LARGE GAPS BETWEEN CONSECUTIVE ZEROS OF THE RIEMANN ZETA-FUNCTION. II

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Abstract. Assuming the Riemann Hypothesis we show that there exist infinitely many consecutive zeros of the Riemann zeta-function whose gaps are greater than 2.9 times the average spacing.

1. Introduction

Subject to the truth of the Riemann Hypothesis (RH), the nontrivial zeros of the Riemann zeta-function can be written as $\rho = \frac{1}{2} + i\gamma$, where $\gamma \in \mathbb{R}$. Denote consecutive ordinates of zeros by $0 < \gamma \leq \gamma'$, we define the normalized gap

$$\delta(\gamma) := (\gamma' - \gamma)\frac{\log \gamma}{2\pi}.$$

It is well-known that

$$N(T) := \sum_{0<\gamma\leq T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

for $T \geq 10$. Hence $\delta(\gamma)$ is 1 on average. It is expected that there are arbitrarily large and arbitrarily small (normalized) gaps between consecutive zeros of the Riemann zeta-function on the critical line, i.e.

$$\lambda := \limsup_{\gamma} \delta(\gamma) = \infty \quad \text{and} \quad \mu := \liminf_{\gamma} \delta(\gamma) = 0.$$

In this article, we focus only on the large gaps, and prove the following theorem.

Theorem 1.1. *Assuming RH. Then we have* $\lambda > 2.9$*.*

Very little is known about λ unconditionally. Selberg [**16**] remarked that he could prove $\lambda > 1$. Conditionally, Bredberg [**2**] showed that $\lambda > 2.766$ under the assumption of RH (see also [**14**,**13**,**7**,**12**,**6**,**10**] for work in this direction), and on the Generalized Riemann Hypothesis (GRH) it is known that $\lambda > 3.072$ [**11**] (see also [**8**,**15**,**3**]). These results either use Hall's approach using Wirtinger's inequality, or exploit the following idea of Mueller [**14**].

Let $H : \mathbb{C} \to \mathbb{C}$ and consider the following functions

$$\mathcal{M}_1(H,T) = \int_0^T \big|H(\tfrac{1}{2}+it)\big|^2 dt$$

and

$$\mathcal{M}_2(H,T;c) = \int_{-c/L}^{c/L} \sum_{0<\gamma\leq T} \big|H(\tfrac{1}{2}+i(\gamma+\alpha))\big|^2 d\alpha,$$

where $L = \log\frac{T}{2\pi}$. We note that if

$$h(c) := \frac{\mathcal{M}_2(H,T;c)}{\mathcal{M}_1(H,T)} < 1$$

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as $T\to\infty$, then $\lambda>c/\pi$, and if $h(c)>1$ as $T\to\infty$, then $\mu<c/\pi$.

Mueller [**14**] applied this idea to $H(s)=\zeta(s)$. Using $H(s)=\sum_{n\leq T^{1-\varepsilon}}d_{2.2}(n)n^{-s}$, where the arithmetic function $d_k(n)$ is defined in terms of the Dirichlet series

$$\zeta(s)^k=\sum_{n=1}^{\infty}\frac{d_k(n)}{n^s}\qquad(\sigma>1)$$

for any real number k, Conrey, Ghosh and Gonek [**9**] showed that $\lambda>2.337$. Later, assuming GRH, they applied to $H(s)=\zeta(s)\sum_{n\leq T^{1/2-\varepsilon}}n^{-s}$ and obtained $\lambda>2.68$ [**8**]. By considering a more general choice

$$H(s)=\zeta(s)\sum_{n\leq T^{1/2-\varepsilon}}\frac{d_r(n)P(\frac{\log y/n}{\log y})}{n^s},$$

where $P(x)$ is a polynomial, Ng [**15**] improved that result to $\lambda>3$ (using $r=2$ and $P(x)=(1-x)^{30}$). In the last two papers, GRH is needed to estimate some certain exponential sums resulting from the evaluation of the discrete mean value over the zeros in $\mathcal{M}_2(H,T;c)$. Recently, Bui and Heath-Brown [**5**] showed how one can use a generalization of the Vaughan identity and the hybrid large sieve inequality to circumvent the assumption of GRH for such exponential sums. Here we use that idea to obtain a weaker version of Ng's result without provoking GRH. It is possible that Feng and Wu's result $\lambda>3.072$ can also be obtained just assuming RH by this method. However, we opt to work on Ng's result for simplicity.

Instead of using the divisor function $d(n)=d_2(n)$, we choose

$$H(s)=\zeta(s)\sum_{n\leq y}\frac{h(n)P(\frac{\log y/n}{\log y})}{n^s},$$

where $y=T^{\vartheta}$, $P(x)$ is a polynomial and $h(n)$ is a multiplicative function satisfying

$$h(n)=\begin{cases}d(n) & \text{if } n \text{ is square-free,}\\ 0 & \text{otherwise.}\end{cases}\tag{1}$$

In Section 3 and Section 4 we shall prove the following two key lemmas.

Lemma 1.1. *Suppose* $0<\vartheta<\frac12$. *We have*

$$\mathcal{M}_1(H,T)=\frac{AT(\log y)^9}{6}\int_0^1(1-x)^3\Big(\vartheta^{-1}P_1(x)^2-2P_1(x)P_2(x)\Big)dx+O(TL^8),$$

where

$$A=\prod_p\Big(1+\frac8p\Big)\Big(1-\frac1p\Big)^8$$

and

$$P_r(x)=\int_0^x t^rP(x-t)dt.$$

Lemma 1.2. *Suppose* $0<\vartheta<\frac12$ *and* $P(0)=P'(0)=0$. *We have*

$$\sum_{0<\gamma\leq T}H(\rho+i\alpha)H(1-\rho-i\alpha)=\frac{ATL(\log y)^9}{6\pi}\int_0^1(1-x)^3\mathrm{Re}\Big\{\sum_{j=1}^{\infty}(i\alpha\log y)^jB(j;x)\Big\}dx+O_\varepsilon(TL^{9+\varepsilon})$$

uniformly for $\alpha \ll L^{-1}$, *where*

$$\begin{aligned}B(j;u) &= -\frac{2P_1(u)P_{j+2}(u)}{(j+2)!} + \frac{2\vartheta P_2(u)P_{j+2}(u)}{(j+2)!} + \frac{4\vartheta P_1(u)P_{j+3}(u)}{(j+3)!}\\ &\quad -\frac{\vartheta}{(j+2)!}\int_0^u t(\vartheta^{-1}-t)^{j+2}P_1(u)P(u-t)dt\\ &\quad +\frac{\vartheta}{(j+1)!}\int_0^u t(\vartheta^{-1}-t)^{j+1}P_2(u)P(u-t)dt - \frac{\vartheta}{6j!}\int_0^u t(\vartheta^{-1}-t)^{j}P_3(u)P(u-t)dt.\end{aligned}$$

Proof of Theorem 1.1. We take $\vartheta = \frac{1}{2}^{-}$. On RH we have

$$\sum_{0<\gamma\le T}\big|H(\tfrac12 + i(\gamma+\alpha))\big|^2 = \sum_{0<\gamma\le T} H(\rho+i\alpha)H(1-\rho-i\alpha).$$

