Truth and Provability – in Faith, Philosophy and Mathematics

Quotations from the New Jerusalem Bible

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What is Truth?

John 18, 33-38:

33 So Pilate went back into the Praetorium and called Jesus to him and asked him, 'Are you the king of the Jews?'
34 Jesus replied, 'Do you ask this of your own accord, or have others said it to you about me?'
35 Pilate answered, 'Am I a Jew? It is your own people and the chief priests who have handed you over to me: what have you done?'
36 Jesus replied, 'Mine is not a kingdom of this world; if my kingdom were of this world, my men would have fought to prevent my being surrendered to the Jews. As it is, my kingdom does not belong here.'
37 Pilate said, 'So, then you are a king?' Jesus answered, 'It is you who say that I am a king. I was born for this, <u>I came into the world for this, to bear witness to the truth</u>; and all who <u>are on the side of truth listen to my voice.'</u>
38 <u>'Truth?'</u> said Pilate. <u>'What is that?'</u> And so saying he went out again to the Jews and said, 'I find no case against him.

Pilate asks: *„Truth? What is that?“* This question is also ours: We seek for truth – a truth we can completetily understand with our human mind and which is in accordance with our thoughts. This is the type of truth we are faced with in Practical Life, in Science and Philosophy. Often we claim a proof for this kind of truth, a proof which gives evidence or even proves a true statement by logical deduction from basic assumptions we believe to hold.

Pilate has this merely *human* or *natural* view of *truth*, but he also feels that *Jesus bears witness* for another, *supra-natural truth* which *transcends human thinking* and which can be found in Jesus' words:

John 3,16:
„For this is how God loved the world: he gave his only Son, so that everyone who believes in him may not perish but may have eternal life.“

This truth indeed goes beyond any purely human thinking!

Natural and Supra-Natural Truth

Natural Truth: **(A) This is the truth which refers to Created Things and the truth directly accessible to Human Comprehension – the truth we are mostly faced with in our everydays lives. In paricular Scientific Truth and a great deal of Philosophical Truth fall under the notion of Natural Truth.**

(B) Natural Truth is closely related to the Notion of Proof. In *Science* – in particular in *Mathematics* – Proofs are the basic tool to decide on Thruth or Falseness of Claims.

Supra-Natural Truth: **(A) This is the truth which refers to God, the Creator, and transcends mere Human Comprehension. To make it meaningful for Humans it must be *related to Faith*. This is the Truth we find in the Bible, the Truth through which God reveals Himself to Humans.**

(B) Also in the Bible we find Proofs. But they are of different nature than in Science. They are Supra-Natural, although they manifest themselves by Naturally visible Effects – often called Miracles or Signs: They cannot be explained by the Laws of Nature.

Aim: (A) We begin with an example of Supra-Natural Proof from the Bible.

(B) We consider Natural Truth and Proofs from the Point of View of Logics and Mathematics – looking at Antinomies, Formalization and Indecidability in Mathematics. We also briefly look at the Formalization of Natural Language.

(C) We try to draw a final Conclusion.

Jesus Gives a Supra-Natural Proof

Mark 2, 3-12:

3 *when some people came bringing him a paralytic carried by four men,*

4 *but as they could not get the man to him through the crowd, they stripped the roof over the place where Jesus was; and when they had made an opening, they lowered the stretcher on which the paralytic lay.*

5 *Seeing their faith, Jesus said to the paralytic, 'My child, your sins are forgiven.'*

6 *Now some scribes were sitting there, and they thought to themselves,*

7 *'How can this man talk like that? He is being blasphemous. Who but God forgives sins?'*

8 *And at once, Jesus, inwardly aware that this is what they were thinking, said to them, 'Why do you have these thoughts in your hearts?*

9 *Which of these is easier: to say to the paralytic, "Your sins are forgiven" or to say, "Get up, pick up your stretcher and walk"?*

10 ***But to prove to you that the Son of man has authority to forgive sins on earth'*** **–**
11 ***he said to the paralytic-'I order you: get up, pick up your stretcher, and go off home.'***
12 ***And the man got up, and at once picked up his stretcher and walked out in front of everyone, so that they were all astonished and praised God saying, 'We have never seen anything like this.'***

The Inner Healing – a Sign for those who Believe:

By the way, the four men brought the paralyitic to Jesus, they witnessed their faith in Him. In reward of this, Jesus forgives the paralytic his sins.

