A NOTE ON THE FOURTH MOMENT OF DIRICHLET L-FUNCTIONS

H. M. BUI AND D. R. HEATH-BROWN

Abstract. We prove an asymptotic formula for the fourth power mean of Dirichlet L-functions averaged over primitive characters to modulus q and over $t \in [0, T]$ which is particularly effective when $q \geq T$. In this range the correct order of magnitude was not previously known.

1. Introduction

For χ a Dirichlet character (mod q), the moments of $L(s, \chi)$ have many applications, for example to the distribution of primes in the arithmetic progressions to modulus q. The asymptotic formula of the fourth power moment in the q-aspect has been obtained by Heath-Brown [**1**], for q prime, and more recently by Soundararajan [**5**] for general q. Following Soundararajan's work, Young [**7**] pushed the result much further by computing the fourth moment for prime moduli q with a power saving in the error term. The problem essentially reduces to the analysis of a particular divisor sum. To this end, Young used various techniques to estimate the off-diagonal terms.

In the case that the t-aspect is also included, a result of Montgomery [**2**] states that

$$\sum_{\chi(\text{mod } q)}^{*} \int_0^T |L(\tfrac{1}{2} + it, \chi)|^4 dt \ll \varphi(q) T (\log qT)^4$$

for $q, T \geq 2$, where $\sum^*_{\chi(\text{mod } q)}$ indicates that the sum is restricted to the primitive characters modulo q. As we shall see, the upper bound is too large by a factor $(q/\varphi(q))^5$. A second result of relevance is due to Rane [**4**]. After correcting a misprint it states that

$$\sum_{\chi(\text{mod } q)}^{*} \int_T^{2T} |L(\tfrac{1}{2} + it, \chi)|^4 dt$$
$$= \frac{\varphi^*(q) T}{2\pi^2} \prod_{p|q} \frac{(1 - p^{-1})^3}{(1 + p^{-1})} (\log qT)^4 + O(2^{\omega(q)} \varphi^*(q) T (\log qT)^3 (\log\log 3q)^5),$$

where $\varphi^*(q)$ is the number of primitive characters modulo q and $\omega(q)$ is the number of distinct prime factors of q. This can only give an asymptotic relation when $2^{\omega(q)} \leq \log q$, which holds for some values of q, but not others. Finally we mention the work of Wang [**6**], where an asymptotic formula is proved for $q \leq T^{1-\delta}$, for any fixed $\delta > 0$.

The goal of the present note is to establish an asymptotic formula, valid for all $q, T \geq 2$, as soon as $q \to \infty$.

Theorem 1. *For $q, T \geq 2$ we have, in the notation above,*

$$\sum_{\chi(\text{mod } q)}^{*} \int_0^T |L(\tfrac{1}{2} + it, \chi)|^4 dt$$
$$= \left(1 + O\left(\frac{\omega(q)}{\log q} \sqrt{\frac{q}{\varphi(q)}}\right)\right) \frac{\varphi^*(q) T}{2\pi^2} \prod_{p|q} \frac{(1 - p^{-1})^3}{(1 + p^{-1})} (\log qT)^4 + O(qT (\log qT)^{\frac{7}{2}}).$$

Our proof uses ideas from the works of Heath-Brown [**1**] and Soundararajan [**5**], but there is extra work to do to handle the integration over t.

Remark 1. It is possible, with only a little more effort, to extend the range to cover all $T > 0$. In this case the term $\varphi^*(q)T$ in the main term remains the same, as does the factor qT in the error term, but one must replace $\log qT$ by $\log q(T+2)$ both in the main term and in the error term.

Remark 2. One may readily verify that our result provides an asymptotic formula, as soon as $q \to \infty$, with an error term which saves at least a factor $O((\log\log q)^{-1/2})$.