Note that this is the only place we need to assume RH. Lemma 1.2 then implies that

$$\int_{-c/L}^{c/L}\sum_{0<\gamma\le T}\big|H(\tfrac12 + i(\gamma+\alpha))\big|^2 d\alpha \sim \frac{AT(\log y)^9}{6\pi}\sum_{j=1}^{\infty}\frac{(-1)^j c^{2j+1}}{2^{2j-1}(2j+1)}\int_0^1 (1-x)^3 B(2j;x)dx.$$

Hence

$$h(c) = \frac{1}{2\pi}\frac{\sum_{j=1}^{\infty}\frac{(-1)^j c^{2j+1}}{2^{2j-1}(2j+1)}\int_0^1(1-x)^3B(2j;x)dx}{\int_0^1(1-x)^3(P_1(x)^2 - P_1(x)P_2(x))dx} + o(1),$$

as $T\to\infty$. Consider the polynomial $P(x) = \sum_{j=2}^M c_j x^j$. Choosing $M=6$ and running Mathematica's Minimize command, we obtain $\lambda > 2.9$. Precisely, with

$$P(x) = 1000x^2 - 9332x^3 + 30134x^4 - 40475x^5 + 19292x^6,$$

we have

$$h(2.9\pi) = 0.99725\ldots < 1,$$

and this proves the theorem.

Remark 1.1. The above lemmas are unconditional. We note that in the case $r=2$ apart from the arithmetical factor a_3 being replaced by A, Lemma 1.1 is the same as what stated in [**15**; Lemma 2.1] (see also [**3**; Lemma 2.3]), while Lemma 1.2, under the additional condition $P(0) = P'(0) = 0$, recovers Theorem 2 of Ng [**15**] (and also Lemma 2.6 of Bui [**3**]) without assuming GRH, though the latters are written in a slightly different and more complicated form. This is as expected because replacing the divisor function $d(n)$ by the arithmetic function $h(n)$ (as defined in (1)) in the definition of $H(s)$ only changes the arithmetical factor in the resulting mean value estimates. This substitution, however, makes our subsequent calculations much easier. Our arguments also work if we set $h(n) = d_r(n)$ when n is square-free for some $r\in\mathbb{N}$ without much changes, but we choose $r=2$ to simplify various statements and expressions in the paper.

Remark 1.2. In the course of evaluating $\mathcal{M}_2(H,T;c)$, we encounter an exponential sum of type (see Section 4.2)

$$\sum_{n\le y}\frac{h(n)P(\frac{\log y/n}{\log y})}{n}\sum_{m\le nT/2\pi}a(m)e\Big(-\frac{m}{n}\Big)$$

for some arithmetic function $a(m)$. At this point, assuming GRH, Ng [**15**] applied Perron's formula to the sum over m, and then moved the line of integration to $\mathrm{Re}(s) = 1/2+\varepsilon$. The main term arises from the residue at $s=1$ and the error terms in this case are easy to handle. To avoid being subject to GRH, we instead use the ideas in [**9**] and [**5**]. That leads to a sum of type

$$\sum_{n\leq y}\frac{\mu(n)h(n)P(\frac{\log y/n}{\log y})}{n}.$$

This is essentially a variation of the prime number theorem, and here the polynomial $P(x)$ is required to vanish with order at least 2 at $x=0$ (see Lemma 2.6). As a result, we cannot take the choice $P(x)=(1-x)^{30}$ as in [**15**]. Here it is not clear how to choose a "good" polynomial $P(x)$. Our theorem is obtained by numerically optimizing over polynomials $P(x)$ with degree less than 7. It is probable that by considering higher degree polynomials, we can establish Ng's result $\lambda>3$ under only RH.

Notation. Throughout the paper, we denote

$$[n]_y:=\frac{\log y/n}{\log y}.$$

For $Q,R\in C^\infty[(0,1)]$ we define

$$Q_r(x)=\int_0^x t^rQ(x-t)dt\qquad\text{and}\qquad R_r(x)=\int_0^x t^rR(x-t)dt.$$

We let $\varepsilon>0$ be an arbitrarily small positive number, and can change from time to time.

2. Various lemmas

The following two lemmas are in [**9**; Lemma 2 and Lemma 3].

Lemma 2.1. *Suppose that* $A(s)=\sum_{m=1}^\infty a(m)m^{-s}$*, where* $a(m)\ll_\varepsilon m^\varepsilon$*, and* $B(s)=\sum_{n\leq y}b(n)n^{-s}$*, where* $b(n)\ll_\varepsilon n^\varepsilon$*. Then we have*

$$\frac{1}{2\pi i}\int_{a+i}^{a+iT}\chi(1-s)A(s)B(1-s)ds=\sum_{n\leq y}\frac{b(n)}{n}\sum_{m\leq nT/2\pi}a(m)e\Big(-\frac{m}{n}\Big)+O_\varepsilon(yT^{1/2+\varepsilon}),$$

where $a=1+L^{-1}$.

Lemma 2.2. *Suppose that* $A_j(s)=\sum_{n=1}^\infty a_j(n)n^{-s}$ *is absolutely convergent for* $\sigma>1$, $1\leq j\leq k$*, and that*

$$A(s)=\sum_{n=1}^\infty\frac{a(n)}{n^s}=\prod_{j=1}^kA_j(s).$$

Then for any $l\in\mathbb{N}$*, we have*

$$\sum_{n=1}^\infty\frac{a(ln)}{n^s}=\sum_{l=l_1\ldots l_k}\prod_{j=1}^k\bigg(\sum_{\substack{n\geq1\\(n,\prod_{i<j}l_i)=1}}\frac{a_j(l_jn)}{n^s}\bigg).$$

We shall need estimates for various divisor-like sums. Throughout the paper, we let

$$F_\tau(n)=\prod_{p|n}\big(1+O(p^{-\tau})\big),$$

for $\tau>0$ and the constant in the O-term is implicit and independent of τ.

Lemma 2.3. *For any $Q \in C^\infty([0,1])$, there exists an absolute constant $\tau_0 > 0$ such that*

$$\text{(i)}\ \sum_{an\le y}\frac{h(an)Q([an]_y)}{n} = C(\log y)^2 h(a)\prod_{p|a}\left(1+\frac{2}{p}\right)^{-1}Q_1([a]_y) + O(d(a)F_{\tau_0}(a)L),$$

$$\text{(ii)}\ \sum_{an\le y}\frac{h(an)Q([an]_y)\log n}{n} = C(\log y)^3 h(a)\prod_{p|a}\left(1+\frac{2}{p}\right)^{-1}Q_2([a]_y) + O(d(a)F_{\tau_0}(a)L^2),$$

where

$$C = \prod_p\left(1+\frac{2}{p}\right)\left(1-\frac{1}{p}\right)^2.$$

Proof. By a method of Selberg [**16**] we have

$$\sum_{n\le t}\frac{h(an)}{n} = \frac{C(\log t)^2}{2}h(a)\prod_{p|a}\left(1+\frac{2}{p}\right)^{-1} + O(d(a)F_{\tau_0}(a)L)$$

for any $t \le T$. The first statement then follows from partial summation.