This is a *sign for the ones who believe in Him* : Not the expected physical healing of the paralytic takes the first place, but the *salvation of his soul to everlasting life*.

The Haeling of the Body – a Sign for Those who do not believe: **The physical healing, which comes at the second place, is a *sign for those who do not believe in Him :* It should prove to them *that the Son of man has authority to forgive sins on earth*.**

The scribes where correct when saying „*Who but God can forgive sins?“* And by giving his proof, Jesus wants to show to them, that the *Son of man* is also the *Son of God* – the *Messiah.*

Proof and Faith: **The Proof given by Jesus aims for a S*upra-Natural Truth* and His proof is *Supra-Natural*, too. A natural proof on its own, cannot give the Grace of Faith needed to believe in a Truth revealed by God. Jesus expresses this in his word to the Apostle Thomas:**

John 20, 29b: ***Blessed are those who do not see and though believe.***

A Special Aspect of Natural Truth: Antinomies

A Short Speech held at Speakers Corner (Hyde Park, London): A young man placed his tribune box at Speaker's Corner and climbed on it.
When sufficiently many people stood around him, he said:

„*I lie !*"

Then he took his box and disappeared in the puzzled croud.
After a few comments, like: „A short speech, indeed !" people started to bet.
Some werde betting: „He did say the Truth." Others were betting: „He did not say the Truth."

Which Party was right? Hint: The Speech is an Antinomy !

Reminder: (A) A ***Set*** is understood to be a collection ***IM*** of objects – at least in *Naive Set Theory.*
(B) The objects which belong to the set ***IM*** are called the ***Elements*** of ***IM***. If ***E*** is an element of ***IM*** we say „***E*** in ***IM***“. Otherwise we say „***E*** not in ***IM***“.
(C) If *P* is a *Property* we write {***E*** : ***E*** has property *P*} for the set of all objects ***E*** which satisfy the Property *P.*

Russell's Antinomy: In 1905 the great British Philosopher and Logician *Bertrand Arthur William Russell* (1872-1970) suggested to form the set:

IA := { ***IM*** : ***IM*** is a set and „***IM not in IM***“ }, *the Set of all Sets which are not Elements of themselves*.

Assume that ***IA*** in ***IA***. Then ***IA*** is an element of ***IA*** thus ***IA*** not in ***IA*** – a contradiction ! Assume that ***IA*** not in ***IA***. Then ***IA*** is an element of ***IA*** thus ***IA*** in ***IA*** – a contradiction, too! ***Antinomy !!***

An Arithmetic Antinomy

Convention: **We write down all *Arithmetic Propositions* A(x) which say something on a positive integer x and which are *Decidable*: For each choice of x, the proposition A(x) is either true or else false.**

Examples **(for A(x)): „x is an odd number", „x is a prime number" , „2(x^3 + 3) = 17x + x^2", „2^x = x^2", „x is squarefree", „x is an even number > 3 which is not the sum of two primes", ...**

Remark: **(A) All propositions A(x) are written down as a finite string of symbols from the finite list: " A, a, B, b, C, c, ..., X, x, Y, y, Z, z" of letters, a finite list „(,), [,], ¬" of auxiliary symbols, a finite list „+ , – , … , ^" of arithmetic symbols and the list of all non-negative integers "0, 1, 2, ...".**
(B) To each proposition A(x) one may assign a positive integer and hence get an *Enumeration of all Propositions* A(x). If n is the number assigned to the proposition A(x), we write A(x) = A_n(x). If A(x) is a proposition we write ¬A(x) for the *negation* of A(x).

