A SHORT COURSE ON WEYL ALGEBRAS AND D-MODULES

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1. Introduction

These Notes base on two short courses and series of lectures held at the University of Thai Nguyen and the Vietnam Institute for Advanced Study in Mathematics Hanoi in October – December 2013. For readers, who aim to get a more complete presentation of the subject, we recommend to consider the notes [6] or [7], which also are available on the authors homepage at the Institute of Mathematics of the University of Zürich or on request by e-mail.

Weyl algebras, sometimes called algebras of differential operators, are a fascinating and important subject, which relates Non-Commutative and Commutative Algebra, Algebraic Geometry and Analysis in very appealing way. The bridging nature of the Theory of Weyl Algebras and D-modules shows uo in a number of surprising applications. Let us mention as an example Luybeznik's finiteness results for local cohomology modules of regular local rings in characteristic 0 (see [15] and [16]), which brought a break-through in Commutative Algebra, as they base on the use of D-modules – and hence present a very important link between these two fields. A further example is an application to Mathematical Physics and relating characteristic varieties of D-modules with Castelnuovo-Mumford regularity (see [2]) – an application lead to the result presented in Section 14 of this course.
Another interesting relation between Weyl algebras and Commutative Algebra is the Theory of Gröbner bases. Indeed, the Theory of Gröbner bases in Weyl algebras proves to be a very fertile tool in the study of D-modules and their characteristic varieties. This lead to the Master and PhD theses of Boldini (see [3], resp. [4]) and the resulting article [5]. We present some of the basic results in these papers in Sections 12 and 13 of this course.

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2. Filtered Algebras

Our first preliminary theme are filtered algebras over a field. Throughout $\mathbb{N}$ is understood to be the set of Positive integers and $\mathbb{N}_0$ is understood to be the set of non-negative integers.

2.1. **Definition and Remark.** (A) Let K be a field and let A be an associative unital K-algebra. By a *filtration* of A we mean a family

$$A_\bullet = (A_i)_{i\in\mathbb{N}_0}$$

such that the following conditions hold:

(a) Each A_i is a K-vector subspace of A;
(b) $A_i \subseteq A_{i+1}$ for all $i \in \mathbb{N}_0$;
(c) $1 \in A_0$;
(d) $A = \bigcup_{i\in\mathbb{N}_0} A_i$;
(e) $A_iA_j := \sum_{(f,g)\in A_i\times A_j} Kfg \subseteq A_{i+j}$ for all $i, j \in \mathbb{N}_0$.

To simplify notation, we also often set

$$A_i = 0 \text{ for all } i < 0$$

and then write our filtration in the form

$$A_\bullet = (A_i)_{i\in\mathbb{Z}}.$$

If a filtration of A is given, we say that $(A, A_\bullet)$ or – by abuse of language – that A is a *filtered K-algebra.*

(B) Let $(A, A_\bullet)$ be a filtered K-algebra. Then, the *degree* of an element $f \in A$ is defined by:

$$\deg_{A_\bullet}(f) := \begin{cases} \min\{i \in \mathbb{N}_0 \mid a \in A_i\} & \text{if } f \neq 0, \\ -\infty & \text{if } f = 0. \end{cases}$$

Observe that we have

$$A_i = \{f \in A \mid \deg_{A_\bullet}(f) \leq i\}.$$

(C) Keep the above notations and hypotheses and let $A_\bullet = (A_i)_{i\in\mathbb{Z}}$ be a filtered K-algebra. Observe that we have the following statements:

(a) A_0 is a K-sub-algebra of A.
(b) For all $i \in \mathbb{Z}$ the K-vector space A_i is a left- and a right- A_0-submodule of A.

2.2. **Example.** (A) *(Weighted degree filtrations of a commutative polynomial ring)* Let $n \in \mathbb{N}$ and let $A = K[X_1, X_2, \ldots, X_n]$ be the commutative polynomial algebra over the field K in the indeterminates $X_1, X_2, \ldots, X_n$. Let

$$\underline{\omega} := (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{N}_0^n \setminus \{\underline{0}\}.$$

Now, for each $i \in \mathbb{Z}$ we set

$$A_i^{\underline{\omega}} := \bigoplus_{\nu_1,\nu_2,\ldots\nu_n\in\mathbb{N}_0:\quad \sum_{j=1}^n \nu_j\omega_j \leq i} KX_1^{\nu_1}X_2^{\nu_2}\ldots X_n^{\nu_n} = \bigoplus_{\underline{\nu}\in\mathbb{N}_0:\quad \underline{\nu}\cdot\underline{\omega}\leq i} K\underline{X}^{\underline{\nu}}.$$

Then

$$A^{\underline{\omega}}_{\bullet} = \left(A^{\underline{\omega}}_i\right)_{i\in\mathbb{Z}}$$

defines a filtration on A.
With the convention that $\sup(\emptyset) = -\infty$ and using the standard notations

$$f = \sum_{\underline{\nu}\in\mathbb{N}_0} c^{(f)}_{\underline{\nu}} \underline{X}^{\underline{\nu}} \quad (c^{(f)}_{\underline{\nu}} \in K, \forall \underline{\nu} \in \mathbb{N}_0^n) \text{ and } \operatorname{supp}(f) := \{\underline{\nu} \in \mathbb{N}_0^n \mid c^{(f)}_{\underline{\nu}} \neq 0\}$$

we have

$$\deg^{\underline{\omega}}(f) := \deg_{A^{\underline{\omega}}_{\bullet}}(f) = \sup\{\underline{\nu}\cdot\underline{\omega} \mid \underline{\nu} \in \operatorname{supp}(f)\}.$$

The map

$$\deg^{\underline{\omega}} : A \longrightarrow \mathbb{N}_0 \cup \{\infty\}, \quad f \mapsto \deg^{\underline{\omega}}(f)$$

is called *degree with weight* $\underline{\omega}$ and the filtration $A^{\underline{\omega}}_{\bullet}$ is called the *weighted degree filtration* of the polynomial algebra $A = K[X_1, X_2, \ldots, X_n]$ with respect to $\underline{\omega}$.
Choosing $\underline{\omega} = \underline{1} := (1, 1, \ldots, 1)$ we get the *standard degree* respectively the *standard degree filtration* of $A = K[X_1, X_2, \ldots, X_n]$.

Clearly filtrations also may occur in non-commutative algebras. The next example presents somehow the "generic occurrence" of this.

2.3. **Example.** *(The standard filtration of a free associative algebra)* Let $n \in \mathbb{N}$, let K be field and let $A = K\langle X_1, X_2, \ldots, X_n\rangle$ be the free central associative algebra over K in the indeterminates $X_1, X_2, \ldots, X_n$. We suppose in particular that $cX_i = X_ic$ for all $c \in K$ and all $i = 1, 2, \ldots, n$, so that $cf = fc$ for all $c \in K$ and all $f \in A$. Let $i \in \mathbb{N}_0$. If

$$\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_i) \in \{1, 2, \ldots, n\}^i$$

we write

$$\underline{X}_{\underline{\sigma}} := \prod_{j=1}^{i} X_{\sigma_j} = X_{\sigma_1}X_{\sigma_2}\ldots X_{\sigma_i}.$$

Then, with the usual convention that the product $\prod_{j\in\emptyset} X_j$ of an empty family of factors equals 1 and using the notation

$$\mathbb{S}_n := \dot{\bigcup}_{i\in\mathbb{N}_0} \{1, 2, \ldots, n\}^i$$

we can write A as a K-space over its *monomial basis* as follows:

$$A := K\langle X_1, X_2, \ldots, X_n\rangle = \bigoplus_{\underline{\sigma}\in\mathbb{S}_n} K\underline{X}_{\underline{\sigma}}.$$

Clearly, as in the case of a commutative polynomial ring, each $f \in A$ may be written uniquely in the form

$$f = \sum_{\underline{\sigma}\in\mathbb{S}_n} c^{(f)}_{\underline{\sigma}} \underline{X}_{\underline{\sigma}} = \sum_{\underline{\sigma}\in\operatorname{supp}(f)} c^{(f)}_{\underline{\sigma}} \underline{X}_{\underline{\sigma}} \qquad (c^{(f)}_{\underline{\nu}} \in K, \forall \underline{\nu} \in \mathbb{N}_0^n)$$

with finite *support* $\operatorname{supp}(f) := \{\underline{\sigma} \in \mathbb{S}_n \mid c^{(f)}_{\underline{\sigma}} \neq 0\}$. Finally, we get a filtration

$$A_{\bullet} \text{ defined by } A_i := \bigoplus_{\sigma\in\bigcup_{j\leq i}\{1,2,\ldots,n\}^j} K\underline{X}_{\underline{\sigma}} \text{ for all } i \in \mathbb{Z}$$

with corresponding degree

$$\deg_{A^{\omega}_{\bullet}} : A \longrightarrow \mathbb{N}_0 \cup \{\infty\}, \quad f \mapsto \sup\{i \in \mathbb{N}_0 \mid \exists \underline{\sigma} \in \operatorname{supp}(f) \cap \{1, 2, \ldots, n\}^i\}.$$

This filtration and its degree are called the *standard filtration* respectively the *standard degree* for $A = K\langle X_1, X_2, \ldots, X_n\rangle$.

3. Associated Graded Rings

3.1. **Remark and Definition.** (A) Let K be a field and let $A = (A, A_\bullet)$ be a filtered K-algebra. We consider the K-vector space

$$\operatorname{Gr}(A) = \operatorname{Gr}_{A_\bullet}(A) = \bigoplus_{i \in \mathbb{N}_0} A_i / A_{i-1}.$$

For all $i \in \mathbb{N}_0$ we also use the notation

$$\operatorname{Gr}(A)_i = \operatorname{Gr}_{A_\bullet}(A)_i := A_i / A_{i-1},$$

so that we may write

$$\operatorname{Gr}(A) = \operatorname{Gr}_{A_\bullet}(A) = \bigoplus_{i \in \mathbb{N}_0} \operatorname{Gr}_{A_\bullet}(A)_i.$$

(B) Let $i, j \in \mathbb{N}_0$, let $f, f' \in A_i$ and let $g, g' \in A_j$ such that $f - f' \in A_{i-1}$ and $g - g' \in A_{j-1}$. Then we have $fg - f'g' \in A_{i+j-1}$. So in $A_{i+j}/A_{i+j-1} = \operatorname{Gr}_{A_\bullet}(A)_{i+j} \subset \operatorname{Gr}_{A_\bullet}(A)$ we get the relation $fg + A_{i+j-1} = f'g' + A_{i+j-1}$. This allows to define a *multiplication* on the K-space $\operatorname{Gr}_{A_\bullet}(A)$ which is induced by

$$(f + A_{i-1})(g + A_{j-1}) := fg + A_{i+j-1} \text{ for all } i, j \in \mathbb{N}_0, \text{ all } f \in A_i \text{ and all } g \in A_j.$$

(C) Keep the above notations and hypotheses. Observe in particular, that $\operatorname{Gr}_{A_\bullet}(A)_0$ is a K-subalgebra of $\operatorname{Gr}_{A_\bullet}(A)$, and that there is an isomorphism of K-algebras

$$\operatorname{Gr}_{A_\bullet}(A)_0 \cong A_0.$$

Moreover, with respect to our multiplication on $\operatorname{Gr}_{A_\bullet}(A)$ we have the relations

$$\operatorname{Gr}_{A_\bullet}(A)_i \operatorname{Gr}_{A_\bullet}(A)_j \subseteq \operatorname{Gr}_{A_\bullet}(A)_{i+j} \text{ for all } i, j \in \mathbb{Z}.$$

So, the K-vector space $\operatorname{Gr}_{A_\bullet}(A)$ is turned into a (positively) graded ring

$$\operatorname{Gr}_{A_\bullet}(A) = \big(\operatorname{Gr}_{A_\bullet}(A), (\operatorname{Gr}_{A_\bullet}(A)_i)_{i \in \mathbb{N}_0}\big) = \bigoplus_{i \in \mathbb{N}_0} \operatorname{Gr}_{A_\bullet}(A)_i$$

by means of the above multiplication. We call this ring the *associated graded ring* of A with respect to the filtration $A_\bullet$. From now on, we always furnish $\operatorname{Gr}_{A_\bullet}(A)$ with this multiplication.

We now introduce a class of filtrations, which will be of particular interest for our lectures.

3.2. **Definition.** Let K be a field and let $A = (A, A_\bullet)$ be a filtered K-algebra. The filtration $A_\bullet$ is said to be *commutative* if

$$fg - gf \in A_{i+j-1} \text{ for all } i, j \in \mathbb{N}_0 \text{ and for all } f \in A_i \text{ and all } g \in A_j.$$

It is equivalent to say that the associated graded ring $\operatorname{Gr}_{A_\bullet}(A)$ is commutative.

We now shall define three special types of commutative filtrations, which will play a particularly important rôle in Weyl algebras.

3.3. **Definition and Remark.** (A) Let $(A, A_\bullet)$ be a filtered K-algebra. The filtration $A_\bullet$ is said to be *of finite type* if it satisfies the following conditions:

(a) The filtration $A_\bullet$ is commutative;
(b) A_0 is a K-algebra of finite type;
(c) There is an integer $\delta \in \mathbb{N}$ such that A_j is finitely generated as a (left-)module over A_0 for all $j \leq \delta$ and
(d) $A_i = \sum_{j=1}^{\delta} A_j A_{i-j}$ for all $i > \delta$.

In this situation, we call the minimal number $\delta = \delta_{A_\bullet} \in \mathbb{N}$ the *generating degree* of the filtration $A_\bullet$.
Observe that now, the associated graded ring $\mathrm{Gr}_{A_\bullet}(A)$ is a commutative Noetherian graded A_0-algebra, which is generated by finitely many homogeneous elements of degree $\leq \delta$.
If $A_\bullet$ is a filtration of A, which is of finite type, we say that $(A, A_\bullet)$ is a *filtered algebra of finite type.*

(B) A filtration $A_\bullet$ of a K-algebra A is called a *good filtration*, if it is of finite type with $\delta = \delta_{A_\bullet} = 1$. If in addition it holds $A_0 = K$, we speak of a *very good filtration.* We then also say that $(A, A_\bullet)$ is a *well-filtered* respectively a *very well-filtered* K-Algebra.

3.4. **Example and Exercise.** (A) Let $n \in \mathbb{N}$, let $\underline{\omega} = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{N}_0^n \setminus \{\underline{0}\}$, Let K be a field and consider the commutative polynomial ring $A = K[X_1, X_2, \ldots, X_n]$, furnished with its weighted degree filtration $A_\bullet^{\underline{\omega}}$ with respect to $\underline{\omega}$. Then, it is easy to see, that A is a very well filtered K-algebra if and only if $\underline{\omega} = \underline{(1)}$.
More generally A is a well-filtered K-algebra if and only if $\omega_i \in \{0, 1\}$ for all $i \in \{1, 2, \ldots, n\}$. In this situation we then have $K[X_j \mid \omega_j = 0] = A_0$ and an isomorphism of graded rings

$$\varepsilon : A \xrightarrow{\cong} \mathrm{Gr}_{A_\bullet^{\underline{\omega}}}(A), \text{ induced by } X_i \mapsto X_i + A_{\delta_{1,\omega_i}-1} \text{ for all } i \in \{1, 2, \ldots, n\},$$

where $\delta_{i,j}$ denotes the *Kronecker symbol.*
(B) Let $n \in \mathbb{N}$, with $n > 1$, let K be a field and consider the free associative K-algebra $A = K\langle X_1, X_2, \ldots, X_n\rangle$, furnished with its standard filtration $A_\bullet$. For each $i \in \{1, 2, \ldots, n\}$, let $\overline{X}_i := (X_i + A_0) \in A_1/A_0 = \mathrm{Gr}_{A_\bullet}(A)_1 \subset \mathrm{Gr}_{A_\bullet}(A)$. Then it is easy to see that $\overline{X}_i\overline{X}_j = \overline{X}_j\overline{X}_i$ if and only if $i = j$. Therefore the filtration $A_\bullet$ cannot be commutative.

4. Derivations

Derivations (or derivatives) are also a basic ingredient for the theory of Weyl algebras. The present section is devoted to this subject.

4.1. **Definition and Remark.** (A) Let K be a field, let A be a commutative K-algebra and let M be an A-module. A *K-derivation* (or *K-derivative*) on A with values in M is a map $d : A \longrightarrow M$ such that:

(a) d is K-linear: $d(\alpha a + \beta b) = \alpha d(a) + \beta d(b)$ for all $\alpha, \beta \in K$ and all $a, b \in A$.
(b) d satisfies the *Leibniz Product Rule*: $d(ab) = ad(b) + bd(a)$ for all $a, b \in A$.

We denote the set of all K-derivations on A with values in M by $\mathrm{Der}_K(A, M)$, thus:

$$\mathrm{Der}_K(A, M) := \{d \in \mathrm{Hom}_K(A, M) \mid d(ab) = ad(b) + bd(a) \text{ for all } a, b \in A\}.$$

To simplify notations, we also write

$$\mathrm{Der}_K(A, A) =: \mathrm{Der}_K(A).$$

(B) Keep in mind, that $\mathrm{Hom}_K(A, M)$ carries a natural structure of A-module, with scalar multiplication given by

$$(ah)(x) := a(h(x)) \text{ for all } a \in A, \text{ all } h \in \mathrm{Hom}_K(A, M) \text{ and all } x \in A.$$

It is easy to verify:

$$\mathrm{Der}_K(A, M) \text{ is a submodule of the } A\text{-module } \mathrm{Hom}_K(A, M).$$

It is also easy to verify that "derivations vanish on constants", thus if we identify $c \in K$ with $c1_A \in A$ we have:

$$d(c) = 0 \text{ for all } c \in K.$$

Next, we shall look at the arithmetic properties of derivations and gain an important embedding proceedure for modules of derivations of K-algebras of finite type.

4.2. **Exercise and Definition.** (A) Let K be a field, let A be a commutative K-algebra and let M be an A-module. Let $d \in \mathrm{Der}_K(A, M)$, let $r \in \mathbb{N}$, let $\nu_1, \nu_2, \ldots, \nu_r \in \mathbb{N}_0$ and let $a_1, a_2, \ldots, a_r \in A$. Use induction on r to prove the *Generalized Product Rule*

$$d\Big(\prod_{j=1}^{r} a_j^{\nu_j}\Big) = \sum_{i \in \{1,2,\ldots,r \mid \nu_i > 0\}} \nu_i a_i^{\nu_i - 1} \Big(\prod_{j \neq i} a_j^{\nu_j}\Big) d(a_i)$$

and the resulting *Power Rule*

$$d(a^r) = r a^{r-1} d(a) \text{ for all } a \in A.$$

(B) Let the notations and hypotheses be as in part (A). Assume in addition that $A = K[a_1, a_2, \ldots, a_r]$. Let $d, e \in \mathrm{Der}_K(A, M)$. Prove that

$$e = d \text{ if and only if } e(a_i) = d(a_i) \text{ for all } i = 1, 2, \ldots, r.$$

(C) Yet assume that $A = K[a_1, a_2, \ldots, a_r]$. Prove that there is a monomorphism of A-modules

$$\Theta^M_{\underline{a}} = \Theta^M_{(a_1, a_2 \ldots, a_r)} : \mathrm{Der}_K(A, M) \longrightarrow M^r, \text{ given by } d \mapsto \big(d(a_1), d(a_2), \ldots, d(a_r)\big).$$

This monomorphism $\Theta^M_{\underline{a}}$ is called the *canonical embedding* of $\mathrm{Der}_K(A, M)$ with respect to the generators $a_1, a_2, \ldots, a_r$.

(D) Let the notations and hypotheses be as in part (C). Assume that M is finitely generated. Prove, that the A-module $\mathrm{Der}_K(A, M)$ is finitely generated.

Now, we turn to derivatives in polynomial algebras.

4.3. **Exercise and Definition.** *(Partial Derivatives in Polynomial Rings)* (A) Let $n \in \mathbb{N}$, let K be a field and consider the polynomial algebra $K[X_1, X_2, \ldots, X_n]$. Fix $i \in \{1, 2, \ldots, n\}$. Then, using the monomial basis of $K[X_1, X_2, \ldots, X_n]$ we see that there is a unique K-linear map

$$\partial_i = \frac{\partial}{\partial X_i} : K[X_1, X_2, \ldots, X_n] \longrightarrow K[X_1, X_2, \ldots, X_n]$$

such that for all $\underline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}_0$ we have

$$\partial_i(\underline{X}^{\underline{\nu}}) = \frac{\partial}{\partial X_i}\big(\prod_{j=1}^{n} X_j^{\nu_j}\big) = \begin{cases} \nu_i X_i^{\nu_i - 1} \prod_{j \neq i} X_j^{\nu_j}, & \text{if } \nu_i > 0 \\ 0, & \text{if } \nu_i = 0. \end{cases}$$

(B) Keep the notations and hypotheses of part (A). Let

$$\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_n), \quad \underline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}_0^n$$

and prove that

$$\partial_i\big(\underline{X}^{\underline{\mu}} \underline{X}^{\underline{\nu}}\big) = \underline{X}^{\underline{\mu}} \partial_i\big(\underline{X}^{\underline{\nu}}\big) + \underline{X}^{\underline{\nu}} \partial_i\big(\underline{X}^{\underline{\mu}}\big).$$

Use the K-linearity of ∂_i to conclude that

$$\partial_i = \frac{\partial}{\partial X_i} \in \mathrm{Der}_K\big(K[X_1, X_2, \ldots, X_n]\big) \text{ for all } 1 = 1, 2 \ldots, n.$$

The derivation $\partial_i = \frac{\partial}{\partial X_i}$ is called the *i-th partial derivative* in $K[X_1, X_2, \ldots, X_n]$.

As we shall see in the proposition below, the canonical embedding introduced in Exercise and Definition 4.2 (C) takes a particularly favorable shape in the case of polynomial algebras. The exercise to come is aimed to prepare the proof of this.

4.4. **Exercise.** Let the notations and hypotheses be as in Exercise and Definition 4.3. Check that $\partial_i(X_j) = \delta_{i,j}$, for all $i, j \in \{1, 2 \ldots, n\}$. and show that:

(a) For each $i \in \{1, 2, \ldots, n\}$ it holds $K[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n] \subseteq \mathrm{Ker}(\partial_i)$ with equality if and only if $\mathrm{Char}(K) = 0$.

(b) $K \subseteq \bigcap_{i=1}^{n} \mathrm{Ker}(\partial_i)$ with equality if and only if $\mathrm{Char}(K) = 0$.

4.5. **Proposition.** ***(The Canonical Basis for the Derivations of a Polynomial Ring)*** *Let $n \in \mathbb{N}$, let K be a field and consider the polynomial algebra $K[X_1, X_2, \ldots, X_n]$. Then the canonical embedding of $\mathrm{Der}_K\big(K[X_1, X_2, \ldots, X_n]$ into $K[X_1, X_2, \ldots, X_n]^n$ with respect to $X_1, X_2, \ldots, X_n$ (see Exercise and Definition 4.2 (C)) yields an isomorphism of $K[X_1, X_2, \ldots, X_n]$-modules*

$$\Theta := \Theta_{X_1, X_2, \ldots, X_n} : \mathrm{Der}_K\big(K[X_1, X_2, \ldots, X_n]\big) \xrightarrow{\cong} K[X_1, X_2, \ldots, X_r]^n,$$

given by

$$d \mapsto \Theta(d) := \Theta_{X_1, X_2, \ldots, X_n}(d) = \big(d(X_1), d(X_2), \ldots, d(X_n)\big),$$
$$\textit{for all } d \in \mathrm{Der}_K\big(K[X_1, X_2, \ldots, X_n]\big).$$

In particular, the n partial derivatives $\partial_1, \partial_2, \ldots, \partial_n$ form a free basis of the $K[X_1, X_2, \ldots, X_n]$-module $\mathrm{Der}_K\big(K[X_1, X_2, \ldots, X_n]\big)$.

Proof. We suggest the proof as an exercise. If you need hints, consult [6]. □

5. Weyl Algebras

Now, we are ready to introduce Weyl algebras.

5.1. **Reminder and Remark.** Let K be a field and let A be a commutative K-algebra, let M be an A-module and consider the *endomorphism ring*

$$\mathrm{End}_K(M) := \mathrm{Hom}_K(M, M)$$

of M together with the *canonical homomorphism* of rings

$$\epsilon_M : A \longrightarrow \mathrm{End}_K(M) \text{ given by } a \mapsto \epsilon_M(a) := a\mathrm{id}_M \text{ for all } a \in A,$$

which becomes injective if $M = A$ and hence allows to consider A as a *sub-algebra* of its endomorphism ring $\mathrm{End}_K(A)$.

5.2. **Remark and Definition.** (A) Let K be a field and let A be a commutative K-algebra. We obviously also have $\mathrm{Der}_K(A) \subseteq \mathrm{End}_K(A)$. So we may consider the K-algebra

$$W_K(A) := K[A, \mathrm{Der}_K(A)] = A[\mathrm{Der}_K(A)] \subseteq \mathrm{End}_K(A),$$

which is called the *Weyl algebra of the K-algebra A*.

(B) Keep the hypotheses and notations of part (A). Assume in addition, that the commutative K-algebra A is of finite type, so that we find some $r \in \mathbb{N}_0$ and elements $a_1, a_2, \ldots, a_r \in A$ such that $A = K[a_1, a_2, \ldots, a_r]$. Then according to Exercise and Definition 4.2 (D), the A-module $\mathrm{Der}_K(A)$ is finitely generated. We thus find some $s \in \mathbb{N}_0$ and derivations $d_1, d_2, \ldots, d_s \in \mathrm{Der}_K(A)$ such that

$$\mathrm{Der}_K(A) = \sum_{i=1}^{s} Ad_i.$$

A straight forward computation now allows to see, that

$$W_K(A) = K[a_1, a_2 \ldots, a_r; d_1, d_2, \ldots, d_s] \subseteq \mathrm{End}_K(A).$$

In particular we may conclude, that the K-algebra $W_K(A)$ is finitely generated.

(C) Keep the above notations and let $n \in \mathbb{N}$. The *n-th standard Weyl algebra* $\mathbb{W}(K, n)$ over the field K is defined as the Weyl algebra of the polynomial ring $K[X_1, X_2, \ldots, X_n]$, thus

$$\mathbb{W}(K, n) := W_K\big(K[X_1, X_2, \ldots, X_n]\big) \subseteq \mathrm{End}_K\big(K[X_1, X_2, \ldots, X_n]\big).$$

Observe, that by Propsition 4.5 and according to the observations made in part (B) we may write

$$\mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n] \subseteq \mathrm{End}_K\big(K[X_1, X_2, \ldots, X_n]\big).$$

The elements of $\mathbb{W}(K, n)$ are called *polynomial differential operators* in the indeterminates $X_1, X_2, \ldots, X_n$ over the field K. They are all K-linear combinations of products of

indeterminates X_i and partial derivatives ∂_j.
The differential operators of the form

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} := X_1^{\nu_1} \ldots X_n^{\nu_n} \partial_1^{\mu_1} \ldots \partial_n^{\mu_n} = \prod_{i=1}^{n} X^{\nu_i} \prod_{j=1}^{n} \partial^{\mu_j} \in \mathbb{W}(K, n)$$

with

$$\underline{\nu} := (\nu_1, \ldots, \nu_n), \quad \underline{\mu} := (\mu_1, \ldots, \mu_n) \in \mathbb{N}_0^n$$

are called *elementary differential operators* in the indeterminates $X_1, X_2, \ldots, X_n$ over the field K.

Next, we aim to consider a class of important relations in standard Weyl algebras: the so-called Heisenberg relations. We begin with the following preparations.

5.3. **Remark and Exercise.** (A) If K is a field and B is a K-algebra, we introduce the *Poisson operation*, that is the map

$$[\bullet, \bullet] : B \times B \longrightarrow B, \text{ defined by } [a, b] := ab - ba \text{ for all } a, b \in B.$$

Show, that the Poisson operation is *anti-commutative* and *K-bilinear*, thus:

(a) $[a, b] = -[b, a]$ for all $a, b \in B$.
(b) $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ for all $a, b, c \in B$.
(c) $[\alpha a + \alpha' a', \beta b + \beta' b'] = \alpha\beta[a, b] + \alpha\beta'[a, b'] + \alpha'\beta[a', b] + \alpha'\beta'[a', b']$ for all $\alpha, \alpha', \beta, \beta' \in K$ and all $a, a', b, b' \in B$.

(B) Now, let K be a field, let A be a commutative K-algebra and consider the Weyl algebra $W_K(A) := K[A, \mathrm{Der}_K(A)]$. Show that the following relations hold:

(a) $[a, b] = 0$ for all $a, b \in A$.
(b) $[a, d] = -d(a)$ for all $a \in A$ and all $d \in \mathrm{Der}_K(A)$.
(c) $[d, e] \in \mathrm{Der}_K(A)$ for all $d, e \in \mathrm{Der}_K(A)$.

(C) Let the notations and hypotheses be as in part (B). Let $d, e \in \mathrm{Der}_K(A)$, let $r \in \mathbb{N}$, let $\nu_1, \nu_2, \ldots, \nu_r \in \mathbb{N}_0$ and let $a_1, a_2, \ldots, a_r \in A$. Use statement (c) of part (B) and the Generalized Product Rule of Exercise and Definition 4.2 (A) to prove that

$$[d, e]\Big(\prod_{j=1}^{r} a_j^{\nu_j}\Big) = \sum_{i:\nu_i>0} \nu_i a_i^{\nu_i - 1}\Big(\prod_{j\neq i} a_j^{\nu_j}\Big)[d, e](a_i).$$

Give an alternative proof of this equality, which uses Exercise (c) of the above part (B).

5.4. **Proposition.** ***(The Heisenberg Relations)*** *Let $n \in \mathbb{N}$, and let K be a field. Then, in the standard Weyl algebra*

$$\mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n]$$

the following relations hold:

(a) $[X_i, X_j] = 0,$ *for all* $i, j \in \{1, 2, \ldots, n\}$;

(b) $[X_i, \partial_j] = -\delta_{i,j}$, *for all* $i, j \in \{1, 2, \ldots, n\}$;

(c) $[\partial_i, \partial_j] = 0,$ *for all* $i, j \in \{1, 2, \ldots, n\}$.

Proof. (a): This is clear on application of Remark and Exercise 5.3 (B)(a) with $a = X_i$ and $b = X_j$.

(b): If we apply Remark and Exercise 5.3 (B)(b) with $a = X_i$ and $d = \partial_j$, and observe that $\partial_j(X_i) = \delta_{j,i} = \delta_{i,j}$ we get our claim.

(c): Observe that for all $i, k \in \{1, 2, \ldots, n\}$ we have $\partial_i(X_k) \in \{0, 1\} \subseteq K$. So for all $i, j, k \in \{1, 2, \ldots, n\}$ we obtain

$$[\partial_i, \partial_j](X_k) = \partial_i\big(\partial_j(X_k)\big) - \partial_j\big(\partial_i(X_k)\big) \in \partial_i(K) + \partial_j(K) = 0 + 0 = 0.$$

Now, we get our claim by Exercise and Definition 4.2 (B) and Remark and Exercise 5.3 (B) (c) and (C). □

In the next section we shall establish an important product formula for elementary differential operators. The exercise to come is is aimed to prepare this.

5.5. **Exercise.** (A) Let $n \in \mathbb{N}$, let K be a field, let B be a K-algebra and let

$$a_1, a_1, \ldots, a_n; d_1, d_2, \ldots, d_n \in B$$

be elements *mimicking the Heisenberg relations*, which means:

(1) $[a_i, a_j] = 0$, for all $i, j \in \{1, 2, \ldots, n\}$;
(2) $[a_i, d_j] = -\delta_{i,j}$, for all $i, j \in \{1, 2, \ldots, n\}$;
(3) $[d_i, d_j] = 0$, for all $i, j \in \{1, 2, \ldots, n\}$.

Let $\mu, \nu \in \mathbb{N}_0$. To simplify notations, we set

$$0b^k := 0 \text{ for all } b \in B \text{ and all } k \in \mathbb{Z}.$$

prove the following statements (using induction on μ and ν):

(a) $a_i^\mu a_j^\nu = a_j^\nu a_i^\mu$;
(b) $d_i^\mu d_j^\nu = d_j^\nu d_i^\mu$;
(c) $d_i^\mu a_j^\nu = a_j^\nu d_i^\mu$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.
(d) $d_i a_i^\nu = a_i^\nu d_i + \nu a_i^{\nu-1}$ for all $i \in \{1, 2, \ldots, n\}$.

(B) Keep the notations and hypotheses of part (A). For all $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}_0$ and each sequence $(b_1, b_2, \ldots, b_n) \in B^n$ we use again our earlier standard notation

$$\underline{\lambda} := (\lambda_1, \lambda_2, \ldots, \lambda_n) \text{ and } \underline{b}^{\underline{\lambda}} := b_1^{\lambda_1} b_2^{\lambda_2} \ldots b_n^{\lambda_n} = \prod_{i=1}^n b_i^{\lambda_i}.$$

Now, let $\underline{\mu}, \underline{\nu}, \underline{\mu}', \underline{\nu}' \in \mathbb{N}_0^n$ and prove that the following relations hold:

(a) $(\underline{a}^{\underline{\nu}} \underline{d}^{\underline{\mu}})(\underline{a}^{\underline{\nu}'} \underline{d}^{\underline{\mu}'}) = \big(\prod_{i=1}^n a_i^{\nu_i} \prod_{j=1}^n d_i^{\mu_j}\big)\big(\prod_{i=1}^n a_i^{\nu_i'} \prod_{j=1}^n d_i^{\mu_j'}\big) = \prod_{i=1}^n a_i^{\nu_i} d_i^{\mu_i} a_i^{\nu_i'} d_i^{\mu_i'}$.
(b) $\underline{a}^{\underline{\nu}} \underline{d}^{\underline{\mu}} = \prod_{i=1}^n a_i^{\nu_i} \prod_{j=1}^n d_i^{\mu_j} = \prod_{i=1}^n a_i^{\nu_i} d_i^{\mu_i}$.

