

CENTRAL VALUES OF DERIVATIVES OF DIRICHLET L -FUNCTIONS

H. M. BUI AND M. B. MILINOVICH

ABSTRACT. Let \mathcal{C}_q^+ be the set of even, primitive Dirichlet characters (mod q). Using the mollifier method we show that $L^{(k)}(\frac{1}{2}, \chi) \neq 0$ for almost all the characters $\chi \in \mathcal{C}_q^+$ when k and q are large. Here $L(s, \chi)$ is the Dirichlet L -function associated to the character χ .

1. INTRODUCTION & STATEMENT OF THE MAIN RESULT

An important topic in number theory is the behavior of families of L -functions and their derivatives inside the critical strip. In particular, questions concerning the order of vanishing of L -functions at special points on the critical line have received a great deal of attention. In the case of Dirichlet L -functions, it is widely believed that $L(\frac{1}{2}, \chi) \neq 0$ for all primitive characters χ . For quadratic characters χ , this appears to have been first conjectured by Chowla (see Chapter 8 of [3]).

Though a proof of the non-vanishing of Dirichlet L -functions at the central point, $s = 1/2$, has remained elusive, there has been considerable progress in showing that $L(\frac{1}{2}, \chi)$ is very often non-zero within various families of characters χ . In [10], Iwaniec and Sarnak show that at least $1/3$ of Dirichlet L -functions in the family of even primitive characters, to a large modulus q , do not vanish at the central point. This improves upon earlier work of Balasubramanian and Murty [1]. Soundararajan [16] has shown that at least $7/8$ of the central values in the family of quadratic Dirichlet L -functions are non-zero. More recently, Baier and Young [2] consider the family of Dirichlet L -functions associated to cubic and sextic characters and show that infinitely many (though not a positive proportion) of these functions are not zero at the central point.

In [15], Michel and VanderKam consider the behavior of the derivatives of completed Dirichlet L -functions, $\Lambda(s, \chi)$, at the central point. (See §2, below, for a definition.) In particular, they show that for $\varepsilon > 0$ and q sufficiently large depending on ε , the inequality

$$\sum_{\substack{\chi \in \mathcal{C}_q^+ \\ \Lambda^{(k)}(\frac{1}{2}, \chi) \neq 0}} 1 \geq (P_k - \varepsilon) \cdot \sum_{\chi \in \mathcal{C}_q^+} 1 \tag{1}$$

holds, where the proportion

$$P_k = \frac{2}{3} - \frac{1}{36k^2} - \frac{c}{k^4}$$

for some absolute constant $c > 0$. As k tends to infinity, the proportion P_k approaches two thirds. This is analogous to a result of Conrey [4], who shows that almost all of the zeros of the k -th derivative of the Riemann ξ -function are on the critical line, and to a result of Kowalski, Michel and VanderKam [13] who show that almost half of the set $\{\Lambda^{(k)}(\frac{1}{2}, f)\}$ is non-zero, where f runs over the set of primitive Hecke eigenforms of weight 2 relative to $\Gamma_0(q)$. This last result is best possible because half of these forms are even and half are odd. However, unlike the results in [4] and [13], the inequality in (1) is not best possible since it is expected that $P_k = 1$ for every positive integer k .

In contrast to [15], we study the behavior of the functions $L^{(k)}(s, \chi)$, the derivatives of Dirichlet L -functions, at $s = \frac{1}{2}$. When k and q are sufficiently large, we show that $L^{(k)}(\frac{1}{2}, \chi) \neq 0$ for almost all of the even, primitive characters χ . As is the case in [4] and [13], our result is asymptotically best possible as k tends to infinity.

Theorem 1.1. *Let $k \in \mathbb{N}$. Then, for $\varepsilon > 0$ and q sufficiently large (depending on ε), we have*

$$\sum_{\substack{\chi \in \mathcal{C}_q^+ \\ L^{(k)}(\frac{1}{2}, \chi) \neq 0}} 1 \geq (P_k^* - \varepsilon) \cdot \sum_{\chi \in \mathcal{C}_q^+} 1, \quad (2)$$

where the proportion

$$P_k^* = 1 - \frac{1}{16k^2} - \frac{c}{k^4} \quad (3)$$

for some absolute constant $c > 0$. In particular, $P_1^* \geq .7544$, $P_2^* \geq .9083$, $P_3^* \geq .9642$, $P_4^* \geq .9853$, $P_5^* \geq .9935$, and $P_{25}^* \geq .9999$.

Theorem 1.1 confirms a prediction of Conrey and Snaith which arises from the L -functions Ratios Conjectures (see §8.1 of [8]). Their heuristic is based upon studying the behavior of the mollified moments of the derivatives of the Riemann zeta-function in t -aspect which they conjecture should behave similarly to the mollified moments of the derivatives of Dirichlet L -functions at the central point in q -aspect. This is in agreement with the conjectures of Keating and Snaith [11, 12] that suggest that both of these families of L -functions, the Riemann zeta-function in t -aspect and Dirichlet L -functions in q -aspect, should have the same underlying “unitary” symmetry and so their (mollified) moments should behave similarly. See [5] for a detailed discussion of these ideas. In particular, our Proposition 2.2 is a q -analogue of a result of Conrey and Ghosh¹ who computed the mollified moments of the derivatives of the Riemann zeta-function on the critical line.

We remark that Theorem 1.1 does not improve upon the main result of [15]. In fact, for $k \in \mathbb{N}$, the zeros of the functions $L^{(k)}(s, \chi)$ and $\Lambda^{(k)}(s, \chi)$ are expected to behave quite differently. To illustrate this point, let χ be a primitive character and assume that the Riemann Hypothesis (RH_χ) holds for the function $L(s, \chi)$. Then all the non-trivial zeros of $L(s, \chi)$ and all the zeros of $\Lambda(s, \chi)$ lie on the critical line $\text{Re } s = \frac{1}{2}$. In addition, $L(s, \chi)$ has an infinite number of trivial zeros on the negative real axis. Under the RH_χ , one can prove that all the zeros of $\Lambda^{(k)}(s, \chi)$ lie on the line $\text{Re } s = \frac{1}{2}$. In contrast, it can be shown that all but possibly a finite number of the non-real zeros of $L^{(k)}(s, \chi)$ are forced to lie in the half-plane $\text{Re } s \geq \frac{1}{2}$ and it is very likely the case that none of these zeros lie on the critical line.² In particular, it is reasonable to conjecture that $L^{(k)}(\frac{1}{2}, \chi) \neq 0$ for all primitive characters χ and all $k \in \mathbb{N}$. However, if χ is an even, real-valued, primitive (i.e. quadratic) character, then the functional equation for $L(s, \chi)$ states that $\Lambda(s, \chi) = \Lambda(1 - s, \chi)$. It follows from this that $\Lambda^{(k)}(\frac{1}{2}, \chi) = 0$ whenever k is odd. Thus, the analogous conjecture for $\Lambda^{(k)}(\frac{1}{2}, \chi)$ fails for infinitely many values of k and infinitely many characters χ .