Remark 3. The literature appears not to contain a precise analogue of this for the second moment. However Motohashi [**3**] has considered a uniform mean value in t-aspect. He proved that if χ is a primitive character modulo a prime q, then

$$\int_0^T |L(\tfrac{1}{2}+it,\chi)|^2 dt = \frac{\varphi(q)T}{q}\bigg(\log\frac{qT}{2\pi} + 2\gamma + 2\sum_{p|q}\frac{\log p}{p-1}\bigg) + O((q^{\frac{1}{3}}T^{\frac{1}{3}} + q^{\frac{1}{2}})(\log qT)^4),$$

for $T \geq 2$. This provides an asymptotic formula when $q \leq T^{2-\delta}$, for any fixed $\delta > 0$. Our theorem does not give a power saving in the error term, but it yields an asymptotic formula without any restrictions on q and T.

2. Auxiliary lemmas

Lemma 1. *Let χ be a primitive character* (mod q) *such that $\chi(-1) = (-1)^{\mathfrak{a}}$ with $\mathfrak{a} = 0$ or 1. Then we have*

$$|L(\tfrac{1}{2}+it,\chi)|^2 = 2\sum_{a,b\geq 1}\frac{\chi(a)\overline{\chi(b)}}{\sqrt{ab}}\Big(\frac{a}{b}\Big)^{-it} W_{\mathfrak{a}}\Big(\frac{\pi ab}{q};t\Big),$$

where

$$W_{\mathfrak{a}}(x;t) = \frac{1}{2\pi i}\int_{(2)}\frac{\Gamma(\frac{1}{4}+\frac{it}{2}+\frac{z}{2}+\frac{\mathfrak{a}}{2})\Gamma(\frac{1}{4}-\frac{it}{2}+\frac{z}{2}+\frac{\mathfrak{a}}{2})}{|\Gamma(\frac{1}{4}+\frac{it}{2}+\frac{\mathfrak{a}}{2})|^2}e^{z^2}x^{-z}\frac{dz}{z}.$$

Proof. Let

$$I := \frac{1}{2\pi i}\int_{(2)}\frac{\Lambda(\frac{1}{2}+it+z,\chi)\Lambda(\frac{1}{2}-it+z,\overline{\chi})}{|\Gamma(\frac{1}{4}+\frac{it}{2}+\frac{\mathfrak{a}}{2})|^2}e^{z^2}\frac{dz}{z},$$

where

$$\Lambda(\tfrac{1}{2}+s,\chi) = \Big(\frac{q}{\pi}\Big)^{s/2}\Gamma\Big(\frac{1}{4}+\frac{s}{2}+\frac{\mathfrak{a}}{2}\Big)L(\tfrac{1}{2}+s,\chi).$$

We recall the functional equation

$$\Lambda(\tfrac{1}{2}+s,\chi) = \frac{\tau(\chi)}{i^{\mathfrak{a}}\sqrt{q}}\Lambda(\tfrac{1}{2}-s,\overline{\chi}).$$

Hence, moving the line of integration to $\Re z = -2$ and applying Cauchy's Theorem, we obtain $|L(\frac{1}{2}+it,\chi)|^2 = 2I$. Finally, expanding $L(\frac{1}{2}+it+z,\chi)L(\frac{1}{2}-it+z,\overline{\chi})$ in a Dirichlet series and integrating termwise we obtain the lemma. □

We decompose $|L(\frac{1}{2}+it,\chi)|^2$ as $2(A(t,\chi)+B(t,\chi))$, where

$$A(t,\chi) = \sum_{ab\leq Z}\frac{\chi(a)\overline{\chi(b)}}{\sqrt{ab}}\Big(\frac{a}{b}\Big)^{-it} W_{\mathfrak{a}}\Big(\frac{\pi ab}{q};t\Big),$$

and

$$B(t,\chi) = \sum_{ab>Z}\frac{\chi(a)\overline{\chi(b)}}{\sqrt{ab}}\Big(\frac{a}{b}\Big)^{-it} W_{\mathfrak{a}}\Big(\frac{\pi ab}{q};t\Big),$$

with $Z = qT/2^{\omega(q)}$. In the next two sections, we evaluate the second moments of $A(t,\chi)$ and $B(t,\chi)$ after which our theorem will be an easy consequence.

