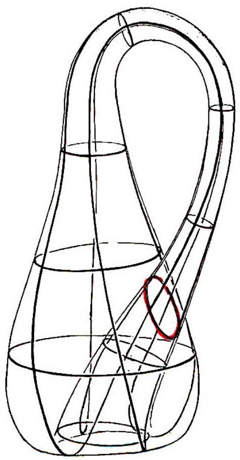
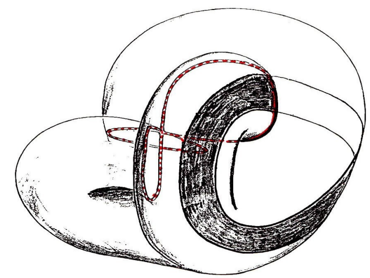
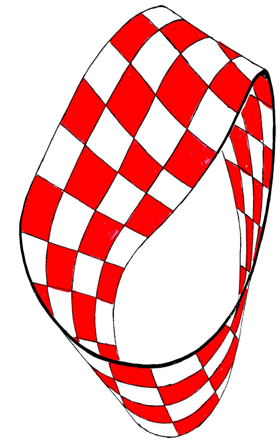
STRIPS, TUBES, BOTTLES, CAPS, UMBRELLAS- A FEW EXAMPLES OF VISUALZATIONS OF SURFACES



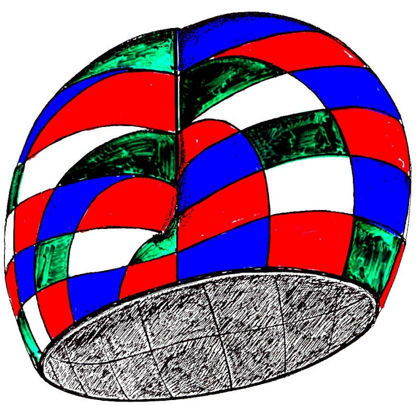


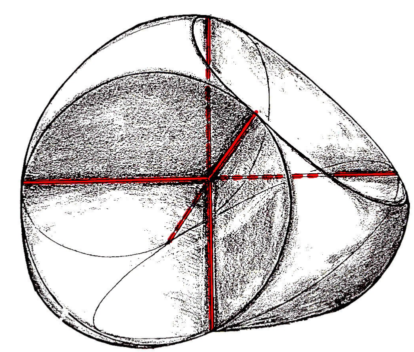


Presentation to the course

Surface Topology-in the Class Room and Beyond

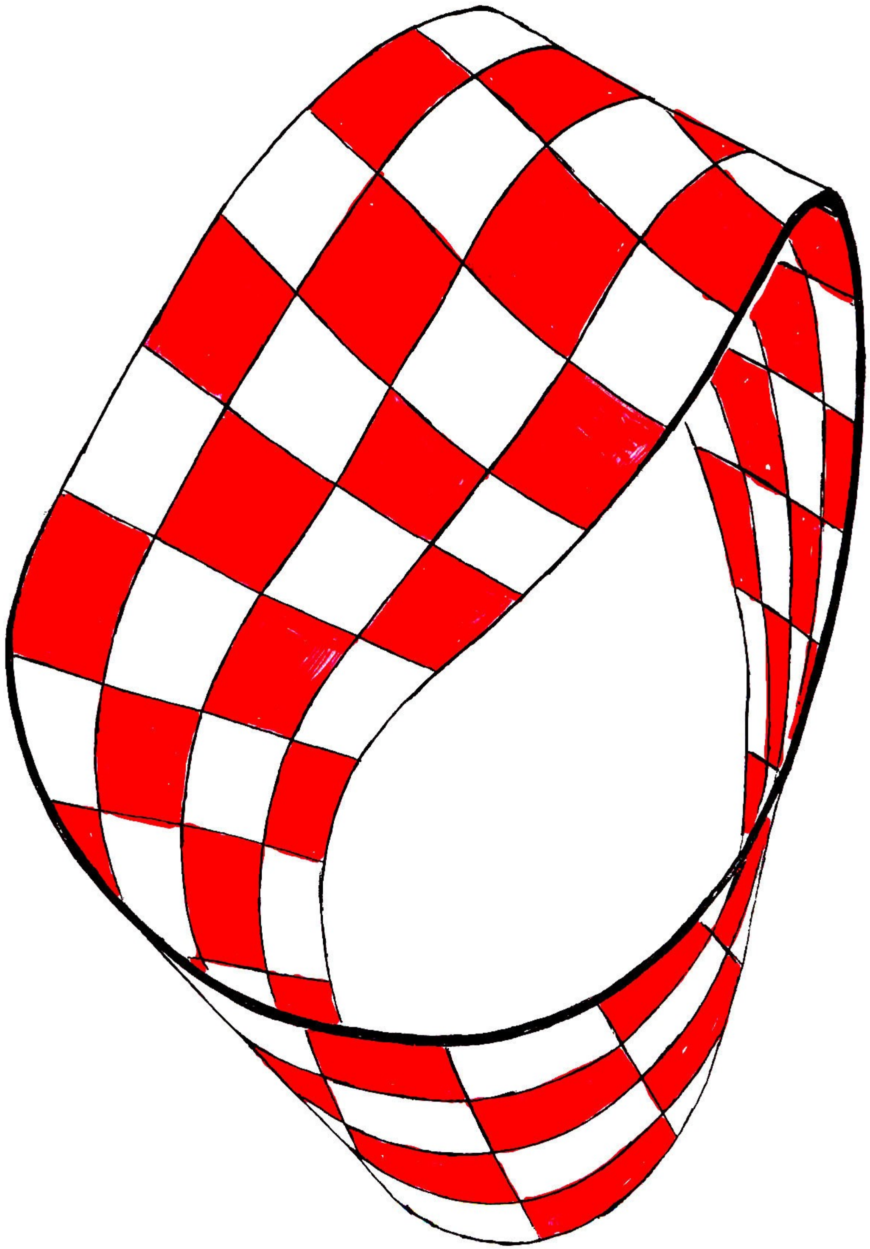
Summer School in the Framework of the North-South Cooperation between the University of Zürich and Kabale University





7.-12. January 2013
Kabale University Uganda

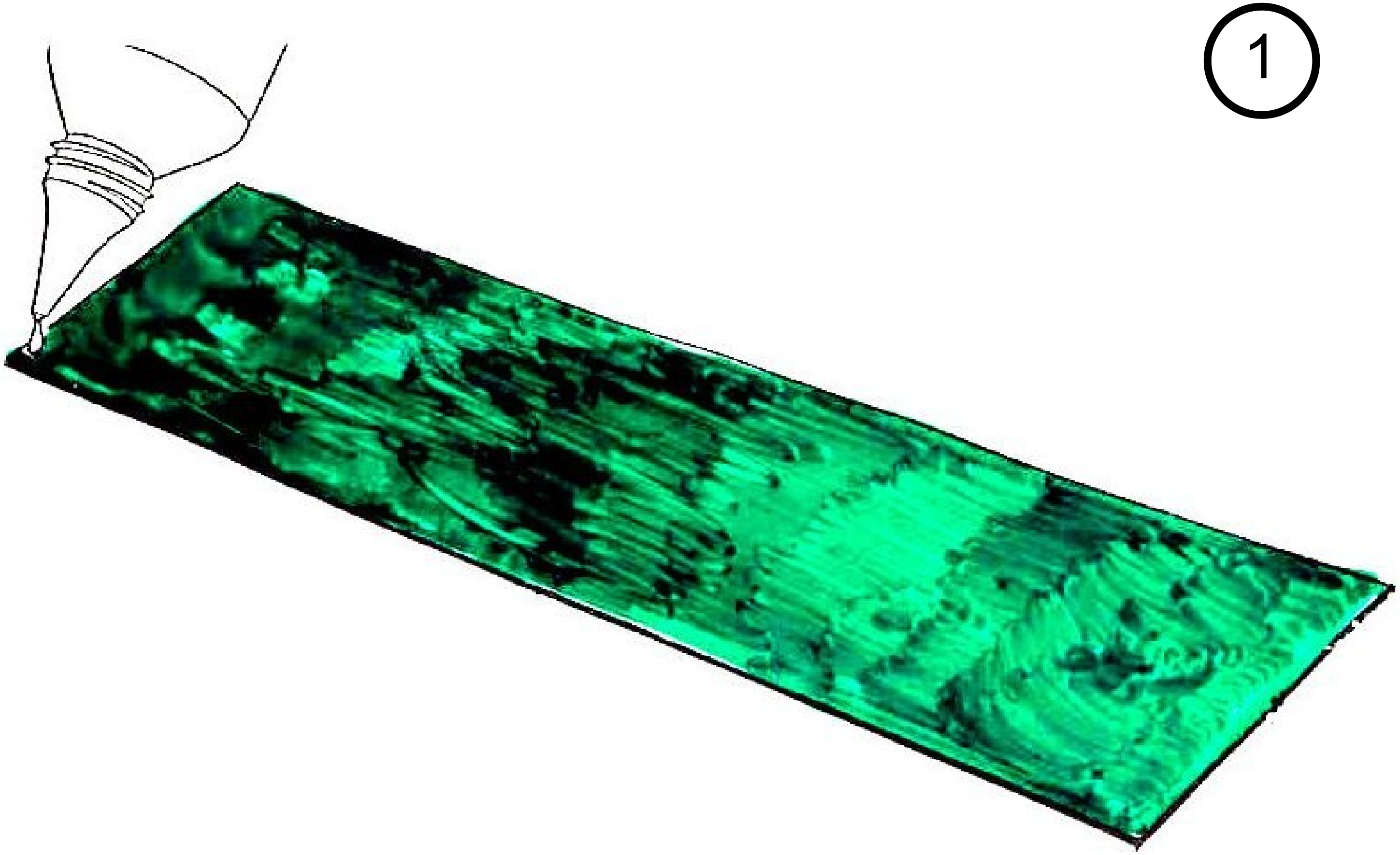
Markus Brodmann
Institute of Mathematics
University of Zürich



The Moebius Strip

Named after
A. F. Möbius
1790 – 1868
Astronomer in
Leipzig, Germany

We make it from a rectangular strip of paper...



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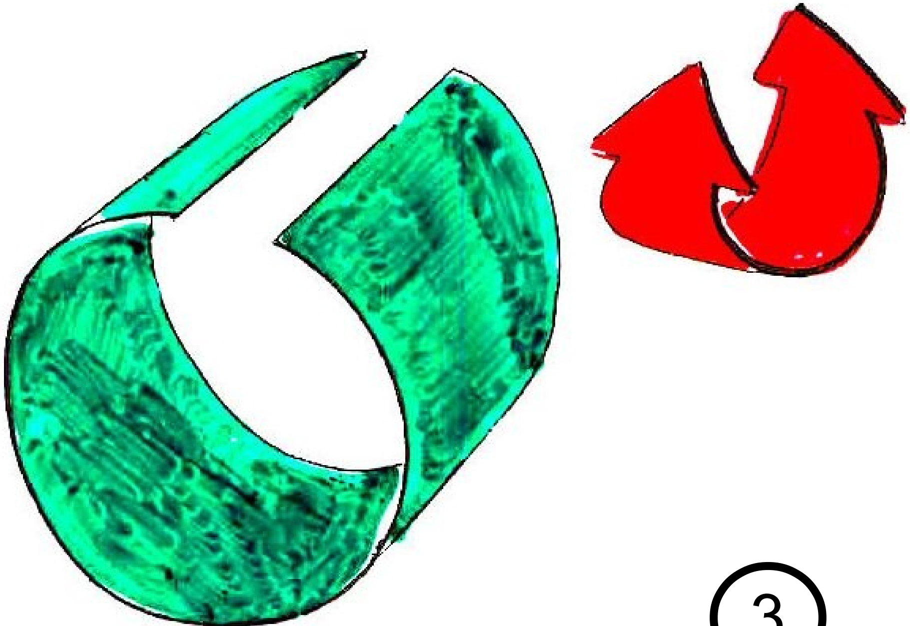

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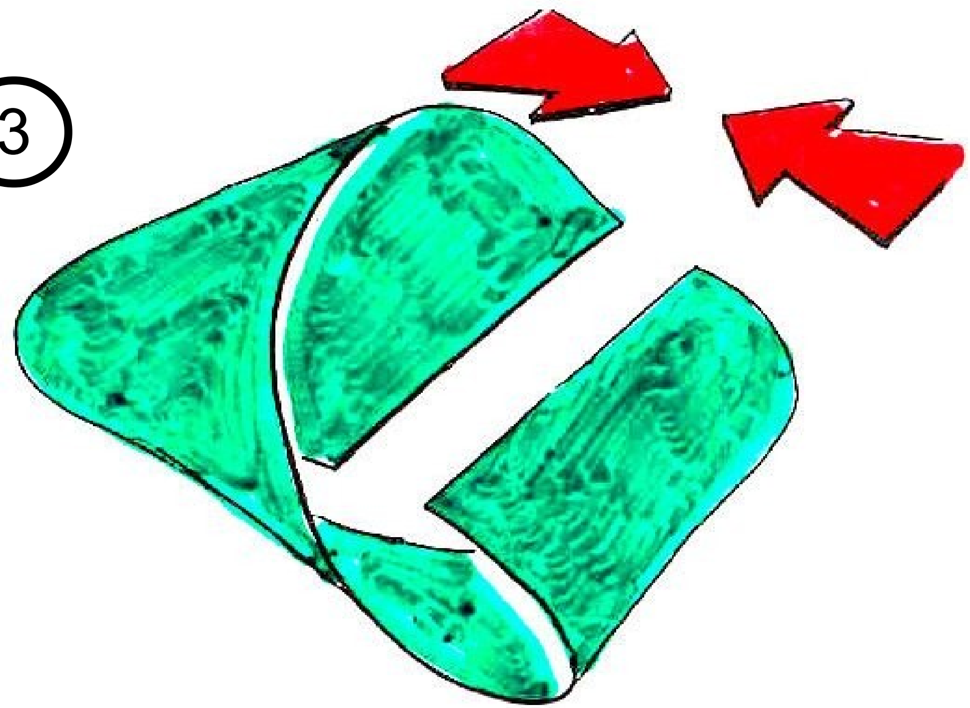

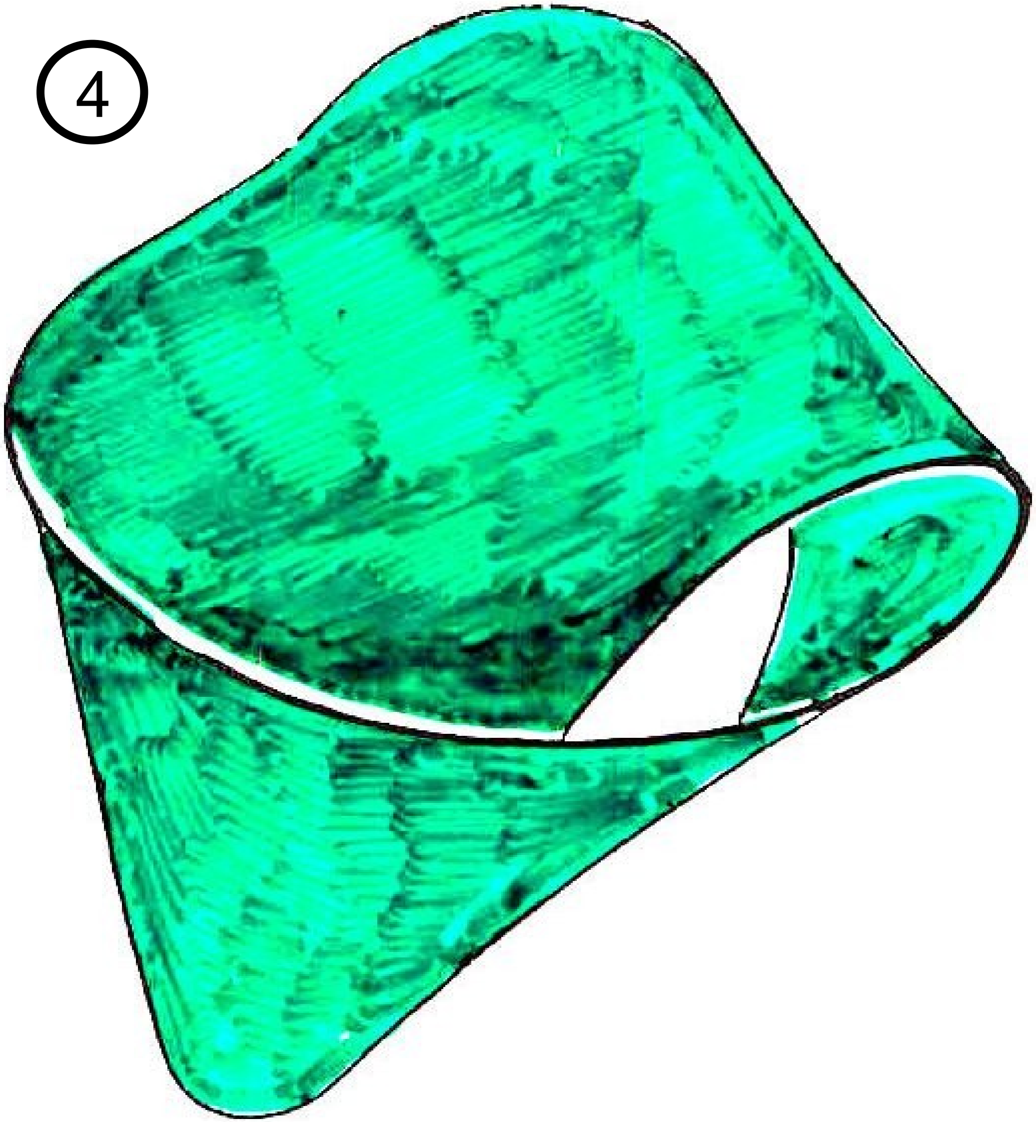
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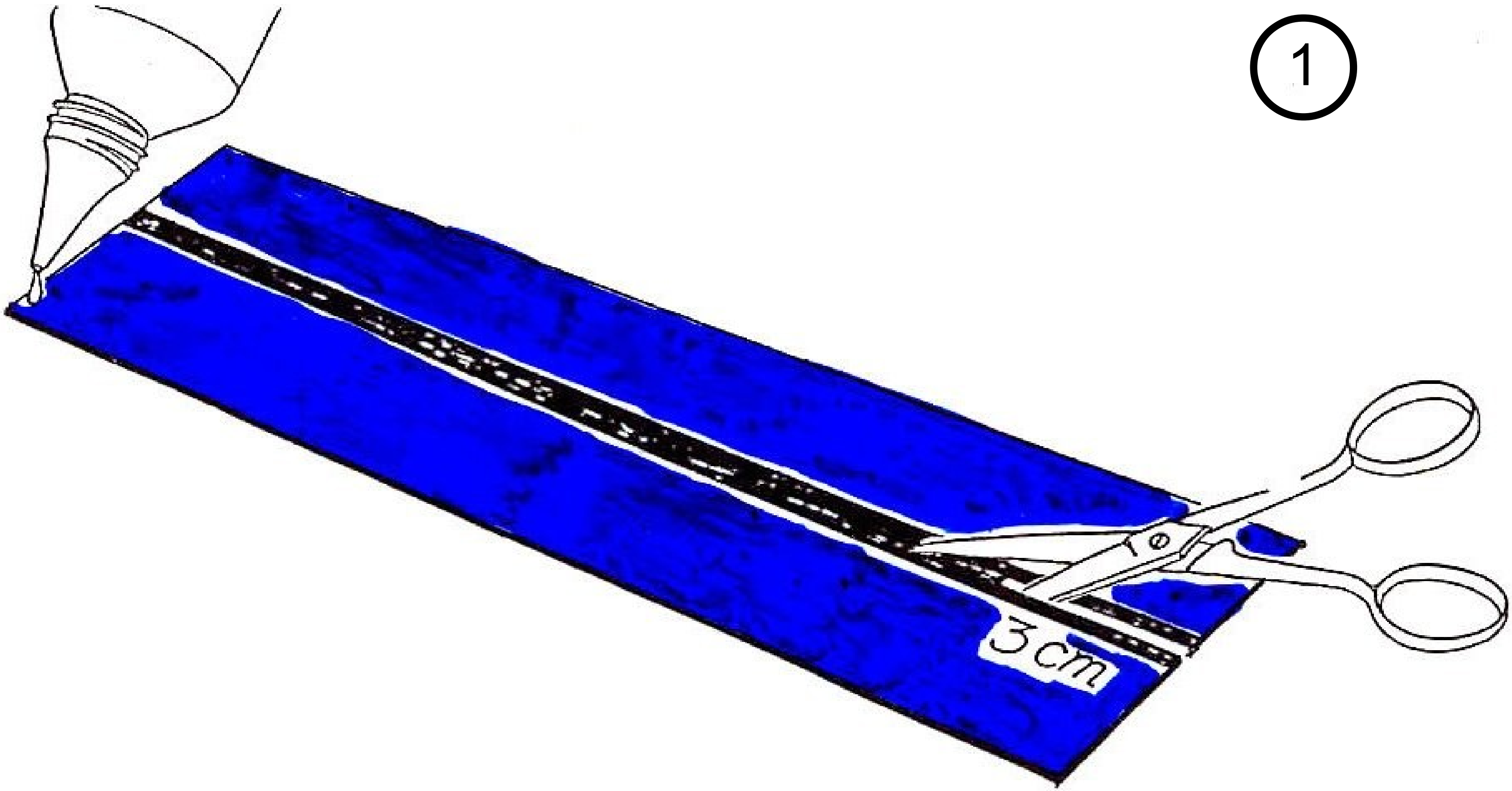
We cut the Moebius Strip....