6. Arithmetic in Weyl Algebras

This section provides the basic tools for computations with differential operators. We begin with the following auxiliary result.

6.1. **Lemma.** *Let $n \in \mathbb{N}$, let K be a field, let B be a K-algebra and let*

$$a_1, a_2, \ldots, a_n; d_1, d_2, \ldots, d_n \in B$$

such that:

(1) $[a_i, a_j] = 0,$ *for all* $i, j \in \{1, 2, \ldots, n\}$;
(2) $[a_i, d_j] = -\delta_{i,j},$ *for all* $i, j \in \{1, 2, \ldots, n\}$;
(3) $[d_i, d_j] = 0,$ *for all* $i, j \in \{1, 2, \ldots, n\}$.

Then, the following statements hold:

(a) *For all* $\mu, \nu \in \mathbb{N}_0$ *and all* $i \in \{1, 2, \ldots, n\}$ *we have*

$$d_i^\mu a_i^\nu = \sum_{k=0}^{\min\{\mu,\nu\}} \binom{\mu}{k} \prod_{p=0}^{k-1} (\nu - p) a_i^{\nu-k} d_i^{\mu-k}.$$

(b) *Let* $\underline{\mu}, \underline{\nu}, \underline{\mu}', \underline{\nu}' \in \mathbb{N}_0^n$, *set*

$$\mathbb{I} := \{\underline{k} := (k_1, k_2, \ldots, k_n) \in \mathbb{N}_0^n \mid k_i \leq \min\{\mu_i, \nu_i'\} \text{ for } i = 1, 2, \ldots, n\},$$

and let

$$\lambda_{\underline{k}} := \Big(\prod_{i=1}^n \binom{\mu_i}{k_i}\Big)\Big(\prod_{i=1}^n \prod_{p=0}^{k_i-1} (\nu_i' - p)\Big) \quad \text{for all } \underline{k} \in \mathbb{I}.$$

Then, we have the relation

$$(\underline{a}^{\underline{\nu}} \underline{d}^{\underline{\mu}})(\underline{a}^{\underline{\nu}'} \underline{d}^{\underline{\mu}'}) = \underline{a}^{\underline{\nu}+\underline{\nu}'} \underline{d}^{\underline{\mu}+\underline{\mu}'} + \sum_{\underline{k} \in \mathbb{I} \setminus \{\underline{0}\}} \lambda_{\underline{k}} \underline{a}^{\underline{\nu}+\underline{\nu}'-\underline{k}} \underline{d}^{\underline{\mu}+\underline{\mu}'-\underline{k}}.$$

Proof. The proof consist of tedious computation making repeatedly use of the exercises of the previous section. We recommend to consult [6]. □

As an application we now get the announced product formula for elementary differential operators.

6.2. **Proposition.** ***(The Product Formula for Elementary Differential Operators)*** *Let $n \in \mathbb{N}$, let K be a field and consider the standard Weyl algebra*

$$\mathbb{W}(K, n) = K[X_1, X_2, \ldots X_n; \partial_1, \partial_2 \ldots, \partial_n].$$

Moreover, let all further notations be as in Lemma 6.1. Then, we have the equality

$$(\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}})(\underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}) = \underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'} + \sum_{\underline{k} \in \mathbb{I} \setminus \{\underline{0}\}} \lambda_{\underline{k}} \underline{X}^{\underline{\nu}+\underline{\nu}'-\underline{k}} \underline{\partial}^{\underline{\mu}+\underline{\mu}'-\underline{k}}.$$

Proof. It suffices to apply Lemma 6.1 (b) with $a_i := X_i$ and $d_i := \partial_i$ for $i = 1, 2 \ldots, n$. □

To approach the main result of this section, we need some more preparations.

6.3. **Notation and Remark.** (A) Let $n \in \mathbb{N}$. For $\underline{\kappa} := (\kappa_1, \kappa_2, \ldots, \kappa_n)$, $\underline{\lambda} := (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}_0^n$ we define

$$\underline{\kappa} \leq \underline{\lambda} :\Leftrightarrow \big(\forall i = 1, 2, \ldots, n : \kappa_i \leq \lambda_i\big) \quad \text{and} \quad \underline{\kappa} < \underline{\lambda} :\Leftrightarrow \big(\underline{\kappa} \leq \underline{\lambda} \quad \wedge \quad \underline{\kappa} \neq \underline{\lambda}\big).$$

(B) Keep the notations of part (A). Observe that

$$\underline{\kappa} \leq \underline{\lambda} \Leftrightarrow \big(\underline{\lambda} - \underline{\kappa} \in \mathbb{N}_0^n\big) \quad \text{and} \quad \underline{\kappa} < \underline{\lambda} \Leftrightarrow \big(\underline{\lambda} - \underline{\kappa} \in \mathbb{N}_0^n \setminus \{\underline{0}\}\big).$$

(C) We now introduce a few notations, which we will have to use later very frequently. Namely, for $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n), \underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}_0^n$ we set

$\mathbb{M}(\underline{\alpha}, \underline{\beta}) := \{(\underline{\alpha} - \underline{k}, \underline{\beta} - \underline{k}) \mid \underline{k} \in \mathbb{N}_0^n \setminus \{\underline{0}\} \text{ with } \underline{k} \leq \underline{\alpha}, \underline{\beta}\}$ and $\overline{\mathbb{M}}(\underline{\alpha}, \underline{\beta}) := \mathbb{M}(\underline{\alpha}, \underline{\beta}) \cup \{(\underline{\alpha}, \underline{\beta})\}$.

Moreover, we write

$$\mathbb{M}_{\leq}(\underline{\alpha}, \underline{\beta}) := \{(\underline{\lambda}, \underline{\kappa}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid \underline{\lambda} \leq \underline{\nu} \text{ and } \underline{\kappa} \leq \underline{\mu} \text{ for some } (\underline{\nu}, \underline{\mu}) \in \mathbb{M}(\underline{\alpha}, \underline{\beta})\}.$$

Observe that $\mathbb{M}(\underline{\alpha}, \underline{\beta}) \subseteq \mathbb{M}_{\leq}(\underline{\alpha}, \underline{\beta})$.

6.4. **Exercise.** (A) Let $n \in \mathbb{N}$, let K be a field and consider the standard Weyl algebra $\mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n]$. In addition, let $\underline{\mu}, \underline{\nu}, \underline{\mu}', \underline{\nu}' \in \mathbb{N}_0^n$. Prove that

$$(\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}})(\underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}) - \underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'} \in \sum_{(\underline{\lambda}, \underline{\kappa}) \in \mathbb{M}(\underline{\nu}+\underline{\nu}', \underline{\mu}+\underline{\mu}')} \mathbb{Z} \underline{X}^{\underline{\lambda}} \underline{\partial}^{\underline{\kappa}}.$$

and

$$(\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}})(\underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}) \in \sum_{(\underline{\lambda}, \underline{\kappa}) \in \overline{\mathbb{M}}(\underline{\nu}+\underline{\nu}', \underline{\mu}+\underline{\mu}')} \mathbb{Z} \underline{X}^{\underline{\lambda}} \underline{\partial}^{\underline{\kappa}}.$$

(B) Let the notations be as in part (A). Prove that

$$\big[\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}, \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}\big] \in \sum_{(\underline{\lambda}, \underline{\kappa}) \in \mathbb{M}(\underline{\nu}+\underline{\nu}', \underline{\mu}+\underline{\mu}')} \mathbb{Z} \underline{X}^{\underline{\lambda}} \underline{\partial}^{\underline{\kappa}}.$$

6.5. **Theorem.** ***(The Reduction Principle)*** *Let $n \in \mathbb{N}$, let K be a field and consider the standard Weyl algebra $\mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n]$. Let $r \in \mathbb{N}$ and let $\underline{\nu}^{(i)}, \underline{\mu}^{(i)} \in \mathbb{N}_0^n \quad (i = 1, 2, \ldots, r)$. Moreover set*

$$\mathbb{M} := \mathbb{M}_{\leq}\big(\sum_{i=1}^{r} \underline{\nu}^{(i)}, \sum_{i=1}^{r} \underline{\mu}^{(i)}\big) \subset \mathbb{N}_0^n \times \mathbb{N}_0^n$$

Then, we have

$$\prod_{i=1}^{r} \underline{X}^{\underline{\nu}^{(i)}} \underline{\partial}^{\underline{\mu}^{(i)}} - \underline{X}^{\sum_{i=1}^{r} \underline{\nu}^{(i)}} \underline{\partial}^{\sum_{i=1}^{r} \underline{\mu}^{(i)}} \in \sum_{(\underline{\kappa}, \underline{\lambda}) \in \mathbb{M}} \mathbb{Z} \underline{X}^{\underline{\lambda}} \underline{\partial}^{\underline{\kappa}} \subseteq \sum_{\underline{\lambda} < \sum_{i=1}^{r} \underline{\nu}^{(i)}, \quad \underline{\kappa} < \sum_{i=1}^{r} \underline{\mu}^{(i)}} \mathbb{Z} \underline{X}^{\underline{\lambda}} \underline{\partial}^{\underline{\kappa}}.$$

Proof. We proceed by induction on r. The case $r = 1$ is obvious. The case $r = 2$ follows from Exercise 6.4 (A). So, let $r > 2$. We set

$$\mathbb{M}' := \mathbb{M}_{\leq}\big(\sum_{i=1}^{r-1} \underline{\nu}^{(i)}, \sum_{i=1}^{r-1} \underline{\mu}^{(i)}\big).$$

By induction we have

$$\varrho := \prod_{i=1}^{r-1} \underline{X}^{\underline{\nu}^{(i)}} \underline{\partial}^{\underline{\mu}^{(i)}} - \underline{X}^{\sum_{i=1}^{r-1} \underline{\nu}^{(i)}} \underline{\partial}^{\sum_{i=1}^{r-1} \underline{\mu}^{(i)}} \in \sum_{(\underline{\lambda}',\underline{\kappa}')\in\mathbb{M}'} \mathbb{Z}\underline{X}^{\underline{\lambda}'}\underline{\partial}^{\underline{\kappa}'} =: N.$$

By the case $r = 2$ we have (see once more Exercise 6.4 (A))

$$\sigma := \big(\underline{X}^{\sum_{i=1}^{r-1} \underline{\nu}^{(i)}} \underline{\partial}^{\sum_{i=1}^{r-1} \underline{\mu}^{(i)}}\big)\underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}} - \underline{X}^{\sum_{i=1}^{r} \underline{\nu}^{(i)}} \underline{\partial}^{\sum_{i=1}^{r} \underline{\mu}^{(i)}} \in \sum_{(\underline{\lambda},\underline{\kappa})\in\mathbb{M}} \mathbb{Z}\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}} =: M.$$

As

$$\prod_{i=1}^{r} \underline{X}^{\underline{\nu}^{(i)}} \underline{\partial}^{\underline{\mu}^{(i)}} - \underline{X}^{\sum_{i=1}^{r} \underline{\nu}^{(i)}} \underline{\partial}^{\sum_{i=1}^{r} \underline{\mu}^{(i)}} = \sigma + \varrho \underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}},$$

it remains to show that $\varrho \underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}} \in M$. Observe that

$$\varrho \underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}} \in N \underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}} = \sum_{(\underline{\lambda}',\underline{\kappa}')\in\mathbb{M}'} \mathbb{Z}\underline{X}^{\underline{\lambda}'}\underline{\partial}^{\underline{\kappa}'}\underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}}.$$

Observe also that $(\underline{\lambda}'+\underline{\nu}^{(r)}, \underline{\kappa}'+\underline{\mu}^{(r)}) \in \mathbb{M}$ for all $(\underline{\lambda}', \underline{\kappa}') \in \mathbb{M}'$ Therefore we have the inclusion $\overline{\mathbb{M}}(\underline{\lambda}' + \underline{\nu}^{(r)}, \underline{\kappa}' + \underline{\mu}^{(r)}) \subseteq \mathbb{M}$ for all $(\underline{\lambda}', \underline{\kappa}') \in \mathbb{M}'$. Thus, on application of Exercise 6.4 (A) it follows that

$$\underline{X}^{\underline{\lambda}'}\underline{\partial}^{\underline{\kappa}'}\underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}} \in \sum_{(\underline{\lambda},\underline{\kappa})\in\overline{\mathbb{M}}(\underline{\lambda}'+\underline{\nu}^{(r)},\underline{\kappa}'+\underline{\mu}^{(r)})} \mathbb{Z}\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}} \subseteq \sum_{(\underline{\lambda},\underline{\kappa})\in\mathbb{M}} \mathbb{Z}\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}} = M,$$

and this shows that indeed $\varrho \underline{X}^{\underline{\nu}^{(r)}} \underline{\partial}^{\underline{\mu}^{(r)}} \in M$. □

To prepare what we aim for in the next section, we suggest the following exercise.

6.6. **Exercise.** (A) Let $n \in \mathbb{N}$ and consider the polynomial ring $K[X_1, X_2, \ldots, X_n]$. Moreover, let $\underline{\mu} := (\mu_1, \mu_1, \ldots, \mu_n), \underline{\nu} := (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}_0^n$. Fix $i \in \{1, 2, \ldots, n\}$ and prove by induction on μ_i, that

$$\partial_i^{\mu_i}\big(\underline{X}^{\underline{\nu}}\big) = \partial_i^{\mu_i}\big(\prod_{j=1}^{i} X_j^{\nu_j}\big) = \begin{cases} \prod_{k=0}^{\mu_i-1}(\nu_i - k)X_i^{\nu_i-\mu_i}\prod_{j\neq i} X_j^{\nu_j}, & \text{if } \nu_i \geq \mu_i; \\ 0, & \text{if } \nu_i < \mu_i. \end{cases}$$

(B) Let the notations and hypotheses be as in part (A) and use what you have shown there to prove that

$$\underline{\partial}^{\underline{\mu}}\big(\underline{X}^{\nu}\big) = \prod_{i=1}^{n} \partial_i^{\mu_i}\big(\prod_{j=1}^{n} X^{\nu_j}\big) = \begin{cases} \prod_{i=1}^{n}\prod_{k=0}^{\mu_i-1}(\nu_i - k)\underline{X}^{\underline{\nu}-\underline{\mu}}, & \text{if } \underline{\nu} \geq \underline{\mu}; \\ 0, & \text{otherwise }. \end{cases}$$

7. The Standard Basis

Now, we are aim prove that – over a base field of characteristic 0 – the elementary differential operators form a vector space basis of the standard Weyl algebra.

7.1. **Theorem.** ***(The Standard Basis)*** *Let $n \in \mathbb{N}$ and let K be a field of characteristic 0. Then, the elementary differential operators*

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} = \prod_{i=n}^{n} X_i^{\nu_i} \prod_{i=1}^{n} \partial_i^{\mu_i}$$

form a K-vector space basis of the standard Weyl algebra

$$\mathbb{W}(K,n) = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n].$$

So, in particular we can say

(a) $\mathbb{W}(K,n) = \bigoplus_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}$.

(b) *Each differential operator $d \in \mathbb{W}(K,n)$ can be written in the form*

$$d = \sum_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n} c^{(d)}_{\underline{\nu},\underline{\mu}} \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}$$

with a unique family

$$\big(c^{(d)}_{\underline{\nu},\underline{\mu}}\big)_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n} \in \prod_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n} K = K^{\mathbb{N}_0^n\times\mathbb{N}_0^n},$$

whose support

$$\operatorname{supp}(d) = \operatorname{supp}\big((c^{(d)}_{\underline{\nu},\underline{\mu}})_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0}\big) := \{(\underline{\nu},\underline{\mu}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid c^{(d)}_{\underline{\nu},\underline{\mu}} \neq 0\}$$

is a finite set.

Proof. We first show that

$$\mathbb{W}(K,n) = \sum_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} =: M.$$

Indeed, each $d \in \mathbb{W}(K,n)$ is a K-linear combination of products of elementary differential operators. But by the Reduction Principle of Theorem 6.5 each product of elementary differential operators is contained in the K-vector space M.
It remains to show, that the elementary differential operators are linearly independent among each other. Assume to the contrary, that there are linearly dependent elementary differential operators in $\mathbb{W}(K,n)$. Then, we find a positive integer $r \in \mathbb{N}$, families

$$\underline{\mu}^{(i)}, \underline{\nu}^{(i)} \in \mathbb{N}_0^n, \quad (i = 1,2,\ldots,r) \text{ with } (\underline{\mu}^{(i)}, \underline{\nu}^{(i)}) \neq (\underline{\mu}^{(j)}, \underline{\nu}^{(j)}) \text{ if } i \neq j,$$

and elements

$$c^{(i)} \in K \setminus \{0\} \quad (i = 1,2,\ldots,r) \text{ such that } d := \sum_{i=1}^{r} c^{(i)} \underline{X}^{\underline{\nu}^{(i)}} \underline{\partial}^{\underline{\mu}^{(i)}} = 0.$$

Using the standard notation $|\underline{\mu}| := \sum_{k=1}^{n} \mu_k$, for all $\underline{\mu} \in \mathbb{N}_0^n$, we may assume, that

$$|\underline{\mu}^{(r)}| = \max\{|\underline{\mu}^{(i)}| \mid i = 1,2,\ldots,r\}, \underline{\mu}^{(i)} \neq \underline{\mu}^{(r)} \text{ for all } i < s \text{ and } \underline{\mu}^{(i)} = \underline{\mu}^{(r)} \text{ for all } i \geq s$$

for some $s \in \{1, 2, \ldots, r\}$. Then, it follows easily by what we have seen in Exercise 6.6 (B), that

$$\underline{X}^{\underline{\nu}^{(i)}}\underline{\partial}^{\underline{\mu}^{(i)}}\big(\underline{X}^{\underline{\mu}^{(r)}}\big) = \begin{cases} \prod_{j=1}^{n} \mu_j^{(r)}!\underline{X}^{\underline{\nu}^{(r)}}, & \text{if } s \leq i \leq r \\ 0, & \text{if } i < s. \end{cases}$$

So, we get

$$0 = d\big(\underline{X}^{\underline{\mu}^{(r)}}\big) = \sum_{i=1}^{r} c^{(i)}\underline{X}^{\underline{\nu}^{(i)}}\underline{\partial}^{\underline{\mu}^{(i)}}\big(\underline{X}^{\underline{\mu}^{(r)}}\big) = \sum_{i=s}^{r} c^{(i)}\prod_{j=1}^{n}\mu_j^{(r)}!\underline{X}^{\underline{\nu}^{(i)}}.$$

As $\mathrm{Char}(K) = 0$, and as the monomials $\underline{X}^{\underline{\nu}^{(i)}}$ are pairwise different for $i = s, s+1, \ldots, r$, the last sum does not vanish, and we have a contradiction. □

7.2. **Definition and Remark.** (A) Let the notations and hypotheses be as in Theorem 6.5. We call the basis of $\mathbb{W}(K,n)$ which consists of all elementary differential operators the *standard basis.* If we present a differential operator $d \in \mathbb{W}(K,n)$ with respect to the standard basis and write

$$d = \sum_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n} c_{\underline{\nu},\underline{\mu}}^{(d)}\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}$$

as in statement (b) of Theorem 6.5, we say that d is written in *standard form.* The support of a differential operator d in $\mathbb{W}(K,n)$ is always defined with respect to the standard form as in statement (b) of Theorem 7.1. We therefore call the support of d also the *standard support* of d.

(B) Keep the above notations and hypotheses. It is a fundamental task, to write an arbitrarily given differential operator $d \in \mathbb{W}(K,n)$ in standard form. This task actually is reduced by the Reduction Principle of Theorem 6.5 to make explicit the coefficients of the differences

$$\Delta_{\underline{\nu}^{(\bullet)}\underline{\mu}^{(\bullet)}} := \prod_{i=1}^{r}\underline{X}^{\underline{\nu}^{(i)}}\underline{\partial}^{\underline{\mu}^{(i)}} - \underline{X}^{\sum_{i=1}^{r}\underline{\nu}^{(i)}}\underline{\partial}^{\sum_{i=1}^{r}\underline{\mu}^{(i)}} \in \sum_{(\underline{\lambda},\underline{\kappa})\in\mathbb{M}}\mathbb{Z}\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}}.$$

This task can be solved by a repeated application of the Product Formula of Propsition 6.2 or – directly – by a repeated application of the Heisenberg relations. Clearly, today this task usually is performed by means of Computer Algebra systems.

As an application, one has the following result on supports of differential operators:

7.3. **Proposition.** ***(Behavior of Supports)*** *Let $n \in \mathbb{N}$, let K be a field of characteristic 0 and consider the differential operators*

$$d, e \in \mathbb{W}(K,n) = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n].$$

For all $(\underline{\alpha}, \underline{\beta}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$, let the sets

$$\mathbb{M}(\underline{\alpha}, \underline{\beta}) \subset \overline{\mathbb{M}}(\underline{\alpha}, \underline{\beta}) \subset \mathbb{N}_0^n \times \mathbb{N}_0^n$$

be defined according to Notation and Remark 6.3 (C). Then, we have

(a) $\big(\mathrm{supp}(d) \cup \mathrm{supp}(e)\big) \setminus \big(\mathrm{supp}(d) \cap \mathrm{supp}(e)\big) \subseteq \mathrm{supp}(d+e) \subseteq \mathrm{supp}(d) \cup \mathrm{supp}(e)$.

(b) $\mathrm{supp}(cd) = \mathrm{supp}(d)$ *for all* $c \in K \setminus \{0\}$.

(c) $\mathrm{supp}(de) \subseteq \bigcup_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} \overline{\mathbb{M}}(\underline{\nu}+\underline{\nu}', \underline{\mu}+\underline{\mu}')$.

(d) $\operatorname{supp}\big([d,e]\big) \subseteq \bigcup_{(\underline{\nu},\underline{\mu})\in \operatorname{supp}(d),(\underline{\nu}',\underline{\mu}')\in\operatorname{supp}(e)} \mathbb{M}(\underline{\nu}+\underline{\nu}',\underline{\mu}+\underline{\mu}')$.

Proof. Statements (a) and (b) are straight forward from our definition of support. Statements (c) and (d) follow by Theorem 7.1, Exercise 6.4 (A) respectively Exercise 6.4 (B) and an repeated application of statements (a) and (b). For further hints see [6]. □

7.4. **Exercise.** (A) Let $n \in \mathbb{N}$, let K be a field of characteristic 0 and consider the standard Weyl algebra $\mathbb{W}(K,n) = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n]$. Prove in detail statements (a), (b), (c) and (d) of Proposition 7.3.

(B) Let the notations and hypotheses be as in part (A). Present in standard form the following differential operators:

$$\partial_1^2 X_1^2 - X_1\partial_1 X_1 - 1, \quad \partial_1^2 X_1^2 \partial_1^2 - \partial_1 X_1^2, \quad \partial_2 X_1 X_2 \partial_1 + \partial_1 X_1 X_2 \in \mathbb{W}(K,n).$$

(C) Keep the notations of part (A), but assume that $n = 1$ and $\operatorname{Char}(K) = 2$. Compute $\partial_1(X_1^\nu)$ for all $\nu \in \mathbb{N}_0$ and comment your findings in view of the Standard Basis Theorem.

As another application of the Standard Basis Theorem we now can prove

7.5. **Corollary.** ***(The Universal Property of Weyl Algebras)*** *Let the notations and hypotheses be as in Theorem 7.1. Let B be a K-algebra and let*

$$\phi : \{X_1, X_2, \ldots, X_n, \partial_1, \partial_2, \ldots, \partial_n\} \longrightarrow B$$

be a map "which respects the Heisenberg relations" and hence satisfies the requirements

(1) $[\phi(X_i), \phi(X_j)] = 0,$ *for all* $i, j \in \{1, 2, \ldots, n\}$*;*
(2) $[\phi(X_i), \phi(\partial_j)] = -\delta_{i,j},$ *for all* $i, j \in \{1, 2, \ldots, n\}$*;*
(3) $[\phi(\partial_i), \phi(\partial_j)] = 0,$ *for all* $i, j \in \{1, 2, \ldots, n\}$.

Then, there is a unique homomorphism of K-algebras

$$\widetilde{\phi} : \mathbb{W}(K,n) \longrightarrow B$$

such that

$$\widetilde{\phi}(X_i) = \phi(X_i) \text{ and } \widetilde{\phi}(\partial_i) = \phi(\partial_i) \text{ for all } i = 1, 2, \ldots, n.$$

Proof. According to Theorem 7.1 there is a K-linear map

$$\widetilde{\phi} : \mathbb{W}(K,n) \longrightarrow B \text{ given by } \widetilde{\phi}\big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big) = \prod_{i=1}^n \phi(X_i)^{\nu_i} \prod_{i=1}^n \phi(\partial_i)^{\mu_i}, \quad (\underline{\mu}, \underline{\nu} \in \mathbb{N}_0^n).$$

Next, we show, that $\widetilde{\phi}$ is multiplicative, hence satisfies the condition that

$$\widetilde{\phi}(de) = \widetilde{\phi}(d)\widetilde{\phi}(e) \text{ for all } d, e \in \mathbb{W}(K,n).$$

As the multiplication maps

$$\mathbb{W}(K,n) \times \mathbb{W}(K,n) \longrightarrow \mathbb{W}(K,n), (d,e) \mapsto de \quad \text{ and } \quad B \times B \longrightarrow B, (a,b) \mapsto ab$$

are both K-bilinear, it suffices to verify the above multiplicativity condition in the special case where

$$d := \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \quad \text{ and } \quad e := \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} \quad \text{ with } \underline{\mu}, \underline{\nu}, \underline{\mu}', \underline{\nu}' \in \mathbb{N}_0^n.$$

But this can be done by a straight forward computation, on use of the Product Formula of Proposition 6.2 and on application of Lemma 6.1 with $a_i : \phi(X_i)$ and $d_i := \phi(\partial_i)$ for all $i = 1, 2, \ldots, n$.
It remains to show, that $\widetilde{\phi} : \mathbb{W}(K, n) \longrightarrow B$ is the only homomorphism of K algebras which satisfies the requirement that

$$\widetilde{\phi}(X_i) = \phi(X_i) \quad \text{and} \quad \widetilde{\phi}(\partial_i) = \phi(\partial_i) \text{ for all } i = 1, 2, \ldots, n.$$

But indeed, if a map $\widetilde{\phi}$ satisfies this requirement and is multiplicative, it must be defined on the elementary differential operators as suggested above. This proves the requested uniqueness. □

7.6. **Exercise.** (A) Let $n \in \mathbb{N}$, let K be a field of characteristic 0. Show, that there is a unique automorphism of K-algebras

$$\alpha : \mathbb{W}(K, n) \xrightarrow{\cong} \mathbb{W}(K, n) \text{ with } \alpha(X_i) = \partial_i \text{ and } \alpha(\partial_i) = -X_i \text{ for all } i = 1, 2, \ldots, n.$$

(B) Keep the notations and hypotheses of part (A). Present in standard form all elements $\alpha(X_i^\nu \partial_i^\mu) \in \mathbb{W}(K, n)$ with $\mu, \nu \in \mathbb{N}_0$.

8. Weighted Degrees and Filtrations

We now study a class of particularly important filtrations of standard Weyl algebras, induced by weighted degrees.

8.1. **Convention.** Throughout this section we fix a positive integer n, a field K of characteristic 0 and we consider the standard Weyl algebra

$$\mathbb{W} := \mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n].$$

8.2. **Definition and Remark.** (A) By a *weight* we mean a pair

$$(\underline{v}, \underline{w}) = \big((v_1, v_2, \ldots, v_n), (w_1, w_2, \ldots, w_n)\big) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \text{ with } (v_i, w_i) \neq (0, 0) \quad (i = 1, 2, \ldots, n).$$

For $\underline{a} := (a_1, a_2, \ldots, a_n)$, $\underline{b} := (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ we frequently shall use the *scalar product*

$$\underline{a} \cdot \underline{b} := \sum_{i=1}^{n} a_i b_i.$$

(B) Fix a weight $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$. We define the *degree associated to the weight* $(\underline{v}, \underline{w})$ (or just the *weighted degree*) of a differential form $d \in \mathbb{W}$ by

$$\deg^{\underline{vw}}(d) := \sup\{\underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} \mid (\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(d)\}.$$

with the usual convention that $\sup(\emptyset) = -\infty$. Observe that for all $d \in \mathbb{W}$ and all $\underline{\mu}, \underline{\nu} \in \mathbb{N}_0$ – and using the notations of Notation and Remark 6.3 (C)– we can say:

(a) $\deg^{\underline{vw}}(d) \in \mathbb{N}_0 \cup \{-\infty\}$ with $\deg^{\underline{vw}}(d) = -\infty$ if and only if $d = 0$.
(b) If $\underline{\lambda} \leq \underline{\nu}$ and $\underline{\kappa} \leq \underline{\mu}$ for all $(\underline{\lambda}, \underline{\mu}) \in \operatorname{supp}(d)$, then $\deg^{\underline{vw}}(d) \leq \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu}$.
(c) If $\operatorname{supp}(d) \subseteq \mathbb{M}_{\leq}(\underline{\nu}, \underline{\mu})$, then $\deg^{\underline{vw}}(d) < \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu}$.

(C) Keep the notations and hypotheses of part (B). We fix some non-negative integer $i \in \mathbb{N}_0$ and set

$$\mathbb{W}_i^{\underline{vw}} := \{d \in \mathbb{W} \mid \deg^{\underline{vw}}(d) \leq i\} = \bigoplus_{\underline{\nu},\underline{\mu} \in \mathbb{N}_0^n : \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} \leq i} K \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}.$$

8.3. **Lemma.** *Let $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ be a weight and let $d, e \in \mathbb{W}$. Then we have*

(a) $\deg^{\underline{vw}}(d+e) \leq \max\{\deg^{\underline{vw}}(d), \deg^{\underline{vw}}(d)\}$, *with equality if* $\deg^{\underline{vw}}(d) \neq \deg^{\underline{vw}}(e)$;
(b) $\deg^{\underline{vw}}(cd) = \deg^{\underline{vw}}(d)$ *for all* $c \in K \setminus \{0\}$.
(c) $\deg^{\underline{vw}}(de) \leq \deg^{\underline{vw}}(d) + \deg^{\underline{vw}}(e)$;
(d) $\deg^{\underline{vw}}\big([d, e]\big) < \deg^{\underline{vw}}(d) + \deg^{\underline{vw}}(e)$.

Proof. We Leave the proof to the reader, with the hint to eventually consult [6]. □

8.4. **Theorem.** ***(Weighted Filtrations)*** *Let $\big((v_1, v_2, \ldots, v_n), (w_1, w_2, \ldots, w_n)\big) = (\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ be a weight. Then, the family*

$$\mathbb{W}_\bullet^{\underline{vw}} := \big(\mathbb{W}_i^{\underline{vw}} = \{d \in \mathbb{W} \mid \deg^{\underline{vw}}(d) \leq i\}\big)_{i \in \mathbb{N}_0}$$

is a commutative filtration of the the K-algebra $\mathbb{W} = \mathbb{W}(K, n)$. Moreover, the following statements hold:

(a) $\mathbb{W}_0^{\underline{vw}} = K[X_i, \partial_j \mid v_i = 0, w_j = 0]$.
(b) *If* $i > \delta := \max\{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$, *then* $\mathbb{W}_i^{\underline{vw}} = \sum_{j=1}^{\delta} \mathbb{W}_j^{\underline{vw}} \mathbb{W}_{i-j}^{\underline{vw}}$.
(c) *The filtration* $\mathbb{W}_\bullet^{\underline{vw}} = \big(\mathbb{W}_i^{\underline{vw}}\big)_{i \in \mathbb{N}_0}$ *is of finite type.*

Proof. It is clear from our definitions and by Lemma 8.3 (c) that for all $i, j \in \mathbb{N}_0$ we have:

$$1 \in \mathbb{W}_0^{\underline{vw}}, \quad \mathbb{W}_i^{\underline{vw}} \subseteq \mathbb{W}_{i+1}^{\underline{vw}}, \quad \mathbb{W} = \bigcup_{i \in \mathbb{N}_0} \mathbb{W}_i^{\underline{vw}}, \quad \mathbb{W}_i^{\underline{vw}} \mathbb{W}_j^{\underline{vw}} \subseteq \mathbb{W}_{i+j}^{\underline{vw}}.$$

So the family $\big(\mathbb{W}_i^{\underline{vw}}\big)_{i \in \mathbb{N}_0}$ constitutes indeed a filtration on the K-algebra $\mathbb{W}$.

Now, let $i, j \in \mathbb{N}_0$, let $d \in \mathbb{W}_i^{\underline{vw}}$ and let $e \in \mathbb{W}_j^{\underline{vw}}$. Then by Lemma 8.3 (d) we have

$$\deg^{\underline{vw}}\big(de - ed\big) \leq \deg^{\underline{vw}}(d) + \deg^{\underline{vw}}(e) - 1 \leq i + j - 1, \text{ hence } de - ed \in \mathbb{W}_{i+j-1}^{\underline{vw}}.$$

This proves, that our filtration is commutative (see Definition 3.2).

