ON SIMPLE ZEROS OF THE RIEMANN ZETA-FUNCTION

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Abstract. We show that at least 19/27 of the zeros of the Riemann zeta-function are simple, assuming the Riemann Hypothesis (RH). This was previously established by Conrey, Ghosh and Gonek [Proc. London Math. Soc. **76** (1998), 497–522] under the additional assumption of the Generalised Lindelöf Hypothesis (GLH). We are able to remove this hypothesis by careful use of the generalised Vaughan identity.

1. Introduction

An important question in number theory is to understand the distribution of the zeros of the Riemann zeta-function. In this paper, we study the simple zeros on the critical line.

Let $N(T)$ denote the number of zeros $\rho = \beta + i\gamma$ with $0 < \gamma < T$, where each zero is counted with multiplicity, denoted by $m(\rho)$. Let $N^*(T)$ denote the number of such zeros which are simple ($m(\rho) = 1$), and let $N_d(T)$ denote the number of such distinct zeros (i.e. each zero is counted precisely once without regard to its multiplicity). Define κ^* and κ_d by

$$\kappa^* := \liminf_{T\to\infty} \frac{N^*(T)}{N(T)}, \qquad \kappa_d := \liminf_{T\to\infty} \frac{N_d(T)}{N(T)}.$$

Unconditionally, it is known that $\kappa^* \geq 0.4058$ (see [**9**, **7**, **4**, **1**, **5**, **2**] for results in this direction). Conditionally, using the pair correlation of the zeros of the Riemann zeta-function Montgomery [**11**] showed that $\kappa^* \geq 2/3$ and $\kappa_d \geq 5/6$ on RH. This was later improved by Cheer and Goldston [**3**] to $\kappa^* \geq 0.6727$ under the same condition. Assuming RH and GLH, Conrey, Ghosh and Gonek [**6**] showed that $\kappa^* \geq 19/27$ and $\kappa_d \geq 0.84568$. Their paper used the mollifier method (described in the next section). We also note that Montgomery's pair correlation conjecture implies that almost all the zeros are simple.

In this paper, we use Heath-Brown's generalisation of the Vaughan identity [**8**] (see Lemma 3 in Section 3 below) to remove the GLH assumption in the paper of Conrey, Ghosh and Gonek. As a result, we obtain

Theorem. *Assuming RH we have*

$$\kappa^* \geq \frac{19}{27}.$$

Corollary. *Assuming RH we have*

$$\kappa_d \geq 0.84665.$$

The corollary is a consequence of our theorem following an observation of Montgomery [**11**] that

$$2N^*(T) \leq \sum_{0<\gamma\leq T} \frac{(m(\rho)-2)(m(\rho)-3)}{m(\rho)} \leq \sum_{0<\gamma\leq T} m(\rho) - 5N(T) + 6N_d(T).$$

Cheer and Goldston [3] also showed that

$$\sum_{0<\gamma\le T} m(\rho) \le \big(1.3275 + o(1)\big)N(T).$$

Hence

$$\kappa_d \ge \frac{5 + 2\kappa^* - 1.3275}{6} \ge 0.84665.$$

Before embarking on the proof we record one piece of notation that we will use throughout the paper, namely that we will write $\mathscr{L} = \log T/2\pi$.

2. The setup

To get a lower bound for κ^*, it suffices to consider the first and second mollified moments of the derivative of the Riemann zeta-function. This section is mostly a summary of [6].

We first note that ρ is a simple zero if and only if $\zeta'(\rho) \ne 0$. Hence it follows from Cauchy's inequality that

$$N^*(T) \ge \frac{\big|\sum_{0<\gamma\le T} B\zeta'(\rho)\big|^2}{\sum_{0<\gamma\le T}\big|B\zeta'(\rho)\big|^2}, \tag{1}$$

for any regular function $B(s)$. Here we shall take $B(s)$ to be a mollifier of the form

$$B(s) = \sum_{k\le y} \frac{b(k)}{k^s},$$

where

$$b(k) = \mu(k)P\bigg(\frac{\log y/k}{\log y}\bigg), \tag{2}$$

with $P(x)$ being a polynomial with real coefficients satisfying $P(0) = 0$, $P(1) = 1$, and $y = T^{\vartheta}$, $0 < \vartheta < 1/2$.

The following result is essentially in [6] (see (3.13), (3.21), (3.26), (3.27), (5.1), (5.4), (5.5)).

