}^{\underline{\omega}}(fg) = \mathrm{LF}^{\underline{\omega}}(f)\mathrm{LF}^{\underline{\omega}}(g)$.
(f) $\mathrm{LF}(f) = f$ if and only if f is homogeneous with respect to the $\underline{\omega}$-grading of $\mathbb{P}$.
(g) If $d \in \mathbb{W}$, then $\sigma^{\underline{\omega}}(d) = \mathrm{LF}^{\underline{\omega}}\big(\Phi(d)\big)$.

(B) *(Leading Form Ideals)* Keep the notations and hypotheses of part (A). If $S \subset \mathbb{P}$ is any subset, we define the *leading form ideal* of S with respect to $\underline{\omega}$ by

$$\mathrm{LFI}^{\underline{\omega}}(S) := \sum_{f\in S} \mathbb{P}\mathrm{LF}^{\underline{\omega}}(f).$$

Let $S \subseteq T \subseteq \mathbb{P}$ and $\leq \in \mathrm{AO}_(\mathbb{E})$. Prove the following statements:

(a) $\mathrm{LFI}^{\underline{\omega}}(S) \subseteq \mathrm{LFI}^{\underline{\omega}}(T)$.
(b) If for each $t \in T \setminus \{0\}$ there is some monomial $m = \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \in \mathbb{M} \subset \mathbb{P}$ and some $s \in S$ such that $\mathrm{LM}_{\leq^{\underline{\omega}}}\big(\Phi^{-1}(t)\big) = m\mathrm{LM}_{\leq^{\underline{\omega}}}\big(\Phi^{-1}(s)\big)$, then $\mathrm{LFI}^{\underline{\omega}}(S) = \mathrm{LFI}^{\underline{\omega}}(T)$.
(c) For each ideal $I \subseteq \mathbb{P}$ it holds

$$\sqrt{\mathrm{LFI}^{\underline{\omega}}(I)} = \sqrt{\mathrm{LFI}^{\underline{\omega}}(\sqrt{I})}.$$

(d) If $I, J \subseteq \mathbb{P}$ are ideals, then
(1) $\mathrm{LFI}^{\underline{\omega}}(I \cap J) \subseteq \mathrm{LFI}^{\underline{\omega}}(I) \cap \mathrm{LFI}^{\underline{\omega}}(I)$ and $\mathrm{LFI}^{\underline{\omega}}(I)\mathrm{LFI}^{\underline{\omega}}(J) \subseteq \mathrm{LFI}^{\underline{\omega}}(IJ)$;
(2) $\sqrt{\mathrm{LFI}^{\underline{\omega}}(I \cap J)} = \sqrt{\mathrm{LFI}^{\underline{\omega}}(I) \cap \mathrm{LFI}^{\underline{\omega}}(J)} = \sqrt{\mathrm{LFI}^{\underline{\omega}}(I)} \cap \sqrt{\mathrm{LFI}^{\underline{\omega}}(J)}$.

The announced Stability Theorem for Characteristic Varieties we are heading for, concerns the behavior of characteristic varieties under certain changes of the involved weights. To prepare this new type of considerations, we suggest the following exercise.

13.12. **Exercise.** (A) Prove that for all $d \in \mathbb{W}$, all $i, j \in \mathbb{N}_0$, all $s \in \mathbb{N}$ and for all weights $\underline{\alpha} = (\underline{a}, \underline{b}), \underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ the following statements hold (For the unexplained notations see Definition and Exercise 13.3):

(a) $\mathrm{supp}\big([d_i^{\underline{\omega}}]_j^{\underline{\alpha}}\big) = \mathrm{supp}_i^{\underline{\omega}}(d) \bigcap \mathrm{supp}_j^{\underline{\alpha}}(d)$.
(b) $\mathrm{supp}\big([d_i^{\underline{\omega}}]_j^{\underline{\alpha}}\big) \subseteq \mathrm{supp}_{j+si}^{\underline{\alpha}+s\underline{\omega}}(d)$.
(c) If $i \geq \deg^{\underline{\omega}}(d)$, $j \geq \deg^{\underline{\alpha}}(d_i^{\underline{\omega}})$ and $s > \deg^{\underline{\alpha}}(d) - j$, then the inclusion of statement (b) becomes an equality.
(d) If $i \geq \deg^{\underline{\omega}}(d)$, $j \geq \deg^{\underline{\alpha}}\left(d_i^{\underline{\omega}}\right)$ and $s > \deg^{\underline{\alpha}}(d) - j$, then

$$[d_i^{\underline{\omega}}]_j^{\underline{\alpha}} = d_{j+si}^{\underline{\alpha}+s\underline{\omega}}.$$

(B) Prove on use of statements (a)–(d) of part (A) that for all $d \in \mathbb{W}$, all $i, j \in \mathbb{N}_0$, all $s \in \mathbb{N}$ and for all weights $\underline{\omega} = (\underline{v}, \underline{w}), \underline{\alpha} = (\underline{a}, \underline{b}) \in \Omega$ the following statements hold:

(a) $\sigma_j^{\underline{\alpha}}(d_i^{\underline{\omega}}) = \sum_{(\underline{\nu},\underline{\mu}) \in \mathrm{supp}_i^{\underline{\omega}}(d) \cap \mathrm{supp}_j^{\underline{\alpha}}(d)} c_{\underline{\nu}\underline{\mu}}^{(d)} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}} = \sigma_i^{\underline{\omega}}(d_j^{\underline{\alpha}})$.
(b) If $i \geq \deg^{\underline{\omega}}(d)$, $j \geq \deg^{\underline{\alpha}}(d_i^{\underline{\omega}})$ and $s > \deg^{\underline{\alpha}}(d) - j$, then

$$[\sigma_i^{\underline{\omega}}(d)]_j^{\underline{\alpha}} = \sigma_{j+si}^{\underline{\alpha}+s\underline{\omega}}(d).$$

The next two auxiliary results are of fairly technical nature. But they will play a crucial role in the proof of our Stability Theorem.

13.13. **Lemma.** *Let $\underline{\alpha}, \underline{\omega} \in \Omega$, let $d \in \mathbb{W} \setminus \{0\}$ and let $s \in \mathbb{N}$ such that*

$$s > \deg^{\underline{\alpha}}(d) - \deg^{\underline{\alpha}}\left(\sigma^{\underline{\omega}}(d)\right).$$

Then, the following statements hold:

(a) $\deg^{\underline{\alpha}+s\underline{\omega}}(d) = \deg^{\underline{\alpha}}\left(\sigma^{\underline{\omega}}(d)\right) + s\deg^{\underline{\omega}}(d)$.
(b) $\mathrm{LF}^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(d)\big) = \sigma^{\underline{\alpha}+s\underline{\omega}}(d)$.