1.1. Notation & Conventions. We say a Dirichlet character $\chi \pmod{q}$ is even if $\chi(-1) = 1$. We let \mathcal{C}_q denote the set of primitive characters \pmod{q} and let \mathcal{C}_q^+ denote the subset of characters in \mathcal{C}_q which are even. We put $\varphi^+(q) = \frac{1}{2}\varphi^*(q)$ where

$$\varphi^*(q) = \sum_{k|q} \varphi(k) \mu\left(\frac{q}{k}\right) = |\mathcal{C}_q|;$$

¹See equation (7) of [6].

²We can show that if q is sufficiently large, then the only zeros of $L'(s, \chi)$ on the critical line are the multiple zeros of $L(s, \chi)$. However, it is believed that the zeros of $L(s, \chi)$ are simple.

the proof of this appears in Lemma 4.1, below. It is not difficult to show that $|\mathcal{C}_q^+| = \varphi^+(q) + O(1)$. In addition, we write $\sum_{\chi(\bmod q)}^+$ to indicate that the summation is restricted to $\chi \in \mathcal{C}_q^+$ and we write $\sum_{a(\bmod q)}^*$ and \sum_n^* to indicate that the summation is restricted to the residues $a(\bmod q)$ which are coprime to q and to n which are relatively prime to q , respectively.

2. THE MOLLIFIED MOMENTS OF $L^{(k)}(\frac{1}{2}, \chi)$

As may be expected, we prove Theorem 1.1 by computing certain mollified first and second moments of $L^{(k)}(\frac{1}{2}, \chi)$ over the characters $\chi \in \mathcal{C}_q^+$ and then we use Cauchy's inequality.

For $\chi \in \mathcal{C}_q^+$, the Dirichlet L -function $L(s, \chi)$ satisfies the functional equation

$$\Lambda(s, \chi) := \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi) = \varepsilon_\chi \Lambda(1-s, \bar{\chi}), \quad (4)$$

where $\bar{\chi}$ is conjugate character of χ , $\varepsilon_\chi = \tau(\chi)q^{-1/2}$, and $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{a(\bmod q)} \chi(a)e\left(\frac{a}{q}\right); \quad e(x) = e^{2\pi i x}.$$

Note that $|\varepsilon_\chi| = 1$ and, since χ is even, $\overline{\tau(\chi)} = \tau(\bar{\chi})$. For each $\chi \in \mathcal{C}_q^+$, we let

$$M(\chi) = M(\chi, P, y) := \sum_{n \leq y} \frac{\mu(n)\chi(n)}{\sqrt{n}} P\left(\frac{\log y/n}{\log y}\right), \quad (5)$$

where P is an arbitrary polynomial satisfying the conditions $P(0) = 0$ and $P(1) = 1$. The purpose of the function $M(\chi)$ is to smooth out or ‘‘mollify’’ the large values of $L^{(k)}(\frac{1}{2}, \chi)$ as we average over $\chi \in \mathcal{C}_q^+$. Since $|\varepsilon_\chi L^{(k)}(\frac{1}{2}, \bar{\chi})| = |L^{(k)}(\frac{1}{2}, \chi)|$, if we let

$$S_1(k, q) = \sum_{\chi(\bmod q)}^+ \varepsilon_\chi L^{(k)}(\frac{1}{2}, \bar{\chi}) M(\chi) \quad (6)$$

and

$$S_2(k, q) = \sum_{\chi(\bmod q)}^+ |L^{(k)}(\frac{1}{2}, \chi)|^2 |M(\chi)|^2, \quad (7)$$

then Cauchy's inequality implies that

$$\sum_{\substack{\chi(\bmod q) \\ L^{(k)}(\frac{1}{2}, \chi) \neq 0}}^+ 1 \geq \frac{|S_1(k, q)|^2}{S_2(k, q)}. \quad (8)$$

Thus, we require a lower bound for $|S_1(k, q)|$ and an upper bound for $S_2(k, q)$. The following propositions provide such estimates.

Proposition 2.1. *Let $k \in \mathbb{N}$. Then, for $y = q^\vartheta$ and $0 < \vartheta < 1$, we have*

$$S_1(k, q) = (-1)^k \varphi^+(q) \log^k q (1 + O((\log q)^{-1})),$$

where the implied constant depends on ϑ and k .

Proposition 2.2. *Let k be a positive integer and $\varepsilon > 0$ be arbitrary. Then, for $y = q^\vartheta$ and $0 < \vartheta < \frac{1}{2}$, we have*

$$S_2(k, q) = C_k(\vartheta) \varphi^+(q) \log^{2k} q (1 + O((\log q)^{-1+\varepsilon})),$$

where

$$C_k(\vartheta) = \frac{\vartheta^{-1}}{2k+1} \int_0^1 P'(x)^2 dx + \frac{1}{2} + \frac{\vartheta k^2}{2k-1} \int_0^1 P(x)^2 dx,$$

and the implied constant depends on ϑ , ε , and k .

It is clear from (8) and the propositions that in order to prove Theorem 1.1 we need to choose the polynomial P , for each $k \geq 1$, which minimizes the constant $\mathcal{C}_k(\vartheta)$. This is done in §6. It turns out that except for a term which is exponentially small (as a function of k), the optimal choice of P is independent of the choice of ϑ . This is not surprising, since similar phenomena have been observed when mollifying high derivatives of the Riemann zeta-function and the Riemann ξ -function on the critical line, and also when mollifying high derivatives of families of L -functions at the central point (see [4, 6, 13, 15]).