Lemma 1. *For any fixed $\varepsilon > 0$ we have*

$$\begin{aligned} S_1 &:= \sum_{0<\gamma\le T} B\zeta'(\rho) \\ &= \frac{T\mathscr{L}^2}{2\pi} - \overline{\mathcal{M}_1} + O(T\mathscr{L}) + O_\varepsilon(yT^{1/2+\varepsilon}) \end{aligned}$$

and

$$\begin{aligned} S_2 &:= \sum_{0<\gamma\le T} B\zeta'(\rho)B\zeta'(1-\rho) \\ &= \frac{T\mathscr{L}^3}{2\pi}\bigg(\tfrac{1}{2} + 3\vartheta\int_0^1 P(u)^2 du\bigg) - 2\mathrm{Re}(\mathcal{M}_2) + O_\varepsilon(T\mathscr{L}^{2+\varepsilon}) + O_\varepsilon(yT^{1/2+\varepsilon}), \end{aligned}$$

where

$$\mathcal{M}_\nu = \sum_{k\le y}\sum_{m\le kT/2\pi} \frac{a_\nu(m)b(k)}{k} e\bigg(-\frac{m}{k}\bigg). \tag{3}$$

Here $\nu = 1$ *or* 2*, and the coefficients* $a_\nu(m)$ *are defined by*

$$\frac{\zeta'}{\zeta}(s)\zeta'(s) = \sum_{n=1}^{\infty} \frac{a_1(n)}{n^s}, \qquad \frac{\zeta'}{\zeta}(s)\zeta'(s)^2 B(s) = \sum_{n=1}^{\infty} \frac{a_2(n)}{n^s}. \tag{4}$$

The main difficulty in the paper of Conrey, Ghosh and Gonek is to extract the main terms and estimate the error terms in $\mathcal{M}_\nu$. At this point, if we assume the Generalised Riemann Hypothesis (GRH), we can apply Perron's formula to the sum over m in (3), and then move the line of integration to $\mathrm{Re}(s) = 1/2 + \varepsilon$. The main terms arise from the residues of the pole at $s = 1$ and the error terms in this case are easy to handle. To avoid assuming GRH, however, we first need to express the additive character $e(-m/k)$ in (3) in terms of multiplicative characters, and then write $\mathcal{M}_\nu$ in the following form (see [**6**; (5.12) and (5.14)])

$$\mathcal{M}_\nu = \sum_{q \le y} \sideset{}{^*}\sum_{\psi (\mathrm{mod}\ q)} \tau(\overline{\psi}) \sum_{k \le y/q} \frac{b(kq)}{kq} \sum_{d | kq} \delta(q, kq, d, \psi) \sum_{m \le kqT/2\pi d} a_\nu(md)\psi(m), \tag{5}$$

where $\sum^*$ denotes summation over all primitive characters $\psi (\mathrm{mod}\ q)$, $\tau(\psi)$ is the Gauss sum, and

$$\delta(q, kq, d, \psi) = \sum_{l | (d,k)} \frac{\mu(d/l)}{\varphi(kq/l)} \overline{\psi}\Big(\frac{-k}{l}\Big)\psi\Big(\frac{d}{l}\Big)\mu\Big(\frac{k}{l}\Big). \tag{6}$$

Following [**6**] we choose a large constant A and set $\eta = \mathscr{L}^A$. We then split the q-summation into three cases: $q = 1$, $1 < q \le \eta$, and $\eta < q \le y$. We write $\mathcal{M}_\nu = \mathcal{M}_{\nu,1} + \mathcal{M}_{\nu,2} + \mathcal{M}_{\nu,3}$ accordingly. The case $q = 1$ gives rise to the main terms (see [**6**; Section 8])

$$\mathcal{M}_{1,1} = \frac{T\mathscr{L}^2}{2\pi}\Big(\tfrac12 - \vartheta \int_0^1 P(u)du\Big) + O(T\mathscr{L}) \tag{7}$$

and

$$\begin{aligned}\mathcal{M}_{2,1} \;=\; \frac{T\mathscr{L}^3}{2\pi}\Big(&\tfrac{1}{12} - \tfrac{\vartheta}{2}\int_0^1 P(u)du + \tfrac{3\vartheta}{2}\int_0^1 P(u)^2du \\ &- \tfrac{\vartheta^2}{2}\Big(\int_0^1 P(u)du\Big)^2 - \tfrac{1}{24\vartheta}\int_0^1 P'(u)^2du\Big) + O(T\mathscr{L}^2).\end{aligned} \tag{8}$$

The terms with $1 < q \le \eta$ are handled using Siegel's theorem on exceptional real zeros of L-functions, (see [**6**; (5.15) and (6.14)]) to give

$$\mathcal{M}_{\nu,2} \ll_A T \exp\big(-c(A)\sqrt{\log T}\big) \qquad (\nu = 1, 2), \tag{9}$$

where $c(A)$ is a positive function of A.