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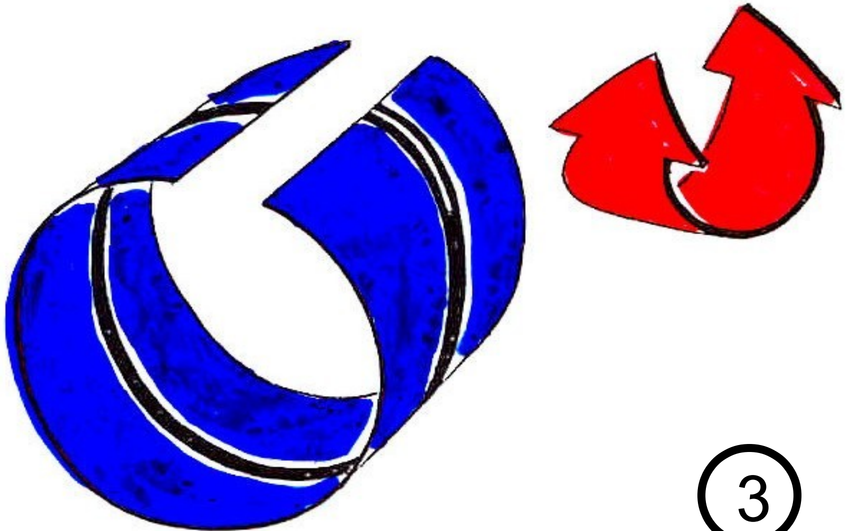

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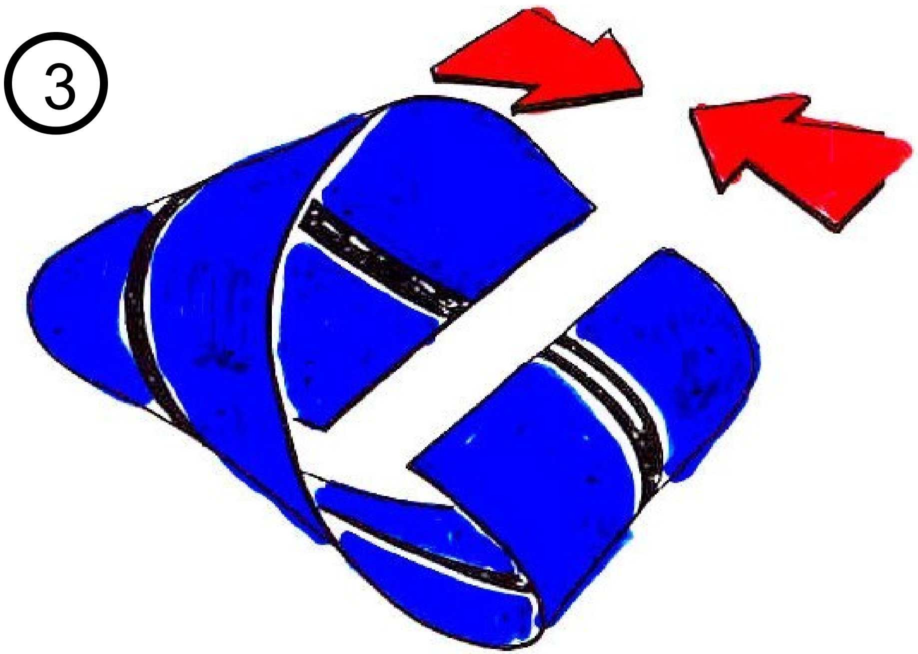
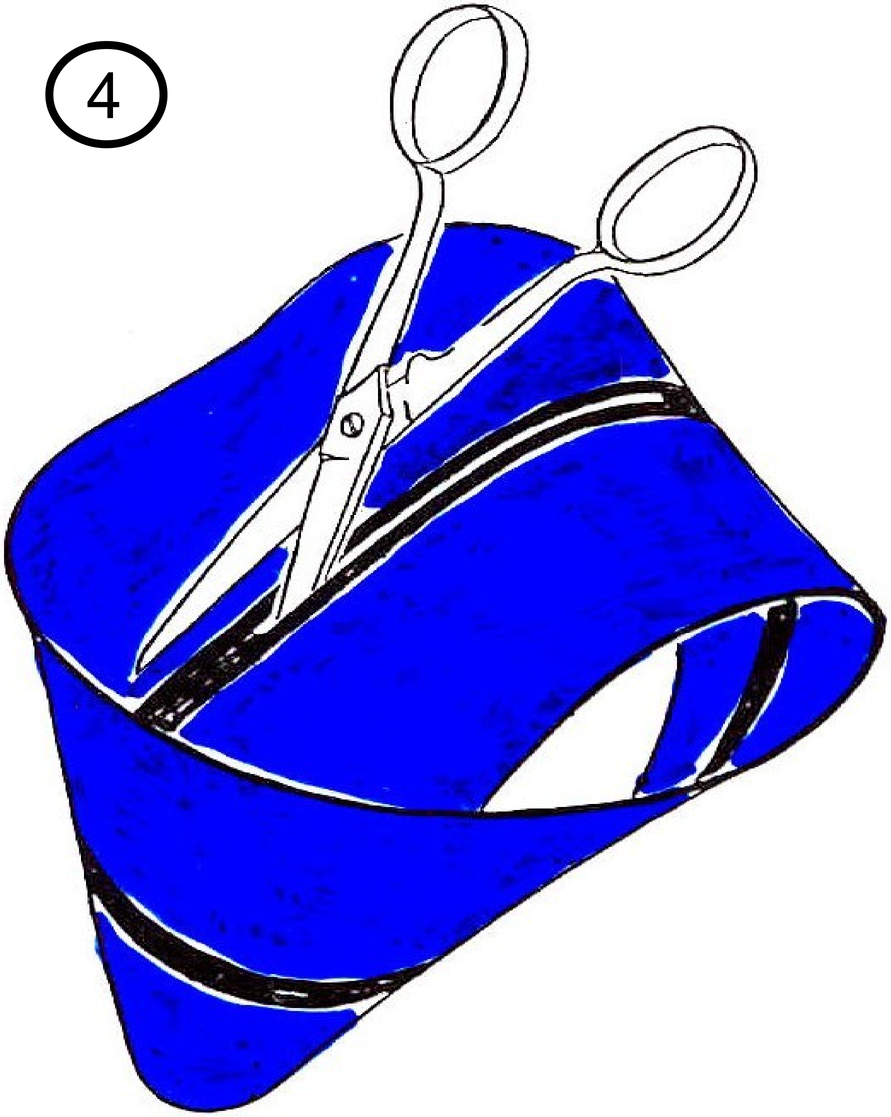
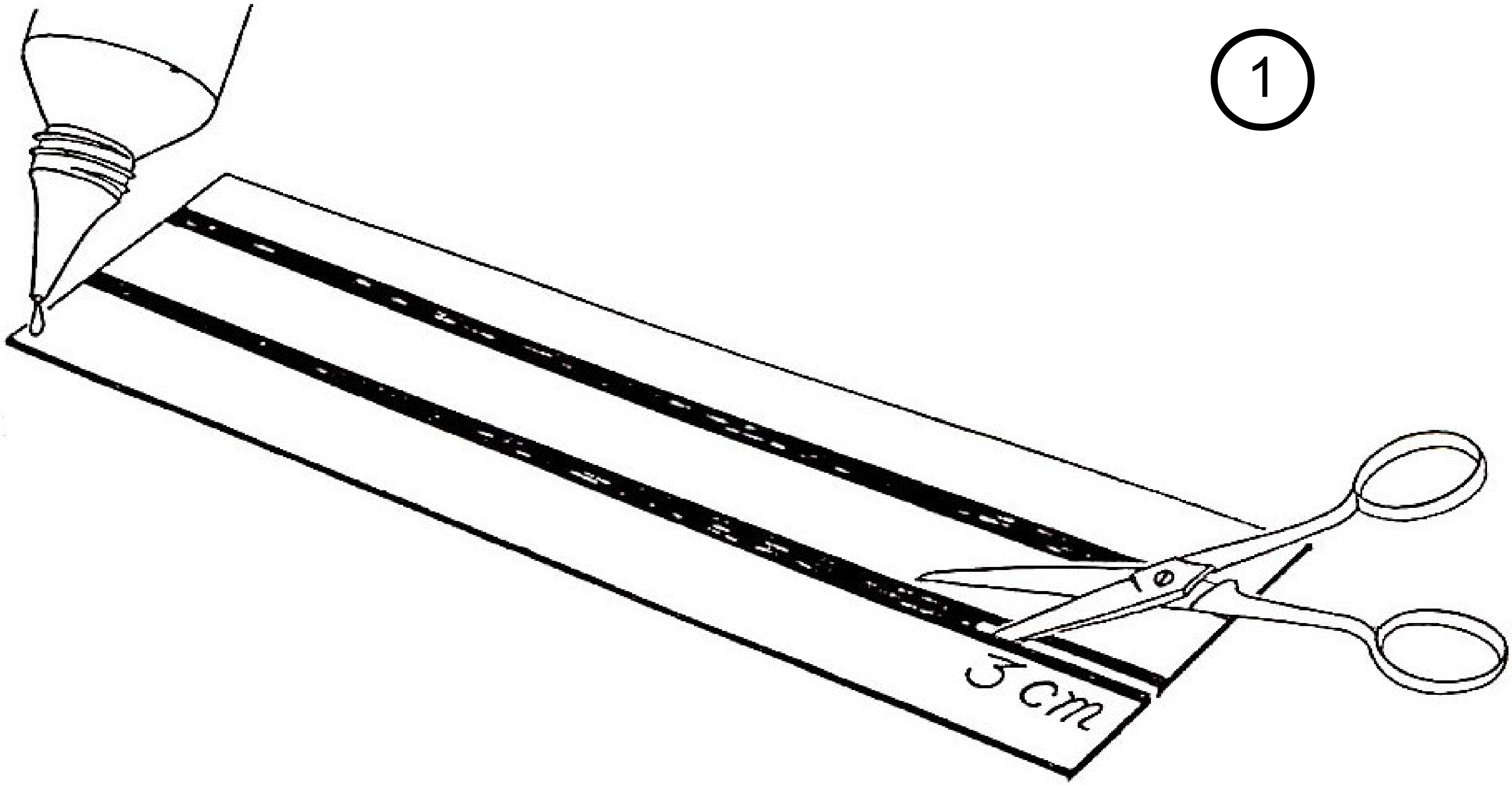


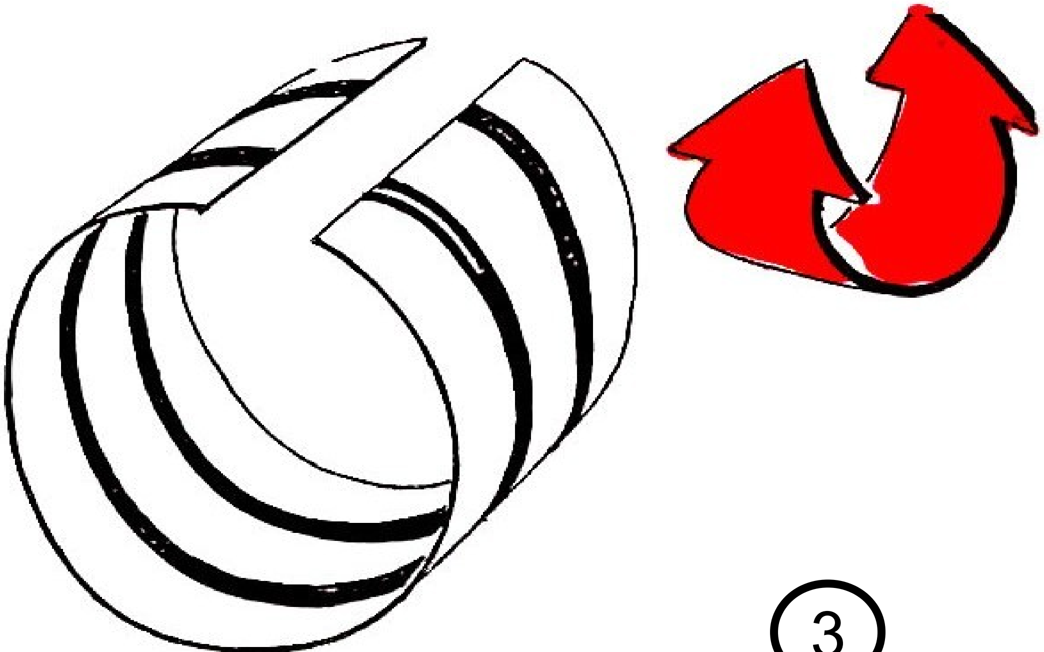


?

Guess first what comes out – then start cutting !

Now, a bit more complicatet





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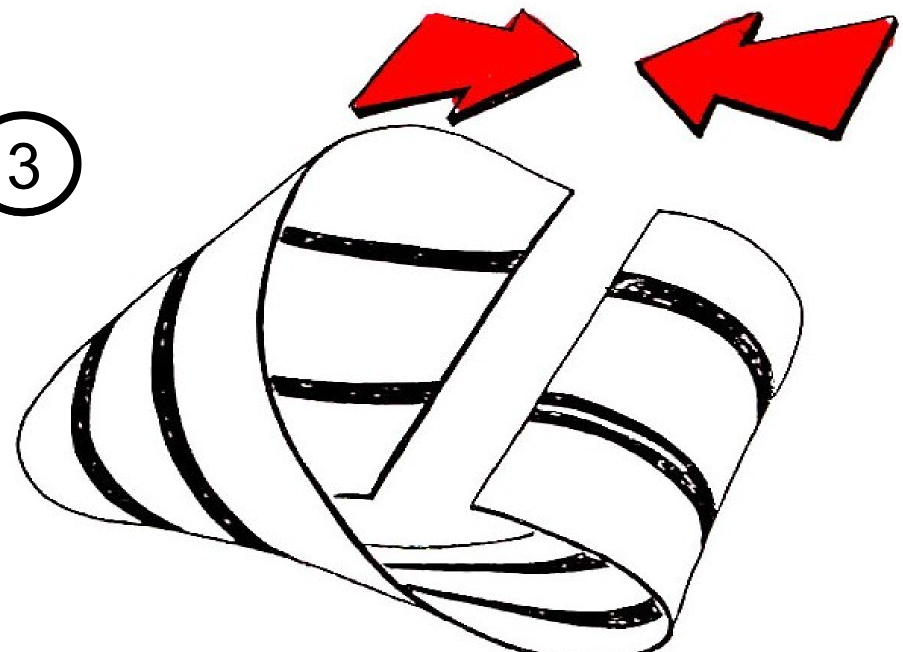
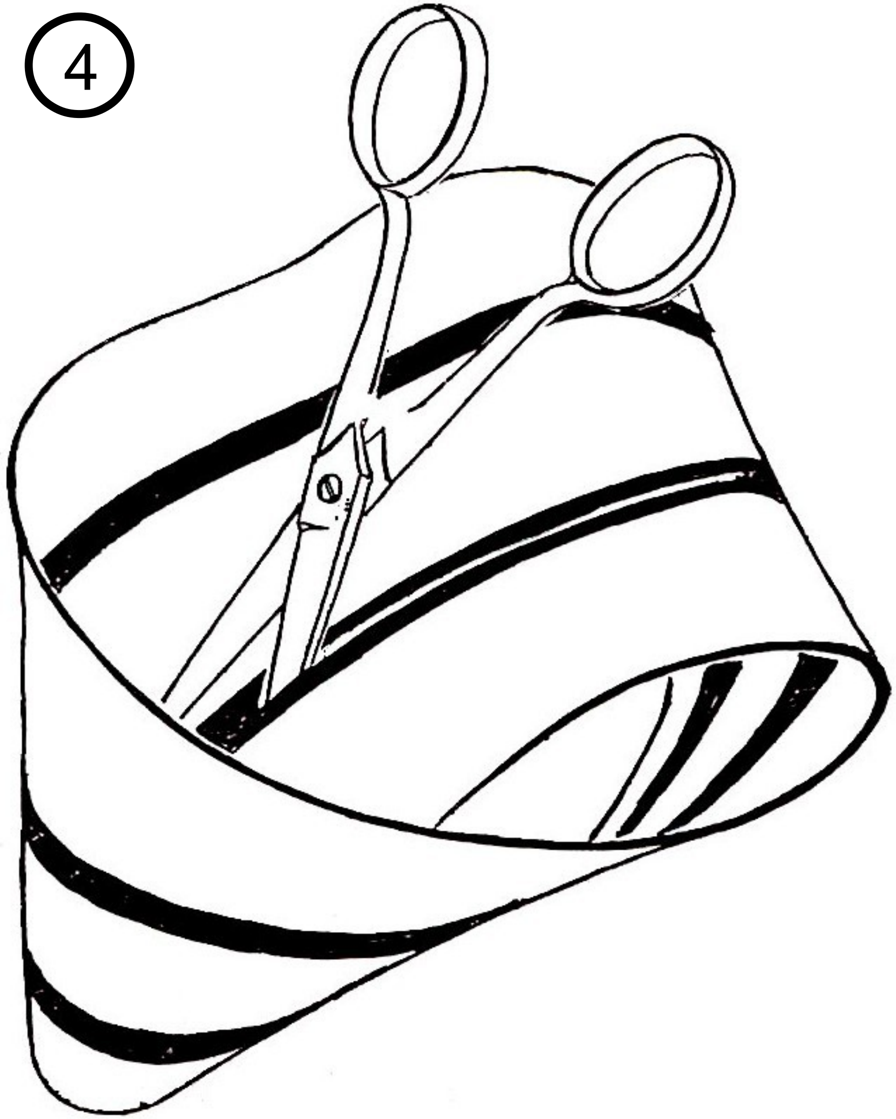


④



?

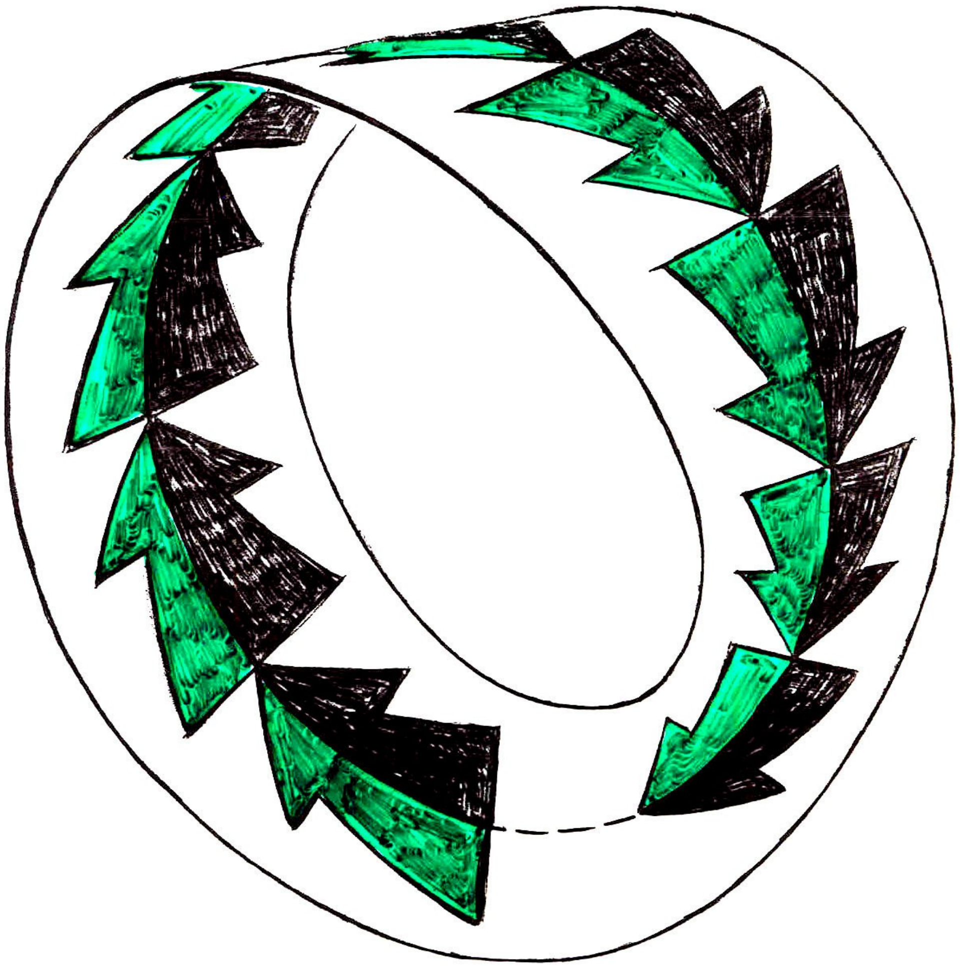
Here, too: first guess, then cut !