3. PROOF OF PROPOSITION 2.1

In this section we establish Proposition 2.1. The result we require is implicit in [15] (see §3, page 135) where it is shown that³

$$\sum_{\chi(\bmod q)}^+ \Lambda^{(k)}\left(\frac{1}{2}, \chi\right) M(\chi) = \varphi^+(q) \Gamma\left(\frac{1}{4}\right) \hat{q}^{1/2} \log^k \hat{q} \left(1 + O((\log q)^{-1})\right) \quad (9)$$

for $k \in \mathbb{N}$ and $0 < \vartheta < 1$. Here $\hat{q} = \sqrt{q/\pi}$ and the implied constant depends on ϑ . From (4), we see that

$$\varepsilon_\chi L(s, \bar{\chi}) = H_q(s) \Lambda(1-s, \chi), \quad \text{where } H_q(s) = \frac{\hat{q}^{-s}}{\Gamma\left(\frac{s}{2}\right)}. \quad (10)$$

Using well-known estimates for the gamma function, it follows that

$$H_q^{(k)}\left(\frac{1}{2}\right) = (-1)^k \frac{\hat{q}^{-1/2}}{\Gamma\left(\frac{1}{4}\right)} \log^k \hat{q} \left(1 + O_k((\log q)^{-1})\right) \quad (11)$$

for each $k \in \mathbb{N}$. Now, combining (9), (10), (11) and using the Leibniz formula for differentiation, we find that

$$\begin{aligned} \sum_{\chi(\bmod q)}^+ \varepsilon_\chi L^{(k)}\left(\frac{1}{2}, \bar{\chi}\right) M(\chi) &= \sum_{\chi(\bmod q)}^+ \sum_{\ell=0}^k \binom{k}{\ell} H_q^{(\ell)}\left(\frac{1}{2}\right) (-1)^{k-\ell} \Lambda^{(k-\ell)}\left(\frac{1}{2}, \chi\right) M(\chi) \\ &= \sum_{\ell=0}^k \binom{k}{\ell} (-1)^{k-\ell} H_q^{(\ell)}\left(\frac{1}{2}\right) \sum_{\chi(\bmod q)}^+ \Lambda^{(k-\ell)}\left(\frac{1}{2}, \chi\right) M(\chi) \\ &= (-1)^k \sum_{\ell=0}^k \binom{k}{\ell} \varphi^+(q) \log^k \hat{q} \left(1 + O((\log q)^{-1})\right) \\ &= (-1)^k 2^k \varphi^+(q) \log^k \hat{q} \left(1 + O((\log q)^{-1})\right), \end{aligned}$$

where the implied constant depends on ϑ and k . Since $2 \log \hat{q} = \log q + O(1)$, we can conclude that

$$\sum_{\chi(\bmod q)}^+ \varepsilon_\chi L^{(k)}\left(\frac{1}{2}, \bar{\chi}\right) M(\chi) = (-1)^k \varphi^+(q) \log^k q \left(1 + O((\log q)^{-1})\right).$$

This establishes Proposition 2.1.

³It follows from the functional equation for $\Lambda(s, \chi)$ that the quantity $\mathcal{L}(P_k)$ in §3 of [15] is equal to

$$2 \sum_{\chi(\bmod q)}^+ \Lambda^{(k)}\left(\frac{1}{2}, \chi\right) M(\chi).$$

4. SOME PRELIMINARY RESULTS

In this section we collect some preliminary results which we will use to establish Proposition 2.2. In what follows, q is a large positive integer and $\alpha, \beta \in \mathbb{C}$ are taken to be small shifts satisfying $|\alpha|, |\beta| \leq 2(\log q)^{-1}$.

Our first lemma concerns the orthogonality of primitive characters.

Lemma 4.1. *For $(mn, q) = 1$ we have*

$$\sum_{\chi(\bmod q)}^+ \chi(m)\bar{\chi}(n) = \frac{1}{2} \sum_{\substack{q=dr \\ r|m \pm n}} \mu(d)\varphi(r),$$

where the sums for the different signs \pm are to be taken separately.

Proof. Let

$$f(h) = \sum_{\chi(\bmod h)}^* \chi(m)\bar{\chi}(n)$$

where \sum^* denotes summation over primitive characters χ . Then for $(mn, q) = 1$ we have

$$\sum_{h|q} f(h) = \sum_{\chi(\bmod q)} \chi(m)\bar{\chi}(n) = \begin{cases} \varphi(q), & \text{if } m \equiv n \pmod{q}, \\ 0, & \text{otherwise.} \end{cases}$$

Using the Möbius inversion we obtain

$$\sum_{\chi(\bmod q)}^* \chi(m)\bar{\chi}(n) = f(q) = \sum_{\substack{h|q \\ h|m-n}} \varphi(h)\mu(q/h).$$

It follows from this identity that

$$|\mathcal{C}_q| = \sum_{\chi(\bmod q)}^* 1 = \sum_{k|q} \varphi(k)\mu\left(\frac{q}{k}\right),$$

which justifies an above remark. Our lemma now follows by noting that

$$\sum_{\substack{\chi(\bmod q) \\ \chi(-1)=1}}^* \chi(m)\bar{\chi}(n) = \sum_{\chi(\bmod q)}^* \left[\frac{1 + \chi(-1)}{2} \right] \chi(m)\bar{\chi}(n).$$

□

Lemma 4.2. *Let $G(s)$ be an even, entire function with rapid decay as $|s| \rightarrow \infty$ in any fixed vertical strip $A \leq \sigma \leq B$ and with $G(0) = 1$. Let*

$$W_{\alpha, \beta}^{\pm}(x) = \frac{1}{2\pi i} \int_{(1)} G(s)H(s)g_{\alpha, \beta}^{\pm}(s)x^{-s} \frac{ds}{s}, \quad (12)$$

where

$$g_{\alpha, \beta}^+(s) = \frac{\Gamma(\frac{1/2+\alpha+s}{2})\Gamma(\frac{1/2+\beta+s}{2})}{\Gamma(\frac{1/2+\alpha}{2})\Gamma(\frac{1/2+\beta}{2})}, \quad g_{\alpha, \beta}^-(s) = \frac{\Gamma(\frac{1/2-\alpha+s}{2})\Gamma(\frac{1/2-\beta+s}{2})}{\Gamma(\frac{1/2+\alpha}{2})\Gamma(\frac{1/2+\beta}{2})},$$

and

$$H(s) = \frac{(\frac{\alpha+\beta}{2})^2 - s^2}{(\frac{\alpha+\beta}{2})^2} \quad (\alpha + \beta \neq 0).$$

Then for $\chi_1, \chi_2 \in \mathcal{C}_q^+$, $\alpha \neq -\beta$ we have that

$$\begin{aligned} L(\tfrac{1}{2} + \alpha, \chi_1)L(\tfrac{1}{2} + \beta, \chi_2) &= \sum_{m,n} \frac{\chi_1(m)\chi_2(n)}{m^{1/2+\alpha}n^{1/2+\beta}} W_{\alpha,\beta}^+ \left(\frac{\pi mn}{q} \right) \\ &\quad + \varepsilon_{\chi_1} \varepsilon_{\chi_2} \left(\frac{q}{\pi} \right)^{-\alpha-\beta} \sum_{m,n} \frac{\overline{\chi_1}(m)\overline{\chi_2}(n)}{m^{1/2-\alpha}n^{1/2-\beta}} W_{\alpha,\beta}^- \left(\frac{\pi mn}{q} \right). \end{aligned}$$

Some Remarks.