Up to this point, all the analysis is unconditional. To study the remaining case, in which $\mathscr{L}^A = \eta < q \le y$, Conrey, Ghosh and Gonek used the Vaughan identity and the large sieve. Their approach requires the assumption of GLH (or precisely, an upper bound for averages of sixth moments of Dirichlet L-functions). In the next section, we shall illustrate how Heath-Brown's generalisation of the Vaughan identity can be used to obtain unconditionally the following estimate.

Lemma 2. *We have*

$$\mathcal{M}_{\nu,3} \ll_\varepsilon y^{1/3}T^{5/6+\varepsilon} + \eta^{-1/2}T\mathscr{L}^C \qquad (\nu = 1, 2),$$

for some absolute constant $C > 0$ and for any fixed $\varepsilon > 0$.

We finish the section with the deduction of our theorem. Given $\vartheta < 1/2$, Lemma 1, Lemma 2 and (5)–(7) give

$$S_1 \sim \frac{T\mathscr{L}^2}{2\pi}\bigg(\tfrac12 + \vartheta\int_0^1 P(u)du\bigg)$$

and

$$S_2 \sim \frac{T\mathscr{L}^3}{2\pi}\bigg(\tfrac13 + \vartheta\int_0^1 P(u)du + \vartheta^2\bigg(\int_0^1 P(u)du\bigg)^2 + \tfrac{1}{12\vartheta}\int_0^1 P'(u)^2du\bigg).$$

Choosing $P(x) = -\vartheta x^2 + (1+\vartheta)x$ and letting $\vartheta \to 1/2^-$ we obtain

$$S_1 \sim \frac{19}{24}\frac{T\mathscr{L}^2}{2\pi} \qquad \text{and} \qquad S_2 \sim \frac{57}{64}\frac{T\mathscr{L}^3}{2\pi}. \tag{10}$$

Assuming RH we have $S_2 = \sum_{0<\gamma\le T}|B\zeta'(\rho)|^2$. Note that this is the only place we need RH. The theorem then follows from (1) and (10).

3. Proof of Lemma 2

We shall prove Lemma 2 for $\mathcal{M}_{2,3}$, the treatment of $\mathcal{M}_{1,3}$ being similar.

3.1. **Initial cleaning.** There are problems arising with the condition $d|kq$ in (5), since we would like to be able to separate the variables d and q. Indeed it appears that Conrey, Ghosh and Gonek run into difficulties at this point in deducing [**6**; (7.2)] from [**6**; (5.15)]. To circumvent such problems we begin by observing that the function $b(*)$ given by (2) is supported on squarefree values. It follows that we can restrict k and q in (5) to be coprime. Thus the variable l in (6) will be coprime to q. One then sees that the term $\psi(d/l)$ will vanish unless d is also coprime to q, since ψ is a character to modulus q. Finally, if d is coprime to q, the condition $d|kq$ reduces to $d|k$. We therefore conclude that

$$\mathcal{M}_{2,3} = \sum_{\eta<q\le y}\ \sideset{}{^*}\sum_{\psi(\text{mod } q)} \tau(\overline{\psi})\sum_{k\le y/q}\frac{b(kq)}{kq}\sum_{d|k}\delta(q,kq,d,\psi)\sum_{m\le kqT/2\pi d} a_2(md)\psi(m). \tag{11}$$

We divide the summation over k, q, d in (11) into dyadic intervals

$$K/2 < k \le K, \quad Q/2 < q \le Q, \quad D < d \le 2D,$$

where

$$Q > \eta = \mathscr{L}^A, \quad D \le K \quad \text{and} \quad KQ \le 4y. \tag{12}$$