Proof. We write

$$i := \deg^{\underline{\omega}}(d) \text{ and } j := \deg^{\underline{\alpha}}\left(\sigma^{\underline{\omega}}(d)\right).$$

Observe, that $\sigma^{\underline{\omega}}(d) = \sigma_i^{\underline{\omega}}(d) = \Phi(d_i^{\underline{\omega}})$, so that

$$j = \deg^{\underline{\alpha}}\left(\sigma^{\underline{\omega}}(d)\right) = \deg^{\underline{\alpha}}(d_i^{\underline{\omega}}) \text{ and also } s > \deg^{\underline{\alpha}}(d) - j.$$

Now, by Exercise 13.12 (B)(b) we obtain

$$\mathrm{LF}^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(d)\big) = [\sigma_i^{\underline{\omega}}(d)]_j^{\underline{\alpha}} = \sigma_{j+si}^{\underline{\alpha}+s\underline{\omega}}(d).$$

It remains to show that

$$j + si = \deg^{\underline{\alpha}+s\underline{\omega}}(d).$$

As $\mathrm{LF}^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(d)\big) \neq 0$ we have $\sigma_{j+si}^{\underline{\alpha}+s\underline{\omega}}(d) \neq 0$ and hence $j + si \leq \deg^{\underline{\alpha}+s\underline{\omega}}(d)$ (see Definition and Exercise 13.3 (B)(b)).

Assume that $j + si > \deg^{\underline{\alpha}+s\underline{\omega}}(d)$. Then, we may write $\deg^{\underline{\alpha}+s\underline{\omega}}(d) = k + si$, with $k > j$. It follows, that $s > \deg^{\underline{\alpha}}(d) - k$. On application of Exercise 13.12 (B)(b) we get that

$$[\sigma_i^{\underline{\omega}}(d)]_k^{\underline{\alpha}} = \sigma_{k+si}^{\underline{\alpha}+s\underline{\omega}}(d) = \sigma^{\underline{\alpha}+s\underline{\omega}}(d) \neq 0.$$

As $k > j = \deg^{\underline{\alpha}}\left(\sigma^{\underline{\omega}}(d)\right)$ we have $[\sigma_i^{\underline{\omega}}(d)]_k^{\underline{\alpha}} = 0$ (see Definition and Exercise 13.11 (A)(a)). This contradiction completes our proof. □

13.14. **Lemma.** *Let $L \subseteq \mathbb{W}$ be a left ideal, let $\underline{\alpha}, \underline{\omega} \in \Omega$, let $\leq \in \mathrm{AO}(\mathbb{E})$ and let G be a $(\leq^{\underline{\alpha}})^{\underline{\omega}}$-Gröbner basis of L. Then*

$$\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \mathrm{LFI}^{\underline{\alpha}}\big(\{\sigma^{\underline{\omega}}(g) \mid g \in G\}\big).$$

Proof. By Reminder, Definition and Exercise 13.5 (B)(a) we have

$$S := \{\sigma^{\underline{\omega}}(g) \mid g \in G \setminus \{0\}\} \subseteq \overline{\mathbb{G}}^{\underline{\omega}}(L) =: T$$

If we apply Proposition 13.6 (b) with $\leq^{\underline{\alpha}}$ instead of $\leq$, we see that for all $t \in T$ there is some monomial $m = \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \in \mathbb{M} \subset \mathbb{P}$ and some $s \in S$ such that $\mathrm{LM}_{\leq^{\underline{\alpha}}}\big(\Phi^{-1}(t)\big) = m\mathrm{LM}_{\leq^{\underline{\alpha}}}\big(\Phi^{-1}(s)\big)$. By Definition and Exercise 13.11 (B)(b) it follows that

$$\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \mathrm{LFI}^{\underline{\alpha}}(S) = \mathrm{LFI}^{\underline{\alpha}}(T) = \mathrm{LFI}^{\underline{\alpha}}\big(\{\sigma^{\underline{\omega}}(g) \mid g \in G\}\big).$$

□

Now, we are ready to formulate and to prove the announced stability result.

13.15. **Theorem.** ***(Stability of Induced Graded Ideals, Boldini** [11]**,** [12]**)** Let $L \subseteq \mathbb{W}$ be a left ideal and let $\underline{\alpha} \in \Omega$. Then, there exists an integer $\overline{s} = \overline{s}(\underline{\alpha}, L) \in \mathbb{N}_0$ such that for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$ we have*

$$\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \overline{\mathbb{G}}^{\underline{\alpha}+s\underline{\omega}}(L).$$

Proof. Let G be a universal Gröbner basis of L. Then, by Lemma 13.14, for each $\underline{\omega} \in \Omega$ we have

$$\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \mathrm{LFI}^{\underline{\alpha}}\big(\{\sigma^{\underline{\omega}}(g) \mid g \in G\}\big) = \sum_{g \in G} \mathbb{P}\mathrm{LF}^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(g)\big).$$

Now, we set

$$\overline{s} := \max\{\deg^{\underline{\alpha}}(g) \mid g \in G \setminus \{0\}\}.$$

By Lemma 13.13 it follows that $\mathrm{LF}^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(g)\big) = \sigma^{\underline{\alpha}+s\underline{\omega}}(g)$ for all $s \in \mathbb{N}$ with $s > \overline{s}$, all $\underline{\omega} \in \Omega$ and all $g \in G \setminus \{0\}$. So, for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$ we have

$$\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \sum_{g \in G} \mathbb{P}\sigma^{\underline{\alpha}+s\underline{\omega}}(g).$$

If we apply Proposition 13.6 (a) with $\underline{\alpha} + s\underline{\omega}$ instead of $\underline{\omega}$ we also get

$$\overline{\mathbb{G}}^{\underline{\alpha}+s\underline{\omega}}(L) = \sum_{g \in G} \mathbb{P}\sigma^{\underline{\alpha}+s\underline{\omega}}(g)$$

for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$. This completes our proof. □

13.16. **Notation.** If $\mathfrak{Z} \subseteq \mathrm{Spec}(\mathbb{P})$ is a closed set we denote the *vanishing ideal* of $\mathfrak{Z}$ by $I_{\mathfrak{Z}}$, thus:

$$I_{\mathfrak{Z}} := \bigcap_{\mathfrak{p}\in\mathfrak{Z}} \mathfrak{p} = \sqrt{J}, \text{ for all ideals } J \subseteq \mathbb{P} \text{ with } \mathfrak{Z} = \mathrm{Var}(J).$$

13.17. **Theorem.** ***(Stability of Characteristic Varieties, Boldini*** [11], [12]***)*** *Let U be a D-module, and let $\underline{\alpha} \in \Omega$. Then, there exists an integer $\overline{s} = \overline{s}(\underline{\alpha}, U) \in \mathbb{N}_0$ such that for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$ we have*

$$\mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}\big(I_{\mathbb{V}^{\underline{\omega}}(U)}\big)\big) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(U).$$

Proof. We proceed by induction on the number r of generators of U. First, let $r = 1$. Then we have $U \cong \mathbb{W}/L$ for some left ideal $L \subseteq \mathbb{W}$. By Theorem 13.15 we find some $\overline{s} \in \mathbb{N}_0$ such that for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$ we have

$$\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \overline{\mathbb{G}}^{\underline{\alpha}+s\underline{\omega}}(L).$$

By Reminder, Definition and Exercise 13.5 (C)(c) we have

$$\mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(U) = \mathrm{Var}\big(\overline{\mathbb{G}}^{\underline{\alpha}+s\underline{\omega}}(L)\big) \text{ and } I_{\mathbb{V}^{\underline{\omega}}(U)} = \sqrt{\overline{\mathbb{G}}^{\underline{\omega}}(L)}.$$