Using the above concepts, in 1905, the French Mathematician *Jules Richard* (1862-1956) suggested the following idea:

Richard's Antinomy: **Form the *<u>Anti-Diagonal Proposition</u>***

$$\neg A_x(x)$$

For any number x it holds: $\neg A_x(x)$ *is true precisely if* $A_x(x)$ *is wrong.*

Let m be the *number of the proposition* $\neg A_x(x)$, hence: $\neg A_x(x) = A_m(x)$.

The proposition **$A_m(m) = \neg A_m(m)$** ***is neither true nor wrong, <u>it is an Antinomy</u> !!***

Question: ***„Is Elementary Arithmetic Self-Contradictory ?"***

Mathematicians Belief: ***„Surely not !"*** **…**

But: ***„What is wrong then ?"***

What is an Arithmetic Proposition?

Definition: To each Arithmetic Proposition A(x) we associate its ***Truth Function*** $f_{A(x)}$, which assigns to each positive integer n either 1 or 0, according to whether A(n) is true or not. Thus:

$f_{A(x)}(n) = 1$, *if* A(n) *is true !* $f_{A(x)}(n) = 0$, *if* A(n) *is false !*

Conclusion: *For all* n, the value $f_{\neg A_x(x)}(n)$ *of the truth function of the Anti-Diagonal Proposition* $\neg A_x(x)$ *is different from the value* $f_{A_n(x)}(n)$, thus:

The Anti-Diagonal Proposition $\neg A_x(x)$ ***is different from all Arithmetic Propositions*** $A_n(x)$ (n = 1,2,3,...). ***So we cannot write*** $\neg A_x(x) = A_m(x)$, ***as we previously did*** *!*

Critical Considerations: **(A) *<u>The notion of Decidable Arithmetic Proposition is not well defined</u>*: The Proposition „x is a green number" is not a decidable arithmetic proposition: In Arithmetics, one does not speak on colours of numbers.**

(B) The sentence A(x): „All even numbers > x are sums of two prime numbers" looks like a decidable arithmetic propostion. But, until today, one cannot decide whether it is true or false if x > 3: The *Problem of Goldbach* (named so after the German Lawyer and Mathematician *Christian Goldbach* (1690-1764)) is not solved yet. It asks: *„Is each even number* > 3 *the sum of two primes ?"*

(C) Observe: *The Anti-Diagonal Proposition* $\neg A_x(x)$ *should be written down with finitely many signs and store at the same time the content of the infinitely many statements* $\neg A_n(n)$ *with* n=1,2,3,... !!

Avoiding Richard's Antinomy: **(A)** ***Arithmetic Propositions must be written down in a Formal Language !*** **The Anti-Diagonal Proposition was formulated on use of natural language based on the doubtable notion of decidability!**

(B) Instead, *Arithmetic Propositions should be strings of symbols* (from a finite Alphabet of given signs) *which are build up according to purely syntactic rules.* These *Formal Arithmetic Propositions* should have an *Interpretation* in Arithmetics. But this interpretation must not be understood nor used to form them.

(C) Hence, a computer could check whether a string of signs is such a formal arithmetic proposition. Moreover the handling of these formal arithmetic propositions should rely only on their syntactic structure and not on their content.

Principia Arithmetica: Formalizing Arithmetics

Using Predicate Calculus: (A) Richard's Antinomie teaches us, that Propositions about Arithmetics must be written down in a *Formal Language*.

(B) *Predicate Calculus* (PC) is approporiate. Below we list the symbols it uses. *Formal Propositions* are finite strings of these symbols, formed according to certain *Syntactic Rules.*

(C) PC was introduced by the German Mathematician and Philosopher *Friedrich Ludwig Gottlob Frege* (1848 - 1925).