The function $W_{\mathfrak{a}}(x;t)$ approximates the characteristic function of the interval $[0,|t|]$. Indeed, we have the following.

Lemma 2. *The function $W_{\mathfrak{a}}(x;t)$ satisfies*

$$W_{\mathfrak{a}}(x;t) = \begin{cases} O((\tau/x)^2) & \text{for } x \geq \tau, \\ 1 + O((x/\tau)^{1/4}) & \text{for } 0 < x < \tau, \end{cases}$$

and

$$\frac{\partial}{\partial t} W_{\mathfrak{a}}(x;t) \ll \begin{cases} \tau^{-1}(\tau/x)^2 & \text{for } x \geq \tau, \\ \tau^{-1}(x/\tau)^{1/4} & \text{for } 0 < x < \tau, \end{cases}$$

where $\tau = |t| + 2$.

Proof. The first estimate is a direct consequence of Stirling's formula, while for the second one merely shifts the line of integration to $\Re z = -1/4$ before employing Stirling's formula. To handle the derivative one proceeds as before, differentiates under the integral sign and uses the estimate

$$\frac{\Gamma'(w)}{\Gamma(w)} = \log w + O(|w|^{-1}),$$

which holds for $1/8 \leq \Re w \leq 2$ □

The next lemma concerns the orthogonality of primitive Dirichlet characters.

Lemma 3. *For $(mn,q) = 1$, we have*

$$\sum_{\chi (\mathrm{mod}\ q)}^{*} \chi(m)\overline{\chi}(n) = \sum_{k|(q,m-n)} \varphi(k)\mu(q/k).$$

Moreover

$$\sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi(-1)=(-1)^{\mathfrak{a}}}}^{*} \chi(m)\overline{\chi}(n) = \frac{1}{2} \sum_{k|(q,m-n)} \varphi(k)\mu(q/k) + \frac{(-1)^{\mathfrak{a}}}{2} \sum_{k|(q,m+n)} \varphi(k)\mu(q/k).$$

Proof. This follows from [**1**; page 27]. □

To handle the off-diagonal term we shall use the following bounds.

Lemma 4. *Let k be a positive integer and $Z_1, Z_2 \geq 2$. If $Z_1 Z_2 \leq k^{\frac{19}{10}}$ then*

$$E := \sum_{\substack{Z_1 \leq ab < 2Z_1 \\ Z_2 \leq cd < 2Z_2 \\ ac \equiv \pm bd (\mathrm{mod}\ k) \\ ac \neq bd \\ (abcd,k)=1}} \frac{1}{|\log \frac{ac}{bd}|} \ll \frac{(Z_1 Z_2)^{1+\varepsilon}}{k}$$

for any fixed $\varepsilon > 0$, while if $Z_1 Z_2 > k^{\frac{19}{10}}$ then

$$E \ll \frac{Z_1 Z_2}{k} (\log Z_1 Z_2)^3. \tag{1}$$

Proof. We note that in each case the contribution of the terms with $|\log ac/bd| > \log 2$ is satisfactory, by the corresponding lemma of Soundararajan [**5**; Lemma 3]. Thus, by symmetry, it is enough to consider the terms with $bd < ac \leq 2bd$. We shall show how to handle the terms in which $ac \equiv bd \,(\mathrm{mod}\ k)$, the alternative case being dealt with similarly. We write $n = bd$ and

$ac = kl + bd$ and observe that $kl \le bd$. We deduce that $n \le 2\sqrt{Z_1Z_2}$ and $1 \le l \le 2\sqrt{Z_1Z_2}/k$. Since $\log ac/bd \gg kl/n$ the contribution of these terms to E is

$$\ll \frac{1}{k}\sum_{l\le 2\sqrt{Z_1Z_2}/k}\frac{1}{l}\sum_{\substack{n\le 2\sqrt{Z_1Z_2}\\(n,k)=1}} nd(n)d(kl+n).$$