By Definition and Exercise 13.11 (B)(c) we thus get

$$\sqrt{\mathrm{LFI}^{\underline{\alpha}}\big(I_{\mathbb{V}^{\underline{\omega}}(U)}\big)} = \sqrt{\mathrm{LFI}^{\underline{\alpha}}\big(\sqrt{\overline{\mathbb{G}}^{\underline{\omega}}(L)}\big)} = \sqrt{\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big)},$$

so that indeed – for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$ – we have

$$\mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}\big(I_{\mathbb{V}^{\underline{\omega}}(U)}\big)\big) = \mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \mathrm{Var}\big(\overline{\mathbb{G}}^{\underline{\alpha}+s\underline{\omega}}(L)\big) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(U).$$

Now, let $r > 1$. Then, we find a short exact of D-modules

$$0 \longrightarrow Q \xrightarrow{\iota} U \xrightarrow{\pi} P \longrightarrow 0$$

such that Q and P are generated by less than r elements. By induction, we thus find a number $\overline{s} \in \mathbb{N}_0$, such that for all $\underline{\omega} \in \Omega$ and all $s \in \mathbb{N}$ with $s > \overline{s}$ it holds

$$\mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}(I_{\mathbb{V}^{\underline{\omega}}(Q)})\big) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(Q) \text{ and } \mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}(I_{\mathbb{V}^{\underline{\omega}}(P)})\big) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(P).$$

By Proposition 13.9 we have

$$\mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(U) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(Q) \cup \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(P)$$

and hence, moreover

$$I_{\mathbb{V}^{\underline{\omega}}(U)} = I_{\mathbb{V}^{\underline{\omega}}(Q)\cup\mathbb{V}^{\underline{\omega}}(Q)} = I_{\mathbb{V}^{\underline{\omega}}(Q)} \cap I_{\mathbb{V}^{\underline{\omega}}(P)}.$$

By Definition and Exercise 13.11 (B)(d)(2) it follows from the last equality that

$$\sqrt{\mathrm{LFI}^{\underline{\alpha}}\big(I_{\mathbb{V}^{\underline{\omega}}(U)}\big)} = \sqrt{\mathrm{LFI}^{\underline{\alpha}}\big(I_{\mathbb{V}^{\underline{\omega}}(Q)}\big)} \cap \sqrt{\mathrm{LFI}^{\underline{\alpha}}\big(I_{\mathbb{V}^{\underline{\omega}}(P)}\big)}.$$

Therefore

$$\mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}(I_{\mathbb{V}^{\underline{\omega}}(U)})\big) = \mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}(I_{\mathbb{V}^{\underline{\omega}}(Q)})\big) \cup \mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}(I_{\mathbb{V}^{\underline{\omega}}(P)})\big).$$

It follows, that

$$\mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}(I_{\mathbb{V}^{\underline{\omega}}(U)})\big) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(Q) \cup \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(P) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(U)$$

for all $\underline{\omega} \in \Omega$ and all $s \in \mathbb{N}$ with $s > \overline{s}$. This completes the step of induction and hence proves our claim. □

To formulate our Stability Theorem in a more geometric manner, we introduce the following notion.

13.18. **Definition.** *(The Critical Cone)* Let $\mathfrak{Z} \subseteq \mathrm{Spec}(\mathbb{P})$ be a closed set. Then, the *critical cone* of $\mathfrak{Z}$ is defined as

$$\mathrm{CCone}(\mathfrak{Z}) := \mathrm{Var}\big(\mathrm{LFI}^{\underline{1}}(I_{\mathfrak{Z}})\big),$$

where $\underline{1} = (\underline{1}, \underline{1}) \in \Omega$ denotes the *standard weight.*

On use of the introduced terminology, we now can define our Stability Theorem as follows.

13.19. **Corollary.** ***(Affine Deformation of Characteristic Varieties to Critical Cones, Boldini*** [11]***,*** [12]***)*** *Let U be a D-module. Then, there is an integer $\overline{s} = \overline{s}(U) \in \mathbb{N}_0$ such that for all $\underline{\omega} \in \Omega$ and all $s \in \mathbb{N}$ with $s > \overline{s}$ it holds*

$$\mathbb{V}^{\underline{1}+s\underline{\omega}}(U) = \mathrm{CCone}\big(\mathbb{V}^{\underline{\omega}}(U)\big).$$

Proof. This is immediate by Theorem 13.17. □

14. Standard Degree and Hilbert Polynomials

In this section, we give an outlook to the relation between D-modules and Castelnuovo-Mumford regularity, which we mentioned in the introduction. We shall consider a situation, which is exclusively related to the standard degree filtration $\mathbb{W}_\bullet = \mathbb{W}_\bullet^{\deg} = \mathbb{W}_\bullet^{\underline{11}}$ of the underlying Weyl algebra $\mathbb{W}$. Having in mind to approach the bounding result for the degree of defining equations of characteristic varieties mentioned in the introduction, we shall restrict ourselves to consider D-modules U endowed with filtrations $V\mathbb{W}_\bullet$ induced by a finite-dimensional generating vector space V of U.

14.1. **Preliminary Remark.** (A) Let $n \in \mathbb{N}$, let K be a field of characteristic 0 and consider the standard Weyl algebra $\mathbb{W} = \mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n]$. Moreover let $\mathcal{A}$ be a ring of smooth functions in $X_1, X_2, \ldots, X_n$ over K (see Remark and Definition 11.11 (A)). One concern of Analysis is to study whole *families of differential equations. So for fixed $r, s \in \mathbb{N}$ one chooses a family $\mathbb{F} \subseteq \mathbb{W}^{s\times r}$ of matrices of differential operators. Then one studies all systems of equations (see Remark and Definition 11.11 (B))*

$$\mathcal{D}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ with } \mathcal{D} \in \mathbb{F}.$$

(B) Let the notations and hypotheses by as in part (A). One aspect of the above approach is to study the behavior of the characteristic varieties $\mathbb{V}^{\deg}(\mathcal{D}) := \mathbb{V}_{\mathbb{W}_\bullet^{\deg}}\big(U_{\mathcal{D}}\big)$ with respect to the degree filtration (see Definition and Remark 8.6 and Definition and Remark 11.2 (D)) of the D-module $U_{\mathcal{D}}$ defined by the matrix $\mathcal{D}$ (see Remark and Definition 11.11 (C)) if this latter runs through the family $\mathbb{F}$.

The goal of this section is to prove that the degree of hypersurfaces which cut out set-theoretically the characteristic variety $\mathbb{V}^{\deg}(\mathcal{D})$ is bounded, if $\mathcal{D}$ runs through appropriate families $\mathbb{F}$.

Below, we recall a few notions from Commutative Algebra.