(a): We define $\mathbb{S} := \{i = 1, 2, \ldots, n \mid v_i \neq 0\}$, $\mathbb{T} := \{j = 1, 2, \ldots, n \mid w_j \neq 0\}$, $\overline{\mathbb{S}} := \{1, 2, \ldots, n\} \setminus \mathbb{S}$, $\overline{\mathbb{T}} := \{1, 2, \ldots, n\} \setminus \mathbb{T}$ and chose $\underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n$. Then

$$\underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} = 0 \text{ if and only if } \nu_i = 0 \text{ for all } i \in \mathbb{S} \text{ and } \mu_j = 0 \text{ for all } j \in \mathbb{T}.$$

But this means that

$$\mathbb{W}_0^{\underline{vw}} = \sum_{(\nu_i)_{i \in \overline{\mathbb{S}}} \in \mathbb{N}_0^{\overline{\mathbb{S}}} \text{ and } (\mu_j)_{j \in \overline{\mathbb{T}}} \in \mathbb{N}_0^{\overline{\mathbb{T}}}} K \prod_{i \in \overline{\mathbb{S}} \text{ and } j \in \overline{\mathbb{T}}} X_i^{\nu_i} \partial_j^{\mu_j} = K[X_i, \partial_j \mid v_i = 0, w_j = 0].$$

(b): Let $i > \delta$. Let $\underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n$ with $\sigma := \deg^{\underline{vw}}\big(\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}\big) = \underline{v} \cdot \underline{\nu} + \underline{w} \cdot \underline{\mu} \leq i$. We aim to show that

$$\underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \in \sum_{j=1}^{\delta} \mathbb{W}_j^{\underline{vw}} \mathbb{W}_{i-j}^{\underline{vw}} =: M.$$

If $\sigma \leq 0$ this is clear as

$$\mathbb{W}_0^{\underline{vw}} = \mathbb{W}_0^{\underline{vw}}\mathbb{W}_0^{\underline{vw}} \subseteq \mathbb{W}_1^{\underline{vw}}\mathbb{W}_{i-1}^{\underline{vw}} \subseteq M.$$

So, let $\sigma > 0$. Then either

(1) there is some $p \in \{1, 2, \ldots, n\}$ with $v_p > 0$ and $\nu_p > 0$, or else,
(2) there is some $q \in \{1, 2, \ldots, n\}$ with $w_q > 0$ and $\mu_q > 0$.

In the above case (1) we can write

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} = X_p d, \text{ with } d := \big(\prod_{k=1}^{n} X_k^{\nu_k - \delta_{k,p}}\big)\underline{\partial}^{\underline{\mu}}.$$

As $\deg^{\underline{vw}}(X_p) = v_p \leq \delta$ and $\deg^{\underline{vw}}(d) = \sigma - v_p$ it follows that

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} = X_p d \in \mathbb{W}_{v_p}^{\underline{vw}}\mathbb{W}_{\sigma - v_p}^{\underline{vw}} \subseteq \mathbb{W}_{v_p}^{\underline{vw}}\mathbb{W}_{i-v_p}^{\underline{vw}} \subseteq M.$$

In the above case (2) we may first assume, that we are not in the case (1). This means in particular that either $v_q = 0$ or $\nu_q = 0$, hence $v_q\nu_q = 0$, so that $\deg^{\underline{vw}}(X_q^{\nu_q}\partial_q) = w_q \leq \delta$. Now, in view of the Heisenberg relations, we may write

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} = X_q^{\nu_q}\partial_q e \text{ with } e := \prod_{s \neq q} X_s^{\nu_s} \prod_{k=1}^{n} \partial_k^{\mu_k - \delta_{k,q}}.$$

As $v_q\nu_q = 0$, we have $\deg^{\underline{vw}}(e) = \sigma - w_q$, and it follows that

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} = X_q^{\nu_q}\partial_q e \in \mathbb{W}_{w_q}^{\underline{vw}}\mathbb{W}_{\sigma - w_q}^{\underline{vw}} \subseteq \mathbb{W}_{w_q}^{\underline{vw}}\mathbb{W}_{i-w_q}^{\underline{vw}} \subseteq M.$$

But this shows that $\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \in M$.

(c): This is an immediate consequence of statements (a) and (b) (see Definition and Remark 3.3 (C)). $\square$

8.5. **Definition.** Let the notations and hypotheses be as in Theorem 8.4. Then, the filtration

$$\mathbb{W}_\bullet^{\underline{vw}} = \big(\mathbb{W}_i^{\underline{vw}}\big)_{i \in \mathbb{N}_0} = \big(\{d \in \mathbb{W} \mid \deg^{\underline{vw}}(d) \leq i\}\big)_{i \in \mathbb{N}_0}$$

is called the *filtration induced by the weight* $(\underline{v}, \underline{w})$. Generally, we call *weighted filtrations* all fltrations which are induced in this way by a weight.

8.6. **Definition and Remark.** (A) We consider the strings

$$\underline{0} := (0, 0, \ldots, 0), \quad \underline{1} := (1, 1, \ldots, 1) \in \mathbb{N}_0^n$$

and a differential form $d \in \mathbb{W}$. We define the *standard degree* or just the *degree* $\deg(d)$ of d as the weighted degree with respect to the weight $(\underline{1}, \underline{1}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$, hence

$$\deg(d) := \deg^{\underline{11}}(d).$$

Observe that

$$\deg(d) := \sup\{|\underline{\nu}| + |\underline{\mu}| \mid (\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(d)\}.$$

The corresponding induced weighted filtration

$$\mathbb{W}_\bullet^{\deg} := \mathbb{W}_\bullet^{\underline{11}} = \big(\mathbb{W}_i^{\underline{11}}\big)_{i \in \mathbb{N}_0} = \big(\{d \in \mathbb{W} \mid \deg(d) \leq i\}\big)_{i \in \mathbb{N}_0}$$

is called the *standard degree filtration* or just the *degree filtration* of $\mathbb{W}$.

(B) Keep the notations and hypotheses of part (A). The *order* of the differential operator d is defined by

$$\mathrm{ord}(d) := \deg^{\underline{01}}(d).$$

Observe that

$$\mathrm{ord}(d) = \sup\{|\underline{\mu}| \mid (\underline{\nu}, \underline{\mu}) \in \mathrm{supp}(d)\}.$$

The corresponding induced weighted filtration

$$\mathbb{W}^{\mathrm{ord}}_{\bullet} := \mathbb{W}^{\underline{01}}_{\bullet} = \left(\mathbb{W}^{\underline{01}}_{i}\right)_{i\in\mathbb{N}_0} = \left(\{d \in \mathbb{W} \mid \mathrm{ord}(d) \leq i\}\right)_{i\in\mathbb{N}_0}$$

is called the *order filtration* of $\mathbb{W}$.

Now, as an immediate application of Theorem 8.4 we obtain:

8.7. **Corollary.** *Let the notations be as in Convention 8.1. Then it holds*

(a) *The degree filtration $\mathbb{W}^{\deg}_{\bullet}$ is very good.*
(b) *The order filtration $\mathbb{W}^{\mathrm{ord}}_{\bullet}$ is good and $\mathbb{W}^{\mathrm{ord}}_{0} = K[X_1, X_2, \ldots, X_n]$.*

Proof. In the notations of Theorem 8.4 (b) we have

$$\delta(\underline{1}, \underline{1}) = 1 \text{ and } \delta(\underline{0}, \underline{1}) = 1.$$

Moreover, by Theorem 8.4 (a) we have

$$\mathbb{W}^{\underline{11}}_{0} = K \text{ and } \mathbb{W}^{\underline{01}}_{0} = K[X_1, X_2, \ldots, X_n]$$

This proves our claim (see Definition and Remark 3.3 (C)). □

8.8. **Exercise.** (A) Show that the degree filtration is the only very good filtration on $\mathbb{W}$.

(B) Write down all weights $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ for which the induced filtration $\mathbb{W}^{\underline{vw}}_{\bullet}$ is good.

9. Weighted Associated Graded Rings

This Section is devoted to the study of the associated graded rings of weighted filtrations of standard Weyl algebras.

9.1. **Convention.** Again, throughout this section we fix a positive integer n, a field K of characteristic 0 and consider the standard Weyl algebra

$$\mathbb{W} := \mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n].$$

In addition, we introduce the polynomial ring

$$\mathbb{P} := K[Y_1, Y_2, \ldots, Y_n; Z_1, Z_2, \ldots, Z_n]$$

in the indeterminates $Y_1, Y_2, \ldots, Y_n; Z_1, Z_2, \ldots, Z_n$ with coefficients in the field K.

9.2. **Definition and Remark.** (A) Fix a weight $(\underline{v},\underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ and consider the induced weighted filtration $\mathbb{W}_\bullet^{\underline{v}\underline{w}}$. To write down the corresponding associated graded ring, we introduce the following notation:

$$\mathbb{G}^{\underline{v}\underline{w}} = \bigoplus_{i\in\mathbb{N}_0} \mathbb{G}_i^{\underline{v}\underline{w}} := \mathrm{Gr}_{\mathbb{W}_\bullet^{\underline{v}\underline{w}}}\big(\mathbb{W}^{\underline{v}\underline{w}}\big) = \bigoplus_{i\in\mathbb{N}_0} \mathrm{Gr}_{\mathbb{W}_\bullet^{\underline{v}\underline{w}}}\big(\mathbb{W}^{\underline{v}\underline{w}}\big)_i.$$

(B) Keep the above notations and hypotheses. For each $j \in \mathbb{Z}$ we introduce the notations:

$$\mathbb{I}_{\leq j}^{\underline{v}\underline{w}} := \{(\underline{\nu},\underline{\mu}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid \underline{v}\cdot\underline{\nu} + \underline{w}\cdot\underline{\mu} \leq j\};$$
$$\mathbb{I}_{=j}^{\underline{v}\underline{w}} := \{(\underline{\nu},\underline{\mu}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid \underline{v}\cdot\underline{\nu} + \underline{w}\cdot\underline{\mu} = j\}.$$

Fix some $i \in \mathbb{N}_0$. Observe that

$$\mathbb{G}_i^{\underline{v}\underline{w}} = \mathbb{W}_i^{\underline{v}\underline{w}}/\mathbb{W}_{i-1}^{\underline{v}\underline{w}} = \Big(\big(\bigoplus_{(\underline{\nu},\underline{\mu})\in\mathbb{I}_{\leq i-1}^{\underline{v}\underline{w}}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big) \oplus \big(\bigoplus_{(\underline{\nu},\underline{\mu})\in\mathbb{I}_{=i}^{\underline{v}\underline{w}}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big)\Big)\Big/\Big(\bigoplus_{(\underline{\nu},\underline{\mu})\in\mathbb{I}_{\leq i-1}^{\underline{v}\underline{w}}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\Big).$$

As a consequence, we get an isomorphism of K-vector spaces

$$\epsilon_i^{\underline{v}\underline{w}} : \bigoplus_{(\underline{\nu},\underline{\mu})\in\mathbb{I}_{=i}^{\underline{v}\underline{w}}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \xrightarrow{\cong} \mathbb{G}_i^{\underline{v}\underline{w}}$$

given by

$$\epsilon_i^{\underline{v}\underline{w}}\big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big) = \big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} + \mathbb{W}_{i-1}^{\underline{v}\underline{w}}\big) \in \mathbb{W}_i^{\underline{v}\underline{w}}/\mathbb{W}_{i-1}^{\underline{v}\underline{w}} = \mathbb{G}_i^{\underline{v}\underline{w}}, \quad \big((\underline{\nu},\underline{\mu}) \in \mathbb{I}_{=i}^{\underline{v}\underline{w}}\big).$$

In particular we can say:

$$\text{The family } \big((\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})^* := \epsilon_i^{\underline{v}\underline{w}}(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})\big)_{(\underline{\nu},\underline{\mu})\in\mathbb{I}_{=i}^{\underline{v}\underline{w}}} \text{ is a } K\text{-basis of } \mathbb{G}_i^{\underline{v}\underline{w}}.$$

We call this basis the *standard basis* of $\mathbb{G}_i^{\underline{v}\underline{w}}$. Its elements are called *standard basis elements* of the associated graded ring $\mathbb{G}^{\underline{v}\underline{w}}$.

(C) Keep the previously introduced notation. We add a few more useful observations on standard basis elements. First, observe that we may write

(a) $(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})^* \in \mathbb{G}_{\underline{v}\cdot\underline{\nu}+\underline{w}\cdot\underline{\mu}}^{\underline{v}\underline{w}}$ for all $(\underline{\nu},\underline{\mu}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$.

(b) $X_i^* \in \mathbb{G}_{v_i}^{\underline{v}\underline{w}}$ and $\partial_j^* \in \mathbb{G}_{w_j}^{\underline{v}\underline{w}}$ for all $i,j \in \{1,2,\ldots,n\}$.

Moreover, by the observations made in part (B) we also can say that all standard basis elements form a basis of the whole associated graded ring, thus:

(c) The family $\big((\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})^*\big)_{(\underline{\nu},\underline{\mu})\in\mathbb{N}_0^n\times\mathbb{N}_0^n}$ is a K-basis of $\mathbb{G}^{\underline{v}\underline{w}}$.

Finally, as the associated graded ring is commutative, and keeping in mind how the multiplication in this ring is defined (see Remark and Definition 3.1 (B)) we get the following product formula

(d) $(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}})^* = \big(\prod_{i=1}^n X_i^{\nu_i} \prod_{j=1}^n \partial^{\mu_j}\big)^* = \prod_{i=1}^n \big(X_i^*\big)^{\nu_i} \prod_{j=1}^n \big(\partial_j^*\big)^{\mu_j} =: \big(\underline{X}^*\big)^{\underline{\nu}}\big(\underline{\partial}^*\big)^{\underline{\mu}}$.

9.3. **Exercise and Definition.** (A) We fix a weight $(\underline{v},\underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$, use the notations of Definition and Remark 9.2 (A) and consider the K-subspace

$$\mathbb{P}_i^{\underline{v}\underline{w}} := \bigoplus_{(\underline{\nu},\underline{\mu})\in\mathbb{I}_{=i}^{\underline{v}\underline{w}}} K\underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \subseteq \mathbb{P} \text{ for all } i \in \mathbb{N}_0.$$

Prove the following statements:

(a) $K \in \mathbb{P}_0^{\underline{v}\underline{w}}$;
(b) $\mathbb{P}_i^{\underline{v}\underline{w}} \mathbb{P}_j^{\underline{v}\underline{w}} \subseteq \mathbb{P}_{i+j}^{\underline{v}\underline{w}}$ for all $i, j \in \mathbb{N}_0$.
(c) $\mathbb{P} = \bigoplus_{i \in \mathbb{N}_0} \mathbb{P}_i^{\underline{v}\underline{w}}$.

(B) Let the hypotheses and notations be as in part (A). Conclude that the family

$$\left(\mathbb{P}_i^{\underline{v}\underline{w}}\right)_{i \in \mathbb{N}_0} \text{ defines a grading of the ring } \mathbb{P}.$$

We call this grading the *grading induced by the weight* $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$. If we endow our polynomial ring with this grading we write it as $\mathbb{P}^{\underline{v}\underline{w}}$, thus

$$\mathbb{P} = \mathbb{P}^{\underline{v}\underline{w}} = \bigoplus_{i \in \mathbb{N}_0} \mathbb{P}_i^{\underline{v}\underline{w}}.$$

9.4. **Theorem.** ***(Structure of Weighted Associated Graded Rings)*** *Let $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ be a weight. Then there exists an isomorphism of K-algebras, which preserves gradings*

$$\eta^{\underline{v}\underline{w}} : \mathbb{P} = \mathbb{P}^{\underline{v}\underline{w}} \xrightarrow{\cong} \mathbb{G}^{\underline{v}\underline{w}}, \text{ given by } \eta^{\underline{v}\underline{w}}(Y_i) := X_i^*, \quad \eta^{\underline{v}\underline{w}}(Z_j) := \partial_j^*, \quad (i, j = 1, 2, \ldots, n).$$

Proof. We leave this as an exercise, giving eventually hint to [6]. □

9.5. **Corollary.** ***(Additivity of Weighted Degrees)*** *Let $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ be a weight and let $d, e \in \mathbb{W}$. Then*

$$\deg^{\underline{v}\underline{w}}(de) = \deg^{\underline{v}\underline{w}}(d) + \deg^{\underline{v}\underline{w}}(e).$$

Proof. (We slightly shorten some arguments with respect to the original exposition in [6]). If $d = 0$ or $e = 0$ our claim is clear. So let $d, e \neq 0$ and observe that $i := \deg^{\underline{v}\underline{w}}(d), j := \deg^{\underline{v}\underline{w}}(e) \in \mathbb{N}_0$. Using our previously introduced notations, we set

$$M := \bigoplus_{(\underline{\nu}, \underline{\mu}) \in \mathbb{I}_{=i}^{\underline{v}\underline{w}}} K \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \quad \text{and } N := \bigoplus_{(\underline{\nu}, \underline{\mu}) \in \mathbb{I}_{=j}^{\underline{v}\underline{w}}} K \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}.$$

We then may write

$$d = a + r \text{ with } a \in M \setminus \{0\}, \deg^{\underline{v}\underline{w}}(r) < i \text{ and } e = b + s \text{ with } b \in N \setminus \{0\}, \deg^{\underline{v}\underline{w}}(s) < j.$$

We thus have $de = ab + (as + br + rs)$. By what we know already about degrees we have $\deg^{\underline{v}\underline{w}}(as + br + rs) < i + j$ (see Lemma 8.3 (a), (c)). So, in view of Lemma 8.3 (a) it suffices to show that $\deg^{\underline{v}\underline{w}}(ab) = i + j$. To do so, we write

$$a = \sum_{(\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(a)} c_{\underline{\nu}, \underline{\mu}}^{(a)} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}, \text{ with } c_{\underline{\nu}, \underline{\mu}}^{(a)} \in K \setminus \{0\} \text{ for all } (\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(a) \text{ and}$$

$$b = \sum_{(\underline{\nu}', \underline{\mu}') \in \operatorname{supp}(b)} c_{\underline{\nu}', \underline{\mu}'}^{(b)} \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}, \text{ with } c_{\underline{\nu}', \underline{\mu}'}^{(b)} \in K \setminus \{0\} \text{ for all } (\underline{\nu}', \underline{\mu}') \in \operatorname{supp}(b).$$

It follows that

$$ab = \sum_{(\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(a) \text{ and } (\underline{\nu}', \underline{\mu}') \in \operatorname{supp}(b)} c_{\underline{\nu}, \underline{\mu}}^{(a)} c_{\underline{\nu}', \underline{\mu}'}^{(b)} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'}.$$

By Exercise 6.4 (A) and Definition and Remark 8.2 (B)(c) we have $\deg^{\underline{v}\underline{w}}\big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} - \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'}\big) < i+j$ for all $(\underline{\nu},\underline{\mu}) \in \mathrm{supp}(a)$ and all $(\underline{\nu}',\underline{\mu}') \in \mathrm{supp}(b)$. If we set

$$h := \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a) \text{ and } (\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'}.$$

and on repeated use of Lemma 8.3 (a) and (b) we thus get

$$\deg^{\underline{v}\underline{w}}(ab-h) =$$
$$\deg^{\underline{v}\underline{w}}\Big[\sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a) \text{ and } (\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}'} (\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} - \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'})\Big] < i+j.$$

So, we may write

$$ab = h + u \text{ with } \deg^{\underline{v}\underline{w}}(u) < i+j.$$

By Lemma 8.3 (a) it thus suffices to show that $\deg^{\underline{v}\underline{w}}(h) = i+j$. As

$$h = \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a) \text{ and } (\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'} \in \bigoplus_{(\underline{\nu},\underline{\mu})\in\mathbb{I}^{\underline{v}\underline{w}}_{=i+j}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}$$

It suffices to show that $h \neq 0$. To do so, we consider the two polynomials

$$f := \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a)} c^{(a)}_{\underline{\nu},\underline{\mu}} \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \in \mathbb{P}^{\underline{v}\underline{w}}_i, \quad g := \sum_{(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \underline{Y}^{\underline{\nu}'}\underline{Z}^{\underline{\mu}'} \in \mathbb{P}^{\underline{v}\underline{w}}_j.$$

As $\mathrm{supp}(a)$ and $\mathrm{supp}(b)$ are non-empty, and all coefficients of f and g are non-zero. As $\mathbb{P}$ is an integral domain, it follows that $fg \neq 0$. We set

$$h^* := (h + \mathbb{W}^{\underline{v}\underline{w}}_{i+j-1}) = \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a) \text{ and } (\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \big(\underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'}\big)^* \in \mathbb{G}^{\underline{v}\underline{w}}_{i+j}.$$

Applying the isomorphism $\eta^{\underline{v}\underline{w}} : \mathbb{P} = \mathbb{P}^{\underline{v}\underline{w}} \xrightarrow{\cong} \mathbb{G}^{\underline{v}\underline{w}}$ of Theorem 9.4, we now get

$$0 \neq \eta^{\underline{v}\underline{w}}(fg) = \eta^{\underline{v}\underline{w}}\Big(\Big[\sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a)} c^{(a)}_{\underline{\nu},\underline{\mu}}\underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}}\Big]\Big[\sum_{(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(b)}_{\underline{\nu}',\underline{\mu}'}\underline{Y}^{\underline{\nu}'}\underline{Z}^{\underline{\mu}'}\Big]\Big) =$$
$$= \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(a) \text{ and } (\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(b)} c^{(a)}_{\underline{\nu},\underline{\mu}} c^{(b)}_{\underline{\nu}',\underline{\mu}'} \big(\underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'}\big)^* = h^*.$$

But this clearly implies that $h \neq 0$. □

9.6. **Corollary.** ***(Integrity of Standard Weyl Algebras)*** *The standard Weyl algebra* $\mathbb{W}$ *is an integral ring: If* $d, e \in \mathbb{W}\setminus\{0\}$, *then* $de \neq 0$.

Proof. Apply Theorem 9.4. □

9.7. **Exercise.** (A) We fix a weight $(\underline{v},\underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ and set $\Gamma^{\underline{v},\underline{w}} := \{\underline{v}\cdot\underline{\nu} + \underline{w}\cdot\underline{\mu} \mid \underline{\nu},\underline{\mu} \in \mathbb{N}_0^n\}$. Prove the following statements

(a) $0 \in \Gamma^{\underline{v}\underline{w}} \subseteq \mathbb{N}_0$.
(b) If $i, j \in \Gamma^{\underline{v}\underline{w}}$, then $i+j \in \Gamma^{\underline{v}\underline{w}}$.
(c) $\mathbb{G}^{\underline{v}\underline{w}}_i \neq 0 \Leftrightarrow \mathbb{P}^{\underline{v}\underline{w}}_i \neq 0 \Leftrightarrow i \in \Gamma^{\underline{v}\underline{w}}$.

$\Gamma^{\underline{v},\underline{w}}$ is called the *degree semigroup* associated to the weight $(\underline{v}, \underline{w})$.

(B) Let $n = 1$, $\underline{v} = (p)$ and $\underline{w} = (q)$, where $p, q \in \mathbb{N}$ are two distinct prime numbers. Determine $\Gamma^{\underline{v},\underline{w}}$ and the standard bases of all K-vector spaces

$$\mathbb{P}_i^{\underline{vw}} \text{ and } \mathbb{G}_i^{\underline{vw}} \text{ for } i \in \Gamma^{\underline{vw}},$$

at least for some specified pairs like $(p, q) = (2, 3), (2, 5), (5, 7), \ldots$

(C) Show, that the ring $\mathrm{End}_K\big(K[X_1, X_2, \ldots, X_n]\big)$ is not integral.

10. Filtered Modules

10.1. **Definition and Remark.** (A) Let K be a field and let $A = (A, A_\bullet)$ be a filtered K algebra. Let U be a left-module over A. By a *filtration of U compatible with $A_\bullet$* or just an *$A_\bullet$-filtration* of U we mean a family $U_\bullet = (U_i)_{i\in\mathbb{Z}}$ such that the following conditions hold:

(a) Each U_i is a K-vector subspace of U;
(b) $U_i \subseteq U_{i+1}$ for all $i \in \mathbb{Z}$;
(c) $U = \bigcup_{i\in\mathbb{Z}} U_i$;
(d) $A_iU_j := \sum_{(f,u)\in A_i\times U_j} Kfu \subseteq U_{i+j}$ for all $i \in \mathbb{N}_0$ and all $j \in \mathbb{Z}$.

If an $A_\bullet$-filtration $U_\bullet$ of U is given, we say that $(U, U_\bullet)$ or – by abuse of language – that U is a $A_\bullet$ *filtered A- module* or just that U is a *filtered A-module.*

(B) Keep the notations and hypotheses of part (A) and let $U_\bullet = (U_i)_{i\in\mathbb{Z}}$ be a filtered A-module. Observe that

For all $i \in \mathbb{Z}$ the K-vector space U_i is a leftA_0-submodule of U.

(C) We say that two $A_\bullet$-filtrations $U_\bullet^{(1)}, U_\bullet^{(2)}$ are *equivalent* if there is some $r \in \mathbb{N}_0$ such that

(a) $U_{i-r}^{(1)} \subseteq U_i^{(2)} \subseteq U_{i+r}^{(1)}$ for all $i \in \mathbb{Z}$.

Later, we shall use the following observation.

Assume that the above condition (a) holds, let $i \in \mathbb{N}$ and let $a \in A_i$. Then we have the following statements, whose prove is left as an exercise to the reader (see [6] if necessary).

(b) $aU_j^{(1)} \subseteq U_{j+i-1}^{(1)}$ for all $j \in \mathbb{Z} \quad \Rightarrow \quad a^kU_j^{(1)} \subseteq U_{j+k(i-1)}^{(1)}$ for all $j \in \mathbb{Z}$ and all $k \in \mathbb{N}_0$.
(c) $aU_j^{(1)} \subseteq U_{j+i-1}^{(1)}$ for all $j \in \mathbb{Z} \quad \Rightarrow \quad a^{2r+1}U_j^{(2)} \subseteq U_{j+(2r+1)i-1}^{(2)}$ for all $j \in \mathbb{Z}$.

10.2. **Remark and Definition.** (A) Let K be a field and let $A = (A, A_\bullet)$ be a filtered K-algebra and let $U = (U, U_\bullet)$ be an $A_\bullet$-filtered A-module. We consider the corresponding associated graded ring

$$\mathrm{Gr}(A) = \mathrm{Gr}_{A_\bullet}(A) = \bigoplus_{i\in\mathbb{N}_0} \mathrm{Gr}_{A_\bullet}(A)_i, \quad \big(\mathrm{Gr}_{A_\bullet}(A)_i := A_i/A_{i-1}\big).$$

and the K-vector space

$$\mathrm{Gr}(U) = \mathrm{Gr}_{U_\bullet}(U) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Gr}(U)_i, \quad \big(\mathrm{Gr}(U)_i := U_i/U_{i-1}\big).$$

(B) Let $i \in \mathbb{N}_0$, let $j \in \mathbb{Z}$ let $f, f' \in A_i$ and let $g, g' \in U_j$ such that $h := f - f' \in A_{i-1}$ and $k := g - g' \in U_{j-1}$. It follows that $fg - f'g' - fk - hg - hk \subseteq U_{i+j-1}$. So in $U_{i+j}/U_{i+j-1} = \mathrm{Gr}_{U_\bullet}(U)_{i+j} \subset \mathrm{Gr}_{U_\bullet}(U)$ we get the relation $fg + U_{i+j-1} = f'g' + U_{i+j-1}$. This allows to define a $\mathrm{Gr}_{A_\bullet}(A)$-*scalar multiplication* on the K-space $\mathrm{Gr}_{U_\bullet}(U)$ which is induced by

$$(f + A_{i-1})(g + U_{j-1}) := fg + U_{i+j-1} \quad \big(i \in \mathbb{N}_0, j \in \mathbb{Z}, f \in A_i, g \in U_j\big).$$

(C) Keep the above notations and hypotheses. With respect to our scalar multiplication on $\mathrm{Gr}_{U_\bullet}(U)$ we have the relations

$$\mathrm{Gr}_{A_\bullet}(A)_i \mathrm{Gr}_{U_\bullet}(U)_j \subseteq \mathrm{Gr}_{U_\bullet}(U)_{i+j} \text{ for all } i, j \in \mathbb{Z}.$$

So, the K-vector space $\mathrm{Gr}_{U_\bullet}(U)$ is turned into a graded $\mathrm{Gr}_{A_\bullet}(A)$-module

$$\mathrm{Gr}_{U_\bullet}(U) = \big(\mathrm{Gr}_{U_\bullet}(U), (\mathrm{Gr}_{U_\bullet}(U)_i)_{i\in\mathbb{Z}}\big) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Gr}_{U_\bullet}(U)_i$$

by means of the above multiplication. We call this $\mathrm{Gr}_{A_\bullet}(A)$-module $\mathrm{Gr}_{U_\bullet}(U)$ the *associated graded module* of U with respect to the filtration $U_\bullet$. From now on, we always furnish $\mathrm{Gr}_{U_\bullet}(U)$ with this structure of graded $\mathrm{Gr}_{A_\bullet}(A)$-module.

10.3. **Definition.** Let K be a field and let $A = (A, A_\bullet)$ be a filtered K-algebra. Assume that the filtration $A_\bullet$ is commutative. Moreover, let $U = (U, U_\bullet)$ be an $A_\bullet$-filtered A-module and consider the corresponding associated graded module $\mathrm{Gr}(U) = \mathrm{Gr}_{U_\bullet}(U)$. Finally, consider the *annihilator* ideal

$$\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet}(U)\big) := \{f \in \mathrm{Gr}_{A_\bullet}(A) \mid f\mathrm{Gr}_{U_\bullet}(U) = 0\}$$

of the $\mathrm{Gr}_{A_\bullet}(A)$-module $\mathrm{Gr}_{U_\bullet}(U)$. We define the *characteristic variety* $\mathbb{V}_{U_\bullet}(U)$ of the $A_\bullet$-filtered A-module $U = (U, U_\bullet)$ as the *prime variety* of the annihilator ideal of $\mathrm{Gr}_{U_\bullet}(U)$, hence

$$\mathbb{V}_{U_\bullet}(U) := \mathrm{Var}\big(\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}(\mathrm{Gr}_{U_\bullet}(U))\big) \subseteq \mathrm{Spec}(\mathrm{Gr}_{A_\bullet}(A)).$$

We also call this variety the *characteristic variety of the left A-module U with respect to the $A_\bullet$ filtration $U_\bullet$* or just the *characteristic variety of U with respect to $U_\bullet$*.

10.4. **Proposition.** ***(Equality of Characteristic Varieties for Equivalent Filtrations)*** *Let K be a field and let $A = (A, A_\bullet)$ be a filtered K-algebra. Assume that the filtration $A_\bullet$ is commutative. Let U be an A-module which is endowed with two equivalent $A_\bullet$-filtrations $U_\bullet^{(1)}$ and $U_\bullet^{(2)}$. Then*

$$\mathbb{V}_{U_\bullet^{(1)}}(U) = \mathbb{V}_{U_\bullet^{(2)}}(U).$$

Proof. We have to show that

$$\sqrt{\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(1)}}(U)\big)} = \sqrt{\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(2)}}(U)\big)}.$$

By symmetry, and in view of the fact that the formation of radicals of ideals is idempotent, it suffices even to show that

$$\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(1)}}(U)\big) \subseteq \sqrt{\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U_\bullet^{(2)}}(U)\big)}.$$

As $\mathrm{Gr}_{U^{(1)}_\bullet}(U)$ is a graded $\mathrm{Gr}_{A_\bullet}(A)$-module, its annihilator is a graded ideal of $\mathrm{Gr}_{A_\bullet}(A)$. So, it finally is enough to show, that

$$\overline{a} \in \sqrt{\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U^{(2)}_\bullet}(U)\big)} \text{ for all } i \in \mathbb{N}_0 \text{ and all } \overline{a} \in \mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U^{(1)}_\bullet}(U)\big)_i.$$

So, fix some $i \in \mathbb{N}_0$ and some $\overline{a} \in \mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U^{(1)}_\bullet}(U)\big)_i \subseteq \mathrm{Gr}_{A_\bullet}(A)_i = A_i/A_{i-1}$. We chose some $a \in A_i$ with $\overline{a} = a + A_{i-1} \in A_i/A_{i-1}$.. For all $j \in \mathbb{Z}$ we have in $\mathrm{Gr}_{U_\bullet}(U)$ the relation

$$aU^{(1)}_j + U^{(1)}_{j+i-1} = (a + A_{i-1})(U^{(1)}_j/U^{(1)}_{j-1}) = \overline{a}(U^{(1)}_j/U^{(1)}_{j-1}) = \overline{a}\mathrm{Gr}_{U_\bullet}(U)_j = 0,$$

and hence $aU^{(1)}_j \subseteq U^{(1)}_{j+i-1}$ for all $j \in \mathbb{Z}$. According to our hypotheses we find some $r \in \mathbb{N}_0$ such that $U^{(1)}_{k-r} \subseteq U^{(2)}_k \subseteq U^{(1)}_{k+r}$ for all $k \in \mathbb{Z}$. Thus, by Definition and Remark 10.1 (C)(c) we therefore have

$$a^{2r+1}U^{(2)}_j \subseteq U^{(2)}_{j+(2r+1)i-1} \text{ for all } j \in \mathbb{Z}.$$

So, for all $j \in \mathbb{Z}$ we get in $U^{(2)}_{j+(2r+1)i}/U^{(2)}_{j+(2r+1)i-1} = \mathrm{Gr}_{U_\bullet}(U)_{j+(2r+1)i}$ the relation:

$$\overline{a}^{2r+1}\mathrm{Gr}_{U_\bullet}(U)_j = (a^{2r+1} + A_{(2r+1)i-1})(U^{(2)}_j/U^{(2)}_{j-1}) \subseteq a^{2r+1}U^{(2)}_j + U^{(2)}_{j+(2r+1)i-1} = 0.$$

This shows that $\overline{a}^{2r+1} \in \mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U^{(2)}_\bullet}(U)\big)$, whence $\overline{a} \in \sqrt{\mathrm{Ann}_{\mathrm{Gr}_{A_\bullet}(A)}\big(\mathrm{Gr}_{U^{(2)}_\bullet}(U)\big)}$. □

The previous result allows us to define in an intrinsic way the notion of characteristic variety of a finitely generated (left-) module over a (commutatively) filtered ring A. We work this out in the following combined exercise and definition.