- (1) An admissible choice of G in the above lemma is $G(s) = \exp(s^2)$.
- (2) The purpose of the function $H(s)$ in the above lemma is to cancel the poles of the functions $\zeta_q(1 \pm (\alpha + \beta) + 2s)$ at $s = \mp(\alpha + \beta)/2$ which appear in the proof of the next lemma. This substantially simplifies our later calculations. A similar effect has been observed by Conrey, Iwaniec, and Soundararajan (see §3 of [7]).

Proof. Consider the integral

$$I_{\alpha,\beta} = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} G(s)H(s) \frac{\Lambda(1/2 + \alpha + s, \chi_1)\Lambda(1/2 + \beta + s, \chi_2)}{\Gamma(\frac{1/2+\alpha}{2})\Gamma(\frac{1/2+\beta}{2})} \frac{ds}{s}.$$

Shifting the line of integration to $\operatorname{Re} s = -1$ and using Cauchy's theorem, it follows that

$$I_{\alpha,\beta} = R_0 + \frac{1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} G(s)H(s) \frac{\Lambda(1/2 + \alpha + s, \chi_1)\Lambda(1/2 + \beta + s, \chi_2)}{\Gamma(\frac{1/2+\alpha}{2})\Gamma(\frac{1/2+\beta}{2})} \frac{ds}{s},$$

where R_0 is the residue of the integrand at $s = 0$. Evidently,

$$R_0 = \left(\frac{q}{\pi} \right)^{(1+\alpha+\beta)/2} L(\tfrac{1}{2} + \alpha, \chi_1)L(\tfrac{1}{2} + \beta, \chi_2).$$

By making the change of variables s to $-s$ and using (4), we have that

$$R_0 = I_{\alpha,\beta} + \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} G(s)H(s) \frac{\Lambda(1/2 - \alpha + s, \overline{\chi_1})\Lambda(1/2 - \beta + s, \overline{\chi_2})}{\Gamma(\frac{1/2+\alpha}{2})\Gamma(\frac{1/2+\beta}{2})} \frac{ds}{s}.$$

The lemma now follows by using (4) to express the Λ -functions in terms of Dirichlet series and then integrating term-by-term. \square

Lemma 4.3. *Let*

$$S_{\alpha,\beta}^+(x) = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{W_{\alpha,\beta}^+(n^2/x)}{n^{1+\alpha+\beta}} \quad \text{and} \quad S_{\alpha,\beta}^-(x) = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{W_{\alpha,\beta}^-(n^2/x)}{n^{1-\alpha-\beta}}.$$

Then, for any $\varepsilon > 0$ and $\alpha \neq -\beta$, we have that

$$S_{\alpha,\beta}^+(x) = \zeta_q(1 + \alpha + \beta) + O(\tau(q)x^{-1/2+\varepsilon})$$

and

$$S_{\alpha,\beta}^-(x) = g_{\alpha,\beta}^-(0)\zeta_q(1 - \alpha - \beta) + O(\tau(q)x^{-1/2+\varepsilon}),$$

where $\tau(q)$ is the number of divisors of q and the function $\zeta_q(s)$ is defined by

$$\zeta_q(s) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s} \right).$$

Proof. From (12), we observe that

$$S_{\alpha,\beta}^+(x) = \frac{1}{2\pi i} \int_{(1)} G(s)H(s)g_{\alpha,\beta}^+(s)x^s \zeta_q(1 + \alpha + \beta + 2s) \frac{ds}{s}.$$

We now shift the line of integration left to $\operatorname{Re} s = -1/2 + \varepsilon$, encountering only a simple pole of the integrand at $s = 0$. We note that the simple pole of $\zeta_q(1 + \alpha + \beta + 2s)$ at $s = -(\alpha + \beta)/2$ is canceled by a zero $H(s)$. The residue of the integrand at $s = 0$ is $\zeta_q(1 + \alpha + \beta)$. Also, the integral along the new contour is trivially $\ll \tau(q)x^{-1/2+\varepsilon}$. This implies the first claim of the lemma. The second claim can be proved in a similar manner. \square

Lemma 4.4. *Assume $\alpha \neq -\beta$ and let*

$$\mathcal{B}(m_1, n_1; \alpha, \beta) = \sum_{\chi \pmod{q}}^+ L\left(\frac{1}{2} + \alpha, \chi\right) L\left(\frac{1}{2} + \beta, \bar{\chi}\right) \chi(m_1) \bar{\chi}(n_1).$$

Then for $(m_1, n_1) = 1$ and $(m_1 n_1, q) = 1$ we have

$$\begin{aligned} \mathcal{B}(m_1, n_1; \alpha, \beta) &= \frac{\varphi^+(q)}{\sqrt{m_1 n_1}} \left(\frac{\zeta_q(1 + \alpha + \beta)}{m_1^\beta n_1^\alpha} + \left(\frac{q}{\pi}\right)^{-\alpha-\beta} g_{\alpha,\beta}^-(0) \frac{\zeta_q(1 - \alpha - \beta)}{m_1^{-\alpha} n_1^{-\beta}} \right) \\ &\quad + O(\beta(m_1, n_1) + q^{1/2+\varepsilon}), \end{aligned}$$

where $\beta(m_1, n_1)$ satisfies

$$\sum_{m_1, n_1 \leq y} \frac{\beta(m_1, n_1)}{\sqrt{m_1 n_1}} \ll yq^{1/2+\varepsilon}.$$