14.2. **Reminder, Definition and Exercise.** *(Hilbert Functions, Hilbert Polynomials and Hilbert Coefficients for Modules over Very Well Filtered Algebras)* (A) Let K be a field and let $R = \bigoplus_{i\in\mathbb{N}_0} R_i$ be a homogeneous Noetherian K-algebra (see Conventions, Reminders and Notations 1.1 (I) for this notion) , so that $R_0 = K$ and $R = K[x_1, x_2, \ldots, x_r]$ with finitely many elements $x_1, x_2, \ldots, x_r \in R_1$. Moreover, let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded R-module. Then we denote the *Hilbert function* of M by h_M, so that $h_M(i) := \dim_K(M_i)$ for all $i \in \mathbb{Z}$. We denote by $P_M(X)$ the *Hilbert polynomial* of M, so that $h_M(i) = P_M(i)$ for all $i \gg 0$. Keep in mind that $\dim(M) = \dim\big(R/\mathrm{Ann}_R(M)\big)$ and

$$\deg\big(P_M(X)\big) = \begin{cases} \dim(M) - 1, & \text{if } \dim(M) > 0 \\ -\infty, & \text{if } \dim(M) \leq 0. \end{cases}.$$

The Hilbert polynomial $P_M(X)$ has a *binomial presentation*:

$$P_M(X) = \sum_{k=0}^{\dim(M)-1} (-1)^k e_k(M) \binom{X + \dim(M) - k - 1}{\dim(M) - k - 1} \quad \big(e_k(M) \in \mathbb{Z}, e_0(M) \geq 0\big).$$

The integer $e_k(M)$ is called the k-th *Hilbert coefficient* of M. If $\dim(M) > 0$, $e_0(M) > 0$ is called the *multiplicity* of M. Finally let us also introduce the *postulation number* of M, thus the number $\mathrm{pstln}(M) := \sup\{i \in \mathbb{Z} \mid h_M(i) \neq P_M(i)\}$.

(B) Now, let $(A, A_\bullet)$ be a very well filtered K-algebra (see Definition and Remark **??** (A)). Let U be a finitely generated (left) A-module. Chose a vector space $V \subseteq U$ of finite dimension such that $AV = U$. Then, the graded $\mathrm{Gr}_{A_\bullet}(A)$-module $\mathrm{Gr}_{A_\bullet V}(U)$ is generated by finitely many homogeneous elements of degree 0 (see Exercise and Definition 10.5 (B)(c)). So, by part (A) this graded module admits a Hilbert function $h_{U,A_\bullet V} := h_{\mathrm{Gr}_{A_\bullet V}(U)}$ with $h_{U,A_\bullet V}(i) := \dim_K\big(\mathrm{Gr}_{A_\bullet V}(U)_i\big)$ for all $i \in \mathbb{Z}$, the *Hilbert function* of U with respect to the filtration induced by V. Moreover, by part (A), the module $\mathrm{Gr}_{A_\bullet V}(U)$ admits a Hilbert polynomial, thus a polynomial $P_{U,A_\bullet V}(X) := P_{\mathrm{Gr}_{A_\bullet V}(U)}(X) \in \mathbb{Q}[X]$ with $h_{U,A_\bullet V}(i) = P_{U,A_\bullet V}(i)$ for all $i \gg 0$. We call this polynomial the *Hilbert polynomial* of U with respect to the filtration induced by V. Keep in mind that according to part (A) we have $d_{A_\bullet}(U) := \dim\big(\mathrm{Gr}_{A_\bullet V}(U)\big) = \dim\big(\mathbb{V}_{A_\bullet}(U)\big)$. Moreover the polynomial $P_{U,A_\bullet V}(X)$ has a binomial presentation:

$$P_{U,A_\bullet V}(X) = \sum_{k=0}^{d_{A_\bullet}(U)-1} (-1)^k e_k\big(U, A_\bullet V\big) \binom{X + d_{A_\bullet}(U) - k - 1}{d_{A_\bullet}(U) - k - 1} \quad \big(e_k(U, A_\bullet V) \in \mathbb{Z}\big).$$

The integer $e_k(U, A_\bullet V)$ is called the k-th *Hilbert coefficient* of U with respect to the filtration induced by V. Finally, keep in mind, that by part (A) we have $e_0(U, A_\bullet V) > 0$ if $d_{A_\bullet}(U) > 0$. In this situation the number $e_0(U, A_\bullet V)$ is called the *multiplicity* of U with respect to the filtration induced by V. For the sake of completeness, we set $e_0(U, A_\bullet V) := 0$

if $d_{A_\bullet}(U) \leq 0$. Finally, according to part (A) we define the *postulation number* of U with respect to the filtration induced by V:

$$\text{pstln}_{U,A_\bullet V}(U) := \text{pstln}(\text{Gr}_{A_\bullet V}(U)) := \sup\{i \in \mathbb{Z} \mid h_{U,A_\bullet V}(i) \neq P_{U,A_\bullet V}(i)\}.$$

(C) Keep the notations and hypotheses of part (B) and assume that $d_{A_\bullet}(U) > 0$. Prove the following claims.

(a) There is a polynomial $Q_{U,A_\bullet V}(X) \in \mathbb{Q}[X]$ such that:
   (1) $\deg\big(Q_{U,A_\bullet V}(X)\big) = d_{A_\bullet}(U)$,
   (2) $\Delta\big(Q_{U,A_\bullet V}(X)\big) := Q_{U,A_\bullet V}(X) - Q_{U,A_\bullet V}(X-1) = P_{U,A_\bullet V}(X)$ and
   (3) $\dim_K(A_iV) = Q_{U,A_\bullet V}(i)$ for all $i \gg 0$.
   (4) For each $t \in \mathbb{Z}$ the polynomial $Q_{U,A_\bullet V}(X+t) \in \mathbb{Q}[X]$ has leading term $\frac{e_0(U,A_\bullet V)}{d_{A_\bullet}(U)!}X^{d_{A_\bullet}(U)}$.

(Hint: Observe that for all $i \in \mathbb{N}$ we have $\dim_K(A_iV) = \sum_{j=0}^{i} \dim_K\big(\text{Gr}_{A_\bullet V}(U)_j\big) = \sum_{j=0}^{i} h_{U,A_\bullet}(j)\big)$.)

(b) The multiplicity $e_{A_\bullet}(U) := e_0(U, A_\bullet V)$ is the same for each finite dimensional K-subspace $V \subseteq U$ with $AV = U$.
(Hint: Let $V^{(1)}, V^{(2)} \subset U$ be two finite dimensional K-subspaces such that $AV^{(1)} = AV^{(2)} = U$. Use Exercise and Definition 10.5 (C)(a) and Definition and Remark 10.1 (C)(a) to find some $r \in \mathbb{N}_0$ such that for all $i \in \mathbb{Z}$ it holds $A_{i-r}V^{(1)} \subseteq A_iV^{(2)} \subseteq A_{i+r}V^{(1)}$. Then apply (a).)

(D) Let $A = \mathbb{W} = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n]$ and let $A_\bullet = \mathrm{W}_\bullet = \mathrm{W}_\bullet^{\underline{11}}$ be the standard degree filtration of $\mathbb{W}$ (see Definition and Remark 8.6). Let $U = K[X_1, X_2, \ldots, X_n]$ be the D-module of Example 11.9. Compute the two polynomials $P_{U,A_\bullet K}(X)$ and $Q_{U,A_\bullet K}(X)$.

The next Exercise and Remark intends to present the *Bernstein Inequality* and the related notion of *holonomic D-module*. For those readers, who aim to learn more abour these important subjects, we recommend to consult one of [9], [8], [24], [37] or [38].