The Moebius strip is one-sided

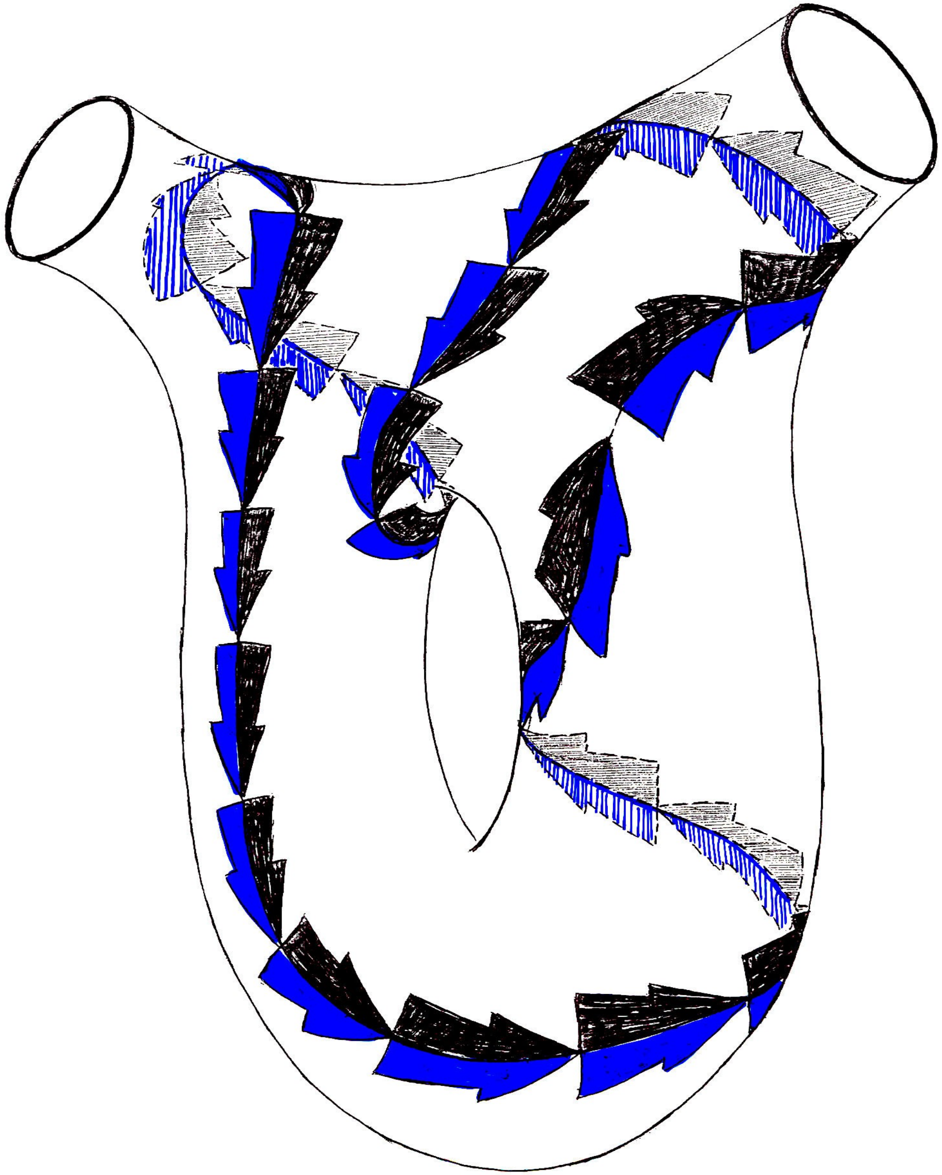
Each try, to paint the „two sides“ of the Moebius strip with different colours leads as into troubles...

The Moebius strip is not orientable...

There is a closed path on the Moebius strip such that a small geometric figure returns with reverse orientation, if moved along this path....



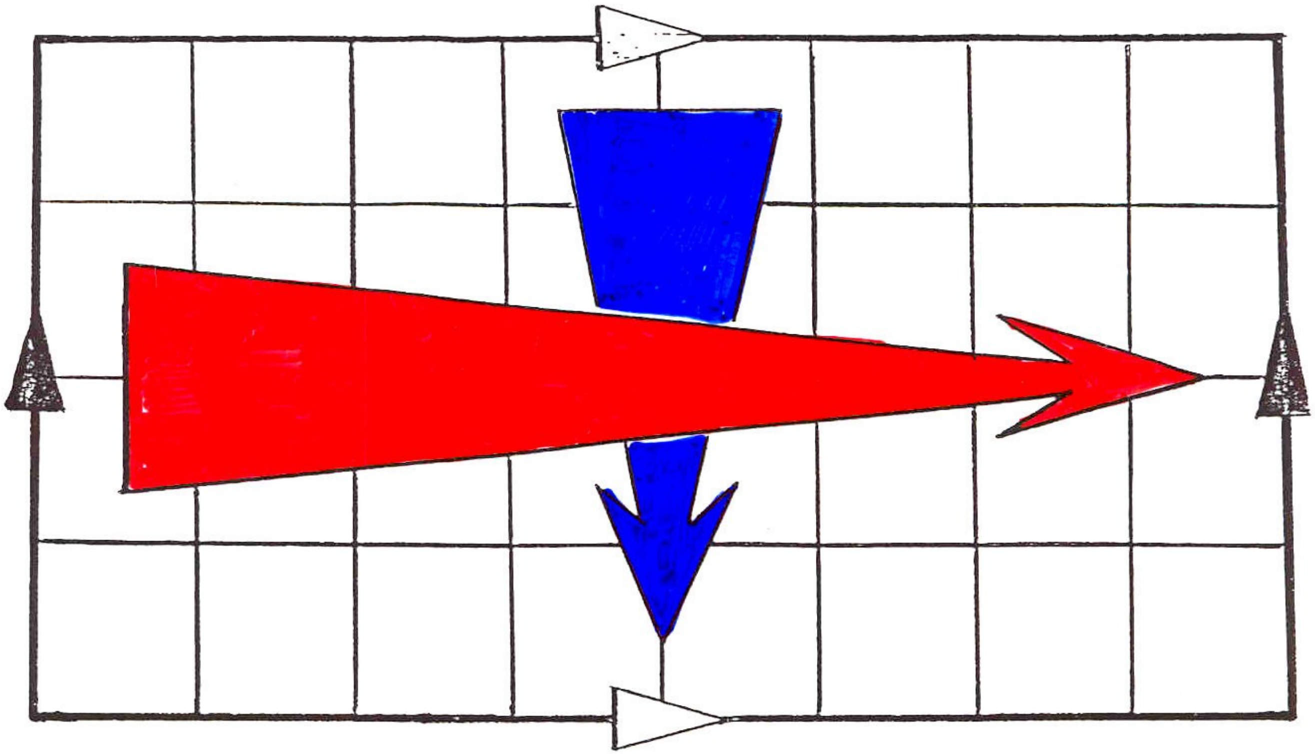
An ordinary band is orientable...



This surface is orientable, too.

We produce a new surface by though...

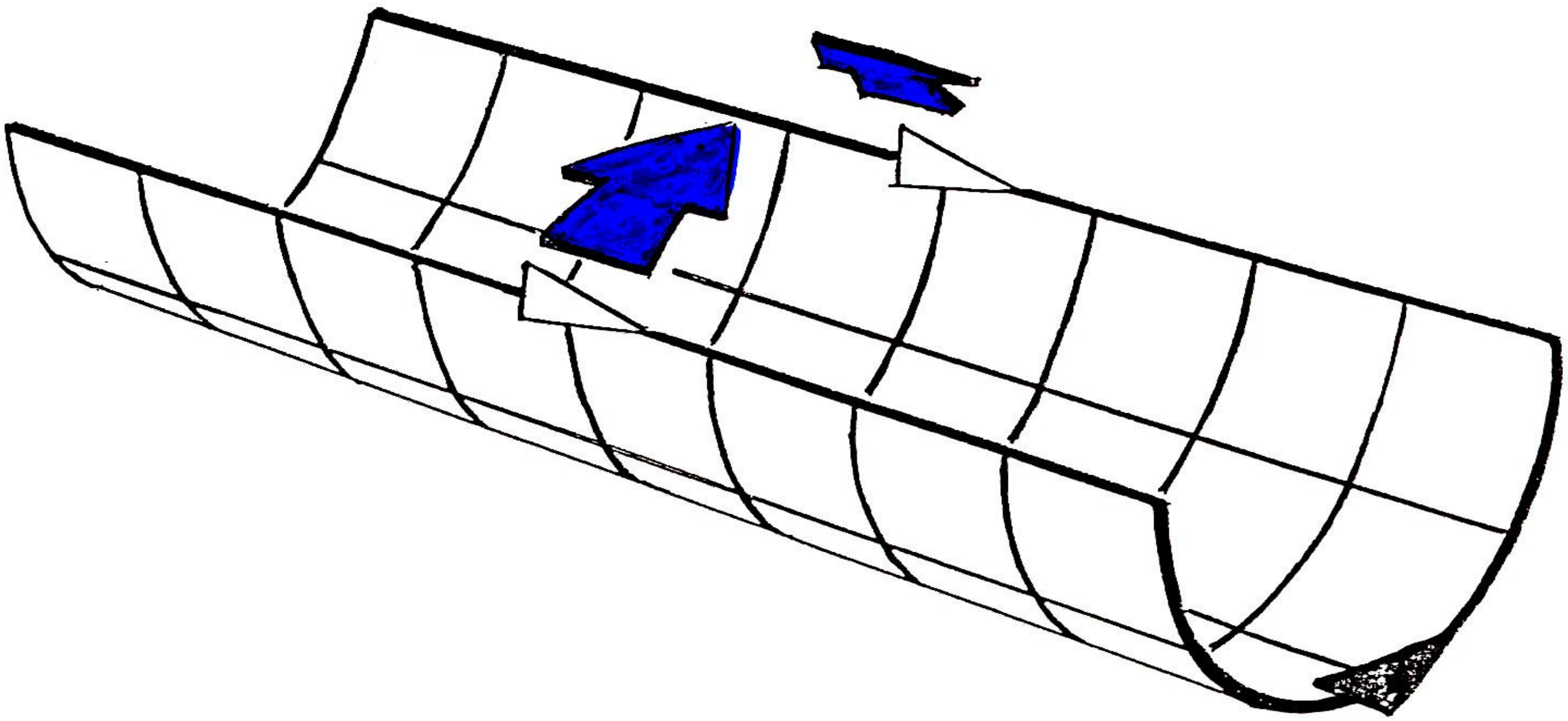
The opposite edges of the rectangle are to be glued as indicated below



Hint: perform first the gluing indicated in blue colour !

1

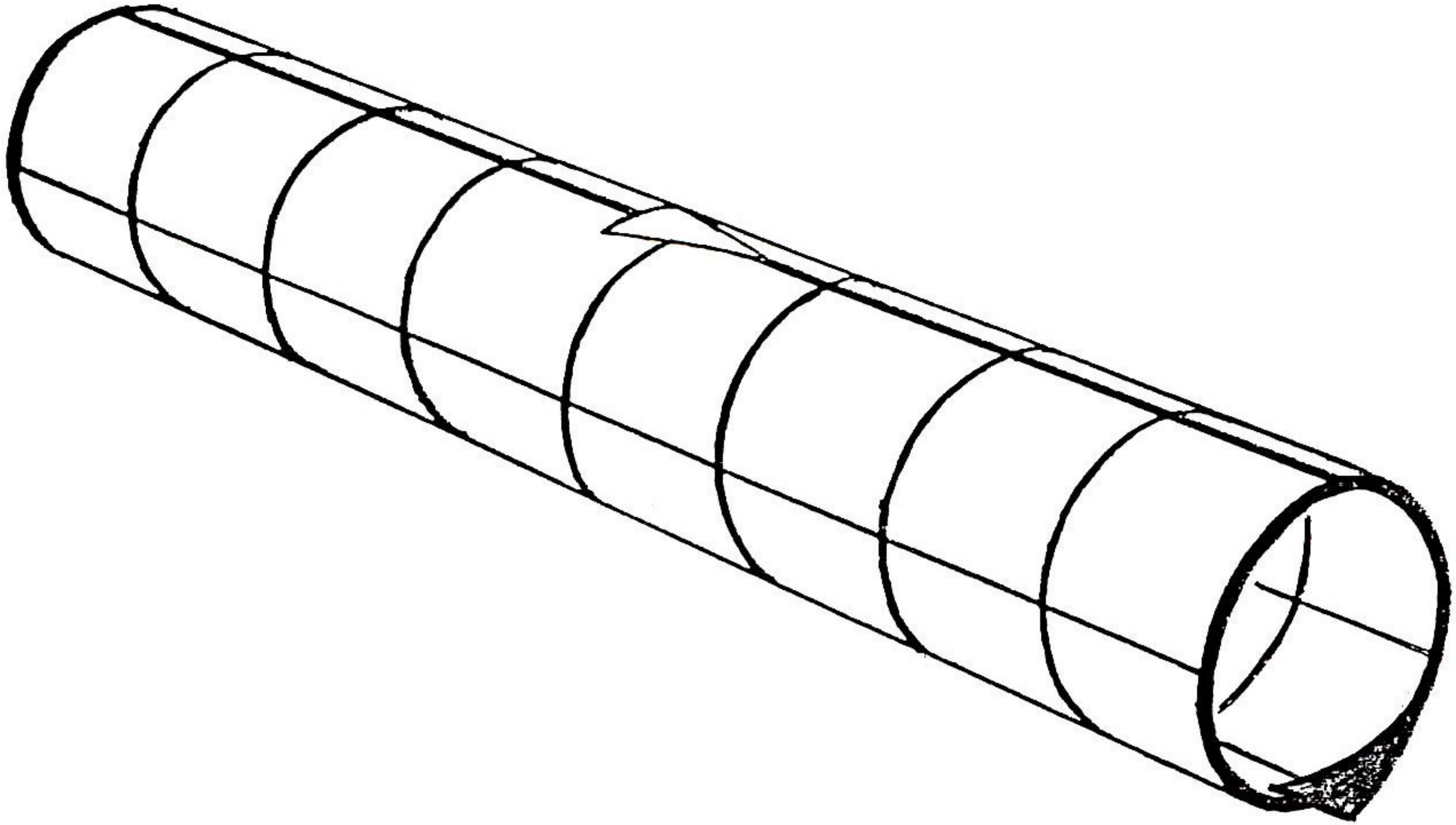
a half tube...



Can you already guess what comes out ?

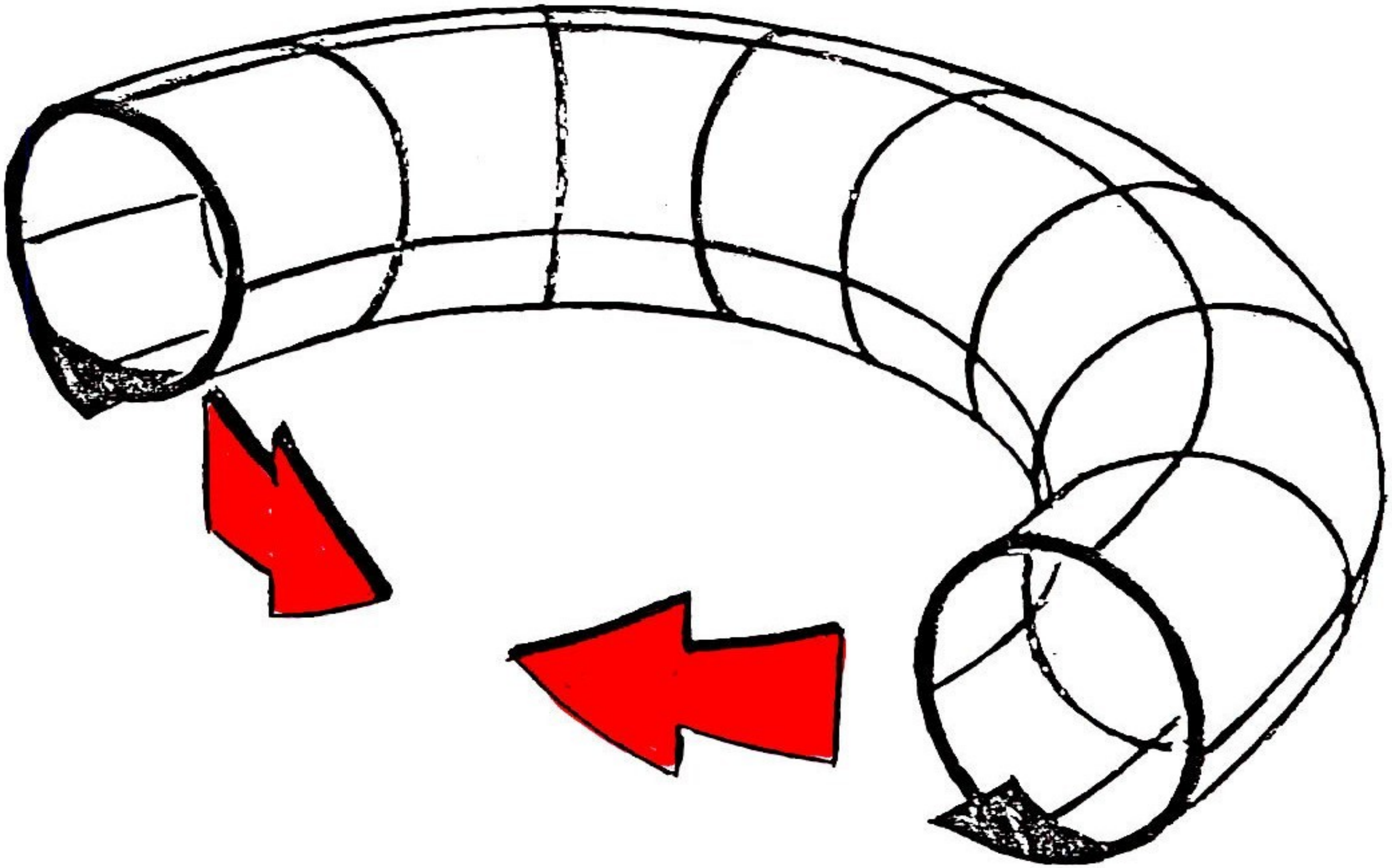
2

a tube...



3

A bended tube...



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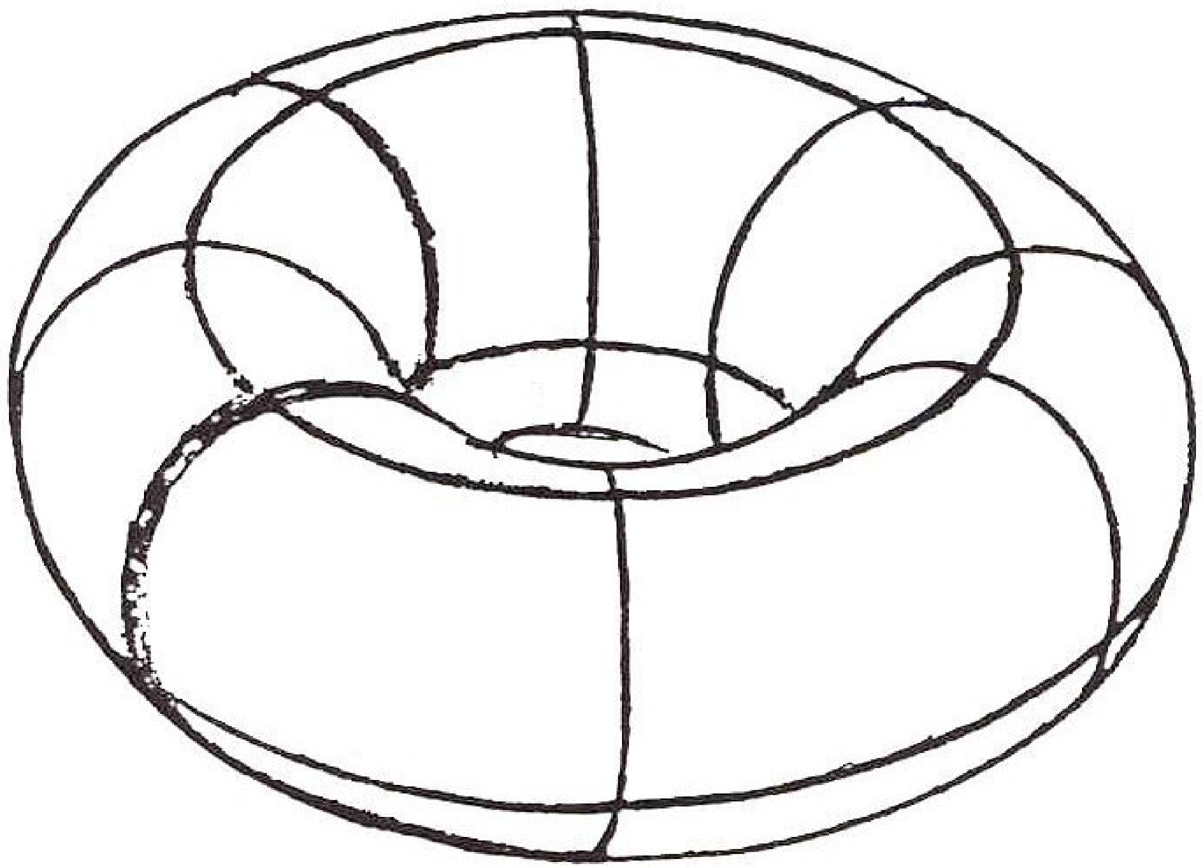
4 ...a „swimming tube"



... usually called **Torus** in Mathematics.