10.5. **Exercise and Definition.** (A) Let $(A, A_\bullet)$ be a filtered K-algebra and let U be a (left) module over A. Let $V \subseteq U$ be a K-subspace such that $U = AV$. Prove the following claims:

(a) $A_iV = 0$ for all $i < 0$.
(b) The family $A_\bullet V := \big(A_iV\big)_{i\in\mathbb{Z}}$ is an $A_\bullet$-filtration of U.

The above filtration $A_\bullet V$ is called the $A_\bullet$-filtration of U *induced* by the subspace V.

(B) Let the notations and hypotheses be as above and assume in addition that $s := \dim_K(V) < \infty$. Prove that

(a) U is finitely generated an an A-module;
(b) A_iV is a finitely generated (left-) module over A_0.
(c) The graded $\mathrm{Gr}_{A_\bullet}(A)$-module $\mathrm{Gr}_{A_\bullet V}(U)$ is generated by finitely many elements $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_s \in \mathrm{Gr}_{A_\bullet V}(U)_0$.

Keep in mind that we can always find a vector space $V \subseteq U$ of finite dimension with $AV = U$ if the A-module U is finitely generated.

(C) Let the notations and hypotheses be as above. Let $V^{(1)}, V^{(2)} \subseteq U$ be two K-subspaces such that $AV^{(1)} = AV^{(2)} = U$ and $\dim_K(V^{(1)}), \dim_K(V^{(2)}) < \infty$. Prove that

(a) The two induced $A_\bullet$-filtrations $A_\bullet V^{(1)}$ and $A_\bullet V^{(2)}$ are equivalent.

(b) If the filtration $A_\bullet$ is commutative, it holds

$$\mathbb{V}_{A_\bullet V^{(1)}}(U) = \mathbb{V}_{A_\bullet V^{(2)}}(U).$$

(D) Keep the above notations and hypotheses. Assume that the filtration $A_\bullet$ is commutative and that the (left) A-module U is finitely generated. By what we have learned by the previous considerations, we find a K-subspace $V \subseteq U$ of finite dimension such that $AV = U$, and the characteristic variety $\mathbb{V}_{A_\bullet V}(U)$ of U with respect to the induced filtration $A_\bullet V$ is independent of the choice of V. So, we may just write

$$\mathbb{V}_{A_\bullet}(U) := \mathbb{V}_{A_\bullet V}(U),$$

and we call $\mathbb{V}_{A_\bullet}(U)$ the *characteristic variety of* U *with respect to the (commutative !) filtration* $A_\bullet$ of A. This is the announced notion of *intrinsic characteristic variety.*

(E) Keep the above notations. Assume that the filtration $A_\bullet$ is of finite type (see Definition and Remark 3.3 (C)) and that the (left) A-module U is finitely generated. The $A_\bullet$ filtration $U_\bullet$ of U is said to be *of finite type* if

(a) $U_i = 0$ for all $i \ll 0$;
(b) There is an integer σ such that U_j is finitely generated as a (left) A_0-module for all $j \leq \sigma$ and
(c) $U_i = \sum_{j \leq \sigma} A_j U_{i-j}$ for all $i > \sigma$.

In this situation σ is again called a *generating degree* of the $A_\bullet$-filtration $U_\bullet$ (compare Definition and Remark 3.3 (C)). In this situation, we also may chose a K-subspace $V \subseteq U$ such that $\dim_K(V) < \infty$ and $A_0 V = U_\sigma$. For this choice of V one now can say: $U = AV$ and the filtrations $U_\bullet$ and $A_\bullet V$ are equivalent. As a consequence it follows by Proposition 10.4 and the observations made in part (D), that

$$\mathbb{V}_{U_\bullet}(U) = \mathbb{V}_{A_\bullet}(U) \text{ for each } A_\bullet\text{-filtration } U_\bullet \text{ which is of finite type.}$$

11. D-Modules

11.1. **Convention.** (A) As in section 9, we fix a positive integer n, a field K of characteristic 0 and the standard Weyl algebra $\mathbb{W} := \mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n]$, together with the polynomial ring $\mathbb{P} := K[Y_1, Y_2, \ldots, Y_n; Z_1, Z_2, \ldots, Z_n]$.

(B) Let $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ be a weight. We consider the induced weighted filtration $\mathbb{W}_\bullet^{\underline{v}\underline{w}}$ and also the corresponding associated graded ring $\mathbb{G}^{\underline{v}\underline{w}}$ (see Definition and Remark 9.2 (A)).

(C) Moreover, we shall consider the polynomial ring $\mathbb{P} = \mathbb{P}^{\underline{v}\underline{w}} = \bigoplus_{i \in \mathbb{N}_0} \mathbb{P}_i^{\underline{v}\underline{w}}$. furnished with the grading induced by our given weight $(\underline{v}, \underline{w})$ (see Exercise and Definition 9.3 (B)), as well as the canonical isomorphism of graded rings (see Theorem 9.4) $\eta^{\underline{v}\underline{w}} : \mathbb{P} = \mathbb{P}^{\underline{v}\underline{w}} \xrightarrow{\cong} \mathbb{G}^{\underline{v}\underline{w}}$.

11.2. **Definition and Remark.** (A) By a *D-module* we mean a finitely generated left module over the standard Weyl algebra $\mathbb{W}$.

(B) Let U be a D-module. If $U_\bullet$ is a $\mathbb{W}^{\underline{vw}}_\bullet$-filtration of U, we may again introduce the corresponding *associated graded module* $\mathrm{Gr}_{U_\bullet}(U)$ of U with respect to the filtration $U_\bullet$ (see Definition 10.3). which is indeed a graded module over the associated graded ring $\mathbb{G}^{\underline{vw}}$ and hence a graded $\mathbb{P}^{\underline{vw}}$-module by means of the canonical isomorphism $\eta^{\underline{vw}} : \mathbb{P} = \mathbb{P}^{\underline{vw}} \xrightarrow{\cong} \mathbb{G}^{\underline{vw}}$.

(C) Keep the notations and hypotheses of part (B). Then, we may again consider the *characteristic variety* of U with respect to the filtration $U_\bullet$, but under the previous view, that $\mathrm{Gr}_{U_\bullet}(U)$ is a graded module over the graded polynomial ring $\mathbb{P} = \mathbb{P}^{\underline{vw}}$. So, we define this characteristic variety by

$$\mathbb{V}_{U_\bullet}(U) := \mathrm{Var}\big(\mathrm{Ann}_{\mathbb{P}^{\underline{vw}}}(\mathrm{Gr}_{U_\bullet}(U))\big) = \mathrm{Var}\big((\eta^{\underline{vw}})^{-1}\big[\mathrm{Ann}_{\mathbb{G}^{\underline{vw}}}(\mathrm{Gr}_{U_\bullet}(U))\big]\big) \subseteq \mathrm{Spec}(\mathbb{P}).$$

Observe in particular, that the ideal

$$\mathrm{Ann}_{\mathbb{P}^{\underline{vw}}}\big(\mathrm{Gr}_{U_\bullet}(U)\big) = (\eta^{\underline{vw}})^{-1}\big[\mathrm{Ann}_{\mathbb{G}^{\underline{vw}}}\big(\mathrm{Gr}_{U_\bullet}(U)\big)\big] \subseteq \mathbb{P}^{\underline{vw}}$$

is graded.

(D) Finally, as U is finitely generated, we may again chose a finite dimensional K-subspace $V \subseteq U$ such that $\mathbb{W}V = U$, and then consider the induced filtration $\mathbb{W}^{\underline{vw}}_\bullet V$ of U and the corresponding *intrinsic characteristic variety* (see Exercise and Definition 10.5 (D)) of U with respect to the weight $(\underline{v}, \underline{w})$, hence:

$$\mathbb{V}^{\underline{vw}}(U) := \mathbb{V}_{\mathbb{W}^{\underline{vw}}_\bullet}(U) = \mathbb{V}_{\mathbb{W}^{\underline{vw}}_\bullet V}(U).$$

11.3. **Example.** (A) Keep the above notations and let

$$d := \sum_{(\underline{\nu},\underline{\mu})\in \mathrm{supp}(d)} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \in \mathbb{W} \setminus \{0\} \text{ and } \delta := \deg^{\underline{vw}}(d),$$

with $c^{(d)}_{\underline{\nu}\underline{\mu}} \in K \setminus \{0\}$ for all $(\underline{\nu}, \underline{\mu}) \in \mathrm{supp}(d)$. We also consider the so-called *leading differential* form of d with respect to the weight $(\underline{v}, \underline{w})$, which is given by

$$h^{\underline{vw}} := \sum_{(\underline{\nu},\underline{\mu})\in \mathrm{supp}(d): \underline{v}\cdot\underline{\nu}+\underline{w}\cdot\underline{\mu}=\delta} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \in \mathbb{W} \setminus \{0\}.$$

Moreover, we introduce the polynomial

$$f^{\underline{vw}} := \sum_{(\underline{\nu},\underline{\mu})\in \mathrm{supp}(d): \underline{v}\cdot\underline{\nu}+\underline{w}\cdot\underline{\mu}=\delta} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}} \in \mathbb{P} \setminus \{0\}.$$

Now, consider the cyclic left $\mathbb{W}$-module $U := \mathbb{W}/\mathbb{W}d$, furnished with the filtration

$$U_\bullet := \mathbb{W}^{\underline{vw}}_\bullet K(1 + \mathbb{W}d/\mathbb{W}d) = \big(U_i := (\mathbb{W}^{\underline{vw}}_i + \mathbb{W}d/\mathbb{W}d)\big)_{i\in\mathbb{Z}}.$$

(B) Keep the above notations and hypotheses. Observe first, that for all $i \in \mathbb{Z}$ we may write $U_i/U_{i-1} = \mathbb{W}^{\underline{vw}}_i/\mathbb{W}^{\underline{vw}}_{i-1} + \mathbb{W}d \cap \mathbb{W}^{\underline{vw}}_i$. By the additivity of weighted degrees (see Corollary 9.5) we have $\mathbb{W}d \cap \mathbb{W}^{\underline{vw}}_i = \mathbb{W}^{\underline{vw}}_{i-\delta}d$ for all $i \in \mathbb{Z}$. So, we obtain

$$\mathrm{Gr}_{U_\bullet}(U)_i = U_i/U_{i-1} = \mathbb{W}^{\underline{vw}}_i/\big(\mathbb{W}^{\underline{vw}}_{i-1} + \mathbb{W}^{\underline{vw}}_{i-\delta}d\big) \text{ for all } i \in \mathbb{N}_0.$$

Consequently, there is a surjective homomorphism of graded $\mathbb{G}^{\underline{vw}}$-modules

$$\pi : \mathbb{G}^{\underline{vw}} = \bigoplus_{i\in\mathbb{Z}} \mathbb{W}_i^{\underline{vw}}/\mathbb{W}_{i-1}^{\underline{vw}} \twoheadrightarrow \mathrm{Gr}_{U_\bullet}(U) = \bigoplus_{i\in\mathbb{Z}} \mathbb{W}_i^{\underline{vw}}/\big(\mathbb{W}_{i-1}^{\underline{vw}} + \mathbb{W}_{i-\delta}^{\underline{vw}} d\big).$$

If we set $\overline{h}^{\underline{vw}} := h^{\underline{vw}} + \mathbb{W}_{\delta-1}^{\underline{vw}} \in \mathbb{W}_\delta^{\underline{vw}}/\mathbb{W}_{\delta-1}^{\underline{vw}} = \mathbb{G}_\delta^{\underline{vw}}$ it follows that

$$\mathrm{Ann}_{\mathbb{G}^{\underline{vw}}}\big(\mathrm{Gr}_{U_\bullet}(U)\big) = \mathrm{Ker}(\pi) = \bigoplus_{i\in\mathbb{Z}} \big(\mathbb{W}_{i-1}^{\underline{vw}} + \mathbb{W}_{i-\delta}^{\underline{vw}} d\big)/\mathbb{W}_{i-1}^{\underline{vw}} =$$
$$= \bigoplus_{i\in\mathbb{Z}} \big(\mathbb{W}_{i-1}^{\underline{vw}} + \mathbb{W}_{i-\delta}^{\underline{vw}} h^{\underline{vw}}\big)/\mathbb{W}_{i-1}^{\underline{vw}} = \mathbb{G}^{\underline{vw}}\overline{h}^{\underline{vw}}.$$

Consequently we get $\mathrm{Gr}_{U_\bullet}(U) \cong \mathbb{G}^{\underline{vw}}/\mathbb{G}^{\underline{vw}}\overline{h}^{\underline{vw}}$. As $\eta^{\underline{vw}}(f^{\underline{vw}}) = \overline{h}^{\underline{vw}}$ and if we consider $\mathrm{Gr}_{U_\bullet}(U)$ as a graded $\mathbb{P}^{\underline{vw}}$-module by means of $\eta^{\underline{vw}}$, we thus may write $\mathrm{Gr}_{U_\bullet}(U) \cong \mathbb{P}^{\underline{vw}}/\mathbb{P}^{\underline{vw}} f^{\underline{vw}}$ and $\mathrm{Ann}_{\mathbb{P}}\big(\mathrm{Gr}_{U_\bullet}(U)\big) = \mathbb{P} f^{\underline{vw}}$. In particular we obtain:

$$\mathbb{V}_{U_\bullet}(U) = \mathbb{V}^{\underline{vw}}(U) = \mathbb{V}^{\underline{vw}}\big(\mathbb{W}/\mathbb{W}d\big) = \mathrm{Var}(\mathbb{P} f^{\underline{vw}}) \subseteq \mathrm{Spec}(\mathbb{P}).$$

11.4. **Exercise.** (A) Let $n = 1$, $K = \mathbb{R}$ and let $d := X_1^4 + \partial_1^2 - X_1^2\partial_1^2$. Determine the two characteristic varieties

$$\mathbb{V}^{\underline{vw}}(\mathbb{W}/\mathbb{W}d) \text{ for } (\underline{v}, \underline{w}) = (1, 1) \text{ and } (\underline{v}, \underline{w}) = (0, 1).$$

(B) To make more apparent what you have done in part (A), determine and sketch the *real traces*

$$\mathbb{V}_{\mathbb{R}}^{\underline{vw}}(\mathbb{W}/\mathbb{W}d) := \{(y, z) \in \mathbb{R}^2 \mid (Y_1 - y, Z_1 - z)K[Y_1, Z_1] \in \mathbb{V}^{\underline{vw}}(\mathbb{W}/\mathbb{W}d)\}$$

for $(\underline{v}, \underline{w}) = (1, 1)$ and $(\underline{v}, \underline{w}) = (0, 1)$. Comment your findings.

Now, we will show that standard Weyl algebras are left Noetherian. We begin with the following preparation.

11.5. **Definition and Remark.** (A) Let $I \subseteq \mathbb{W}$ be a left ideal. We consider the following K-subspace of $\mathbb{G}^{\underline{vw}}$:

$$\mathbb{G}^{\underline{vw}}(I) := \bigoplus_{i\in\mathbb{N}_0} \big(I \cap \mathbb{W}_i^{\underline{vw}} + \mathbb{W}_{i-1}^{\underline{vw}}\big)/\mathbb{W}_{i-1}^{\underline{vw}} \subseteq \bigoplus_{i\in\mathbb{N}_0} \mathbb{W}_i^{\underline{vw}}/\mathbb{W}_{i-1}^{\underline{vw}} = \mathbb{G}^{\underline{vw}}.$$

It is immediate to see, that $\mathbb{G}^{\underline{vw}}(I) \subseteq \mathbb{G}^{\underline{vw}}$ is a graded ideal. We call this ideal the *graded ideal induced by* I in $\mathbb{G}^{\underline{vw}}$.

(B) Let the notations and hypotheses as in part (A). It is straight forward to see, that the family

$$I_\bullet^{\underline{vw}} := \big(I \cap \mathbb{W}_i^{\underline{vw}}\big)_{i\in\mathbb{Z}}$$

is a filtration of the (left) $\mathbb{W}$-module I, which we call the *filtration induced by* $\mathbb{W}_\bullet^{\underline{vw}}$. Observe, that for all $i \in \mathbb{Z}$ we have a canonical isomorphism of K-vector spaces

$$\mathbb{G}^{\underline{vw}}(I)_i := \big(I \cap \mathbb{W}_i^{\underline{vw}} + \mathbb{W}_{i-1}^{\underline{vw}}\big)/\mathbb{W}_{i-1}^{\underline{vw}} \cong I \cap \mathbb{W}_i^{\underline{vw}}/I \cap \mathbb{W}_{i-1}^{\underline{vw}} = I_i^{\underline{vw}}/I_{i-1}^{\underline{vw}} = \mathrm{Gr}_{I_\bullet^{\underline{vw}}}(I)_i.$$

It is easy to see, that these isomorphisms of K-vector spaces actually give rise to an isomorphism of graded $\mathbb{G}^{\underline{vw}}$-modules

$$\mathbb{G}^{\underline{vw}}(I) := \bigoplus_{i\in\mathbb{Z}} \big(I \cap \mathbb{W}_i^{\underline{vw}} + \mathbb{W}_{i-1}^{\underline{vw}}\big)/\mathbb{W}_{i-1}^{\underline{vw}} \cong \bigoplus_{i\in\mathbb{Z}} I_i^{\underline{vw}}/I_{i-1}^{\underline{vw}} = \mathrm{Gr}_{I_\bullet^{\underline{vw}}}(I).$$

So, by means of this canonical isomorphism we may identify $\mathbb{G}^{\underline{vw}}(I) = \mathrm{Gr}_{I_\bullet^{\underline{vw}}}(I)$.

11.6. **Lemma.** *Let $I, J \subseteq \mathbb{W}$ be two left ideals with $I \subseteq J$. Then we have*

(a) $\mathbb{G}^{\underline{vw}}(I) \subseteq \mathbb{G}^{\underline{vw}}(J)$.

(b) *If $\mathbb{G}^{\underline{vw}}(I) = \mathbb{G}^{\underline{vw}}(J)$, then $I = J$.*

Proof. (a): This is immediate by Definition and Remark 11.5 (A).

(b): Assume that $I \subsetneq J$. Then, there is a least integer $i \in \mathbb{N}_0$ such that $I_i^{\underline{vw}} = I \cap \mathbb{W}_i^{\underline{vw}} \subsetneq J_i^{\underline{vw}} = J \cap \mathbb{W}_i^{\underline{vw}}$. As $I_{i-1}^{\underline{vw}} = J_{i-1}^{\underline{vw}}$ it follows that $\mathbb{G}^{\underline{vw}}(I)_i = I_i^{\underline{vw}}/I_{i-1}^{\underline{vw}}$ is not isomorphic to $I_i^{\underline{vw}}/I_{i-1}^{\underline{vw}} = \mathbb{G}^{\underline{vw}}(J)_i$, so that indeed $\mathbb{G}^{\underline{vw}}(I) \neq \mathbb{G}^{\underline{vw}}(J)$. □

11.7. **Theorem.** ***(Noetherianness of Weyl Algebras)*** *The Weyl algebra $\mathbb{W}$ is left Noetherian.*

Proof. : This is immediate by Lemma 11.6 as $\mathbb{G}^{\underline{vw}} \cong \mathbb{P}^{\underline{vw}} = \mathbb{P}$ is Noetherian. □

11.8. **Corollary.** ***(Finite Presentability of D-Modules)*** *Each D-module U admits a finite presentation*

$$\mathbb{W}^s \longrightarrow \mathbb{W}^r \longrightarrow U \longrightarrow 0.$$

Proof. This follows immediately by Theorem 11.7. □

11.9. **Example.** (A) Consider the polynomial ring $U := K[X_1, X_2, \ldots, X_n]$. As $\mathbb{W} \subseteq \mathrm{End}_K(U)$, this polynomial ring can be viewed in a canonical way as a left module over $\mathbb{W}$, the scalar multiplication being given by $d \cdot f := d(f)$ for all $d \in \mathbb{W}$ and all $f \in U$. As $f \cdot 1 = f$ for all $f \in U$ it follows that $U = \mathbb{W}1_U$. So, the $\mathbb{W}$-module $U := K[X_1, X_2, \ldots, X_n]$ is generated by a single element, and hence in particular is a D-module.

(B) Keep the previous notations and hypotheses. Observe that

$$\sum_{i=1}^n \mathbb{W}\partial_i = \bigoplus_{\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n:\underline{\mu}\neq\underline{0}} K\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \text{ and hence } \mathbb{W} = U \oplus \sum_{i=1}^n \mathbb{W}\partial_i.$$

We thus have an exact sequence of K-vector spaces

$$0 \longrightarrow \sum_{i=1}^n \mathbb{W}\partial_i \longrightarrow \mathbb{W} \xrightarrow{\pi} U \longrightarrow 0,$$

in which $\mathbb{W} \xrightarrow{\pi} U$ is the *canonical projection* map given by

$$\pi\big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big) = \begin{cases} \underline{X}^{\underline{\nu}}, & \text{if } \underline{\mu} = \underline{0}, \\ 0, & \text{if } \underline{\mu} \neq \underline{0}. \end{cases}$$

Our aim is to show:

$$\mathbb{W} \xrightarrow{\pi} U \text{ is a homomorphism of left } \mathbb{W}\text{-modules.}$$

To do so, it suffices to show that for all $\underline{\nu},\underline{\mu},\underline{\nu}',\underline{\mu}' \in \mathbb{N}_0^n$ it holds

$$\pi(dd') = d\pi(d'), \text{ where } d := \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \text{ and } d' := \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}.$$

If $\underline{\mu} = \underline{\mu}' = \underline{0}$, we have

$$\pi(dd') = \pi\big(\underline{X}^{\underline{\nu}}\underline{X}^{\underline{\nu}'}\big) = \pi\big(\underline{X}^{\underline{\nu}+\underline{\nu}'}\big) = \underline{X}^{\underline{\nu}+\underline{\nu}'} = \underline{X}^{\underline{\nu}}\underline{X}^{\underline{\nu}'} = \underline{X}^{\underline{\nu}}\pi\big(\underline{X}^{\underline{\nu}'}\big) = d\pi(d').$$

If $\underline{\mu} = 0$ and $\underline{\mu}' \neq 0$ we have

$$\pi(dd') = \pi\big(\underline{X}^{\underline{\nu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) = \pi\big(\underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) = 0 = \underline{X}^{\underline{\nu}}\pi\big(\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) = d\pi(d').$$

So, let $\underline{\mu} \neq \underline{0}$. By the Product Formula of Proposition 6.2 we have

$$dd' = \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} = \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'} + s,$$

with

$$s := \sum_{\underline{k}\in\mathbb{N}_0^n:\underline{0}<\underline{k}\leq\underline{\mu},\underline{\nu}'} \lambda_{\underline{k}}\underline{X}^{\underline{\nu}+\underline{\nu}'-\underline{k}}\underline{\partial}^{\underline{\nu}+\underline{\nu}'-\underline{k}} \text{ and } \lambda_{\underline{k}} = \Big(\prod_{i=1}^n \binom{\mu_i}{k_i}\Big)\Big(\prod_{i=1}^n\prod_{p=0}^{k_i-1}(\nu_i'-p)\Big).$$

Assume first, that $\underline{\mu}' \neq 0$. Then we have

$$\pi\big(\underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'}\big) = 0 \text{ and } \pi\big(\underline{X}^{\underline{\nu}+\underline{\nu}'-\underline{k}}\underline{\partial}^{\underline{\nu}+\underline{\nu}'-\underline{k}}\big) = 0 \text{ for all } \underline{k} \in \mathbb{N}_0^n \text{ with } \underline{0} < \underline{k} \leq \underline{\mu},\underline{\nu}'.$$

It thus follows, that $\pi(dd') = 0 = d0 = d\pi\big(\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) = d\pi(d')$. So, finally let $\underline{\mu}' = 0$. Then $dd' = \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} + s$, and

$$s = \begin{cases} \prod_{i=1}^n \prod_{p=0}^{\mu_i-1}(\nu_i'-p)\underline{X}^{\underline{\nu}+\underline{\nu}'-\underline{\mu}}, & \text{if } \underline{\mu} \leq \underline{\nu}'; \\ 0, & \text{otherwise.} \end{cases}$$

So, by what we have learned in Exercise 6.6 (B), we have $s = \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big(\underline{X}^{\underline{\nu}'}\big)$. As s is a K-multiple of a monomial in the X_i's we have $\pi(s) = s$. It thus follows

$$\pi(dd') = \pi\big(\underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) + \pi(s) = s = \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\big(\underline{X}^{\underline{\nu}'}\big) = \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'} = d\pi(d').$$

This proves, that π is indeed a homomorphism of left $\mathbb{W}$-modules.

(C) Keep the previous notations and hypotheses. Then, according the above observations, we have an exact sequence of left $\mathbb{W}$-modules

$$0 \longrightarrow \mathbb{W}^n \xrightarrow{h} \mathbb{W} \xrightarrow{\pi} U \longrightarrow 0,$$

in which h is given by

$$(d_1, d_2, \ldots, d_n) \mapsto h(d_1, d_2, \ldots, d_n) = \sum_{i=1}^n d_i\partial_i.$$

This sequence clearly constitutes a presentation of the left $\mathbb{W}$-module U. and the corresponding presentation matrix for U is the row

$$\partial := \begin{pmatrix} \partial_1 \\ \partial_2 \\ \dots \\ \partial_n \end{pmatrix} \in \mathbb{W}^{n\times 1}.$$

11.10. **Exercise.** (A) We consider the polynomial ring $U = K[X_1, X_2, \dots, X_n]$ canonically as a D-module, as done in Example 11.9. Fix a weight $(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$. Consider the K-suspace $K \subset U$, observe that $\mathbb{W}K = U$ and endow U with the induced filtration $U_\bullet := \mathbb{W}_\bullet^{\underline{v}\underline{w}}K$. Show, that there is an isomorphism of graded $\mathbb{P}$-modules

$$\mathrm{Gr}_{U_\bullet}(U) = \mathrm{Gr}_{\mathbb{W}^{\underline{v}\underline{w}}_\bullet K}(U) \cong U^{\underline{v}} = \bigoplus_{i\in\mathbb{N}_0} U_i^{\underline{v}}, \text{ with } U_i^{\underline{v}} := \sum_{\underline{v}\cdot\underline{\nu}=i} K\underline{X}^{\underline{\nu}} \text{ for all } i \in \mathbb{N}_0.$$

Determine the characteristic variety $\mathbb{V}^{\underline{v}\underline{w}}(U) \subseteq \mathrm{Spec}(\mathbb{P})$.

(B) Keep the notations and hypotheses of part (A). Show, the left $\mathbb{W}$-module U is simple.

11.11. **Remark and Definition.** (A) We furnish the polynomial ring $K[X_1, X_2, \dots, X_n]$ with its *canonical structure of D-module* (see Example 11.9). We now consider a ring $\mathcal{O}$ with the following properties

(1) $\mathcal{O}$ is commutative;
(2) $\mathcal{O}$ is a left $\mathbb{W}$-module;
(3) $K[X_1, X_2, \dots, X_n] \subseteq \mathcal{O}$ is a left submodule.

In this situation, we call $\mathcal{O}$ a ring of *$\mathcal{C}^\infty$-functions* (or a ring of *smooth functions*) in $X_1, X_2, \dots, X_n$ over K.
The idea covered by this concept is that for all $d \in \mathbb{W}$ and all $f \in \mathcal{O}$ the product $df \in \mathcal{O}$ should be viewed as the result of the application of the differential operator d to the function f. Therefore, one often writes

$$d(f) := df \text{ for all } d \in \mathbb{W} \text{ and all } f \in \mathcal{O}.$$

(B) Let the notations and hypotheses be as in part (A). By a *system of polynomial differential equations* in $\mathcal{O}$ we mean a system of equations

$$\begin{aligned} d_{11}(f_1) + d_{12}(f_2) + \dots + d_{1r}(f_r) &= 0 \\ d_{21}(f_1) + d_{22}(f_2) + \dots + d_{2r}(f_r) &= 0 \\ \dots\dots\dots\dots\dots\dots \\ d_{s1}(f_1) + d_{s2}(f_2) + \dots + d_{sr}(f_r) &= 0 \end{aligned}$$

with $r, s \in \mathbb{N}$ such that $d_{ij} \in \mathbb{W}$ and $f_j \in \mathcal{O}$ for all $i, j \in \mathbb{N}$ with $i \le s$ and $j \le r$.
The above system of differential equations can be understood as a linear system of equations over the ring $\mathcal{O}$. We namely may consider the matrix

$$\mathcal{D} := \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1r} \\ d_{21} & d_{22} & \dots & d_{2r} \\ \dots & \dots & \dots & \dots \\ d_{s1} & d_{s2} & \dots & d_{sr} \end{pmatrix} \in \mathbb{W}^{s\times r}.$$

Then, the above system may be written in matrix form as

$$\mathcal{D}\begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ f_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix}.$$

We call $\mathcal{D}$ the *matrix of differential operators* associated to our system of differential equations. So, systems of differential equations correspond to matrices with entries in a standard Weyl algebra.

(C) Keep the previous notations and hypotheses, then the matrix of differential operators $\mathcal{D} \in \mathbb{W}^{s\times r}$ gives rise to an exact sequence of left $\mathbb{W}$-modules

$$0 \longrightarrow \mathbb{W}^s \xrightarrow{h_{\mathcal{D}}} \mathbb{W}^r \xrightarrow{\pi_{\mathcal{D}}} U_{\mathcal{D}} \longrightarrow 0 \quad \big((a_1, a_2, \ldots, a_s) \overset{h_{\mathcal{D}}}{\mapsto} (a_1, a_2, \ldots, a_s)\mathcal{D}\big).$$

In particular $U_{\mathcal{D}}$ is a D-module and the previous sequence is a finite presentation of $U_{\mathcal{D}}$. We call this presentation the *presentation induced by the matrix* $\mathcal{D}$ and we call $U_{\mathcal{D}}$ the D-module *defined* by the matrix $\mathcal{D}$ – or the D-module associated with our system of differential equations. So, each system of differential equations defines a D-module. Obviously, one is particularly interested in the *solution space* of our system of differential equations, hence in the space

$$\mathbb{S}_{\mathcal{D}}(\mathcal{O}) := \{(f_1, f_2, \ldots, f_r) \in \mathcal{O}^r \mid \mathcal{D}\begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ f_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix}\}.$$

Observe, that $\mathbb{S}(\mathcal{D})$ is a K-subspace of $\mathcal{O}^r$.