We estimate the sum over n using a bound from Heath-Brown's paper [**1**; (17)]. This shows that the above expression is

$$\ll \frac{Z_1Z_2(\log Z_1Z_2)^2}{k}\sum_{l\le 2\sqrt{Z_1Z_2}/k}\frac{1}{l}\sum_{d|l}d^{-1} \ll \frac{Z_1Z_2}{k}(\log Z_1Z_2)^3.$$

This suffices to complete the proof. The reader will observe that when $Z_1Z_2 \le k^{\frac{19}{10}}$ it is only the terms with $|\log ac/bd| > \log 2$ which prevent us from achieving the bound (1). □

Finally we shall require the following two lemmas [**5**; Lemmas 4 and 5].

Lemma 5. *For $q \ge 2$ we have*

$$\sum_{\substack{n\le x\\(n,q)=1}}\frac{1}{n} = \frac{\varphi(q)}{q}\big(\log x + O\big(1+\log\omega(q)\big)\big) + O\bigg(\frac{2^{\omega(q)}\log x}{x}\bigg).$$

Lemma 6. *For $x \ge \sqrt{q}$ we have*

$$\sum_{\substack{n\le x\\(n,q)=1}}\frac{2^{\omega(n)}}{n} \ll \bigg(\frac{\varphi(q)}{q}\bigg)^2(\log x)^2,$$

and

$$\sum_{\substack{n\le x\\(n,q)=1}}\frac{2^{\omega(n)}}{n}\bigg(\log\frac{x}{n}\bigg)^2 = \bigg(1+O\bigg(\frac{1+\log\omega(q)}{\log q}\bigg)\bigg)\frac{(\log x)^4}{12\zeta(2)}\prod_{p|q}\frac{1-1/p}{1+1/p}.$$

3. The main term

Applying Lemma 3 we have

$$\sum_{\chi(\mathrm{mod}\ q)}{}^{*}\int_0^T A(t,\chi)^2dt = M + E,$$

where

$$M = \frac{\varphi^*(q)}{2}\sum_{\mathfrak{a}=0,1}\sum_{\substack{ab,cd\le Z\\ac=bd\\(abcd,q)=1}}\frac{1}{\sqrt{abcd}}\int_0^T W_{\mathfrak{a}}\Big(\frac{\pi ab}{q};t\Big)W_{\mathfrak{a}}\Big(\frac{\pi cd}{q};t\Big)dt,$$

and

$$E = \sum_{k|q}\varphi(k)\mu(q/k)E(k),$$

with

$$E(k) = \sum_{\mathfrak{a}=0,1}\sum_{\substack{ab,cd\le Z\\ac\equiv\pm bd(\mathrm{mod}\ k)\\ac\ne bd\\(abcd,q)=1}}\frac{1}{\sqrt{abcd}}\int_0^T\Big(\frac{ac}{bd}\Big)^{-it}W_{\mathfrak{a}}\Big(\frac{\pi ab}{q};t\Big)W_{\mathfrak{a}}\Big(\frac{\pi cd}{q};t\Big)dt.$$

We first estimate the error term E. We integrate by parts, using Lemma 2. This produces

$$E(k) \ll \sum_{\substack{ab,cd\leq Z\\ ac\equiv\pm bd(\mathrm{mod}\ k)\\ ac\neq bd\\ (abcd,q)=1}} \frac{1}{\sqrt{abcd}|\log\frac{ac}{bd}|}.$$