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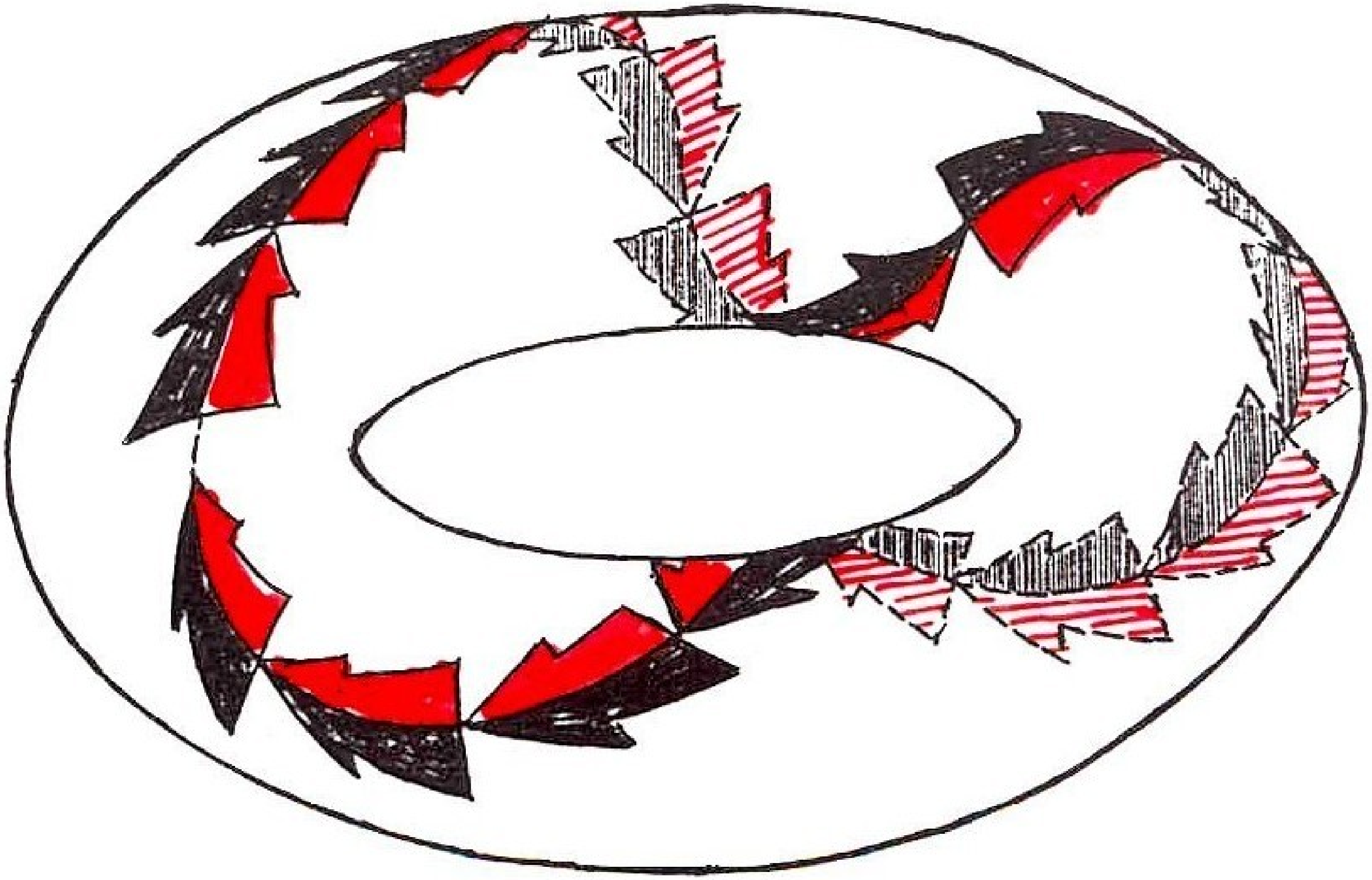
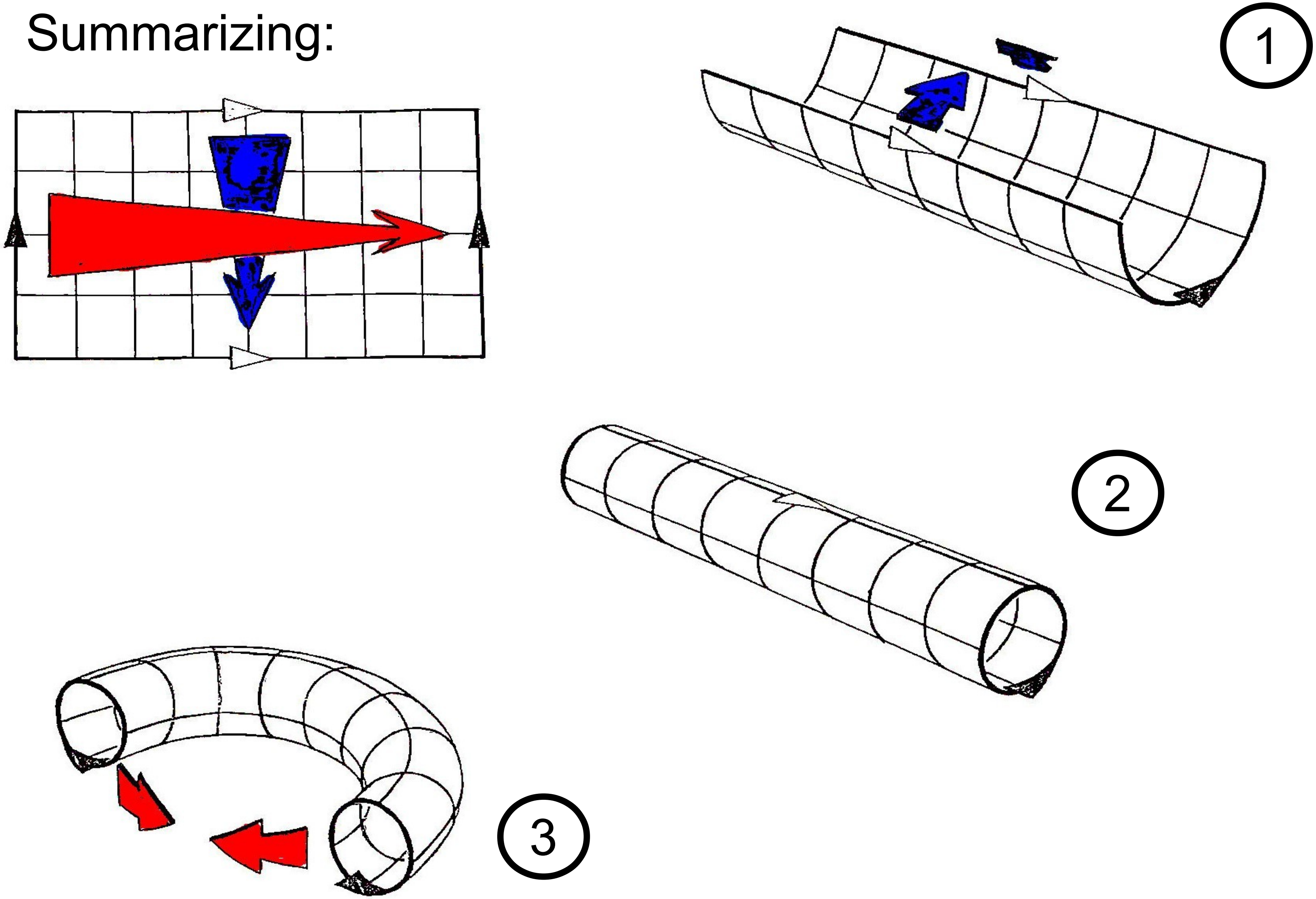
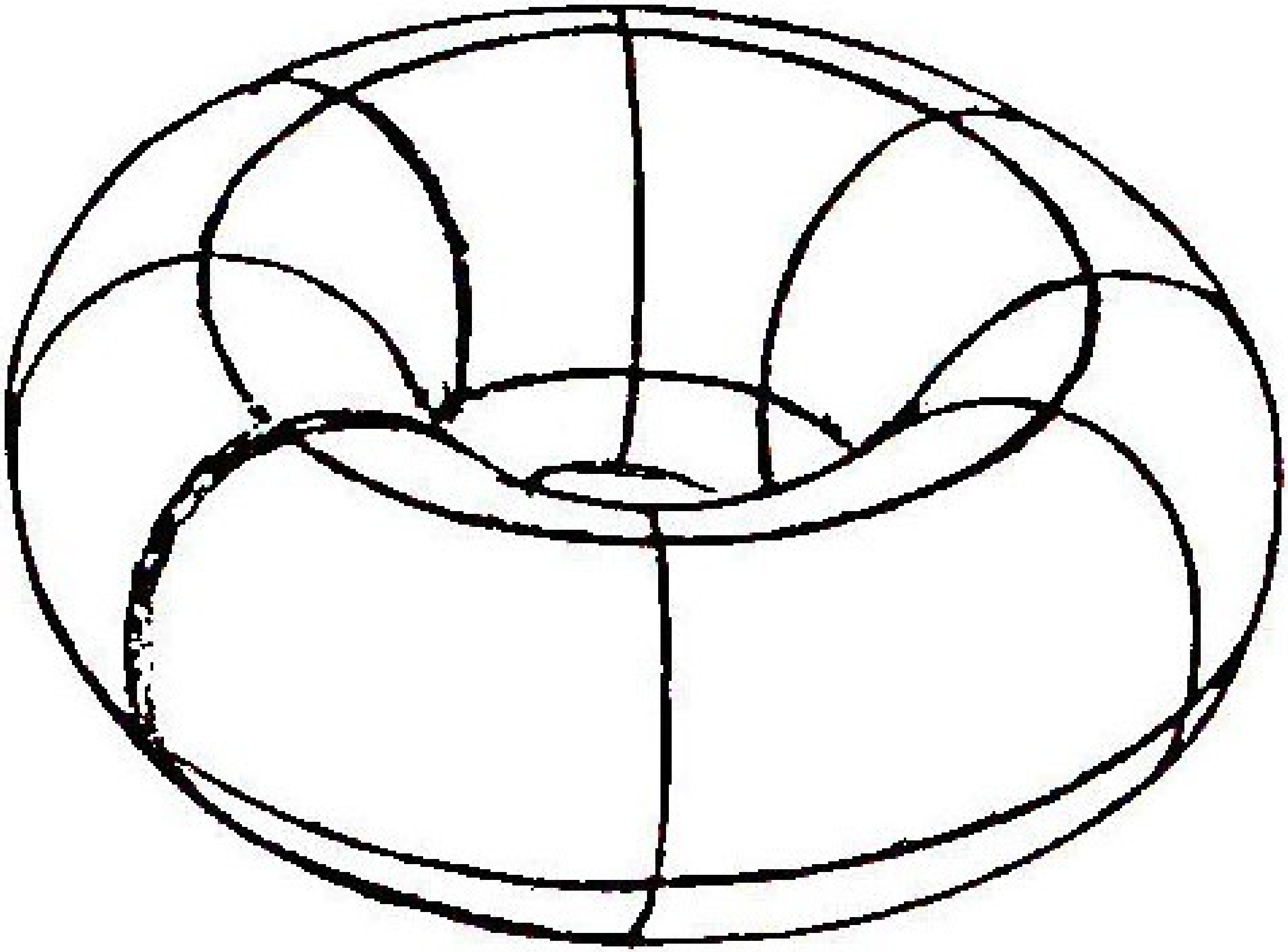


The torus is a closed and orientable surface !

Summarizing:

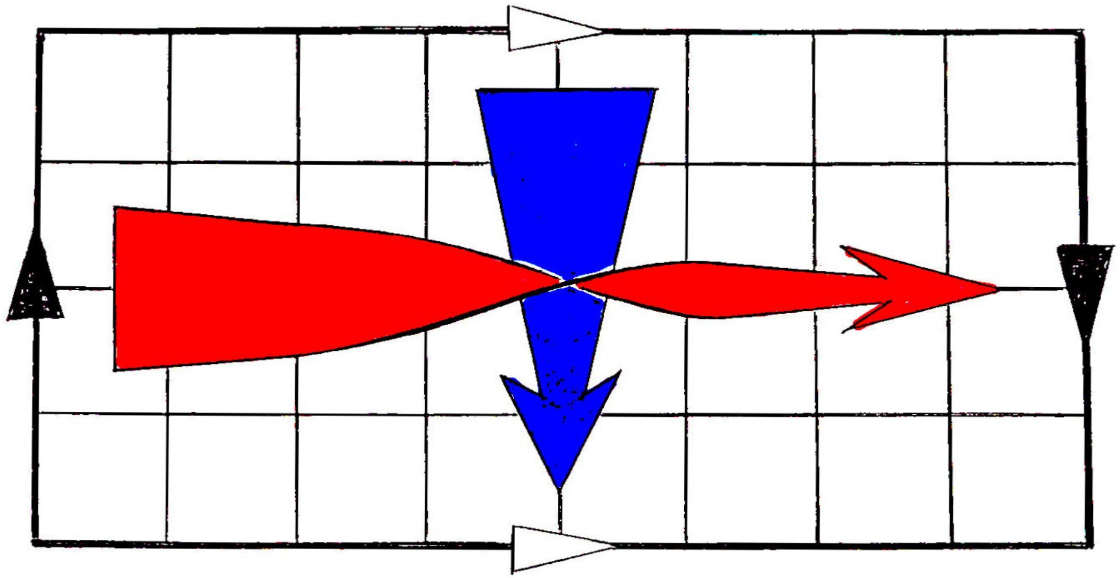


4



A Glance Beyond 3-Space...

We glue the opposite edges of a rectangle as indicated below

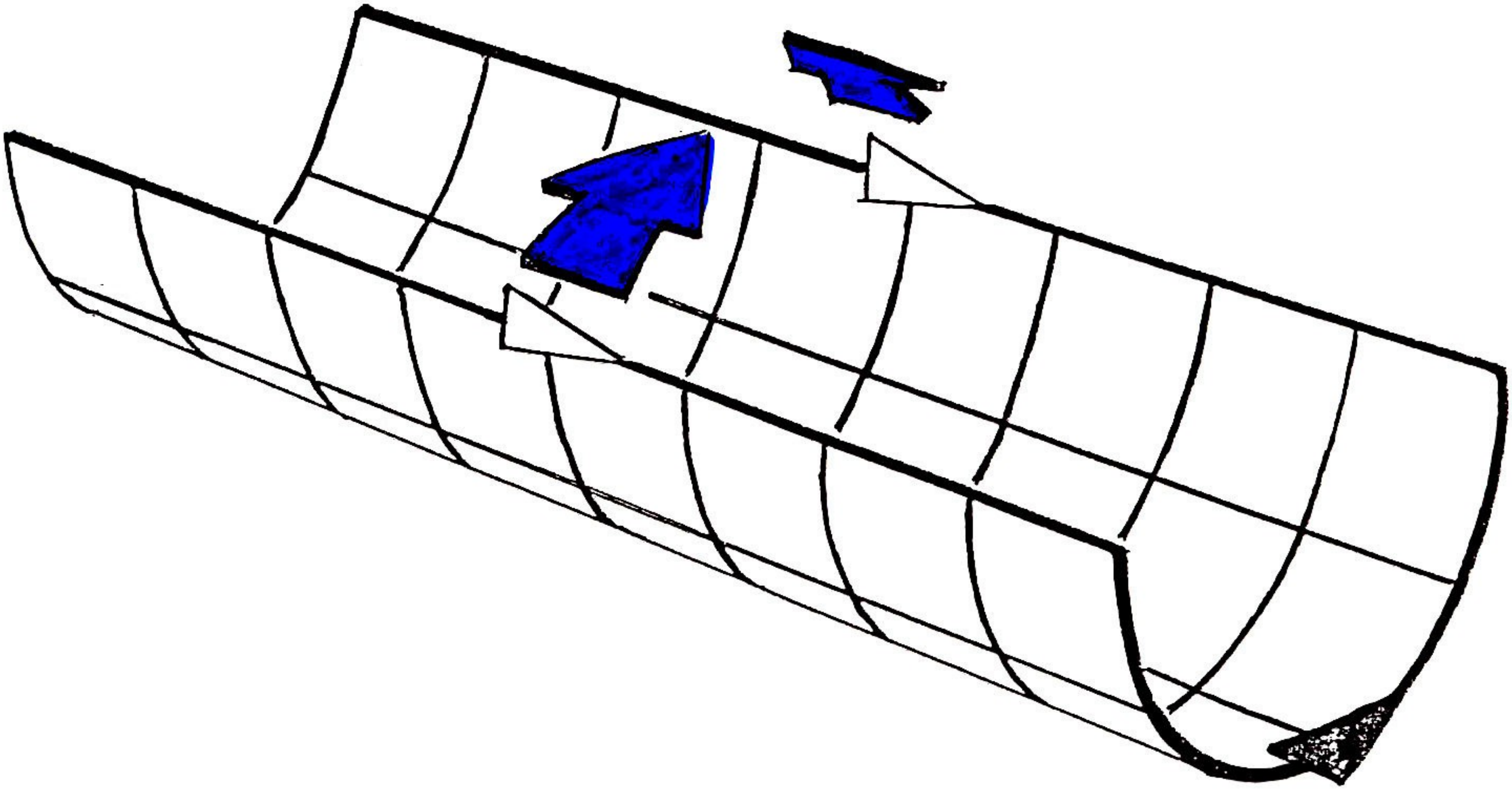


Observe the difference to the Torus case: one of the shorter edges is reversed before it is glued to its opponent !

Hint: perform first the gluing indicated in blue colour !

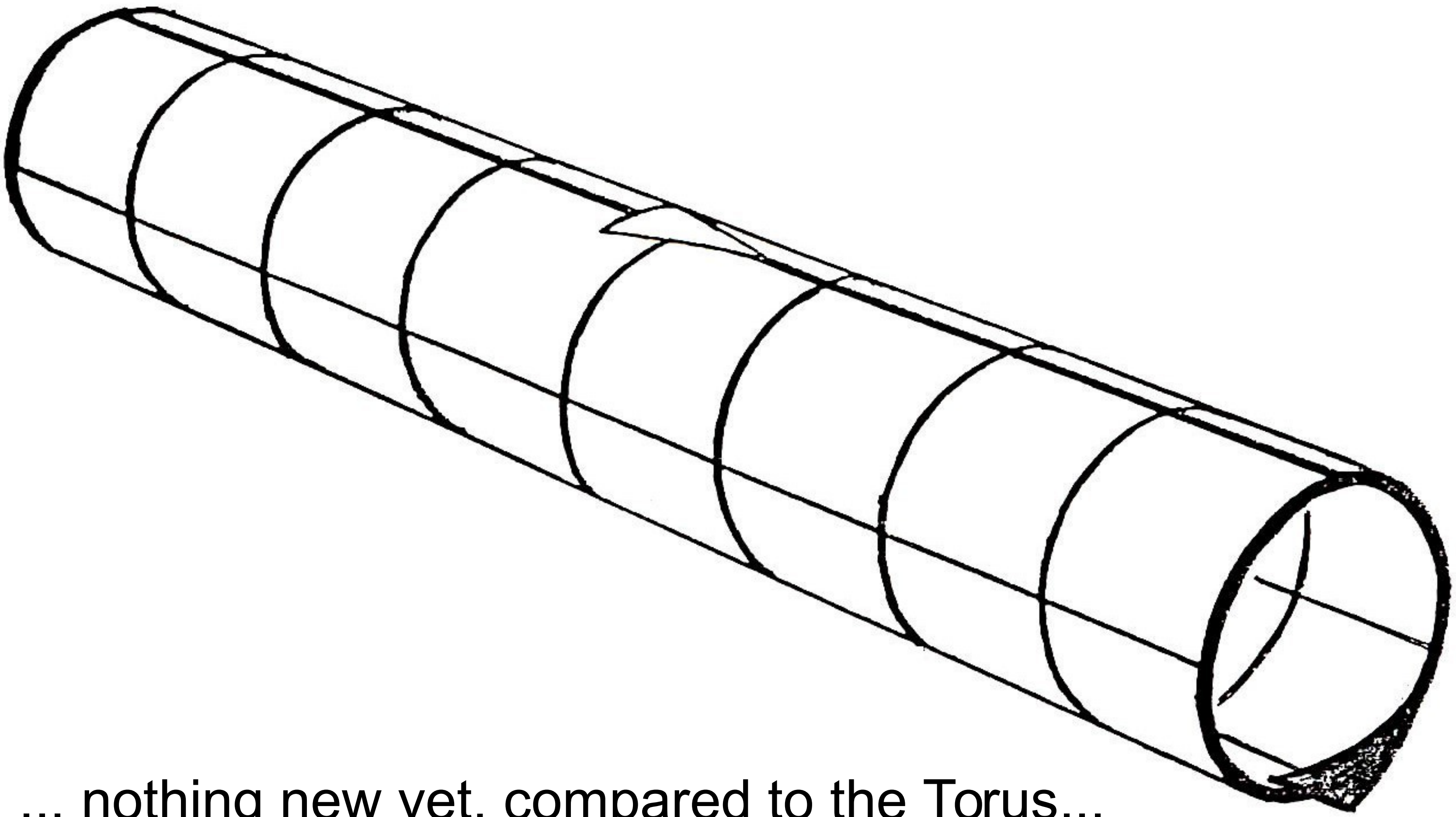
1

A half tube...