11.12. **Proposition.** *Let $r, s \in \mathbb{N}$, let*

$$\mathcal{D} = \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1r} \\ d_{21} & d_{22} & \ldots & d_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ d_{s1} & d_{s2} & \ldots & d_{sr} \end{pmatrix} \in \mathbb{W}^{s\times r}$$

be a matrix of differential operators, consider the induced presentation

$$0 \longrightarrow \mathbb{W}^s \xrightarrow{h=h_{\mathcal{D}}} \mathbb{W}^r \xrightarrow{\pi=\pi_{\mathcal{D}}} U_{\mathcal{D}} \longrightarrow 0$$

and the corresponding solution space $\mathbb{S}_{\mathcal{D}}(\mathcal{O})$. For all $i = 1, 2, \ldots, r$ let $\mathrm{e}_i := (\delta_{i,j})_{j=1}^r \in \mathbb{W}^r$ be the i-th canonical basis element. Then, there is an isomorphism

$$\varepsilon_{\mathcal{D}} : \mathrm{Hom}_{\mathbb{W}}\big(U_{\mathcal{D}}, \mathcal{O}\big) \xrightarrow{\cong} \mathbb{S}_{\mathcal{D}}(\mathcal{O}),$$

given by

$$m \mapsto \varepsilon_{\mathcal{D}}(m) := \big(m(\pi(\mathrm{e}_1)), m(\pi(\mathrm{e}_2)), \ldots, m(\pi(\mathrm{e}_r))\big) \text{ for all } m \in \mathrm{Hom}_{\mathbb{W}}\big(U_{\mathcal{D}}, \mathcal{O}\big).$$

Proof. Observe, that there is indeed a K-linear map

$$\varepsilon := \varepsilon_{\mathcal{D}} : \mathrm{Hom}_{\mathbb{W}}\big(U_{\mathcal{D}}, \mathcal{O}\big) \longrightarrow \mathcal{O}^r, m \mapsto \varepsilon_{\mathcal{D}}(m) := \big(m(\pi(\mathrm{e}_1)), m(\pi(\mathrm{e}_2)), \ldots, m(\pi(\mathrm{e}_r))\big).$$

If $\varepsilon(m) = 0$, then $m(\pi(\mathrm{e}_i)) = 0$ for all $i = 1, 2, \ldots, r$. As π is surjective, the elements $\pi(\mathrm{e}_i) \quad (i = 1, 2, \ldots, r)$ generate the left $\mathbb{W}$-module $U = U_{\mathcal{D}}$. So, it follows that $m = 0$ and this proves, that the map ε is injective. It remains to show that $\varepsilon\big(\mathrm{Hom}_{\mathbb{W}}\big(U_{\mathcal{D}}, \mathcal{O}\big)\big) = \mathbb{S}_{\mathcal{D}}(\mathcal{O}) =: \mathbb{S}(\mathcal{O})$. Do do so, let $\mathrm{b}_j := (\delta_{j,k})_{k=1}^s \in \mathbb{W}^s \quad (j = 1, 2, \ldots, s)$ be the canonical basis elements of $\mathbb{W}^s$. First, let $m \in \mathrm{Hom}_{\mathbb{W}}\big(U_{\mathcal{D}}, \mathcal{O}\big)$. We aim to show, that $\varepsilon(m) \in \mathbb{S}_{\mathcal{D}}(\mathcal{O})$. We have to check that the column

$$\begin{pmatrix} g_1 \\ g_2 \\ \cdot \\ g_s \end{pmatrix} := \mathcal{D} \begin{pmatrix} m(\mathrm{e}_1) \\ m(\mathrm{e}_2) \\ \cdot \\ m(\mathrm{e}_r) \end{pmatrix}$$

vanishes. For each $i = 1, 2, \ldots, s$ we can write $\sum_{j=1}^r d_{ij}\mathrm{e}_j = \mathrm{b}_i\mathcal{D} = h(\mathrm{b}_i)$, and hence get indeed

$$g_i = \sum_{j=1}^r d_{ij} m(\pi(\mathrm{e}_j)) = m\big(\sum_{j=1}^r d_{ij}\pi(\mathrm{e}_j)\big) = m\big(\pi(\sum_{j=1}^r d_{ij}\mathrm{e}_j)\big) = m\big(\pi(h(\mathrm{b}_i))\big) = m(0) = 0.$$

Conversely, let $(f_1, f_2, \ldots, f_r) \in \mathbb{S}(\mathcal{O})$, so that $\sum_{i=1}^r d_{ij} f_j = 0$. We aim to show that $(f_1, f_2, \ldots, f_r) \in \varepsilon\big(\mathrm{Hom}_{\mathbb{W}}(U, \mathcal{O})\big)$.
To this end, we consider the homomorphism of left $\mathbb{W}$-modules

$$k : \mathbb{W}^r \longrightarrow \mathcal{O}, \text{ given by } (u_1, u_2, \ldots, u_r) \mapsto \sum_{j-1}^r u_j f_j.$$

Observe that $k(h(\mathrm{b}_i)) = k(\mathrm{b}_i\mathcal{D}) = k(d_{i1}, d_{i2}, \ldots, d_{ir}) = \sum_{j=1}^r d_{ij} f_j = 0$ for all $i = 1, 2, \ldots, s$. It follows that $k \circ h = 0$. Therefore k induces a homomorphism of left $\mathbb{W}$-modules $m : U \longrightarrow \mathcal{O}$, such that $m \circ \pi = k$. It follows that $m(\pi(\mathrm{e}_j)) = k(\mathrm{e}_j) = f_j$ for all $j = 1, 2, \ldots, r$. But this means that $(f_1, f_2, \ldots, f_r) = \varepsilon(m) \in \varepsilon\big(\mathrm{Hom}_{\mathbb{W}}(U, \mathcal{O})\big)$. $\square$

11.13. **Exercise.** (A) Let $n = 1$, $K = \mathbb{R}$ and let $\mathcal{O} := \mathcal{C}^\infty(\mathbb{R})$ be set of smooth functions on $\mathbb{R}$. Fix $d \in \mathbb{W} = \mathbb{W}(\mathbb{R}, 1) = \mathbb{R}[X, \partial]$ and consider the matrix $\mathcal{D} = (d) \in \mathbb{W}^{1\times 1}$. Determine $U_{\mathcal{D}}, \mathbb{S}_{\mathcal{D}}(\mathcal{O})$ and $\mathbb{V}^{\underline{v},\underline{w}}(U_{\mathcal{D}})$ for all weights $(\underline{v}, \underline{w}) = (v, w) \in \mathbb{N}_0 \times \mathbb{N}_0 \setminus \{(0, 0)\}$ and for

$d = \partial_1, \quad d = \partial^2 - 1, \quad d = \partial^2 - 1, \quad d = \partial - x^2$ and $d = \partial^2 + c\partial - b$ with $c, b \in \mathbb{R} \setminus \{0\}$.

(B) Let $n, m \in \mathbb{N}$, $\mathcal{O} := K[X_1, X_2, \ldots, X_n]$ and consider the matrix

$$\mathcal{D} := \begin{pmatrix} \partial_1^m \\ \partial_2^m \\ \cdot \\ \partial_n^m \end{pmatrix} \in \mathbb{W}^{n\times 1}.$$

Determine $U_{\mathcal{D}}, \mathbb{S}_{\mathcal{D}}(\mathcal{O})$ and $\mathbb{V}^{\underline{1}\underline{1}}(U_{\mathcal{D}})$.

12. Gröbner Bases

In this section, we introduce and treat Gröbner bases of left ideals in standard Weyl algebras with respect to so-called admissible orderings of the set of elementary differential operators. Indeed a great deal of what we shall present in the sequel could also be deduced from the theory of Gröbner bases in commutative polynomial rings. Nevertheless, we prefer to introduce the subject in a self contained way for Weyl algebras. As for Gröbner bases in polynomial rings and their applications, there are is a number of sources, for example [1], [11], [12], [13], [14], [17] and [20].
The main goal of the present section is to prove that left ideals in Weyl algebras admit so-called universal Gröbner bases. This existence result can actually be proved in the more general setting of admissible algebras. Readers, who are interested in this, should consult for example Boldini's thesis [3] or else [18], [19] or [21].

12.1. **Convention.** (A) As previously, we fix a positive integer n, a field K of characteristic 0 and consider the standard Weyl algebra $\mathbb{W} := K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n]$ and the polynomial ring $\mathbb{P} := K[Y_1, Y_2, \ldots, Y_n; Z_1, Z_2, \ldots, Z_n]$ in the indeterminates $Y_1, Y_2, \ldots, Y_n; Z_1, Z_2, \ldots, Z_n$ over K.

(B) In addition, we fix the isomorphism of K-vector spaces

$$\Phi : \mathbb{W} \xrightarrow{\cong} \mathbb{P} \text{ given by } \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \mapsto \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \text{ for all all } \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n$$

and the sets:

$$\mathbb{E} := \{\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \mid \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n\} \text{ and } \mathbb{M} := \{\underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \mid \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n\} = \Phi(\mathbb{E}).$$

12.2. **Definition, Reminder and Exercise.** (A) *(Total Orderings)* Let S be any set. A *total ordering* of S is a binary relation $\leq$ on S such that for all $a, b, c \in S$ the following requirements are satisfied:

(a) *(Reflexivity)* $a \leq a$.
(b) *(Antisymmetry)* If $a \leq b$ and $b \leq a$, then $a = b$.
(c) *(Transitivity)* If $a \leq b$ and $b \leq c$, then $a \leq c$.
(b) *(Totality)* Either $a \leq b$ or $b \leq a$.

We write $\mathrm{TO}(S)$ for the set of total orderings on S. If $\leq \in \mathrm{TO}(S)$ and $a, b \in S$, we write

$$a < b \text{ if } a \leq b \text{ and } a \neq b, \quad b \geq a \text{ if } a \leq b, \quad b < a \text{ if } a < b.$$

(B) *(Well Orderings)* Keep the above notations and hypotheses. A total ordering $\leq \in \mathrm{TO}(S)$ is said to be a *well ordering* of S, if it satisfies the following additional requirement:

(e) *(Existence of Least Elements)* For each non-empty subset $T \subseteq S$ there is an element (clearly unique) $t := \min_{\leq}(T) \in T$ such that $t \leq t'$ for all $t' \in T$ – called *least element* or the *minimum* of T with respect to $\leq$.

We write $\mathrm{WO}(S)$ for the set of all well orderings of S.

(C) *(Admissible Orderings)* A total ordering $\leq \in \mathrm{TO}(\mathbb{E})$ of the set of all elementary differential operators is called an *admissible ordering* of $\mathbb{E}$ if it satisfies the following requirements:

(a) *(Foundedness)* $1 \leq \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}$ for all $\underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n$

(b) *(Compatibility)* For all $\underline{\lambda}, \underline{\lambda}', \underline{\kappa}, \underline{\kappa}', \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n$ we have the implication:

$$\text{If } \underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}} \leq \underline{X}^{\underline{\lambda}'}\underline{\partial}^{\underline{\kappa}'}, \text{ then } \underline{X}^{\underline{\lambda}+\underline{\nu}}\underline{\partial}^{\underline{\kappa}+\underline{\mu}} \leq \underline{X}^{\underline{\lambda}'+\underline{\nu}}\underline{\partial}^{\underline{\kappa}'+\underline{\mu}}.$$

We write $\mathrm{AO}(\mathbb{E})$ for the set of all admissible orderings of $\mathbb{E}$. Prove the following facts:

(c) If $\underline{\nu}, \underline{\nu}', \underline{\mu}, \underline{\mu}', \underline{\lambda}, \underline{\lambda}', \underline{\kappa}, \underline{\kappa}', \in \mathbb{N}_0^n$ with $\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \leq \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}$ and $\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}} < \underline{X}^{\underline{\lambda}'}\underline{\partial}^{\underline{\kappa}'}$, then

$$\underline{X}^{\underline{\lambda}+\underline{\nu}}\underline{\partial}^{\underline{\kappa}+\underline{\mu}} < \underline{X}^{\underline{\lambda}'+\underline{\nu}'}\underline{\partial}^{\underline{\kappa}'+\underline{\mu}'}.$$

(d) $\mathrm{AO}(\mathbb{E}) \subseteq \mathrm{WO}(\mathbb{E})$.

(D) *(Leading Elementary Differential Operators and Related Concepts)* Keep the above notations and hypotheses. If $\leq \in \mathrm{AO}(\mathbb{E})$ and $d \in \mathbb{W}\setminus\{0\}$, we define the *leading elementary differential operator* of d with respect to $\leq$ by:

$$\mathrm{LE}_{\leq}(d) := \max_{\leq} \mathrm{supp}(d), \text{ so that } \mathrm{LE}_{\leq}(d) \in \mathrm{supp}(d) \text{ and } e \leq \mathrm{LE}_{\leq}(d) \text{ for all } e \in \mathrm{supp}(d).$$

Moreover, we define the *leading coefficient* $\mathrm{LC}_{\leq}(d)$ of d with respect to $\leq$ as the coefficient of d with respect to $\mathrm{LE}_{\leq}(d)$ and – correspondingly the *leading differential operator* $\mathrm{LD}_{\leq}(d)$ of d with respect to $\leq$. Hence:

(a) $\mathrm{LC}_{\leq}(d) \in K \setminus \{0\}$ with $\mathrm{LE}_{\leq}\big(d - \mathrm{LC}_{\leq}(d)\mathrm{LE}_{\leq}(d)\big) < \mathrm{LE}_{\leq}(d)$.

(b) $\mathrm{LD}_{\leq}(d) = \mathrm{LC}_{\leq}(d)\mathrm{LE}_{\leq}(d)$.

(c) $\mathrm{LE}_{\leq}\big(d - \mathrm{LD}_{\leq}(d)\big) < \mathrm{LE}_{\leq}(d)$.

Finally, we define the *leading monomial* and the *leading term* of d with respect to $\leq$ respectively by

$$\mathrm{LM}_{\leq}(d) := \Phi\big(\mathrm{LE}_{\leq}(d)\big) \text{ and } \mathrm{LT}_{\leq}(d) := \Phi\big(\mathrm{LD}_{\leq}(d)\big) = \mathrm{LC}_{\leq}(d)\mathrm{LM}_{\leq}(d).$$

Prove the following statements:

(d) If $d, e \in \mathbb{W} \setminus \{0\}$, with $d \neq -e$, then $\mathrm{LE}_{\leq}(d + e) \leq \max_{\leq}\{\mathrm{LE}_{\leq}(d), \mathrm{LE}_{\leq}(e)\}$, with equality if and only if $\mathrm{LD}_{\leq}(d) \neq -\mathrm{LD}_{\leq}(e)$.

12.3. **Examples and Exercises.** (A) *(Well Orderings)* Keep the above notations and hypotheses. Prove the following statements:

(a) Let $\varphi : \mathbb{N}_0 \longrightarrow \mathbb{N}_0^n \times \mathbb{N}_0^n$ be a bijective map. Show that the binary relation $\leq_\varphi$ defined on on $\mathbb{E}$ by $\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \leq_\varphi \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} \Leftrightarrow \varphi^{-1}(\underline{\nu}, \underline{\mu}) \leq \varphi^{-1}(\underline{\nu}, \underline{\mu})$ $(\forall \underline{\nu}, \underline{\mu}, \underline{\nu}', \underline{\mu}' \in \mathbb{N}_0^n)$ is a well ordering.

(b) Show that in the notations of exercise (a) the well ordering $\leq_\varphi$ is *discrete*, which means that the set $\{e \in \mathbb{E} \mid e \leq_\varphi d\}$ is finite for all $d \in \mathbb{E}$.

(c) Show, that there uncountably many discrete well orderings of $\mathbb{E}$.

(d) Let $n = 1$, set $X_1 =: X, \partial_1 =: \partial$ and define the binary relation $\leq$ on the set of elementary differential operators $\mathbb{E} = \{X^\nu\partial^\mu \mid \nu, \mu \in \mathbb{N}_0\}$ by

$$X^\nu\partial^\mu \leq X^{\nu'}\partial^{\mu'} \text{ if either } \begin{cases} \nu < \nu' \text{ or else} \\ \nu = \nu' \text{ and } \mu < \mu' \end{cases}$$

for all $\nu, \mu \in \mathbb{N}_0$. Show, that $\leq$ is a non-discrete well ordering of $\mathbb{E}$.

(B) *(Admissible Orderings)* Keep the above notations and hypotheses.

(a) We define the binary relation $\leq_{\text{lex}} \subseteq \mathbb{E}\times\mathbb{E}$ by setting (again for all $\underline{\nu}, \underline{\mu}, \underline{\nu}', \underline{\mu}' \in \mathbb{N}_0^n$):

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \leq_{\text{lex}} \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} \text{ if either}$$

(1) $\underline{\nu} = \underline{\nu}'$ and $\underline{\mu} = \underline{\mu}'$, or
(2) $\underline{\nu} = \underline{\nu}'$ and $\exists j \in \{1, 2, \ldots, n\} : \left[\mu_j < \mu_j' \text{ and } \mu_k = \mu_k', \forall k < j\right]$, or else
(3) $\exists i \in \{1, 2, \ldots, n\} : \left[\nu_i < \nu_i' \text{ and } \nu_k = \nu_k', \forall k < i\right]$.

Prove that $\leq_{\text{lex}} \in \mathrm{AO(E)}$. The admissible ordering $\leq_{\text{lex}}$ is called the *lexicographic ordering* of the set of elementary differential operators.

(b) Set $n = 1$, $X_1 =: X$, $\partial_1 =: \partial$ and write down the first 20 elementary differential operators $d \in \mathbb{E} = \{X^\nu \partial^\mu \mid \nu, \mu \in \mathbb{N}_0\}$ with respect to the ordering $\leq_{\text{lex}}$.

(c) Solve the similar task as in exercise (b), but with $n = 2$ instead of $n = 1$ and with 30 instead of 20.

(d) We define another binary relation $\leq_{\text{deglex}} \subseteq \mathbb{E} \times \mathbb{E}$ by setting

$$d \leq_{\text{deglex}} e \text{ if either } \begin{cases} \deg(d) < \deg(e) \text{ or else} \\ \deg(d) = \deg(e) \text{ and } d \leq_{\text{lex}} e. \end{cases}$$

Show, that $\leq_{\text{deglex}} \in \mathrm{AO}(\mathbb{E})$. This admissible ordering is called the *degree-lexicographic ordering* of the set of elementary differential operators.

(e) Solve the previous exercises (b) and (c) but this time with the ordering $\leq_{\text{deglex}}$.

(f) We introduce a further binary relation $\leq_{\text{degrevlex}}$ on $\mathbb{E}$ by: setting ($\underline{\nu}, \underline{\mu}, \underline{\nu}', \underline{\mu}' \in \mathbb{N}_0^n$):

$$\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \leq_{\text{degrevlex}} \underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'} \text{ if either}$$

(1) $\deg\left(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\right) < \deg\left(\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\right)$, or else
(2) $\deg\left(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\right) = \deg\left(\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\right)$ and either
  (i) $\underline{\nu} = \underline{\nu}'$ and $\underline{\mu} = \underline{\mu}'$, or
  (ii) $\underline{\mu} = \underline{\mu}'$ and $\exists i \in \{1, 2, \ldots, n\} : \left[\nu_i > \nu_i' \text{ and } \nu_k = \nu_k', \forall k > i\right]$, or else
  (iii) $\exists j \in \{1, 2, \ldots, n\} : \left[\mu_j > \mu_j' \text{ and } \mu_k = \mu_k', \forall k > j\right]$.

Prove, that $\leq_{\text{degrevlex}} \in \mathrm{AO}(\mathbb{E})$. This admissible ordering is called the *degree-reverse-lexicographic ordering* of the set of elementary differential operators.

(g) Solve the previous exercise (e) but with $\leq_{\text{degrevlex}}$ instead of $\leq_{\text{deglex}}$.

(h) An *admissible ordering* of the set $\mathbb{M} = \{\underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \mid \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n\}$ of all monomials in $\mathbb{P}$ is a total ordering $\leq$ of $\mathbb{M}$ which satisfies the requirements
(1) *(Foundedness)* $1 \leq m$ for all $m \in \mathbb{M}$.
(2) *(Compatibility)* For all m, m' and $t \in \mathbb{M}$ we have the implication:

$$\text{If } m \leq m', \text{ then } mt \leq m't.$$

For any $\leq \in \mathrm{AO}(\mathbb{E})$ we define the binary relation $\leq_\Phi$ on $\mathbb{M}$ by:

$$m \leq_\Phi m' \Leftrightarrow \Phi^{-1}(m) \leq \Phi^{-1}(m') \text{ for all } m, m' \in \mathbb{M}.$$

Prove, that $\leq_\Phi \in \mathrm{AO}(\mathbb{M})$ and that there is indeed a bijection

$$\bullet_\Phi : \mathrm{AO}(\mathbb{E}) \xrightarrow{\cong} \mathrm{AO}(\mathbb{M}), \text{ given by } \leq \mapsto \leq_\Phi .$$

The names given in the previous exercises (a), (d) and (f) to the three admissible orderings of $\mathbb{E}$ introduced in these exercises are "inherited" from the "classical" designations used in polynomial rings, via the above bijection.

(i) Prove, that $\leq_{\text{deglex}}$ and $\leq_{\text{degrevlex}}$ are both discrete in the sense of Exercise (A) (b), where as $\leq_{\text{lex}}$ is not.

(C) *(Leading Elementary Differential Operators and Related Concepts)* Keep the previous notations and hypotheses.

(a) Let $n = 1$, set $X_1 =: X$, $\partial_1 =: \partial$, $Y_1 =: Y$ and $Z_1 =: Z$. Write down the leading elementary differential operator, the leading differential operator, the leading coefficient, the leading monomial and the leading term of each of the following differential operators, with respect to each of the admissible orderings $\leq_{\text{lex}}$, $\leq_{\text{deglex}}$ and $\leq_{\text{degrevlex}}$:
  (1) $5X^6 + 4X^4\partial - 2X^2\partial^3 + X\partial^4 - 3\partial^6$.
  (2) $\partial^4 - 4X\partial^3 + 6X^2\partial^2 - 4X\partial + X^4$.
  (3) $\partial^{12} - X^5\partial^7 + X^7\partial^5 - X^9\partial^3 + X^{12}$.

(b) Let $n = 2$ solve the task corresponding to exercise (a) above for the differential operators
  (1) $X_1^3X_2^2 + 2\partial_1^3\partial_2^2$.
  (2) $X_1^2X_2^3\partial_1^2\partial_2^3 - \partial_1^4\partial_2^6$.
  (3) $X_1^k + X_2^k + \partial_1^k + \partial_2^k$ with $k \in \mathbb{N}$.

The next proposition will play a crucial role for our further considerations.

12.4. **Proposition.** ***(Multiplicativity of Leading Terms)*** *Let* $\leq \in \operatorname{AO}(\mathbb{E})$ *and let* $d, e \in \mathbb{W} \setminus \{0\}$. *Then it holds*

(a) $\operatorname{LT}_{\leq}(de) = \operatorname{LT}_{\leq}(d)\operatorname{LT}_{\leq}(e)$.
(b) $\operatorname{LM}_{\leq}(de) = \operatorname{LM}_{\leq}(d)\operatorname{LM}_{\leq}(e)$.

Proof. The product formula for elementary differential operators of Proposition 6.2 yields that $\operatorname{LE}_{\leq}\big(\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}\big) = \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}'+\underline{\mu}'}$ for all $\underline{\nu}, \underline{\nu}', \underline{\mu}, \underline{\mu}' \in \mathbb{N}_0^n$. We may write

$$d = \sum_{(\underline{\nu},\underline{\mu})\in\operatorname{supp}(d)} c^{(d)}_{\underline{\nu}\underline{\mu}}\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \text{ and } e = \sum_{(\underline{\nu}',\underline{\mu}')\in\operatorname{supp}(e)} c^{(e)}_{\underline{\nu}'\underline{\mu}'}\underline{X}^{\underline{\nu}'}\underline{\partial}^{\underline{\mu}'}$$

with $c^{(d)}_{\underline{\nu}\underline{\mu}}, c^{(e)}_{\underline{\nu}'\underline{\mu}'} \in K \setminus \{0\}$ for all $(\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(d)$ and all $(\underline{\nu}', \underline{\mu}') \in \operatorname{supp}(e)$. With appropriate pairs $(\underline{\nu}^{(0)}, \underline{\mu}^{(0)}) \in \operatorname{supp}(d)$ and $(\underline{\nu}'^{(0)}, \underline{\mu}'^{(0)}) \in \operatorname{supp}(e)$ we also may write

$$\operatorname{LE}_{\leq}(d) = \underline{X}^{\underline{\nu}^{(0)}}\underline{\partial}^{\underline{\mu}^{(0)}}, \operatorname{LE}_{\leq}(e) = \underline{X}^{\underline{\nu}'^{(0)}}\underline{\partial}^{\underline{\mu}'^{(0)}}, \operatorname{LC}_{\leq}(d) = c^{(d)}_{\underline{\nu}^{(0)}\underline{\mu}^{(0)}} \text{ and } \operatorname{LC}_{\leq}(e) = c^{(e)}_{\underline{\nu}'^{(0)}\underline{\mu}'^{(0)}}.$$

Now, bearing in mind the previous observation on leading elementary differential operators we may write

$$\begin{aligned} de &= \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} c^{(e)}_{\underline{\nu}'\underline{\mu}'} \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'} = \\ &= \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} c^{(d)}_{\underline{\nu}\underline{\mu}} c^{(e)}_{\underline{\nu}'\underline{\mu}'} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \underline{X}^{\underline{\nu}'} \underline{\partial}^{\underline{\mu}'} = \\ &= \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}(e)} \big[c^{(d)}_{\underline{\nu}\underline{\mu}} c^{(e)}_{\underline{\nu}'\underline{\mu}'} \underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'} + r_{\underline{\nu}\underline{\nu}'\underline{\mu}\underline{\mu}'} \big], \end{aligned}$$

with $r_{\underline{\nu}\underline{\nu}'\underline{\mu}\underline{\mu}'} \in \mathbb{W}$, such that for all $(\underline{\nu},\underline{\mu}) \in \mathrm{supp}(d)$ and all $(\underline{\nu}',\underline{\mu}') \in \mathrm{supp}(e)$ it holds

$$\mathrm{LE}_{\leq}(r_{\underline{\nu}\underline{\nu}'\underline{\mu}\underline{\mu}'}) < \underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'}, \text{ whenever } r_{\underline{\nu}\underline{\nu}'\underline{\mu}\underline{\mu}'} \neq 0.$$

By Definition, Reminder and Exercise 12.2 (C)(c) we have

$$\underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'} < \underline{X}^{\underline{\nu}^{(0)}+\underline{\nu}'^{(0)}} \underline{\partial}^{\underline{\mu}^{(0)}+\underline{\mu}'^{(0)}}, \text{ for all}$$
$$\big((\underline{\nu},\underline{\mu}),(\underline{\nu}',\underline{\mu}')\big) \in \mathrm{supp}(d)\times\mathrm{supp}(e) \setminus \{\big((\underline{\nu}^{(0)},\underline{\mu}^{(0)}),(\underline{\nu}'^{(0)},\underline{\mu}'^{(0)})\big)\}.$$

By Definition, Reminder and Exercise 12.2 (D)(d) it now follows easily that

$$\mathrm{LE}_{\leq}(de) = \underline{X}^{\underline{\nu}^{(0)}+\underline{\nu}'^{(0)}} \underline{\partial}^{\underline{\mu}^{(0)}+\underline{\mu}'^{(0)}} \text{ and } \mathrm{LC}_{\leq}(de) = c^{(d)}_{\underline{\nu}^{(0)}\underline{\mu}^{(0)}} c^{(e)}_{\underline{\nu}'^{(0)}\underline{\mu}'^{(0)}} = \mathrm{LC}_{\leq}(d)\mathrm{LC}_{\leq}(e).$$

We thus obtain $\mathrm{LM}_{\leq}(de) = \Phi\big(\underline{X}^{\underline{\nu}^{(0)}+\underline{\nu}'^{(0)}} \underline{\partial}^{\underline{\mu}^{(0)}+\underline{\mu}'^{(0)}}\big) = \underline{Y}^{\underline{\nu}^{(0)}+\underline{\nu}'^{(0)}} \underline{Z}^{\underline{\mu}^{(0)}+\underline{\mu}'^{(0)}} =$
$= \underline{Y}^{\underline{\nu}^{(0)}} \underline{Z}^{\underline{\mu}^{(0)}} \underline{Y}^{\underline{\nu}'^{(0)}} \underline{Z}^{\underline{\mu}'^{(0)}} = \Phi\big(\underline{X}^{\underline{\nu}^{(0)}} \underline{\partial}^{\underline{\mu}^{(0)}}\big)\Phi\big(\underline{X}^{\underline{\nu}'^{(0)}} \underline{\partial}^{\underline{\mu}'^{(0)}}\big) = \Phi\big(\mathrm{LE}_{\leq}(d)\big)\Phi\big(\mathrm{LE}_{\leq}(e)\big) =$
$= \mathrm{LM}_{\leq}(d)\mathrm{LM}_{\leq}(e)$. But now it follows immediately that $\mathrm{LT}_{\leq}(de) = \mathrm{LC}_{\leq}(de)\mathrm{LM}_{\leq}(de) =$
$= \mathrm{LC}_{\leq}(d)\mathrm{LC}_{\leq}(e)\mathrm{LM}_{\leq}(d)\mathrm{LM}_{\leq}(e) = \mathrm{LC}_{\leq}(d)\mathrm{LM}_{\leq}(d)\mathrm{LC}_{\leq}(e)\mathrm{LM}_{\leq}(e) = \mathrm{LT}_{\leq}(d)\mathrm{LT}_{\leq}(e)$. □

The next result may be understood as an extension of the classical division algorithms of Euclid for uni-variate polynomials to the case of differential operators.

12.5. **Proposition.** ***(The Division Property)*** *Let* $\leq \in \mathrm{AO}(\mathbb{E})$, *let* $d \in \mathbb{W}$ *and let* $F \subset \mathbb{W}$ *be a finite set. Then, there is an element* $r \in \mathbb{W}$ *and a family* $(q_f)_{f\in F} \in \mathbb{W}^F$ *such that*

(a) $d = \sum_{f\in F} q_f f + r$;
(b) $\Phi(s) \notin \mathbb{P}\mathrm{LM}_{\leq}(f)$ *for all* $f \in F \setminus \{0\}$ *and all* $s \in \mathrm{supp}(r)$.
(c) $\mathrm{LE}_{\leq}(q_f f) \leq \mathrm{LE}_{\leq}(d)$ *for all* $f \in F$ *with* $q_f f \neq 0$.

Proof. We clearly may assume that $F \subset \mathbb{W}\setminus\{0\}$. If $d = 0$, we choose $r = 0$ and $q_f = 0$ for all $f \in F$. Assume, that our claim is wrong, and let $U \subsetneq \mathbb{W}$ be the set of all differential operators $d \in \mathbb{W}$ which do not admit a presentation of the requested form. As $\leq \in \mathrm{WO}(\mathbb{E})$ and $U \subset \mathbb{W}\setminus\{0\}$, we find some $d \in U$ such that

$$\mathrm{LE}_{\leq}(d) = \min_{\leq}\{\mathrm{LE}_{\leq}(u) \mid u \in U\}.$$

We distinguish the following two cases:

(1) There is some $f \in F$ such that $\mathrm{LM}_{\leq}(d) \in \mathbb{P}\mathrm{LM}_{\leq}(f)$.
(2) $f \notin \bigcup_{f\in F} \mathbb{P}\mathrm{LM}_{\leq}(f)$.

In the case (1) we find some $e \in \mathbb{E}$ such that $\mathrm{LM}_\leq(d) = \Phi(e)\mathrm{LM}_\leq(f)$ and so we can introduce the element

$$d' := d - \frac{\mathrm{LC}_\leq(d)}{\mathrm{LC}_\leq(f)} ef \in \mathbb{W}.$$

If $d' = 0$, we set

$$r = 0, \quad q_f := \frac{\mathrm{LC}_\leq(d)}{\mathrm{LC}_\leq(f)} e, \text{ and } q_{f'} = 0 \text{ for all } f' \in F \setminus \{f\}.$$

But then

$$d = \frac{\mathrm{LC}_\leq(d)}{\mathrm{LC}_\leq(f)} ef = q_f f + r$$

is a presentation of d with the requested properties. So, let $d' \neq 0$. Observe, that by Proposition 12.4 (a) we can write

$$\mathrm{LT}_\leq\big(\frac{\mathrm{LC}_\leq(d)}{\mathrm{LC}_\leq(f)} ef\big) = \frac{\mathrm{LC}_\leq(\mathrm{d})}{\mathrm{LC}_\leq(f)}\mathrm{LT}_\leq(ef) = \frac{\mathrm{LC}_\leq(d)}{\mathrm{LC}_\leq(f)}\mathrm{LT}_\leq(e)\mathrm{LT}_\leq(f) =$$
$$\mathrm{LC}_\leq(d)\mathrm{LM}_\leq(e)\mathrm{LM}_\leq(f) = \mathrm{LC}_\leq(d)\Phi(e)\mathrm{LM}_\leq(f) = \mathrm{LC}_\leq(d)\mathrm{LM}_\leq(d) = \mathrm{LT}_\leq(d).$$

If follows that $\mathrm{LD}_\leq\big(\frac{\mathrm{LC}_\leq(d)}{\mathrm{LC}_\leq(f)} ef\big) = \mathrm{LD}_\leq(d)$, and hence by Definition, Reminder and Exercise 12.2 (D)(d) we obtain that $\mathrm{LE}_\leq(d') < \mathrm{LE}_\leq(d) = \min_\leq\{\mathrm{LE}_\leq(u) \mid u \in U\}$. Therefore, $d' \notin U$ and so we find an element $r' \in \mathbb{W}$ and a family $(q'_{f'})_{f' \in F} \in \mathbb{W}^F$ with

(a)$'$ $d' = \sum_{f' \in F} q'_{f'} f' + r'$;
(b)$'$ $\Phi(s') \notin \mathbb{P}\mathrm{LM}_\leq(f')$ for all $f' \in F$ and all $s' \in \mathrm{supp}(r')$.
(c)$'$ $\mathrm{LE}_\leq(q'_{f'} f') \leq \mathrm{LE}_\leq(d')$ for all $f' \in F$ with $q'_{f'} \neq 0$.

Now, we set

$$r := r' \text{ and } q_f := \begin{cases} q'_{f'} & \text{if } f' \neq f, \\ q'_f + \frac{\mathrm{LC}_\leq(d)}{\mathrm{LC}_\leq(f)} e & \text{if } f = f'. \end{cases}$$

As

$$\mathrm{LE}_\leq(q'_f) \leq \mathrm{LE}_\leq(d') < \mathrm{LE}_\leq(d) \text{ and } \mathrm{LE}_\leq\big(\frac{\mathrm{LC}_\leq(d)}{\mathrm{LC}_\leq(f)} e\big) = \mathrm{LE}_\leq(e) \leq \mathrm{LE}_\leq(d),$$

we get

$$\mathrm{LE}_\leq(q_f) = \mathrm{LE}_\leq\big(q'_f + \frac{\mathrm{LC}_\leq(d)}{\mathrm{LC}_\leq(f)} e\big) \leq \mathrm{LE}_\leq(d).$$

Now, it follows easily, that the requirements (a),(b) and (c) of our proposition are satisfied in the case (1).

So, let us assume that we are in the case (2). We set $d' := d - \mathrm{LD}_\leq(d)$. If $d' = 0$ we have $d' = \mathrm{LD}_\leq(d)$ and it suffices to choose $q_f := 0$ for all $f \in F$ and $r = d$.

So, let $d' \neq 0$. Then, we have $\mathrm{LE}_\leq(d') < \mathrm{LE}_\leq(d)$ (see Definition, Reminder and Exercise‘12.2 (D)(c)), so that again $d' \notin U$. But this means once more, that we get elements r' and $q'_{f'} \in \mathbb{W}$ (for all $f' \in F$) such that the above conditions (a)$'$, (b)$'$ and (c)$'$ are satisfied. Now, we set $r := r' + \mathrm{LD}_\leq(d)$ and $q_f := q'_f$ for all $f \in F$. As $\mathrm{supp}(r) \subseteq \mathrm{supp}(r') \cup \{\mathrm{LE}_\leq(d)\}$ and $\mathrm{LE}_\leq(q_f f) \leq \mathrm{LE}(d') \leq \mathrm{LE}_\leq(d)$ for all $f \in F$ with $q_f \neq 0$ the requirements (a),(b) and (c) are again satisfied for the suggested choice. $\square$

Now, we are ready to introduce the basic notion of this section: the concept of Gröbner basis.