We divide the terms $ab, cd \leq Z$ into dyadic blocks $Z_1 \leq ab < 2Z_1$ and $Z_2 \leq cd < 2Z_2$. From Lemma 4, the contribution of this range to $E(k)$ is

$$\ll \frac{1}{\sqrt{Z_1Z_2}}\frac{Z_1Z_2}{k}(\log Z_1Z_2)^3 = \frac{\sqrt{Z_1Z_2}}{k}(\log Z_1Z_2)^3,$$

if $Z_1Z_2 > k^{\frac{19}{10}}$, and is $O((Z_1Z_2)^{\frac{1}{2}+\varepsilon}k^{-1})$ if $Z_1Z_2 \leq k^{\frac{19}{10}}$. Summing over all such dyadic blocks we have

$$E(k) \ll \frac{Z}{k}(\log Z)^3 + k^{-\frac{1}{20}+2\varepsilon}.$$

Thus

$$E \ll Z2^{\omega(q)}(\log Z)^3 \ll qT(\log qT)^3. \tag{2}$$

We now turn to the main term M. Since $ac = bd$, we can write $a = gr$, $b = gs$, $c = hs$ and $d = hr$, where $(r,s) = 1$. We put $n = rs$. Hence

$$M = \frac{\varphi^*(q)}{2}\sum_{\mathfrak{a}=0,1}\sum_{\substack{n\leq Z\\ (n,q)=1}}\frac{2^{\omega(n)}}{n}\sum_{\substack{g,h\leq\sqrt{Z/n}\\ (gh,q)=1}}\frac{1}{gh}\int_0^T W_{\mathfrak{a}}\Big(\frac{\pi g^2 n}{q};t\Big)W_{\mathfrak{a}}\Big(\frac{\pi h^2 n}{q};t\Big)dt.$$

From Lemma 2 we have $W_{\mathfrak{a}}(\pi g^2 n/q;t) = 1 + O(g^{1/2}(n/qt)^{\frac{1}{4}})$, whence

$$M = \varphi^*(q)T\sum_{\substack{n\leq Z\\ (n,q)=1}}\frac{2^{\omega(n)}}{n}\Big(\sum_{\substack{g\leq\sqrt{Z/n}\\ (g,q)=1}}\frac{1}{g} + O(1)\Big)^2.$$

We split the terms $n \leq Z$ into the cases $n \leq Z_0$ and $Z_0 < n \leq Z$, where $Z_0 = Z/9^{\omega(q)}$. In the first case, from Lemma 5 the sum over g is

$$= \frac{\varphi(q)}{2q}\log\frac{Z_0}{n} + O(1+\log\omega(q)),$$

since the first error term in Lemma 5 dominates the second. Hence the contribution of such values of n to M is

$$\varphi^*(q)T\Big(\frac{\varphi(q)}{2q}\Big)^2\sum_{\substack{n\leq Z_0\\ (n,q)=1}}\frac{2^{\omega(n)}}{n}\Big(\Big(\log\frac{Z_0}{n}\Big)^2 + O(\omega(q)\log Z)\Big).$$

Here we use the fact that $q/\varphi(q) \ll 1 + \log\omega(q)$. This estimate will be employed a number of times in what follows, without further comment. In view of Lemma 6 the contribution from terms with $n \leq Z_0$ is now seen to be

$$\frac{\varphi^*(q)T}{8\pi^2}\prod_{p|q}\frac{(1-1/p)^3}{(1+1/p)}(\log Z_0)^4\Big(1 + O\Big(\frac{\omega(q)}{\log q}\Big)\Big). \tag{3}$$

For $Z_0 \leq n \leq Z$, we extend the sum over g to all $g \leq 3^{\omega(q)}$ that are coprime to q. By Lemma 5, this sum is $\ll \omega(q)\varphi(q)/q$. Hence the contribution of these terms to M is

$$\ll \varphi^*(q)T\Big(\omega(q)\frac{\varphi(q)}{q}\Big)^2\sum_{Z_0\leq n\leq Z}\frac{2^{\omega(n)}}{n} \ll \varphi^*(q)T\Big(\frac{\varphi(q)}{q}\Big)^4\omega(q)^2(\log Z)^2.$$