Proof. With $\chi_1 = \chi$, $\chi_2 = \bar{\chi}$, Lemma 4.1 and Lemma 4.2 imply that

$$\begin{aligned} \mathcal{B}(m_1, n_1; \alpha, \beta) &= \frac{1}{2} \sum_{q=dr} \mu(d)\varphi(r) \sum_{r|mm_1 \pm nn_1}^* \frac{W_{\alpha,\beta}^+(\frac{\pi mn}{q})}{m^{1/2+\alpha} n^{1/2+\beta}} \\ &\quad + \frac{1}{2} \left(\frac{q}{\pi}\right)^{-\alpha-\beta} \sum_{q=dr} \mu(d)\varphi(r) \sum_{r|mn_1 \pm nm_1}^* \frac{W_{\alpha,\beta}^-(\frac{\pi mn}{q})}{m^{1/2-\alpha} n^{1/2-\beta}}, \end{aligned} \quad (13)$$

where \sum^* denotes summation over all $(mn, q) = 1$. The main contribution to $\mathcal{B}(m_1, n_1; \alpha, \beta)$ comes from the diagonal terms $mm_1 = nn_1$ and $mn_1 = nm_1$ in the first and second sums on the right-hand side of (13), respectively. For $(m_1, n_1) = 1$, this contribution is

$$\begin{aligned} &\varphi^+(q) \left(\sum_{mm_1=nn_1}^* \frac{W_{\alpha,\beta}^+(\frac{\pi mn}{q})}{m^{1/2+\alpha} n^{1/2+\beta}} + \left(\frac{q}{\pi}\right)^{-\alpha-\beta} \sum_{mn_1=nm_1}^* \frac{W_{\alpha,\beta}^-(\frac{\pi mn}{q})}{m^{1/2-\alpha} n^{1/2-\beta}} \right) \\ &= \varphi^+(q) \left(\frac{S_{\alpha,\beta}^+(\frac{q}{\pi m_1 n_1})}{n_1^{1/2+\alpha} m_1^{1/2+\beta}} + \left(\frac{q}{\pi}\right)^{-\alpha-\beta} \frac{S_{\alpha,\beta}^-(\frac{q}{\pi m_1 n_1})}{m_1^{1/2-\alpha} n_1^{1/2-\beta}} \right), \end{aligned}$$

where $S_{\alpha,\beta}^\pm(x)$ are defined in Lemma 4.3. By Lemma 4.3 the above expression is equal to

$$\frac{\varphi^+(q)}{\sqrt{m_1 n_1}} \left(\frac{\zeta_q(1 + \alpha + \beta)}{m_1^\beta n_1^\alpha} + \left(\frac{q}{\pi}\right)^{-\alpha-\beta} g_{\alpha,\beta}^-(0) \frac{\zeta_q(1 - \alpha - \beta)}{m_1^{-\alpha} n_1^{-\beta}} \right) + O(q^{1/2+\varepsilon}).$$

All the other terms in (13) contribute at most

$$\beta(m_1, n_1) = \sum_{mm_1 \neq nn_1} \frac{(mm_1 \pm nn_1, q)}{\sqrt{mn}} \left| W_{\alpha,\beta}^\pm \left(\frac{\pi mn}{q} \right) \right|.$$

Using the estimate $|W_{\alpha,\beta}^{\pm}(x)| \ll (1 + |x|)^{-1}$ one can show that (see [10], Section 4)

$$\sum_{m_1, n_1 \leq y} \frac{\beta(m_1, n_1)}{\sqrt{m_1 n_1}} \ll yq^{1/2+\varepsilon} (\log yq)^4.$$

The lemma now follows from the above estimates. \square

Lemma 4.5. *Let*

$$S_j(d) = \sum_{\substack{n \leq y/d \\ (n,dq)=1}} \frac{\mu(n)}{n} (\log n)^j P\left(\frac{\log y/dn}{\log y}\right).$$

Then $S_j(d) = M_j(d) + O(E_j(d))$ uniformly for $d \leq y$, where

$$M_0(d) = \frac{dq}{\varphi(dq) \log y} P'\left(\frac{\log y/d}{\log y}\right), \quad M_1(d) = -\frac{dq}{\varphi(dq)} P\left(\frac{\log y/d}{\log y}\right),$$

$M_j(d) = 0$ ($j \geq 2$), and

$$E_j(d) = (\log y)^{j-2} (\log \log y)^2 (1 + (d/y)^\theta \log y) \prod_{p|dq} \left(1 + \frac{1}{p^{1-2\delta}}\right)$$

with $\theta \gg 1/\log \log y$ and $\delta = 1/\log \log y$.

Proof. Consider the Dirichlet polynomial

$$G(z) = \sum_{\substack{n \leq y/d \\ (n,dq)=1}} \frac{\mu(n)}{n^{1+z}} P\left(\frac{\log y/dn}{\log y}\right).$$

Since, for $n \leq y/d$, we have

$$P\left(\frac{\log y/dn}{\log y}\right) = \sum_{\ell \geq 1} \frac{a_\ell}{(\log y)^\ell} (\log y/dn)^\ell = \sum_{\ell \geq 1} \frac{a_\ell \ell!}{(\log y)^\ell} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(\frac{y}{dn}\right)^s \frac{ds}{s^{\ell+1}},$$

we can express $G(z)$ as

$$G(z) = \sum_{\ell \geq 1} \frac{a_\ell \ell!}{(\log y)^\ell} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(\frac{y}{d}\right)^s A(s+z) \frac{ds}{\zeta(1+z+s) s^{\ell+1}}$$

where

$$A(s) = \prod_{p|dq} \left(1 - \frac{1}{p^{1+s}}\right)^{-1}.$$

We note that $G(z)$ is precisely $G_j(1+z)$ in Lemma 10 of Conrey [4] (see the first expression in the proof), with x being replaced by y/d and $1/F(j, s)$ being replaced by $A(s-1)$. Using this, we obtain that

$$G^{(j)}(z) = M_j(d; z) + O(E_j(d)) \tag{14}$$

uniformly for $0 < |z| \ll 1/\log y$, where

$$M_0(d; z) = A(z) \left[z P\left(\frac{\log y/d}{\log y}\right) + \frac{1}{\log y} P'\left(\frac{\log y/d}{\log y}\right) \right],$$

$$M_1(d; z) = A(z) P\left(\frac{\log y/d}{\log y}\right), \text{ and}$$

$$M_j(d; z) = 0 \quad \text{for } j \geq 2.$$

Since $G(z)$ and $M_j(d; z)$ are both holomorphic in z , (14) also holds for $z = 0$. Observing that $S_j(d) = (-1)^j G^{(j)}(0)$ and $A(0) = dq/\varphi(dq)$, the lemma follows. \square

Lemma 4.6. *Suppose that $f(d) = \prod_{p|d} f(p)$ with $f(p) = 1 + O(p^{-c})$ for some $c > 0$ and that*

$$J_j(y) = \sum_{d \leq y}^* \frac{\mu(d)^2}{d} f(d) \left(\log \frac{y}{d} \right)^j.$$