12.6. **Definition, Reminder and Exercise.** (A) *(Monomial Ideals)* An ideal $I \subseteq \mathbb{P}$ is called a *monomial ideal* if there is a set $S \subset \mathbb{M} = \{\underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \mid \underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n\}$ such that

$$I = \sum_{s \in S} \mathbb{P}s.$$

Show that in this situation for all $m \in \mathbb{M} \setminus \{0\}$ we have

(a) If $m = \sum_{i=1}^{t} f_i s_i$ with $s_1, s_2, \ldots, s_t \in S$ and $f_1, f_2, \ldots, f_t \in \mathbb{P}$, then there is some $i \in \{1, 2, \ldots, t\}$ and some $n_i \in \mathrm{supp}(f_i)$ such that $m = n_i s_i$.
(b) $m \in I$ if and only if there are $n \in \mathbb{M}$ and some $s \in S$ such that $m = ns$.

(B) *(Leading Monomial Ideals)* Let $\leq \in \mathrm{AO}(\mathbb{E})$ and $T \subset \mathbb{W}$. Then, the ideal

$$\mathrm{LMI}_{\leq}(T) := \sum_{d \in T \setminus \{0\}} \mathbb{P}\mathrm{LM}_{\leq}(d)$$

is called the *leading monomial ideal of* T with respect to $\leq T$.
Prove that for all $m \in \mathbb{M}$, we have the following statements.

(a) If $m = \sum_{i=1}^{s} f_i \mathrm{LM}_{\leq}(t_i)$ with $t_1, t_2, \ldots, t_s \in T$ and $f_1, f_2, \ldots, f_s \in \mathbb{P}$, then there is some $i \in \{1, 2, \ldots, s\}$ and some $n_i \in \mathrm{supp}(f_i)$ such that $t_i \neq 0$ and $m = n_i \mathrm{LM}_{\leq}(t_i)$.
(b) $m \in \mathrm{LMI}_{\leq}(T)$ if and only if there are elements $u \in \mathbb{E}$ and $t \in T$ such that $m = \mathrm{LM}_{\leq}(u)\mathrm{LM}_{\leq}(t)$.

(C) *(Gröbner Bases)* Let $\leq \in \mathrm{AO}(\mathbb{E})$ and let $L \subseteq \mathbb{W}$ be a left ideal. A *Gröbner basis* of L with respect to $\leq$ (or a $\leq$-*Gröbner basis* of L) is a subset $G \subseteq L$ such that

$$\#G < \infty \text{ and } \mathrm{LMI}_{\leq}(L) = \mathrm{LMI}_{\leq}(G).$$

Prove the following facts:

(a) If G is a $\leq$-Gröbner basis of L and $G \subseteq H \subseteq L$ with $\#H < \infty$, then H is a $\leq$-Gröbner basis of L.
(b) If G is a $\leq$-Gröbner basis of L, then for each $d \in L \setminus \{0\}$ there is some $u \in \mathbb{E}$ and some $g \in G \setminus \{0\}$ such that $\mathrm{LM}_{\leq}(d) = \mathrm{LM}_{\leq}(u)\mathrm{LM}_{\leq}(g) = \mathrm{LM}_{\leq}(ug)$.
(c) If G is a $\leq$-Gröbner basis of L, then for each $d \in L \setminus \{0\}$ there is some monomial $m = \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \in \mathbb{P}$ and some $g \in G \setminus \{0\}$ such that $\mathrm{LM}_{\leq}(d) = m\mathrm{LM}_{\leq}(g)$.

Now, we prove that Gröbner bases always exist, and that they deserve the name of "basis", as they generate the involved left ideal.

12.7. **Proposition.** ***(Existence and Generating Property of Gröbner Bases)*** *Let $\leq \in \mathrm{AO(E)}$ and let $L \subseteq \mathbb{W}$ be a left ideal. Then the following statements hold.*

(a) *L admits a $\leq$-Gröbner basis.*
(b) *If G is any $\leq$-Gröbner basis of L, then $L = \sum_{g \in G} \mathbb{W}g$.*

Proof. (a): This is clear as the ideal $\mathrm{LMI}_{\leq}(L)$ is generated by finitely many elements of the form $\mathrm{LM}_{\leq}(g)$ with $g \in L$.

(b): Let $G \subseteq L$ be a $\leq$-Gröbner basis of L and assume that $\sum_{g \in G} \mathbb{W}g \subsetneq L$. As $\leq \in \mathrm{WO}(\mathbb{E})$, we find some $e \in \mathbb{W} \setminus \sum_{g \in G} \mathbb{W}g$ such that

$$\mathrm{LE}(e) = \min_{\leq}\{\mathrm{LE}_{\leq}(d) \mid d \in L \setminus \sum_{g \in G} \mathbb{W}g\}.$$

By Definition, Reminder and Exercise 12.6 (C)(b) we find some $u \in \mathbb{E}$ and some $g \in G$ such that $\mathrm{LM}_{\leq}(e) = \mathrm{LM}_{\leq}(u)\mathrm{LM}_{\leq}(g)$. Setting

$$v := -\frac{\mathrm{LC}_{\leq}(e)}{\mathrm{LC}_{\leq}(g)}u$$

we now get on use of Proposition 12.4 (a) that

$$\begin{aligned}\mathrm{LT}_{\leq}(e) &= \mathrm{LC}_{\leq}(e)\mathrm{LM}_{\leq}(e) = \mathrm{LC}_{\leq}(e)\mathrm{LM}_{\leq}(u)\mathrm{LM}_{\leq}(g) = \\ &= \mathrm{LC}_{\leq}(e)\mathrm{LT}_{\leq}(u)\frac{1}{\mathrm{LC}_{\leq}(g)}\mathrm{LT}_{\leq}(g) = \frac{\mathrm{LC}_{\leq}(e)}{\mathrm{LC}_{\leq}(g)}\mathrm{LT}_{\leq}(u)\mathrm{LT}_{\leq}(g) = \\ &= -\mathrm{LT}_{\leq}(v)\mathrm{LT}_{\leq}(g) = -\mathrm{LT}_{\leq}(ve).\end{aligned}$$

As $e \notin \sum_{g \in G} \mathbb{W}g$ and $g \in G$, we have $e + vg \in L \setminus \sum_{g \in G} \mathbb{W}g$. In particular $e + vg \neq 0$. So by Definition, Reminder and Exercise 12.2 (D)(d) we get the contradiction

$$\mathrm{LE}_{\leq}(e + vg) < \mathrm{LE}_{\leq}(e)\mathrm{LE}(e) = \min_{\leq}\{\mathrm{LE}_{\leq}(d) \mid d \in L \setminus \sum_{g \in G} \mathbb{W}g\}.$$

□

12.8. **Examples and Exercises.** (A) *(Leading Monomial Ideals)* Keep the above notations and hypotheses. Prove the following statements:

(a) Let $d \in \mathbb{W} \setminus \{0\}$ and $\leq \in \mathrm{AO}(\mathbb{E})$. Prove that $\mathrm{LMI}_{\leq}(\mathbb{W}d)$ is a principal ideal.
(b) Let $n = 1$, $X_1 =: X$ and $\partial_1 =: \partial$. Set $L := \mathbb{W}(X^2 - \partial) + \mathbb{W}(X\partial)$ and determine $\mathrm{LMI}_{\leq}(L)$ for $\leq := \leq_{\mathrm{lex}}, \leq_{\mathrm{deglex}}$ and $\leq := \leq_{\mathrm{degrevlex}}$.

(B) *(Gröbner Bases)* Keep the above notations and hypotheses. Prove the following statements:

(a) Let the notations be as in exercise (a) of part (A) and prove that $\{cd\}$ is a $\leq$-Gröbner basis of $\mathbb{W}d$ for all $c \in K \setminus \{0\}$, and that any singleton $\leq$-Gröbner bases of $\mathbb{W}d$ is of the above form.
(b) Let the notations and hypotheses be as in exercise (b) of part (A) and compute a $\leq$-Gröbner basis for $\leq := \leq_{\mathrm{lex}}, \leq_{\mathrm{deglex}}$ and $\leq := \leq_{\mathrm{degrevlex}}$

For our next main result, we need some notation.

12.9. **Notation.** (A) For any set $S \subseteq \mathbb{W}$ we write $\mathrm{supp}(S) := \bigcup_{s \in S} \mathrm{supp}(s)$.

(B) Let $\leq \in \mathrm{TO}(\mathrm{E})$ (see Definition, Reminder and Exercise 12.2 (A)) and let $T \subset \mathbb{E}$. We write $\leq\restriction_T$ for the *restriction* of $\leq$ to T, thus – if we interpret binary relations on a set S as subsets of $S \times S$:

$$\leq\restriction_T \;:=\; \leq \cap (T \times T), \text{ so that } : d \leq\restriction_T e \Leftrightarrow d \leq e \text{ for all } d, e \in T.$$

12.10. **Proposition.** ***(The Restriction Property of Gröbner Bases)*** *Let $L \subseteq \mathbb{W}$ be a left ideal. Let $\leq, \leq' \in \mathrm{AO}(\mathbb{E})$ and let G be a $\leq$-Gröbner basis of L. Assume that*

$$\leq\restriction_{\mathrm{supp}(G)} \quad = \quad \leq'\restriction_{\mathrm{supp}(G)} .$$

Then G is also a $\leq'$-Gröbner basis of L.

Proof. Let $d \in L \setminus \{0\}$. We have to show that $\mathrm{LM}_{\leq'}(d) \in \mathrm{LMI}_{\leq'}(G)$. We may assume that $0 \notin G$. If we apply Proposition 12.5 to the ordering $\leq'$, we find an element r and a family $(q_g)_{g \in G} \in \mathbb{W}^G$ such that

(1) $d = \sum_{g \in G} q_g g + r$;
(2) $\Phi(s) \notin \mathbb{P}\mathrm{LM}_{\leq'}(g)$ for all $g \in G$ and all $s \in \mathrm{supp}(r)$.
(3) $\mathrm{LE}_{\leq'}(q_g g) \leq' \mathrm{LE}_{\leq'}(d)$ for all $g \in G$ with $q_g \neq 0$.

Our immediate aim is to show that $r = 0$. Assume to the contrary that $r \neq 0$. As $r \in L$ and G is a $\leq$-Gröbner basis of L, we get $\mathrm{LM}_{\leq}(r) \in \mathrm{LMI}_{\leq}(G)$. So, there is some $g \in G$ such that $q_g \neq 0$ and $\mathrm{LM}_{\leq}(r) = n\mathrm{LM}_{\leq}(g)$ for some $n \in \mathbb{M}$ (see Definition, Reminder and Exercise 12.6 (B)(a)). As $\leq\restriction_{\mathrm{supp}(G)} \quad = \quad \leq'\restriction_{\mathrm{supp}(G)}$ it follows that

$$\Phi\big(\mathrm{LT}_{\leq}(r)\big) = \mathrm{LM}_{\leq}(r) \in \mathbb{P}\mathrm{LM}_{\leq'}(g).$$

As $\mathrm{LT}_{\leq}(r) \in \mathrm{supp}(r)$, this contradicts the above condition (2). Therefore $r = 0$.
But now, we may write

$$d = \sum_{g \in G^*} q_g g, \text{ whith } G^* := \{g \in G \mid q_g \neq 0\}.$$

By the above condition (3) we have $\mathrm{LE}_{\leq'}(q_g g) \leq' \mathrm{LE}_{\leq'}(d)$ for all $g \in G^*$. So, there is some $g \in G^*$ such that $\mathrm{LE}_{\leq'}(d) = \mathrm{LE}_{\leq'}(q_g g)$ (see Definition, Reminder and Exercise 12.2 (D)(d)), and hence $\mathrm{LM}_{\leq'}(d) = \mathrm{LM}_{\leq'}(q_g g)$. Thus, on use of Proposition 12.4 (b) we get indeed $\mathrm{LM}_{\leq'}(d) = \mathrm{LM}_{\leq'}(q_g)\mathrm{LM}_{\leq'}(g) \in \mathrm{LMI}_{\leq'}(G)$. □

Now, we shall introduce the central concept of this section.

12.11. **Definition.** *(Universal Gröbner Bases)* Let $L \subseteq \mathbb{W}$ be a left ideal. A *universal Gröbner basis* of L is a (finite) subset $G \subset \mathbb{W}$ which is a $\leq$-Gröbner basis for all $\leq \in \mathrm{AO}(\mathbb{E})$.

Following an approach which relies on an idea of Sikora, and performed in greater generality in Boldini's thesis [4] we now aim to prove that universal Gröbner bases in Weyl algebras always exist. We begin with a few preparations.

12.12. **Definition, Exercise and Convention.** (A) *(The Natural Metric on the Set* $\mathrm{TO}(\mathbb{E})$*)* For all $i \in \mathbb{Z}$ we introduce the notation

$$\mathbb{E}_i := \{e \in \mathbb{E} \mid \deg(e) \leq i\} = \{\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \mid |\underline{\nu}| + |\underline{\mu}| \leq i\}.$$

We define a map

$$\mathrm{dist} : \mathrm{TO}(\mathbb{E}) \times \mathrm{TO}(\mathbb{E}) \longrightarrow \mathbb{R}, \text{ given by for all } \leq, \leq' \in \mathrm{TO}(\mathbb{E}) \text{ by}$$

$$\mathrm{dist}(\leq, \leq') := \begin{cases} 2^{-\sup\{r \in \mathbb{N}_0 \mid \leq\restriction_{\mathbb{E}_r} \quad = \quad \leq'\restriction_{\mathbb{E}_r}\}}, & \text{if } \leq \neq \leq', \\ 0, & \text{if } \leq = \leq' . \end{cases}$$

Prove that

(a) For all $\leq, \leq' \in \mathrm{TO}(\mathbb{E})$ and all $r \in \mathbb{N}_0$ we have

$$\mathrm{dist}(\leq, \leq') < \frac{1}{2^r} \text{ if and only if } \leq\restriction_{\mathbb{E}_{r+1}} \;=\; \leq'\restriction_{\mathbb{E}_{r+1}} .$$

(b) The map $\mathrm{dist} : \mathrm{TO}(\mathbb{E}) \times \mathrm{TO}(\mathbb{E}) \longrightarrow \mathbb{R}$ is a *metric* on $\mathrm{TO}(\mathbb{E})$.

From now on, we always endow $\mathrm{TO}(\mathbb{E})$ with this metric and the induced *Hausdorff topology.*

(B) *(Completeness of the Metric Space* $\mathrm{TO}(\mathbb{E})$*)* Let $(\leq_i)_{i\in\mathbb{N}_0}$ be a *Cauchy sequence* in $\mathrm{TO}(\mathbb{E})$. This means:

For all $r \in \mathbb{N}_0$ there is some $n(r) \in \mathbb{N}_0$ such that $\mathrm{dist}(\leq_i, \leq_j) < \frac{1}{2^r}$ for all $i, i \geq n(r)$.

We introduce the binary relation $\leq$ on $\mathbb{E}$ given for all $d, e \in \mathbb{E}$ by

$$d \leq e \text{ if and only if } d \leq_i e \text{ for all } i \gg 0.$$

Prove the following statements:

(a) If $r \in \mathbb{N}_0$, $d, e \in \mathbb{E}_{r+1}$, and $i, j \geq n(r)$, then $d \leq_i e$ if and only if $d \leq_j e$.
(b) If $r \in \mathbb{N}_0$, $d, e \in \mathbb{E}_{r+1}$, and $i \geq n(r)$, then $d \leq_i e$ if and only if $d \leq e$.
(c) $\leq \in \mathrm{TO}(\mathbb{E})$.
(d) If $r \in \mathbb{N}_0$, and $i \geq n(r)$, then $\mathrm{dist}(\leq_i, \leq) \leq \frac{1}{2^r}$.
(e) $\lim_{i\to\infty} \leq_i = \leq$.
(f) $\mathrm{TO}(\mathbb{E})$ is a *complete* metric space.

12.13. **Proposition.** ***(Compactness of the Space of Total Orderings)*** *The space* $\mathrm{TO}(\mathbb{E})$ *is compact.*

Proof. Let $(\leq_i)_{i\in\mathbb{N}_0}$ be a sequence in $\mathrm{TO}(\mathbb{E})$. It suffices to show, that $(\leq_i)_{i\in\mathbb{N}_0}$ has a convergent subsequence. Bearing in mind Definition, Exercise and Convention 12.12 (B)(f) (or (e)), it suffices to find a subsequence of $(\leq_i)_{i\in\mathbb{N}_0}$ which is a Cauchy sequence. Observe that all the sets $\mathbb{E}_r$ are finite. We want to construct a sequence $(\mathbb{S}_r)_{r\in\mathbb{N}_0}$ of infinite subsets $\mathbb{S}_r \subseteq \mathbb{N}_0$ such that for all $s \in \mathbb{N}_0$ we have

(1) $\mathbb{S}_{s+1} \subseteq \mathbb{S}_s$.
(2) $\leq_j\restriction_{\mathbb{E}_{s+1}} \;=\; \leq_k\restriction_{\mathbb{E}_{s+1}}$ for all $j, k \in \mathbb{S}_s$.

We construct the members $\mathbb{S}_r$ of the sequence $(\mathbb{S}_r)_{r\in\mathbb{N}_0}$ by induction r. As $\mathbb{E}_1$ is finite, we can find an infinite set $\mathbb{S}_0 \subseteq \mathbb{N}_0$ such that requirement (2) is satisfied with $s = 0$. Now, let $r > 0$ and assume that the sets $\mathbb{S}_0, \mathbb{S}_1, \ldots, \mathbb{S}_r$ are already defined such that requirement (1) holds for all $s < r$ and requirement (2) holds for all $s \leq r$.

As $\mathbb{E}_{r+2}$ is finite, we find an infinite subset $\mathbb{S}_{r+1} \subseteq \mathbb{S}_r$ (which hence satisfies requirement (1) for $s = r$) such that requirement (2) is also satisfied with $s = r+1$. This completes the step of induction and hence proves that a sequence $(\mathbb{S}_r)_{r\in\mathbb{N}_0}$ with the requested properties exists.

Now, we may choose a sequence $(i_k)_{k\in\mathbb{N}_0}$ in $\mathbb{N}_0$, such that $i_r < i_{r+1}$ and $i_r \in \mathbb{S}_r$ for all $r \in \mathbb{N}_0$. In particular it follows that $\leq_{i_j}\restriction_{\mathbb{E}_{r+1}} \;=\; \leq_{i_k}\restriction_{\mathbb{E}_{r+1}}$ for all $j, k \geq r$ and hence (see Definition, Exercise and Convention 12.12 (A)(a))

$$\mathrm{dist}(\leq_{i_j}, \leq_{i_k}) < \frac{1}{2^r} \text{ for all } j, k \geq r.$$

So, the constructed subsequence $(\leq_{i_k})_{k\in\mathbb{N}_0}$ of our original sequence $(\leq_i)_{i\in\mathbb{N}_0}$ is indeed a Cauchy sequence. □

12.14. **Proposition.** ***(Compactness of the Space of Admissible Orderings)*** *The set* $\mathrm{AO}(\mathbb{E})$ *is a closed subset of* $\mathrm{TO}(\mathbb{E})$ *and hence compact.*

Proof. Let $(\leq_i)_{i\in\mathbb{N}_0}$ be sequence in $\mathrm{AO}(\mathbb{E})$, which converges in $\mathrm{TO}(\mathbb{E})$ and let

$$\lim_{i\to\infty}\leq_i \quad = \quad \leq .$$

We aim to show, that $\leq\in \mathrm{AO}(\mathbb{E})$. According to Definition, Reminder and Exercise 12.2 (C), we must show, that for all $\underline{\lambda},\underline{\lambda}',\underline{\kappa},\underline{\kappa}',\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n$ the following statements hold.

(1) $1\leq \underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}}$.
(2) If $\underline{X}^{\underline{\lambda}}\underline{\partial}^{\underline{\kappa}}\leq \underline{X}^{\underline{\lambda}'}\underline{\partial}^{\underline{\kappa}'}$ then $\underline{X}^{\underline{\lambda}+\underline{\nu}}\underline{\partial}^{\underline{\kappa}+\underline{\mu}}\leq \underline{X}^{\underline{\lambda}'+\underline{\nu}}\underline{\partial}^{\underline{\kappa}'+\underline{\mu}}$.

So, fix $\underline{\lambda},\underline{\lambda}',\underline{\kappa},\underline{\kappa}',\underline{\nu},\underline{\mu}\in\mathbb{N}_0^n$. Then we find some $r\in\mathbb{N}_0$ such that all the elementary differential operators which occur in (1) and (2) belong to $\mathbb{E}_{r+1}$. Now, we find some $i\in\mathbb{N}_0$ such that $\mathrm{dist}(\leq_i,\leq)<\frac{1}{2^r}$, hence such that $\leq\upharpoonright_{\mathbb{E}_{r+1}} \; = \; \leq_i\upharpoonright_{\mathbb{E}_{r+1}}$. As $\leq_i\in \mathrm{AO}(\mathbb{E})$ the required inequalities hold for $\leq_i$. But then, by the coincidence of $\leq$ and $\leq_i$ on $\mathbb{E}_{r+1}$, they hold also for $\leq$. □

Now, after having established the following auxiliary result, we are ready to prove the announced main result.

12.15. **Lemma.** *Let* $L\subset\mathbb{W}$ *be a left ideal and let* $G\subseteq L$ *be a finite subset. Then, the set*

$$\mathbb{U}_L(G):=\{\leq\in \mathrm{AO}(\mathbb{E})\mid G \textit{ is a } \leq - \text{ Gröbner } \textit{basis of } L\} \textit{ is open in } \mathrm{AO}(\mathbb{E}).$$

Proof. We may assume that $\mathbb{U}_L(G)$ is not empty and choose $\leq\in\mathbb{U}_L(G)$. We find some $r\in\mathbb{N}_0$ with $\mathrm{supp}(G)\subseteq\mathbb{E}_{r+1}$. Let $\leq'\in \mathrm{AO}(\mathbb{E})$ such that $\mathrm{dist}(\leq,\leq')<\frac{1}{2^r}$. So, we obtain that $\leq'\upharpoonright_{\mathbb{E}_{r+1}} \; = \; \leq'\upharpoonright_{\mathbb{E}_{r+1}}$ and hence in particular that $\leq'\upharpoonright_{\mathrm{supp}(G)} \; = \; \leq'\upharpoonright_{\mathrm{supp}(G)}$. By Proposition 12.10 it follows that G is a $\leq'$-Gröbner basis of L and hence that $\leq'\in\mathbb{U}_L(G)$. But this means, that the open neighborhood $\{\leq'\in \mathrm{AO}(\mathbb{E})\mid \mathrm{dist}(\leq',\leq)<2^{-r}\}$ of $\leq$ belongs to $\mathbb{U}_L(G)$. □

12.16. **Theorem.** ***(Existence of Universal Gröbner Bases)*** *Each left ideal* L *of* $\mathbb{W}$ *admits a universal Gröbner basis.*

Proof. Let $L\subseteq\mathbb{W}$ be a left ideal. For each $\leq\in \mathrm{AO}(\mathrm{E})$ we choose a $\leq$-Gröbner basis $G_\leq$ of L. In the notations of Lemma 12.15 we have $\leq\in\mathbb{U}_L(G_\leq)$. So, by this same Lemma the family

$$\big(\mathbb{U}_L(G_\leq)\big)_{\leq\in \mathrm{AO}(\mathbb{E})}$$

is an open covering of $\mathrm{AO}(\mathbb{E})$. By Proposition 12.14 we thus find finitely many elements $\leq_1,\leq_2,\ldots,\leq_r\in \mathrm{AO}(\mathbb{E})$ such that

$$\mathrm{AO}(\mathbb{E})=\bigcup_{i=1}^{r}\mathbb{U}_L(G_{\leq_i}).$$

Let $\leq\in \mathrm{AO}(\mathbb{E})$. Then $\leq\in\mathbb{U}_L(G_{\leq_i})$ for some $i\in\{1,2,\ldots,r\}$. Therefore $G_{\leq_i}$ is a $\leq$-Gröbner basis of L. So $\bigcup_{i=1}^{r}G_{\leq_i}$ is a Gröbner basis of L for all $\leq\in \mathrm{AO}(\mathbb{E})$. □

12.17. **Corollary.** ***(Finiteness of the Set of Leading Monomial Ideals)*** *Let $L \subseteq \mathbb{W}$ be a left ideal. Then the set $\{\mathrm{LMI}_{\leq}(L) \mid \leq \in \mathrm{AO}(\mathbb{E})\}$ of all leading monomial ideals of L with respect to admissible orderings of $\mathbb{E}$ is finite.*

Proof. Let $G \subseteq L$ be a universal Gröbner basis of L. Then $\{\mathrm{LMI}_{\leq}(L) \mid \leq \in \mathrm{AO}(\mathbb{E})\} = \{\mathrm{LMI}_{\leq}(G) \mid \leq \in \mathrm{AO}(\mathbb{E})\}$. Therefore

$$\#\{\mathrm{LMI}_{\leq}(L) \mid \leq \in \mathrm{AO}(\mathbb{E})\} \leq \#\{\sum_{h \in H} \mathbb{P}\Phi(h) \mid H \subseteq \mathrm{supp}(G)\} \leq$$
$$\leq \#\{H \subseteq \mathrm{supp}(G)\} = 2^{\#\mathrm{supp}(G)}.$$

□

13. Weighted Orderings

This section is devoted to the study of admissible orderings which are compatible with a given weight. Our principal goal is to prove a certain stability result for characteristic varieties found in Boldini's thesis [4], published in [5].

13.1. **Notation.** (A) As previously, we fix a positive integer n, a field K of characteristic 0 and consider the standard Weyl algebra $\mathbb{W} = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n]$, the polynomial ring $\mathbb{P} := K[Y_1, Y_2, \ldots, Y_n; Z_1, Z_2, \ldots, Z_n]$ over K and the isomorphism of K-vector spaces $\Phi : \mathbb{W} \xrightarrow{\cong} \mathbb{P}$, given by $\underline{X}^{\underline{\nu}}\underline{\partial}^{\underline{\mu}} \mapsto \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}}$ for all $\underline{\nu}, \underline{\mu} \in \mathbb{N}_0^n$.
(B) We also write

$$\Omega := \{(\underline{v}, \underline{w}) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid (v_i, w_i) \neq (0,0) \text{ for all } i = 1, 2, \ldots, n\} \subset \mathbb{N}_0^n \times \mathbb{N}_0^n$$

for the set of all weights. If $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ we also use the suffix $\underline{\omega}$ instead of the suffix $\underline{vw}$ in all the previously introduced notations. Observe, that

$$\underline{\omega} + \underline{\alpha} \in \Omega, \quad s\underline{\omega} \in \Omega, \quad (\underline{\omega}, \underline{\alpha} \in \Omega, s \in \mathbb{N}).$$

13.2. **Definition and Exercise.** (A) *(Weight Compatible Orderings)* We fix $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ and $\leq \in \mathrm{AO}(\mathbb{E})$ (see Definition, Reminder and Exercise 12.2 (C)). We say that $\leq$ is *compatible* with the weight $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ (or $\underline{\omega}$*-compatible*), if for all $d, e \in \mathbb{E}$ we have:

$$\text{If } \deg^{\underline{\omega}}(d) < \deg^{\underline{\omega}}(e), \text{ then } d < e.$$

We set

$$\mathrm{AO}^{\underline{\omega}}(\mathbb{E}) = \mathrm{AO}^{\underline{vw}}(\mathbb{E}) := \{\leq \in \mathrm{AO}(\mathbb{E}) \mid \ \leq \text{ is compatible with } \underline{\omega} = (\underline{v}, \underline{w})\}.$$

(B) *(Weighted Admissible Orderings)* Keep the notations and hypotheses of part (A). We define a new binary relation $\leq^{\underline{\omega}} = \leq^{\underline{vw}}$ on $\mathbb{E}$ by setting, for all $d, e \in \mathbb{E}$:

$$d \leq^{\underline{\omega}} e \text{ if } \begin{cases} \text{either} & \deg^{\underline{\omega}}(d) < \deg^{\underline{\omega}}(e) \\ \text{or else} & \deg^{\underline{\omega}}(d) = \deg^{\underline{\omega}}(e) \text{ and } d < e. \end{cases}$$

Prove that for each $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ and each $\leq \in \mathrm{AO}(\mathbb{E})$ the following statements hold.
(a) $\leq^{\underline{\omega}} \in \mathrm{AO}^{\underline{\omega}}(\mathbb{E})$.
(b) $(\leq^{\underline{\omega}})^{\underline{\omega}} = \leq^{\underline{\omega}}$.
(c) $\leq \in \mathrm{AO}^{\underline{\omega}}(\mathbb{E})$ if and only if $\leq = \leq^{\underline{\omega}}$.

The admissible ordering $\leq^{\underline{\omega}} \in \mathrm{AO}(\mathbb{E})$ is called the $\underline{\omega}$*-weighted ordering associated to* $\leq$.

13.3. **Definition and Exercise.** (A) Let $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$, let $i \in \mathbb{N}_0$ and let

$$d = \sum_{(\underline{\nu},\underline{\mu})\in \operatorname{supp}(d)} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}} \in \mathbb{W} \quad \text{with } c^{(d)}_{\underline{\nu}\underline{\mu}} \in K \setminus \{0\} \text{ for all } (\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(d).$$

We set

$$\operatorname{supp}^{\underline{\omega}}_i(d) := \{(\underline{\nu}, \underline{\mu}) \in \operatorname{supp}(d) \mid \underline{\mu}\underline{v} + \underline{\mu}\underline{w} = i\} \text{ and } d^{\underline{\omega}}_i = d^{\underline{v}\underline{w}}_i := \sum_{(\underline{\nu},\underline{\nu})\in \operatorname{supp}^{\underline{\omega}}_i(d)} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}.$$

Prove that for all $d, e \in \mathbb{W}$, all $i, j \in \mathbb{N}_0$ and for all $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ it holds:

(a) If $i > \deg^{\underline{\omega}}(d)$, then $d^{\underline{\omega}}_i = 0$.
(b) $d^{\underline{\omega}}_i = [d^{\underline{\omega}}_i]^{\underline{\omega}}_i$.
(c) $(d + e)^{\underline{\omega}}_i = d^{\underline{\omega}}_i + e^{\underline{\omega}}_i$.
(d) If $d, e \neq 0$, $i := \deg^{\underline{\omega}}(d)$ and $j := \deg^{\underline{\omega}}(e)$, then

$$\operatorname{supp}^{\underline{\omega}}_{i+j}(de) = \{(\underline{\nu} + \underline{\nu}', \underline{\mu} + \underline{\mu}') \mid (\underline{\nu}, \underline{\mu}) \in \operatorname{supp}^{\underline{\omega}}_i(d) \text{ and } (\underline{\nu}', \underline{\mu}') \in \operatorname{supp}^{\underline{\omega}}_j(e)\}.$$

(e) If $d, e \neq 0$, $i := \deg^{\underline{\omega}}(d)$ and $j := \deg^{\underline{\omega}}(e)$, then

$$(de)^{\underline{\omega}}_{i+j} = \sum_{(\underline{\nu},\underline{\mu})\in \operatorname{supp}^{\underline{\omega}}_i(d),(\underline{\nu}',\underline{\mu}')\in \operatorname{supp}^{\underline{\omega}}_j(e)} c^{(d)}_{\underline{\nu}\underline{\mu}} c^{(e)}_{\underline{\nu}\underline{\mu}} \underline{X}^{\underline{\nu}+\underline{\nu}'} \underline{\partial}^{\underline{\mu}+\underline{\mu}'}.$$

(B) Keep the notations and hypotheses of part (A). We set

$$\sigma^{\underline{\omega}}_i(d) := \Phi\big(d^{\underline{\omega}}_i\big) = \sum_{(\underline{\nu},\underline{\nu})\in \operatorname{supp}^{\underline{\omega}}_i(d)} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}}.$$

Prove on use of statements (a)–(e) of part (A) that for all $d, e \in \mathbb{W}$, all $i, j \in \mathbb{N}_0$ and for all weights $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ the following statements hold:

(a) $\sigma^{\underline{\omega}}_i(d) := \sigma^{\underline{\omega}}_i(d^{\underline{\omega}}_i)$.
(b) If $i > \deg^{\underline{\omega}}(d)$, then $\sigma^{\underline{\omega}}_i(d) = 0$.
(c) $\sigma^{\underline{\omega}}_i(d) = \sigma^{\underline{\omega}}_i(d^{\underline{\omega}}_i)$.
(d) $\sigma^{\underline{\omega}}_i(d + e) = \sigma^{\underline{\omega}}_i(d) + \sigma^{\underline{\omega}}_i(e)$.

(C) *(The Symbol of a Differential operator with Respect to a Weight)* Keep the notations of part (A) and (B). We define the $\underline{\omega} = (\underline{v}, \underline{w})$-*symbol* of the differential operator $d \in \mathbb{W}$ by

$$\sigma^{\underline{\omega}}(d) := \begin{cases} 0 & \text{if } d = 0, \\ \sigma^{\underline{\omega}}_{\deg^{\underline{\omega}}}(d) & \text{if } d \neq 0. \end{cases}$$

Prove that for all $d, e \in \mathbb{W} \setminus \{0\}$ the following statements hold.