Then there will be some such triple K, Q, D for which we have

$$\mathcal{M}_{2,3} \ll \mathscr{L}^3\sum_{d\sim D}\sum_{\substack{k\sim K\\ d|k}}\sum_{q\sim Q}|\tau(\overline{\psi})|\frac{|b(kq)|}{kq}|\delta(q,kq,d,\psi)|\ \sideset{}{^*}\sum_{\psi(\text{mod } q)}\bigg|\sum_{m\le kqT/2\pi d} a_2(md)\psi(m)\bigg|.$$

We now note that $|\tau(\psi)| = q^{1/2}$ and

$$\delta(q,kq,d,\psi) \ll \sum_{l|d}\frac{1}{\varphi(kq/l)} \ll \mathscr{L}dk^{-1}q^{-1},$$

since $\varphi(n) \gg n/\log\log n$ and $\sigma(n) \ll n\log\log n$. This allows us to write

$$\begin{aligned}\mathcal{M}_{2,3} &\ll K^{-2}Q^{-3/2}D\mathscr{L}^4 \sum_{d\sim D}\sum_{\substack{k\sim K\\ d|k}}\sum_{q\sim Q}\sideset{}{^*}\sum_{\psi(\text{mod } q)}\left|\sum_{m\le kqT/2\pi d} a_2(md)\psi(m)\right| \\ &\ll K^{-2}Q^{-3/2}D\mathscr{L}^4 \sum_{d\sim D}\sum_{\substack{k\sim K\\ d|k}} S(Q,X,d),\end{aligned}$$

where we have defined

$$X = KQT/\pi D$$

and

$$S(Q,X,d) = \sum_{q\sim Q}\sideset{}{^*}\sum_{\psi(\text{mod } q)}\max_{M\le X}\left|\sum_{m\le M} a_2(md)\psi(m)\right|.$$

Since the number of available values for k is $\ll K/d \ll K/D$ we conclude that

$$\mathcal{M}_{2,3} \ll K^{-1}Q^{-3/2}\mathscr{L}^4\sum_{d\sim D} S(Q,X,d). \tag{13}$$

The sum $S(Q,X,d)$ would be in a suitable form to apply the maximal large sieve, if it involved the square of the innermost sum. However X is too large compared with Q for one merely to apply Cauchy's inequality. Thus the strategy is to use a generalisation of the Vaughan identity to write the function a_2 as a convolution, thereby enabling us to replace the innermost sum by a product of two Dirichlet polynomials. Providing these two polynomials are of suitable lengths a satisfactory estimate will emerge. The details of our implementation differ from those of Conrey, Ghosh and Gonek in two important ways. Firstly, by using the identity in Lemma 3 we produce more flexibility in the choice of lengths for our Dirichlet polynomials. Secondly, Conrey, Ghosh and Gonek used $L(s,\psi)$ where we employ a finite Dirichlet polynomial of the type $\sum_{h\sim H} h^{-s}\psi(m)$. This is clearly advantageous if H is small.

There are two inconvenient technical problems which need to be dealt with. Firstly, since we have $a_2(md)$ rather than merely $a_2(m)$ we have to handle the dependence on d. Secondly, when we replace a_2 by a convolution we need to eliminate the condition $m \le M$. We do this in the standard way by using Perron's formula, which introduces a further variable, and a further averaging, into our analysis.

3.2. **The generalised Vaughan identity.**

Heath-Brown's version [**8**] of the Vaughan identity comes from the following trivial lemma.

Lemma 3. *For any integer $r \ge 1$ we have*

$$\zeta'(s)/\zeta(s) = \sum_{j=1}^{r}(-1)^{j-1}\binom{r}{j}\zeta(s)^{j-1}\zeta'(s)M(s)^j + \big(1-\zeta(s)M(s)\big)^r\zeta'(s)/\zeta(s), \tag{14}$$

where

$$M(s) = \sum_{n\le X}\frac{\mu(n)}{n^s}.$$

We apply Lemma 3 to the sum $S(Q,X,d)$, where the coefficients $a_2(n)$ are defined in (4), so that $a_2=-\Lambda*\log*\log*b$. We choose $r=3$, $X=T^{1/2}$, and pick out the relevant coefficients of n^{-s} with $n=md$. Since

$$Md\leq KQT/\pi\leq 4yT/\pi<T^{3/2} \tag{15}$$

for large T, we see that the last term on the right hand side of (14) makes no contribution. On splitting each range of summation into dyadic intervals, we find that $a_2(md)$ is a linear combination of $O(\mathscr{L}^9)$ expressions of the form $(f_1*\ldots*f_9)(md)$, where the functions f_i are independent of m and d, and are each supported on a dyadic interval $(N_i/2,N_i]$, say. For terms in which the function f_i is absent we set $N_i=1$ and take the corresponding function f_i to be the identity for the Dirichlet convolution, so that $f_i(1)=1$ and $f_i(m)=0$ for $m\geq 2$. Whenever $N_i>1$ we can take

$$f_1=f_2=f_3=\log,\quad f_4=b,\quad f_5=f_6=1,\quad\text{and}\quad f_7=f_8=f_9=\mu.$$