14.3. **Exercise and Remark.** (A) Endow the Weyl algebra

$$\mathbb{W} = \mathbb{W}(K,n) = K[X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n]$$

with its standard degree filtration $\mathbb{W}_\bullet := \mathbb{W}_\bullet^{\deg}$ (see Definition and Remark 8.6). If $d \in \mathbb{W}$ write $\deg(d)$ for the standard degree $\deg^{\underline{11}}(d)$ of d. Use Exercise 6.4 (D) to prove the following statement:

If $d \in \mathbb{W} \setminus K$, then there is some $i \in \{1, 2, \ldots, n\}$ such that
$\deg\big([X_i, d]\big) = \deg(d) - 1$ or else $\deg\big([\partial_i, d]\big) = \deg(d) - 1$.

(B) *(The Bernstein Monomorphisms)* Keep the notations of part (A) and let U be a non-zero D-module over the Weyl algebra $\mathbb{W}$. Let $V \subseteq U$ be a K-vector space of finite dimension and endow U with the induced filtration $U_\bullet := \mathbb{W}_\bullet V$ (see Exercise and Definition 10.5 (A),(B) and Definition and Remark 11.2 (D)). Let $k \in \mathbb{N}_0$, let $d \in \mathbb{W}$ with $\deg(d) = k$ and let $i \in \{1, 2, \ldots, n\}$. Prove the following statement

(a) If $k > 0$ and $dU_k = 0$, then $[X_i, d]U_{k-1} = [\partial_i, d]U_{k-1} = 0$.

Use part (A) and statement (B)(a) to prove the following claim by induction on k:

(b) For each $k \in \mathbb{N}_0$ there is a K-linear injective map $\phi_k : \mathbb{W}_k \longrightarrow \mathrm{Hom}_K(U_k, U_{2k})$, given by $\phi_k(d)(u) := du$, for all $d \in \mathbb{W}_k$ and all $u \in U_k$.

(Hint: The existence of the linear map ϕ_k is easy to verify. The injectivity of ϕ_0 is obvious. If $k > 0$ and ϕ_k is not injective, part (A) and statement (B)(b) imply that ϕ_{k-1} is not injective.)

(C) *(The Bernstein Inequality)* Keep the previous notations. Use statement (B)(b) to prove

(a) For all $k \in \mathbb{N}_0$ it holds $\binom{k+2n}{2n} \leq \dim_K(U_k)\dim_K(U_{2k})$.

(Hint: Determine $\dim_K(\mathbb{W}_k)$ for all $k \in \mathbb{N}_0$ and keep in mind that for any two K-vector spaces S, T of finite dimension one has $\dim\big(\mathrm{Hom}_K(S,T)\big) = \dim_K(S)\dim_K(T)$.)
Use statement (a) and Reminder, Definition and Exercise 14.2 (C)(a) to prove Bernstein's Inequality:

(b) If $U \neq 0$, then $d_{\mathbb{W}_\bullet}(U) = d_{\mathbb{W}_\bullet^{1,1}}(U) \geq n$.

(D) *(Holonomic D-Modules)* Keep the above notations. It is immediate from the definition, that one always has the inequality $d_{\mathbb{W}_\bullet}(U) \leq 2n$. The D-module U is called *holonomic* if $d_{\mathbb{W}_\bullet}(U) \leq n$, hence if $U = 0$ or else (by Bernsteins' Inequality) $U \neq 0$ and $d_{\mathbb{W}_\bullet}(U) = n$. Holonomic D-modules are of particular interest and play a crucial role in many applications of D-modules. The result of Reminder, Definition and Exercise 14.2 (D) shows that the (simple!) D-module $U = K[X_1, X_2, \ldots, X_n]$ be the D-module of Example 11.9 is holonomic.
Use Proposition 13.9 to prove the following result:

(a) If $0 \longrightarrow Q \longrightarrow U \longrightarrow P \longrightarrow 0$ is an exact sequence of D-modules, then U is holonomic if and only Q and P are holonomic.

Accepting without proof the fact that all simple D-modules are holonomic, one can prove by statement (a) that a D-module U is holonomic if and only if it is of finite length, hence if and only if it admits a finite ascending chain $0 = U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_{l-1} \subsetneq U_l = U$ of submodules, such that U_i/U_{i-1} is simple for all all $i = 1, \ldots, l$.

We now recall some basics facts on Local Cohomology Theory. As a reference we suggest [18].

14.4. **Reminder.** *(Local Cohomology Modules)* (A) Let R be a commutative Noetherian ring and let $\mathfrak{a} \subset R$ be an ideal. The $\mathfrak{a}$-*torsion submodule* of an R-module M is given by

$$\Gamma_{\mathfrak{a}}(M) := \bigcup_{n \in \mathbb{N}_0} (0 :_M \mathfrak{a}^n) \cong \lim_{\overrightarrow{n}} \mathrm{Hom}_R(R/\mathfrak{a}^n, M).$$

Observe, that the assignment $M \mapsto \Gamma_{\mathfrak{a}}(M)$ gives rise to a covariant left-exact functor of R-modules (indeed a sub-functor of the identity functor) – called the $\mathfrak{a}$-*torsion functor* – so that for each short exact sequence of R-modules $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ we naturally have an exact sequence $0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow \Gamma_{\mathfrak{a}}(P)$.
If $i \in \mathbb{N}_0$, the *i-th local cohomology functor* $H^i_{\mathfrak{a}}(\bullet)$ with respect to the ideal $\mathfrak{a}$ can be defined as the i-th right derived functor $\mathcal{R}^i\Gamma_{\mathfrak{a}}(\bullet)$ of the $\mathfrak{a}$-torsion functor, so that for each

R-module M one has:

$$H^i_{\mathfrak{a}}(M) = \mathcal{R}^i\Gamma_{\mathfrak{a}}(M) \cong \lim_{\overrightarrow{n}} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M).$$

For each short exact sequence of R-modules $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ there is a natural exact sequence of R-modules

$$\begin{aligned} 0 \longrightarrow H^0_{\mathfrak{a}}(N) \longrightarrow H^0_{\mathfrak{a}}(M) \longrightarrow H^0_{\mathfrak{a}}(P) \longrightarrow H^1_{\mathfrak{a}}(N) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow H^1_{\mathfrak{a}}(P) \longrightarrow \\ \longrightarrow H^2_{\mathfrak{a}}(N) \longrightarrow H^2_{\mathfrak{a}}(M) \longrightarrow H^2_{\mathfrak{a}}(P) \longrightarrow H^3_{\mathfrak{a}}(N) \longrightarrow H^3_{\mathfrak{a}}(M) \longrightarrow H^3_{\mathfrak{a}}(P) \cdots, \end{aligned}$$

the *cohomology sequence* associated to the given short exact sequence. In particular, local cohomology commutes with finite direct sums.
Moreover, we have

(a) If $\sqrt{\mathfrak{a}} = \sqrt{\sum_{i=1}^r Rx_i}$ for some elements $x_1, x_2, \ldots, x_r \in R$, then $H^i_{\mathfrak{a}}(M) = 0$ for all $i > r$ and all R-modules M.
(b) $H^i_{\mathfrak{a}}(M) = 0$ for all $i > \dim(M)$ and all (finitely generated) R-modules M.