The second statement is an easy consequence of the first one. □

Lemma 2.4. *For any $Q \in C^\infty([0,1])$, we have*

$$\sum_{n\le y}\frac{h(n)^2\varphi(n)Q([n]_y)}{n^2}\prod_{p|n}\left(1+\frac{2}{p}\right)^{-2} = \frac{D(\log y)^4}{6}\int_0^1(1-x)^3Q(x)dx + O(L^3),$$

where

$$D = \prod_p\left[1+\frac{4(p-1)}{p^2}\left(1+\frac{2}{p}\right)^{-2}\right]\left(1-\frac{1}{p}\right)^4.$$

Proof. The proof is similar to the above lemma. □

We need a lemma concerning the size of the function $F_{\tau_0}(n)$ on average.

Lemma 2.5. *Suppose $-1 \le \sigma \le 0$. We have*

$$\sum_{n\le y}\frac{d_k(n)F_{\tau_0}(n)}{n}\left(\frac{y}{n}\right)^\sigma \ll_k L^{k-1}\min\left\{|\sigma|^{-1}, L\right\}.$$

Proof. We use Lemma 4.6 in [**4**] that

$$\sum_{n\le y}\frac{d_k(n)}{n}\left(\frac{y}{n}\right)^\sigma \ll_k L^{k-1}\min\left\{|\sigma|^{-1}, L\right\}.$$

We have

$$F_{\tau_0}(n) \le \prod_{p|n}\left(1+Ap^{-\tau_0}\right) = \sum_{l|n} l^{-\tau_0}A^{w(l)}$$

for some $A > 0$, where $w(n)$ is the number of prime factors of n. Hence

$$\sum_{n\le y}\frac{d_k(n)F_{\tau_0}(n)}{n}\left(\frac{y}{n}\right)^\sigma \ll \sum_{l\le y}\frac{d_k(l)A^{w(l)}}{l^{1+\tau_0}}\sum_{n\le y/l}\frac{d_k(n)}{n}\left(\frac{y/l}{n}\right)^\sigma \ll_k L^{k-1}\min\left\{|\sigma|^{-1}, L\right\},$$

since $d_k(l)A^{w(l)} \ll l^{\tau_0/2}$ for sufficiently large l. □

Lemma 2.6. *Let $F(n)=F(n,0)$, where*

$$F(n,\alpha)=\prod_{p|n}\bigg(1-\frac{1}{p^{1+\alpha}}\bigg).$$

For any $Q\in C^\infty([0,1])$ satisfying $Q(0)=Q'(0)=0$, there exist an absolute constant $\tau_0>0$ and some $\nu\asymp(\log\log y)^{-1}$ such that

$$\begin{aligned}\mathcal{A}_1(y,Q;a,b,\underline{\alpha}) &= \sum_{\substack{an\leq y\\(n,b)=1}}\frac{\mu(n)h(n)Q([an]_y)}{\varphi(n)n^{\alpha_1}}F(n,\alpha_2)F(n,\alpha_3)\\ &= U_1V_1(b)\bigg(\frac{Q''([a]_y)}{(\log y)^2}+\frac{2\alpha_1Q'([a]_y)}{\log y}+\alpha_1^2Q([a]_y)\bigg)\\ &\qquad +O(F_{\tau_0}(b)L^{-3})+O_\varepsilon\bigg(F_{\tau_0}(b)\Big(\frac{y}{a}\Big)^{-\nu}L^{-2+\varepsilon}\bigg)\end{aligned}$$

uniformly for $\alpha_j\ll L^{-1}$, $1\leq j\leq 3$, where $U_1=U_1(0,\underline{0})$ and $V_1(n)=V_1(0,n,\underline{0})$, with

$$U_1(s,\underline{\alpha})=\prod_p\bigg[1-\frac{2F(p,\alpha_2)F(p,\alpha_3)}{\varphi(p)p^{s+\alpha_1}}\bigg]\bigg(1-\frac{1}{p^{1+s+\alpha_1}}\bigg)^{-2}$$

and

$$V_1(s,n,\underline{\alpha})=\prod_{p|n}\bigg[1-\frac{2F(p,\alpha_2)F(p,\alpha_3)}{\varphi(p)p^{s+\alpha_1}}\bigg]^{-1}.$$

Proof. This is essentially a variation of the prime number theorem.

It suffices to consider $Q(x)=\sum_{j\geq2}a_jx^j$. We have

$$\mathcal{A}_1(y,Q;a,b,\underline{\alpha})=\sum_{j\geq2}\frac{a_jj!}{(\log y)^j}\sum_{(n,b)=1}\frac{1}{2\pi i}\int_{(2)}\Big(\frac{y}{a}\Big)^s\frac{\mu(n)h(n)}{\varphi(n)n^{s+\alpha_1}}F(n,\alpha_2)F(n,\alpha_3)\frac{ds}{s^{j+1}}.$$

The sum over n converges absolutely. Hence

$$\mathcal{A}_1(y,Q;a,b,\underline{\alpha})=\sum_{j\geq2}\frac{a_jj!}{(\log y)^j}\frac{1}{2\pi i}\int_{(2)}\Big(\frac{y}{a}\Big)^s\sum_{(n,b)=1}\frac{\mu(n)h(n)}{\varphi(n)n^{s+\alpha_1}}F(n,\alpha_2)F(n,\alpha_3)\frac{ds}{s^{j+1}}.$$

The sum in the integrand equals

$$\prod_{p\nmid b}\bigg(1-\frac{2F(p,\alpha_2)F(p,\alpha_3)}{\varphi(p)p^{s+\alpha_1}}\bigg)=\frac{U_1(s,\underline{\alpha})V_1(s,b,\underline{\alpha})}{\zeta(1+s+\alpha_1)^2}.$$

Let $Y=o(T)$ be a large parameter to be chosen later. By Cauchy's theorem, $\mathcal{A}_1(y,Q;a,b,\underline{\alpha})$ is equal to the residue at $s=0$ plus integrals over the line segments $\mathcal{C}_1=\{s=it,t\in\mathbb{R},|t|\geq Y\}$, $\mathcal{C}_2=\{s=\sigma\pm iY,-\frac{c}{\log Y}\leq\sigma\leq0\}$, and $\mathcal{C}_3=\{s=-\frac{c}{\log Y}+it,|t|\leq Y\}$, where c is some fixed positive constant such that $\zeta(1+s+\alpha_1)$ has no zeros in the region on the right hand side of the contour determined by the $\mathcal{C}_j$'s. Furthermore, we require that for such c we have $1/\zeta(\sigma+it)\ll\log(2+|t|)$ in this region [see **17**; Theorem 3.11]. Then the integral over $\mathcal{C}_1$ is

$$\ll F_{\tau_0}(b)L^{-j}(\log Y)^2/Y^j\ll_\varepsilon F_{\tau_0}(b)L^{-2}Y^{-2+\varepsilon},$$

since $j\geq2$. The integral over $\mathcal{C}_2$ is

$$\ll F_{\tau_0}(b)L^{-j}(\log Y)/Y^{j+1}\ll_\varepsilon F_{\tau_0}(b)L^{-2}Y^{-3+\varepsilon}.$$

Finally, the contribution from $\mathcal{C}_3$ is

$$\ll F_{\tau_0}(b)L^{-j}(\log Y)^j\Big(\frac{y}{a}\Big)^{-c/\log Y}\ll_\varepsilon F_{\tau_0}(b)\Big(\frac{y}{a}\Big)^{-c/\log Y}L^{-2+\varepsilon}.$$

Choosing $Y \asymp L$ gives an error so far of size $O_\varepsilon\big(F_{\tau_0}(b)(y/a)^{-\nu}L^{-2+\varepsilon}\big) + O_\varepsilon(F_{\tau_0}(b)L^{-4+\varepsilon})$.