List of Symbols for Principia Arithmetica (Simplified)

(1) **Variables:** $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$

(2) **Auxiliary Symbols:** $(,), [,], \{, \}, \ldots$

(3) **Logical Symbols:**
- (a) *Conjunction "and":* $\wedge$
- (b) *Implication "implies":* $\Rightarrow$
- (c) *Negation "non":* $\neg$
- (d) *Existence-Quantifier "there is":* $\exists$
- (e) *All-Quantifier "for all":* $\forall$

(4) **Arithmetic Symbols:**
- (a) *Successor "plus one":* *
- (b) *Sum "plus":* $+$
- (c) *Product "times"* $\cdot$
- (d) *Equality "equal":* $=$
- (e) *Nil-Symbol "zero":* $\mathbf{0}$

Natural Numbers are symbolically written by repeated application of the successor operation:

$$\mathbf{1} := \mathbf{0}^*, \mathbf{2} := \mathbf{1}^* = (\mathbf{0}^*)^*, \mathbf{3} := \mathbf{2}^* = ((\mathbf{0})^*)^*)^*, \mathbf{4} := \mathbf{3}^* = (((\mathbf{0}^*)^*)^*)^*, \ldots$$

Instead of *Natural Numbers* we shall speak just of Numbers. Variables $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ can be replaced by numbers.

Formal Axioms and Formal Proofs

The Formal Peano Axioms: To complete the Formal System *<u>Principia Arithmetica</u>*, we need a *System of Formal Axioms*.

The Peano Axioms for Principia Arithmetica (Modified)

(1) Initiality of Zero: $(\forall x)\big(\neg(x^* = 0)\big)$
(2) Uniqueness of Predecessors: $(\forall x)(\forall y)\big((x^* = y^*) \Rightarrow (x = y)\big)$
(3) Neutrality of Zero: $(\forall x)(x + 0 = x)$
(4) Pre-Associativity: $(\forall x)(\forall y)(x + (y^*) = (x + y)^*)$
(5) Triviality of Zero: $(\forall x)(x \cdot 0 = 0)$
(6) Pre-Distributivity: $(\forall x)(\forall y)\big(x \cdot (y^*) = (x \cdot y) + x\big)$
(7) Principle of Induction: If A(x) is a formal proposition of Principia Arithmetica in which x occurs, but the strings $\forall x$ and $\exists x$ do not occur, then the following formal proposition is an Axiom:

$$\big[A(0) \wedge (\forall x)\big(A(x) \Rightarrow A(x^*)\big)\big] \Rightarrow (\forall x)A(x)$$

We chose formal propositions whose interpretations are the *Peano Axioms* for Elementary Arithmetics. These originally were introduced by the great Italian Mathematician *Giuseppe Peano* (1858-1932). In Predicate Calculus Peano's Axioms look as shown above.

Formal Proofs: A *Formal Proof* of a *Formal Proposition P (in Principia Arithmetica)* is a sequence of Formal Propositions which consecutively follow each other according to the *Formal Rules* of *Predicate Logic* from the above *Formal Peano Axioms* and which ends with the given formal proposition *P.*

Comparison: (A) We can say: „A computer can verify whether a certain sequence P_1, P_2, P_3, … , P_n of formal proposisitions is a formal proof of its last formal proposition P_n."

(B) This means: ***The verification of the correctness of a formal proof is a purely algorithmic issue ! No understanding of the interpretation of the occuring formal propositions is needed !***

Definition and Remark: (A) A Formal Proposition which admits a Formal Proof is called *Provable*.

(B) ***The Interpretation of a Provable Formal Proposition is True !***

Gödel's Uncompleteness Theorem

The Gödel Enumeration: (A) In 1931 the Cech-Austrian-American Mathematician *Kurt Gödel* (1906-1978) suggested an ingeneous *System of Enumeration* of all *Formal Propositions* of Principia Arithmetica and of all *Formal Proofs* in Principia Arithmetica – the *Gödel Enumeration*.

(B) To each formal proposition *A,* this enumeration assigns a natural number – called the *Gödel Number* of *A.* Similarly, to each formal proof *Q,* the Gödel enumeration assigns a natural number – called the *Gödel Number* of *Q.*

(C) After a change of variables we may assume that the variable x occurs in any given formal proposition *A* and hence write $A = A(x)$. If n is the Gödel number of *A*, we write $A = A_n(x)$. Now, for the number n we may form the formal proposition $A_n(n)$ which is obtained by substituting n = ((((...(((0)*)*)* ...)*)*)* for x in $A_n(x)$.