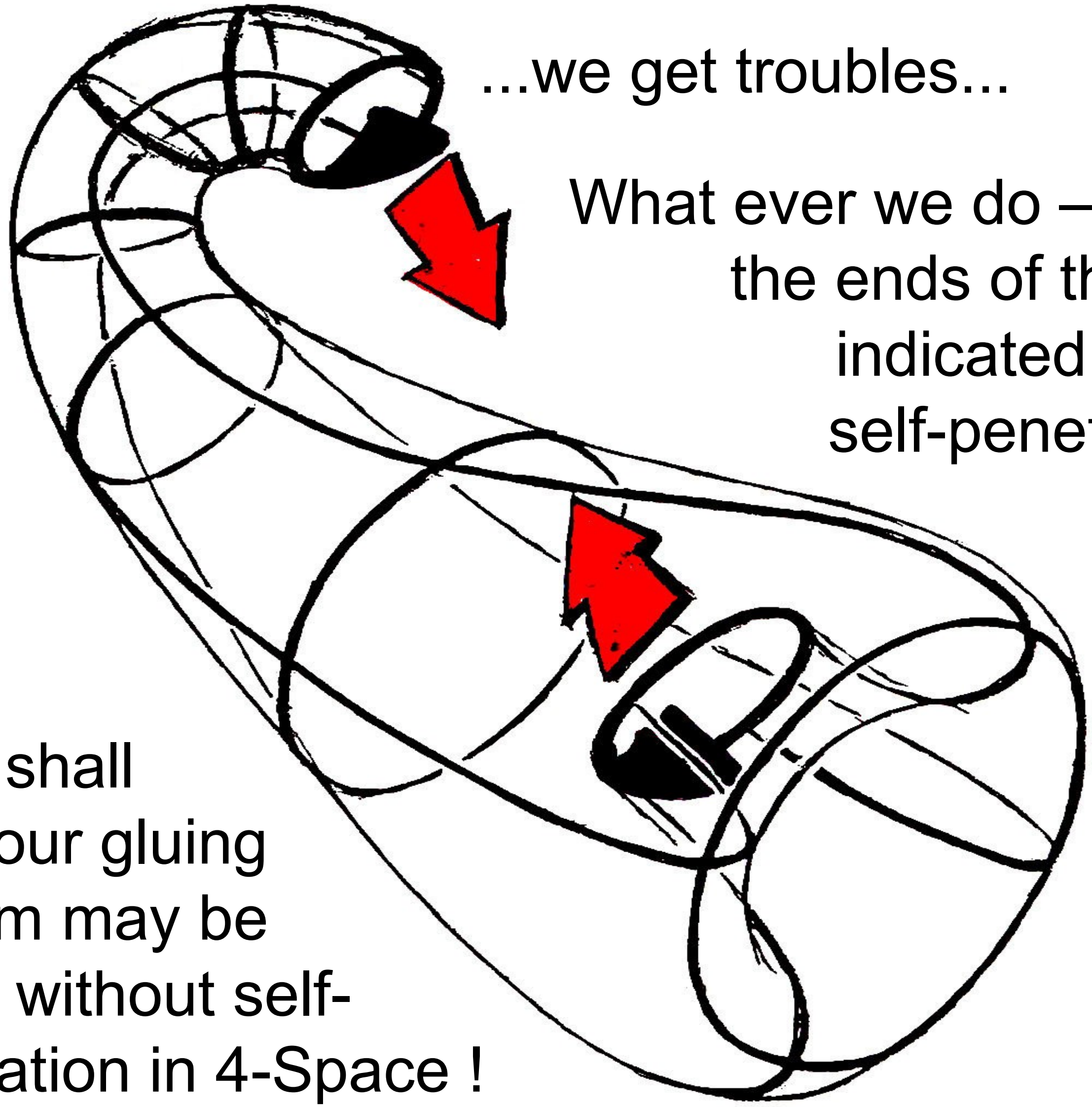
2

a tube...



... nothing new yet, compared to the Torus...

3



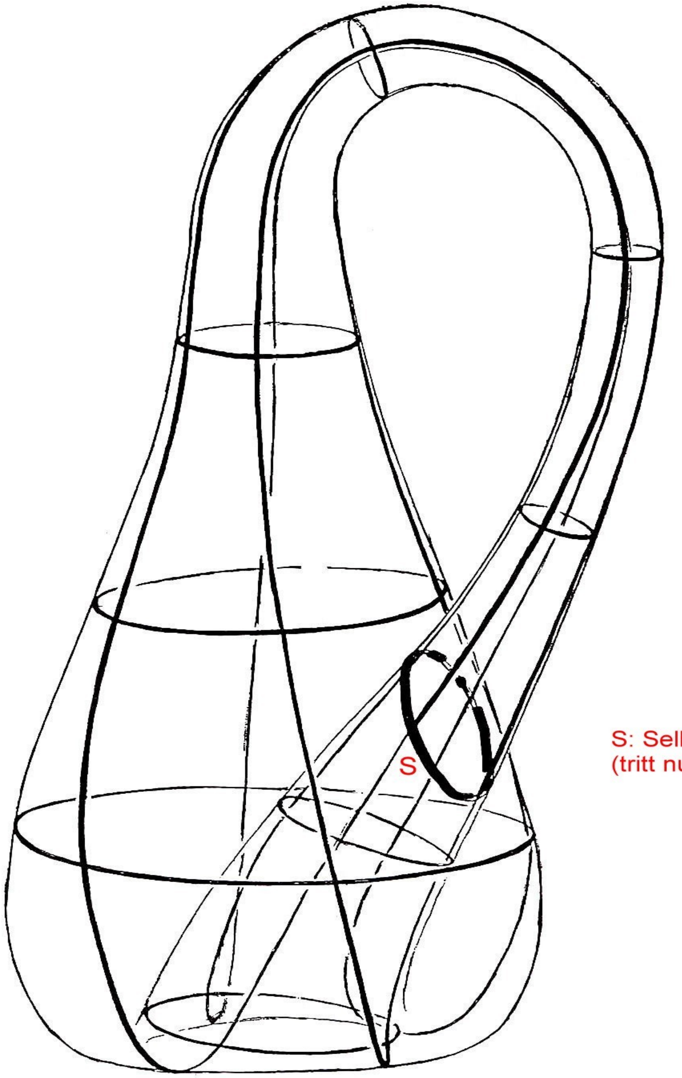
...we get troubles...

What ever we do – if we want to glue the ends of the bended tube as indicated, we must admit a self-penetration of our tube!

As we shall learn, our gluing problem may be solved without self-penetration in 4-Space !

(4) But for the moment, we go on in 3-Space...

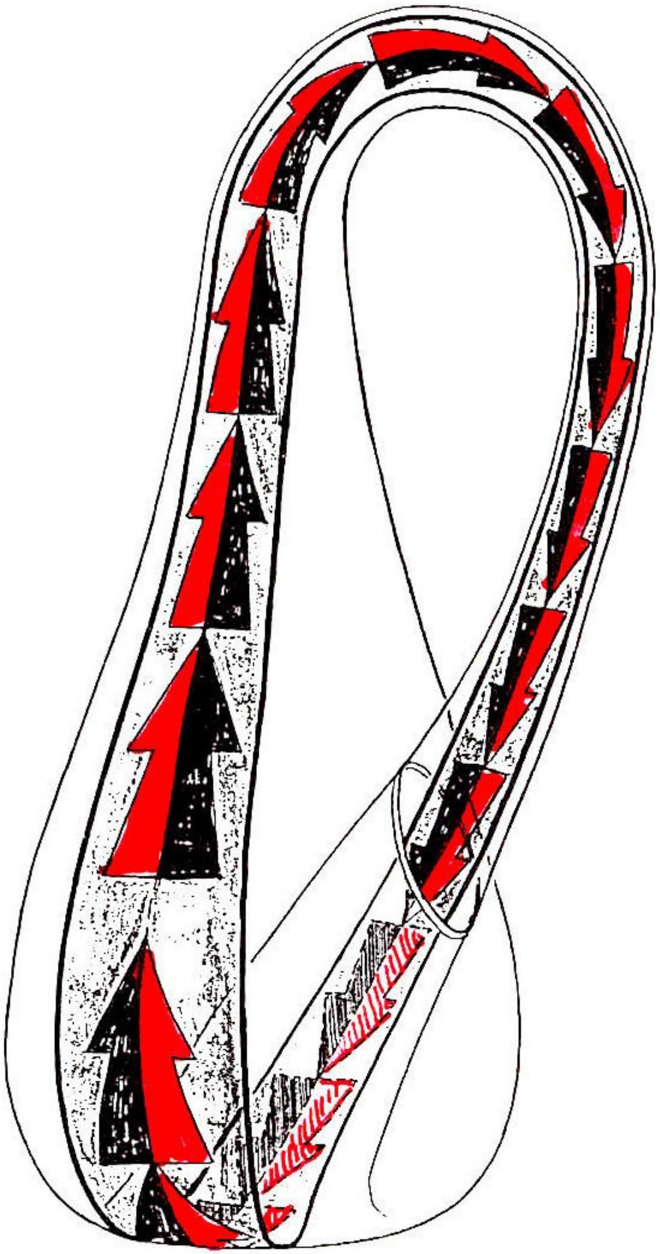
... and obtain a so-called
Model of a closed surface
which is embedded into 4-Space:



The Klein Bottle

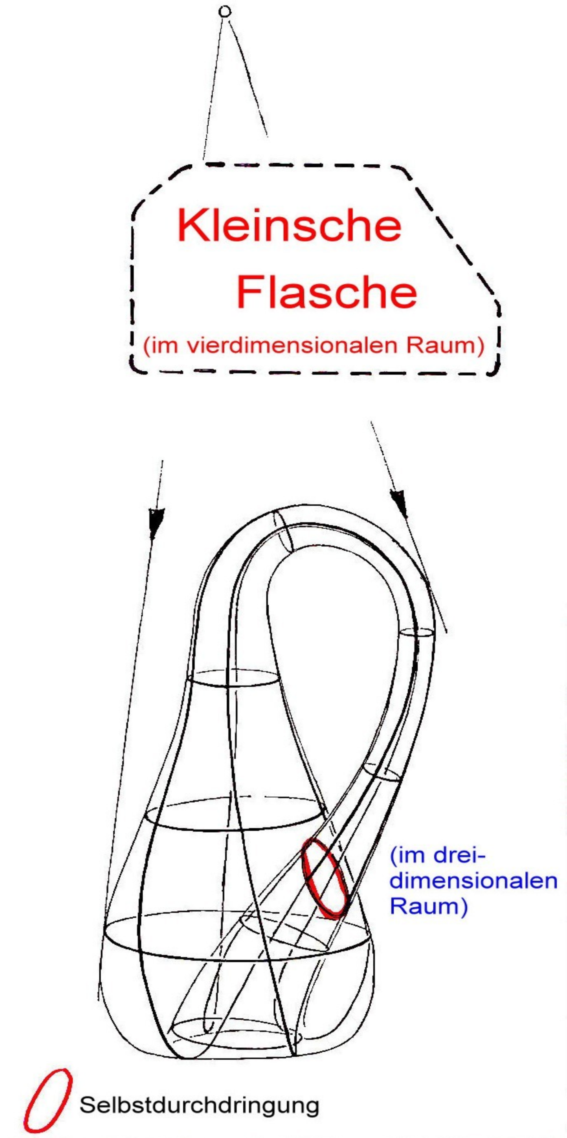
...named after:

Felix Klein
1849 – 1924
Mathematcian in Göttingen, Germany



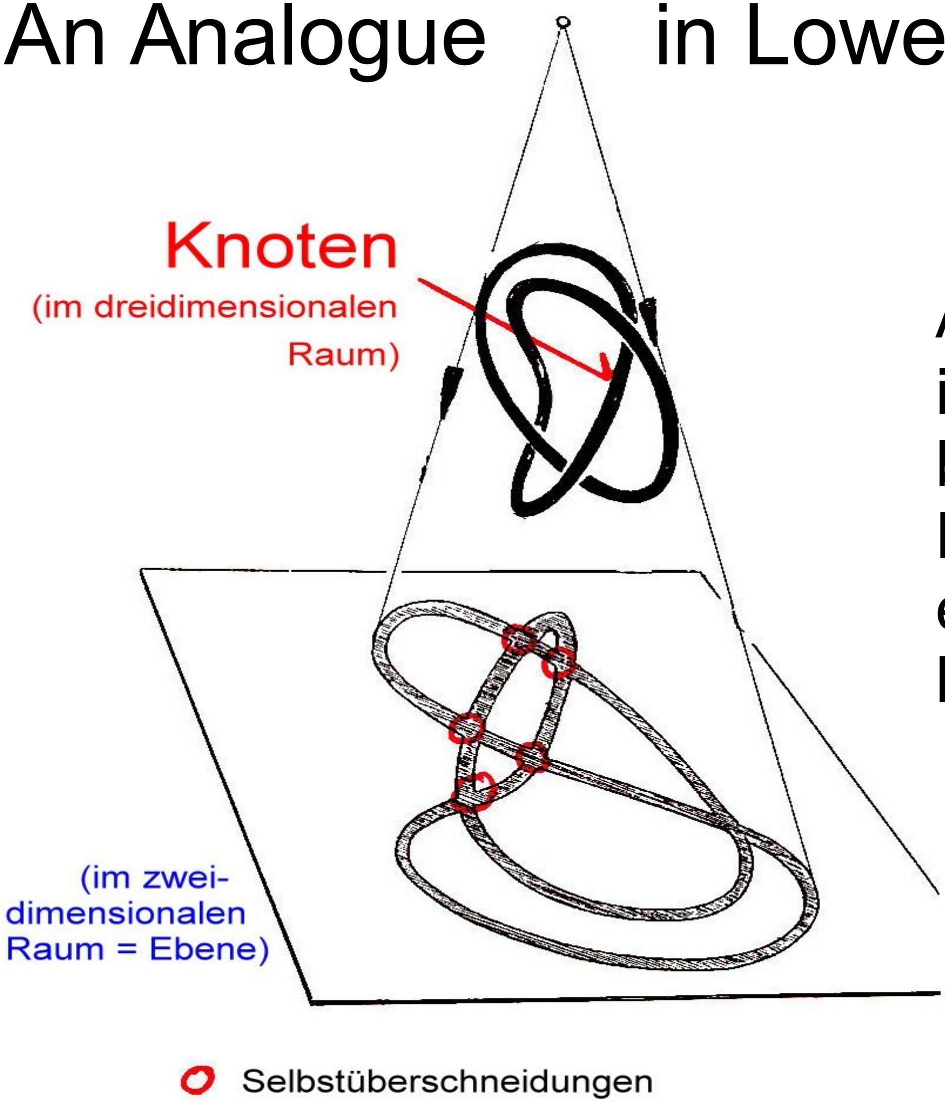
The Klein Bottle is not orientable...

... the model shows, that the Klein Bottle contains a Moebius strip !



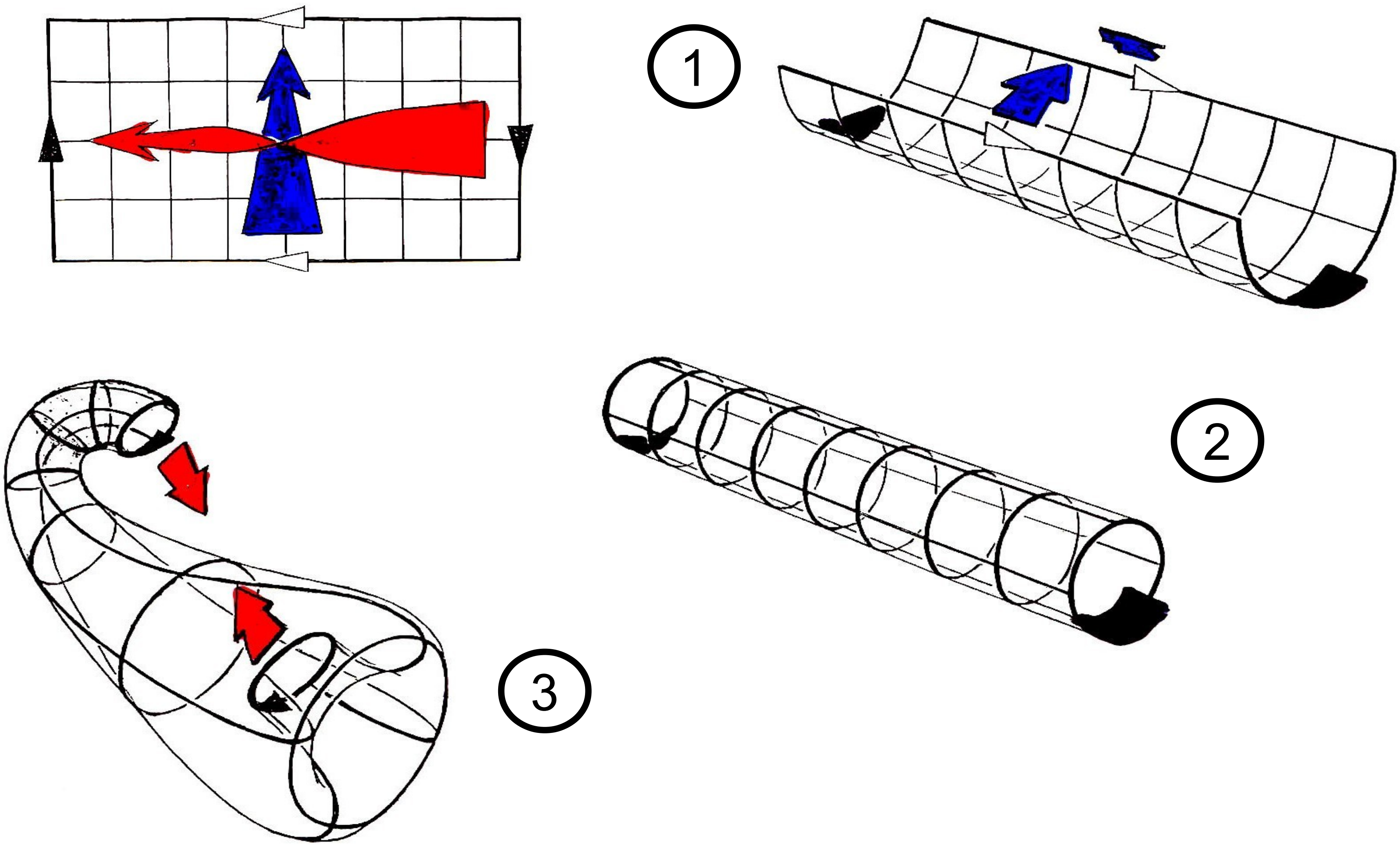
A Theorem says: A closed non-orientable surface (as the Klein Bottle for example) cannot be properly realized in 3-Space: all models of such a surface in 3-space must have self-penetrations.

An Analogue in Lower Dimensions...



A **Knot** is a closed curve in 3-Space, which cannot be properly realized in a 2-Plane:
each plane model of the knot has self-intersections.

Summarizing: the Making of the klein Bottle...