For the residue at $s = 0$, we write this as

$$\sum_{j\geq 2} \frac{a_j j!}{(\log y)^j} \frac{1}{2\pi i} \oint \left(\frac{y}{a}\right)^s \frac{U_1(s,\underline{\alpha})V_1(s,b,\underline{\alpha})}{\zeta(1+s+\alpha_1)^2} \frac{ds}{s^{j+1}},$$

where the contour is a circle of radius $\asymp L^{-1}$ around the origin. This integral is trivially bounded by $O(L^{-2})$ so that taking the first term in the Taylor series of $\zeta(1+s+\alpha_1)$ finishes the proof. □

Lemma 2.7. *For any $Q, R \in C^\infty([0,1])$, there exists an absolute constant $\tau_0 > 0$ such that*

$$\begin{aligned}\mathcal{A}_2(y,Q,R;a_1,a_2,\alpha_1) &= \sum_{\substack{a_1a_2l\leq y\\ a_1m\leq y}} \frac{h(a_1a_2l)h(a_1m)Q([a_1m]_y)R([a_1a_2l]_y)V_1(a_1a_2lm)}{lm^{1+\alpha_1}}\\ &= U_2(\log y)^4 h(a_1a_2)h(a_1)V_1(a_1a_2)V_2(a_1)V_3(a_2)V_4(a_1a_2)\\ &\qquad \int_0^{[a_1]_y} y^{-\alpha_1 t} tQ([a_1]_y - t)R_1([a_1a_2]_y)dt + O(d_4(a_1)d(a_2)F_{\tau_0}(a_1a_2)L^3)\end{aligned}$$

uniformly for $\alpha_1 \ll L^{-1}$, where

$$U_2 = \prod_p \left(1 + \frac{2V_1(p)}{p}\right)\left[1 + \frac{2V_1(p)}{p}\left(1+\frac{2}{p}\right)\left(1 + \frac{2V_1(p)}{p}\right)^{-1}\right]\left(1 - \frac{1}{p}\right)^4,$$

$$V_2(n) = \prod_{p|n}\left(1 + \frac{2V_1(p)}{p}\right)^{-1}, \qquad V_3(n) = \prod_{p|n}\left(1+\frac{2}{p}\right)\left(1 + \frac{2V_1(p)}{p}\right)^{-1}$$

and

$$V_4(n) = \prod_{p|n}\left[1 + \frac{2V_1(p)}{p}\left(1+\frac{2}{p}\right)\left(1 + \frac{2V_1(p)}{p}\right)^{-1}\right]^{-1}.$$

Proof. The proof uses Selberg's method [**16**] similarly to Lemma 2.3. One first executes the sum over m, and then the sum over l. □

Lemma 2.8. *For any $Q, R \in C^\infty([0,1])$, we have*

(i)
$$\begin{aligned}&\sum_{l_1l_2\leq y} \frac{h(l_1l_2)h(l_1)Q([l_1]_y)R([l_1l_2]_y)}{l_1l_2^{1+\alpha_1}} F(l_1,\alpha_2)F(l_1l_2,\alpha_3)V_1(l_1l_2)V_2(l_1)V_3(l_2)V_4(l_1l_2)\\ &\quad = \frac{W(\log y)^6}{6}\int_0^1\int_0^x (1-x)^3 y^{-\alpha_1 t_1} t_1 Q(x)R(x-t_1)dt_1dx + O(L^5),\end{aligned}$$

(ii)
$$\begin{aligned}&\sum_{pl_1l_2\leq y} \frac{\log p}{(p^{1+\alpha_4}-1)p^{\alpha_5}} \frac{h(pl_1l_2)h(l_1)Q([l_1]_y)R([pl_1l_2]_y)}{l_1l_2^{1+\alpha_1}}\\ &\qquad F(pl_1,\alpha_2)F(pl_1l_2,\alpha_3)V_1(pl_1l_2)V_2(l_1)V_3(pl_2)V_4(pl_1l_2)\\ &\quad = \frac{W(\log y)^7}{3}\int_0^1 \int_{\substack{t_j\geq 0\\ t_1+t_2\leq x}} (1-x)^3 y^{-\alpha_1 t_1 - (\alpha_4+\alpha_5)t_2} t_1 Q(x)R(x-t_1-t_2)dt_1dt_2dx\\ &\qquad + O(L^6)\end{aligned}$$

uniformly for $\alpha_j \ll L^{-1}$, $1 \leq j \leq 5$, where

$$W = \prod_p \left(1 + \frac{2F(p)V_1(p)V_3(p)V_4(p)}{p} + \frac{4F(p)^2V_1(p)V_2(p)V_4(p)}{p}\right)\left(1 - \frac{1}{p}\right)^6.$$

Proof. We consider the first statement. We start with the sum over l_2 on the left hand side of (i), which is

$$\sum_{\substack{l_2\leq y/l_1\\(l_2,l_1)=1}}\frac{h(l_2)R([l_1l_2]_y)}{l_2^{1+\alpha_1}}F(l_2,\alpha_3)V_1(l_2)V_3(l_2)V_4(l_2).$$

As in how we prove Lemma 2.3, this equals

$$\prod_p\bigg\{W_1(p)^{-1}\Big(1-\frac{1}{p}\Big)^2\bigg\}(\log y)^2W_1(l_1)\int_0^{[l_1]_y}y^{-\alpha_1t_1}t_1R([l_1]_y-t_1)dt_1+O(L), \qquad (2)$$

where

$$W_1(n)=\prod_{p|n}\bigg(1+\frac{2F(p)V_1(p)V_3(p)V_4(p)}{p}\bigg)^{-1}.$$

Hence the required expression is

$$\prod_p\bigg\{W_1(p)^{-1}\Big(1-\frac{1}{p}\Big)^2\bigg\}(\log y)^2\sum_{l_1\leq y}\frac{h(l_1)^2Q([l_1]_y)}{l_1} \qquad (3)$$
$$F(l_1,\alpha_2)F(l_1,\alpha_3)V_1(l_1)V_2(l_1)V_4(l_1)W_1(l_1)\int_0^{[l_1]_y}y^{-\alpha_1t_1}t_1R([l_1]_y-t_1)dt_1+O(L^5).$$

Using Selberg's method [**16**] again we have

$$\sum_{l_1\leq t}\frac{h(l_1)^2}{l_1}F(l_1,\alpha_2)F(l_1,\alpha_3)V_1(l_1)V_2(l_1)V_4(l_1)W_1(l_1)$$
$$=\prod_p\bigg\{W_2(p)^{-1}\Big(1-\frac{1}{p}\Big)^4\bigg\}\frac{(\log t)^4}{24}+O(L^3)$$

for any $t\leq T$, where

$$W_2(n)=\prod_{p|n}\bigg\{1+\frac{4F(p)^2V_1(p)V_2(p)V_4(p)W_1(p)}{p}\bigg\}^{-1}.$$

Partial summation then implies that (3) is equal to

$$\prod_p\bigg\{W_1(p)^{-1}W_2(p)^{-1}\Big(1-\frac{1}{p}\Big)^6\bigg\}\frac{(\log y)^4}{6}\int_0^1\int_0^x(1-x)^3y^{-\alpha_1t_1}t_1Q(x)R(x-t_1)dt_1dx+O(L^5).$$

It is easy to check that the arithmetical factor is W, and we obtain the first statement.