Moreover

$$N_4\leq y\quad\text{and}\quad N_7,N_8,N_9\leq T^{1/2}.$$

We observe that the numbers N_i run over powers of 2 or, in the case of N_4 over numbers $2^{-h}y$. Since these are independent of q we can estimate $S(Q,X,d)$ as

$$S(Q,X,d)\ll\sum_{N_i}\sum_{q\sim Q}\sideset{}{^*}\sum_{\psi(\bmod q)}\max_{M\leq X}\Big|\sum_{m\leq M}(f_1*\ldots*f_9)(md)\psi(m)\Big|,$$

where the sum over N_i runs through $O(\mathscr{L}^9)$ sets of values with $\prod N_i\ll Xd$.

To evaluate $(f_1*\ldots*f_9)(md)$ we call on Lemma 3 of Conrey, Ghosh and Gonek [**6**], which shows that

$$(f_1*\ldots*f_9)(md)=\sum_{d=d_1\ldots d_9}(g_1*\ldots*g_9)(m),$$

with

$$g_i(m)=g_i(m;d_1,\ldots,d_i)=\begin{cases}f_i(md_i), & \text{if } (m,d_1\ldots d_{i-1})=1,\\ 0, & \text{otherwise.}\end{cases}$$

Each g_i is now supported on a dyadic interval $(M_i/2,M_i]$ with $M_i=N_i/d_i$, so that $\prod M_i\ll X$.

This allows us to estimate $S(Q,X,d)$ as

$$S(Q,X,d)\ll\sum_{N_i}\sum_{d=d_1\ldots d_9}\sum_{q\sim Q}\sideset{}{^*}\sum_{\psi(\bmod q)}\max_{M\leq X}\Big|\sum_{m\leq M}(g_1*\ldots*g_9)(m)\psi(m)\Big|.$$

We begin by disposing of the case in which $M_i>yT^{1/2}$ for some index i, which will necessarily be 1, 2, 3, 5 or 6. For $B\ll X$ we can use partial summation to show that

$$\sum_{A<h\leq B}g_i(h)\psi(h)\ll\mathscr{L}\max_{Y\leq B}\left|\sum_{\substack{h\leq Y\\(h,D_i)=1}}\psi(h)\right|,$$

with $D_i=d_1\ldots d_{i-1}$. Moreover

$$\sum_{\substack{h\leq Y\\(h,D_i)=1}}\psi(h)=\sum_{e|D_i}\mu(e)\psi(e)\sum_{k\leq Y/e}\psi(k)\ll_\varepsilon\tau(D_i)q^{1/2}\log q$$

by the Pólya–Vinogradov inequality. It follows that

$$\sum_{A<h\le B} g_i(h)\psi(h) \ll_\varepsilon Q^{1/2}T^\varepsilon$$

for $B \ll X$.

We now write g for the convolution of the 8 functions g_j with $j \ne i$, so that g is supported on integers $n \ll X/M_i$, and $g(n) \ll_\varepsilon T^\varepsilon$. Then

$$\begin{aligned}\sum_{m\le M}(g_1*\ldots*g_9)(m)\psi(m) &= \sum_n g(n)\psi(n)\sum_{\substack{h\sim M_i\\ h\le M/n}} g_i(h)\psi(h)\\ &\ll_\varepsilon XM_i^{-1}Q^{1/2}T^{2\varepsilon}\\ &\ll_\varepsilon XQ^{1/2}y^{-1}T^{-1/2+2\varepsilon}.\end{aligned}$$

The contribution to $S(Q,X,d)$ when $M_i > yT^{1/2}$ is therefore

$$\begin{aligned}&\ll_\varepsilon \mathscr{L}^9\tau_9(d)Q^2.XQ^{1/2}y^{-1}T^{-1/2+2\varepsilon}\\ &\ll_\varepsilon KQ^{7/2}D^{-1}y^{-1}T^{1/2+3\varepsilon} \qquad (16)\end{aligned}$$

by (15) and (12).