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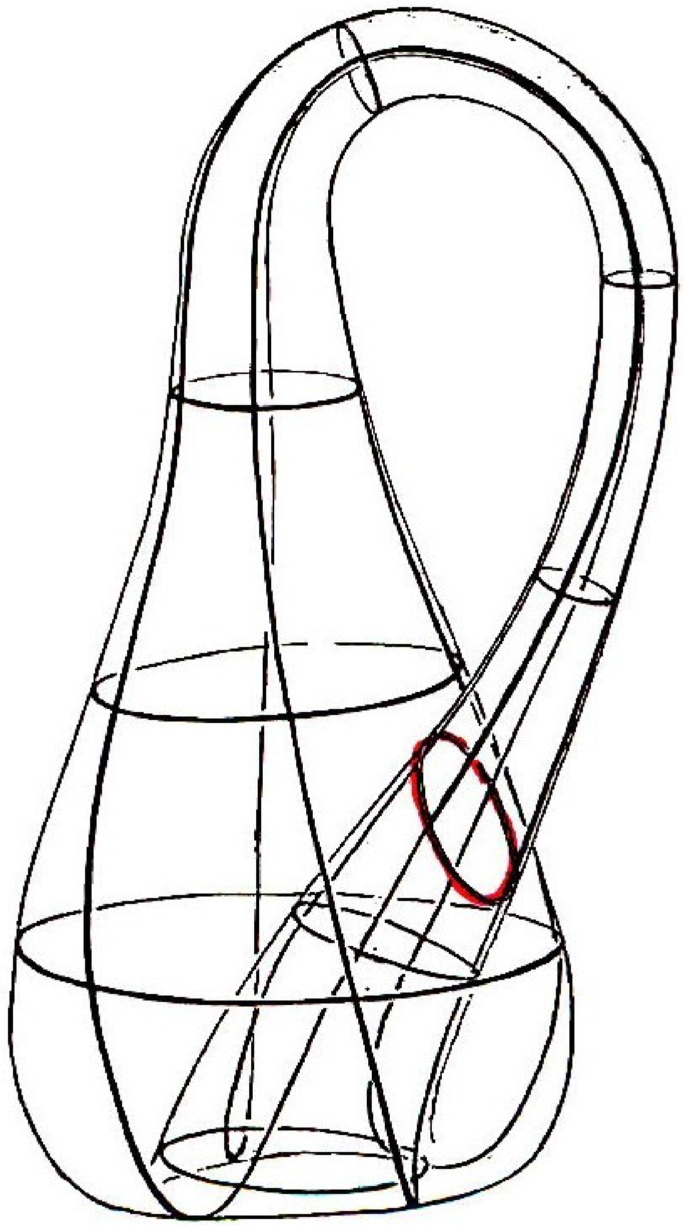
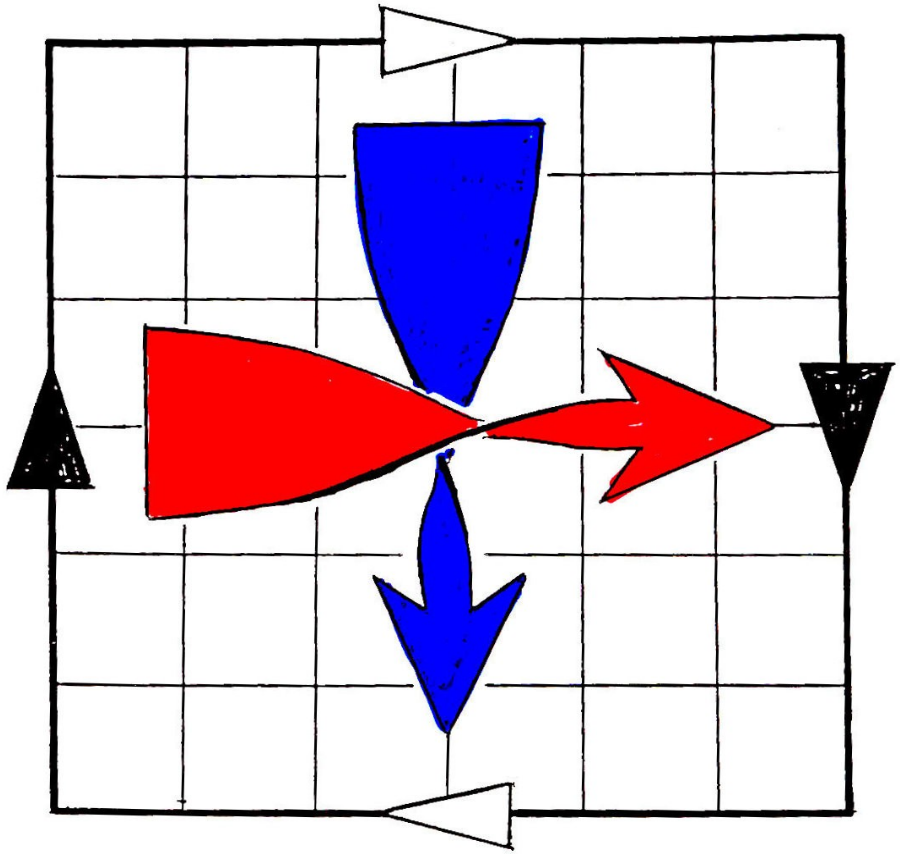
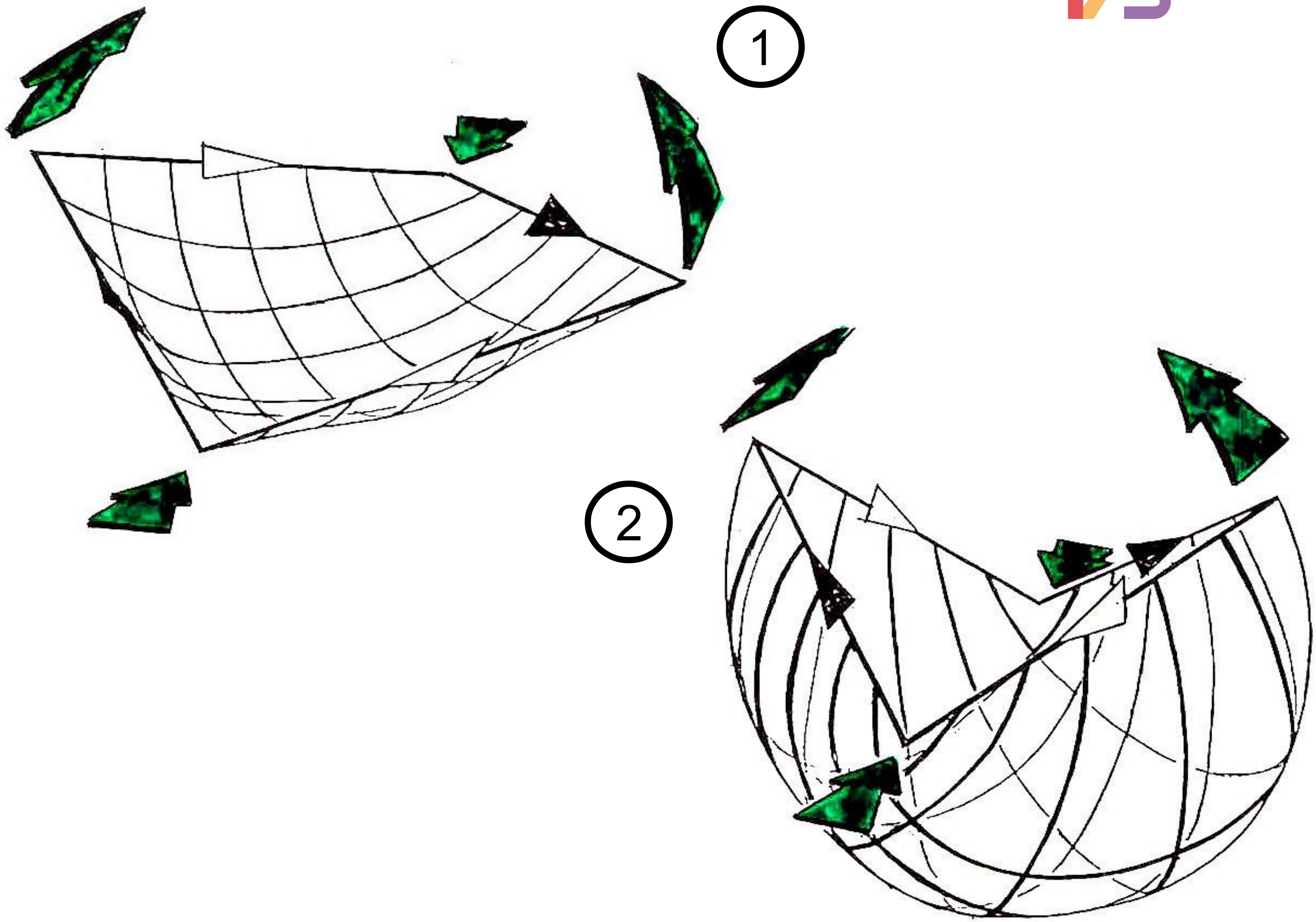


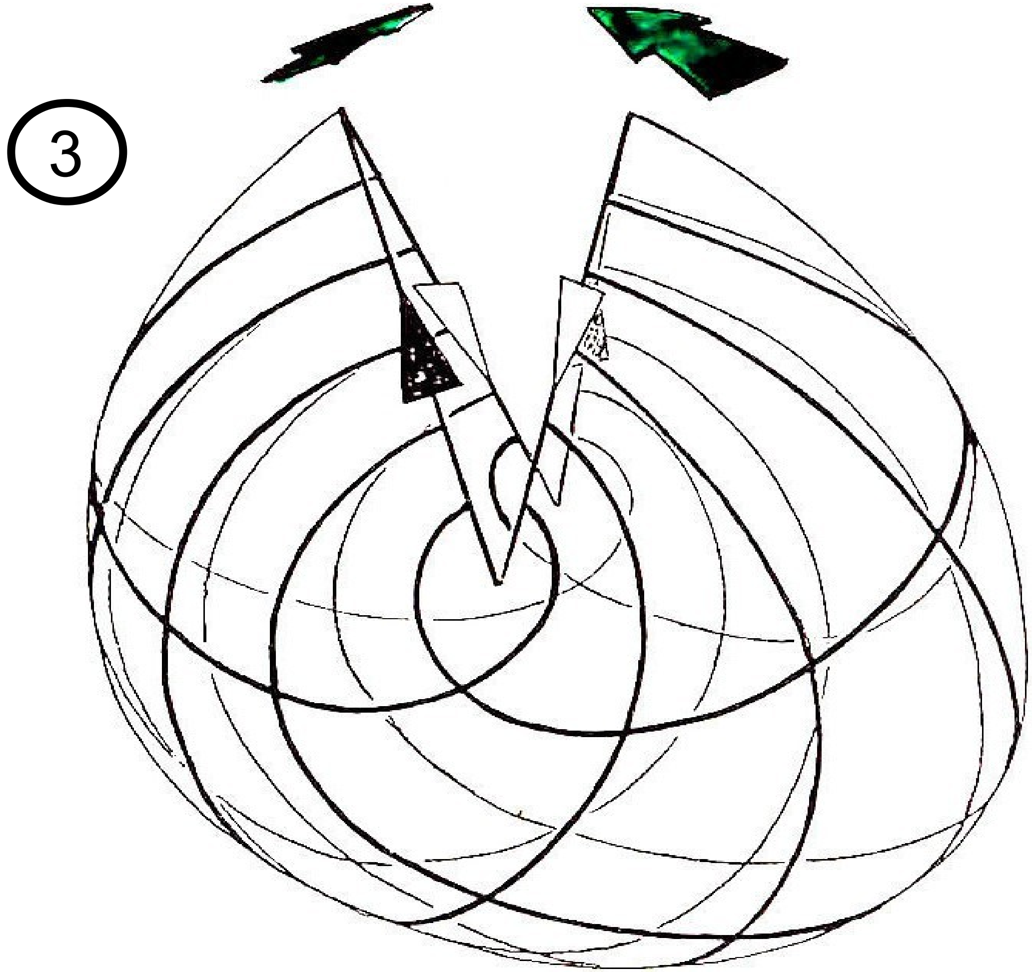
Another closed, non-orientable surface...

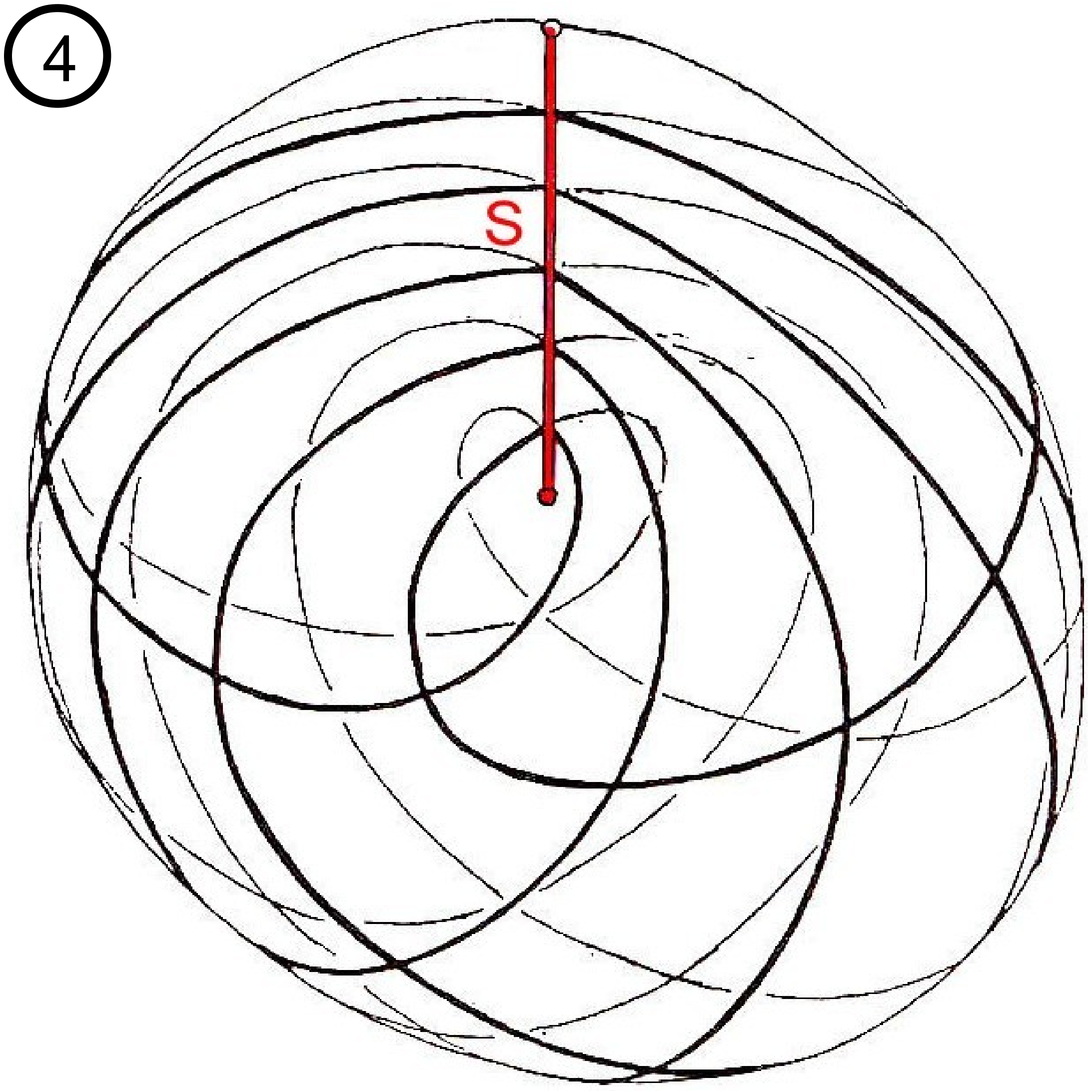
We glue the opposite edges of a square as indicated:




1
2

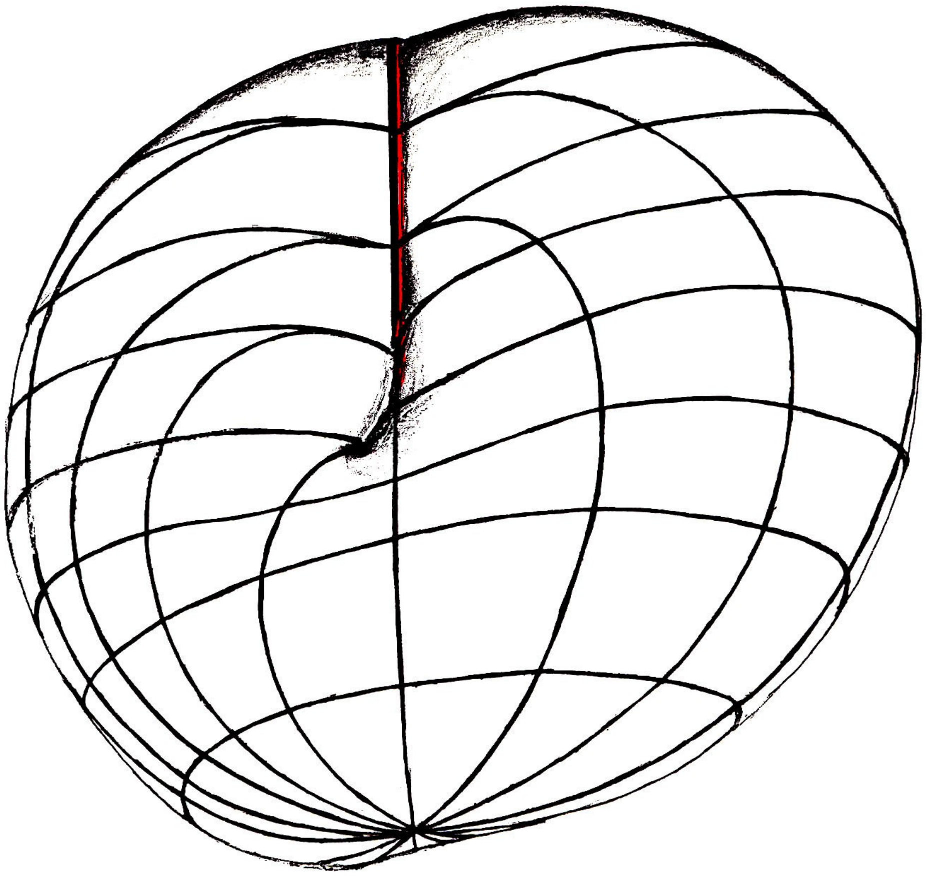
... and have again troubles at step 3: an un-avoidable self-penetration.





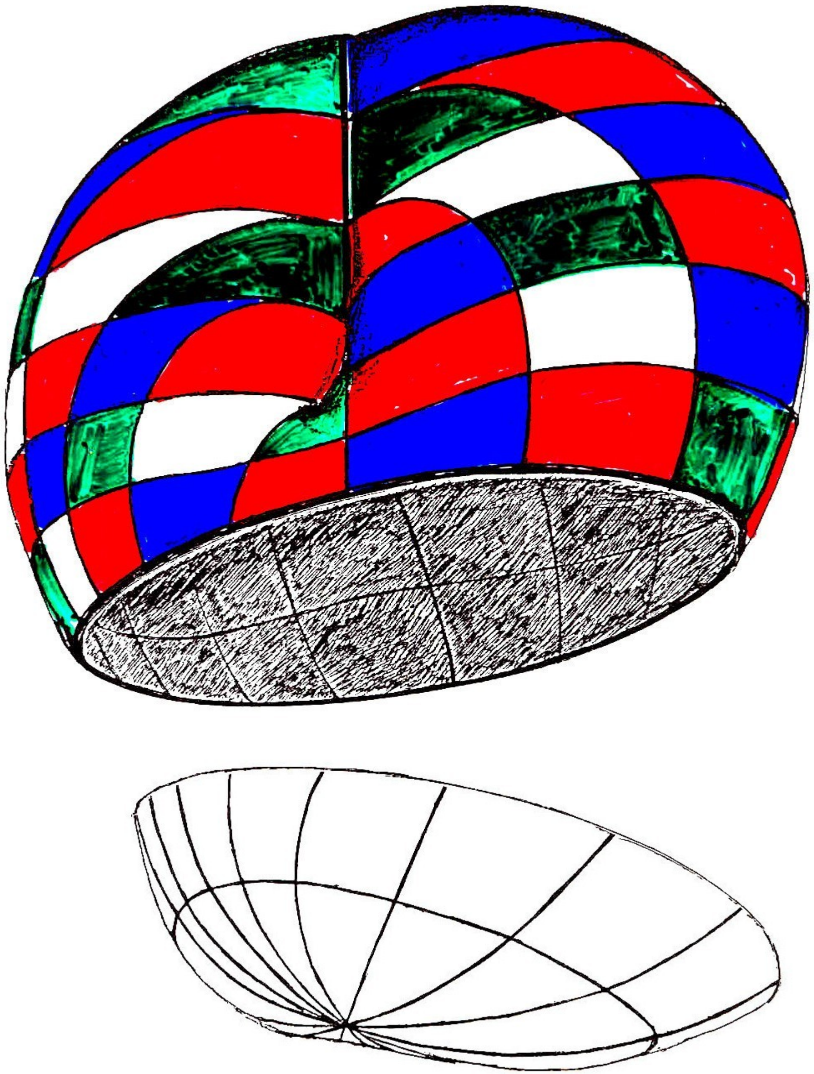
If we admit the self-penetration, we get a closed surface at step 4 ...