For the second statement, we first notice that the contribution of the terms involving p^{-s} with $\mathrm{Re}(s)>1$ is $O(L^6)$. Hence the left hand side of (ii) is

$$2\sum_{l_1l_2\leq y}\frac{h(l_1l_2)h(l_1)Q([l_1]_y)}{l_1l_2^{1+\alpha_1}}F(l_1,\alpha_2)F(l_1l_2,\alpha_3)V_1(l_1l_2)V_2(l_1)V_3(l_2)V_4(l_1l_2)$$
$$\sum_{\substack{p\leq y/l_1l_2\\(p,l_1l_2)=1}}\frac{(\log p)R([pl_1l_2]_y)}{p^{1+\alpha_4+\alpha_5}}+O(L^6).$$

The same argument shows that we can include the terms $p|l_1l_2$ in the innermost sum with an admissible error $O(L^6)$, so that the above expression is equal to

$$2\sum_{p\leq y}\frac{\log p}{p^{1+\alpha_4+\alpha_5}}\sum_{l_1l_2\leq y/p}\frac{h(l_1l_2)h(l_1)Q([l_1]_y)R([pl_1l_2]_y)}{l_1l_2^{1+\alpha_1}}$$
$$F(l_1,\alpha_2)F(l_1l_2,\alpha_3)V_1(l_1l_2)V_2(l_1)V_3(l_2)V_4(l_1l_2)+O(L^6).$$

We have

$$\sum_{p\le t}\frac{\log p}{p}=\log t+O(1)$$

for any $t\le T$. The result follows by using Part (i) and partial summation. □

3. Proof of Lemma 1.1

To evaluate $\mathcal{M}_1(H,T)$, we first appeal to Theorem 1 of [**1**] and obtain

$$\begin{aligned}\mathcal{M}_1(H,T) &= T\sum_{m,n\le y}\frac{h(m)h(n)P([m]_y)P([n]_y)(m,n)}{mn}\bigg(\log\frac{T(m,n)^2}{2\pi mn}+2\gamma-1\bigg)\\ &\qquad +O_B(TL^{-B})+O_\varepsilon(y^2T^\varepsilon)\end{aligned}$$

for any $B>0$, where γ is the Euler constant. Using the Möbius inversion formula

$$f\big((m,n)\big)=\sum_{\substack{l|m\\ l|n}}\sum_{d|l}\mu(d)f\Big(\frac{l}{d}\Big),$$

we can write the above as

$$T\sum_{l\le y}\sum_{d|l}\frac{\mu(d)}{dl}\sum_{m,n\le y/l}\frac{h(lm)h(ln)P([lm]_y)P([ln]_y)}{mn}\bigg(\log\frac{T}{2\pi d^2mn}+2\gamma-1\bigg)+O_B(TL^{-B}).$$

We next replace the term in the bracket by $\log\frac{T}{2\pi mn}$. This produces an error of size

$$\ll T\sum_{l\le y}\frac{d(l)^2}{l}\bigg(\sum_{n\le y/l}\frac{d(n)}{n}\bigg)^2\sum_{d|l}\frac{\log d}{d}\ll TL^8.$$

Hence

$$\begin{aligned}\mathcal{M}_1(H,T) &= T\sum_{l\le y}\frac{\varphi(l)}{l^2}\sum_{m,n\le y/l}\frac{h(lm)h(ln)P([lm]_y)P([ln]_y)}{mn}\big(L-\log m-\log n\big)+O(TL^8)\\ &= TL\sum_{l\le y}\frac{\varphi(l)}{l^2}\bigg(\sum_{n\le y/l}\frac{h(ln)P([ln]_y)}{n}\bigg)^2\\ &\qquad -2T\sum_{l\le y}\frac{\varphi(l)}{l^2}\sum_{m,n\le y/l}\frac{h(lm)h(ln)P([lm]_y)P([ln]_y)\log n}{mn}+O(TL^8).\end{aligned}$$

The result follows by using Lemma 2.3, Lemma 2.4 and Lemma 2.5. Here we use a fact which is easy to verify that $C^2D=A$.

4. Proof of Lemma 1.2

We denote $H(s)=\zeta(s)G(s)$, i.e.

$$G(s)=\sum_{n\le y}\frac{h(n)P([n]_y)}{n^s}.$$

By Cauchy's theorem we have

$$\sum_{0<\gamma\le T}H(\rho+i\alpha)H(1-\rho-i\alpha)=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\zeta'}{\zeta}(s)\zeta(s+i\alpha)\zeta(1-s-i\alpha)G(s+i\alpha)G(1-s-i\alpha)ds,$$

where $\mathcal{C}$ is the positively oriented rectangle with vertices at $1-a+i$, $a+i$, $a+iT$ and $1-a+iT$. Here $a=1+L^{-1}$ and T is chosen so that the distance from T to the nearest γ

is $\gg L^{-1}$. It is standard that the contribution from the horizontal segments of the contour is $O_\varepsilon(yT^{1/2+\varepsilon})$.

We denote the contribution from the right edge by $\mathcal{N}_1$, where

$$\mathcal{N}_1=\frac{1}{2\pi i}\int_{a+i}^{a+iT}\chi(1-s-i\alpha)\frac{\zeta'}{\zeta}(s)\zeta(s+i\alpha)^2G(s+i\alpha)G(1-s-i\alpha)ds. \tag{4}$$

From the functional equation we have

$$\frac{\zeta'}{\zeta}(1-s)=\frac{\chi'}{\chi}(1-s)-\frac{\zeta'}{\zeta}(s).$$

Hence the contribution from the left edge, by substituting s by $1-s$, is

$$\begin{aligned}&\frac{1}{2\pi i}\int_{a-i}^{a-iT}\frac{\zeta'}{\zeta}(1-s)\zeta(1-s+i\alpha)\zeta(s-i\alpha)G(1-s+i\alpha)G(s-i\alpha)ds\\ =&\ \frac{1}{2\pi i}\int_{a-i}^{a-iT}\Big(\frac{\chi'}{\chi}(1-s)-\frac{\zeta'}{\zeta}(s)\Big)\zeta(1-s+i\alpha)\zeta(s-i\alpha)G(1-s+i\alpha)G(s-i\alpha)ds\\ =&\ -\overline{\mathcal{N}_2}+\overline{\mathcal{N}_1}+O_\varepsilon(yT^{1/2+\varepsilon}),\end{aligned}$$

where

$$\mathcal{N}_2(\beta,\gamma)=\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\chi'}{\chi}(1-s)\zeta(1-s+i\alpha)\zeta(s-i\alpha)G(1-s+i\alpha)G(s-i\alpha)ds. \tag{5}$$

Thus

$$\sum_{0<\gamma\leq T}H(\rho+i\alpha)H(1-\rho-i\alpha)=2\text{Re}\big(\mathcal{N}_1\big)-\overline{\mathcal{N}_2}+O_\varepsilon(yT^{1/2+\varepsilon}). \tag{6}$$

4.1. **Evaluate $\mathcal{N}_2$.** We move the line of integration in (5) to the $\frac12$-line. As before, this produces an error of size $O_\varepsilon(yT^{1/2+\varepsilon})$. Hence we get

$$\mathcal{N}_2=\frac{1}{2\pi}\int_{1-\alpha}^{T-\alpha}\frac{\chi'}{\chi}(\tfrac12-it-i\alpha)\big|H(\tfrac12+it)\big|^2dt+O_\varepsilon(yT^{1/2+\varepsilon}).$$