(B) *(Graded Local Cohomology)* Assume from now on, that the ring R of part (A) is (positively) graded and that the ideal $\mathfrak{a} \subseteq R$ is graded, so that

$$R = \bigoplus_{j \in \mathbb{N}_0} R_j \text{ and } \mathfrak{a} = \bigoplus_{j \in \mathbb{N}_0} \mathfrak{a}_j, \text{ with } \mathfrak{a}_j = \mathfrak{a} \cap R_j \quad (\forall j \in \mathbb{N}_0).$$

If $M = \bigoplus_{k \in \mathbb{Z}} M_k$ is a graded R-module then, for each $i \in \mathbb{N}_0$, the local cohomology module of M with respect to $\mathfrak{a}$ carries a *natural grading*:

$$H^i_{\mathfrak{a}}(M) = \bigoplus_{j \in \mathbb{Z}} H^i_{\mathfrak{a}}(M)_j.$$

Moreover, if $h : M \longrightarrow N$ is a homomorphism of graded R-modules, then the induced homomorphism in cohomology $H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(N)$ is a homomorphism of graded R modules. If $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ is an exact sequence of graded R-modules, then so is its associated cohomology sequence (see part (A)).

(C) *(Graded Local Cohomology with Respect to the irrelevant Ideal)* Let $R = \bigoplus_{j \in \mathbb{N}_0} R_j$ be as in part (B). The *irrelevant ideal* of R is defined by

$$R_+ := \bigoplus_{j \in \mathbb{N}} R_j.$$

The graded components of local cohomology modules of finitely generated graded R modules with respect to the irrelevant ideal R_+ behave particularly well, namely:

(a) Let $i \in \mathbb{N}_0$ and let $M = \bigoplus_{j \in \mathbb{Z}} M_j$ be a finitely generated graded R-module. Then:
  (1) $H^i_{R_+}(M)_j$ is a finitely generated R_0-module for all $j \in \mathbb{Z}$.
  (2) $H^i_{R_+}(M)_j = 0$ for all $j \gg 0$.

Bearing in mind what we just said in Part (C), we no can introduce the cohomological invariant which plays the crucial rôle in this Section: Castelnuovo-Mumford regularity. As a reference we suggest Chapter 17 of [18].

14.5. **Reminder, Remark and Exercise.** *(Castelnuovo-Mumford Regularity)* (A) Keep the notations and hypotheses of Reminder, Definition and Exercise 14.2(A) and of Reminder 14.4. For each finitely generated graded module $M = \bigoplus_{j\in\mathbb{Z}} M_j$ over the homogeneous Noetherian K-algebra $R = \bigoplus_{j\in\mathbb{N}_0} R_j = K[x_1, x_2, \ldots, x_r]$ and for each $k \in \mathbb{N}_0$ by Reminder 14.4 (A)(a),(b) and (C)(a)(2) we now can define the *Castelnuovo-Mumford regularity at and above level* k of M by

$$\operatorname{reg}^k(M) := \sup\{a_i(M) + i \mid i \geq k\} = \max\{a_i(M) + i \mid i = k, k+1, \ldots, \dim(M)\}$$

with

$$a_i(M) := \sup\{j \in \mathbb{Z} \mid H^i_{R_+}(M)_j \neq 0\} \text{ for all } i \in \mathbb{N}_0,$$

where $H^i_{R_+}(M)_j$ denotes the j-th graded component of the i-th local cohomology module $H^i_{R_+}(M) = \bigoplus_{k\in\mathbb{Z}} H^i_{R_+}(M)_k$ of M with respect to the irrelevant ideal $R_+ := \bigoplus_{j\in\mathbb{N}} R_j = \sum_{m=1}^r Rx_m$ (see Reminder 14.4 (B),(C)).
Keep in mind that the *Castelnuovo-Mumford regularity* of M is defined by

$$\operatorname{reg}(M) := \operatorname{reg}^0(M) = \sup\{a_i(M) + i \mid i \in \mathbb{N}_0\} = \max\{a_i(M) + i \mid i = 0, 1, \ldots, \dim(M)\}$$

and keep in mind the fact that

$$\operatorname{reg}^1(M) = \operatorname{reg}(M/\Gamma_{R_+}(M)) \text{ and } P_{M/\Gamma_{R_+}(M)}(X) = P_M(X).$$

(B) Keep the notations and hypotheses of part (A). Let

$$\operatorname{gendeg}(M) := \inf\{m \in \mathbb{Z} \mid M = \sum_{k\leq m} RM_k\} \quad \big(\leq \operatorname{reg}(M)\big)$$

denote the *generating degree* of M. Keep in mind, that the ideal $\operatorname{Ann}_R(M) \subseteq R$ is homogeneous. Use the previous inequality to prove the following claims:

(a) If $b \in \mathbb{Z}$ such that $\operatorname{reg}\big(\operatorname{Ann}_R(M)\big) \leq b$, there are elements

$$f_1, f_2, \ldots, f_s \in \operatorname{Ann}_R(M) \cap \big(\bigcup_{i\leq b} R_i\big) \text{ with } \operatorname{Var}\big(\operatorname{Ann}_R(M)\big) = \bigcap_{i=1}^{s} \operatorname{Var}(f_i).$$

(C) We recall a few basic facts on Castelnuovo-Mumford regularity.

(a) If $r \in \mathbb{N}$ and $R = K[T_1, T_2, \ldots, T_r]$ is a polynomial ring over the field K, then $\operatorname{reg}(R) = \operatorname{reg}\big(K[T_1, T_2, \ldots, T_r]\big) = 0$.

(b) If $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ is a short exact of finitely generated graded R -modules, then we have the equality $\operatorname{reg}(N) \leq \max\{\operatorname{reg}(M), \operatorname{reg}(P) + 1\}$.

(c) If $r \in \mathbb{N}$ and if $M^{(1)}, M^{(2)}, \ldots, M^{(r)}$ are finitely generated graded R-modules, then we have the equality $\operatorname{reg}\big(\bigoplus_{i=1}^r M^{(i)}\big) = \max\{\operatorname{reg}(M^{(i)}) \mid i = 1, 2, \ldots, r\}$.

(D) We mention the following bounding result (see Corollary 17.4.2 of [18]):

(a) Let $R = \bigoplus_{j\in\mathbb{N}_0} R_j$ be a Noetherian homogeneous ring (see Conventions, Reminders and Notations 1.1 (I) for this notion) such that R_0 is Artinian and local. Let $W = \bigoplus_{j\in\mathbb{Z}} W_j$ be a finitely generated graded R-module and let $P \in \mathbb{Q}[X]\setminus\{0\}$. Then, there is an integer G such that for each R-homomorphism $f : W \longrightarrow M$ of finitely generated graded R-modules, which is surjective in all large degrees and such that $P_M = P$, we have $\operatorname{reg}^1(M) \leq G$.

Use the bounding result of statement (a) to prove the following result.