Before handling the remaining terms we must eliminate the condition $m \le M$, which may be done via Perron's formula. Let $M_0 = M + 1/2$ and $\delta = (\log M)^{-1}$, and take $U > 0$. Then

$$\frac{1}{2\pi i}\int_{\delta-iU}^{\delta+iU}\left(\frac{M_0}{m}\right)^s\frac{ds}{s} = \left\{\begin{array}{ll}1 & \text{if } m\le M\\ 0 & \text{if } m>M\end{array}\right\} + O(MU^{-1}).$$

Thus

$$S(Q,X,d)$$
$$\ll\ 1+\sum_{N_i}\sum_{d=d_1\ldots d_9}\int_{-U}^{U}\frac{\log X}{1+|t|}\sum_{q\sim Q}\ \sideset{}{^*}\sum_{\psi(\mathrm{mod}\ q)}\left|\sum_m (g_1*\ldots*g_9)(m)\psi(m)m^{-\delta-it}\right|dt,$$

provided that $U \gg Q^2X^2T^{\delta'}$ for some fixed $\delta' > 0$. The reader should note here that the sum over m is finite, being supported on values

$$m = m_1\ldots m_9 \le M_1\ldots M_9 \ll X.$$

We now choose $U = T^5$, which is more than sufficient when $\delta' = 1/2$, say. We proceed to define functions $h_j(m) = g_j(m)m^{-\delta}$, and set

$$H_j(\psi,t) = \sum_{m\sim M_j} h_j(m)\psi(m)m^{-it},$$

which allows us to conclude that

$$S(Q,X,d) \ll 1+\mathscr{L}^2\sum_{N_i}\sum_{d=d_1\ldots d_9}\max_{1\le V\le T^5} V^{-1}T(Q,V), \qquad (17)$$

where

$$T(Q,V) = T(Q,V;d_1,\ldots,d_9) = \sum_{q\sim Q}\ \sideset{}{^*}\sum_{\psi(\mathrm{mod}\ q)}\int_{-V}^{V}|H_1(\psi,t)\ldots H_9(\psi,t)|dt.$$

3.3. **Estimating** $T(Q,V)$**.** We may now suppose that $M_i \le yT^{1/2}$ for every index i. Our strategy is to split $\prod H_i(\psi,t)$ into a product $\mathscr{A}(\psi,t)\mathscr{B}(\psi,t)$ of Dirichlet polynomials of approximately equal lengths A and B respectively. They will take the form

$$\sum_{m\le A} a_m\psi(m)m^{-it}, \quad \text{and} \quad \sum_{m\le B} b_m\psi(m)m^{-it}$$

with coefficients such that

$$|a_m|, |b_m| \ll \mathscr{L}^3\tau_9(m). \tag{18}$$

Giving $\mathscr{A}(\psi,t)$ and $\mathscr{B}(\psi,t)$ approximately equal lengths will optimise our eventual application of the hybrid large sieve. To achieve this we introduce a parameter $A_0 \ge yT^{1/2}$, to be specified in due course, with the aim of making $\max\{A,B\} \ll A_0$. We recall that

$$X = KQT/\pi D. \tag{19}$$

Now, if there is a factor $H_i(\psi,t)$ of length $M_i \ge KQT(DA_0)^{-1}$ we can merely take $\mathscr{A}(\psi,t) = H_i(\psi,t)$. We will then have $A = M_i \le yT^{1/2} \le A_0$. Moreover, since $A = M_i \ge KQT(DA_0)^{-1}$, the corresponding factor $\mathscr{B}(\psi,t)$ will have $B \ll X/A \ll KQT(DA)^{-1} \ll A_0$ as required. We can therefore assume that each $H_i(\psi,t)$ has length $M_i \le KQT(DA_0)^{-1}$.

In this remaining case we define J as the largest integer for which $\prod_{j\le J} M_j \le A_0$. We then set

$$\mathscr{A}(\psi,t) = \prod_{j\le J} H_j(\psi,t), \quad \text{and} \quad \mathscr{B}(\psi,t) = \prod_{J<j\le 9} H_j(\psi,t)$$

so that $A \le A_0$. Moreover our construction implies that $AM_{J+1} > A_0$, and since we are assuming that $M_i \le KQT(DA_0)^{-1}$ for every index i we see that

$$A \gg \frac{A_0}{KQT(DA_0)^{-1}},$$

whence

$$B \ll \frac{X}{A} \ll \frac{KQT/D}{A} \ll \left(\frac{KQT}{DA_0}\right)^2.$$

We therefore see that if we set

$$A_0 = \max\left\{yT^{1/2}, \, (KQT/D)^{2/3}\right\} \tag{20}$$

then we can always produce a factorisation with $A, B \ll A_0$.