From Stirling's approximation we have

$$\frac{\chi'}{\chi}(\tfrac12-it)=-\log\frac{t}{2\pi}+O(t^{-1})\qquad(t\geq 1).$$

Combining this with Lemma 1.1 and integration by parts, we easily obtain

$$\mathcal{N}_2=-\frac{ATL(\log y)^9}{12\pi}\int_0^1(1-x)^3\Big(\vartheta^{-1}P_1(x)^2-2P_1(x)P_2(x)\Big)dx+O(TL^9). \tag{7}$$

4.2. **Evaluate $\mathcal{N}_1$.** It is easier to start with a more general sum

$$\begin{aligned}\mathcal{N}_1(\beta,\gamma)\ \ =\ \ &\frac{1}{2\pi i}\int_{a+i(1+\alpha)}^{a+i(T+\alpha)}\chi(1-s)\bigg(\frac{\zeta'}{\zeta}(s+\beta)\zeta(s+\gamma)\zeta(s)\sum_{m\leq y}\frac{h(m)P([m]_y)}{m^s}\bigg)\\ &\bigg(\sum_{n\leq y}\frac{h(n)P([n]_y)}{n^{1-s}}\bigg)ds,\end{aligned}$$

so that $\mathcal{N}_1=\mathcal{N}_1(-i\alpha,0)$. From Lemma 2.1, we obtain

$$\mathcal{N}_1(\beta,\gamma)=\sum_{n\leq y}\frac{h(n)P([n]_y)}{n}\sum_{m\leq nT/2\pi}a(m)e\Big(-\frac{m}{n}\Big)+O_\varepsilon(yT^{1/2+\varepsilon}),$$

where the arithmetic function $a(m)$ is defined by

$$\frac{\zeta'}{\zeta}(s+\beta)\zeta(s+\gamma)\zeta(s)\sum_{m\leq y}\frac{h(m)P([m]_y)}{m^s}=\sum_{m=1}^{\infty}\frac{a(m)}{m^s}. \tag{8}$$

By the work of Conrey, Ghosh and Gonek [**9**; Sections 5–6 and (8.2)], and the work of Bui and Heath-Brown [**5**], we can write

$$\mathcal{N}_1(\beta,\gamma)=\mathcal{Q}(\beta,\gamma)+E+O_\varepsilon(yT^{1/2+\varepsilon}),$$

where

$$\mathcal{Q}(\beta,\gamma)=\sum_{ln\leq y}\frac{h(ln)P([ln]_y)}{ln}\frac{\mu(n)}{\varphi(n)}\sum_{\substack{m\leq nT/2\pi\\(m,n)=1}}a(lm) \tag{9}$$

and

$$E\ll_{B,\varepsilon}T\mathscr{L}^{-B}+y^{1/3}T^{5/6+\varepsilon}$$

for any $B>0$.

Let

$$\frac{\zeta'}{\zeta}(s+\beta)\zeta(s+\gamma)\zeta(s)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}. \tag{10}$$

From (8) and Lemma 2.2 we have

$$a(lm)=\sum_{\substack{l=l_1l_2\\m=m_1m_2\\l_1m_1\leq y\\(m_2,l_1)=1}}h(l_1m_1)P([l_1m_1]_y)g(l_2m_2).$$

Hence

$$\mathcal{Q}(\beta,\gamma)=\sum_{l_1l_2n\leq y}\frac{h(l_1l_2n)P([l_1l_2n]_y)}{l_1l_2n}\frac{\mu(n)}{\varphi(n)}\sum_{\substack{l_1m_1\leq y\\(m_1,n)=1}}h(l_1m_1)P([l_1m_1]_y)\sum_{\substack{m_2\leq nT/2\pi m_1\\(m_2,l_1n)=1}}g(l_2m_2). \tag{11}$$

Lemma 4.1. *Suppose a and b are coprime, squarefree integers. Then we have*

$$\begin{aligned}
G(x;a,b)\ &:=\ \sum_{\substack{n\leq x\\(n,b)=1}}g(an)\\
&=\ -\frac{x^{1-\beta}}{1-\beta}\sum_{a=a_2a_3}\frac{1}{a_2^{\gamma}}\zeta(1-\beta+\gamma)\zeta(1-\beta)F(b,-\beta+\gamma)F(a_2b,-\beta)\\
&\quad+\frac{x^{1-\gamma}}{1-\gamma}\sum_{a=a_2a_3}\frac{1}{a_2^{\gamma}}\bigg(\frac{\zeta'}{\zeta}(1+\beta-\gamma)+\sum_{p|b}\frac{\log p}{p^{1+\beta-\gamma}-1}\bigg)\zeta(1-\gamma)F(b)F(a_2b,-\gamma)\\
&\quad-\frac{x^{1-\gamma}}{1-\gamma}\sum_{a=pa_2a_3}\frac{1}{p^{\beta}a_2^{\gamma}}\frac{\log p}{1-p^{-(1+\beta-\gamma)}}\zeta(1-\gamma)F(pb)F(pa_2b,-\gamma)\\
&\quad+x\sum_{a=a_2a_3}\frac{1}{a_2^{\gamma}}\bigg(\frac{\zeta'}{\zeta}(1+\beta)+\sum_{p|b}\frac{\log p}{p^{1+\beta}-1}\bigg)\zeta(1+\gamma)F(b,\gamma)F(a_2b)\\
&\quad-x\sum_{a=pa_2a_3}\frac{1}{p^{\beta}a_2^{\gamma}}\frac{\log p}{1-p^{-(1+\beta)}}\zeta(1+\gamma)F(pb,\gamma)F(pa_2b)\\
&\qquad+O_{B,\varepsilon}\big((\log ab)^{1+\varepsilon}x(\log x)^{-B}\big).
\end{aligned}$$

Proof. It is standard that up to an error term of size $O_{B,\varepsilon}\big((\log ab)^{1+\varepsilon}x(\log x)^{-B}\big)$ for any $B>0$, $G(x;a,b)$ is the sum of the residues at $s=1-\beta$, $s=1-\gamma$ and $s=1$ of

$$\frac{x^s}{s}\sum_{(n,b)=1}\frac{g(an)}{n^s}.$$

Combining (10) and Lemma 2.2, the above expression is

$$\frac{x^s}{s}\sum_{a=a_1a_2a_3}\bigg(-\sum_{(n,b)=1}\frac{\Lambda(a_1n)}{(a_1n)^\beta n^s}\bigg)\bigg(\sum_{(n,a_1b)=1}\frac{1}{(a_2n)^\gamma n^s}\bigg)\bigg(\sum_{(n,a_1a_2b)=1}\frac{1}{n^s}\bigg)$$
$$=\frac{x^s}{s}\sum_{a=a_1a_2a_3}\frac{1}{a_1^\beta a_2^\gamma}\bigg(-\sum_{(n,b)=1}\frac{\Lambda(a_1n)}{n^{s+\beta}}\bigg)\zeta(s+\gamma)\zeta(s)F(a_1b,s+\gamma-1)F(a_1a_2b,s-1).$$

We have

$$-\sum_{(n,b)=1}\frac{\Lambda(a_1n)}{n^{s+\beta}}=\begin{cases}\frac{\zeta'}{\zeta}(s+\beta)+\sum_{p|b}\frac{\log p}{p^{s+\beta}-1} & \text{if } a_1=1,\\ -\frac{\log p}{1-p^{-(s+\beta)}} & \text{if } a_1=p,\\ 0 & \text{otherwise.}\end{cases}$$

The result follows. □

In view of the above definition, the innermost sum in (11) is

$$G(nT/2\pi m_1;l_2,l_1n).$$

We then write

$$\mathcal{Q}(\beta,\gamma)=\sum_{j=1}^{6}\mathcal{Q}_j(\beta,\gamma)$$

corresponding to the decomposition of $G(x;a,b)$ in Lemma 4.1.