(a) $\sigma^{\underline{\omega}}(d) = \Phi(d^{\underline{\omega}}_{\deg^{\underline{\omega}}(d)}) = \sigma^{\underline{\omega}}\big(d^{\underline{\omega}}_{\deg^{\underline{\omega}}}(d)\big)$.
(b) $\sigma^{\underline{\omega}}(d + e) = \begin{cases} \sigma^{\underline{\omega}}(d) + \sigma^{\underline{\omega}}(e) & \text{if } \deg^{\underline{\omega}}(d) = \deg^{\underline{\omega}}(e), \\ \sigma^{\underline{\omega}}(d) & \text{if } \deg^{\underline{\omega}}(d) > \deg^{\underline{\omega}}(e). \end{cases}$

13.4. **Proposition.** ***(Multiplicativity of Symbols)*** *Let $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ and let $d, e \in \mathbb{W}$. Then*

$$\sigma^{\underline{\omega}}(de) = \sigma^{\underline{\omega}}(d)\sigma^{\underline{\omega}}(e).$$

Proof. If $d = 0$ or $e = 0$, our claim is obvious. So, let $d, e \neq 0$. We write $i := \deg^{\underline{\omega}}(d)$ and $j := \deg^{\underline{\omega}}(e)$. Observe that $\deg^{\underline{\omega}}(de) = i + j$. So, by Definition and Exercise 13.3 (A)(e) we have

$$\begin{aligned}\sigma^{\underline{\omega}}(de) &= \sigma^{\underline{\omega}}_{i+j}(de) = \Phi\big((de)^{\underline{\omega}}_{i+j}\big) = \\ &= \Phi\Big(\sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}^{\underline{\omega}}_i(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}^{\underline{\omega}}_j(e)} c^{(d)}_{\underline{\nu}\underline{\mu}} c^{(e)}_{\underline{\nu}'\underline{\mu}'} \underline{X}^{\underline{\nu}+\underline{\nu}'}\underline{\partial}^{\underline{\mu}+\underline{\mu}'}\Big) = \\ &= \sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}^{\underline{\omega}}_i(d),(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}^{\underline{\omega}}_j(e)} c^{(d)}_{\underline{\nu}\underline{\mu}} c^{(e)}_{\underline{\nu}'\underline{\mu}'} \underline{Y}^{\underline{\nu}+\underline{\nu}'}\underline{Z}^{\underline{\mu}+\underline{\mu}'} = \\ &= \Big(\sum_{(\underline{\nu},\underline{\mu})\in\mathrm{supp}^{\underline{\omega}}_i(d)} c^{(d)}_{\underline{\nu}\underline{\mu}} \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}}\Big)\Big(\sum_{(\underline{\nu}',\underline{\mu}')\in\mathrm{supp}^{\underline{\omega}}_j(e)} c^{(e)}_{\underline{\nu}'\underline{\mu}'} \underline{Y}^{\underline{\nu}'}\underline{Z}^{\underline{\mu}'}\Big) = \\ &= \Phi(d^{\underline{\omega}}_i)\Phi(e^{\underline{\omega}}_j) = \sigma^{\underline{\omega}}_i(d)\sigma^{\underline{\omega}}_j(e) = \sigma^{\underline{\omega}}(d)\sigma^{\underline{\omega}}(e).\end{aligned}$$

□

13.5. **Reminder, Definition and Exercise.** (A) *(Induced Graded Ideals)* Let $L \subset \mathbb{W}$ be a left ideal, let $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ be a weight and let us consider the $\underline{\omega}$-graded induced ideal (see Definition and Remark 11.5)

$$\mathbb{G}^{\underline{\omega}}(L) := \bigoplus_{i\in\mathbb{Z}} \big(L \cap \mathbb{W}^{\underline{\omega}}_i + \mathbb{W}^{\underline{\omega}}_{i-1}\big)/\mathbb{W}^{\underline{\omega}}_{i-1} \cong \bigoplus_{i\in\mathbb{Z}} L^{\underline{\omega}}_i/L^{\underline{\omega}}_{i-1} = \mathrm{Gr}_{L^{\underline{\omega}}_\bullet}(L) \subseteq \mathbb{G}^{\underline{\omega}}(L),$$

where

$$L^{\underline{\omega}}_\bullet = L \cap \mathbb{W}^{\underline{\omega}}_\bullet := \big(L \cap \mathbb{W}^{\underline{\omega}}_i\big)_{i\in\mathbb{N}_0}$$

is the filtration induced on L by the weighted filtration $\mathbb{W}^{\underline{\omega}}_\bullet$. We also consider the $\underline{\omega}$-graded ideal of $\mathbb{P}^{\underline{\omega}} = \mathbb{P}$ given by $\overline{\mathbb{G}}^{\underline{\omega}}(L) := (\eta^{\underline{\omega}})^{-1}\big(\mathbb{G}^{\underline{\omega}}(L)\big)$, where $\eta^{\underline{v}\underline{w}} = \eta^{\underline{\omega}} : \mathbb{P} = \mathbb{P}^{\underline{\omega}} \xrightarrow{\cong} \mathbb{G}^{\underline{\omega}}$ is the canonical isomorphism of graded rings of Theorem 9.4. We call $\overline{\mathbb{G}}^{\underline{\omega}}(L)$ the ($\underline{\omega}$-graded) *ideal induced by* L in $\mathbb{P}$.

(B) Let the notations and hypotheses be as part (A). Fix $i \in \mathbb{N}_0$ and consider the i-th $\underline{\omega}$-graded part $\overline{\mathbb{G}}^{\underline{\omega}}(L)_i = \overline{\mathbb{G}}^{\underline{\omega}}(L) \cap \mathbb{P}^{\underline{\omega}}_i = (\eta^{\underline{\omega}})^{-1}\big(\mathbb{G}^{\underline{\omega}}_i\big)$ of the ideal $\overline{\mathbb{G}}^{\underline{\omega}}(L) \subseteq \mathbb{P}$. Prove the following statements:

(a) Let $d \in L$ with $\deg^{\underline{\omega}}(d) = i$ and let $\overline{d} := d + \mathbb{W}^{\underline{\omega}}_{i-1} \in \mathbb{G}^{\underline{\omega}}(L)_i$. Then it holds

$$(\eta^{\underline{\omega}})^{-1}(\overline{d}) = \Phi(d^{\underline{\omega}}_i) = \sigma^{\underline{\omega}}(d) \in \overline{\mathbb{G}}^{\underline{\omega}}(L)_i.$$

(b) Each element $h \in \mathbb{G}^{\underline{\omega}}(L)_i \setminus \{0\}$ can be written as

$$h = \sigma^{\underline{\omega}}(d), \text{ with } d \in L \text{ and } \deg^{\underline{\omega}}(d) = i.$$

(C) *(The Induced Exact Sequence Associated to a Left Ideal with Respect to a Weight)* Keep the above notations and hypotheses. Prove the following statements:

(a) There is a short exact sequence of graded of graded $\mathbb{P}^{\underline{\omega}}$-modules

$$0 \longrightarrow \overline{\mathbb{G}}^{\underline{\omega}}(L) \longrightarrow \mathbb{G}^{\underline{\omega}} \longrightarrow \mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet K\overline{1}}(\mathbb{W}/L) \longrightarrow 0,$$

where $\overline{1} := 1 + L \in \mathbb{W}/L$ and $\mathbb{W}^{\underline{\omega}}_\bullet K\overline{1}$ is the $\underline{\omega}$ -filtration induced on the cyclic D-module $\mathbb{W}/L$ by its subspace $K\overline{1}$.

(b) $\mathrm{Ann}_{\mathbb{P}}\big(\mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_{\bullet}K\overline{1}}(\mathbb{W}/L)\big) = \overline{\mathbb{G}}^{\underline{\omega}}(L)$.
(c) $\mathbb{V}^{\underline{\omega}}(\mathbb{W}/L) = \mathrm{Var}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big)$.

We call this sequence the *short exact sequence associated to the left ideal L with respect to the weight $\underline{\omega}$.*

13.6. **Proposition.** ***(Generation of the Induced Ideal by the Symbols of a Gröbner Basis)*** *Let $\underline{\omega} \in \Omega$, let $L \subseteq \mathbb{W}$ be a left ideal, let $\leq \in \mathrm{AO}(\mathbb{E})$ and let G be a $\leq^{\underline{\omega}}$-Gröbner basis if L. Then it holds*

(a) $\overline{\mathbb{G}}^{\underline{\omega}}(L) = \sum_{g\in G} \mathbb{P}\sigma^{\underline{\omega}}(g)$.
(b) *For each $h \in \overline{\mathbb{G}}^{\underline{\omega}}(L) \setminus \{0\}$ there is some $g \in G \setminus \{0\}$ and some monomial $m = \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \in \mathbb{P}$ such that*

$$\mathrm{LM}_{\leq}\big(\Phi^{-1}(h)\big) = m\mathrm{LM}_{\leq}\big(\Phi^{-1}(\sigma^{\underline{\omega}}(g))\big).$$

Proof. (a): As the ideal $\overline{\mathbb{G}}^{\underline{\omega}}(L) \subseteq \mathbb{P}^{\underline{\omega}}$ is graded, it suffices to show, that for each $i \in \mathbb{N}_0$ and each $h \in \overline{\mathbb{G}}^{\underline{\omega}}(L)_i \setminus \{0\}$ we have $h \in \sum_{g\in G} \mathbb{P}\sigma^{\underline{\omega}}(g)$. So, fix $i \in \mathbb{N}_0$ and assume that $h \notin \sum_{g\in G} \mathbb{P}\sigma^{\underline{\omega}}(g)$ for some $h \in \overline{\mathbb{G}}^{\underline{\omega}}(L)_i \setminus \{0\}$. Then, by Reminder, Definition and Exercise 13.5 (B)(b), the set

$$\mathfrak{S} := \{e \in L \mid \deg^{\underline{\omega}}(e) = i \text{ and } \sigma^{\underline{\omega}}(e) \notin \sum_{g\in G} \mathbb{P}\sigma^{\underline{\omega}}(g)\}$$

is not empty. Choose $d \in \mathfrak{S}$ such that $\mathrm{LE}_{\leq^{\underline{\omega}}}(d) = \min_{\leq^{\underline{\omega}}}\{\mathrm{LE}_{\leq^{\underline{\omega}}}(e) \mid e \in \mathfrak{S}\}$. As G is a $\leq^{\underline{\omega}}$-Gröbner basis of L we find some $g \in G$ and some $u \in \mathbb{E}$ such that $\mathrm{LM}_{\leq^{\underline{\omega}}}(d) = \mathrm{LM}_{\leq^{\underline{\omega}}}(ug)$ (see Definition, Reminder and Exercise 12.6 (C)(b)), or – equivalently – such that $\mathrm{EM}_{\leq^{\underline{\omega}}}(d) = \mathrm{EM}_{\leq^{\underline{\omega}}}(ug)$. With

$$v := \frac{\mathrm{LC}_{\leq^{\underline{\omega}}}(d)}{\mathrm{LC}_{\leq^{\underline{\omega}}}(ug)}u$$

it follows that

$$\mathrm{LD}_{\leq^{\underline{\omega}}}(d) = \mathrm{LC}_{\leq^{\underline{\omega}}}(d)\mathrm{LE}_{\leq^{\underline{\omega}}}(d) = \mathrm{LC}_{\leq^{\underline{\omega}}}(d)\mathrm{LE}_{\leq^{\underline{\omega}}}(ug) = \frac{\mathrm{LC}_{\leq^{\underline{\omega}}}(d)}{\mathrm{LC}_{\leq^{\underline{\omega}}}(ug)}\mathrm{LD}_{\leq^{\underline{\omega}}}(ug) =$$
$$= \mathrm{LD}_{\leq^{\underline{\omega}}}\Big(\frac{\mathrm{LC}_{\leq^{\underline{\omega}}}(d)}{\mathrm{LC}_{\leq^{\underline{\omega}}}(ug)}ug\Big) = \mathrm{LD}_{\leq^{\underline{\omega}}}(vg) \quad \text{and } \deg^{\underline{\omega}}(vg) = i.$$

So, by Definition, Reminder and Exercise 12.2 (D)(d) we may conclude that either

(1) $\deg^{\underline{\omega}}(d - vg) < i$, or else
(2) $\deg^{\underline{\omega}}(d - vg) = i$ and $\mathrm{LE}_{\leq^{\underline{\omega}}}(d - vg) < \mathrm{LE}_{\leq^{\underline{\omega}}}(d)$.

In the case (1) we have (see Definition and Exercise 13.3 (C)(b) and Proposition 13.4)

$$\sigma^{\underline{\omega}}(d) = \sigma^{\underline{\omega}}(d - (d - vg)) = \sigma^{\underline{\omega}}(vg)) = \sigma^{\underline{\omega}}(v)\sigma^{\underline{\omega}}(g) \in \sum_{g\in G} \mathbb{P}\sigma^{\underline{\omega}}(g) \text{ – a contradiction.}$$

So, assume that we are in the case (2). As $d - vg \in L$ it follows by our choice of d, that $\sigma^{\underline{\omega}}(d - vg) \in \sum_{g\in G} \mathbb{P}\sigma^{\underline{\omega}}(g)$. But now, by Definition and Exercise 13.3 (C)(b) and by

Proposition 13.4 we get the contradiction

$$\sigma^{\underline{\omega}}(d) = \sigma^{\underline{\omega}}(d - vg) + \sigma^{\underline{\omega}}(vg) = \sigma^{\underline{\omega}}(d - vg) + \sigma^{\underline{\omega}}(v)\sigma^{\underline{\omega}}(g) \in \sum_{g \in G} \mathbb{P}\sigma^{\underline{\omega}}(g).$$

(b): We find some $i \in \mathbb{N}_0$ such that $\mathrm{LM}_{\leq}\big(\Phi^{-1}(h)\big) = \mathrm{LM}_{\leq}\big(\Phi^{-1}(h_i^{\underline{\omega}}(h))\big)$. As the ideal $\overline{\mathbb{G}}^{\underline{\omega}}(L) \subseteq \mathbb{P}^{\underline{\omega}}$ is graded, we have $h_i^{\underline{\omega}}(h) \in \overline{\mathbb{G}}^{\underline{\omega}}(L)$. So we may assume, that $h \in \overline{\mathbb{G}}^{\underline{\omega}}(L)_i \setminus \{0\}$. Now, by Reminder, Definition and Exercise 13.5 (B), we find some $d \in L$ with $\deg^{\underline{\omega}}(d) = i$ and $\Phi^{-1}(h) = d_i^{\underline{\omega}}$, whence $\mathrm{LM}_{\leq}\big(\Phi^{-1}(h)\big) = \mathrm{LM}_{\leq}(d_i^{\underline{\omega}}) = \mathrm{LM}_{\leq^{\underline{\omega}}}(d)$. As G is a $\leq^{\underline{\omega}}$-Gröbner basis of L, we find some $g \in G \setminus \{0\}$ with $\deg^{\underline{\omega}}(g) = j$ and some monomial $m = \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \in \mathbb{P}$ such that (see Definition, Reminder and Exercise 12.6 (C)(c) and also Definition and Exercise 13.3 (C)(a)) $\mathrm{LM}_{\leq^{\underline{\omega}}}(d) = m\mathrm{LM}_{\leq^{\underline{\omega}}}(g) = m\mathrm{LM}_{\leq}(g_j^{\underline{\omega}}) = m\mathrm{LM}_{\leq}\big(\Phi^{-1}(\sigma_j^{\underline{\omega}})\big)$. □

13.7. **Corollary.** ***(Finiteness of the Set of Induced Ideals, cf. Boldini*** [4]***)*** *Let $L \subseteq \mathbb{W}$ be a left ideal. Then, the following statements hold:*

(a) $\#\{\overline{\mathbb{G}}^{\underline{\omega}}(L) \mid \underline{\omega} \in \Omega\} < \infty$.
(b) $\#\{\mathbb{V}^{\underline{\omega}}(\mathbb{W}/L) \mid \underline{\omega} \in \Omega\} < \infty$.

Proof. (a): Let G be an universal Gröbner basis of L. Then, by Proposition 13.6, for each $\underline{\omega} \in \Omega$ we have $\overline{\mathbb{G}}^{\underline{\omega}}(L) = \sum_{g \in G} \mathbb{P}\sigma^{\underline{\omega}}(g)$. For each $g \in G$ we write

$$g = \sum_{(\underline{\nu},\underline{\mu}) \in \mathrm{supp}(g)} c^{(g)}_{\underline{\nu}\underline{\mu}} \underline{X}^{\underline{\nu}} \underline{\partial}^{\underline{\mu}}.$$

Then, for each $\underline{\omega} \in \Omega$ we have $\sigma^{\underline{\omega}}(g) = \Phi(g^{\underline{\omega}}_{\deg_{\underline{\omega}}(g)}) = \sum_{(\underline{\nu},\underline{\mu}) \in \mathrm{supp}^{\underline{\omega}}_{\deg^{\underline{\omega}}}(g)} c^{(g)}_{\underline{\nu}\underline{\mu}} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}}$. Therefore $\#\{\sigma^{\underline{\omega}}(g) \mid \underline{\omega} \in \Omega\} \leq \#\{H \subseteq \mathrm{supp}(g)\} = 2^{\#\mathrm{supp}(g)}$. It follows that

$$\#\{\overline{\mathbb{G}}^{\underline{\omega}}(L) = \sum_{g \in G} \mathbb{P}\sigma^{\underline{\omega}}(g) \mid \underline{\omega} \in \Omega\} \leq \#\{\big(\sigma^{\underline{\omega}}(g)\big)_{g \in G} \in \mathbb{P}^G \mid \underline{\omega} \in \Omega\} \leq 2^{\#\mathrm{supp}(G)}.$$

(b): This follows immediately from statement (a) on use of Reminder, Definition and Exercise 13.5 (C)(c). □

To apply this result, we need a few more preparations.

13.8. **Exercise and Definition.** (A) Let $\underline{\omega} \in \Omega$ and let

$$0 \longrightarrow Q \xrightarrow{\iota} U \xrightarrow{\pi} P \longrightarrow 0$$

be an exact sequence of D-modules. Let $V \subseteq U$ be a finitely generated K-vector subspaces such that $U = \mathbb{W}V$. We endow Q with the filtration

$$Q_\bullet := \big(\iota^{-1}(\mathbb{W}_i^{\underline{\omega}} V)\big)_{i \in \mathbb{N}_0}.$$

Prove the following statements:

(a) For each $i \in \mathbb{N}_0$ there is a K-linear map

$$\bar{\iota}_i : Q_i/Q_{i-1} \longrightarrow \mathbb{W}_i^{\underline{\omega}} V/\mathbb{W}_{i-1}^{\underline{\omega}} V, \quad q + Q_{i-1} \mapsto \iota(q) + \mathbb{W}_{i-1}^{\underline{\omega}} V.$$

(b) For each $i \in \mathbb{N}_0$ there is a K-linear map

$$\overline{\pi}_i : \mathbb{W}_i^{\underline{\omega}} V/\mathbb{W}_{i-1}^{\underline{\omega}} V \longrightarrow \mathbb{W}_i^{\underline{\omega}} \pi(V)/\mathbb{W}_{i-1}^{\underline{\omega}} \pi(V), \quad d + \mathbb{W}_{i-1}^{\underline{\omega}} V \mapsto \pi(q) + \mathbb{W}_{i-1}^{\underline{\omega}} \pi(V).$$

(c) For each $i \in \mathbb{N}_0$ it holds
$$\pi^{-1}\big(\mathbb{W}^{\underline{\omega}}_{i-1}\pi(V)\big) = \iota(Q_i) + \mathbb{W}^{\underline{\omega}}_{i-1}V.$$
(d) For each $i \in \mathbb{N}_0$ there is a short exact sequence of K-vector spaces
$$0 \longrightarrow Q_i/Q_{i-1} \xrightarrow{\overline{\iota}_i} \mathbb{W}^{\underline{\omega}}_i V/\mathbb{W}^{\underline{\omega}}_{i-1}V \xrightarrow{\overline{\pi}_i} \mathbb{W}^{\underline{\omega}}_i\pi(V)/\mathbb{W}^{\underline{\omega}}_{i-1}\pi(V) \longrightarrow 0.$$

(B) *(The Graded Exact Sequence associated to a Short Exact Sequence of D-Modules)* Keep the hypotheses and notations of part (A). Prove the following statements:

(a) For each $i \in \mathbb{N}_0$ there is a short exact sequence of K-vector spaces
$$0 \longrightarrow \mathrm{Gr}_{Q_\bullet}(Q)_i \xrightarrow{\overline{\iota}_i} \mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet V}(U)_i \xrightarrow{\overline{\pi}_i} \mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet \pi(V)}(P)_i \longrightarrow 0.$$
(b) There is an exact sequence of graded $\mathbb{P}^{\underline{\omega}}$-modules
$$0 \longrightarrow \mathrm{Gr}_{Q_\bullet}(Q) \xrightarrow{\overline{\iota}} \mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet V}(U) \xrightarrow{\overline{\pi}} \mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet \pi(V)}(P) \longrightarrow 0,$$
with $\overline{\iota} := \bigoplus_{i\in\mathbb{N}_0} \overline{\iota}_i$ and $\overline{\pi} := \bigoplus_{i\in\mathbb{N}_0} \overline{\pi}_i$.

The exact sequence of statement (b) is called the *exact sequence induced by the exact sequence* $0 \to Q \xrightarrow{\iota} U \xrightarrow{\pi} P \to 0$ and the generating vector space V of U.

(C) Keep the previous notations and hypotheses. Prove the following statements:

(a) For each finitely generated K-vector subspace $T \subseteq Q$ with $Q = \mathbb{W}T$ and $V \subseteq \iota(T)$, the two filtrations $Q_\bullet$ and $\mathbb{W}^{\underline{\omega}}_\bullet T$ of Q are equivalent.

(b) $\mathrm{Var}\big(\mathrm{Ann}_{\mathbb{P}}(\mathrm{Gr}_{Q_\bullet}(Q))\big) = \mathbb{V}^{\underline{\omega}}(Q)$.

13.9. **Proposition.** ***(Additivity of Characteristic Varieties)*** *Let $\underline{\omega} \in \Omega$ and let*
$$0 \longrightarrow Q \xrightarrow{\iota} U \xrightarrow{\pi} P \longrightarrow 0$$
be an exact sequence of D-modules. Then it holds
$$\mathbb{V}^{\underline{\omega}}(U) = \mathbb{V}^{\underline{\omega}}(Q) \cup \mathbb{V}^{\underline{\omega}}(P).$$

Proof. We fix a finitely generated K-vector subspace $V \subseteq U$ with $\mathbb{W}V = U$ and consider the corresponding induced short exact sequence (see Exercise and Definition 13.8 (B))
$$0 \longrightarrow \mathrm{Gr}_{Q_\bullet}(Q) \xrightarrow{\overline{\iota}} \mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet V}(U) \xrightarrow{\overline{\pi}} \mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet \pi(V)}(P) \longrightarrow 0.$$
On use of Exercise and Definition 13.8 (C)(b) we obtain
$$\mathbb{V}^{\underline{\omega}}(U) = \mathrm{Var}\big(\mathrm{Ann}_{\mathbb{P}}(\mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet V}(U))\big) =$$
$$= \mathrm{Var}\big(\mathrm{Ann}_{\mathbb{P}}(\mathrm{Gr}_{Q_\bullet}(Q))\big) \cup \mathrm{Var}\big(\mathrm{Ann}_{\mathbb{P}}(\mathrm{Gr}_{\mathbb{W}^{\underline{\omega}}_\bullet \pi(V)}(P))\big) = \mathbb{V}^{\underline{\omega}}(Q) \cup \mathbb{V}^{\underline{\omega}}(P).$$
□

Now, we are ready to prove the first main result of this section.

13.10. **Theorem.** ***(Finiteness of the Set of Characteristic Varieties)*** *Let U be a D-module. Then*
$$\#\{\mathbb{V}^{\underline{\omega}}(U) \mid \underline{\omega} \in \Omega\} < \infty.$$

Proof. We proceed by induction on the number r of generators of U. If $r = 1$ we have $U \cong \mathbb{W}/L$ for some left ideal $L \subseteq \mathbb{W}$. In this case, we may conclude by Corollary 13.7 (b). So, let $r > 1$. Then, we find a short exact of D-modules

$$0 \longrightarrow Q \xrightarrow{\iota} U \xrightarrow{\pi} P \longrightarrow 0$$

such that Q and P are generated by less than r elements. By induction, we have

$$\#\{\mathbb{V}^{\underline{\omega}}(Q) \mid \underline{\omega} \in \Omega\} < \infty \text{ and } \#\{\mathbb{V}^{\underline{\omega}}(P) \mid \underline{\omega} \in \Omega\} < \infty.$$

By Proposition 13.9 we also have

$$\{\mathbb{V}^{\underline{\omega}}(U) \mid \underline{\omega} \in \Omega\} = \{\mathbb{V}^{\underline{\omega}}(Q) \cup \mathbb{V}^{\underline{\omega}}(P) \mid \underline{\omega} \in \Omega\},$$

hence

$$\#\{\mathbb{V}^{\underline{\omega}}(U) \mid \underline{\omega} \in \Omega\} \leq \#\{\mathbb{V}^{\underline{\omega}}(Q) \mid \underline{\omega} \in \Omega\} + \#\{\mathbb{V}^{\underline{\omega}}(P) \mid \underline{\omega} \in \Omega\} < \infty.$$

□

To prove the second main result of this section, we need some more preparations.

13.11. **Definition and Exercise.** (A) *(Leading Forms)* We consider the polynomial ring $\mathbb{P}$. Let

$$f = \sum_{(\underline{\nu},\underline{\mu}) \in \mathrm{supp}(f)} c^{(f)}_{\underline{\nu}\underline{\mu}} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}} \in \mathbb{P} \quad \text{with } c^{(f)}_{\underline{\nu}\underline{\mu}} \in K \setminus \{0\} \text{ for all } (\underline{\nu},\underline{\mu}) \in \mathrm{supp}(f).$$

We set

$$\mathrm{supp}^{\underline{\omega}}_i(f) := \{(\underline{\nu},\underline{\mu}) \in \mathrm{supp}(f) \mid \underline{\nu}\underline{v} + \underline{\mu}\underline{w} = i\}$$

and consider the $i-th$ *homogeneous component* of f with respect to ω, thus the polynomial

$$f^{\underline{\omega}}_i = f^{\underline{v}\underline{w}}_i := \sum_{(\underline{\nu},\underline{\nu}) \in \mathrm{supp}^{\underline{\omega}}_i(f)} c^{(f)}_{\underline{\nu}\underline{\mu}} \underline{Y}^{\underline{\nu}} \underline{Z}^{\underline{\mu}}.$$

The *leading form* of f with respect to the weight $\underline{\omega}$ is defined by

$$\mathrm{LF}^{\underline{\omega}}(f) := \begin{cases} 0 & \text{if } f = 0, \\ f^{\underline{\omega}}_{\deg^{\underline{\omega}}(f)} & \text{if } f \neq 0. \end{cases}$$

Prove that for all $f, g \in \mathbb{P}$, all $i, j \in \mathbb{N}_0$ and for all weights $\underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ the following statements hold:

(a) If $i > \deg^{\underline{\omega}}(f)$, then $f^{\underline{\omega}}_i = 0$.
(b) $f^{\underline{\omega}}_i = \big[f^{\underline{\omega}}_i\big]^{\underline{\omega}}_i$.
(c) $(f+g)^{\underline{\omega}}_i = f^{\underline{\omega}}_i + g^{\underline{\omega}}_i$.
(d) $(fg)^{\underline{\omega}}_i = \sum_{j+k=i} f^{\underline{\omega}}_j g^{\underline{\omega}}_k$.
(e) $\mathrm{LF}^{\underline{\omega}}(fg) = \mathrm{LF}^{\underline{\omega}}(f)\mathrm{LF}^{\underline{\omega}}(g)$.
(f) $\mathrm{LF}(f) = f$ if and only if f is homogeneous with respect to the $\underline{\omega}$-grading of $\mathbb{P}$.
(g) If $d \in \mathbb{W}$, then $\sigma^{\underline{\omega}}(d) = \mathrm{LF}^{\underline{\omega}}\big(\Phi(d)\big)$.

(B) *(Leading Form Ideals)* Keep the notations and hypotheses of part (A). If $S \subset \mathbb{P}$ is any subset, we define the *leading form ideal* of S with respect to $\underline{\omega}$ by

$$\mathrm{LFI}^{\underline{\omega}}(S) := \sum_{f \in S} \mathbb{P}\mathrm{LF}^{\underline{\omega}}(f).$$

Let $S \subseteq T \subseteq \mathbb{P}$ and $\leq \in \mathrm{AO}(\mathbb{E})$. Prove the following statements:

(a) $\mathrm{LFI}^{\underline{\omega}}(S) \subseteq \mathrm{LFI}^{\underline{\omega}}(T)$.

(b) If for each $t \in T \setminus \{0\}$ there is some monomial $m = \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \in \mathbb{M} \subset \mathbb{P}$ and some $s \in S$ such that $\mathrm{LM}_{\leq^{\underline{\omega}}}\big(\Phi^{-1}(t)\big) = m\mathrm{LM}_{\leq^{\underline{\omega}}}\big(\Phi^{-1}(s)\big)$, then $\mathrm{LFI}^{\underline{\omega}}(S) = \mathrm{LFI}^{\underline{\omega}}(T)$.

(c) For each ideal $I \subseteq \mathbb{P}$ it holds

$$\sqrt{\mathrm{LFI}^{\underline{\omega}}(I)} = \sqrt{\mathrm{LFI}^{\underline{\omega}}(\sqrt{I})}.$$

(d) If $I, J \subseteq \mathbb{P}$ are ideals, then

(1) $\mathrm{LFI}^{\underline{\omega}}(I \cap J) \subseteq \mathrm{LFI}^{\underline{\omega}}(I) \cap \mathrm{LFI}^{\underline{\omega}}(I)$ and $\mathrm{LFI}^{\underline{\omega}}(I)\mathrm{LFI}^{\underline{\omega}}(J) \subseteq \mathrm{LFI}^{\underline{\omega}}(IJ)$;

(2) $\sqrt{\mathrm{LFI}^{\underline{\omega}}(I \cap J)} = \sqrt{\mathrm{LFI}^{\underline{\omega}}(I) \cap \mathrm{LFI}^{\underline{\omega}}(J)} = \sqrt{\mathrm{LFI}^{\underline{\omega}}(I)} \cap \sqrt{\mathrm{LFI}^{\underline{\omega}}(J)}$.

13.12. **Exercise.** (A) Prove that for all $d \in \mathbb{W}$, all $i, j \in \mathbb{N}_0$, all $s \in \mathbb{N}$ and for all weights $\underline{\alpha} = (\underline{a}, \underline{b}), \underline{\omega} = (\underline{v}, \underline{w}) \in \Omega$ the following statements hold (For the unexplained notations see Definition and Exercise 13.3):

(a) $\mathrm{supp}\big([d_i^{\underline{\omega}}]_j^{\underline{\alpha}}\big) = \mathrm{supp}_i^{\underline{\omega}}(d) \bigcap \mathrm{supp}_j^{\underline{\alpha}}(d)$.

(b) $\mathrm{supp}\big([d_i^{\underline{\omega}}]_j^{\underline{\alpha}}\big) \subseteq \mathrm{supp}_{j+si}^{\underline{\alpha}+s\underline{\omega}}(d)$.

(c) If $i \geq \deg^{\underline{\omega}}(d)$, $j \geq \deg^{\underline{\alpha}}(d_i^{\underline{\omega}})$ and $s > \deg^{\underline{\alpha}}(d) - j$, then the inclusion of statement (a) becomes an equality.

(d) If $i \geq \deg^{\underline{\omega}}(d)$, $j \geq \deg^{\underline{\alpha}}\big(d_i^{\underline{\omega}}\big)$ and $s > \deg^{\underline{\alpha}}(d) - j$, then

$$[d_i^{\underline{\omega}}]_j^{\underline{\alpha}} = d_{j+si}^{\underline{\alpha}+s\underline{\omega}}.$$

(B) Prove on use of statements (a)–(d) of part (A) that for all $d \in \mathbb{W}$, all $i, j \in \mathbb{N}_0$, all $s \in \mathbb{N}$ and for all weights $\underline{\omega} = (\underline{v}, \underline{w}), \underline{\alpha} = (\underline{a}, \underline{b}) \in \Omega$ the following statements hold:

(a) $\sigma_j^{\underline{\alpha}}(d_i^{\underline{\omega}}) = \sum_{(\underline{\nu},\underline{\mu}) \in \mathrm{supp}_i^{\underline{\omega}}(d) \cap \mathrm{supp}_j^{\underline{\alpha}}(d)} c_{\underline{\nu}\underline{\mu}}^{(d)} \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} = \sigma_i^{\underline{\omega}}(d_j^{\underline{\alpha}})$.

(b) If $i \geq \deg^{\underline{\omega}}(d)$, $j \geq \deg^{\underline{\alpha}}(d_i^{\underline{\omega}})$ and $s > \deg^{\underline{\alpha}}(d) - j$, then

$$[\sigma_i^{\underline{\omega}}(d)]_j^{\underline{\alpha}} = \sigma_{j+si}^{\underline{\alpha}+s\underline{\omega}}(d).$$

13.13. **Lemma.** *Let $\underline{\alpha}, \underline{\omega} \in \Omega$, let $d \in \mathbb{W} \setminus \{0\}$ and let $s \in \mathbb{N}$ with $s > \deg^{\underline{\alpha}}(d) - \deg^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(d)\big)$. Then, the following statements hold:*

(a) $\deg^{\underline{\alpha}+s\underline{\omega}}(d) = \deg^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(d)\big) + s\deg^{\underline{\omega}}(d)$.

(b) $\mathrm{LF}^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(d)\big) = \sigma^{\underline{\alpha}+s\underline{\omega}}(d)$.