Then we have

$$J_j(y) = \frac{1}{j+1} \prod_p \left(1 - \frac{1}{p} \right) \left(1 + \frac{f(p)}{p} \right) \prod_{p|q} \left(1 + \frac{f(p)}{p} \right)^{-1} (\log y)^{j+1} + O((\log y)^j).$$

Proof. We consider only the case where $j \geq 1$. The case $j = 0$ can be handled by following the proof of Lemma 3.11 of Levinson [14]. We first express $J_j(y)$ as a complex integral, namely

$$J_j(y) = \frac{j!}{2\pi i} \int_{2-i\infty}^{2+i\infty} \sum_{(d,q)=1} \frac{\mu(d)^2 f(d)}{d^{1+s}} y^s \frac{ds}{s^{j+1}}.$$

The sum over d is

$$\prod_{p \nmid q} \left(1 + \frac{f(p)}{p^{1+s}} \right) = B(s) \zeta(1+s),$$

where

$$B(s) = \prod_p \left[\left(1 - \frac{1}{p^{1+s}} \right) \left(1 + \frac{f(p)}{p^{1+s}} \right) \right] \prod_{p|q} \left(1 + \frac{f(p)}{p^{1+s}} \right)^{-1}.$$

Since $f(p) = 1 + O(p^{-c})$ for some $c > 0$, $B(s)$ is absolutely and uniformly convergent in some half-plane containing the origin. We now shift the line of integration left to $\operatorname{Re} s = -\delta$, crossing a pole of order $(j+2)$ at $s = 0$. Here $\delta > 0$ is some small, fixed constant chosen so that the arithmetical factor $B(s)$ converges absolutely for $\operatorname{Re} s \geq -\delta$. Using Cauchy's theorem and the bound $\zeta(s) \ll (1+|t|)^{1/2+\delta}$ on the new line of integration, we obtain the estimate

$$J_j(y) = \frac{1}{j+1} B(0) (\log y)^{j+1} + O((\log y)^j).$$

The lemma now follows. \square

5. PROOF OF PROPOSITION 2.2

In this section, we prove Proposition 2.2. Throughout the proof, we let $y = q^\vartheta$ and assume that $0 < \vartheta < \frac{1}{2}$. We begin by considering the mollified ‘‘shifted’’ second moment

$$J_{\alpha,\beta}(q) = \sum_{\chi \pmod{q}}^+ L\left(\frac{1}{2} + \alpha, \chi\right) L\left(\frac{1}{2} + \beta, \bar{\chi}\right) |M(\chi)|^2, \quad (15)$$

where $\alpha, \beta \in \mathbb{C}$ are small shifts satisfying $|\alpha|, |\beta| \leq (\log q)^{-1}$ and $\alpha \neq -\beta$. Applying Lemma 4.4, we have that

$$\begin{aligned} J_{\alpha,\beta}(q) &= \sum_{m,n \leq y} \frac{\mu(m)\mu(n)}{\sqrt{mn}} P\left(\frac{\log y/m}{\log y}\right) P\left(\frac{\log y/n}{\log y}\right) \mathcal{B}(m,n;\alpha,\beta) \\ &= \Sigma_1(\alpha,\beta) + \Sigma_2(\alpha,\beta) + O(yq^{1/2+\varepsilon}), \end{aligned} \quad (16)$$

where

$$\Sigma_1(\alpha,\beta) = \varphi^+(q) \zeta_q(1+\alpha+\beta) \sum_{d \leq y}^* \sum_{\substack{m,n \leq y/d \\ (m,n)=1}}^* \frac{\mu(dm)\mu(dn)}{dm^{1+\beta}n^{1+\alpha}} P\left(\frac{\log y/dm}{\log y}\right) P\left(\frac{\log y/dn}{\log y}\right)$$

and

$$\begin{aligned} \Sigma_2(\alpha, \beta) &= \varphi^+(q) \left(\frac{q}{\pi}\right)^{-\alpha-\beta} g_{\alpha, \beta}^-(0) \zeta_q(1-\alpha-\beta) \times \\ &\quad \sum_{d \leq y}^* \sum_{\substack{m, n \leq y/d \\ (m, n) = 1}}^* \frac{\mu(dm)\mu(dn)}{dm^{1-\alpha}n^{1-\beta}} P\left(\frac{\log y/dm}{\log y}\right) P\left(\frac{\log y/dn}{\log y}\right). \end{aligned}$$

We can remove the restriction $(m, n) = 1$ by writing $K_{\alpha, \beta}(q) := \Sigma_1(\alpha, \beta) + \Sigma_2(\alpha, \beta)$ as

$$\begin{aligned} \varphi^+(q) \sum_{cd \leq y}^* \frac{\mu(c)\mu(cd)^2}{c^2d} \times \\ \sum_{\substack{m, n \leq y/cd \\ (mn, cdq) = 1}} \frac{\mu(m)\mu(n)}{mn} P\left(\frac{\log y/cdm}{\log y}\right) P\left(\frac{\log y/cdn}{\log y}\right) Z_{q, \alpha, \beta}(m, n, c), \end{aligned} \quad (17)$$

where

$$Z_{q, \alpha, \beta}(m, n, c) = \frac{\zeta_q(1+\alpha+\beta)}{c^{\alpha+\beta} m^\beta n^\alpha} + \left(\frac{q}{\pi}\right)^{-\alpha-\beta} g_{\alpha, \beta}^-(0) \frac{\zeta_q(1-\alpha-\beta)}{c^{-\alpha-\beta} m^{-\alpha} n^{-\beta}}. \quad (18)$$

Though the function $\zeta_q(s)$ has a simple pole at $s = 1$, we note that $Z_{q, \alpha, \beta}(m, n, c)$ is holomorphic in both α and β in a small neighborhood of $\alpha = \beta = 0$ (as can be seen, for instance, by computing the Laurent series expansion of each of the terms on the right-hand side of (18) about $\alpha = \beta = 0$). Therefore, the expressions in (15) and (17) provide an analytic continuation of the function $J_{\alpha, \beta}(q) - K_{\alpha, \beta}(q)$ to the region $|\alpha|, |\beta| \leq (\log q)^{-1}$; the function $K_{0,0}(q)$ must be defined in terms of the limit

$$Z_{q, 0, 0}(m, n, c) = \lim_{\alpha \rightarrow 0} \left(\frac{\zeta_q(1+2\alpha)}{(c^2 mn)^\alpha} + \left(\frac{q}{\pi}\right)^{-2\alpha} \frac{\zeta_q(1-2\alpha)}{(c^2 mn)^{-\alpha}} \right).$$