We begin with $\mathcal{Q}_1(\beta,\gamma)$. Writing l_2l_3 for l_2, and m for m_1, we have $\mathcal{Q}_1(\beta,\gamma)$ equals

$$-\frac{(T/2\pi)^{1-\beta}}{1-\beta}\zeta(1-\beta+\gamma)\zeta(1-\beta)\sum_{\substack{l_1l_2l_3\leq y\\ l_1m\leq y}}\frac{h(l_1l_2l_3)h(l_1m)P([l_1m]_y)}{l_1l_2^{1+\gamma}l_3m^{1-\beta}}F(l_1,-\beta+\gamma)F(l_1l_2,-\beta)$$
$$\sum_{\substack{n\leq y/l_1l_2l_3\\ (n,l_1l_2l_3m)=1}}\frac{\mu(n)h(n)P([l_1l_2l_3n]_y)}{\varphi(n)n^\beta}F(n,-\beta+\gamma)F(n,-\beta).$$

From Lemma 2.6, the innermost sum is

$$U_1V_1(l_1l_2l_3m)\bigg(\frac{P''([l_1l_2l_3]_y)}{(\log y)^2}+\frac{2\beta P'([l_1l_2l_3]_y)}{\log y}+\beta^2P([l_1l_2l_3]_y)\bigg)$$
$$+O(F_{\tau_0}(l_1l_2l_3m)L^{-3})+O_\varepsilon\bigg(F_{\tau_0}(l_1l_2l_3m)\Big(\frac{y}{l_1l_2l_3}\Big)^{-\nu}L^{-2+\varepsilon}\bigg).$$

By Lemma 2.5, the contributions of the O-terms to $\mathcal{Q}_1(\beta,\gamma)$ is $O_\varepsilon(TL^{9+\varepsilon})$. Hence

$$\mathcal{Q}_1(\beta,\gamma)=-U_1(T/2\pi)^{1-\beta}\zeta(1-\beta+\gamma)\zeta(1-\beta)\sum_{l_1l_2\leq y}\frac{F(l_1,-\beta+\gamma)F(l_1l_2,-\beta)}{l_1l_2^{1+\gamma}}$$
$$\bigg(\frac{\mathcal{A}_2(y,P,P'';l_1,l_2,-\beta)}{(\log y)^2}+\frac{2\beta\mathcal{A}_2(y,P,P';l_1,l_2,-\beta)}{\log y}+\beta^2\mathcal{A}_2(y,P,P;l_1,l_2,-\beta)\bigg)$$
$$+O_\varepsilon(TL^{9+\varepsilon}).$$

Using Lemmas 2.7–2.8 we obtain

$$\mathcal{Q}_1(\beta,\gamma) = -\frac{A(T/2\pi)^{1-\beta}(\log y)^{10}}{6}\zeta(1-\beta+\gamma)\zeta(1-\beta)\int_0^1\int_0^x\int_0^x (1-x)^3 y^{\beta t-\gamma t_1} t t_1$$
$$P(x-t)\bigg(\frac{P(x-t_1)}{(\log y)^2}+\frac{2\beta P_0(x-t_1)}{\log y}+\beta^2 P_1(x-t_1)\bigg)dt dt_1 dx + O_\varepsilon(TL^{9+\varepsilon}). \quad (12)$$

Here we have used a fact which is easy to verify that $U_1U_2W = A$.

For $\mathcal{Q}_2(\beta,\gamma)$, we write the sum $\sum_{p|l_1 n}$ as $\sum_{p|l_1}+\sum_{p|n}$, since the function $h(n)$ is supported on square-free integers. In doing so we have $\mathcal{Q}_2(\beta,\gamma)$ equals

$$\frac{(T/2\pi)^{1-\gamma}}{1-\gamma}\zeta(1-\gamma)\sum_{\substack{l_1l_2l_3\le y\\ l_1m\le y}}\frac{h(l_1l_2l_3)h(l_1m)P([l_1m]_y)}{l_1l_2^{1+\gamma}l_3m^{1-\gamma}}\bigg(\frac{\zeta'}{\zeta}(1+\beta-\gamma)+\sum_{p|l_1}\frac{\log p}{p^{1+\beta-\gamma}-1}\bigg)$$
$$F(l_1)F(l_1l_2,-\gamma)\sum_{\substack{n\le y/l_1l_2l_3\\ (n,l_1l_2l_3m)=1}}\frac{\mu(n)h(n)P([l_1l_2l_3n]_y)}{\varphi(n)n^\gamma}F(n)F(n,-\gamma)$$
$$+\frac{(T/2\pi)^{1-\gamma}}{1-\gamma}\zeta(1-\gamma)\sum_{\substack{l_1l_2l_3\le y\\ l_1m\le y}}\frac{h(l_1l_2l_3)h(l_1m)P([l_1m]_y)}{l_1l_2^{1+\gamma}l_3m^{1-\gamma}}F(l_1)F(l_1l_2,-\gamma)$$
$$\sum_{\substack{p|n\\ n\le y/l_1l_2l_3\\ (n,l_1l_2l_3m)=1}}\frac{\log p}{p^{1+\beta-\gamma}-1}\frac{\mu(n)h(n)P([l_1l_2l_3n]_y)}{\varphi(n)n^\gamma}F(n)F(n,-\gamma). \quad (13)$$

We consider the contribution from the terms $\sum_{p|l_1}$. From Lemma 2.6, the sum over n is

$$\ll L^{-2}+F_{\tau_0}(l_1l_2l_3m)L^{-3}+O_\varepsilon\bigg(F_{\tau_0}(l_1l_2l_3m)\Big(\frac{y}{l_1l_2l_3}\Big)^{-\nu}L^{-2+\varepsilon}\bigg).$$

Hence the contribution of the terms $\sum_{p|l_1}$ to $\mathcal{Q}_2(\beta,\gamma)$ is

$$\ll_\varepsilon TL^{-1}\sum_{\substack{p|l_1\\ l_1l_2l_3\le y\\ l_1m\le y}}\frac{\log p}{p-1}\frac{d_4(l_1)d(l_2)d(l_3)d(m)}{l_1l_2l_3m}$$
$$\bigg(1+F_{\tau_0}(l_1l_2l_3m)L^{-1}+F_{\tau_0}(l_1l_2l_3m)\Big(\frac{y}{l_1l_2l_3}\Big)^{-\nu}L^{\varepsilon}\bigg)$$
$$\ll_\varepsilon TL^5\sum_{\substack{p|l_1\\ l_1\le y}}\frac{\log p}{p-1}\frac{d_4(l_1)}{l_1}\big(1+F_{\tau_0}(l_1)L^{-1+\varepsilon}\big)\ll_\varepsilon TL^{9+\varepsilon}.$$