Gödels' Undecidable Formal Propostion: **(A) Using his enumeration, Gödel could build up a formal proposition *G*(x,y) with interpretation: *„y is the number of a proof of A_x(x)“.***

(B) On the base of this, he wrote down *the formal propositions*

***P*(x) = (∃ y)*G*(x,y) with interpretation: „A_x(x) *is provable*“ and**

Q*(x) = ¬ *P*(x) with interpretation: „*A_x*(x) *is not provable“.

(C) The formal proposition *Q*(x) has a certain Gödel Number, say n. So: *Q*(x) = *A_n*(x).

(D) One obtains the formal proposition

Q*(n) = *A_n*(n) with interpretation: „*Q*(n) *is not provable“

Gödel's Incompleteness Theorem: ***In Principia Arithmetica there are Formal Propositions with True Interpretation but without Formal Proof – for example Q*(n). Hence:**

<u>In Principia Arithmetica there are True Formal Propositions without a Formal Proof.</u>

„Gödel versus Richard“

Remark: **(A) Gödels formal Proposition *Q*(x) has the interpretation „A_x(x) *is not provable“.* Note: A_x(x) *is* ***not*** *a Formal Arithmetic Proposition in the Variable* x*!* Otherwise the negation ¬A_x(x) of A_x(x) would be a formal arithmetic proposition, too – and thus would have a number m in the Gödel enumeration. But then, we would run again into Richard's Antinomy A_m(m) = ¬A_m(m) !**

(B) Indeed A_x(x) and ¬A_x(x) are *formal propositions only for each choice of number* x = 1,2,3,... . So, for each individual number x the formal proposition A_x(x) has a Gödel Number m(x) which depends on x! Therefore, the above antinomy cannot be achieved in Gödels setting.

Formal Decidability: (A) A formal arithmetic proposition *A* is said to be *formally decidable* – or just *decidable* for short – if either *A* or else ¬*A* is formally provable. In the first case we say, that *A* can be *formally proved*. In the second case we say that *A* *can be formally disproved.*

(B) The terminology introduced in (A) is used in any Formalized Mathematical Theory.

Comment: (A) *The original – not well defined – notion of decidability now is replaced by a well defined notion, which does not refer to „true“ or „false“ anymore !*

(B) *Formal (Arithmetic) Propositions with true interpretation but without formal proof are undecidable !*

The Revision of Hilbert's Program: **The great German Mathematician *David Hilbert* (1862-1943) originally was convinced that any Mathematical Theory could be *Finitely Axiomatisized* such it becames *Decidable* – and hence has no undecidable propositions. This is what he suggested as a basic Program for the *Foundation of Mathematics*. Gödels Result shows, that Hilbert's Program cannot be realized in its original form:**

In most Mathematical Theories there remain propositions on whose truth one cannot decide by an algorithm !

Hilberts original idea reflects his *Neo-Positivistic* View carved in his grave-stone in Göttingen: *„We must know. We shall know.“*

Conclusion: ***Once, proofs were invented to decide on the truth of Mathematical statements. Now, Mathematics itself came to the conclusion, that truth and provability need not be the same.***

Formal Theories of Natural Languages

Transformational and Universal Grammar: ***Rational Linguistics*** **is a formal approach to Natural Languages. Sentences are considered as strings of signs out of a finite alphabet.**

They are formed according to purely *Syntactic Rules* corresponing to the *Grammar* of the underlying natural language. The first such system of formal grammar was suggested in 1965 by the the American Philosopher and Linguist *Noam Chomsky* (*1928): *Transformational Grammar* or *Universal Grammar.*

Its formal rules are formulated such that they should apply to „each natural language“. These systems should allow to build up and to analyze sentences in a purely syntactic way, without use of *semantics* – thus, *without using their content.*

Computer Assisted Handling of Language: ***Possible Applications*** **of Systems of Transformational Grammar are:**

Translation of sentences or full texts by means of computers. automated response systems which answer questions asked in natural language...