Moreover, by the maximum modulus principle and (16), we see that

$$\left| J_{\alpha, \beta}(q) - K_{\alpha, \beta}(q) \right| \ll_\varepsilon yq^{1/2+\varepsilon}$$

uniformly for $|\alpha|, |\beta| \leq (\log q)^{-1}$. Hence, by Cauchy's Integral Theorem,

$$\begin{aligned} \frac{d^{2k}}{d\alpha^k d\beta^k} \left[J_{\alpha, \beta}(q) - K_{\alpha, \beta}(q) \right] \Big|_{\alpha=\beta=0} \\ = \frac{(k!)^2}{(2\pi i)^2} \int_{\mathcal{C}_\alpha} \int_{\mathcal{C}_\beta} \frac{J_{w_\alpha, w_\beta}(q) - K_{w_\alpha, w_\beta}(q)}{(w_\alpha w_\beta)^{k+1}} dw_\alpha dw_\beta \\ \ll_{k, \varepsilon} yq^{1/2+2\varepsilon}, \end{aligned}$$

where \mathcal{C}_α (resp. \mathcal{C}_β) denotes the positively oriented circle in the complex plane centered at $\alpha = 0$ (resp. $\beta = 0$) with radius $(\log q)^{-1}$. Thus, we have shown that

$$S_2(k, q) = \frac{d^{2k}}{d\alpha^k d\beta^k} K_{\alpha, \beta}(q) \Big|_{\alpha=\beta=0} + O_{k, \varepsilon}(yq^{1/2+2\varepsilon}). \quad (19)$$

Writing

$$\begin{aligned} \frac{d^{2k}}{d\alpha^k d\beta^k} Z_{q, \alpha, \beta}(m, n, c) \Big|_{\alpha=\beta=0} \\ = \sum_{h+i+j \leq 2k+1} \left(a_{h, i, j} (\log c)^h + b_{h, i, j} (\log q/c)^h \right) (\log m)^i (\log n)^j \end{aligned}$$

for certain constants $a_{h,i,j}$ and $b_{h,i,j}$, we see that

$$\begin{aligned} & \frac{d^{2k}}{d\alpha^k d\beta^k} K_{\alpha,\beta}(q) \Big|_{\alpha=\beta=0} \\ &= \varphi^+(q) \sum_{h+i+j \leq 2k+1} \sum_{cd \leq y}^* \left(a_{h,i,j} (\log c)^h + b_{h,i,j} (\log cq)^h \right) \frac{\mu(c)\mu(cd)^2}{c^2 d} S_i(cd) S_j(cd), \end{aligned} \quad (20)$$

where S_i and S_j are defined in Lemma 4.5. It follows from Lemma 4.5 that

$$S_i(cd) \ll_i \frac{cdq}{\varphi(cdq)} (\log y)^{i-1},$$

from which it can be seen that the contribution of the terms with $h+i+j \leq 2k$ to the sum on the right-hand side of (20) is

$$\ll_k (\log q)^{2k-1} q \varphi^+(q) / \varphi(q) \ll_{k,\varepsilon} \varphi^+(q) (\log q)^{2k-1+\varepsilon}.$$

The last estimate holds since $q/\varphi(q) \ll \log \log q$. It remains to consider the contribution of the terms with $h+i+j = 2k+1$. In the notation of Lemma 4.5, it can be shown that

$$\sum_{cd \leq y}^* \frac{S_i(cd) E_j(cd)}{c^2 d} \ll_{i,j,\varepsilon} (\log y)^{i+j-2+\varepsilon}$$

and

$$\sum_{cd \leq y}^* \frac{E_i(cd) E_j(cd)}{c^2 d} \ll_{i,j,\varepsilon} (\log y)^{i+j-3+\varepsilon}.$$

Hence the contribution of the error terms E_i and E_j , arising from Lemma 4.5, to the terms in (20) with $h+i+j = 2k+1$ is $\ll_{k,\varepsilon} \varphi^+(q) (\log q)^{2k-1+\varepsilon}$. Thus,

$$\begin{aligned} & \frac{d^{2k}}{d\alpha^k d\beta^k} K_{\alpha,\beta}(q) \Big|_{\alpha=\beta=0} \\ &= \varphi^+(q) \sum_{h+i+j=2k+1} \sum_{cd \leq y}^* \left(a_{h,i,j} (\log c)^h + b_{h,i,j} (\log cq)^h \right) \frac{\mu(c)\mu(cd)^2}{c^2 d} M_i(cd) M_j(cd) \\ & \quad + O_{k,\varepsilon}(\varphi^+(q) (\log q)^{2k-1+\varepsilon}). \end{aligned}$$

Since $M_i(cd) = 0$ for $i > 1$, we need only to consider the terms with $0 \leq i, j \leq 1$. Moreover, the terms involving powers of $\log c$ can be ignored, as they contribute (due to the presence of c^{-2} in the sum) an amount which is $\ll_{k,\varepsilon} (\log q)^{2k-1+\varepsilon}$. Therefore, the above expression simplifies to

$$\frac{d^{2k}}{d\alpha^k d\beta^k} K_{\alpha,\beta}(q) \Big|_{\alpha=\beta=0} = T_1 + 2T_2 + T_3 + O_{k,\varepsilon}(\varphi^+(q) (\log q)^{2k-1+\varepsilon}), \quad (21)$$

where

$$T_1 = \varphi^+(q) \sum_{cd \leq y}^* b_{2k+1,0,0} (\log q)^{2k+1} \frac{\mu(c)\mu(cd)^2}{c^2 d} M_0(cd)^2,$$

$$T_2 = \varphi^+(q) \sum_{cd \leq y}^* b_{2k,1,0} (\log q)^{2k} \frac{\mu(c)\mu(cd)^2}{c^2 d} M_0(cd) M_1(cd),$$

and

$$T_3 = \varphi^+(q) \sum_{cd \leq y}^* b_{2k-1,1,1} (\log q)^{2k-1} \frac{\mu(c)\mu(cd)^2}{c^2 d} M_1(cd)^2.$$

We first evaluate T_1 . Using Lemma 4.5 we have that

$$\begin{aligned} T_1 &= \varphi^+(q) \frac{b_{2k+1,0,0} q^2 (\log q)^{2k+1}}{\varphi(q)^2 (\log y)^2} \sum_{cd \leq y}^* \frac{\mu(c) \mu(cd)^2 d}{\varphi(cd)^2} P' \left(\frac{\log y / cd}{\log y} \right)^2 \\ &= \varphi^+(q) \frac{b_{2k+1,0,0} q^2 (\log q)^{2k+1}}{\varphi(q)^2 (\log y)^2} \sum_{n \leq y}^* \frac{\mu(n)^2}{\varphi(n)} P' \left(\frac{\log y / n}{\log y} \right)^2. \end{aligned}$$