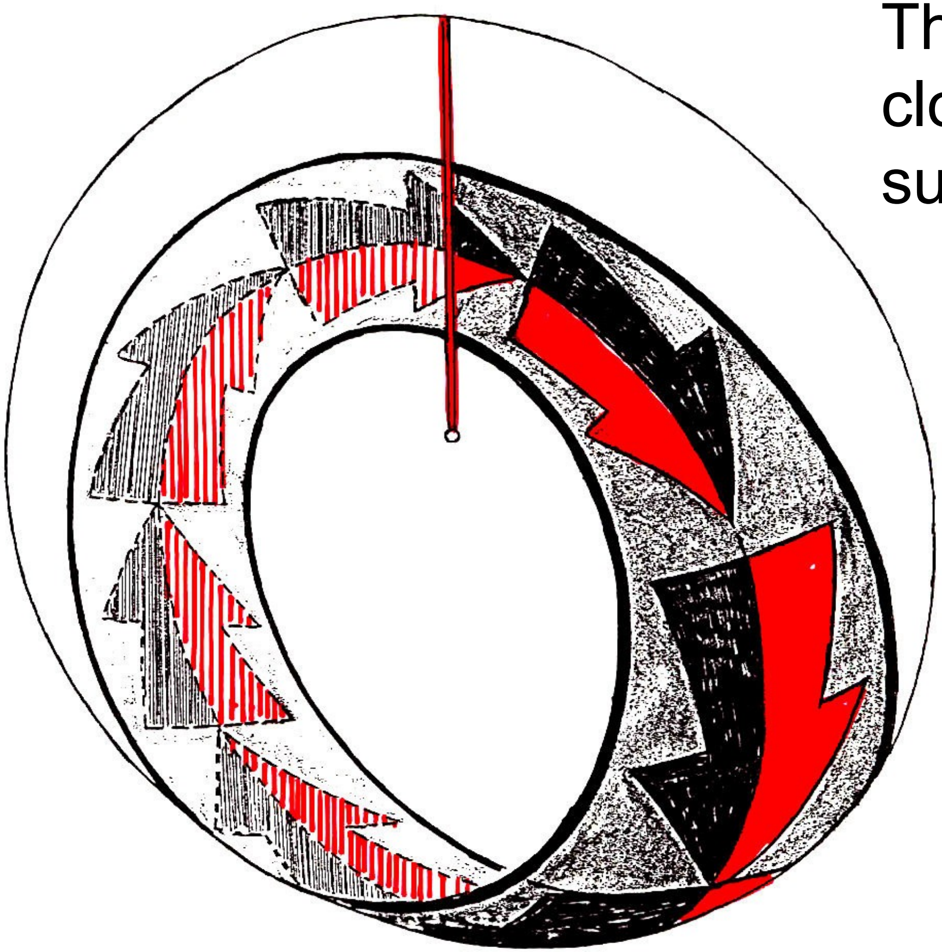
The Cross Cap Model of the Projective Plane



This surface was discovered by Jacob Steiner, 1796 – 1863 pupil at the orphan home of Heinrich Pestalozzi and later Professor of Mathematics at Berlin, Germany.

To explain its name, we cut off the inferior part of the model.

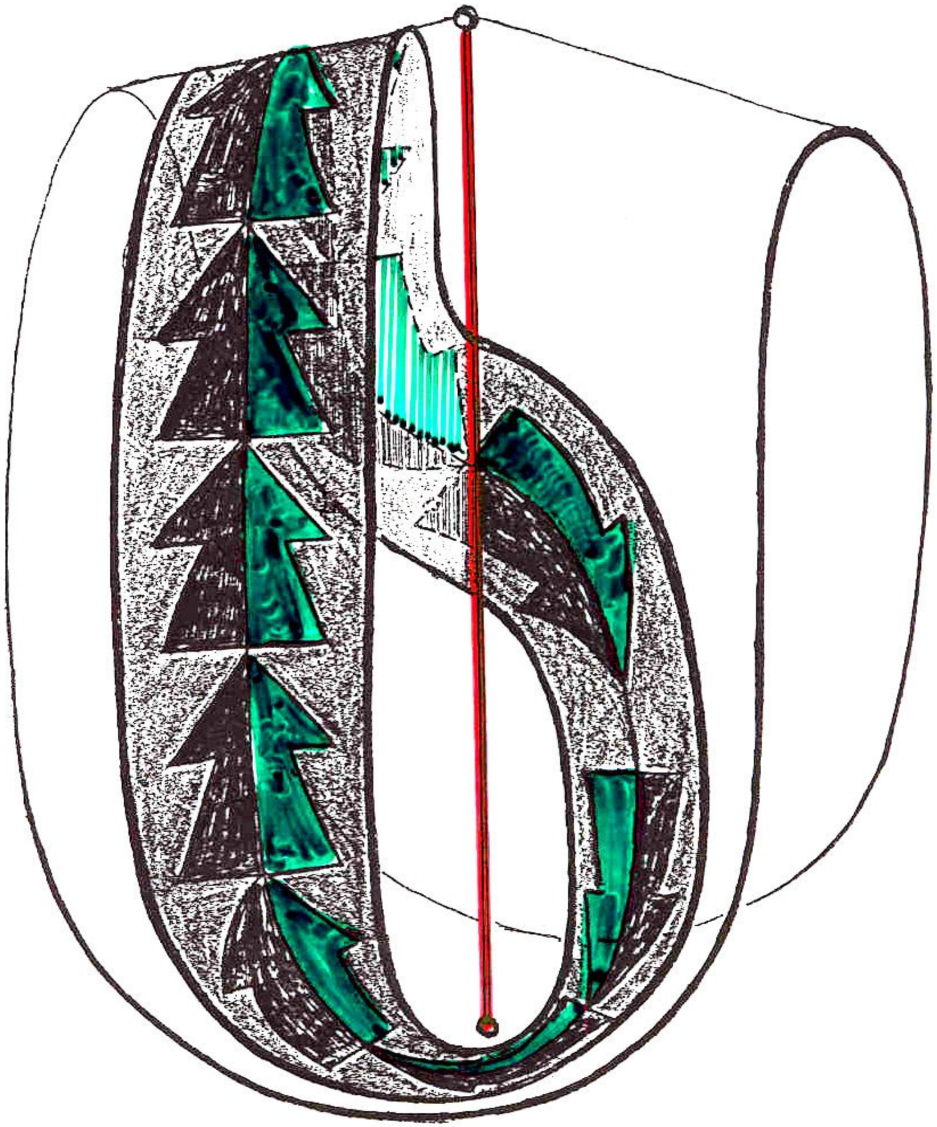




The Projective Plane is a closed and non-orientable surface.

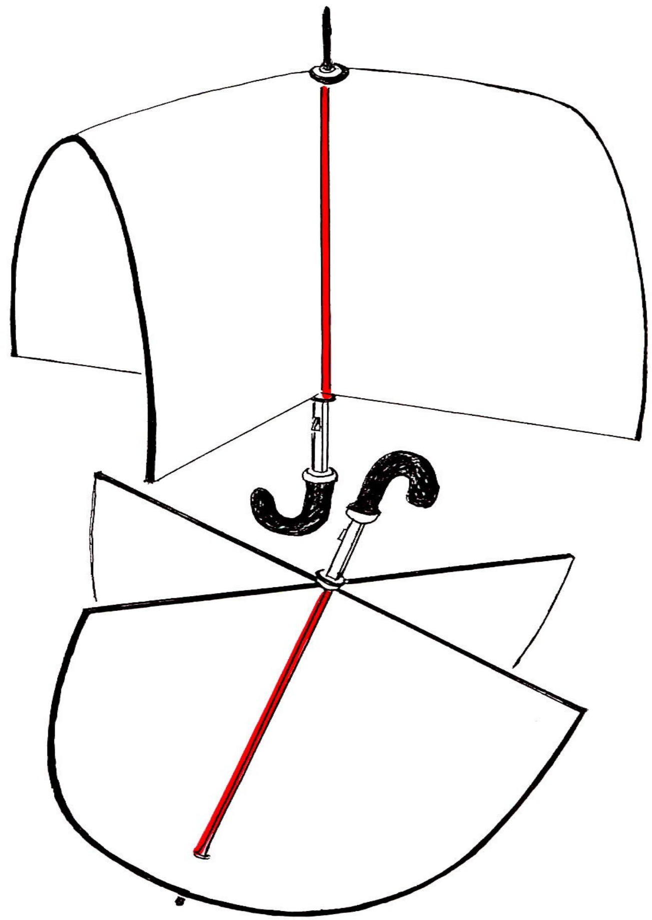
As visualized at the Cross Cap model, the Projective Plane contains a Moebius strip.

We cut out the piece of the Cross Cap model which contains the self-penetration and obtain a **Double Whitney Umbrella.**

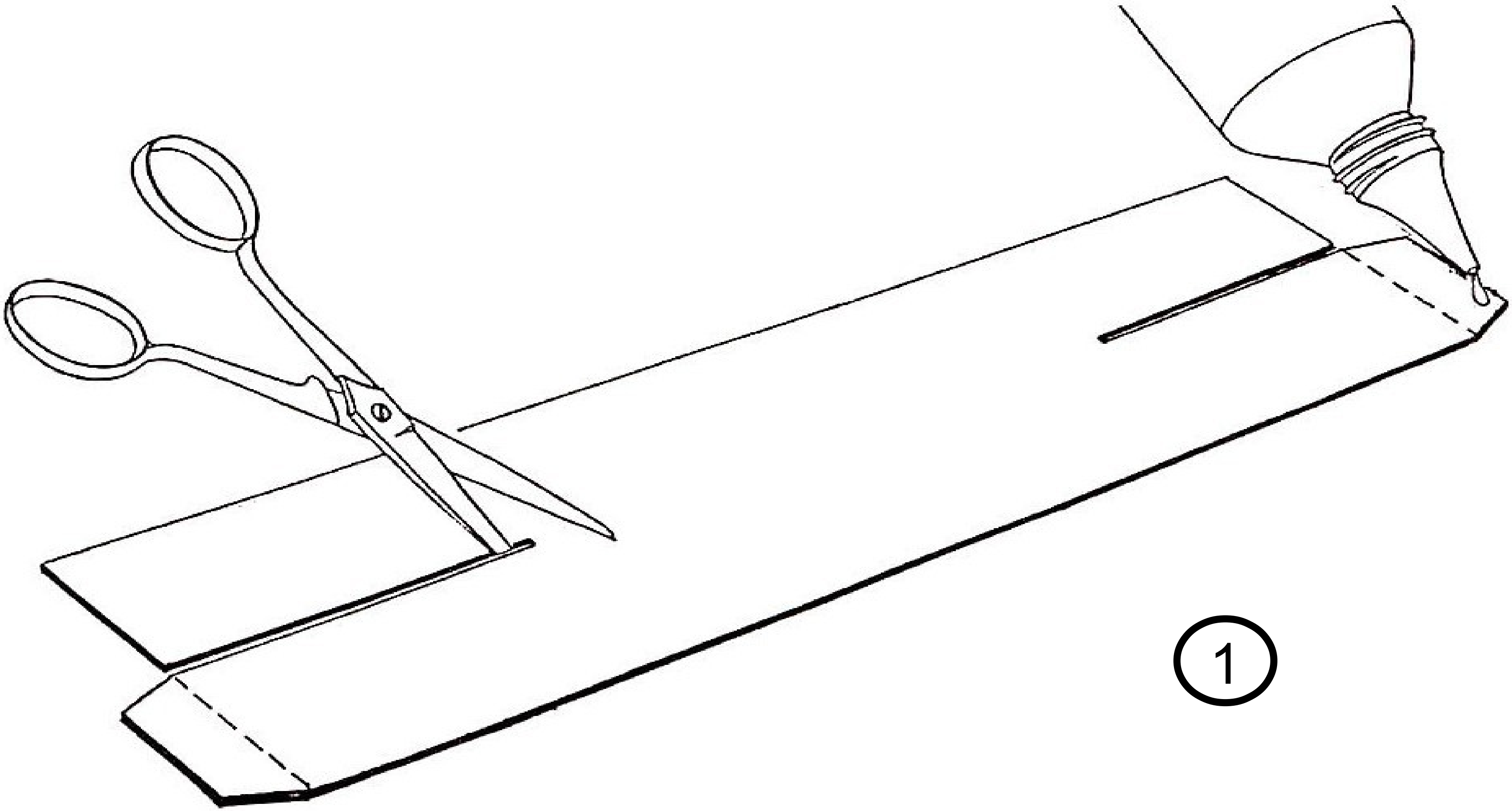


This surface is named after the American Mathematician H. Whitney 1907 – 1989.

By cutting this surface, one obtains two Whitney Umbrellas.



We make a Double Whitney umbrella from a strip of paper.



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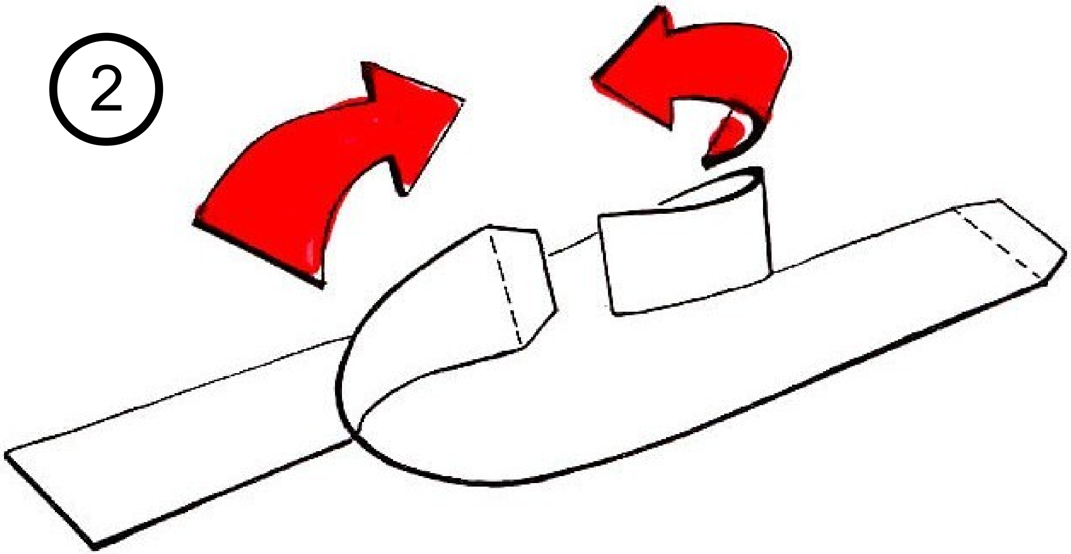
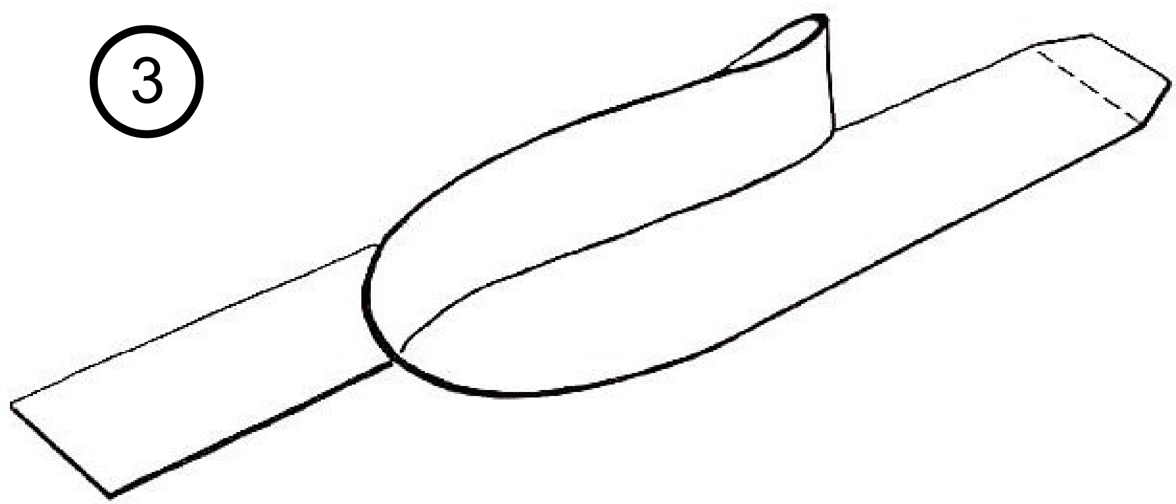
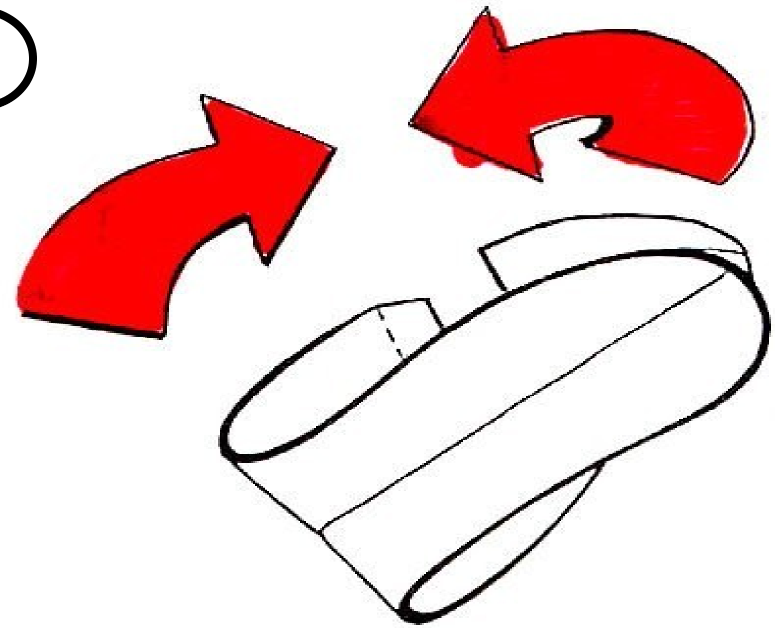