(b) There is a function $\overline{B} : \mathbb{N}_0^2 \times \mathbb{Q}[X] \longrightarrow \mathbb{Z}$ such that for each choice of $r, t \in \mathbb{N}$, for each field K, for each homogeneous Noetherian K-algebra $R = \bigoplus_{i\in\mathbb{N}_0} R_i$ with $h_R(1) \leq t$ and each finitely generated graded R-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ with $M = RM_0$ and $h_M(0) \leq r$ we have

$$\mathrm{reg}^1(M) \leq \overline{B}\big(t, r, P_M\big).$$

Another bounding result, which we shall use later is (see Corollary 6.2 of [17]):

(c) Let $R = K[T_1, T_2, \ldots, T_r]$ be a polynomial ring over the field K, furnished with its standard grading. Let $f : W \longrightarrow V$ be a homomorphism of finitely generated graded R-modules such that $V \neq 0$ is generated by μ homogeneous elements of degree 0. Then

$$\mathrm{reg}\,\big(\mathrm{Im}(f)\big) \leq \big[\max\{\mathrm{gendeg}(W), \mathrm{reg}(V) + 1\} + \mu + 1\big]^{2^{r-1}}.$$

We now prove a special case of Theorem 3.10 of [16].

14.6. **Proposition.** *Let $r \in \mathbb{N}$, let $R := K[T_1, T_2, \ldots, T_r]$ be the polynomial ring over the field K and let $M = \bigoplus_{n\in\mathbb{N}_0} M_n$ be finitely generated graded R-module with $M = RM_0$. Then*

$$\mathrm{reg}\,\big(\mathrm{Ann}_R(M)\big) \leq \big[\,\mathrm{reg}(M) + h_M(0)^2 + 2\big]^{2^{r-1}} + 1.$$

Proof. Observe first, that we have an exact sequence of graded R-modules

$$0 \longrightarrow \mathrm{Ann}_R(M) \longrightarrow R \stackrel{\epsilon}{\longrightarrow} \mathrm{Hom}_R(M, M), \text{ with } x \mapsto \epsilon(x) := x\mathrm{Id}_M, \text{ for all } x \in R.$$

Moreover, there is an epimorphism of graded R-modules

$$\pi : R^{h_M(0)} \longrightarrow M \longrightarrow 0.$$

So, with $g := \mathrm{Hom}_R(\pi, \mathrm{Id}_M)$ we get an induced monomorphism of graded R-modules

$$0 \longrightarrow \mathrm{Hom}_R(M, M) \stackrel{g}{\longrightarrow} \mathrm{Hom}_R\big(R^{h_M(0)}, M\big) \cong M^{h_M(0)}.$$

So, we get a composition map

$$f := g \circ \epsilon : R \longrightarrow M^{h_M(0)} =: V, \text{ with } \mathrm{Im}(f) = \mathrm{Im}(\epsilon) \cong R/\mathrm{Ann}_R(M).$$

Now, observe that $\mathrm{gendeg}(R) = 0$ (see Reminder, Remark and Exercise 14.5 (C)(a)), $\mathrm{reg}(V) = \mathrm{reg}(M)$ (see Reminder, Remark and Exercise 14.5 (C)(c)) and that V is generated by $h_M(0)^2$ homogeneous elements of degree 0. So, by Reminder, Remark and Exercise 14.5 (D)(c) we obtain

$$\mathrm{reg}\,\big(R/\mathrm{Ann}_R(M)\big) = \mathrm{reg}\,\big(\mathrm{Im}(f)\big) \leq \big[\,\mathrm{reg}(M) + h_M(0)^2 + 2\big]^{2^{r-1}}.$$

On application of Reminder, Remark and Exercise 14.5 (C) (b) to the short exact sequence of graded R-modules

$$0 \longrightarrow \mathrm{Ann}_R(M) \longrightarrow R \longrightarrow R/\mathrm{Ann}_R(M) \longrightarrow 0$$

and keeping in mind that $\mathrm{reg}(R) = 0$, we thus get indeed our claim. □

14.7. **Exercise.** Let the notations and hypotheses be as in Proposition 14.6. Show that

(a) $\mathrm{reg}\,\big(\mathrm{Ann}_R(M/\Gamma_{R_+}(M))\big) \leq \big[\,\mathrm{reg}^1(M) + h_M(0)^2 + 2\big]^{2^{r-1}} + 1.$

(b) $\operatorname{Var}\big(\operatorname{Ann}_R(M/\Gamma_{R_+}(M))\big) = \begin{cases} \operatorname{Var}\big(\operatorname{Ann}_R(M)\big), & \text{if } \dim_R(M) > 0 \\ \emptyset, & \text{if } \dim_R(M) = 0. \end{cases}$

14.8. **Notation, Remark and Exercise.** (A) Let $\overline{B} : \mathbb{N}_0^2 \times \mathbb{Q}[X] \longrightarrow \mathbb{Z}$ be the bounding function introduced in Reminder, Remark and Exercise 14.5 (D)(b).
We define a new function

$F : \mathbb{N}^2 \times \mathbb{Q}[X] \longrightarrow \mathbb{Z}$ by $F(t,r,P) := \big[\overline{B}(t,r,P) + r^2 + 2\big]^{2^{r-1}} + 1 \quad \big(t,r \in \mathbb{N}, P \in \mathbb{Q}[X]\big).$

(B) Let the notations as in part (A). Use Proposition 14.6, Reminder, Remark and Exercise 14.5 (B) and Exercise 14.7 to show that for each field K, for each choice of $r,t \in \mathbb{N}$, for each polynomial ring $R = K[T_1, T_2, \ldots, T_t]$ and for each finitely generated graded R-module $M = \bigoplus_{n \in \mathbb{N}_0} M_n$ with $M = RM_0$, $h_M(0) \leq r$ and $P_M = P$, we have the following statements:

(a) $\operatorname{reg}\big(\operatorname{Ann}_R(M/\Gamma_{R_+}(M))\big) \leq F(t,r,P)$.
(b) There are homogeneous polynomials $f_1, f_2, \ldots, f_s \in \operatorname{Ann}_R\big(M/\Gamma_{R_+}(M)\big)$ with
  (1) $\deg(f_i) \leq F(t,r,P)$ for all $i = 1, 2, \ldots, s$.
  (2) $\operatorname{Var}\big(\operatorname{Ann}_R(M)\big) = \operatorname{Var}(f_1, f_2, \ldots, f_s) = \bigcap_{i=1}^{s} \operatorname{Var}(f_i)$.

No, we are ready to prove the main result of this section.

14.9. **Theorem.** ***(Boundedness of the Degrees of Defining Equations of Characteristic Varieties, compare*** [16]***)*** *Let $n \in \mathbb{N}$, let K be a field of characteristic 0, let U be a D-module over the standard Weyl algebra*

$$\mathbb{W} = \mathbb{W}(K,n) = K[X_1, X_2, \ldots, X_n \partial_1, \partial_2, \ldots, \partial_n]$$

and let $V \subseteq U$ be a K-subspace with $\dim_K(V) \leq r < \infty$ and $U = \mathbb{W}V$. Moreover, let

$$F : \mathbb{N}^2 \times \mathbb{Q}[X] \longrightarrow \mathbb{Z}$$

be the bounding function defined in Notation, Remark and Exercise 14.8 (A). Keep in mind that the degree filtration $\mathbb{W}_\bullet^{\deg}$ of $\mathbb{W}$ (see Definition and Remark 8.6) is very good (see Corollary 8.7 (a)) and let

$$P_{U,\mathbb{W}_\bullet^{\deg}V} \in \mathbb{Q}[X]$$

be the Hilbert polynomial of U induced by V with respect to the degree filtration $\mathbb{W}_\bullet^{\deg}$ (see Reminder, Definition and Exercise 14.2 (B)).