Now Lemma 4.6 implies that

$$\sum_{n \leq y}^* \frac{\mu(n)^2}{\varphi(n)} P' \left(\frac{\log y / n}{\log y} \right)^2 = \frac{\varphi(q)}{q} (\log y + O(1)) \int_0^1 P'(x)^2 dx.$$

Hence

$$T_1 = \varphi^+(q) \frac{b_{2k+1,0,0} q (\log q)^{2k+1}}{\varphi(q) \log y} \int_0^1 P'(x)^2 dx + O_{k,\varepsilon}(\varphi^+(q) (\log q)^{2k-1+\varepsilon}). \quad (22)$$

Similarly, it can be shown that

$$\begin{aligned} T_2 &= -\varphi^+(q) \frac{b_{2k,1,0} q (\log q)^{2k}}{\varphi(q)} \int_0^1 P'(x) P(x) dx + O_{k,\varepsilon}(\varphi^+(q) (\log q)^{2k-1+\varepsilon}) \\ &= -\varphi^+(q) \frac{b_{2k,1,0} q (\log q)^{2k}}{2\varphi(q)} + O_{k,\varepsilon}(\varphi^+(q) (\log q)^{2k-1+\varepsilon}) \end{aligned} \quad (23)$$

and that

$$T_3 = \varphi^+(q) \frac{b_{2k-1,1,1} q (\log q)^{2k-1} \log y}{\varphi(q)} \int_0^1 P(x)^2 dx + O_{k,\varepsilon}(\varphi^+(q) (\log q)^{2k-1+\varepsilon}). \quad (24)$$

Thus, combining (19), (21), (22), (23), and (24), and noting that

$$b_{2k+1,0,0} = \frac{\varphi(q)}{q(2k+1)}, \quad b_{2k,0,1} = -\frac{\varphi(q)}{2q}, \quad \text{and} \quad b_{2k-1,1,1} = \frac{\varphi(q)k^2}{q(2k-1)},$$

it follows that, for $y = q^\vartheta$ and $0 < \vartheta < \frac{1}{2}$,

$$\begin{aligned} S_2(k, q) &= \left(\frac{\vartheta^{-1}}{2k+1} \int_0^1 P'(x)^2 dx + \frac{1}{2} + \frac{\vartheta k^2}{2k-1} \int_0^1 P(x)^2 dx \right) \varphi^+(q) (\log q)^{2k} \\ &\quad + O_{k,\varepsilon}(\varphi^+(q) (\log q)^{2k-1+\varepsilon}). \end{aligned}$$

This completes the proof of Proposition 2.2.

6. COMPLETING THE PROOF OF THEOREM 1.1: OPTIMIZING THE MOLLIFIER

We are now in a position to complete the proof of Theorem 1. By Proposition 2.1 and Proposition 2.2, for $0 < \vartheta < \frac{1}{2}$, we see that

$$P_k^* \geq \left[\frac{\vartheta^{-1}}{2k+1} \int_0^1 P'(x)^2 dx + \frac{1}{2} + \frac{\vartheta k^2}{2k-1} \int_0^1 P(x)^2 dx \right]^{-1}. \quad (25)$$

For each choice of $k \in \mathbb{N}$, we wish to find a polynomial P satisfying $P(0) = 0$ and $P(1) = 1$ that maximizes the expression on the right-hand side of the above inequality. Equivalently, we wish to minimize the expression

$$F_k(P) := \frac{\vartheta^{-1}}{2k+1} \int_0^1 P'(x)^2 dx + \frac{\vartheta k^2}{2k-1} \int_0^1 P(x)^2 dx. \quad (26)$$

This optimization problem is solved explicitly in §7 of [15] (and, independently, in [6]; see the remarks on page 97). We recall the argument given by Michel and Vanderkam in [15].

Using a standard approximation argument, the polynomial P can be replaced by any infinitely differentiable function with a rapidly convergent Taylor series on $[0, 1]$. In this case, using the calculus of variations, the optimization problem can be explicitly solved and, for $k > 0$, the optimal choice of P is

$$P(t) = \frac{\sinh(\Lambda t)}{\sinh(\Lambda)}, \quad \text{where } \Lambda = \vartheta k \sqrt{\frac{2k+1}{2k-1}}.$$

With this choice of P , it follows that

$$F_k(P) = \frac{\Lambda \coth \Lambda}{\vartheta(2k+1)} = \frac{k \coth \Lambda}{\sqrt{4k^2-1}}. \quad (27)$$

As k gets large, the function $\coth \Lambda \rightarrow 1$ and so asymptotically (as $k \rightarrow \infty$) we have

$$F_k(P) = \frac{1}{2} + \frac{1}{16k^2} + O\left(\frac{1}{k^4}\right).$$

When combined with (25) and (26), this asymptotic formula is enough to establish the estimate for P_k^* in (3) and, thus, completes the proof of Theorem 1.1.

TABLE 1. In the table below, lower bounds for the proportions P_k and P_k^* , defined in equations (1) and (2), respectively. These calculations were performed by using the expression for $F_k(P)$ given in (27) with $\vartheta = \frac{1}{2} - 1 \times 10^{-8}$.

k	Lower bound for P_k	Lower bound for P_k^*
1	$\frac{2}{3} \times 0.8216 \dots$	0.7544 ...
2	$\frac{2}{3} \times 0.9369 \dots$	0.9083 ...
3	$\frac{2}{3} \times 0.9758 \dots$	0.9642 ...
4	$\frac{2}{3} \times 0.9901 \dots$	0.9853 ...
5	$\frac{2}{3} \times 0.9956 \dots$	0.9935 ...
10	$\frac{2}{3} \times 0.9995 \dots$	0.9993 ...
15	$\frac{2}{3} \times 0.9997 \dots$	0.9997 ...
20	$\frac{2}{3} \times 0.9998 \dots$	0.9998 ...
25	$\frac{2}{3} \times 0.9999 \dots$	0.9999 ...

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MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, OXFORD, OX1 3LB UK

E-mail address: hung.bui@maths.ox.ac.uk

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSISSIPPI, UNIVERSITY, MS 38677 USA

E-mail address: mbmilino@olemiss.edu