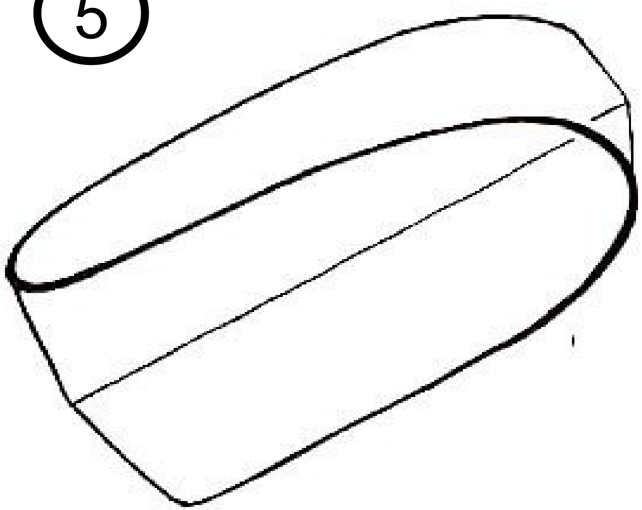
3



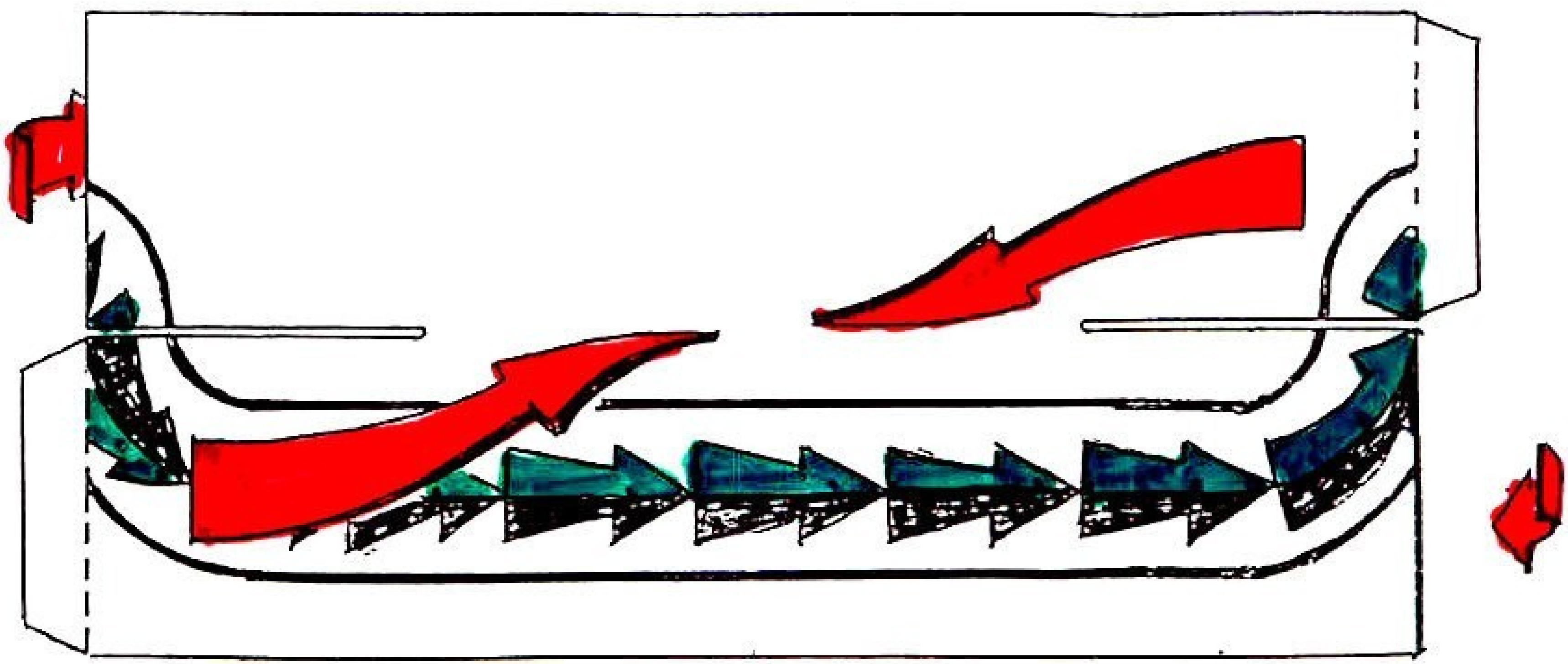
4



5



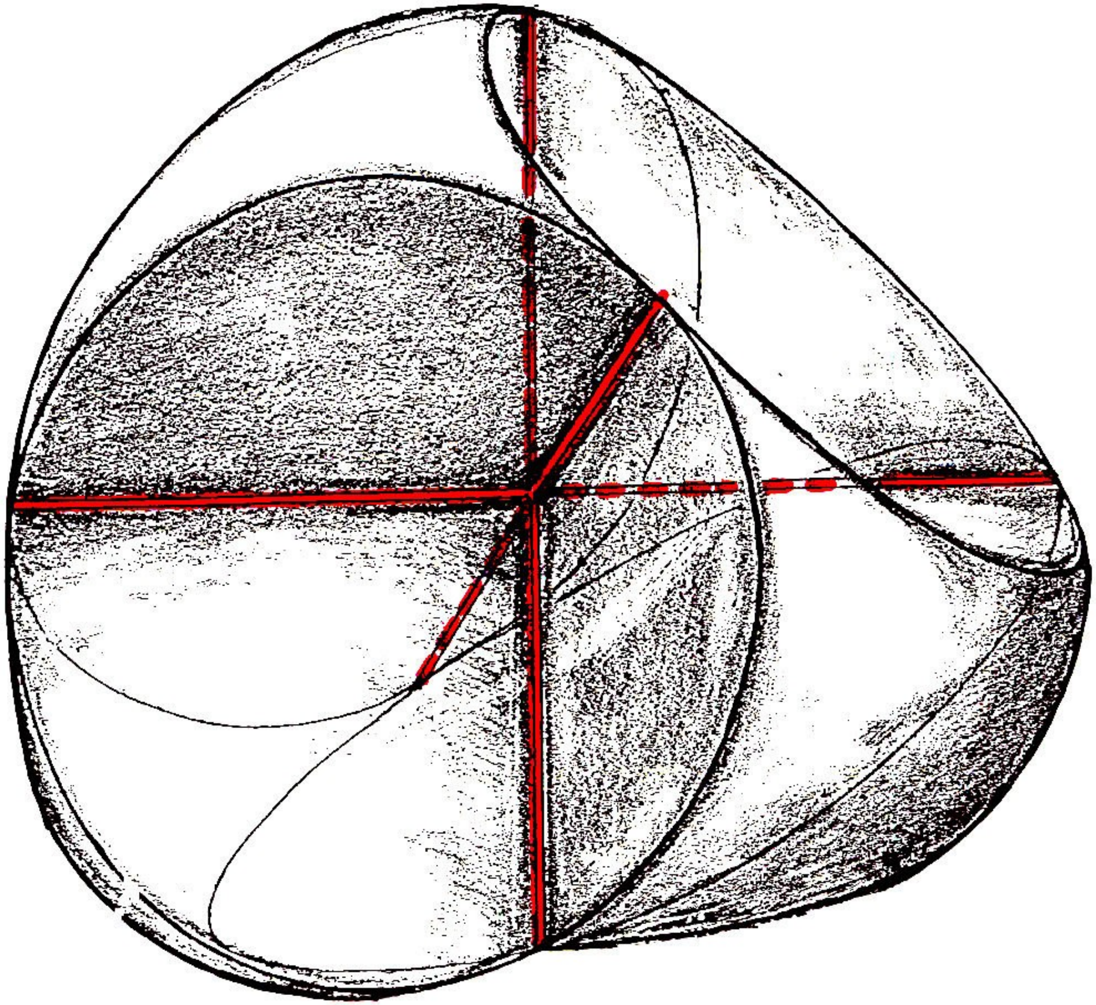
... even with a Moebius strip on it...



Further models of the projective plane are...

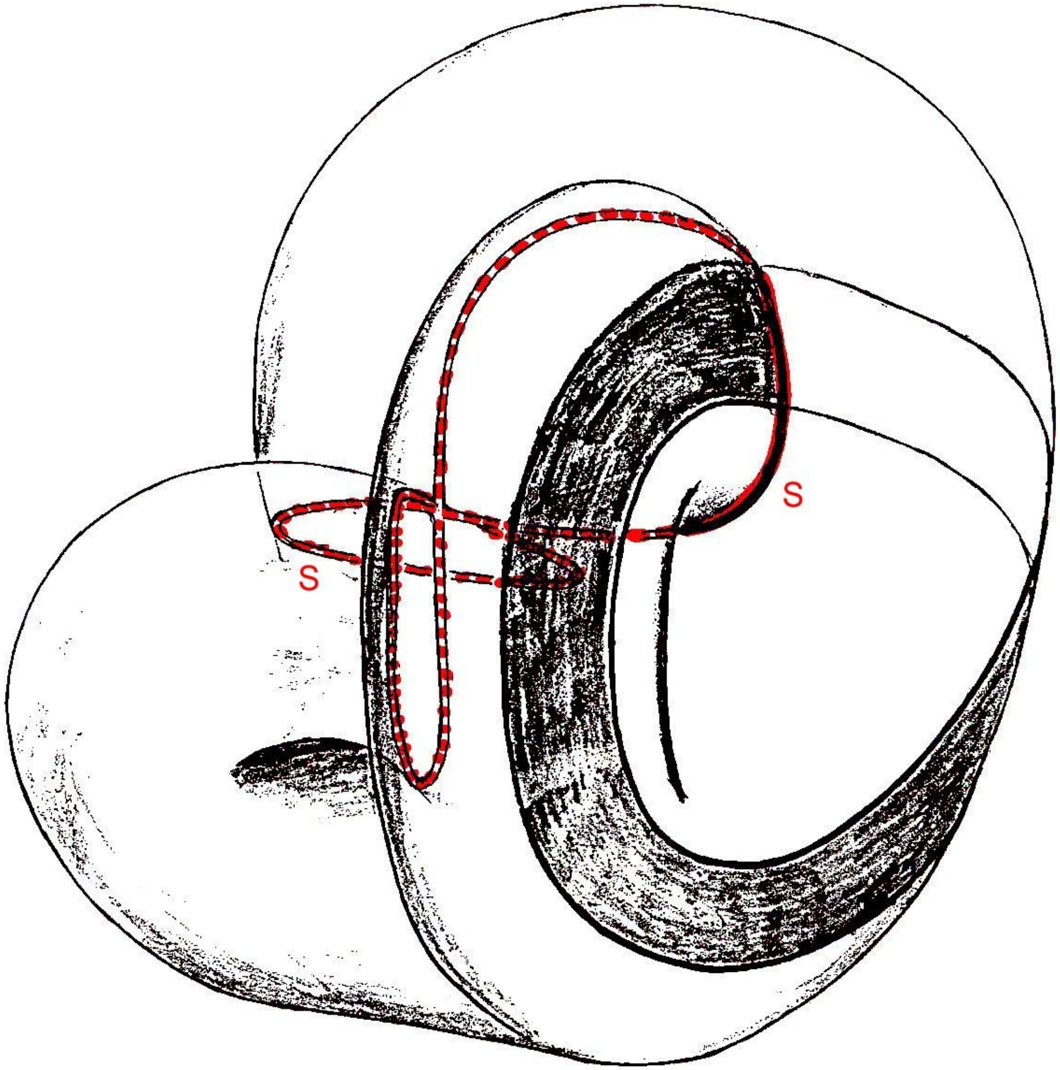
The Steiner Roman surface,

discovered by Jacob Steiner on a journey to Rome in 1844.



The Boy Surface,

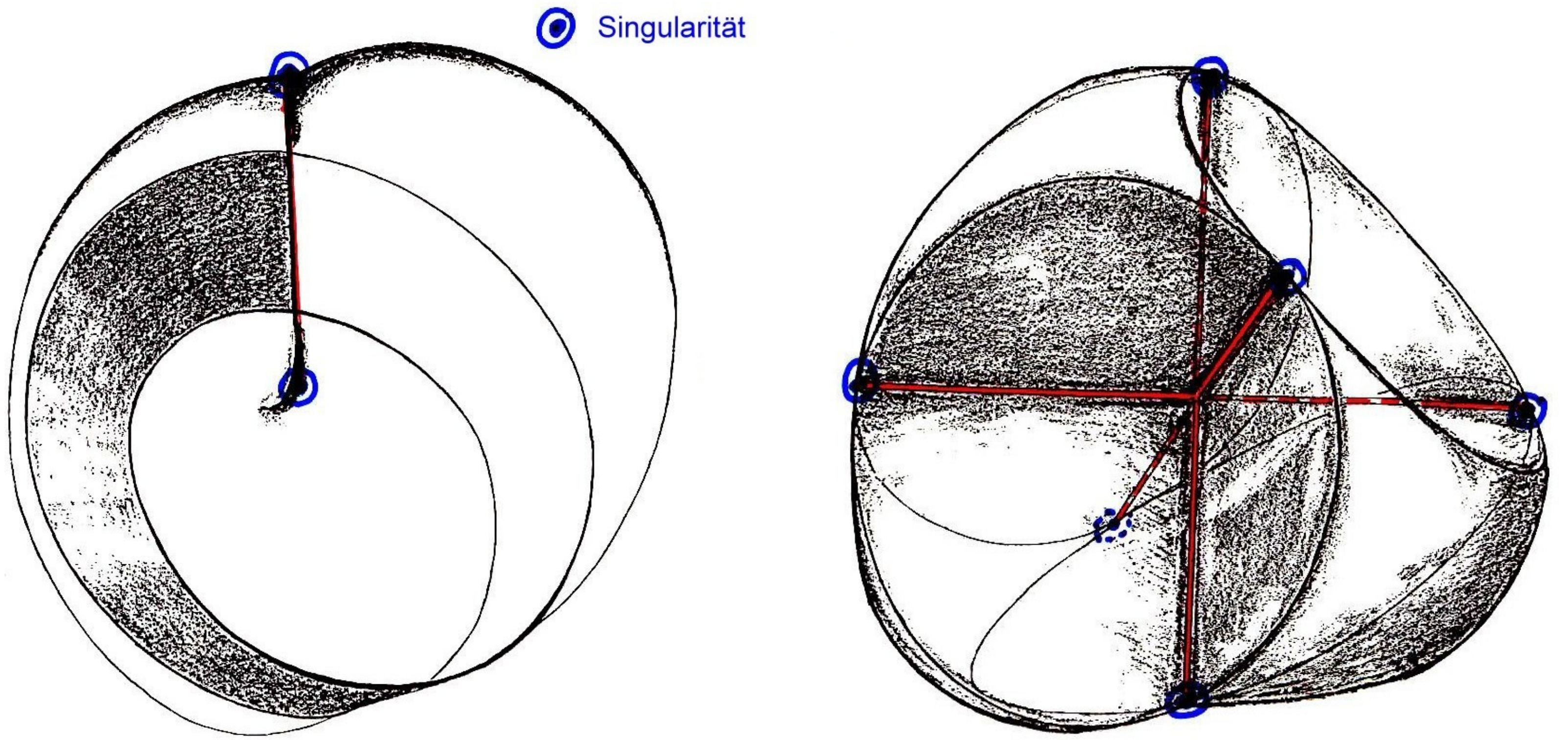
depicted here with a Moebius strip on it.



This surface was discovered in 1919 by Walter Boy (a student of David Hilbert in Göttingen).

An Important Difference...

The Cross Cap model and the Steiner Roman Surface are both models with singularities...



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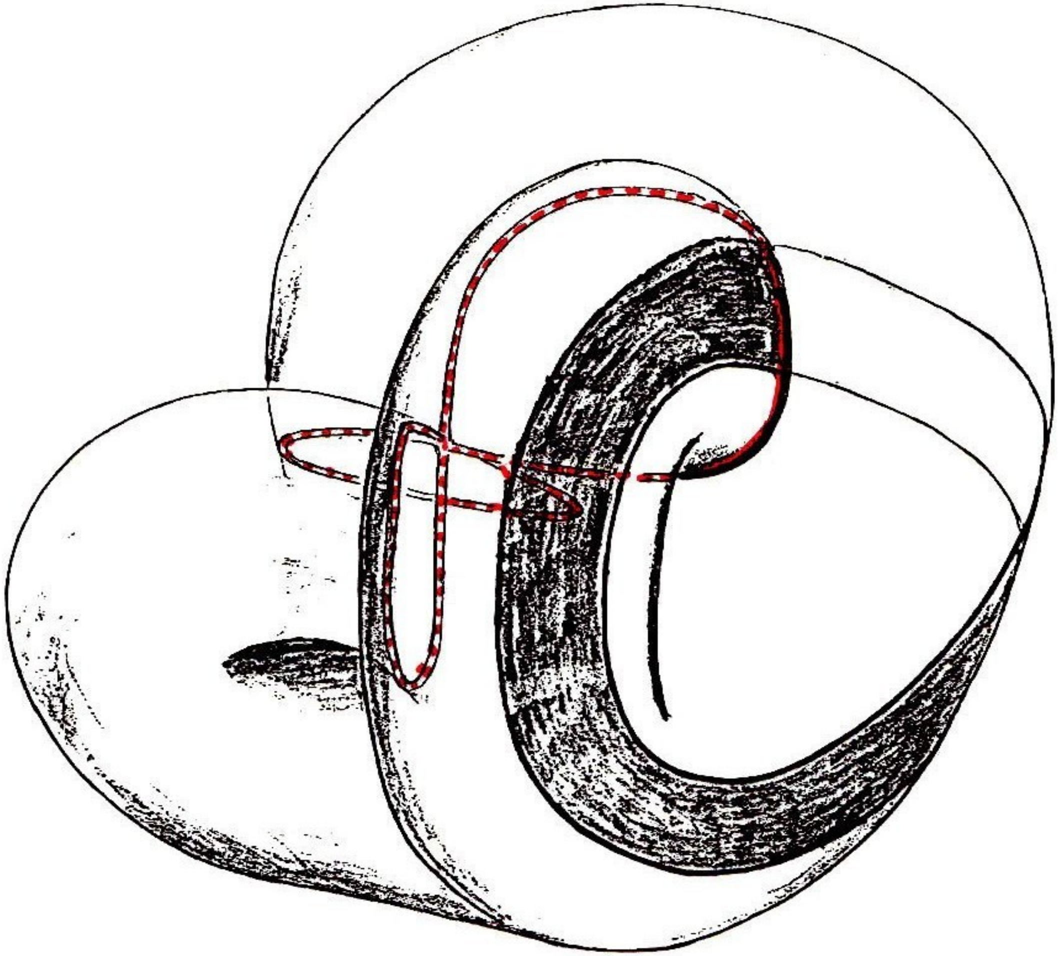


... whereas the Boy Surface is a Non-Singular model of the Projective Plane.

Two Theorems of Whitney:

Each closed surface which is embedded into 4-Space can be projected to 3-Space such that all singularities of the projected surface occur on Whitney umbrellas.

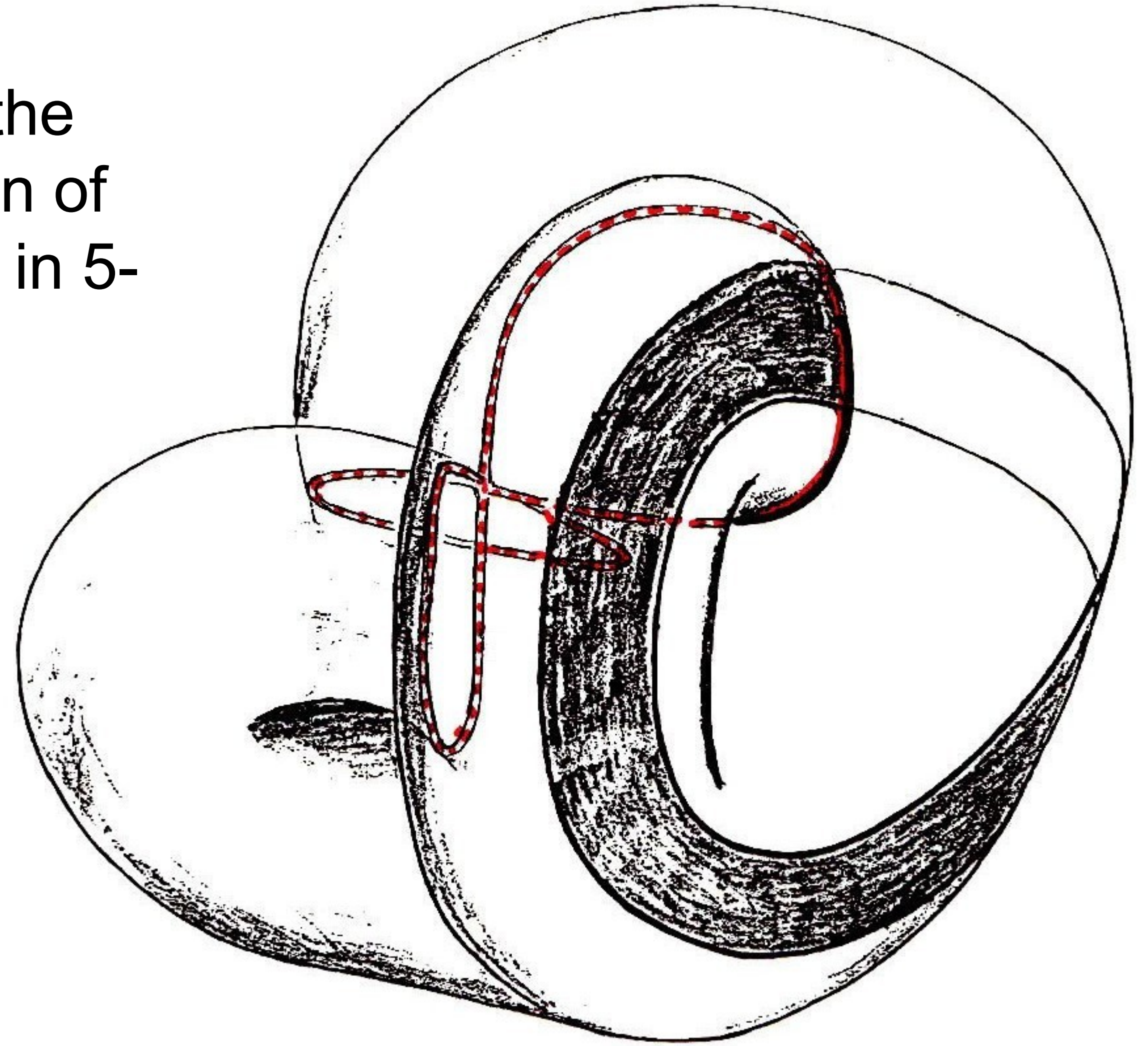
If one realizes the Projective Plane in 4-Space, then each of its projections has singularities.

Corollary:

The Boy Surface can only be obtained by projecting to 3-Space a proper realization of the Projective Plane in a Space of dimension at least 5.

Indeed:

The Boy Surface is the model of a realization of the Projective Plane in 5-Space.



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