Combining this with (2) and (3) we obtain

$$\sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)}\int_0^T A(t,\chi)^2dt=\bigg(1+O\Big(\frac{\omega(q)}{\log q}\Big)\bigg)\frac{\varphi^*(q)T}{8\pi^2}\prod_{p|q}\frac{(1-1/p)^3}{(1+1/p)}(\log qT)^4. \qquad (4)$$

4. The error term

We have

$$\begin{aligned}\sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)}\int_0^T B(t,\chi)^2dt &\leq \sum_{\chi(\mathrm{mod}\ q)}\int_0^T B(t,\chi)^2dt\\ &=\frac{\varphi(q)}{2}\sum_{\mathfrak{a}=0,1}\sum_{\substack{ab,cd>Z\\ ac\equiv\pm bd(\mathrm{mod}\ q)\\ (abcd,q)=1}}\frac{1}{\sqrt{abcd}}\int_0^T\Big(\frac{ac}{bd}\Big)^{-it}W_{\mathfrak{a}}\Big(\frac{\pi ab}{q};t\Big)W_{\mathfrak{a}}\Big(\frac{\pi cd}{q};t\Big)dt. \qquad (5)\end{aligned}$$

Using Lemma 2 and integration by parts, the integral over t is

$$\ll\frac{1}{|\log\frac{ac}{bd}|}\Big(1+\frac{ab}{qT}\Big)^{-2}\Big(1+\frac{cd}{qT}\Big)^{-2}$$

if $ac\neq bd$, and is

$$\ll T\Big(1+\frac{ab}{qT}\Big)^{-2}\Big(1+\frac{cd}{qT}\Big)^{-2}$$

if $ac=bd$. Hence the right hand side of (5) is $O(R_1+R_2)$, where

$$R_1=\varphi(q)T\sum_{\substack{ab,cd>Z\\ ac=bd\\ (abcd,q)=1}}\frac{1}{\sqrt{abcd}}\Big(1+\frac{ab}{qT}\Big)^{-2}\Big(1+\frac{cd}{qT}\Big)^{-2},$$

and

$$R_2=\varphi(q)\sum_{\substack{ab,cd>Z\\ ac\equiv\pm bd(\mathrm{mod}\ q)\\ ac\neq bd\\ (abcd,q)=1}}\frac{1}{\sqrt{abcd}|\log\frac{ac}{bd}|}\Big(1+\frac{ab}{qT}\Big)^{-2}\Big(1+\frac{cd}{qT}\Big)^{-2}.$$

To estimate R_2, we again break the terms into dyadic ranges $Z_1\leq ab<2Z_1$ and $Z_2\leq cd<2Z_2$, where $Z_1,Z_2>Z$. By Lemma 4, the contribution of each such block is

$$\ll\frac{\varphi(q)}{\sqrt{Z_1Z_2}}\Big(1+\frac{Z_1}{qT}\Big)^{-2}\Big(1+\frac{Z_2}{qT}\Big)^{-2}\frac{Z_1Z_2}{q}(\log Z_1Z_2)^3.$$

Summing over all the dyadic ranges we obtain

$$R_2\ll\varphi(q)T(\log qT)^3. \qquad (6)$$

To handle R_1 we argue as in the previous section. We write $a=gr$, $b=gs$, $c=hs$ and $d=hr$, where $(r,s)=1$, and we put $n=rs$. Then

$$R_1\ll\varphi(q)T\sum_{(n,q)=1}\frac{2^{\omega(n)}}{n}\bigg(\sum_{\substack{g>\sqrt{Z/n}\\ (g,q)=1}}\frac{1}{g}\Big(1+\frac{g^2n}{qT}\Big)^{-2}\bigg)^2. \qquad (7)$$

We split the sum over n into the ranges $n \le qT$ and $n > qT$. In the first case, the sum over g is

$$\ll 1 + \sum_{\substack{\sqrt{Z/n} \le g \le \sqrt{qT/n} \\ (g,q)=1}} \frac{1}{g}.$$