The same argument shows that the last term in (13) is also $O_\varepsilon(TL^{9+\varepsilon})$. The remaining terms are

$$\frac{(T/2\pi)^{1-\gamma}}{1-\gamma}\frac{\zeta'}{\zeta}(1+\beta-\gamma)\zeta(1-\gamma)\sum_{\substack{l_1l_2l_3\le y\\ l_1m\le y}}\frac{h(l_1l_2l_3)h(l_1m)P([l_1m]_y)}{l_1l_2^{1+\gamma}l_3m^{1-\gamma}}$$
$$F(l_1)F(l_1l_2,-\gamma)\sum_{\substack{n\le y/l_1l_2l_3\\ (n,l_1l_2l_3m)=1}}\frac{\mu(n)h(n)P([l_1l_2l_3n]_y)}{\varphi(n)n^\gamma}F(n)F(n,-\gamma).$$

Similarly to $\mathcal{Q}_1(\beta,\gamma)$, we thus obtain

$$\mathcal{Q}_2(\beta,\gamma)=\frac{A(T/2\pi)^{1-\gamma}(\log y)^{10}}{6}\frac{\zeta'}{\zeta}(1+\beta-\gamma)\zeta(1-\gamma)\int_0^1\int_0^x\int_0^x(1-x)^3y^{\gamma(t-t_1)}tt_1$$
$$P(x-t)\bigg(\frac{P(x-t_1)}{(\log y)^2}+\frac{2\gamma P_0(x-t_1)}{\log y}+\gamma^2P_1(x-t_1)\bigg)dtdt_1dx+O_\varepsilon(TL^{9+\varepsilon}).\quad (14)$$

The fourth term $\mathcal{Q}_4(\beta,\gamma)$ is in the same form as $\mathcal{Q}_2(\beta,\gamma)$. The same calculations yield

$$\mathcal{Q}_4(\beta,\gamma)=\frac{A(T/2\pi)(\log y)^8}{6}\frac{\zeta'}{\zeta}(1+\beta)\zeta(1+\gamma)\int_0^1\int_0^x(1-x)^3y^{-\gamma t_1}t_1$$
$$P_1(x)P(x-t_1)dt_1dx+O_\varepsilon(TL^{9+\varepsilon}).\quad (15)$$

To evaluate $\mathcal{Q}_3(\beta,\gamma)$, we rearrange the sums and write $\mathcal{Q}_3(\beta,\gamma)$ in the form

$$-\frac{(T/2\pi)^{1-\gamma}}{1-\gamma}\zeta(1-\gamma)\sum_{\substack{pl_1l_2l_3\leq y\\ l_1m\leq y}}\frac{\log p}{(p^{1+\beta-\gamma}-1)p^\gamma}\frac{h(pl_1l_2l_3)h(l_1m)P([l_1m]_y)}{l_1l_2^{1+\gamma}l_3m^{1-\gamma}}$$
$$F(pl_1)F(pl_1l_2,-\gamma)\sum_{\substack{n\leq y/pl_1l_2l_3\\ (n,pl_1l_2l_3m)=1}}\frac{\mu(n)h(n)P([pl_1l_2l_3n]_y)}{\varphi(n)n^\gamma}F(n)F(n,-\gamma).$$

By Lemma 2.6, the innermost sum is

$$U_1V_1(pl_1l_2l_3m)\bigg(\frac{P''([pl_1l_2l_3]_y)}{(\log y)^2}+\frac{2\gamma P'([pl_1l_2l_3]_y)}{\log y}+\gamma^2P([pl_1l_2l_3]_y)\bigg)$$
$$+O(F_{\tau_0}(pl_1l_2l_3m)L^{-3})+O_\varepsilon\bigg(F_{\tau_0}(pl_1l_2l_3m)\Big(\frac{y}{pl_1l_2l_3}\Big)^{-\nu}L^{-2+\varepsilon}\bigg).$$

The contribution of the O-terms, using Lemma 2.5, is $O_\varepsilon(TL^{9+\varepsilon})$. The remaining terms contribute

$$-\frac{U_1(T/2\pi)^{1-\gamma}}{(1-\gamma)}\zeta(1-\gamma)\sum_{pl_1l_2\leq y}\frac{\log p}{(p^{1+\beta-\gamma}-1)p^\gamma}\frac{F(pl_1)F(pl_1l_2,-\gamma)}{l_1l_2^{1+\gamma}}$$
$$\bigg(\frac{\mathcal{A}_2(y,P,P'';l_1,pl_2,-\gamma)}{(\log y)^2}+\frac{2\gamma\mathcal{A}_2(y,P,P';l_1,pl_2,-\gamma)}{\log y}+\gamma^2\mathcal{A}_2(y,P,P;l_1,pl_2,-\gamma)\bigg).$$

In view of Lemma 2.7, this equals

$$-U_1U_2(T/2\pi)^{1-\gamma}(\log y)^4\zeta(1-\gamma)\sum_{pl_1l_2\leq y}\frac{\log p}{(p^{1+\beta-\gamma}-1)p^\gamma}\frac{h(pl_1l_2)h(l_1)}{l_1l_2^{1+\gamma}}$$
$$F(pl_1)F(pl_1l_2,-\gamma)V_1(pl_1l_2)V_2(l_1)V_3(pl_2)V_4(pl_1l_2)\int_0^{[l_1]_y}y^{\gamma t}tP([l_1]_y-t)$$
$$\bigg(\frac{P([pl_1l_2]_y)}{(\log y)^2}+\frac{2\gamma P_0([pl_1l_2]_y)}{\log y}+\gamma^2P_1([pl_1l_2]_y)\bigg)dt+O(TL^9).$$

From Lemma 2.8(ii) we obtain

$$\mathcal{Q}_3(\beta,\gamma)=-\frac{A(T/2\pi)^{1-\gamma}(\log y)^{11}}{3}\zeta(1-\gamma)\int_0^1\int_{\substack{t,t_j\geq0\\ t\leq x\\ t_1+t_2\leq x}}(1-x)^3y^{\gamma(t-t_1)-\beta t_2}tt_1P(x-t)$$
$$\bigg(\frac{P(x-t_1-t_2)}{(\log y)^2}+\frac{2\gamma P_0(x-t_1-t_2)}{\log y}+\gamma^2P_1(x-t_1-t_2)\bigg)dtdt_1dt_2dx+O_\varepsilon(TL^{9+\varepsilon}).(16)$$

The term $\mathcal{Q}_5(\beta,\gamma)$ is in the same form as $\mathcal{Q}_3(\beta,\gamma)$. The same calculations give

$$\mathcal{Q}_5(\beta,\gamma) = -\frac{A(T/2\pi)(\log y)^9}{3}\zeta(1+\gamma)\int_0^1\int_{\substack{t_j\geq 0\\ t_1+t_2\leq x}}(1-x)^3y^{-\gamma t_1-\beta t_2}t_1 P_1(x)P(x-t_1-t_2)dt_1dt_2dx+O_\varepsilon(TL^{9+\varepsilon}). \quad (17)$$

Finally, we have $\mathcal{Q}_6(\beta,\gamma)=O_B(TL^{-B})$ for any $B>0$.

Collecting the estimates (6), (7), (12), (14)–(17), and letting $\beta=-i\alpha$, $\gamma\to 0$ we easily obtain Lemma 1.2.

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