Proof. We write $i := \deg^{\underline{\omega}}(d)$ and $j := \deg^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(d)\big)$. Observe, that $\sigma^{\underline{\omega}}(d) = \sigma_i^{\underline{\omega}}(d) = \Phi(d_i^{\underline{\omega}})$, so that $j = \deg^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(d)\big) = \deg^{\underline{\alpha}}(d_i^{\underline{\omega}})$ and also $s > \deg^{\underline{\alpha}}(d) - j$. Now, by Exercise 13.12 (B)(b) we obtain $\mathrm{LF}^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(d)\big) = [\sigma_i^{\underline{\omega}}(d)]_j^{\underline{\alpha}} = \sigma_{j+si}^{\underline{\alpha}+s\underline{\omega}}(d)$. It remains to show that $j + si = \deg^{\underline{\alpha}+s\underline{\omega}}(d)$. As $\mathrm{LF}^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(d)\big) \neq 0$ we have $\sigma_{j+si}^{\underline{\alpha}+s\underline{\omega}}(d) \neq 0$ and hence $j + si \leq \deg^{\underline{\alpha}+s\underline{\omega}}(d)$ (see Definition and Exercise 13.3 (B)(b)).
Assume that $j + si > \deg^{\underline{\alpha}+s\underline{\omega}}(d)$. Then, we may write $\deg^{\underline{\alpha}+s\underline{\omega}}(d) = k + si$, with $k > j$. It follows, that $s > \deg^{\underline{\alpha}}(d) - k$. On application of Exercise 13.12 (B)(b) we get that $[\sigma_i^{\underline{\omega}}(d)]_k^{\underline{\alpha}} = \sigma_{k+si}^{\underline{\alpha}+s\underline{\omega}}(d) = \sigma^{\underline{\alpha}+s\underline{\omega}}(d) \neq 0$. As $k > j = \deg^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(d)\big)$ we have $[\sigma_i^{\underline{\omega}}(d)]_k^{\underline{\alpha}} = 0$ (see Definition and Exercise 13.11 (A)(a)). This contradiction completes our proof. □

13.14. **Lemma.** *Let $L \subseteq \mathbb{W}$ be a left ideal, let $\underline{\alpha}, \underline{\omega} \in \Omega$, let $\leq \in \mathrm{AO}(\mathbb{E})$ and let G be a $(\leq^{\underline{\alpha}})^{\underline{\omega}}$-Gröbner basis of L. Then* $\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \mathrm{LFI}^{\underline{\alpha}}\big(\{\sigma^{\underline{\omega}}(g) \mid g \in G\}\big)$.

Proof. By Reminder, Definition and Exercise 13.5 (B)(a) we have

$$S := \{\sigma^{\underline{\omega}}(g) \mid g \in G \setminus \{0\}\} \subseteq \overline{\mathbb{G}}^{\underline{\omega}}(L) =: T$$

If we apply Proposition 13.6 (b) with $\leq^{\underline{\alpha}}$ instead of $\leq$, we see that for all $t \in T$ there is some monomial $m = \underline{Y}^{\underline{\nu}}\underline{Z}^{\underline{\mu}} \in \mathbb{M} \subset \mathbb{P}$ and some $s \in S$ such that $\mathrm{LM}_{\leq^{\underline{\alpha}}}\big(\Phi^{-1}(t)\big) = m\mathrm{LM}_{\leq^{\underline{\alpha}}}\big(\Phi^{-1}(s)\big)$. By Definition and Exercise 13.11 (B)(b) it follows that $\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \mathrm{LFI}^{\underline{\alpha}}(S) = \mathrm{LFI}^{\underline{\alpha}}(T) = \mathrm{LFI}^{\underline{\alpha}}\big(\{\sigma^{\underline{\omega}}(g) \mid g \in G\}\big)$. □

13.15. **Theorem.** ***(Stability of Induced Graded Ideals, Boldini*** [5]***,*** [4]***)*** *Let $L \subseteq \mathbb{W}$ be a left ideal and let $\underline{\alpha} \in \Omega$. Then, there exists an integer $\overline{s} = \overline{s}(\underline{\alpha}, L) \in \mathbb{N}_0$ such that for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$ we have*

$$\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \overline{\mathbb{G}}^{\underline{\alpha}+s\underline{\omega}}(L).$$

Proof. Let G be a universal Gröbner basis of L. Then, by Lemma 13.14, for each $\underline{\omega} \in \Omega$ we have

$$\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \mathrm{LFI}^{\underline{\alpha}}\big(\{\sigma^{\underline{\omega}}(g) \mid g \in G\}\big) = \sum_{g\in G} \mathbb{P}\mathrm{LF}^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(g)\big).$$

Now, we set $\overline{s} := \max\{\deg^{\underline{\alpha}}(g) \mid g \in G \setminus \{0\}\}$. By Lemma 13.13 it follows that $\mathrm{LF}^{\underline{\alpha}}\big(\sigma^{\underline{\omega}}(d)\big) = \sigma^{\underline{\alpha}+s\underline{\omega}}(d)$ for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$. So, for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$ we have

$$\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \sum_{g\in G} \sigma^{\underline{\alpha}+s\underline{\omega}}(d).$$

If we apply Proposition 13.6 (a) with $\underline{\alpha} + s\underline{\omega}$ instead of $\underline{\omega}$ we also get

$$\overline{\mathbb{G}}^{\underline{\alpha}+\underline{\omega}}(L) = \sum_{g\in G} \sigma^{\underline{\alpha}+s\underline{\omega}}(d)$$

for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$. This completes our proof. □

13.16. **Notation.** If $\mathfrak{Z} \subseteq \mathrm{Spec}(\mathbb{P})$ is a closed set we denote the *vanishing ideal* of $\mathfrak{Z}$ by $I_{\mathfrak{Z}}$, thus:

$$I_{\mathfrak{Z}} := \bigcap_{\mathfrak{p}\in\mathfrak{Z}} \mathfrak{p} = \sqrt{J}, \text{ for all ideals } J \subseteq \mathbb{P} \text{ with } \mathfrak{Z} = \mathrm{Var}(J).$$

13.17. **Theorem.** ***(Stability of Characteristic Varieties, Boldini*** [5]***,*** [4]***)*** *Let U be a D-module, and let $\underline{\alpha} \in \Omega$. Then, there exists an integer $\overline{s} = \overline{s}(\underline{\alpha}, U) \in \mathbb{N}_0$ such that for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$ we have*

$$\mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}\big(I_{\mathbb{V}^{\underline{\omega}}(U)}\big)\big) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(U).$$

Proof. We proceed by induction on the number r of generators of U. First, let $r = 1$. Then we have $U \cong \mathbb{W}/L$ for some left ideal $L \subseteq \mathbb{W}$. By Theorem 13.15 we find some $\overline{s} \in \mathbb{N}_0$ such that for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$ we have $\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \overline{\mathbb{G}}^{\underline{\alpha}+s\underline{\omega}}(L)$. By Reminder, Definition and Exercise 13.5 (C)(c) we have

$$\mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(U) = \mathrm{Var}\big(\overline{\mathbb{G}}^{\underline{\alpha}+s\underline{\omega}}(L)\big) \text{ and } I_{\mathbb{V}^{\underline{\omega}}(U)} = \sqrt{\overline{\mathbb{G}}^{\underline{\omega}}(L)}.$$

By Definition and Exercise 13.11 (B)(c) we thus get

$$\sqrt{\mathrm{LFI}^{\underline{\alpha}}\big(I_{\mathbb{V}^{\underline{\omega}}(U)}\big)} = \sqrt{\mathrm{LFI}^{\underline{\alpha}}\big(\sqrt{\overline{\mathbb{G}}^{\underline{\omega}}(L)}\big)} = \sqrt{\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big)},$$

so that indeed – for all $s \in \mathbb{N}$ with $s > \overline{s}$ and all $\underline{\omega} \in \Omega$ – we have

$$\mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}\big(I_{\mathbb{V}^{\underline{\omega}}(U)}\big)\big) = \mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}\big(\overline{\mathbb{G}}^{\underline{\omega}}(L)\big) = \mathrm{Var}\big(\overline{\mathbb{G}}^{\underline{\alpha}+s\underline{\omega}}(L)\big) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(U).$$

Now, let $r > 1$. Then, we find a short exact of D-modules

$$0 \longrightarrow Q \xrightarrow{\iota} U \xrightarrow{\pi} P \longrightarrow 0$$

such that Q and P are generated by less than r elements. By induction, we thus find a number $\overline{s} \in \mathbb{N}_0$, such that for all $\underline{\omega} \in \Omega$ and all $s \in \mathbb{N}$ with $s > \overline{s}$ it holds

$$\mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}(I_{\mathbb{V}^{\underline{\omega}}(Q)})\big) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(Q) \text{ and } \mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}(I_{\mathbb{V}^{\underline{\omega}}(P)})\big) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(P).$$

By Proposition 13.9 we have $\mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(U) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(Q) \cup \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(P)$ and hence $I_{\mathbb{V}^{\underline{\omega}}(U)} = I_{\mathbb{V}^{\underline{\omega}}(Q) \cup \mathbb{V}^{\underline{\omega}}(Q)} = I_{\mathbb{V}^{\underline{\omega}}(Q)} \cap I_{\mathbb{V}^{\underline{\omega}}(P)}$. By Definition and Exercise 13.11 (B)(d)(2) it follows from the last equality that

$$\sqrt{\mathrm{LFI}^{\underline{\alpha}}\big(I_{\mathbb{V}^{\underline{\omega}}(U)}\big)} = \sqrt{\mathrm{LFI}^{\underline{\alpha}}\big(I_{\mathbb{V}^{\underline{\omega}}(Q)}\big)} \cap \sqrt{\mathrm{LFI}^{\underline{\alpha}}\big(I_{\mathbb{V}^{\underline{\omega}}(P)}\big)}.$$

Therefore $\mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}(I_{\mathbb{V}^{\underline{\omega}}(U)})\big) = \mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}(I_{\mathbb{V}^{\underline{\omega}}(Q)}\big) \cup \mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}(I_{\mathbb{V}^{\underline{\omega}}(P)}\big)$ and it follows, that

$$\mathrm{Var}\big(\mathrm{LFI}^{\underline{\alpha}}(I_{\mathbb{V}^{\underline{\omega}}(U)})\big) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(Q) \cup \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(P) = \mathbb{V}^{\underline{\alpha}+s\underline{\omega}}(U)$$

for all $\underline{\omega} \in \Omega$ and all $s \in \mathbb{N}$ with $s > \overline{s}$. This completes the step of induction and hence proves our claim. □

13.18. **Definition.** *(The Critical Cone)* Let $\mathfrak{Z} \subseteq \mathrm{Spec}(\mathbb{P})$ be a closed set. Then, the *critical cone* of $\mathfrak{Z}$ is defined as

$$\mathrm{CCone}(\mathfrak{Z}) := \mathrm{Var}\big(\mathrm{LFI}^{\underline{1}}(I_{\mathfrak{Z}})\big),$$

where $\underline{1} = (\underline{1}, \underline{1}) \in \Omega$ denotes the *standard weight.*

13.19. **Corollary.** ***(Affine Deformation of Characteristic Varieties to Critical Cones, Boldini*** [5], [4]***)*** *Let U be a D-module. Then, there is an integer $\overline{s} = \overline{s}(U) \in \mathbb{N}_0$ such that for all $\underline{\omega} \in \Omega$ and all $s \in \mathbb{N}$ with $s > \overline{s}$ it holds*

$$\mathbb{V}^{\underline{1}+s\underline{\omega}}(U) = \mathrm{CCone}\big(\mathbb{V}^{\underline{\omega}}(U)\big).$$

Proof. This is immediate by Theorem 13.17. □

14. Standard Degree and Hilbert Polynomials

In this section, we relate D-modules with Castelnuovo-Mumford regularity.

14.1. **Preliminary Remark.** (A) Let $n \in \mathbb{N}$, let K be a field of characteristic 0 and consider the standard Weyl algebra $\mathbb{W} = \mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n]$. Moreover let $\mathcal{O}$ be a ring of smooth functions in $X_1, X_2, \ldots, X_n$ over K (see Remark and Definition 11.11 (A)). One concern of Analysis is to study whole *families of differential equations. So for fixed $r, s \in \mathbb{N}$ one chooses a family $\mathbb{F} \subseteq \mathbb{W}^{s\times r}$ of matrices of differential* *operators. Then one studies all systems of equations (see Remark and Definition 11.11 (B))*

$$\mathcal{D}\begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ f_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix}, \text{ with } \mathcal{D} \in \mathbb{F}.$$

(B) Let the notations and hypotheses by as in part (A). One aspect of the above approach is to study the behavior of the characteristic varieties $\mathbb{V}^{\deg}(\mathcal{D}) := \mathbb{V}_{\mathbb{W}_\bullet^{\deg}}\big(U_{\mathcal{D}}\big)$ with respect to the degree filtration (see Definition and Remark 8.6 and Definition and Remark 11.2 (D)) of the D-module $U_{\mathcal{D}}$ defined by the matrix $\mathcal{D}$ (see Remark and Definition 11.11 (C)) if this latter runs through the family $\mathbb{F}$.
The goal of this section is to prove that the degree of hypersurfaces which cut out set-theoretically the characteristic variety $\mathbb{V}^{\deg}(\mathcal{D})$ is bounded, if $\mathcal{D}$ runs through appropriate families $\mathbb{F}$.

Below, we recall a few notions from Commutative Algebra.

14.2. **Reminder, Definition and Exercise.** *(Hilbert Functions, Hilbert Polynomials and Hilbert Coefficients for Modules over Very Well Filtered Algebras)* (A) Let K be a field and let $R = \bigoplus_{i\in\mathbb{N}_0} R_i$ be a homogeneous Noetherian K-algebra, so that $R_0 = K$ and $R = K[x_1, x_2, \ldots, x_r]$ with finitely many elements $x_1, x_2, \ldots, x_r \in R_1$. Moreover, let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded R-module. Then we denote the *Hilbert function* of M by h_M, so that $h_M(i) := \dim_K(M_i)$ for all $i \in \mathbb{Z}$. We denote by $P_M(X)$ the *Hilbert polynomial* of M, so that $h_M(i) = P_M(i)$ for all $i \gg 0$. Keep in mind that $\dim(M) = \dim\big(R/\mathrm{Ann}_R(M)\big)$ and

$$\deg\big(P_M(X)\big) = \begin{cases} \dim(M) - 1, & \text{if } \dim(M) > 0 \\ -\infty, & \text{if } \dim(M) \leq 0. \end{cases}.$$

The Hilbert polynomial $P_M(X)$ has a *binomial presentation*:

$$P_M(X) = \sum_{k=0}^{\dim(M)-1} (-1)^k e_k(M)\binom{X + \dim(M) - k - 1}{\dim(M) - k - 1} \quad \big(e_k(M) \in \mathbb{Z}, e_0(M) \geq 0\big).$$

The integer $e_k(M)$ is called the k-th *Hilbert coefficient* of M. If $\dim(M) > 0$, $e_0(M) > 0$ is called the *multiplicity* of M. Finally let us also introduce the *postulation number* of M, thus the number $\mathrm{pstln}(M) := \sup\{i \in \mathbb{Z} \mid h_M(i) \neq P_M(i)\}$.

(B) Now, let $(A, A_\bullet)$ be a very well filtered K-algebra (see Definition and Remark 3.3 (A)). Let U be a finitely generated (left) A-module. Chose a vector space $V \subseteq U$ of finite dimension such that $AV = U$. Then, the graded $\mathrm{Gr}_{A_\bullet}(A)$-module $\mathrm{Gr}_{A_\bullet V}(U)$ is generated by finitely many homogeneous elements of degree 0 (see Exercise and Definition 10.5 (B)(c)). So, by part (A) this graded module admits a Hilbert function $h_{U,A_\bullet V} := h_{\mathrm{Gr}_{A_\bullet V}(U)}$ with $h_{U,A_\bullet V}(i) := \dim_K\big(\mathrm{Gr}_{A_\bullet V}(U)_i\big)$ for all $i \in \mathbb{Z}$, the *Hilbert function* of U with respect to the filtration induced by V. Moreover, by part (A), the module $\mathrm{Gr}_{A_\bullet V}(U)$ admits a Hilbert polynomial, thus a polynomial $P_{U,A_\bullet V}(X) := P_{\mathrm{Gr}_{A_\bullet V}(U)}(X) \in \mathbb{Q}[X]$ with $h_{U,A_\bullet V}(i) =$

$P_{U,A_\bullet V}(i)$ for all $i \gg 0$. We call this polynomial the *Hilbert polynomial* of U with respect to the filtration induced by V. Keep in mind that according to part (A) we have $d_{A_\bullet}(U) := \dim\big(\mathrm{Gr}_{A_\bullet V}(U)\big) = \dim\big(\mathbb{V}_{A_\bullet}(U)\big)$. Moreover the polynomial $P_{U,A_\bullet V}(X)$ has a binomial presentation:

$$P_{U,A_\bullet V}(X) = \sum_{k=0}^{d_{A_\bullet}(U)-1} (-1)^k e_k\big(U, A_\bullet V\big)\binom{X + d_{A_\bullet}(U) - k - 1}{d_{A_\bullet}(U) - k - 1} \quad \big(e_k(U, A_\bullet V) \in \mathbb{Z}\big).$$

The integer $e_k(U, A_\bullet V)$ is called the k-th *Hilbert coefficient* of U with respect to the filtration induced by V. Finally, keep in mind, that by part (A) we have $e_0(U, A_\bullet V) > 0$ if $d_{A_\bullet}(U) > 0$. In this situation the number $e_0(U, A_\bullet V)$ is called the *multiplicity* of U with respect to the filtration induced by V. For the sake of completeness, we set $e_0(U, A_\bullet V) := 0$ if $d_{A_\bullet}(U) \leq 0$. Finally, according to part (A) we define the *postulation number* of U with respect to the filtration induced by V:

$$\mathrm{pstln}_{U,A_\bullet V}(U) := \mathrm{pstln}(\mathrm{Gr}_{A_\bullet V}(U)) := \sup\{i \in \mathbb{Z} \mid h_{U,A_\bullet V}(i) \neq P_{U,A_\bullet V}(i)\}.$$

(C) Keep the notations and hypotheses of part (B). Prove the following claims.

(a) The multiplicity $e_{A_\bullet}(U) := e_0(U, A_\bullet V)$ is the same for each finite dimensional K-subspace $V \subseteq U$ with $U = AV$.

(b) There is a polynomial $Q_{U,A_\bullet V}(X) \in \mathbb{Q}[X]$ such that

(1) $\deg\big(Q_{U,A_\bullet V}(X)\big) = d_{A_\bullet}(U)$,

(2) $\Delta\big(Q_{U,A_\bullet V}(X)\big) := Q_{U,A_\bullet V}(X) - Q_{U,A_\bullet V}(X-1) = P_{U,A_\bullet V}(X)$ and

(3) $\dim_K(A_i V) = Q_{U,A_\bullet V}(i)$ for all $i \gg 0$.

14.3. **Reminder, Remark and Exercise.** *(Castelnuovo-Mumford Regularity)* (A) Keep the notations and hypotheses of Reminder, Definition and Exercise 14.2(A). For each finitely generated graded $R = \bigoplus_{i\in\mathbb{N}_0} R_i = K[x_1, x_2, \ldots, x_r]$-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ and each $k \in \mathbb{N}_0$ let $\mathrm{reg}^k(M)$ denote the *Castelnuovo-Mumford regularity at and above level k* of M, so that

$$\mathrm{reg}^k(M) := \max\{a_i(M) + i \mid i = k, k+1, \ldots, \dim(M)\}$$

with

$$a_i(M) := \sup\{j \in \mathbb{Z} \mid H^i_{R_+}(M)_j \neq 0\} \text{ for all } i \in \mathbb{N}_0,$$

and where $H^i_{R_+}(M)_j$ denotes the j-th graded component of the (naturally graded) i-th *local cohomology module* $H^i_{R_+}(M) = \bigoplus_{k\in\mathbb{Z}} H^i_{R_+}(M)_k$ of M with respect to the *irrelevant ideal* $R_+ := \bigoplus_{m\in\mathbb{N}} R_m$.

Keep in mimd that the *Castelnuovo-Mumford regularity* of M is defined by

$$\mathrm{reg}(M) := \mathrm{reg}^0(M) = \max\{a_i(M) + i \mid i = 0, 1, \ldots, \dim(M)\}$$

and refresh the fact that

$$\mathrm{reg}^1(M) = \mathrm{reg}(M/\Gamma_{R_+}(M)) \text{ and } P_{M/\Gamma_{R_+}(M)}(X) = P_M(X).$$

(B) Keep the notations and hypotheses of part (A). Let

$$\mathrm{gendeg}(M) := \inf\{m \in \mathbb{Z} \mid M = \sum_{k\leq m} RM_k\} \quad \big(\leq \mathrm{reg}(M)\big)$$

denote the *generating degree* of M. Keep in mind, that the ideal $\mathrm{Ann}_R(M) \subseteq R$ is homogeneous. Use the previous inequality to prove the following claims:

(a) If $b \in \mathbb{Z}$ such that $\mathrm{reg}\,\big(\mathrm{Ann}_R(M)\big) \leq b$, there are elements

$$f_1, f_2, \ldots, f_s \in \mathrm{Ann}_R(M) \cap \big(\bigcup_{i \leq b} R_i\big) \text{ with } \mathrm{Var}\big(\mathrm{Ann}_R(M)\big) = \bigcap_{i=1}^{s} \mathrm{Var}(f_i).$$

(C) We recall a few basic facts on Castelnuovo-Mumford regularity.

(a) If $r \in \mathbb{N}$ and $R = K[T_1, T_2, \ldots, T_r]$ is a polynomial ring over the field R, then $\mathrm{reg}(R) = \mathrm{reg}\,\big(K[T_1, T_2, \ldots, T_r]\big) = 0$.

(b) If $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ is a short exact of finitely generated graded R -modules, then we have the equality $\mathrm{reg}(N) \leq \max\{\mathrm{reg}(M), \mathrm{reg}(P) + 1\}$.

(c) If $r \in \mathbb{N}$ and if $M^{(1)}, M^{(2)}, \ldots, M^{(r)}$ are finitely generated graded R-modules, then we have the equality $\mathrm{reg}\,\big(\bigoplus_{i=1}^{r} M^{(i)}\big) = \max\{\mathrm{reg}(M^{(i)}) \mid i = 1, 2, \ldots, r\}$.

(D) We mention the following bounding result (see Corollary 17.4.2 of [10]):

(a) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian Homogeneous ring such that R_0 is Aritinian and local. Let $W = \bigoplus_{n \in \mathbb{Z}} W_n$ be a finitely generated graded R-module and let $P \in \mathbb{Q}[X] \setminus \{0\}$. Then, there is an integer G such that for each homogeneous R-homomorphism $f : W \longrightarrow M$ of finitely generated graded R-modules that is surjective in all large degrees and such that $P_M = P$, we have $\mathrm{reg}^1(M) \leq G$.

Use the bounding result of statement (a) to prove the following result.

(b) There is a function $\overline{B} : \mathbb{N}_0^2 \times \mathbb{Q}[X] \longrightarrow \mathbb{Z}$ such that for each choice of $r, t \in \mathbb{N}$, for each field K, for each homogeneous Noetherian K-algebra $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ with $h_R(1) \leq t$ and each finitely generated graded R-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ with $M = RM_0$ and $h_M(0) \leq r$ we have

$$\mathrm{reg}^1(M) \leq \overline{B}\big(t, r, P_M\big).$$

Another bounding result, which we shall use later is (see Corollary 6.2 of [9]):

(c) Let $R = K[T_1, T_2, \ldots, T_r]$ be a polynomial ring over the field K, furnished with its standard grading. Let $f : W \longrightarrow V$ be a homomorphism of finitely generated graded R-modules such that $V \neq 0$ is generated by μ homogeneous elements of degree 0. Then

$$\mathrm{reg}\,\big(\mathrm{Im}(f)\big) \leq \big[\max\{\mathrm{gendeg}(W), \mathrm{reg}(V) + 1\} + \mu + 1\big]^{2^{r-1}}.$$

We now prove a special case of Theorem 3.10 of [8].

14.4. **Proposition.** *Let $r \in \mathbb{N}$, let $R := K[T_1, T_2, \ldots, T_r]$ be polynomial ring over the field K and let $M = \bigoplus_{n \in \mathbb{N}_0} M_n$ be finitely generated graded R-module with $M = RM_0$. Then*

$$\mathrm{reg}\,\big(\mathrm{Ann}_R(M)\big) \leq \big[\,\mathrm{reg}(M) + h_M(0)^2 + 2\big]^{2^{r-1}} + 1.$$

Proof. Observe first, that we have an exact sequence of graded R-modules

$$0 \longrightarrow \mathrm{Ann}_R(M) \longrightarrow R \xrightarrow{\epsilon} \mathrm{Hom}_R(M, M), \text{ with } x \mapsto \epsilon(x) := x\mathrm{Id}_M, \text{ for all } x \in R.$$

Moreover, there is an epimorphism of graded R-modules

$$\pi : R^{h_M(0)} \longrightarrow M \longrightarrow 0.$$

So, with $g := \mathrm{Hom}_R(\pi, \mathrm{Id}_M)$ we get an induced monomorphism of graded R-modules

$$0 \longrightarrow \mathrm{Hom}_R(M, M) \xrightarrow{g} \mathrm{Hom}_R\big(R^{h_M(0)}, M\big) \cong M^{h_M(0)}.$$

So, we get a composition map

$$f := g \circ \epsilon : R \longrightarrow M^{h_M(0)} =: V, \text{ with } \mathrm{Im}(f) = \mathrm{Im}(\epsilon) \cong R/\mathrm{Ann}_R(M).$$

Now, observe that $\mathrm{gendeg}(R) = 0$ (see Reminder, Remark and Exercise 14.3 (C)(a)), $\mathrm{reg}(V) = \mathrm{reg}(M)$ (see Reminder, Remark and Exercise 14.3 (C)(c)) and that V is generated by $h_M(0)^2$ homogeneous elements of degree 0. So, by Reminder, Remark and Exercise 14.3 (D)(c) we obtain

$$\mathrm{reg}\left(R/\mathrm{Ann}_R(M)\right) = \mathrm{reg}\left(\mathrm{Im}(f)\right) \leq \left[\mathrm{reg}(M) + h_M(0)^2 + 2\right]^{2^{r-1}}.$$

On application of Reminder, Remark and Exercise 14.3 (C) (b) to the short exact sequence of graded R-modules

$$0 \longrightarrow \mathrm{Ann}_R(M) \longrightarrow R \longrightarrow R/\mathrm{Ann}_R(M) \longrightarrow 0$$

and keeping in mind that $\mathrm{reg}(R) = 0$, we thus get indeed our claim. □

14.5. **Exercise.** Let the notations and hypotheses be as in Proposition 14.4. Show that

(a) $\mathrm{reg}\left(\mathrm{Ann}_R(M/\Gamma_{R_+}(M))\right) \leq \left[\mathrm{reg}^1(M) + h_M(0)^2 + 2\right]^{2^{r-1}} + 1.$

(b) $\mathrm{Var}\big(\mathrm{Ann}_R(M/\Gamma_{R_+}(M))\big) = \begin{cases} \mathrm{Var}\big(\mathrm{Ann}_R(M)\big), & \text{if } \dim_R(M) > 0 \\ \emptyset, & \text{if } \dim_R(M) = 0. \end{cases}$

14.6. **Notation, Remark and Exercise.** (A) Let $\overline{B} : \mathbb{N}_0^2 \times \mathbb{Q}[X] \longrightarrow \mathbb{Z}$ be the bounding function introduced in Reminder, Remark and Exercise 14.3 (D)(b).
We define a new function

$$F : \mathbb{N}^2 \times \mathbb{Q}[X] \longrightarrow \mathbb{Z} \text{ by } F(t, r, h) := \left[\overline{B}(t, P) + r^2 + 2\right]^{2^{r-1}} + 1 \quad \big(t, r \in \mathbb{N}, P \in \mathbb{Q}[X]\big).$$

(B) Let the notations as in part (A). Use Proposition 14.4, Reminder, Remark and Exercise 14.3 (B) and Exercise 14.5 to show that for each field K, for each choice of $r, t \in \mathbb{N}$, for each polynomial ring $R = K[T_1, T_2, \ldots, T_t]$ and for each finitely generated graded R-module $M = \bigoplus_{n \in \mathbb{N}_0} M_n$ with $M = RM_0$, $h_M(0) \leq r$ and $P_M = P$, we have the following statements:

(a) $\mathrm{reg}\left(\mathrm{Ann}_R(M/\Gamma_{R_+}(M))\right) \leq F(t, r, P)$.

(b) There are homogeneous polynomials $f_1, f_2, \ldots, f_s \in \mathrm{Ann}_R\big(M/\Gamma_{R_+}(M)\big)$ with
  (1) $\deg(f_i) \leq F(t, r, P)$ for all $i = 1, 2, \ldots, s$.
  (2) $\mathrm{Var}\big(\mathrm{Ann}_R(M)\big) = \mathrm{Var}(f_1, f_2, \ldots, f_s) = \bigcap_{i=1}^{s} \mathrm{Var}(f_i)$.

No, we are ready to prove the main result of this section.

14.7. **Theorem.** ***(Boundedness of the Degrees of Defining Equations of Characteristic Varieties, compare*** [8]***)*** *Let $n \in \mathbb{N}$, let K be a field of characteristic 0, let U be a D-module over the standard Weyl algebra*

$$\mathbb{W} = \mathbb{W}(K, n) = K[X_1, X_2, \ldots, X_n; \partial_1, \partial_2, \ldots, \partial_n]$$

and let $V \subseteq U$ *be a* K*-subspace with* $\dim_K(V) \leq r < \infty$ *and* $U = \mathbb{W}V$*. Morover, let*

$$F : \mathbb{N} \times \mathbb{N}_0^{\mathbb{Z}} \longrightarrow \mathbb{Z}$$

be the bounding function defined in Notation, Remark and Exercise 14.6 (A). Keep in mind that the degree filtration $\mathbb{W}_\bullet^{\deg}$ *of* $\mathbb{W}$ *(see Definition and Remark 8.6) is very good (see Corollary 8.7 (a)) and let*

$$P_{U,\mathbb{W}_\bullet^{\deg}V} \in \mathbb{Q}[X]$$

be the Hilbert polynomial of U *induced by* V *with respect to the degree filtration* $\mathbb{W}_\bullet^{\deg}$ *(see Reminder, Definition and Exercise 14.2 (B)).*

Then, there are homogeneous polynomials

$$f_1, f_2, \ldots, f_s \in \mathbb{P} = K[Y_1, Y_2, \ldots, Y_n; Z_1, Z_2, \ldots, Z_n]$$

such that

(a) $\deg(f_i) \leq F\big(2n, r, P_{U,\mathbb{W}_\bullet^{\deg}}\big)$.
(b) $\mathbb{V}_{\mathbb{W}_\bullet^{\deg}}(U) = \mathrm{Var}(f_1, f_2, \ldots, f_s) = \bigcap_{i=1}^s \mathrm{Var}(f_i)$.

Proof. Observe that (see Definition and Remark 11.2)

$$\mathbb{V}_{\mathbb{W}_\bullet^{\deg}}(U) = \mathrm{Var}\big(\mathrm{Ann}_{\mathbb{P}}(\mathrm{Gr}_{\mathbb{W}_\bullet^{\deg}V}(U)\big).$$

Now, we may conclude by Notation, Remark and Exercise 14.6 (B)(b), applied to the graded $\mathbb{P}$-module $\mathrm{Gr}_{\mathbb{W}_\bullet^{\deg}V}(U)$ and bearing in mind that – by Exercise and Definition 10.5 (B)(c) – this latter graded module is generated in degree 0. □

14.8. **Conclusive Remark.** (A) Keep the above notations. To explain the meaning of this result, we fix $r, s \in \mathbb{N}$ and we fix a polynomial $P \in \mathbb{Q}[X]$. For any matrix

$$\mathcal{D} = \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1r} \\ d_{21} & d_{22} & \ldots & d_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ d_{s1} & d_{s2} & \ldots & d_{sr} \end{pmatrix} \in \mathbb{W}^{s\times r} \quad \big(s \in \mathbb{N}\big)$$

of polynomial partial differential operators we consider the induced epimorphism of D-modules

$$\mathbb{W}^r \xrightarrow{\pi_{\mathcal{D}}} U_{\mathcal{D}} \longrightarrow 0,$$

consider the K-subspace

$$K^r = \big(\mathbb{W}_0^{\deg}\big)^r \subset \mathbb{W}^r$$

and set

$$V_{\mathcal{D}} := \pi_{\mathcal{D}}(K^r).$$

Then, referring to our Preliminary Remark 14.1 we consider the family of systems of differential equations

$$\mathbb{F} = \mathbb{F}^P := \{\mathcal{D} \in \mathbb{W}^{s\times r} \mid P_{U_{\mathcal{D}},\mathbb{W}_\bullet^{\deg}V_{\mathcal{D}}} = P\}$$

whose *canonical Hilbert polynomial* $P_{U_{\mathcal{D}},\mathbb{W}_\bullet^{\deg}V_{\mathcal{D}}}$ equals P. As an immediate application of Theorem 14.7 we can say

The degree of hypersurfaces which cut out set-theoretically the characteristic variety $\mathbb{V}^{\deg}(\mathcal{D})$ is bounded, if $\mathcal{D}$ runs through the family $\mathbb{F}^P$.

Clearly, our results give much more, as they bound the invariant

$$\operatorname{reg}\big(\operatorname{Ann}_{\mathbb{P}}\big[\operatorname{Gr}_{\mathbb{W}^{\deg}_{\bullet}V_{\mathcal{D}}}(U_{\mathcal{D}})/\Gamma_{\mathbb{P}_+}(\operatorname{Gr}_{\mathbb{W}^{\deg}_{\bullet}V_{\mathcal{D}}}(U_{\mathcal{D}})]\big)$$

along the class $\mathbb{F}^P$.

(B) The study of the class $\mathbb{F}^P$ corresponds to the study of the classical Hilbert scheme Hilb^P of closed subschemes $X \subset \mathbb{P}^r_K$ with Hilbert polynomial P. So, it seems a challenging task to approach the classes $\mathbb{F}^P$ from this point of view and to pursue further what one could understand as a *Theory of Hilbert schemes for D-modules.*

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