Then, there are homogeneous polynomials

$$f_1, f_2, \ldots, f_s \in \mathbb{P} = K[Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n]$$

such that

(a) $\deg(f_i) \leq F\big(2n, r, P_{U,\mathbb{W}_\bullet^{\deg}}\big)$.
(b) $\mathbb{V}_{\mathbb{W}_\bullet^{\deg}}(U) = \operatorname{Var}(f_1, f_2, \ldots, f_s) = \bigcap_{i=1}^{s} \operatorname{Var}(f_i)$.

Proof. Observe that (see Definition and Remark 11.2)

$$\mathbb{V}_{\mathbb{W}_\bullet^{\deg}}(U) = \operatorname{Var}\big(\operatorname{Ann}_{\mathbb{P}}(\operatorname{Gr}_{\mathbb{W}_\bullet^{\deg}V}(U)\big).$$

Now, we may conclude by Notation, Remark and Exercise 14.8 (B)(b), applied to the graded $\mathbb{P}$-module $\mathrm{Gr}_{\mathbb{W}_\bullet^{\deg}V}(U)$ and bearing in mind that – by Exercise and Definition 10.5 (B)(c) – this latter graded module is generated in degree 0. □

14.10. **Conclusive Remark.** (A) Keep the above notations. To explain the meaning of this result, we fix $r, s \in \mathbb{N}$ and we fix a polynomial $P \in \mathbb{Q}[X]$. For any matrix

$$\mathcal{D} = \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1r} \\ d_{21} & d_{22} & \dots & d_{2r} \\ \vdots & \vdots & & \vdots \\ d_{s1} & d_{s2} & \dots & d_{sr} \end{pmatrix} \in \mathbb{W}^{s\times r}$$

of polynomial partial differential operators we consider the induced epimorphism of D-modules

$$\mathbb{W}^r \xrightarrow{\pi_\mathcal{D}} U_\mathcal{D} \longrightarrow 0,$$

consider the K-subspace

$$K^r = \left(\mathbb{W}_0^{\deg}\right)^r \subset \mathbb{W}^r$$

and set

$$V_\mathcal{D} := \pi_\mathcal{D}(K^r).$$

Then, referring to our Preliminary Remark 14.1 we consider the family of systems of differential equations

$$\mathbb{F} = \mathbb{F}^P := \{\mathcal{D} \in \mathbb{W}^{s\times r} \mid P_{U_\mathcal{D}, \mathbb{W}_\bullet^{\deg} V_\mathcal{D}} = P\}$$

whose *canonical Hilbert polynomial* $P_{U_\mathcal{D}, \mathbb{W}_\bullet^{\deg} V_\mathcal{D}}$ equals P. As an immediate application of Theorem 14.9 we can say

> *The degree of hypersurfaces which cut out set-theoretically the characteristic variety* $\mathbb{V}^{\deg}(\mathcal{D})$ *is bounded, if* $\mathcal{D}$ *runs through the family* $\mathbb{F}^P$.

Clearly, our results give much more, as they bound the invariant

$$\mathrm{reg}\big(\mathrm{Ann}_\mathbb{P}\big[\mathrm{Gr}_{\mathbb{W}_\bullet^{\deg}V_\mathcal{D}}(U_\mathcal{D})/\Gamma_{\mathbb{P}_+}(\mathrm{Gr}_{\mathbb{W}_\bullet^{\deg}V_\mathcal{D}}(U_\mathcal{D})\big]\big)$$

along the class $\mathbb{F}^P$.

(B) Our motivation to prove Theorem 14.9 was a question arising in relation with the PhD thesis [5], namely: Does the Hilbert function (with respect to an appropriate filtration) of a D-module U over a standard Weyl algebra $\mathbb{W}$ bound the degrees of polynomials which cut out set-theoretically the characteristic variety of U? This leads to the question, whether the Hilbert function h_M of a graded module M which is generated over the polynomial ring $K[X_1, X_2, \dots, X_r]$ by finitely many elements of degree 0 bounds the (Castelnuovo-Mumford) regularity $\mathrm{reg}(\mathrm{Ann}_R(M))$ of the annihilator $\mathrm{Ann}_R(M)$ of M. This latter question was answered affirmatively in the Master thesis [39] and lead to the article [16].

Theorem 14.9 above actually improves what has been shown in [16] and in Theorem 14.6 of [15]. There it is shown, that the degrees of the polynomials $f_1, f_2, , \dots, f_s \in \mathbb{P}$ which occur in Theorem 14.9 are bounded in terms of n and the Hilbert function $h_{U,A_\bullet V} = h_{\mathrm{Gr}_{A_\bullet V}(U)}$ (see Reminder, Definition and Exercise 14.2 (B)). More precisely, in these previous results, the degrees in question are bounded in terms of n, $h_{U,A_\bullet V}(0)$ and the postulation number (see Reminder, Definition and Exercise 14.2 (A))

$$\mathrm{pstln}_{A_\bullet V}(U) := \sup\{i \in \mathbb{Z} \mid h_{U,A_\bullet V}(i) \neq P_{U,A_\bullet V}(0)\} = \mathrm{pstln}\big(\mathrm{Gl}_{A_\bullet V}(U)$$

of U with respect to the filtration $A_\bullet V$. Theorem 14.9 shows, that the postulation number $\mathrm{pstln}_{A_\bullet V}(U)$ is not needed to bound the degrees we are interested in.

(C) We thank the referee for having pointed out to us, that Aschenbrenner and Leykin [2] have proved a result, which is closely related to Theorem 14.9 and which furnishes a bound on the degree of the elements of Gröbner bases of a left ideal $I \subseteq \mathbb{W}$ of our Weyl algebra. More precisely, if $\underline{\omega} \in \Omega$ (see Notation 13.1), if $d \in \mathbb{N}$ and if I is generated by elements whose $\underline{\omega}$-weighted degree $\deg^{\underline{\omega}}(\bullet)$ does not exceed d, then I admits a $\leq^{\underline{\omega}}$-Gröbner basis consisting of elements whose $\underline{\omega}$-weighted degree does not exceed the bound $2\big(\frac{d^2}{2} + d\big)^{2^{2n-1}}$.
As the Castelnuovo-Mumford regularity $\mathrm{reg}(\mathfrak{a})$ of a graded ideal in the polynomial ring $\mathfrak{a} \subseteq K[X_1, x_2, \ldots, X_n]$ over a field K is an upper bound for the degree of the polynomials occuring in some Gröbner basis of $\mathfrak{a}$, the mentioned result in [2] corresponds to the "classical“ regularity bound $\mathrm{reg}(\mathfrak{a}) \leq \big(2\mathrm{gendeg}(\mathfrak{a})\big)^{2^{n-2}}$ for graded ideals in the polynomial ring (see [27], [28], [23], but also [17] and [22]). Via Gröbner bases and Macaulay's Theorem for Hilbert Functions (see [26], for example), this latter regularity bound on its turn, is also related to the module theoretic form of Mumford's regularity bound ([18], Corollary 17.4.2), we were using as an important tool in the proof of Theorem 14.9 (see Reminder, Remark and Exercise 14.5 (D)(a)).

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