Having chosen $\mathscr{A}(\psi,t)$ and $\mathscr{B}(\psi,t)$ we proceed to apply Cauchy's inequality to obtain

$$T(Q,V) \le T(\mathscr{A})^{1/2}T(\mathscr{B})^{1/2},$$

where

$$T(\mathscr{A}) = \sum_{q\sim Q} \sideset{}{^*}\sum_{\psi (\mathrm{mod}\ q)} \int_{-V}^{V} |\mathscr{A}(\psi,t)|^2 dt,$$

and similarly for $T(\mathscr{B})$. We then use the hybrid large sieve in the form

$$\sum_{q\sim Q} \sideset{}{^*}\sum_{\psi (\mathrm{mod}\ q)} \int_{-V}^{V} \left|\sum_{m\le H} h_m\right|^2 dt \ll (Q^2V + H)\sum |h_m|^2,$$

due to Montgomery [**10**; Theorem 7.1]. This produces a bound

$$T(\mathscr{A}) \ll (Q^2V + A) \sum_{m \le A} |a_m|^2 \ll (Q^2V + A)A\mathscr{L}^{86},$$

in view of (18), and similarly for $T(\mathscr{B})$. It follows that

$$\begin{aligned} T(Q,V) &\ll \big\{(Q^2V + A)A\big\}^{1/2}\big\{(Q^2V + B)B\big\}^{1/2}\mathscr{L}^{86} \\ &\ll \big\{Q^2VX^{1/2} + QV^{1/2}X^{1/2}\max\{A^{1/2}, B^{1/2}\} + X\big\}\mathscr{L}^{86} \\ &\ll \big\{Q^2VX^{1/2} + QV^{1/2}X^{1/2}A_0^{1/2} + X\big\}\mathscr{L}^{86}. \end{aligned}$$

Since $V \ge 1$ we now deduce from (19) and (20) that

$$\begin{aligned} V^{-1}T(Q,V) &\ll_\varepsilon K^{1/2}Q^{5/2}D^{-1/2}T^{1/2+\varepsilon} + K^{1/2}Q^{3/2}D^{-1/2}y^{1/2}T^{3/4+\varepsilon} \\ &\quad + K^{5/6}Q^{11/6}D^{-5/6}T^{5/6+\varepsilon} + KQD^{-1}T\mathscr{L}^{86} \end{aligned} \tag{21}$$

for any fixed $\varepsilon > 0$, when $\max M_i \le yT^{1/2}$.

3.4. **Deduction of Lemma 2.** Putting the estimate (21) into (17) and comparing with (16) we get

$$\begin{aligned} S(Q,X,d) &\ll_\varepsilon KQ^{7/2}D^{-1}y^{-1}T^{1/2+3\varepsilon} + K^{1/2}Q^{5/2}D^{-1/2}T^{1/2+2\varepsilon} \\ &\quad + K^{1/2}Q^{3/2}D^{-1/2}y^{1/2}T^{3/4+2\varepsilon} + K^{5/6}Q^{11/6}D^{-5/6}T^{5/6+2\varepsilon} \\ &\quad + KQ\tau_9(d)D^{-1}T\mathscr{L}^{97}, \end{aligned}$$

whence (12) and (13) yield

$$\mathcal{M}_{2,3} \ll_\varepsilon Q^2y^{-1}T^{1/2+4\varepsilon} + QT^{1/2+3\varepsilon} + y^{1/2}T^{3/4+3\varepsilon} + Q^{1/3}T^{5/6+3\varepsilon} + Q^{-1/2}T\mathscr{L}^{109}.$$

Thus since $\eta \ll Q \ll y$ we have

$$\mathcal{M}_{2,3} \ll_\varepsilon yT^{1/2+4\varepsilon} + y^{1/2}T^{3/4+3\varepsilon} + y^{1/3}T^{5/6+3\varepsilon} + \eta^{-1/2}T\mathscr{L}^{109},$$

and since $y \le T^{1/2}$ the bound required for Lemma 2 follows, on re-defining ε.

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