When $n \le Z_0$ this is

$$\ll \frac{\varphi(q)}{q}\,\omega(q).$$

by Lemma 5. In the alternative case $n > Z_0$ we extend the sum over g to include all $g \le 3^{\omega(q)}$ that are coprime to q. Lemma 5 then gives the same bound as before. Thus the contribution of the terms $n \le qT$ to (7), using Lemma 6, is

$$\ll \varphi(q)T\left(\frac{\varphi(q)}{q}\omega(q)\right)^2 \sum_{\substack{n \le qT \\ (n,q)=1}} \frac{2^{\omega(n)}}{n} \ll qT\left(\frac{\varphi(q)}{q}\right)^5 \omega(q)^2 (\log qT)^2. \tag{8}$$

In the remaining case $n > qT$, the sum over g in (7) is $O(q^2T^2/n^2)$. Hence the contribution of such terms is

$$\ll \varphi(q)T \sum_{n > qT} \frac{2^{\omega(n)}}{n}\frac{q^4T^4}{n^4} \ll \varphi(q)T\log qT.$$

In view of (6) and (8) we now have

$$\sideset{}{^*}\sum_{\chi (\bmod\ q)} \int_0^T B(t,\chi)^2 dt \ll qT\left(\frac{\varphi(q)}{q}\right)^5 \omega(q)^2(\log qT)^2 + \varphi(q)T(\log qT)^3. \tag{9}$$

5. Deduction of Theorem 1

From Lemma 1 we have

$$\sideset{}{^*}\sum_{\chi (\bmod\ q)} \int_0^T |L(\tfrac{1}{2}+it,\chi)|^4 dt = 4 \sideset{}{^*}\sum_{\chi (\bmod\ q)} \int_0^T \big(A(t,\chi)^2 + 2A(t,\chi)B(t,\chi) + B(t,\chi)^2\big) dt.$$

The first and third terms on the right hand side are handled by (4) and (9). Also, by Cauchy's inequality we have

$$\sideset{}{^*}\sum_{\chi (\bmod\ q)} \int_0^T A(t,\chi)B(t,\chi)dt \le \left(\sideset{}{^*}\sum_{\chi (\bmod\ q)} \int_0^T A(t,\chi)^2 dt\right)^{\frac{1}{2}} \left(\sideset{}{^*}\sum_{\chi (\bmod\ q)} \int_0^T B(t,\chi)^2 dt\right)^{\frac{1}{2}}.$$

Hence (4) and (9) also yield an estimate for the cross term. Combining these results leads to the theorem.

References

[1] D. R. Heath-Brown, *The fourth power mean of Dirichlet's L-functions*, Analysis **1** (1981), 25–32.
[2] H. Montgomery, *Topics in multiplicative number theory*, Lecture Notes in Mathematics, Vol. 227, (Springer-Verlag, Berlin-New York, 1971).
[3] Y. Motohashi, *A note on the mean value of the zeta and L-functions. II.*, Proc. Japan Acad. Ser. A Math. Sci. **61** (1985), 313–316.
[4] V. V. Rane, *A note on the mean value of L-series*, Proc. Indian Acad. Sci. Math. Sci. **90** (1981), 273–286.
[5] K. Soundararajan, *The fourth moment of Dirichlet L-functions*, Analytic Number Theory: A Tribute to Gauss and Dirichlet, 2007.
[6] W. Wang, *Fourth power mean value of Dirichlet's L-functions*, International Symposium in Memory of Hua Loo Keng, Vol. I (Beijing, 1988), 293–321, (Springer, Berlin, 1991).
[7] M. Young, *The fourth moment of Dirichlet L-functions*, http://arxiv.org/abs/math/0610335.

Mathematical Institute, University of Oxford, OXFORD, OX1 3LB
E-mail address: hung.bui@maths.ox.ac.uk

Mathematical Institute, University of Oxford, OXFORD, OX1 3LB
E-mail address: rhb@maths.ox.ac.uk