General Undecidability and Partial Decidability: (A) By the *Unsolvability of the General Word Problem in Semigroups* Transformational Grammars may not decide whether arbitrary sentences are grammatically equivalent.
(B) On fragments of language, decidability may hold – as the Word Problem is solvable in certain semigroups. This allows computer assisted handling of language within certain limits.

Conclusion: *Language cannot be explained by a Purely Formal Approach !* Language expresses human mind and spirituality – and so it cannot be subject to the limits of a strictly formal system of rules. *The meaning of a sentence is more important than its formal structure*... and moreover:

God reveals himself to Humans through Language.

Language, Truth and Faith

John 8, 56-59:

56 ***Your father Abraham rejoiced to think that he would see my Day; he saw it and was glad.***

57 ***The Jews then said, 'You are not fifty yet, and you have seen Abraham!'***

58 ***Jesus replied: In all truth I tell you, before Abraham ever was, I am.***

59 ***At this they picked up stones to throw at him; but Jesus hid himself and left the Temple.***

These verses are part of a dispute in which Jesus teaches the Pharisees that He is the promised Messiah. The core point of His teaching is the sentence: *„before Abraham ever was, I am“* (s. Verse 58).
From the point of view of Grammar, this sentence is not correct: It does not observe the *accord of time.* But: by its disaccord of time, the *„I am“* of Jesus is a two-fold revelation of the fundamental Truth of Faith that Jesus is the expected Messiah:

(1) The *„I am“* says, that He, Jesus, is *Son of God since Eternity*, above and independendent of time – the One about whom the Pharisees have read in the Thora (Psalm 110, 4): *„Yahweh has sworn an oath he will never retract, you are a priest for ever of the order of Melchizedek.“*

(2) The *„I am“* refers to one of the most Holy Sentences of the Thora: The Revelation of YHWH, [Hebrew: אֶהְיֶה אֲשֶׁר אֶ הְיֶה (*ehyeh ăšer ehyeh*)] the *Name of God* to Mose in the Burning Bush at Mount Horeb (Exodus 3,14): *„God said to Moses, 'I am he who is.' And he said, 'This is what you are to say to the Israelites, "<u>I am</u> has sent me to you." '“* The *„I am“ of Jesus* confirms His Authority to speak in the Name of Yahweh.

<u>If not related to Faith, the Truth taught by Jesus through His „I am“ transcends human mind.</u> **It is not accepted by Jesus' contrahents: They take Him as a blasphemist (Verse 59).**

Conclusion: ***God, who gave us the Language with its rules, breaks these rules if it serves Him to reveal Himself: <u>The Authority of the Creator exceeds the rules given by Him to His creature !</u>***

Beyond the Reach of Man: „Knowing All for Sure“

Conclusive Consideration: By *Mathematics* – more generally by *Science,* but also in *Language* – more generally in *Philosophy* – we were taught that Purely *Formal* and *Algorithmic Approaches* cannot lead to a full understanding of *Truth* and *Meaning*.
Formal Approaches are the most typical expression of the human effort to gain *„Sure Knowledge“*. But they cannot lead to *„Full Knowledge“* and *„Full Truth“.* So – within Human Thinking itself – we are taught that *„Knowing all for Sure“* is beyond the reach of Human Thinking.

Hence, we are taught by Scientific-Analytic Thinking, that there is a Gap between those Truths which are accessible to Human efforts and can be made available and controllable by these efforts – and Truths which cannot be gained by mere Human efforts.

In Expectation of the Final Reconciliation

Scientific Considerations finally lead us to a kind of ***Picture of the relation between Natural and Supra-natural Truth***, we were departing from in our talk.

But... is the Human „Thirst of Knowledge“ not an ***Expression of the deep desire, that once both Truths become one an the same? A desire, which finds its final fulfillment in God.***

1 Corinthians 13, 12-13:

12 *Now we see only reflections in a mirror, mere riddles, but then we shall be seeing face to face. Now I can know only imperfectly; but then I shall know just as fully as I am myself known.*

13 *As it is, these remain: faith, hope and love, the three of them; and the greatest of them is love.*